Configuration: $\Theta \triangleq (t\ \ q_0\ \ q_1\ \cdots\ q_{23})$ local q_{PB}

Genralized Velocity: $\dot\Theta \triangleq (\dot t\ \ \omega_0\ \ \omega_1\ \cdots\ \omega_{23})$

Genralized Acceleration: $\ddot\Theta \triangleq (\ddot t\ \ \dot\omega_0\ \ \dot\omega_1\ \cdots\ \dot\omega_{23})$

Update: $\Theta \leftarrow \Theta \oplus \dot\Theta \Delta t$, $\quad \begin{cases} t \leftarrow t + \dot t\, \Delta t \\ q_i \leftarrow q_i \otimes e^{\frac12 \omega_i \Delta t} \end{cases}$

$\dot\Theta \leftarrow \dot\Theta + \ddot\Theta \Delta t$

Differential Equation: $\dot q_{PB} = \frac12 q_{PB} \otimes (\omega_{BB} - \omega_{BP})$

$\updownarrow \qquad \updownarrow \qquad \updownarrow$

$\dot q_i = \frac12 q_i \otimes \omega_i$

$$\dot q_{PB} = \dot q_{WP}^* \otimes q_{WB} + q_{WP}^* \otimes \dot q_{WB}$$
$$= (q_{WP} \otimes \tfrac12 \omega_{PP})^* \otimes q_{WB} + q_{WP}^* \otimes (q_{WB} \otimes \tfrac12 \omega_{BB})$$
$$= -\tfrac12 \omega_{PP} \otimes q_{PB} + q_{PB} \otimes \tfrac12 \omega_{BB}$$
$$= \tfrac12 q_{PB} \otimes (\omega_{BB} - q_{BP} \otimes \omega_{PP} \otimes q_{BP}^*)$$
$$= \tfrac12 q_{PB} \otimes (\omega_{BB} - \omega_{BP})$$

Velocity kinematics:

assuming $B = parent(O) \Rightarrow t_{BO} \equiv 0$, $P = parent(B)$, ... $R = root$

$$\dot t_{WO} = \dot T_{WB} t_{BO} = T_{WB} [\mathcal{V}_{BB}]_x t_{BO} \triangleq \begin{pmatrix} R_{WB} & t_{WB} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} [\omega_{BB}]_x & v_{BB} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t_{BO} \\ 1 \end{pmatrix}$$
$$= R_{WB}[\omega_{BB}]_x t_{BO} + R_{WB} V_{BB} = -R_{WB}[t_{BO}]_x \omega_{BB} + \dot t_{WB} \quad ①$$
$$= -R_{WB}[t_{BO}]_x \omega_i - R_{WP} R_{PB} [t_{BO}]_x R_{BP} \omega_{PP} + \dot t_{WB} \quad (\omega_i = \omega_{BB} - \omega_{BP})$$
$$= -R_{WB}[t_{BO}]_x \omega_i - R_{WP}[R_{PB} t_{BO}]_x \omega_{PP} - R_{WP}[t_{PB}]_x \omega_{PP} + \dot t_{WP} \quad ②$$
$$= -R_{WB}[t_{BO}]_x \omega_i - R_{WP}[R_{PB} t_{BO} + t_{PB}]_x \omega_{PP} + \dot t_{WP}$$
$$= -R_{WB}[t_{BO}]_x \omega_i - R_{WP}[t_{PO}]_x \omega_{PP} + \dot t_{WP}$$
$$= -R_{WB}[t_{BO}]_x \omega_i - R_{WP}[t_{PO}]_x \omega_j - \dots - R_{WR}[t_{RO}]_x \omega_0 + \dot t_{WR} \ (= \dot t)$$

Jacobian: (derived from velocity kinematics)

$$\dot t_{WO} = (I \ \ -R_{WB}[t_{BO}]_x \ \ 0 \ \ 0 \ \ -R_{WB}[t_{BO}]_x \ \cdots)\, \dot\Theta \triangleq J \dot\Theta, \quad B \in \{ancestors(O)\}$$

Angular Velocity kinematics:

$$\omega_{WO} = \omega_{WB} = R_{WB}\omega_{BB} = R_{WB}\omega_i + R_{WB}\omega_{BP} = R_{WB}\omega_i + \omega_{WP}$$
$$= R_{WB}\omega_i + R_{WP}\omega_j + \dots + R_{WR}\omega_0 + \omega_{WW} \ (=0)$$

Angular Jacobian:

$$\omega_{WO} = (0 \ \ R_{WB} \ \ 0 \ \ 0 \ \ R_{WB} \ \cdots)\, \dot\Theta \triangleq J \dot\Theta, \quad B \in \{ancestors(O)\}$$

$M = \iiint \rho\, dV$

$CoM = \frac{1}{M} \iiint \rho \vec r\, dV$

$I = -\iiint \rho [\vec r]_x^2\, dV$

Jacobian dot:

$$(-R_{WB}[t_{BO}]_x)^{\cdot} = -(R_{WB}[t_{BO}]_x R_{WB}^T R_{WB})^{\cdot} = -([t_{WO} - t_{WB}]_x R_{WB})^{\cdot}$$
$$= -[\dot t_{WO} - \dot t_{WB}]_x R_{WB} - [t_{WO} - t_{WB}]_x [\omega_{WB}]_x R_{WB}$$
$$\dot t_{WO} - \dot t_{WB} = -R_{WC}[t_{CO}]_x \omega_{CC} - R_{WB}[t_{BC}]_x \omega_{BB} \quad \text{(see ①, ②, assuming B–C–O)}$$
$$= -[R_{WC} t_{CO}]_x \omega_{WC} - [R_{WB} t_{BC}]_x \omega_{WB}$$

Angular Jacobian dot: $\dot R_{WB} = [\omega_{WB}]_x R_{WB}$

一个关节在子物体上加
正矩，父物体上加反矩，
因此功率为角速度差*矩

Generalized Force:

$$\begin{cases} \dot t = J \dot\Theta \\ \dot t^T \lambda = \dot\Theta^T \tau \end{cases} \Rightarrow \tau = J^T \lambda \qquad \Sigma(\underbrace{\omega_{BB} - \omega_{BP}}_{\dot\Theta_i})^T \underbrace{\tau_{BB}}_{\tau_i} \Rightarrow \tau = \begin{pmatrix} f \\ \tau_0 \\ \vdots \\ \tau_{23} \end{pmatrix} \begin{cases} f \triangleq f_{WR} \\ \tau_i \triangleq \tau_{BB} \end{cases}$$

Inverse Dynamics (Recursive Newton-Euler Inverse Dynamics):

Algorithm: $\Theta, \dot\Theta, \ddot\Theta \rightarrow \tau$

for $i=1$ to n do: $(\dot V_{WR} = -g + \ddot t)$

$$\omega_{BB} = R_{BP}\omega_{PP} + \omega_i$$
$$\dot\omega_{BB} = R_{BP}\dot\omega_{PP} + [\omega_{BB}]_x \omega_i + \dot\omega_i$$
$$V_{BB} = R_{BP}(V_{PP} + [\omega_{PP}]_x P_{PB})$$
$$\dot V_{BB} = R_{BP}(\dot V_{PP} + [\dot\omega_{PP}]_x P_{PB}) + [V_{BB}]_x \omega_i$$

for $i=n$ to 1 do:

$$f_{BB} = R_{BC} f_{CC} + \underbrace{m(\dot V_{BB} + [\omega_{BB}]_x V_{BB} + [\omega_{BB}]_x^2 P_{B\,com} + [\dot\omega_{BB}]_x P_{B\,com})}_{} + f_{ext}$$
$$\tau_{BB} = R_{BC}\tau_{CC} + [P_{BC}]_x R_{BC} f_{CC} + [P_{B\,com}]_x \bar f_{BB} + I_{BB}\dot\omega_{BB} + [\omega_{BB}]_x I_{BB}\omega_{BB} + r \times f_{ext} + \tau_{ext}$$

Proof:

$$\omega_{BB} = R_{BP}\omega_{PP} + \omega_{BB} - \omega_{BP} \quad \square$$
$$\dot\omega_{BB} = R_{BP}(R_{WP}^T \omega_{WP})^{\cdot} - (R_{WB}^T \omega_{WP})^{\cdot} + [\omega_{BB}]_x(\omega_{BP} - \omega_{BP}) + \dot\omega_{BB}$$
$$= -R_{BP}[\omega_{PP}]_x \omega_{PP} + R_{BP} R_{WP}^T \dot\omega_{WP} + [\omega_{BB}]_x R_{WB}^T \omega_{WP}$$
$$- R_{WB}^T \dot\omega_{WP} - [\omega_{BB}]_x \omega_{BP} + \dot\omega_{BB} = \dot\omega_{PB} \quad \square$$
$$V_{BB} = R_{BP} V_{PB} = R_{BP}([\omega_{PP}]_x P_{PB} + V_{PP})$$
$$\dot V_{BB} = R_{BP}(\dot\Delta) + (R_{WB}^T R_{WP}[\omega_{PP}]_x - [\omega_{BB}]_x R_{WB}^T R_{WP})(\Delta)$$
$$= R_{BP}(\dot\Delta) + [\omega_{BP} - \omega_{BB}]_x R_{BP}(\Delta) = R_{BP}(\dot\Delta) - [\omega_i]_x V_{BB}$$
$$F_{BB} = [Ad_{T_{comB}}]^T F_{comB} \qquad \mathcal{V}_{comB} = [Ad_{T_{comB}}]\mathcal{V}_{BB} = \begin{pmatrix} \omega_{BB} \\ \omega_{BB} \times P_{B\,com} + V_{BB} \end{pmatrix}$$
$$\dot{\mathcal{V}}_{comB} = \begin{pmatrix} \dot\omega_{BB} \\ \dot\omega_{BB} \times P_{B\,com} + \dot V_{BB} \end{pmatrix}$$

Forward Dynamics $(M(\Theta)\ddot\Theta = \tau - c(\Theta, \dot\Theta) - g(\Theta) - J^T(\Theta)\lambda)$:

Algorithm: $\Theta, \dot\Theta, \tau \rightarrow \ddot\Theta$

non-linear effects: $c(\Theta, \dot\Theta) + g(\Theta) + J^T(\Theta)\lambda = ID(\Theta, \dot\Theta, \ddot\Theta = 0)$

mass matrix: $M(\Theta) = (M_1(\Theta) \cdots M_d(\Theta))$, $M_i(\Theta) = ID(\Theta, \dot\Theta = 0, \ddot\Theta = e_i, g = 0, \lambda = 0)$

Mass Matrix:

Continuous-time True-state Kinematics

$$\dot p_t = v_t$$
$$\dot v_t = R_t(a_m - a_{bt} - a_n) + g_t$$
$$\dot q_t = \tfrac{1}{2} q_t \otimes (\omega_m - \omega_{bt} - \omega_n)$$
$$\dot a_{bt} = a_w$$
$$\dot \omega_{bt} = \omega_w$$

Nominal-state Kinematics

$$\dot p = v$$
$$\dot v = R(a_m - a_b) + g$$
$$\dot q = \tfrac{1}{2} q \otimes (\omega_m - \omega_b)$$
$$\dot a_b = 0$$
$$\dot \omega_b = 0$$

Error-state Kinematics

$$\delta\dot p = \delta v$$
$$\delta\dot v = -R[a_m - a_b]_\times \delta\theta - R\delta a_b - a_n + \delta g$$
$$\delta\dot\theta = -[\omega_m - \omega_b]_\times \delta\theta - \delta\omega_b - \omega_n$$
$$\delta\dot a_b = a_w$$
$$\delta\dot\omega_b = \omega_w$$

Proof:

$$\delta\dot v = (\dot v_t - \dot v) = R_t(a_m - a_{bt} - a_n) - R(a_m - a_b) + g_t - g$$
$$= R(I + [\delta\theta]_\times)(a_m - a_{bt} - a_n) - R(a_m - a_b) + \delta g$$
$$= -R\delta a_b + R[\delta\theta]_\times(a_m - a_b) - Ra_n + \delta g$$

$$\delta\dot\theta = [\delta\dot\theta]_\times^\vee = \delta\dot R^\vee = \dot R^{-1}R_t + R^{-1}\dot R_t \overset{\vee}{=} -[\omega]_\times R^{-1}R_t + R^{-1}R_t[\omega_t]_\times^\vee$$
$$= -[\omega]_\times \delta R + \delta R[\omega_t]_\times^\vee = [\delta\omega]_\times + [\delta\theta]_\times[\omega_t]_\times - [\omega]_\times[\delta\theta]_\times^\vee$$
$$= [\delta\omega + [\delta\theta]_\times \omega]_\times^\vee = \delta\omega - [\omega]_\times\delta\theta = -[\omega_m - \omega_b]_\times\delta\theta - \delta\omega_b - \omega_n$$

Discrete-time (first-order)

Nominal-state Kinematics

$$p \leftarrow p + v\Delta t$$
$$v \leftarrow v + (R(a_m - a_b) + g)\Delta t$$
$$q \leftarrow q \otimes q\{(\omega_m - \omega_b)\Delta t\}$$
$$a_b \leftarrow a_b$$
$$\omega_b \leftarrow \omega_b$$

Error-state Kinematics

$$\delta p \leftarrow \delta p + \delta v\Delta t$$
$$\delta v \leftarrow \delta v + (-R[a_m - a_b]_\times\delta\theta - R\delta a_b + \delta g)\Delta t + v_i$$
$$\delta\theta \leftarrow R^T\{(\omega_m - \omega_b)\Delta t\}\delta\theta - \delta\omega_b\Delta t + \theta_i$$
$$\delta a_b \leftarrow \delta a_b + a_i$$
$$\delta\omega_b \leftarrow \delta\omega_b + \omega_i$$

Prediction

given $\delta x \to 0$, $i \to 0$

Error-state: $\delta x \leftarrow f(x, \delta x, u_m, i) = f(x, 0, u_m, 0) + \left.\frac{\partial f}{\partial \delta x}\right|_{x,0,u_m,0} \delta x + \left.\frac{\partial f}{\partial i}\right|_{x,0,u_m,0} i$

$$\begin{pmatrix}\delta p\\ \delta v\\ \delta\theta\\ \delta a_b\\ \delta\omega_b\end{pmatrix} = \begin{pmatrix} I & \Delta t I & -\frac{1}{2}R[a_m - a_b]_\times\Delta t^2 & -\frac{1}{2}R\Delta t^2 & \frac{1}{6}R[a_m - a_b]_\times\Delta t^3\\ 0 & I & -R[a_m - a_b]_\times\Delta t & -R\Delta t & \frac{1}{2}R[a_m - a_b]_\times\Delta t^2\\ 0 & 0 & R^T\{(\omega_m - \omega_b)\Delta t\} & 0 & -I\Delta t\\ 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix}\delta p\\ \delta v\\ \delta\theta\\ \delta a_b\\ \delta\omega_b\end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0\\ I & 0 & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{pmatrix}\begin{pmatrix}v_i\\ \theta_i\\ a_i\\ \omega_i\end{pmatrix}$$

$$\delta x = F_{\delta x}\,\delta x + F_i\, i$$

Prediction:

nominal state update

$$\hat{\delta x} \leftarrow F_x \hat{\delta x} \quad \text{Always } 0$$
$$P \leftarrow F_x P F_x^T + F_i Q_i F_i^T$$

$$\delta x \sim N(\hat{\delta x}, P)$$

$$Q_i = \begin{pmatrix} V_i & & & \\ & \Theta_i & & \\ & & A_i & \\ & & & \Omega_i\end{pmatrix}$$

$\mathbf{V_i} = \sigma_{\tilde{\mathbf{a}}_n}^2 \Delta t^2 \mathbf{I}$	$[m^2/s^2]$
$\mathbf{\Theta_i} = \sigma_{\tilde{\boldsymbol{\omega}}_n}^2 \Delta t^2 \mathbf{I}$	$[rad^2]$
$\mathbf{A_i} = \sigma_{\mathbf{a}_w}^2 \Delta t \mathbf{I}$	$[m^2/s^4]$
$\mathbf{\Omega_i} = \sigma_{\boldsymbol{\omega}_w}^2 \Delta t \mathbf{I}$	$[rad^2/s^2]$

Correction

① $O_3 = R_t(m_m - m_n) - n_t$ earth-mag state

② $O_3 = R_t(a_m - a_{bt} - a_n) + g_t$ acc-less state

③ $O_3 = v_t$ vel-less state

④ $O_3 = p_t - p_m$ p-obs

⑤ $O_3 = v_t - v_m$ v-obs

$$H_x = \begin{array}{c} \begin{array}{ccccc} p_t & v_t & q_t & a_{bt} & \omega_{bt} \end{array}\\ \begin{pmatrix} 0_{3,3} & 0_{3,3} & \frac{\partial R_t(\cdot)}{\partial q_t} & 0_{3,3} & 0_{3,3}\\ 0_{3,3} & 0_{3,3} & \frac{\partial R_t(\cdot)}{\partial q_t} & -R_t & 0_{3,3}\\ 0_{3,3} & I_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3}\\ I_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3}\\ 0_{3,3} & I_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} \end{pmatrix}\end{array}$$

Correction

$$H \triangleq \left.\frac{\partial h}{\partial \delta x}\right|_x = \left.\frac{\partial h}{\partial x_t}\right|_x \left.\frac{\partial x_t}{\partial \delta x}\right|_x = H_x X_{\delta x}$$

$$X_{\delta x} = \begin{pmatrix} \frac{\partial(p+\delta p)}{\partial \delta p} & & & \\ & \frac{\partial(v+\delta v)}{\partial \delta v} & & \\ & & \frac{\partial(q\otimes\delta q)}{\partial\delta\theta} & \\ & & & \ddots \end{pmatrix} = \begin{pmatrix} I_{3,3} & & & & \\ & I_{3,3} & & & \\ & & Q_{\delta\theta} & & \\ & & & I_{3,3} & \\ & & & & \ddots \end{pmatrix}$$

$\mathbf{q} = [w\ \mathbf{v}] = w + \mathbf{v}.$

$$\boxed{\frac{\partial(\mathbf{q}\otimes\mathbf{a}\otimes\mathbf{q}^*)}{\partial\mathbf{q}} = 2\left[\, w\mathbf{a} + \mathbf{v}\times\mathbf{a} \;\middle|\; \mathbf{v}^\top\mathbf{a}\mathbf{I}_3 + \mathbf{v}\mathbf{a}^\top - \mathbf{a}\mathbf{v}^\top - w[\mathbf{a}]_\times \right] \in \mathbb{R}^{3\times4}} \quad (174)$$

$$\boxed{\frac{\partial(\mathbf{q}\otimes\mathbf{a}\otimes\mathbf{q}^*)}{\partial\mathbf{a}} = \frac{\partial(\mathbf{R}\mathbf{a})}{\partial\mathbf{a}} = \mathbf{R}} \quad \mathbf{Q}_{\delta\theta} = \frac{1}{2}\begin{bmatrix} -q_x & -q_y & -q_z\\ q_w & -q_z & q_y\\ q_z & q_w & -q_x\\ -q_y & q_x & q_w \end{bmatrix}$$

$$\delta q \xrightarrow{\delta\theta\to 0} \begin{pmatrix} 1\\ \frac{1}{2}\delta\theta\end{pmatrix}$$

$$\underset{4,3}{Q_{\delta\theta}} \triangleq \left.\frac{\partial(q\otimes\delta q)}{\partial\delta\theta}\right|_q = \left.\frac{\partial(q\otimes\delta q)}{\partial\delta q}\right|_q \left.\frac{\partial\delta q}{\partial\delta\theta}\right|_{\delta\theta=0} = [q]_L \frac{1}{2}\begin{pmatrix}0_{1,3}\\ I_{3,3}\end{pmatrix}$$

$$K = PH^T(HPH^T + V)^{-1}$$
$$\hat{\delta x} \leftarrow K(y - h(\hat x_t)) \qquad \hat x_t = x \oplus \overset{\leftarrow 0}{\hat{\delta x}} = x$$
$$P \leftarrow (I - KH)P$$
$$x \leftarrow x \oplus \hat{\delta x} \quad (q \leftarrow q\otimes q\{\hat{\delta\theta}\},\ \text{others } p \leftarrow p + \hat{\delta p})$$
$$\hat{\delta x} \leftarrow 0$$
$$P \leftarrow GPG^T \qquad G = \begin{pmatrix} I_{3,3} & & & & \\ & I_{3,3} & & & \\ & & I - [\frac{1}{2}\hat{\delta\theta}]_\times & & \\ & & & I_{3,3} & \\ & & & & \ddots\end{pmatrix}$$

Quaternion normalization

Proof: $q_t^+ = q_t \qquad q^+ = q\otimes\hat{\delta q}$

$$q^+\otimes\delta q^+ = q\otimes\delta q$$
$$\Rightarrow \delta q^+ = (q^+)^*\otimes q\otimes\delta q$$
$$= (q\otimes\hat{\delta q})^*\otimes q\otimes\delta q$$
$$= \hat{\delta q}^*\otimes\delta q$$
$$\Rightarrow \begin{pmatrix}1\\ \frac{1}{2}\delta\theta^+\end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2}\hat{\delta\theta}^T\\ -\frac{1}{2}\hat{\delta\theta} & I - [\frac{1}{2}\hat{\delta\theta}]_\times\end{pmatrix}\begin{pmatrix}1\\ \frac{1}{2}\delta\theta\end{pmatrix}$$
$$\Rightarrow \delta\theta^+ = -\hat{\delta\theta} + (I - [\tfrac{1}{2}\hat{\delta\theta}]_\times)\delta\theta$$
$$\frac{\partial\delta\theta^+}{\partial\delta\theta} = I - [\tfrac{1}{2}\hat{\delta\theta}]_\times$$