Last Time:

- Stochastic Optimal Control
- LQG

Today:

- Optimal Estimation
- Finish LQG
- Duality

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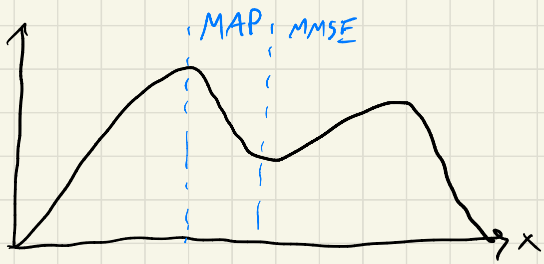
* From Last Time:

  - "Certainty Equivalence"
  - "Separation Principle"
  - Frequently applied to nonlinear systems in practice

* Optimal State Estimation

  - What should I optimize



- Maximum a-posteriori (MAP)

$$\operatorname{argmax} \underbrace{p(x|y)}_{\text{probability of state given measurements}}$$

- Minimum mean squared error (MMSE)

$$\underset{\hat{x}}{\operatorname{argmin}} \; \underbrace{E[(x-\hat{x})^T(x-\hat{x})]}$$ "least squares" "minimum variance"

$$E[tr((x-\hat{x})^T(x-\hat{x}))] = E[tr((x-\hat{x})(x-\hat{x})^T)]$$

$$= tr(E[(x-\hat{x})(x-\hat{x})]) = tr(\Sigma)$$

- These are the same for a Gaussian!

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* Kalman Filter

  - Recursive linear MMSE estimator

  - Assume an estimate of the state that includes all measurements up to the current time:

$$\hat{x}_{k|k} = E[x_k \mid y_{1:k}]$$

  - Assume we also know the error covariance:

$$\Sigma_{k|k} = E[(x_k-\hat{x}_{k|k})(x_k-\hat{x}_{k|k})^T]$$

  - We want to update $\hat{x}$ and Σ to include a new measurement at t_{k+1}

  - The KF can be broken into 2 steps

Prediction:

$$\hat{x}_{n+1|k} = E[Ax_n + Bu_n + w_n \mid y_{1:k}]$$
$$= A\hat{x}_{n|n} + Bu_n$$

$$\Sigma_{n+1|n} = E[(x_{n+1} - \hat{x}_{n+1|n})(\cdots)^T]$$
$$= E[(Ax_n + \cancel{Bu_n} + w_n - A\hat{x}_{n|k} - \cancel{Bu_n})(\cdots)^T]$$
$$= A\underbrace{E[(x_n - \hat{x}_{n|k})(\cdots)^T]}_{\Sigma_{n|k}}A^T + \underbrace{E[w_n w_n^T]}_{W}$$
$$= A\Sigma_{n|k}A^T + W$$

(x_n and w_n are uncorrelated)

Measurement Update:

- Define "innovation"

$$z_{n+1} = y_{n+1} - C\hat{x}_{n+1|k}$$
$$= Cx_{n+1} + v_{n+1} - C\hat{x}_{n+1|n}$$

- Innovation Covariance

$$S_{n+1} = E[z_{n+1} z_{n+1}^T]$$
$$= E[(Cx_{n+1} + v_{n+1} - C\hat{x}_{n+1|n})(\cdots)^T]$$

* v_{n+1} and x_{n+1} are uncorrelated

$$\Rightarrow \quad S_{n+1} = C \underbrace{E[(x_{n+1} - \hat{x}_{n+1|n})(\cdots)^T]}_{\Sigma_{n+1|k}} C^T + \underbrace{E[V_{n+1} V_{n+1}^T]}_{V}$$

$$= C \Sigma_{n+1|n} C^T + V$$

- Innovation is the error signal we feed back into the estimator.

- State Update :

$$\hat{x}_{n+1|n+1} = \hat{x}_{n+1|n} + \underbrace{L_{n+1}}_{\text{“Kalman Gain”}} z_{n+1}$$

- Covariance Update :

$$\Sigma_{n+1|n+1} = E[(x_{n+1} - \hat{x}_{n+1})(\cdots)^T]$$

$$= E[(x_{n+1} - \hat{x}_{n+1|n} - L_{n+1}((x_{n+1} + V_{n+1} - C\hat{x}_{n+1|n}))(\cdots)^T]$$

* V_{n+1} and x_{n+1} are uncorrelated

$$= \underbrace{(I - L_{n+1}C)\Sigma_{n+1|n}(I - L_{n+1}C)^T + L_{n+1} V L_{n+1}^T}_{\text{“Joseph Form”}}$$

- Kalman Gain

$$\text{MMSE} \Rightarrow \text{minimize } E[(x_{n+1} - \hat{x}_{n+1|n+1})^T(\cdots)] = tr(\Sigma_{n+1|n+1})$$

$$\Rightarrow \quad \text{set} \quad \frac{\partial \, tr(\Sigma_{n+1|n+1})}{\partial L_{n+1}} = 0 \quad (\text{and solve for } L_{n+1})$$

$$\Rightarrow \boxed{L_{n+1} = \Sigma_{n+1|n} C^T S_{n+1}^{-1}}$$

* KF Algorithm Summary:

1) Start with $\hat{x}_{0|0}$, $\Sigma_{0|0}$, W, V

2) Predict:

$$\hat{x}_{n+1|n} = A\hat{x}_{n|n} + Bu_n$$

$$\Sigma_{n+1|n} = A\Sigma_{n|n}A^T + W$$

3) Calculate Innovation + Covariance:

$$z_{n+1} = y_{n+1} - C\hat{x}_{n+1|n}$$

$$S_{n+1} = C\Sigma_{n+1|n}C^T + V$$

4) Calculate Kalman Gain:

$$L_{n+1} = \Sigma_{n+1|n} C^T S_{n+1}^{-1}$$

5) Update:

$$\hat{x}_{n+1|n+1} = \hat{x}_{n+1|n} + L_{n+1} z_{n+1}$$

$$\Sigma_{n+1|n+1} = (I - L_{n+1}C)\Sigma_{n+1|n}(I - L_{n+1}C)^T + L_{n+1} V L_{n+1}^T$$

6) GoTo 2

* How do we apply we apply this to nonlinear systems?

  - Extended KF: Linearize about $\hat{x}$ and proceed as in standandard KF

  - Many other generalizations

* Duality + Trajectory Optimization

  - MMSE estimation problem is equivalent to the following optimal control problem:

$$\min_{\substack{x_{1:N} \\ w_{1:N-1}}} \sum_{n=1}^{N-1} \underbrace{\tfrac{1}{2}(y_n - \overbrace{g(x_n)}^{\text{measurement model}})^T V^{-1}(y_n - g(x_n))}_{\text{state cost}} + \underbrace{\tfrac{1}{2} w_n^T W^{-1} w_n}_{\text{"control cost"}}$$

$$\text{s.t. } x_{n+1} = f(x_n) + \underbrace{w_n}_{\text{"controls"}}$$

  - If $f(x) = Ax$ and $g(x) = Cx$, this is an LQR problem