Simulation of Traffic Flow and Congestion on a Circular Road

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Problems and Equation: Problem Definition

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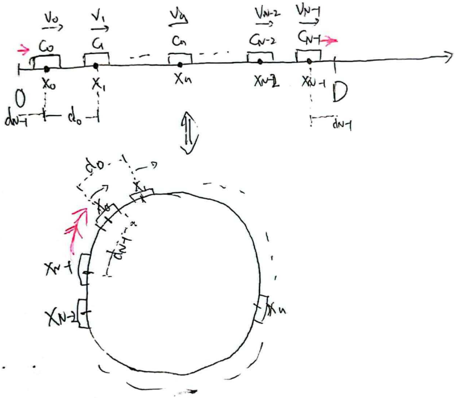

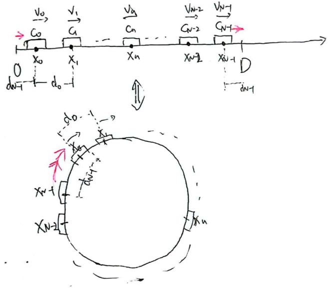

- x_n: Location of vehicle c_n
- v_n: Velocity of vehicle c_n
- d_n: Distance from vehicle c_n to vehicle c_{n+1}

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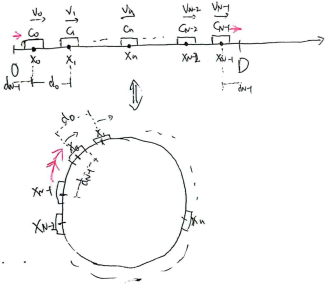

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Let $x_N = x_0$, then

$$d_n = (x_{i+1} - x_i) \bmod D$$

$$
\begin{cases}
\frac{d}{dt}x_n(t) = v_n(t) \\
\frac{d}{dt}v_n(t) = \frac{1}{\tau}(ov(d_n) - v_n(t)) \\
x_n(0) = x_n^{(0)} \\
v_n(0) = 0 & n = 0, ..., N-1
\end{cases}
$$

where we define a distance-based optimal velocity $ov(d)$

$$
ov(d) = \begin{cases}
0 & \text{if } d \leq d_{min} \\
v_{max} * \frac{log(d/d_{min})}{log(d_{max}/d_{min})}) & \text{if } d_{min} < d < d_{max} \\
v_{max} & \text{if } d \geq d_{max}
\end{cases}
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We prefer d_n over x_n because:

1. Easier calculation, ov is a function of d_n
2. d_n has more interesting equilibrium (equilibrium under the formulation using x_n requires $v_n = 0$)

Problems and Equation: Meaning of the Equation

Drivers will adjust the velocity of their vehicles so that they can approximate the optimal velocity $ov(d_n)$.

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Using the method of integral factor, we can find the solution

$$v(t) = e^{-\frac{1}{\tau}t} \int_0^t \frac{1}{\tau} e^{\frac{1}{\tau}s} ov(s)\ ds$$

Let $K_\tau(s) = \frac{1}{\tau}e^{-\frac{1}{\tau}s}$ be a family of kernels parametrized by τ, then we can rewrite the solution as a convolution between K and ov.

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$$\begin{aligned} v(t) &= e^{-\frac{1}{\tau}t}\int_0^t \frac{1}{\tau}e^{\frac{1}{\tau}s} ov(s)\ ds \\ &= \int_0^t \frac{1}{\tau}e^{-\frac{1}{\tau}(t-s)} ov(s)\ ds \\ &= \int_0^t K_\tau(t-s) ov(s)\ ds \\ &= (K_\tau * ov)(t) \end{aligned}$$

$\lim_{\tau \to 0} K_\tau(s) = \delta(s)$ where $\delta(s)$ is the Dirac delta function

$$\delta(s) = \begin{cases} +\infty & s = 0 \\ \\ 0 & s \neq 0 \end{cases}$$

Demo: `https://www.geogebra.org/calculator/manp7fbc`

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Problems and Equation: Equilibrium of the System

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Theorem

The system reaches equilibrium if and only if all vehicles are of equal, constant distance $d_i = d_e = D/N$ *and equal, constant velocity* $v_i = v_e = ov(d_e)$ *for* $i = 0, 1, ..., N-1$

Problems and Equation: Equilibrium of the System

The system reaches an equilibrium if and only if

$$\begin{cases} \frac{d}{dt} d_n(t) = v_{n+1}(t) - v_n(t) = 0 \\ \\ \frac{d}{dt} v_n(t) = \frac{1}{\tau}(ov(d_n) - v_n(t)) = 0 \end{cases}$$

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Assume $D/N \geq d_{min}$, we have $d_n(t) = d_e = D/N$

$$\begin{cases} d_n(t) = d_e = D/N \\ \\ v_n(t) = v_e = ov(d_e) \end{cases}$$

Numerical Methods: Discretization of the System

Let Δt be the time step, T be the duration of simulation. Then, we have

$$\begin{cases} x_n(t+\Delta t) = x_n(t) + (v_{n+1} - v_n(t)) \cdot \Delta t \\ v_n(t+\Delta t) = \frac{(\Delta t \cdot ov(d_n) + \tau \cdot v_n(t))}{\Delta t + \tau} \\ x_n(0) = x_n^{(0)} \\ v_n(0) = 0 \qquad\qquad n = 0, ..., N-1 \end{cases}$$

Numerical Methods: Discretization of the System

The discretization of v_n is obtained by the backward-Euler method:

$$\frac{v_n(t+\Delta t) - v_n(t)}{\Delta t} = \frac{ov(d_n) - v(t+\Delta t)}{\tau}$$

$$v_n(t+\Delta t) = \frac{(\Delta t \cdot ov(d_n) + \tau \cdot v_n(t))}{\Delta t + \tau}$$

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Numerical Methods: Verification

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We test for $\Delta t = 0.0025, 0.005, 0.01, 0.025, 0.05$, and compare the results to that of $\Delta t = 0.0025$ (the most accurate result)

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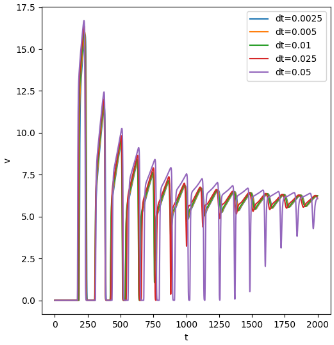

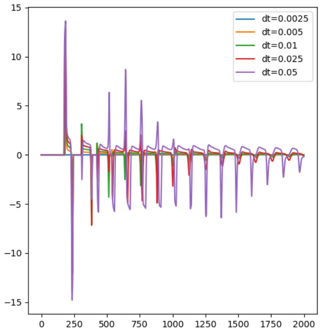

Experiment 1: Stability of the Equilibrium

Let $\vec{y} = (\vec{d}, \vec{v})$. We can rewrite the system (represented by d and v) as

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$$\begin{cases} \frac{d}{dt}\vec{y} = f(\vec{y}) \\ \\ \vec{y}(0) = \vec{y_0} = (\vec{d_0}, \vec{v_0}) \end{cases}$$

where f is a nonlinear function.

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We can study the stability of the equilibrium by computing the Jacobian of f evaluated at $\vec{y_e} = (\vec{d_e}, \vec{v_e})$

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$$Df(y_e) = J = \left[\begin{array}{ccc|c} & 0_n & & M \\ \hline \frac{c}{d_0} & & 0 & \\ & \dots & & -\frac{1}{\tau} I_n \\ 0 & & \frac{c}{d_{N-1}} & \end{array}\right]$$

Experiment 1: Stability of the Equilibrium

where

$$c = \frac{v_{max}}{\tau(log(d_{max}/d_{min}))}$$

$$M = (m_{ij}), m_{i,i} = -1,\ m_{i+1\%N,i} = 1,\ \text{otherwise}\ m_{ij} = 0$$

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For $N = 4$, we have

$$M = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

$$\vec{d}'(t) = M\vec{v}(t) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 - v_0 \\ v_2 - v_1 \\ v_3 - v_2 \\ v_0 - v_3 \end{bmatrix}$$

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- If J has an eigenvalue whose real part is positive, the equilibrium is unstable.

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- If J has an eigenvalue whose real part is positive, the equilibrium is unstable.

We can numerically calculate the eigenvalues of J for different τ.

https://colab.research.google.com/drive/1gD9n_8F9nWiFnkkdR28wrpgU5O6NUqWl?usp=sharing

Experiment 1: Stability of the Equilibrium

To check the asymptoticity of the equilibrium, we can initialize the system with small perturbation from equilibrium

- $v_n = 0$
- $d_n = d_e + \epsilon_n$ where $\epsilon_n \sim Unif[-P, P]$ for some $P \geq 0$

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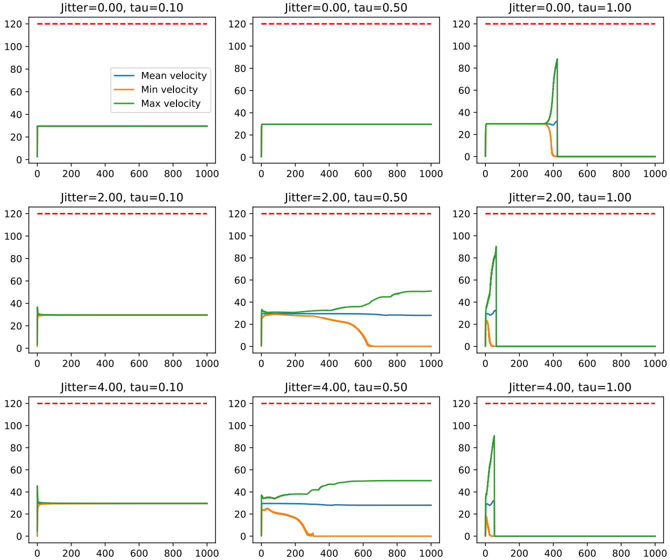
- $d_n = d_e + \epsilon_n$ where $\epsilon_n \sim Unif[-P, P]$ for some $P \geq 0$

We experiments with $P = 0, 2, 4$ and $\tau = 0.1, 0.5, 1.0$


Jitter=0.00, tau=0.10
Jitter=0.00, tau=0.50
Jitter=0.00, tau=1.00
Jitter=2.00, tau=0.10
Jitter=2.00, tau=0.50
Jitter=2.00, tau=1.00
Jitter=4.00, tau=0.10
Jitter=4.00, tau=0.50
Jitter=4.00, tau=1.00
Mean velocity
Min velocity
Max velocity
0
20
40
60
80
100
120
0
200
400
600
800
1000

Experiment 2: Traffic Snake

Traffic snake: A queue of vehicles that are stuck in traffic congestion.

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Numerically: A queue of vehicles whose velocity is less than a small number.

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F2 The traffic snake will grow / shrink / stay still depending on τ, d_{min}, d_{max}, v_{max}.

F3 Depending on the parameters, the traffic snake will either be fully alleviated or stay at a constant length and moves backwards at a constant velocity.

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e.g. 1
`https://xinyu-li-123.github.io/videos/reach_equi_vid.mp4`

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https://xinyu-li-123.github.io/videos/reach_equi_vid.mp4

e.g. 2
https://xinyu-li-123.github.io/videos/not_reach_equi_vid.mp4

F1. The traffic snake moves against the traffic flow

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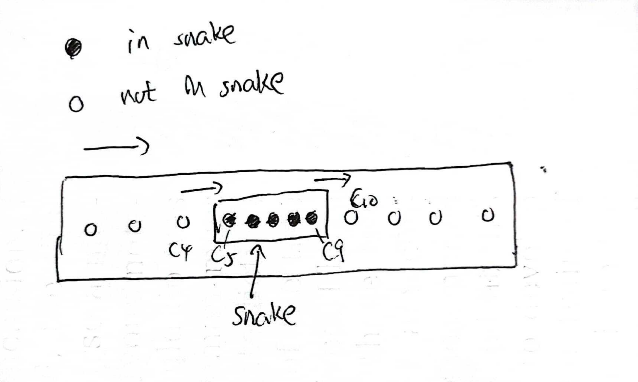

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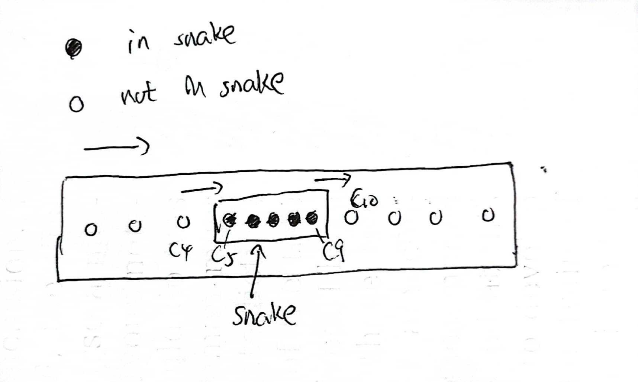

After some time steps, c_9 will leave the queue and c_4 will join the queue. This will move the queue against the direction of the traffic flow.

F2. The traffic snake will grow / shrink / stay still depending on τ, d_{min}, d_{max}, v_{max}.

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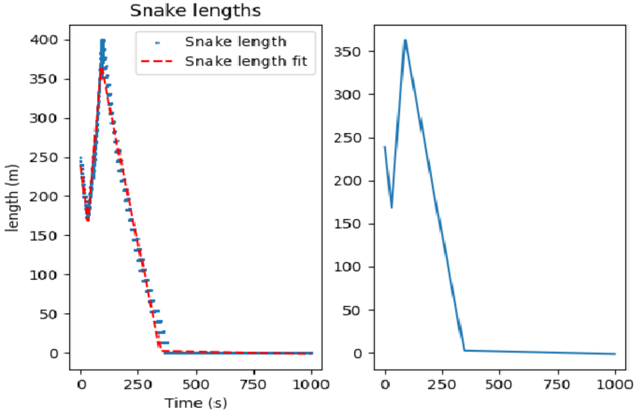

Figure: Length of traffic snake in e.g.1

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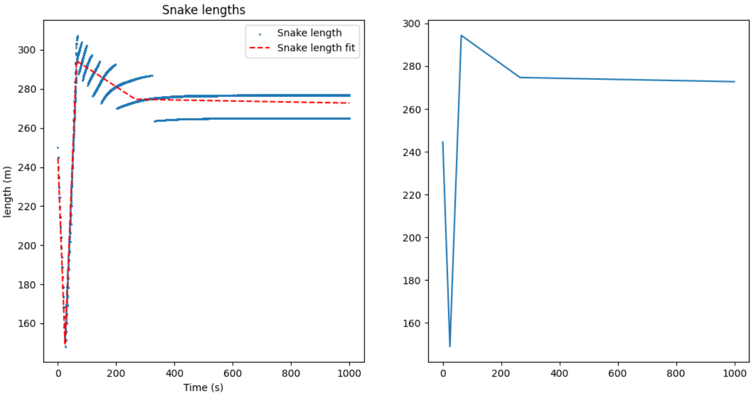

Figure: Length of traffic snake in e.g.2

Experiment 2: Traffic Snake

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d_{min}	d_{max}	v_{max}	reach equilibrium b/f 1000s?
0.2+2l	100+3l	120	True
0.2+3l	/	/	False
/	50+3l	/	False
/	/	150	False
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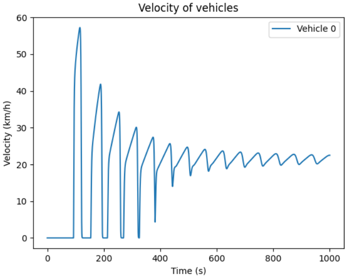
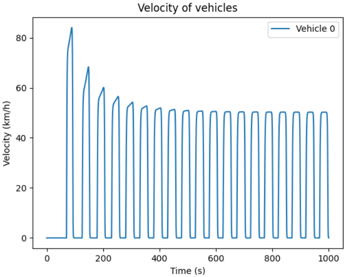
These parameters will affect the derivative of ov at d_e. This demo gives an example `https://www.geogebra.org/calculator/xbd9shvh`

Experiment 2: Traffic Snake

Example of both cases.



(a) $v0 - t$ relation in e.g.1



(b) $v0 - t$ relation in e.g.2