Linear Algebra Exercise #2
Fall 2020

Name:

Instructions:

Please include essential steps in your solution. For most of the problems, answers without essential steps may receive a score of 0.

1. Express the quadratic form as a matrix product involving a symmetric coefficient matrix

   (a) $Q = 8x_1x_2 - x_1^2 - 31x_2^2$

   (b) $Q = 3x_1^2 - 2x_1x_2 + 4x_1x_3 + 5x_2^2 + 4x_3^2 - 2x_2x_3$.

   Solution

   (a) $Q = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 4 & -31 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

   (b) $Q = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

2. Determine the definiteness of the following matrices based on the leading principal minors:

   (a) $\begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

   (c) $\begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 5 \\ 2 & 0 & -5 & 0 \\ 0 & 5 & 0 & -15 \end{bmatrix}$.

   Solution

(a) $|D_1| = -3$, $|D_2| = \begin{vmatrix} -3 & 4 \\ 4 & -5 \end{vmatrix} = -1$. $|D_2| < 0$ doesn't fit any of the patterns for positive and negative definite quadratic forms. It is indefinite.

(b) $|D_1| = 1$, $|D_2| = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 1$, $|D_3| = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 6 \end{vmatrix} == (-1)^{3+3} \times 6 \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 6$.

Hence it is positive definite.

(c) $|D_1| = -1$, $|D_2| = \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} = 2$, $|D_3| = \begin{vmatrix} -1 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & -5 \end{vmatrix} = (-1)^{2+2} \times (-2) \begin{vmatrix} -1 & 2 \\ 2 & -5 \end{vmatrix} = -2$.

$$|D_4| = \begin{vmatrix} -1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 5 \\ 2 & 0 & -5 & 0 \\ 0 & 5 & 0 & -15 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 5 \\ 0 & 0 & -1 & 0 \\ 0 & 5 & 0 & -15 \end{vmatrix} = -\begin{vmatrix} -2 & 0 & 5 \\ 0 & -1 & 0 \\ 5 & 0 & -15 \end{vmatrix} =$$

$-(-1)^{2+2} \times (-1) \begin{vmatrix} -2 & 5 \\ 5 & -15 \end{vmatrix} = 5$. Hence $|D_1|, |D_3|$ are negative and $|D_2|$, $|D_4|$ are positive. It is negative definite.

3. Determine the definiteness of the examples in Problem 2 based on eigenvalues of A. Solution:

(a) $det(\lambda I - A) = \begin{vmatrix} \lambda + 3 & -4 \\ -4 & \lambda + 5 \end{vmatrix} = (\lambda+3)(\lambda+5) - 16 = \lambda^2 + 8\lambda - 1$. Therefore, since $b^2 - 4ac = 8^2 - 4 \times (-1) = 68 > 0$, the characteristic equation have two real roots and there product is -1, which implies that one is negative and the other is positive. Hence it is indefinite.

(b)

$$det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -2 & 0 \\ -2 & \lambda - 5 & 0 \\ 0 & 0 & \lambda - 6 \end{vmatrix} = (-1)^{3+3}(\lambda - 6) \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 5 \end{vmatrix}$$
$$= (\lambda - 6)(\lambda^2 - 6\lambda + 1) = 0$$

The eigenvalues are $\lambda_1 = 6, \lambda_2 = \frac{6+\sqrt{32}}{2}, \lambda_3 = \frac{6-\sqrt{32}}{2}$. Since they are all positive, the quadratic form is positive definite.

(c)

$$det(\lambda I - A) = \begin{vmatrix} \lambda+1 & 0 & -2 & 0 \\ 0 & \lambda+2 & 0 & -5 \\ -2 & 0 & \lambda+5 & 0 \\ 0 & -5 & 0 & \lambda+15 \end{vmatrix}$$

$$= (-1)^{1+1}(\lambda+1) \begin{vmatrix} \lambda+2 & 0 & -5 \\ 0 & \lambda+5 & 0 \\ -5 & 0 & \lambda+15 \end{vmatrix} + (-1)^{3+1}(-2) \begin{vmatrix} 0 & -2 & 0 \\ \lambda+2 & 0 & -5 \\ -5 & 0 & \lambda+15 \end{vmatrix}$$

$$= (\lambda+1)(-1)^{2+2}(\lambda+5) \begin{vmatrix} \lambda+2 & -5 \\ -5 & \lambda+15 \end{vmatrix} - 2 \times (-1)^{1+2}(-2) \begin{vmatrix} \lambda+2 & -5 \\ -5 & \lambda+15 \end{vmatrix}$$

$$= (\lambda+1)(\lambda+5)(\lambda^2+17\lambda+5) - 4(\lambda^2+17\lambda+5)$$

$$= (\lambda^2+6\lambda+1)(\lambda^2+17\lambda+5)$$

The eigenvalues are $\lambda_1 = \frac{-6+\sqrt{32}}{2}$, $\lambda_2 = \frac{6-\sqrt{32}}{2}$, $\lambda_3 = \frac{-17-\sqrt{269}}{2}$, $\lambda_4 = \frac{-17+\sqrt{269}}{2}$. Since they are all negative, the quadratic form is negative definite.

4. For what conditions of a and b is the quadratic form

$$Q(x_1, x_2, x_3, x_4) = ax_1^2 + x_2^2 + bx_3^2 + 2x_1x_4$$

   (a) positive definite.

   (b) negative semidefinite.

5. Find the orthogonal canonical form of the quadratic form

$$Q(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2.$$

In addition, give the associated coordinate transformation, canonical basis and principal axes of the given form.
Solution
orthogonal canonical form $3y_1^2 - y_2^2$; canonical basis:$\{\frac{1}{\sqrt{2}}(1,1), \frac{1}{\sqrt{2}}(-1,1)$; associated coordinate transformation:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$x_1 = \frac{1}{\sqrt{2}}(y_1 - y_2), x_2 = \frac{1}{\sqrt{2}}(y_1 + y_2)$.