Linear Algebra Homework #7
Fall 2020

Name: Due: Wednesday, Dec. 23rdt, 2020

Instructions:

Please include essential steps in your solution. For most of the problems, answers without essential steps may receive a score of 0.

1. Let V be a vector space over the reals with basis $\vec{v}_1, \ldots, \vec{v}_n$. Show that the linear operator $T : R^n \to V$ given by
$$T((c_1, c_2, \ldots, c_n)) = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$
is an isomorphism of vector spaces.

2. Let $V = R^3$ and $\vec{w}_1 = (0, 1, 0)$, $\vec{w}_2 = (1, 1, 1)$, $\vec{v}_1 = (1, 3, 1)$, $\vec{v}_2 = (2, -1, 1)$, $\vec{v}_3 = (1, 0, 1)$. The set $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is a basis of V. Determine which $\vec{v}_j's$ could be replaced by $\vec{w}_1$, and which $\vec{v}_j$ could be replaced by both $\vec{w}_1$ and $\vec{w}_2$, while retaining the basis property.

3. Assume that $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \subset V$, where V is a vector space of dimension n. Answer True/False to the following :
   (a) If S is a basis of V then $k = n$.
   (b) if S spans V, then $k \le n$.
   (c) If S is linearly independent then $k \le n$.
   (d) If S is linearly independent and $k = n$, then S spans V.
   (e) If S spans V and $k = n$ then S is a basis for V.

4. Use the fact that B is a reduced echelon form of A to find bases for the row and column spaces of A with no calculations, and the null space with minimum calculations, where $\begin{bmatrix} 3 & 5 & -1 & 5 & 1 \\ 1 & 2 & -1 & 2 & 0 \\ 2 & 3 & 0 & 3 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

5. Let $A = \begin{bmatrix} 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 2 & 1 & 2 \\ 2 & 2 & 4 & 4 & 8 \end{bmatrix}$. Use the column space algorithm on the matrix $[A\ I_3]$ to find a basis B of $C(A)$ and to expand it to a basis of R^3.