Configuration: $\Theta \triangleq (t \;\; q_0 \;\; q_1 \;\; \cdots \;\; q_{23})$ local q_{PB}

Generalized Velocity: $\dot\Theta \triangleq (\dot t \;\; \omega_0 \;\; \omega_1 \;\; \cdots \;\; \omega_{23})$

Generalized Acceleration: $\ddot\Theta \triangleq (\ddot t \;\; \dot\omega_0 \;\; \dot\omega_1 \;\; \cdots \;\; \dot\omega_{23})$

Update: $\Theta \leftarrow \Theta \oplus \dot\Theta \Delta t$ $\quad \begin{cases} t \leftarrow t + \dot t \Delta t \\ q_i \leftarrow q_i \otimes e^{\frac{1}{2}\omega_i \Delta t} \end{cases}$

$\dot\Theta \leftarrow \dot\Theta + \ddot\Theta \Delta t$

Differential Equation: $\dot q_{PB} = \frac{1}{2} q_{PB} \otimes (\omega_{BB} - \omega_{BP})$

$\updownarrow \quad \updownarrow \quad \updownarrow$

$\dot q_i = \frac{1}{2} q_i \otimes \omega_i$

$$\begin{aligned}
\dot q_{PB} &= \dot q^*_{WP} \otimes q_{WB} + q^*_{WP} \otimes \dot q_{WB} \\
&= (q_{WP} \otimes \tfrac{1}{2}\omega_{PP})^* \otimes q_{WB} + q^*_{WP} \otimes (q_{WB} \otimes \tfrac{1}{2}\omega_{BB}) \\
&= -\tfrac{1}{2}\omega_{PP} \otimes q_{PB} + q_{PB} \otimes \tfrac{1}{2}\omega_{BB} \\
&= \tfrac{1}{2} q_{PB} \otimes (\omega_{BB} - q_{BP} \otimes \omega_{PP} \otimes q^*_{BP}) \\
&= \tfrac{1}{2} q_{PB} \otimes (\omega_{BB} - \omega_{BP})
\end{aligned}$$

Velocity kinematics:

assuming $B = \text{parent}(O) \Rightarrow t_{BO} \equiv 0$, $P = \text{parent}(B)$, $\ldots$ $R = \text{root}$

$$\dot t_{WO} = \dot T_{WB} t_{BO} = T_{WB} [\mathcal{V}_{BB}]_\times t_{BO} \triangleq \begin{pmatrix} R_{WB} & t_{WB} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} [\omega_{BB}]_\times & v_{BB} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t_{BO} \\ 1 \end{pmatrix}$$

$$\begin{aligned}
&= R_{WB}[\omega_{BB}]_\times t_{BO} + R_{WB} V_{BB} = -R_{WB}[t_{BO}]_\times \omega_{BB} + \dot t_{WB} \quad ① \\
&= -R_{WB}[t_{BO}]_\times \omega_i - R_{WP} R_{PB} [t_{BO}]_\times R_{BP} \omega_{PP} + \dot t_{WB} \quad (\omega_i = \omega_{BB} - \omega_{BP}) \\
&= -R_{WB}[t_{BO}]_\times \omega_i - R_{WP}[R_{PB} t_{BO}]_\times \omega_{PP} - R_{WP}[t_{PB}]_\times \omega_{PP} + \dot t_{WP} \quad ② \\
&= -R_{WB}[t_{BO}]_\times \omega_i - R_{WP}[R_{PB} t_{BO} + t_{PB}]_\times \omega_{PP} + \dot t_{WP} \\
&= -R_{WB}[t_{BO}]_\times \omega_i - R_{WP}[t_{PO}]_\times \omega_{PP} + \dot t_{WP} \\
&= -R_{WB}[t_{BO}]_\times \omega_i - R_{WP}[t_{PO}]_\times \omega_j - \ldots - R_{WR}[t_{RO}]_\times \omega_0 + \dot t_{WR} \; (= \dot t)
\end{aligned}$$

Jacobian: (derived from velocity kinematics)

$\dot t_{WO} = (I \;\; -R_{WR}[t_{RO}]_\times \;\; 0 \;\; 0 \;\; -R_{WB}[t_{BO}]_\times \;\; \cdots)\dot\Theta \triangleq J\dot\Theta$, $B \in \{\text{ancestors}(O)\}$

Angular Velocity Kinematics:

$$\begin{aligned}
\omega_{WO} = \omega_{WB} = R_{WB}\omega_{BB} &= R_{WB}\omega_i + R_{WB}\omega_{BP} = R_{WB}\omega_i + \omega_{WP} \\
&= R_{WB}\omega_i + R_{WP}\omega_j + \ldots + R_{WR}\omega_0 + \omega_{WW} \; (=0)
\end{aligned}$$

Angular Jacobian:

$\omega_{WO} = (0 \;\; R_{WB} \;\; 0 \;\; 0 \;\; R_{WB} \;\; \cdots)\dot\Theta \triangleq J\dot\Theta$, $B \in \{\text{ancestors}(O)\}$

Jacobian dot:

$$\begin{aligned}
(-R_{WB}[t_{BO}]_\times)^{\cdot} &= -(R_{WB}[t_{BO}]_\times R_{WB}^T R_{WB})^{\cdot} = -([t_{WO} - t_{WB}]_\times R_{WB})^{\cdot} \\
&= -[\dot t_{WO} - \dot t_{WB}]_\times R_{WB} - [t_{WO} - t_{WB}]_\times [\omega_{WB}]_\times R_{WB}
\end{aligned}$$

$$\begin{aligned}
\dot t_{WO} - \dot t_{WB} &= -R_{WC}[t_{CO}]_\times \omega_{CC} - R_{WB}[t_{BC}]_\times \omega_{BB} \quad (\text{see } ①, ②, \text{ assuming } -B-C-O) \\
&= -[R_{WC} t_{CO}]_\times \omega_{WC} - [R_{WB} t_{BC}]_\times \omega_{WB}
\end{aligned}$$

Angular Jacobian dot: $\dot R_{WB} = [\omega_{WB}]_\times R_{WB}$

一个关节在子物体上加正矩，父物体上加反矩，因此功率为角速度差*矩

Generalized Force:

$\begin{cases} \dot t = J\dot\Theta \\ \dot t^T \lambda = \dot\Theta^T \tau \end{cases} \Rightarrow \tau = J^T\lambda$ $\quad \Sigma \underbrace{(\omega_{BB} - \omega_{BP})}_{\dot\Theta_i}{}^T \underbrace{\tau_{BB}}_{\tau_i}$ $\quad \Rightarrow \tau = \begin{pmatrix} f \\ \tau_0 \\ \vdots \\ \tau_{23} \end{pmatrix}$ $\begin{cases} f \triangleq f_{WR} \\ \tau_i \triangleq \tau_{BB} \end{cases}$

Inverse Dynamics (Recursive Newton-Euler Inverse Dynamics):

Algorithm: $\Theta, \dot\Theta, \ddot\Theta \longrightarrow \tau$

for $i=1$ to n do: $(\dot V_{WR} = -g + \ddot t)$

$$\begin{aligned}
\omega_{BB} &= R_{BP}\omega_{PP} + \omega_i \\
\dot\omega_{BB} &= R_{BP}\dot\omega_{PP} + [\omega_{BB}]_\times \omega_i + \dot\omega_i \\
V_{BB} &= R_{BP}(V_{PP} + [\omega_{PP}]_\times P_{PB}) \\
\dot V_{BB} &= R_{BP}(\dot V_{PP} + [\dot\omega_{PP}]_\times P_{PB}) + [V_{BB}]_\times \omega_i
\end{aligned}$$

for $i=n$ to 1 do:

$$f_{BB} = R_{BC} f_{CC} + \underbrace{m(\dot V_{BB} + [\omega_{BB}]_\times V_{BB} + [\omega_{BB}]_\times^2 P_{B\,com} + [\dot\omega_{BB}]_\times P_{B\,com})}_{\bar f_{BB}} + f_{ext}$$

$$\tau_{BB} = R_{BC}\tau_{CC} + [P_{BC}]_\times R_{BC} f_{CC} + [P_{B\,com}]_\times \bar f_{BB} + I_{BB}\dot\omega_{BB} + [\omega_{BB}]_\times I_{BB}\omega_{BB} + r \times f_{ext} + \tau_{ext}$$

Proof:

$\omega_{BB} = R_{BP}\omega_{PP} + \omega_{BB} - \omega_{BP}$ $\square$

$$\begin{aligned}
\dot\omega_{BB} &= R_{BP}(R_{WP}^T \omega_{WP})^{\cdot} - (R_{WB}^T \omega_{WP})^{\cdot} + [\omega_{BB}]_\times(\omega_{BP} - \omega_{BP}) + \dot\omega_{BB} \\
&= -R_{BP}[\omega_{PP}]_\times \omega_{PP} + R_{BP}R_{WP}^T\dot\omega_{WP} + [\omega_{BB}]_\times R_{WB}^T \omega_{WP} \\
&\quad - R_{WB}^T \dot\omega_{WP} - [\omega_{BB}]_\times \omega_{BP} + \dot\omega_{BB} = \dot\omega_{BB} \quad \square
\end{aligned}$$

$V_{BB} = R_{BP}V_{PB} = R_{BP}([\omega_{PP}]_\times P_{PB} + V_{PP})$

$$\begin{aligned}
\dot V_{BB} &= R_{BP}(\dot\Delta) + (R_{WB}^T R_{WP}[\omega_{PP}]_\times - [\omega_{BB}]_\times R_{WB}^T R_{WP})(\Delta) \\
&= R_{BP}(\dot\Delta) + [\omega_{BP} - \omega_{BB}]_\times R_{BP}(\Delta) = R_{BP}(\dot\Delta) - [\omega_i]_\times V_{BB}
\end{aligned}$$

$F_{BB} = [Ad_{T_{comB}}]^T F_{comB}$ $\quad \mathcal{V}_{comB} = [Ad_{T_{comB}}]\mathcal{V}_{BB} = \begin{pmatrix} \omega_{BB} \\ \omega_{BB} \times P_{B\,com} + V_{BB} \end{pmatrix}$

$\dot{\mathcal{V}}_{comB} = \begin{pmatrix} \dot\omega_{BB} \\ \dot\omega_{BB} \times P_{B\,com} + \dot V_{BB} \end{pmatrix}$

Forward Dynamics $(M(\Theta)\ddot\Theta = \tau - c(\Theta, \dot\Theta) - g(\Theta) - J^T(\Theta)\lambda)$:

Algorithm: $\Theta, \dot\Theta, \tau \rightarrow \ddot\Theta$

non-linear effects: $c(\Theta, \dot\Theta) + g(\Theta) + J^T(\Theta)\lambda = ID(\Theta, \dot\Theta, \ddot\Theta = 0)$

mass matrix: $M(\Theta) = (M_1(\Theta) \cdots M_d(\Theta))$, $M_i(\Theta) = ID(\Theta, \dot\Theta = 0, \ddot\Theta = e_i, g = 0, \lambda = 0)$

这里所有I_{BB}应该是I_{CoMB}，见下一页证明

Mass Matrix:

$M = \iiint \rho \, dV$

$CoM = \frac{1}{M}\iiint \rho \vec r \, dV$

$I = -\iiint \rho [\vec r]_\times^2 \, dV$

$$f_{BB} = R_{BC} f_{CC} + \underbrace{m(\dot{v}_{BB} + [\omega_{BB}]_\times v_{BB} + [\omega_{BB}]_\times^2 p_{B\,com} + [\dot{\omega}_{BB}]_\times p_{B\,com})}_{\bar{f}_{BB}} + f_{ext}$$

$$\tau_{BB} = R_{BC}\tau_{CC} + [p_{BC}]_\times R_{BC} f_{CC} + [p_{B\,com}]_\times \bar{f}_{BB} + I_{BB}\dot{\omega}_{BB} + [\omega_{BB}]_\times I_{BB}\omega_{BB} + r \times f_{ext} + \tau_{ext}$$

Proof: Method 1 (B = body, C = com)

$$\mathcal{F}_{BB} = \mathcal{G}_{BB}\dot{\mathcal{V}}_{BB} - [ad_{\mathcal{V}_{BB}}]^T \mathcal{G}_{BB}\mathcal{V}_{BB}$$

$$\mathcal{G}_{BB} = [Ad_{T_{CB}}]^T \mathcal{G}_{CB} [Ad_{T_{CB}}]$$

$$= \begin{pmatrix} I & [p_{BC}] \\ 0 & I \end{pmatrix}\begin{pmatrix} I_{CB} & \\ & mI \end{pmatrix}\begin{pmatrix} I & 0 \\ [p_{CB}] & I \end{pmatrix}$$

$$= \begin{pmatrix} I_{CB} - m[p_{BC}]^2 & [p_{BC}]m \\ -[p_{BC}]m & mI \end{pmatrix}$$

$$[ad_{\mathcal{V}_{BB}}] = \begin{pmatrix} [\omega_{BB}] & 0 \\ [v_{BB}] & [\omega_{BB}] \end{pmatrix}$$

$$\Rightarrow \mathcal{F}_{BB} = \begin{pmatrix} I_{CB}\dot{\omega}_{BB} - m[p_{BC}]^2\dot{\omega}_{BB} + m[p_{BC}]\dot{v}_{BB} \\ -m[p_{BC}]\dot{\omega}_{BB} + m\dot{v}_{BB} \end{pmatrix} + \begin{pmatrix} [\omega_{BB}] & [v_{BB}] \\ 0 & [\omega_{BB}] \end{pmatrix}\begin{pmatrix} I_{CB}\omega_{BB} - m[p_{BC}]^2\omega_{BB} + m[p_{BC}]v_{BB} \\ -m[p_{BC}]\omega_{BB} + m v_{BB} \end{pmatrix}$$

$$= \begin{pmatrix} I_{CB}\dot{\omega}_{BB} + m[p_{BC}]([\dot{\omega}_{BB}]p_{BC}) + m[p_{BC}]\dot{v}_{BB} + [\omega_{BB}]I_{CB}\omega_{BB} - m[\omega_{BB}][p_{BC}]^2\omega_{BB} + m[\omega_{BB}][p_{BC}]v_{BB} - m[v_{BB}][p_{BC}]\omega_{BB} + 0 \\ m([\dot{\omega}_{BB}]p_{BC} + \dot{v}_{BB} + [\omega_{BB}]^2 p_{BC} + [\omega_{BB}]v_{BB}) \end{pmatrix}$$

Given That:

① $a \times b \times b \times a = -b \times a \times a \times b \Rightarrow m[p_{BC}][\omega_{BB}]^2 p_{BC}$

② $a \times (b \times c) - c \times (b \times a) = b \times (a \times c) \Rightarrow m[p_{BC}][\omega_{BB}]v_{BB}$ □

Method 2 (B = body, C = com)

$$\mathcal{F}_{BB} = [Ad_{T_{CB}}]^T \mathcal{F}_{CB}$$

$$= \begin{pmatrix} I & [p_{BC}] \\ 0 & I \end{pmatrix}\begin{pmatrix} I_{CB}\dot{\bar{\omega}}_{CB} + [\bar{\omega}_{CB}]I_{CB}\bar{\omega}_{CB} \\ m(\dot{\bar{v}}_{CB} + [\bar{\omega}_{CB}]\bar{v}_{CB}) \end{pmatrix}$$

$$= \begin{pmatrix} I_{CB}\dot{\omega}_{BB} + [\omega_{BB}]I_{CB}\omega_{BB} + m[p_{BC}](\dot{\bar{v}}_{CB} + [\omega_{BB}]\bar{v}_{CB}) \\ m(\dot{\bar{v}}_{CB} + [\omega_{BB}]\bar{v}_{CB}) \end{pmatrix}$$

$$\mathcal{V}_{com\,B} = [Ad_{T_{com\,B}}]\mathcal{V}_{BB} = \begin{pmatrix} \omega_{BB} \\ \omega_{BB} \times p_{B\,com} + v_{BB} \end{pmatrix}$$

$$\dot{\mathcal{V}}_{com\,B} = \begin{pmatrix} \dot{\omega}_{BB} \\ \dot{\omega}_{BB} \times p_{B\,com} + \dot{v}_{BB} \end{pmatrix}$$

Continuous-time True-state Kinematics

$$\dot{p}_t = v_t$$
$$\dot{v}_t = R_t(a_m - a_{bt} - a_n) + g_t$$
$$\dot{q}_t = \frac{1}{2} q_t \otimes (\omega_m - \omega_{bt} - \omega_n)$$
$$\dot{a}_{bt} = a_w$$
$$\dot{\omega}_{bt} = \omega_w$$

Nominal-state Kinematics

$$\dot{p} = v$$
$$\dot{v} = R(a_m - a_b) + g$$
$$\dot{q} = \frac{1}{2} q \otimes (\omega_m - \omega_b)$$
$$\dot{a}_b = 0$$
$$\dot{\omega}_b = 0$$

Error-state Kinematics

$$\delta\dot{p} = \delta v$$
$$\delta\dot{v} = -R[a_m - a_b]_\times \delta\theta - R\delta a_b - a_n + \delta g$$
$$\delta\dot{\theta} = -[\omega_m - \omega_b]_\times \delta\theta - \delta\omega_b - \omega_n$$
$$\delta\dot{a}_b = a_w$$
$$\delta\dot{\omega}_b = \omega_w$$

Proof:

$$\delta\dot{v} = (\dot{v}_t - \dot{v}) = R_t(a_m - a_{bt} - a_n) - R(a_m - a_b) + g_t - g$$
$$= R(I + [\delta\theta]_\times)(a_m - a_{bt} - a_n) - R(a_m - a_b) + \delta g$$
$$= -R\delta a_b + R[\delta\theta]_\times(a_m - a_b) - Ra_n + \delta g$$

$$\delta\dot{\theta} = [\delta\dot{\theta}]_\times^\vee = \delta\dot{R}^\vee = \dot{R}^{-1}R_t + R^{-1}\dot{R}_t \overset{\vee}{=} -[\omega]_\times R^{-1}R_t + R^{-1}R_t[\omega_t]_\times^\vee$$
$$= -[\omega]_\times \delta R + \delta R[\omega_t]_\times^\vee = [\delta\omega]_\times + [\delta\theta]_\times[\omega_t]_\times - [\omega]_\times[\delta\theta]_\times^\vee$$
$$= [\delta\omega + [\delta\theta]_\times \omega]_\times^\vee = \delta\omega - [\omega]_\times\delta\theta = -[\omega_m - \omega_b]_\times\delta\theta - \delta\omega_b - \omega_n$$

Discrete-time (first-order)

Nominal-state Kinematics

$$p \leftarrow p + v\Delta t$$
$$v \leftarrow v + (R(a_m - a_b) + g)\Delta t$$
$$q \leftarrow q \otimes q\{(\omega_m - \omega_b)\Delta t\}$$
$$a_b \leftarrow a_b$$
$$\omega_b \leftarrow \omega_b$$

Error-state Kinematics

$$\delta p \leftarrow \delta p + \delta v\Delta t$$
$$\delta v \leftarrow \delta v + (-R[a_m - a_b]_\times\delta\theta - R\delta a_b + \delta g)\Delta t + v_i$$
$$\delta\theta \leftarrow R^T\{(\omega_m - \omega_b)\Delta t\}\delta\theta - \delta\omega_b\Delta t + \theta_i$$
$$\delta a_b \leftarrow \delta a_b + a_i$$
$$\delta\omega_b \leftarrow \delta\omega_b + \omega_i$$

Prediction

Error-state: $\delta x \leftarrow f(x, \delta x, u_m, i) = f(x, 0, u_m, 0) + \left.\frac{\partial f}{\partial \delta x}\right|_{x,0,u_m,0} \delta x + \left.\frac{\partial f}{\partial i}\right|_{x,0,u_m,0} i$

given $\delta x \to 0$, $i \to 0$

$$\underbrace{\begin{pmatrix} \delta p \\ \delta v \\ \delta\theta \\ \delta a_b \\ \delta\omega_b \end{pmatrix}}_{\delta x} = \underbrace{\begin{pmatrix} I & \Delta t I & -\frac{1}{2}R[a_m - a_b]_\times\Delta t^2 & -\frac{1}{2}R\Delta t^2 & \frac{1}{6}R[a_m - a_b]_\times\Delta t^3 \\ 0 & I & -R[a_m - a_b]_\times\Delta t & -R\Delta t & \frac{1}{2}R[a_m - a_b]_\times\Delta t^2 \\ 0 & 0 & R^T\{(\omega_m - \omega_b)\Delta t\} & 0 & -I\Delta t \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}}_{F_{\delta x}} \underbrace{\begin{pmatrix} \delta p \\ \delta v \\ \delta\theta \\ \delta a_b \\ \delta\omega_b \end{pmatrix}}_{\delta x} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}}_{F_i} \underbrace{\begin{pmatrix} v_i \\ \theta_i \\ a_i \\ \omega_i \end{pmatrix}}_{i}$$

Prediction:

nominal state update

$$\hat{\delta x} \leftarrow F_x \hat{\delta x}$$ Always 0

$$P \leftarrow F_x P F_x^T + F_i Q_i F_i^T$$

$$\delta x \sim N(\hat{\delta x}, P)$$

$$Q_i = \begin{pmatrix} V_i & & & \\ & \Theta_i & & \\ & & A_i & \\ & & & \Omega_i \end{pmatrix}$$

$$\mathbf{V_i} = \sigma_{\tilde{\mathbf{a}}_n}^2 \Delta t^2 \mathbf{I} \quad [m^2/s^2]$$
$$\mathbf{\Theta_i} = \sigma_{\tilde{\omega}_n}^2 \Delta t^2 \mathbf{I} \quad [rad^2]$$
$$\mathbf{A_i} = \sigma_{\mathbf{a}_w}^2 \Delta t \mathbf{I} \quad [m^2/s^4]$$
$$\mathbf{\Omega_i} = \sigma_{\omega_w}^2 \Delta t \mathbf{I} \quad [rad^2/s^2]$$

Correction

① $0_3 = R_t(m_m - m_n) - n_t$ earth-mag state

② $0_3 = R_t(a_m - a_{bt} - a_n) + g_t$ acc-less state

③ $0_3 = v_t$ vel-less state

④ $0_3 = p_t - p_m$ p-obs

⑤ $0_3 = v_t - v_m$ v-obs

$$H_x = \begin{array}{c} \begin{array}{ccccc} p_t & v_t & q_t & a_{bt} & \omega_{bt} \end{array} \\ \begin{pmatrix} 0_{3,3} & 0_{3,3} & \frac{\partial R_t(\cdot)}{\partial q_t} & 0_{3,3} & 0_{3,3} \\ 0_{3,3} & 0_{3,3} & \frac{\partial R_t(\cdot)}{\partial q_t} & -R_t & 0_{3,3} \\ 0_{3,3} & I_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} \\ I_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} \\ 0_{3,3} & I_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} \end{pmatrix} \end{array}$$

Correction

$$H \triangleq \left.\frac{\partial h}{\partial \delta x}\right|_x = \left.\frac{\partial h}{\partial x_t}\right|_x \left.\frac{\partial x_t}{\partial \delta x}\right|_x = H_x X_{\delta x}$$

$$X_{\delta x} = \begin{pmatrix} \frac{\partial(p+\delta p)}{\partial \delta p} & & & \\ & \frac{\partial(v+\delta v)}{\partial \delta v} & & \\ & & \frac{\partial(q\otimes\delta q)}{\partial \delta\theta} & \\ & & & \ddots \end{pmatrix} = \begin{pmatrix} I_{3,3} & & & & \\ & I_{3,3} & & & \\ & & Q_{\delta\theta} & & \\ & & & I_{3,3} & \\ & & & & \ddots \end{pmatrix}$$

$$\mathbf{q} = [w\ \mathbf{v}] = w + \mathbf{v}.$$

$$\boxed{\frac{\partial(\mathbf{q}\otimes\mathbf{a}\otimes\mathbf{q}^*)}{\partial\mathbf{q}} = 2\left[\, w\mathbf{a} + \mathbf{v}\times\mathbf{a} \;\middle|\; \mathbf{v}^\top\mathbf{a}\mathbf{I}_3 + \mathbf{v}\mathbf{a}^\top - \mathbf{a}\mathbf{v}^\top - w[\mathbf{a}]_\times \,\right] \in \mathbb{R}^{3\times4}} \quad (174)$$

$$\boxed{\frac{\partial(\mathbf{q}\otimes\mathbf{a}\otimes\mathbf{q}^*)}{\partial\mathbf{a}} = \frac{\partial(\mathbf{R}\,\mathbf{a})}{\partial\mathbf{a}} = \mathbf{R}}$$

$$\mathbf{Q}_{\delta\theta} = \frac{1}{2}\begin{bmatrix} -q_x & -q_y & -q_z \\ q_w & -q_z & q_y \\ q_z & q_w & -q_x \\ -q_y & q_x & q_w \end{bmatrix}$$

$$\delta q \xrightarrow{\delta\theta\to 0} \begin{pmatrix} 1 \\ \frac{1}{2}\delta\theta \end{pmatrix}$$

$$\underset{4,3}{Q_{\delta\theta}} \triangleq \left.\frac{\partial(q\otimes\delta q)}{\partial\delta\theta}\right|_q = \left.\frac{\partial(q\otimes\delta q)}{\partial\delta q}\right|_q \left.\frac{\partial\delta q}{\partial\delta\theta}\right|_{\delta\theta=0} = [q]_L \frac{1}{2}\begin{pmatrix} 0_{1,3} \\ I_{3,3} \end{pmatrix}$$

$$K = PH^T(HPH^T + V)^{-1}$$
$$\hat{\delta x} \leftarrow K(y - h(\hat{x}_t)) \qquad \hat{x}_t = x \oplus \overset{0}{\hat{\delta x}} = x$$
$$P \leftarrow (I - KH)P$$

$x \leftarrow x \oplus \hat{\delta x}$ ($q \leftarrow q \otimes q\{\hat{\delta\theta}\}$, others $p \leftarrow p + \hat{\delta p}$)

$$\hat{\delta x} \leftarrow 0$$
$$P \leftarrow GPG^T \qquad G = \begin{pmatrix} I_{3,3} & & & & \\ & I_{3,3} & & & \\ & & I - [\frac{1}{2}\hat{\delta\theta}]_\times & & \\ & & & I_{3,3} & \\ & & & & \ddots \end{pmatrix}$$

Quaternion normalization

Proof: $q_t^+ = q_t \qquad q^+ = q \otimes \hat{\delta q}$

$$q^+ \otimes \delta q^+ = q \otimes \delta q$$
$$\Rightarrow \delta q^+ = (q^+)^* \otimes q \otimes \delta q$$
$$= (q \otimes \hat{\delta q})^* \otimes q \otimes \delta q$$
$$= \hat{\delta q}^* \otimes \delta q$$
$$\Rightarrow \begin{pmatrix} 1 \\ \frac{1}{2}\delta\theta^+ \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2}\hat{\delta\theta}^T \\ -\frac{1}{2}\hat{\delta\theta} & I - [\frac{1}{2}\hat{\delta\theta}]_\times \end{pmatrix}\begin{pmatrix} 1 \\ \frac{1}{2}\delta\theta \end{pmatrix}$$
$$\Rightarrow \delta\theta^+ = -\hat{\delta\theta} + (I - [\frac{1}{2}\hat{\delta\theta}]_\times)\delta\theta$$
$$\frac{\partial\delta\theta^+}{\partial\delta\theta} = I - [\frac{1}{2}\hat{\delta\theta}]_\times$$