Last Time:

- Iterative Learning Control

Today:

- Stochastic Optimal Control

Stochastic Control:

- So far we have assumed we know the system's state perfectly
- What happens when all we have are noisy measurements of quantities related to the state?

$$\underset{\text{measurements}}{y} = \underbrace{g(x)}_{\text{"measurement model"}}$$

Deterministic		Stochastic
x	$\longrightarrow$	$\underbrace{p(x \mid y)}_{\text{PDF of the state conditioned on the measurements}}$

* Stochastic Optimal Control Problem

$$\min_u E[J(x,u)]$$

- In principle, can solve with DP

- Very hard in general

* LQG

  - Special case we can solve in closed form

    Linear Dynamics

    Quadratic Cost

    Gaussian Noise

- Dynamics

$$x_{n+1} = Ax_n + Bu_n + \underbrace{w_n}_{\text{"Process Noise"}}$$

$$y_n = Cx_n + \underbrace{v_n}_{\text{"measurement noise"}}$$

$$w_n \underset{\text{"drawn from"}}{\sim} \underset{\text{Normal Distribution (Gaussian)}}{N}(0, W) \qquad v_n \sim N(\underset{\text{mean}}{0}, \underset{\text{Covariance}}{V})$$

* Multivariate Gaussian

$$p(x) = \frac{1}{\sqrt{(2\pi)^n \det(S)}} \exp\left(-\tfrac{1}{2}(x-\mu)^T S^{-1}(x-\mu)\right)$$

$$\text{mean: } \quad \mu = \hat{x} = E[x] \in \mathbb{R}^n$$

$$\text{covariance: } \quad S = E[(x-\mu)(x-\mu)^T] \in S^n_{++}$$

$$E[f(x)] = \int_{(\text{All space})} f(x)\, p(x)\, dx$$

$$\text{"Uncorrelated"} \Rightarrow E[(x-\hat{x})(y-\hat{y})^T] = 0$$

- Cost Function

$$J = E\left[\tfrac{1}{2}x_N^T Q x_N + \tfrac{1}{2}\sum_{n=1}^{N-1}(x_n^T Q x_n + u_n^T R u_n)\right]$$

- D.P. Recursion

$$V_N(x) = E[x_N^T Q x_N] = E[x_N^T P_N x_N]$$

$$V_{N-1}(x) = \min_u E\Big[\tfrac{1}{2}x_{N-1}^T Q x_{N-1} + \tfrac{1}{2}u_{N-1}^T R u_{N-1} + \tfrac{1}{2}(Ax_{N-1} + Bu_{N-1} + w_{N-1})^T P_N (Ax_{N-1} + Bu_{N-1} + w_{N-1})\Big]$$

$$= \min_u \underbrace{E\left[\tfrac{1}{2}x_{N-1}^T Q x_{N-1} + \tfrac{1}{2}u_{N-1}^T R u_{N-1} + (Ax_{N-1} + Bu_{N-1})^T P_N (Ax_{N-1} + Bu_{N-1})\right]}_{\text{Standard LQR}}$$

$$+ \underbrace{E\Big[\cancelto{0}{(Ax_{N-1} + Bu_{N-1})^T P_N w_{N-1}} + \cancelto{0}{w_{N-1}^T P_N (Ax_{N-1} + Bu_{N-1})} + \underbrace{w_{N-1}^T P_N w_{N-1}}_{\text{Constant!}}\Big]}_{\text{Noise Terms}}$$

* Noise sample drawn at time k has nothing to do with state (or control) at time k. x_k depends on w_{k-1} (and all past w) but not on w_k or future w.

  ⇒ Uncorrelated ⇒ cross-correlation is zero

⇒ Noise terms have no impact on the controller design! (you just get a higher cost).

* "Certainty-Equivalence Principle"

  - The optimal LQG controller is just LQR with x replaced by $E[x]$

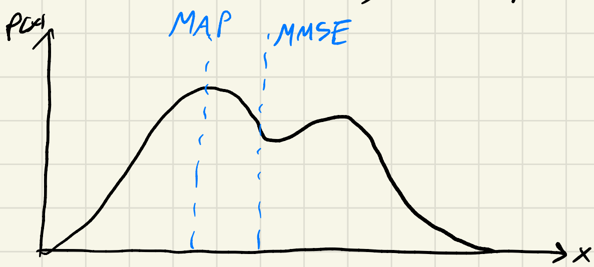
* "Separation Principle"

  - For LQG we can design an optimal feedback controller and an optimal estimator seperately and then hook them together. The resulting feedback policy is optimal.

* Neither of these holds in general, but are still frequently used in practice to design sub-optimal policies.

* Optimal State Estimation:

- What should I try to optimize?



- Maximum a-posteriori (MAP):

$$\arg\max \; p(x|y)$$

- Minimum mean-squared error (MMSE): ← "Least squares"

$$\arg\min \; E\left[(x-\hat{x})^T(x-\hat{x})\right]$$

- These are the same for a Gaussian!