Linear Algebra Exercise #2
Fall 2020

Name:

Instructions:

Please include essential steps in your solution. For most of the problems, answers without essential steps may receive a score of 0.

1. Express the quadratic form as a matrix product involving a symmetric coefficient matrix

   $[x_1\ x_2]\begin{bmatrix}-1 & 4\\ 4 & -31\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

   (a) $Q = 8x_1x_2 - x_1^2 - 31x_2^2$

   (b) $Q = 3x_1^2 - 2x_1x_2 + 4x_1x_3 + 5x_2^2 + 4x_3^2 - 2x_2x_3.$

   $\begin{bmatrix}3 & -1 & 2\\ -1 & 5 & -1\\ 2 & -1 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ $[x_1\ x_2\ x_2]$

2. Determine the definiteness of the following matrices based on the leading principal minors:

   (a) $\begin{bmatrix}-3 & 4\\ 4 & -5\end{bmatrix}$ $D_1 = -3 < 0$ $D_2 = -1 < 0$, So it's indefinite

   (b) $\begin{bmatrix}1 & 2 & 0\\ 2 & 4 & 5\\ 0 & 5 & 6\end{bmatrix}$ $D_1 = 1 > 0$ $D_2 = 0$ $D_3 = -1 - 2\cdot 12 = -25$ So it's indefinite

   (c) $\begin{bmatrix}1 & 0 & 3 & 0\\ 0 & 2 & 0 & 5\\ 3 & 0 & 4 & 0\\ 0 & 5 & 0 & 6\end{bmatrix}.$ $D_1 = 1 > 0$ $D_2 = 2 > 0$ $D_3 = 8 + 3\cdot -6 = -10$

   So it is indefinite

$\lambda^2 + 8\lambda + 15 - 16 = 0$
$\lambda^2 + 8\lambda + 16 = 17$
$(\lambda+4)^2 = \pm\sqrt{17}$

(a) $\begin{pmatrix}\lambda+3 & -4\\ -4 & \lambda+5\end{pmatrix}$
∴ indefinite

(b) $\begin{bmatrix}\lambda-1 & -2 & 0\\ -2 & \lambda-4 & -5\\ 0 & -5 & \lambda-6\end{bmatrix}$

$(\lambda-1)\cdot[\lambda^2 - 10\lambda + 24 - 25]$
$+2\cdot(-2\lambda+12)$
$=$

$\begin{bmatrix}\lambda-1 & 0 & -3 & 0\\ 0 & \lambda-2 & 0 & -5\\ -3 & 0 & \lambda-4 & 0\\ 0 & -5 & 0 & \lambda-6\end{bmatrix}$

$(\lambda-1)\cdot[(\lambda-2)(\lambda-4)(\lambda-6) + 25(\lambda-4)]$

3. Determine the definiteness of the examples in Problem 2 based on eigenvalues of A.

4. For what conditions of a and b is the quadratic form

$$Q(x_1, x_2, x_3, x_4) = ax_1^2 + x_2^2 + bx_3^2 + 2x_1x_4$$

   (a) positive definite.

   (b) negative semidefinite.

$\begin{bmatrix}a & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & b & 0\\ 1 & 0 & 0 & 0\end{bmatrix}$

$\begin{bmatrix}\lambda-a & 0 & 0 & -1\\ 0 & \lambda-1 & 0 & 0\\ 0 & 0 & \lambda-b & 0\\ -1 & 0 & 0 & \lambda\end{bmatrix} = (\lambda-a)\cdot(\lambda-1)(\lambda-b)\lambda) + (-1)^5\cdot(-1)\cdot(\lambda-b)$
$(-1)^3$ $= 0$

$(\lambda-b)[\lambda(\lambda^2 - a\lambda - \lambda + a) + 1]$ $\cancel{b} = 0$

5. Find the orthogonal canonical form of the quadratic form

$$Q(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2.$$

In addition, give the associated coordinate transformation, canonical basis and principal axes of the given form.
Solution
orthogonal canonical form $3y_1^2 - y_2^2$; canonical basis:$\{\frac{1}{\sqrt{2}}(1,1), \frac{1}{\sqrt{2}}(-1,1)$; associated coordinate transformation:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$x_1 = \frac{1}{\sqrt{2}}(y_1 - y_2), x_2 = \frac{1}{\sqrt{2}}(y_1 + y_2).$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad [\lambda I - A] = \begin{bmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{bmatrix} = (\lambda-1)^2 - 4 = 0$$

$\lambda - 1 = \pm 2$

$\cancel{\lambda_1 = 3 \quad \lambda_2 = -1}$

$y_1^2 + 9y_2^2$

$\lambda_1 = -1 \quad \lambda_2 = 3$

$\lambda_1 \leq \lambda_2$

$$\begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$x_1 = \frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_2$

$x_2 = \frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_2$