Last Time:

- How to drive
- Dynamic Games

Today:

- Calculus of Variations
- Physics as Trajectory Optimization
- Lagrange, Hamilton, and Pontryagin

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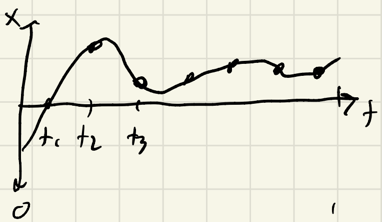
* Trajectories as Infinite Dimensional Vectors

- If we sample a trajectory $x(t)$ on the interval $0-1$:



$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- We recover the continuous $x(t)$ in the limit $n \to \infty$

* Differentiating w.r.t. Functions

- Given a function $f(x): \mathbb{R}^n \to \mathbb{R}$, e.g. $f(x) = \frac{1}{n} x^T x$
- Derivative is the best linear approximation of f.

$$f(x + \Delta x) = \frac{1}{n}(x + \Delta x)^T (x + \Delta x) = \frac{1}{n}(x^T x + x^T \Delta x + \Delta x^T x + \cancel{\Delta x^T \Delta x})$$

$$\approx \underbrace{\frac{1}{n} x^T x}_{f(x)} + \underbrace{\frac{1}{n}(x^T \Delta x + \Delta x^T x)}_{\frac{\partial f}{\partial x}(\Delta x)}$$

- We can write this in components:

$$f(x) = \frac{1}{n}\sum_{n=1}^{n} x_n^2 \Rightarrow f(x+\Delta x) = \frac{1}{n}\sum (x_n + \Delta x_n)^2$$

$$= \frac{1}{n}\sum_{n=1}^{n} x_n^2 + 2x_n \Delta x_n + \cancel{\Delta x_n^2}^{\,0}$$

$$\approx \underbrace{\frac{1}{n}\sum_{n=1}^{n} x_n^2}_{f(x)} + \underbrace{\frac{1}{n}\sum_{n=1}^{n} 2x_n \Delta x_n}_{\frac{\partial f}{\partial x}(\Delta x)}$$

- Now we can take the limit $n \to \infty$

$$F(x(t)) = \int_0^1 x(t)^2 \, dt$$

$$F(x(t) + \Delta x(t)) = \int_0^1 (x(t) + \Delta x(t))^2 \, dt$$

$$= \int_0^1 x(t)^2 + 2x(t)\Delta x(t) + \cancel{\Delta x(t)^2}^{\,0} \, dt$$

$$\approx \underbrace{\int_0^1 x(t)^2 \, dt}_{F(x(t))} + \underbrace{\int_0^1 2x(t)\Delta x(t) \, dt}_{\frac{\partial F}{\partial x}(\Delta x(t))}$$

- In the standard notation of calculus of variations:

$$\underbrace{\delta F}_{\text{"variation of } F\text{"}} \approx \int_0^1 2x(t) \underbrace{\delta x(t)}_{\text{"variation of } x\text{"}} \, dt$$

- It is not common to write $\frac{\delta F}{\delta x}$

* The Least-Action Principle:

- Much of physics can be formulated as optimization
- For rigid-body systems (e.g. robots), we can simulate dynamics by minimizing:

$$\underbrace{S(\overset{\text{Pose trajectory}}{q(t)})}_{\text{"Action"}} = \int_{t_0}^{t_f} \underbrace{\frac{1}{2}\dot{q}(t)^T \overset{\text{mass matrix}}{M(q(t))}\dot{q}(t)}_{\text{Kinetic Energy}} - \underbrace{V(q(t))}_{\text{Potential energy}}\, dt$$

- For simplicity, we'll assume $M(q) = M$ (constant)
- Let's introduce velocity explicitly:

$$\min_{\substack{q(t)\\ v(t)}} S(q(t), v(t)) = \int_{t_0}^{t_f} \frac{1}{2} v(t)^T M v(t) - V(q(t))\, dt$$

$$\text{s.t. } \dot{q} = v$$

- This is now a standard optimal control problem with:

$$x = q, \quad u = v, \quad \dot{x} = f(x,u) = u$$

$$\min_{\substack{x(t)\\ u(t)}} J(x(t), u(t)) = \int_{t_0}^{t_f} \ell(x(t), u(t))\, dt$$

$$\text{s.t. } \dot{x} = f(x,u)$$

- Let's set gradients w.r.t. v and $q = 0$ to find a minimum:

$$S(q, v+\delta v) = \int_{t_0}^{t_f} \frac{1}{2}(v+\delta v)^T M (v+\delta v) - V(q) + \lambda^T(\dot{q} - v - \delta v)\, dt$$

$$\Rightarrow \delta S = \int_{t_0}^{t_f} v^T M \delta v - \lambda^T \delta v \, dt = 0$$

$$= \int_{t_0}^{t_f} (Mv - \lambda)^T \delta v \, dt = 0$$

- Must be true for all possible $\delta v(t) \Rightarrow \boxed{Mv - \lambda = 0}$

$$S(q+\delta q, v) = \int_{t_0}^{t_f} \frac{1}{2} v^T M v - V(q+\delta q) + \lambda^T(\dot{q} + \delta\dot{q} - v)\, dt$$

$$\Rightarrow \delta S = \int_{t_0}^{t_f} -\frac{\partial V}{\partial q}\delta q + \lambda^T \delta \dot{q} \, dt$$

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* Integration by Parts:

$$\frac{d}{dt}(x(t)\,y(t)) = \dot{x}y + x\dot{y}$$

$$\Rightarrow \quad x(t)\,y(t)\Big|_{t_0}^{t_f} = \int_{t_0}^{t_f} \dot{x} y \, dt + \int_{t_0}^{t_f} x \dot{y}\, dt$$

$$\Rightarrow \delta S = \int_{t_0}^{t_f} -\frac{\partial V}{\partial q}\delta q - \dot{\lambda}^T \delta q \, dt$$

$$= \int_{t_0}^{t_f} -(\nabla V(q(t)) + \dot{\lambda}(t))^T \delta q(t) \, dt = 0$$

- Must be true for all possible $\delta q(t) \Rightarrow \boxed{\nabla V + \dot{\lambda} = 0}$

* In Summary:

$$\boxed{\begin{aligned} \lambda &= Mv \\ \dot{\lambda} &= -\nabla V \\ \dot{q} &= v \end{aligned}}$$

- This is Pontryagin's minimum principle
- The Lagrange multiplier is momentum (p)
- If we eliminate v and λ, we get Euler-Lagrange (manipulator) equation:

$$\boxed{M\ddot{q} = -\nabla V(q)}$$

- If we eliminate v but keep λ we get Hamilton's equations:

$$\boxed{\begin{aligned} \lambda &= M\dot{q} \\ \dot{\lambda} &= -\nabla V(q) \end{aligned}}$$

- If we plug this back into the least-action problem:

$$v = M^{-1}\lambda \Rightarrow S = \int_{t_0}^{t_f} \underbrace{-\frac{1}{2}\lambda^T M^{-1}\lambda - V(q) + \lambda^T \dot{q}}_{\text{"Hamiltonian"}} \, dt$$

$$\min_{\substack{q(t) \\ \lambda(t)}} \int_{t_0}^{t_f} H(q,\lambda)\, dt$$

* Dynamics with Contact

  - Impacts and friction are non-smooth

  - Many simulators use "soft" contact models (e.g. MuJoCo) that approximate impact with nonlinear spring/damper

  - A straight-forward way to capture impacts is with a "constrained least-Action" problem:

$$\min_{\substack{q(t)\\ v(t)}} \sum_{n=1}^{N-1} h\tfrac{1}{2}v_n^T M v_n - hV(q_n)$$

$$\text{s.t. } q_{n+1} = q_n + h v_n$$

$$\phi(q_n) \geq 0$$ } "signed distance function"

- Basically collocation applied to least-action

- Coulomb friction (stick-slip) can be modeled by the "Maximum Dissipation Principle"

$$\min_{b} \dot{T}$$

$$\text{s.t. } \|b\| \leq \mu n$$

- To account for both, we solve both optimization problems simultaneously like a Nash equilibrium.