Last Time:

- Infinite-Horizon LQR
- Controllability
- Dynamic Programming

Today:

- Convexity Background
- Convex MPC

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* Convex Model-Predictive Control

- LQR is very powerful but we often need to reason about constraints
- Often these are simple (e.g. torque limits)
- Constraints break the Riccati solution, but we can still solve the QP online.
- Convex MPC has gotten popular as computers have gotten faster

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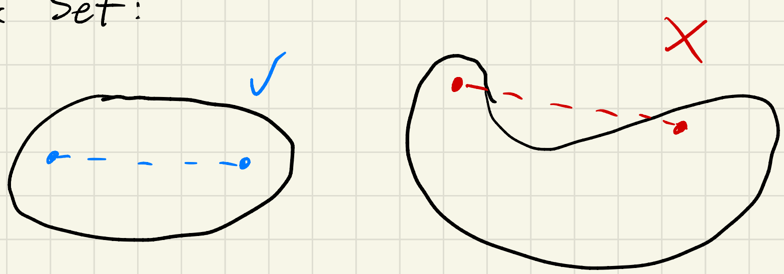
* Background: Convexity

- Convex Set:



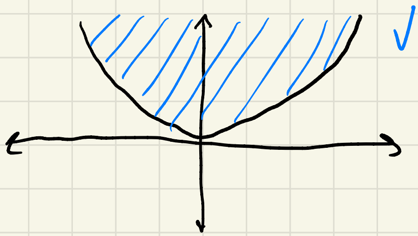
- A line connecting any 2 points in the set is also contained in the set.

- Standard Examples:

  - Linear subspace $(Ax = b)$
  - Half space / Box / Polytopes $(Ax \leq b)$
  - Ellipsoids $(x^T P x \leq 1,\ P \succ 0)$
  - Cones $\underbrace{(\|x_{2:n}\|_2 \leq x_1)}_{\text{"second-order" cone (standard icecream cone)}}$

- Convex Function

  - A convex $f(x) : \mathbb{R}^n \to \mathbb{R}$ who's epigraph is a convex set:



- Standard examples:

  - Linear $f(x) = c^T x$
  - Quadratic $f(x) = \frac{1}{2} x^T Q x + q^T x\ ,\quad Q \succeq 0$
  - Norms $f(x) = |x|$ ← any norm

- Convex optimization problem: minimize a convex function over a convex set

- Standard Examples:
  - Linear Program (LP): Linear $f(x)$, linear $C(x)$
  - Quadratic Program (QP): Quadratic $f(x)$, "
  - Quadratically Constrained QP (QCQP): " , ellipsoid $C(x)$
  - Second-order Cone Program (SOCP): linear $f(x)$, cone $C(x)$

* Convex optimization problems don't have any spurious local optima that satisfy KKT

$\Rightarrow$ If you find a local KKT solution, you have <u>The</u> answer.

- Practically, Newton's method converges really fast and reliably (5~10 iterations max).

$\Rightarrow$ Can bound solution time for real-time control.

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* Convex MPC

- Think "constrained LQR"
- Remember from DP, if we have a cost-to-go function $V(x)$, we can u by solving a one-step problem:

$$u_n = \underset{u}{\operatorname{argmin}} \; l(x,u) + V_{n+1}(f(x_n,u))$$

$$= \underset{u}{\operatorname{argmin}} \; \tfrac{1}{2}u^T R u + (Ax_n + Bu_n)^T P_{n+1} (Ax_n + Bu_n)$$

- We can add constraints on u to this one-step problem but this will perform poorly because $V(x)$ was computed without constraints.

- There's no reason I can't add more steps to the one-step problem:

$$\min_{\substack{x_{1:H} \\ u_{1:H-1}}} \sum_{n=1}^{H-1} \frac{1}{2} x_n^T Q x_n + \frac{1}{2} u_n^T R u_n + \underbrace{x_H^T P_H x_H}_{\text{LQR cost-to-go}}$$

- $H \ll N$ is called "Horizon"

- With no additional constraints, MPC ("receding-horizon") exactly matches LQR for any H

- Intuition: explicit constrained optimization over first H steps gets the state close enough to the reference that the constraints are no longer active and the LQR cost-to-go is valid farther into the future.

- In General:

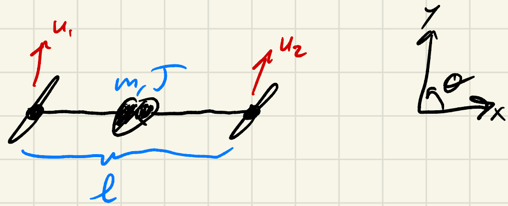
  - A good approximation of $V(x)$ is important for good performance

  - Better $V(x) \Rightarrow$ shorter horizon

  - Longer $H \Rightarrow$ less reliance on $V(x)$

* Example:

- Planar Quadrotor



$$m\ddot{x} = -(u_1 + u_2)\sin(\theta)$$
$$m\ddot{y} = (u_1 + u_2)\cos(\theta) - mg$$
$$J\ddot{\theta} = \frac{1}{2}\ell(u_2 - u_1)$$

- Linearize about hover:

$$\Rightarrow \quad u_1 = u_2 = \frac{1}{2}mg$$

$$\Rightarrow \begin{cases} \Delta\ddot{x} \approx -g\Delta\theta \\ \Delta\ddot{y} \approx \frac{1}{m}(\Delta u_1 + \Delta u_2) \\ \Delta\ddot{\theta} \approx \frac{1}{J}\frac{\ell}{2}(\Delta u_2 - \Delta u_1) \end{cases}$$

$$\underbrace{\begin{bmatrix} \Delta\dot{x} \\ \Delta\dot{y} \\ \Delta\dot{\theta} \\ \Delta\ddot{x} \\ \Delta\ddot{y} \\ \Delta\ddot{\theta} \end{bmatrix}}_{\dot{x}} = \underbrace{\left[\begin{array}{ccc|c} & 0 & & I \\ \hline 0 & 0 & -g & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{array}\right]}_{A} \underbrace{\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta\theta \\ \Delta\dot{x} \\ \Delta\dot{y} \\ \Delta\dot{\theta} \end{bmatrix}}_{x} + \underbrace{\left[\begin{array}{cc} \multicolumn{2}{c}{0} \\ \hline 0 & 0 \\ 1/m & 1/m \\ -\ell/J & \ell/J \end{array}\right]}_{B} \underbrace{\begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix}}_{u}$$

- MPC Cost Function:

$$J = \sum_{n=1}^{H-1} \frac{1}{2}(x_n - x_{ref})^T Q (x_n - x_{ref}) + \frac{1}{2} \Delta u_n^T R \Delta u_n$$
$$+ \frac{1}{2}(x_H - x_{ref})^T P_H (x_H - x_{ref})$$