B0684
Economic Engineering Analysis

Time Value of Money

Learning Objective

- Cash Flow Diagram (CFD)
- Single cash flow calculation
  - Future worth
  - Present worth
- Multiple cash flow calculation
  - Irregular cash flow
  - Uniform series
  - Gradient series
  - Geometric series
- Compounding frequency
  - Period interest rate
  - Effective annual interest rate

This chapter is very important!!

It serves as a foundation for the remainder of the module.

CASH FLOW DIAGRAMS

- A diagram depicting the magnitude and timing of cash flowing in and out of the investment alternative.

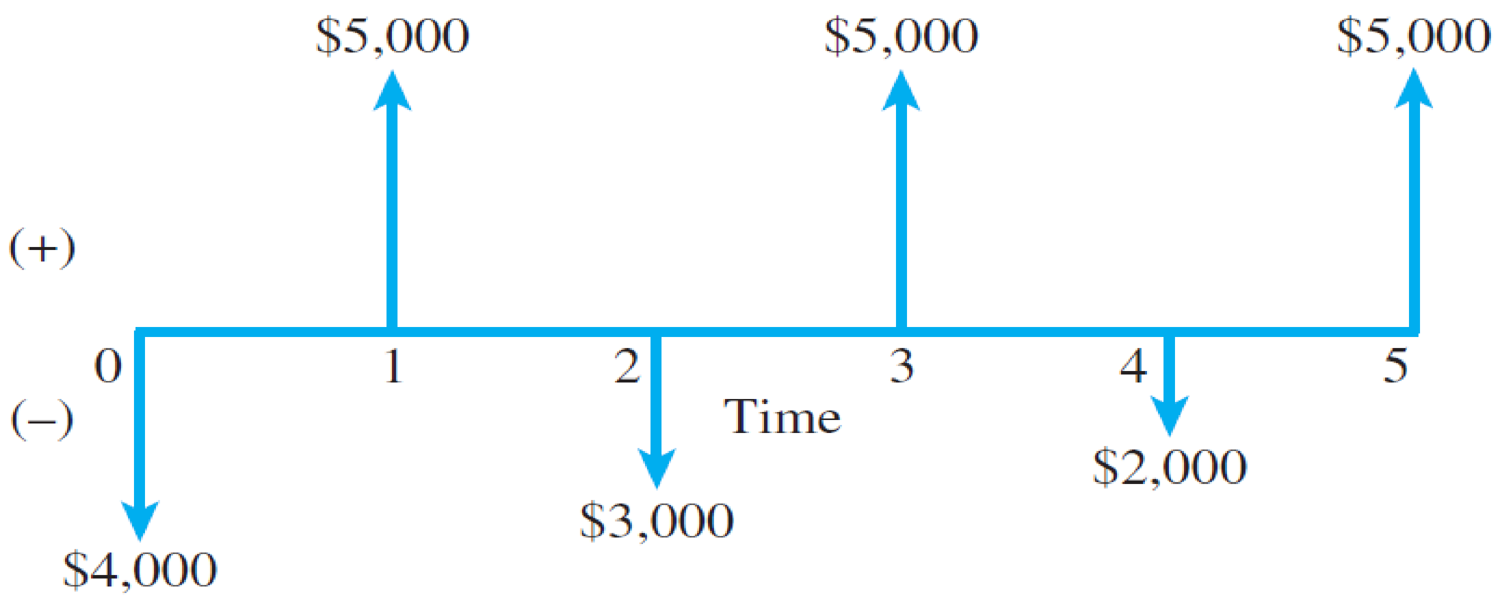
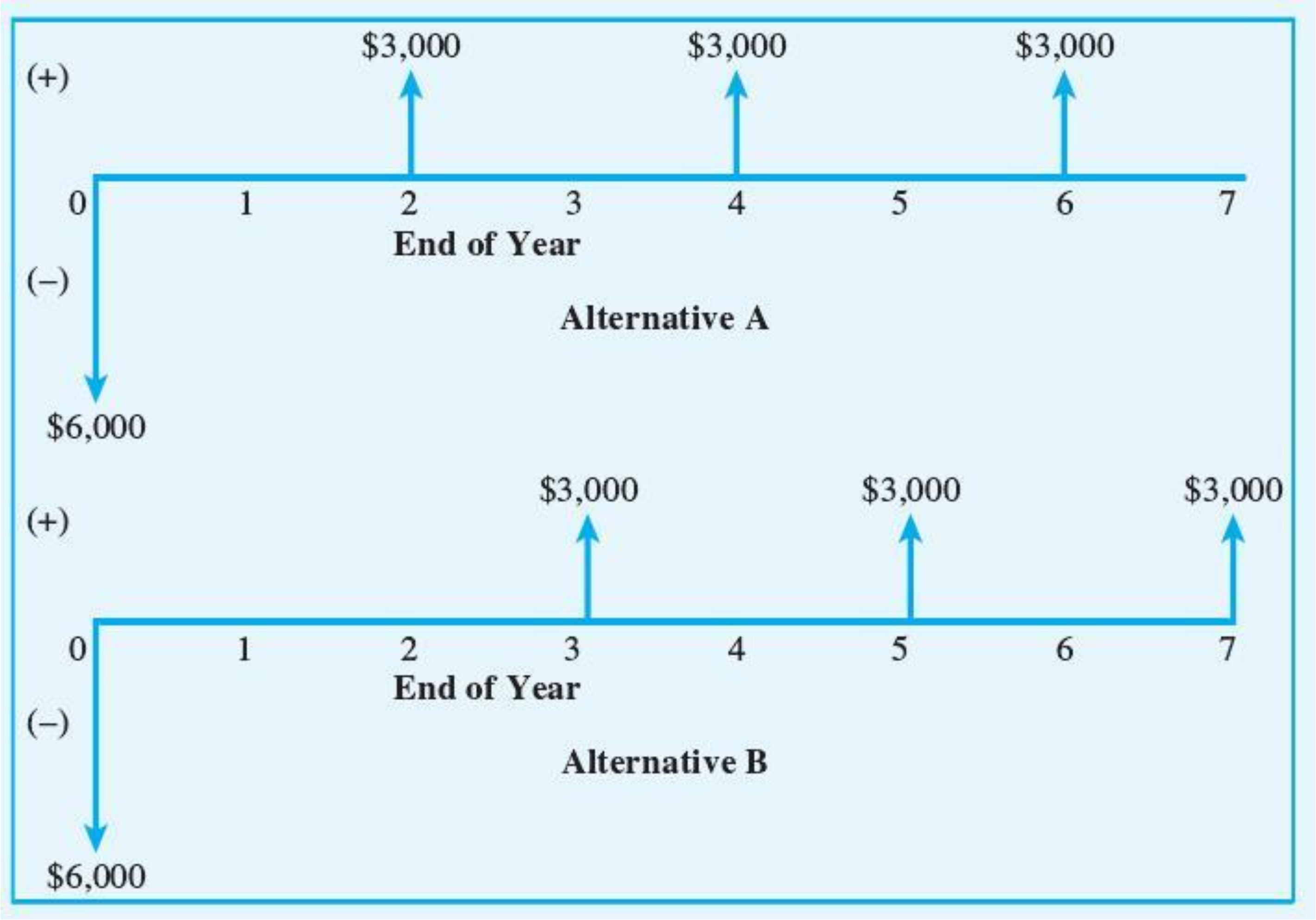


FIGURE 2.1 A Cash Flow Diagram (CFD)

- A horizontal line as a time scale
- Vertical arrows indicating cash flows
  - An upward arrow indicates a cash inflow or positive-valued cash flow (receipts)
  - A downward arrow indicates a cash outflow or negative-valued cash flow (expenditures)
- The lengths of the arrows suggest the magnitudes of cash flows (but not precisely)

Which investment alternative will you choose?



Recall TVOM, when receiving a given sum of money, we prefer to receive it sooner!

Alternative A is preferred to Alternative B

- End-of-period cash flows, end-of-year cash flows, end-of-period compounding, are assumed unless otherwise noted
- The end of period t is the beginning of period t+1
- The role of CFDs:
  - Clearly communicate cash flows (than words do)
  - Identify cash flow patterns (e.g., a uniform/gradient/geometric series)

SINGLE CASH FLOWS

The simplest scenario: there is only one cash flow in the planning horizon.

Compound interest is used in almost all business and lending situations:

> Interest should be charged (or earned) against both the principal and accumulated interest to date.

Future Worth

$$F_n = F_{n-1}(1+i)$$

F_0=P, where P is the present value of a single sum of money; F_n is the accumulated value of P over n years; i – interest rate

$$I_n = \sum_{t=1}^{n} iF_{t-1}$$

In – the accumulated (total) interest over n years

Illustration

Suppose you loan $10,000 for 1 year to an individual who agrees to pay you interest at a compound rate of 10 percent/year. At the end of 1 year, the individual asks to extend the loan period an additional year. The borrower repeats the process several more times. Five years after loaning the person the $10,000, how much would the individual owe you?

Year	Unpaid Balance at the Beginning of the Year	Annual Interest	Payment	Unpaid Balance at the End of the Year
1	$10,000.00	$1,000.00	$0.00	$11,000.00
2	$11,000.00	$1,100.00	$0.00	$12,100.00
3	$12,100.00	$1,210.00	$0.00	$13,310.00
4	$13,310.00	$1,331.00	$0.00	$14,641.00
5	$14,641.00	$1,464.10	$16,105.10	$0.00

$$F=P(1+i)^n$$

P = present worth.

F = future worth. F occurs n periods after P.

i = the interest rate, expressed as a decimal or percentage.

n = the number of interest periods.

- $(1+i)^n$ is referred to as *the single sum, future worth factor*.
- It is denoted (F|P i%, n), and reads "**the F, given P factor at i% for n periods**" Thus,

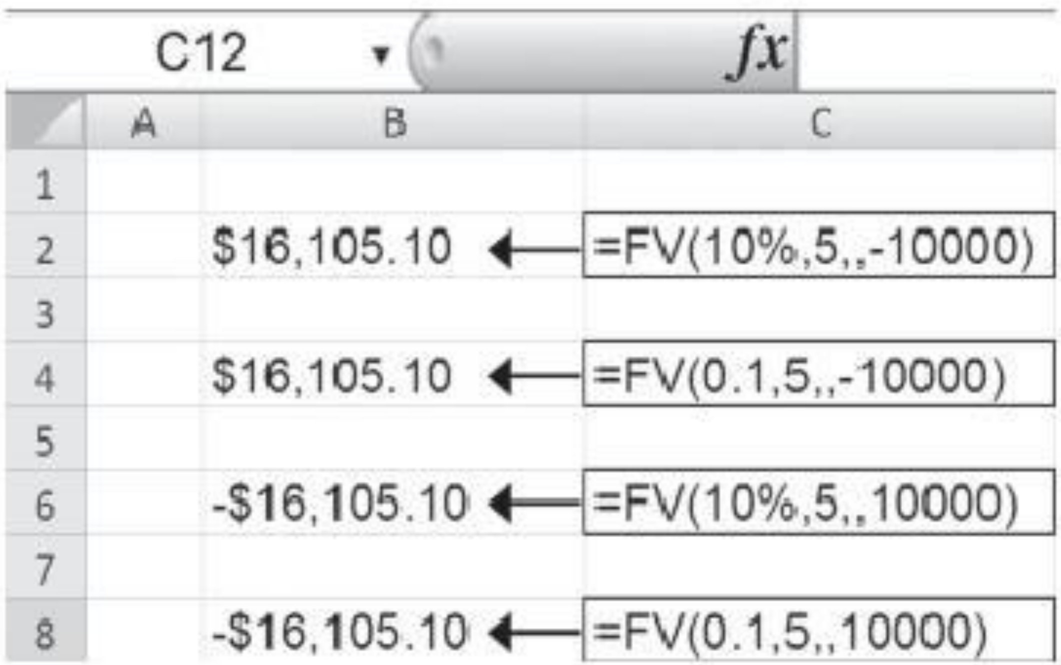
$$F=P(F|P\ i\%, n)$$

Excel financial function FV

- Parameters in order: interest rate (i), number of periods (n), equal-sized cash flow per period (A), present amount (P), and type [either end-of-period cash flows (0 or omitted) or beginning-of-period cash flows (1)].
- Entering the following in any cell in an Excel spreadsheet: =FV(i,n,,-P).
- A negative value is entered for P, since the sign of the value obtained for F by using the FV function will be opposite the sign used for P.
- The FV function was developed for a loan situation where $P are loaned (negative cash flow) in order to receive $F (positive cash flow)
- If you enter P rather than -P, F comes out negative. Remember to change the sign!

Use the Excel function FV to calculate the illustration example.

=FV(0.1,5,,-10000)



Example

Dia St. John borrows $1,000 at 12 percent compounded annually. The loan is to be paid back after 5 years. How much should she repay?

– Solution 1: Using the compound interest tables in Appendix A for 12% and 5 periods, the value of *the single sum, future worth factor* (F|P 12%,5) is shown to be 1.76234. Thus, F = P(F|P 12%,5) = $1,000*1.76234 = $1,762.34

– Solution 2*: Using the Excel FV worksheet function, F = FV(12%, 5,,-1000)=1762.34

Question: how long does it take for an investment to double?

if it earns (a) 2 percent, (b) 4 percent, or (c) 12 percent annual compound interest?

- Solution 1: Rule of 72

The quotient of 72 and the interest rate provides a reasonably good approximation of the number of interest periods required to double the value of an investment.

a. $n \approx 72/2 = 36$ yrs

b. $n \approx 72/4 = 18$ yrs

c. $n \approx 72/12 = 6$ yrs

Question: how long does it take for an investment to double?

if it earns (a) 2 percent, (b) 4 percent, or (c) 12 percent annual compound interest?

- Solution 2: mathematics

solve mathematically for n such that $(1 + i)^n = 2$

gives $n = \log 2/\log(1 + i)$. Therefore,

a. $n = \log 2/\log 1.02 = 35.003$ yrs;

b. $n = \log 2/\log 1.04 = 17.673$ yrs;

c. $n = \log 2/\log 1.12 = 6.116$ yrs.

Question: how long does it take for an investment to double?

if it earns (a) 2 percent, (b) 4 percent, or (c) 12 percent annual compound interest?

- Solution 3*: Excel NPER (Number of periods) function

parameters in order: interest rate, equal-sized cash flow per period, present amount, future amount, and type.

Letting F equal 2 and P equal −1, the NPER function yields

a. n = NPER(2%,, -1,2) = 35.003 yrs

b. n = NPER(4%,, -1,2) = 17.673 yrs

c. n = NPER(12%,, -1,2) = 6.116 yrs

Present Worth

As $F=P(1+i)^n$,

$P=F(1+i)^{-n}$

or, $P=F(P|F\ i\%,n)$

- $(1 + i)^{-n}$ and $(P|F\ i\%,n)$ are referred to as *the single sum, present worth factor*.
- Using Excel financial function PV (present value).
- The parameters in order: interest rate (i), number of periods (n), equal-sized cash flow per period (A), future amount (F), and type.
- To solve for P when given i, n, and F, the following can be entered in any cell in an Excel spreadsheet:

=PV(i,n,,-F)

Illustration

Suppose you wish to accumulate $10,000 in a savings account 4 years from now, and the account pays interest at a rate of 5 percent compounded annually. How much must be deposited today?

- using the Excel PV worksheet function,
- P =PV(5%,4,,-10000) = $8227.02

MULTIPLE CASH FLOWS

- We have considered single cash flow.
- We will extend our analysis to multiple cash flows.
- Begin with multiple cash flows that do not exhibit a pattern.
- Follow with cash flow series that form a pattern(the uniform/gradient /geometric series), allowing the use of shortcuts in determining PV and FV.

Irregular Cash Flows

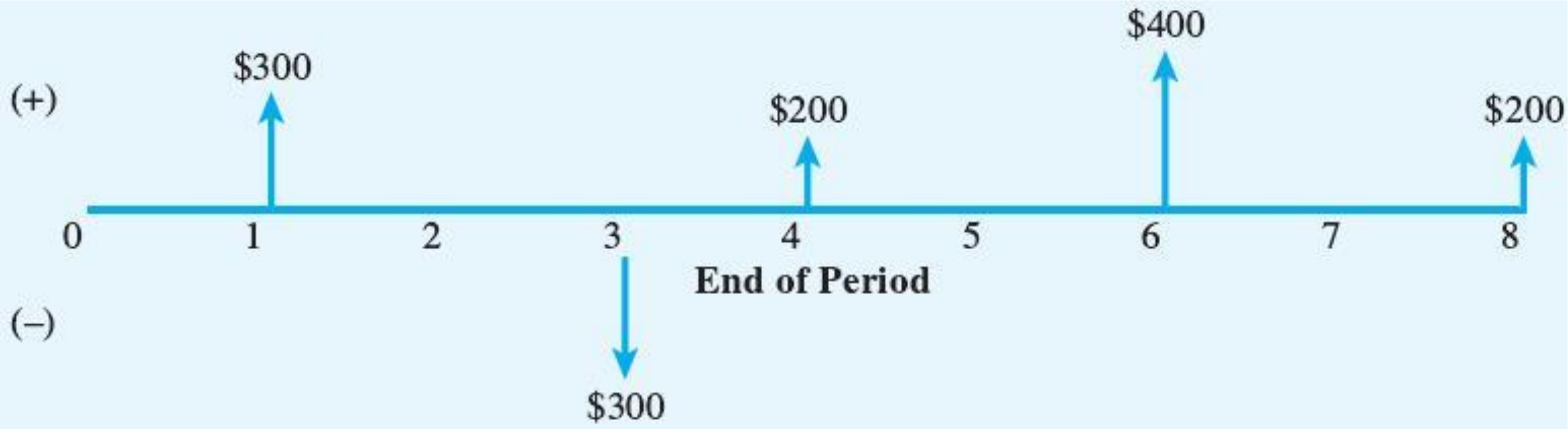
- When there're more than one cash flow,
  - the present worth can be determined by adding the present worth of the individual cash flows.
  - the future worth can be determined by adding the future worth of the individual cash flows.
- let A_t denote a cash flow at the end of time period t, the present worth,

$$P = A_1(1+i)^{-1} + A_2(1+i)^{-2} + A_3(1+i)^{-3} + \cdots + A_{n-1}(1+i)^{-(n-1)} + A_n(1+i)^{-n} \tag{2.8}$$

- or, $P = \sum_{t=1}^{n} A_t(1+i)^{-t}$
- or, $P = \sum_{t=1}^{n} A_t(P|F\ i\%,t)$

Illustration

Consider the following CFD. Using an interest rate of 6 percent per interest period, what is the present worth equivalent of cash flows?



Solution 1: P = \$300(P|F 6%,1)- \$300(P|F 6%,3)+ \$200(P|F 6%,4)+ \$400(P|F 6%,6)+ \$200(P|F 6%,8) = \$597.02

- Solution 2: Using the Excel NPV (net present value) Function
- Parameters in order: interest rate (i), followed by the cash flows value 1, value 2, value 3,……, value n
- Note NPV occurs one time period prior to value 1
- Enter the following into any Excel cell: =NPV(0.06,300,0,-300,200,0,400,0,200)
- NPV = $597.02

- Alternatively, enter value 1 … value n into a column:

C12 ▾ fx =NPV(6%,C4:C11)

	A	B	C	D	E	F
1						
2		**End of Year (n)**	**Cash Flow (CF)**			
3		0	$0			
4		1	$300			
5		2	$0			
6		3	-$300			
7		4	$200			
8		5	$0			
9		6	$400			
10		7	$0			
11		8	$200			
12		**P =**	$597.02		**=NPV(6%,C4:C11)**	

- Note no blank cells can be included, insert 0 if no cash flow occurs
- F_1 = 300, NPV calculates P that occurs one time period prior to F_1 (coincide with F_0 in the timescale)

- What if F_0=100? Please calculate NPV

0	0	100	100
1	300	300	300
2	0	0	0
3	-300	-300	-300
4	200	200	200
5	0	0	0
6	400	400	400
7	0	0	0
8	200	200	200
NPV	\$597.02	\$657.56	\$697.02

- If $F_0 \neq 0$, it should NOT be included in values, but added to the outcome of NPV
- Otherwise you're calculating $P_{0\text{-}1}$

- The future worth,

$$F = A_1(1+i)^{n-1} + A_2(1+i)^{n-2} + A_3(1+i)^{n-3} + \cdots + A_{n-2}(1+i)^2 + A_{n-1}(1+i) + A_n \qquad (2.11)$$

- or, $F = \sum_{t=1}^{n} A_t(1+i)^{n-t}$

- or, $F = \sum_{t=1}^{n} A_t(F|P\ i\%, n-t)$

- Note the last (n^th) cash flow does not draw interest

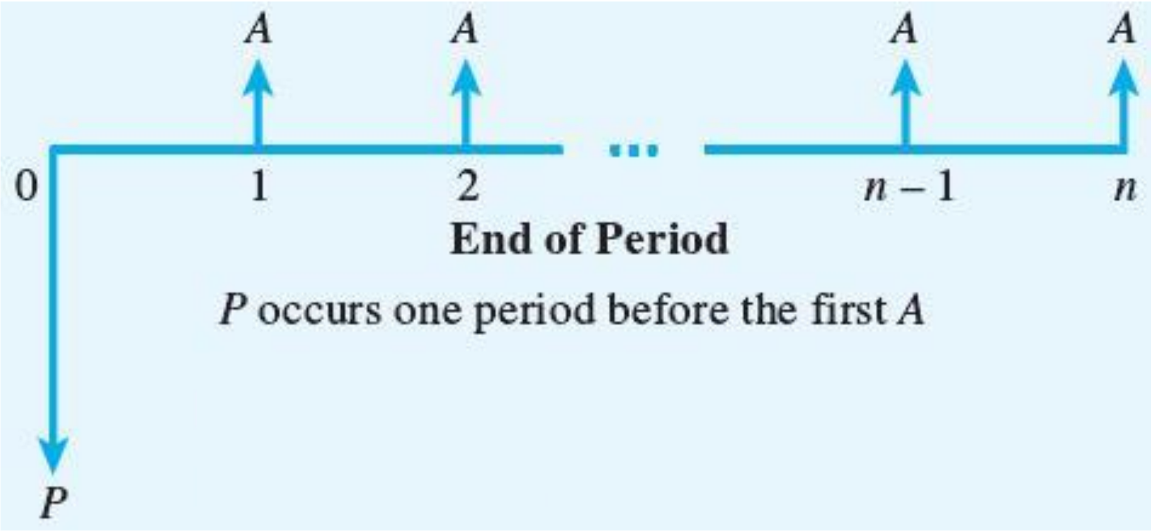
In the previous example, determine the future worth at the end of the eighth period using an interest rate of 6 percent per interest period.

- Solution 1: F = \$300(F|P 6%,7) - \$300(F|P 6%,5) + \$200(F|P 6%,4) + \$400(F|P 6%,2) + \$200 = \$951.56

- Solution 2: since we already know P = \$597.02, F= 597.02(F|P 6%,8)=\$951.56

Uniform Series

When all cash flows in a series are equally sized and spaced.



$$P = A_1(1+i)^{-1} + A_2(1+i)^{-2} + A_3(1+i)^{-3} + \cdots + A_{n-1}(1+i)^{-(n-1)} + A_n(1+i)^{-n} \quad (2.8)$$

Since A1=A2=……An=A

$$P = \sum_{t=1}^{n} A(1+i)^{-t} \quad (2.16)$$

where A is the magnitude of an individual cash flow in the series.

Letting $X = (1+i)^{-1}$ and bringing A outside the summation yields

$$P = A\sum_{t=1}^{n} X^t$$

$$= AX\sum_{t=1}^{n} X^{t-1}$$

Letting $h = t - 1$ gives the geometric series

$$P = AX\sum_{h=0}^{n-1} X^h \quad (2.17)$$

Since the summation in Equation 2.17 represents *the first n terms of a geometric series*

S_n=a_1(1-q^n)/(1-q)

Here a_1=1, q=X, thus S_n=(1-X^n)/(1-X),

Replacing ∑. with S_n in Equation 2.17

$$\sum_{h=0}^{n-1} X^h = \frac{1 - X^n}{1 - X}$$

Replacing X with $(1 + i)^{-1}$, [numerator and denominator multiply by (1+i)^n respectively]

$$P = A\frac{(1 + i)^n - 1}{i(1 + i)^n}$$

or, P = A(P|A i%,n)

(P|A i%,n) is referred to as *the uniform series, present worth factor*

Please note the mathematical deduction is NOT the key point in our module, and I’m not doing it thereafter.

If you’re interested in the deduction process, please consult your MATHS teacher!

Illustration

Troy Long wishes to deposit a single sum of money in a savings account so that five equal annual withdrawals of $2,000 can be made before depleting the fund. If the first withdrawal is to occur 1 year after the deposit and the fund pays interest at a rate of 5 percent compounded annually, how much should he deposit?

- Using the Excel PV function, it prompts you with the parameters in order: interest rate (i), number of periods (n), equal-sized cash flow per period (A), future amount (F), and type.
- Enter in any cell in Excel: =PV(i,n,-A)
- As before, a negative sign is used for A (because the PV function reverses the sign of A when calculating the value of P).
- P =PV(5%,5,-2000)= 8658.95

A Delayed Uniform Series

In the previous example, suppose the first withdrawal does not occur until 3 years after the deposit. How much should be deposited?

Solution:

- Stage 1: $P' = F_2 = PV(5\%,5,-2000) = 8658.95$
- Stage 2: $P = PV(5\%,2,,-8658.95) = 7853.92$

Size of uniform withdraw/payment

Given $P = A\dfrac{(1+i)^n - 1}{i(1+i)^n}$

$$A = P\frac{i(1+i)^n}{(1+i)^n - 1}$$

or as

$$A = P(A|P\ i\%,n)$$

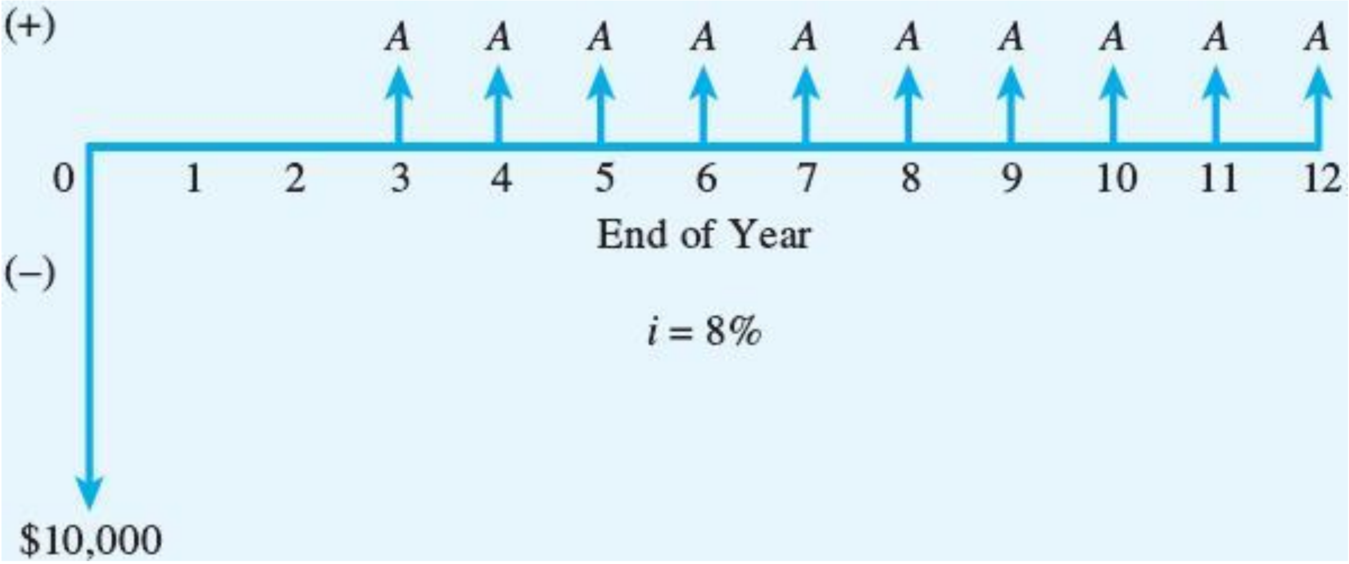
(A|P i%,n) is called *the capital recovery factor*, used frequently in both personal financing and in comparing economic investment alternatives.

Illustration

Suppose Rachel Townsley deposits $10,000 into an account that pays 8 percent interest compounded annually. If she withdraws 10 equal annual amounts from the account, with the first withdrawal occurring 1 year after the deposit, how much can she withdraw each year in order to deplete the fund with the last withdrawal?

- Using the Excel PMT (payment) function
- Parameters in order: interest rate, number of periods, present amount, future amount, and type.
- Enter the following into any Excel cell: =PMT(8%,10,-10000); A= 1490.29

What about the first withdrawal is delayed for 2 years? How much can be withdrawn each of the 10 years?



Solution:

F_2=FV(8%,2,,-10000)= $11,664.00

A=PMT(8%,10,-11664)= $1738.28

The future worth of a uniform series is obtained by recalling that

$$F = P(1 + i)^n \tag{2.23}$$

Substituting Equation 2.19 into Equation 2.23 for P and reducing yields

$$F = A\frac{(1 + i)^n - 1}{i} \tag{2.24}$$

or, equivalently,

$$F = A(F|A\ i\%,n) \tag{2.25}$$

where $(F|A\ i\%,n)$ is referred to as the *uniform series, future worth factor.*

Illustration

If Luis Jimenez makes annual deposits of $1,000 into a savings account for 30 years, how much will be in the fund immediately after his last deposit if the fund pays 6 percent interest compounded annually?

Using the Excel FV worksheet function,

F =FV(6%,30,-1000)=$79,058.19

$$A = F\frac{i}{(1+i)^n - 1}$$

or, equivalently,

$$A = F(A|F\ i\%,n)$$

(A|F i%,n) is referred to as *the sinking fund factor*.

Note the last A or deposit earns no interest.

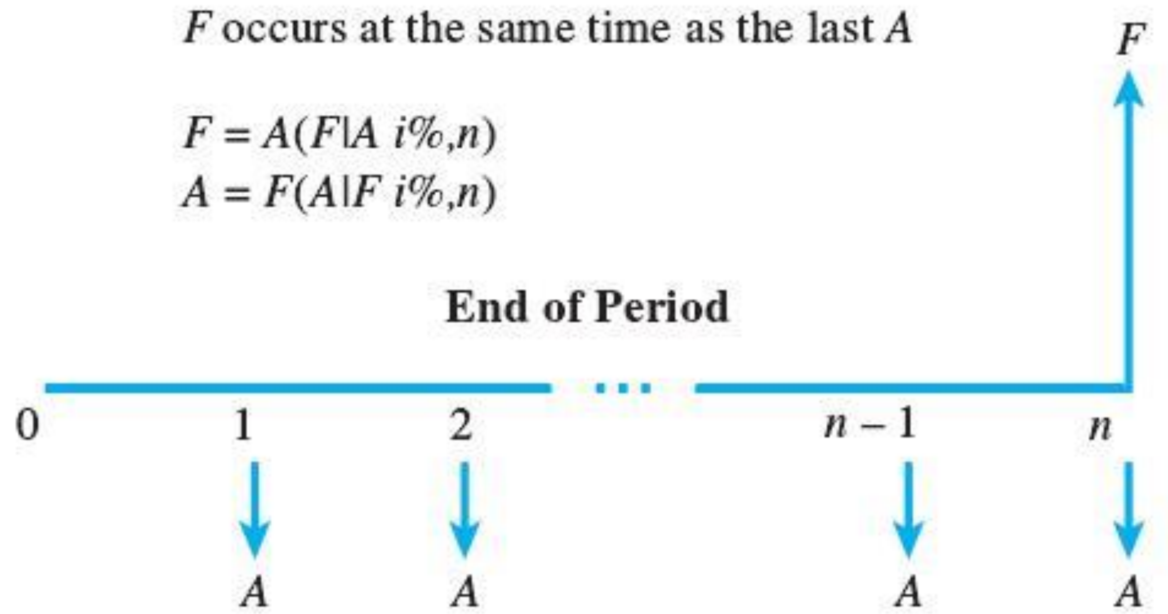


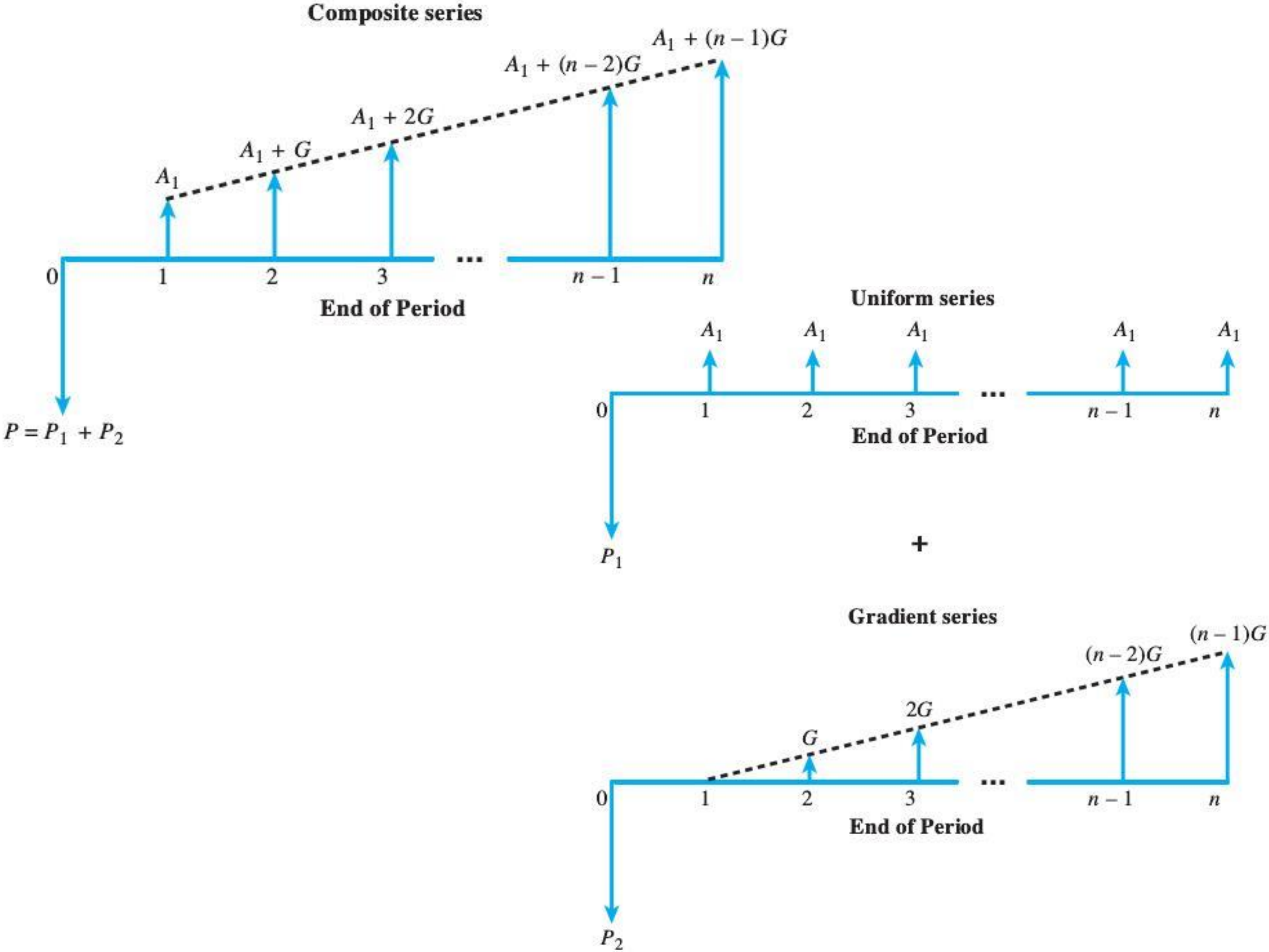
Illustration - Who Wants to Be a Millionaire?

Suppose Crystal Wilson wants to accumulate $1,000,000 by the time she retires in 40 years. If she earns 10 percent on her investments, how much must she invest each year in order to realize her goal?

- Using the Excel PMT function
- A =PMT(10%,40,,-1000000)= $2,259.41/ year

Gradient Series

- When the value of a given cash flow is greater than the value of the previous cash flow by a constant amount, G (the gradient step).
- The series can be represented by the sum of a uniform series and a gradient series.
- The gradient series is defined to have the first positive cash flow occur at the end of the second time period.
- Bonus increases, operating and maintenance costs often are approximated by a gradient series.


Composite series
A1 + (n – 1)G
A1 + (n – 2)G
A1 + 2G
A1 + G
A1
0
1
2
3
n – 1
n
End of Period
P = P1 + P2
Uniform series
A1
A1
A1
A1
A1
0
1
2
3
n – 1
n
End of Period
P1
+
Gradient series
(n – 1)G
(n – 2)G
2G
G
0
1
2
3
n – 1
n
End of Period
P2

- The cash flow in the gradient series occurring at the end of period t is, $A_t=(t-1)G \quad t = 1, \ldots\ldots, n$
- Recall $P = \sum_{t=1}^{n} A_t(1+i)^{-t}$
- Substituting A_t

$$P = G\sum_{t=1}^{n}(t-1)(1+i)^{-t}$$

which reduces to

$$P = G\left[\frac{1-(1+ni)(1+i)^{-n}}{i^2}\right]$$

or, equivalently,

$$P = G(P|G\ i\%,n)$$

- (P|G i%,n) is *the gradient series, present worth factor*

The uniform series equivalent to the gradient series is obtained by multiplying the value of the gradient series, present worth factor by the value of the $(A|P\ i\%,n)$ factor:

$$A = G\left[\frac{1}{i} - \frac{n}{i}(A|F\ i\%,n)\right]$$

or, equivalently,

$$A = G(A|G\ i\%,n) \tag{2.33}$$

- (A|G i%,n) is referred to as *the gradient-to-uniform series conversion factor*

To obtain the future worth equivalent of a gradient series at time n, multiply the value of the $(A|G\ i\%,n)$ factor by the value of the $(F|A\ i\%,n)$ factor to obtain the $(F|G\ i\%,n)$ factor:

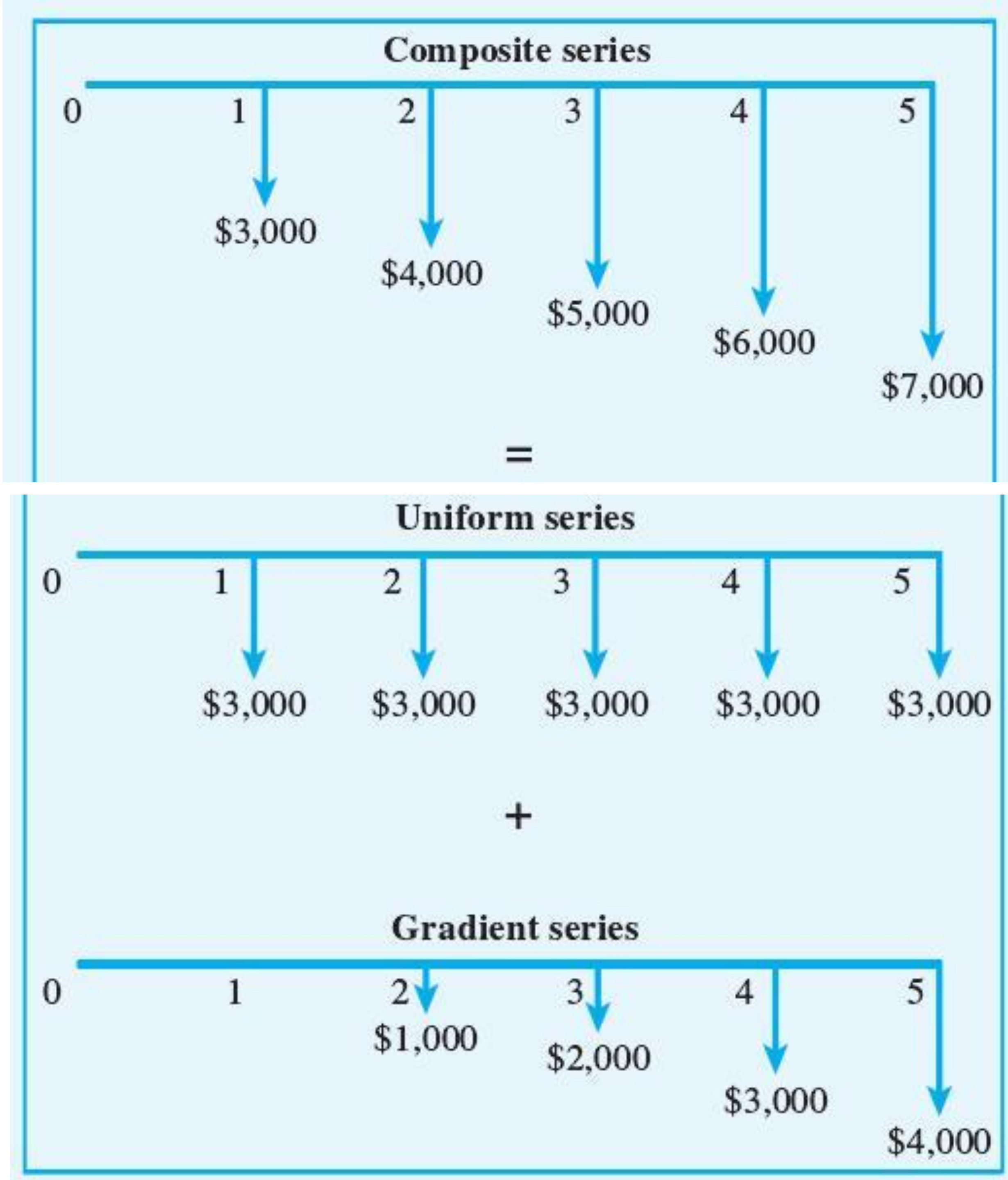
$$F = G\left[\frac{(1+i)^n - (1+ni)}{i^2}\right] \tag{2.34}$$

(F|G i%,n) is referred to as *the gradient series, future worth factor*

Illustration

Maintenance costs for a particular production machine increase by $1,000/year over the 5-year life of the equipment. The initial maintenance cost is $3,000. Using an interest rate of 8 percent compounded annually, determine the present worth equivalent for the maintenance costs.

For this problem, it is very helpful to sketch the cash flow diagram.


Composite series
0
1
2
3
4
5
$3,000
$4,000
$5,000
$6,000
$7,000
=
Uniform series
0
1
2
3
4
5
$3,000
$3,000
$3,000
$3,000
$3,000
+
Gradient series
0
1
2
3
4
5
$1,000
$2,000
$3,000
$4,000

Solution 1: the composite series may be converted to a uniform series plus a gradient series

- A_G= \$1,000(A|G 8%,5)=\$1,000*1.84647=\$1,846.47
- Adding the "converted" uniform series to the "base" uniform series
- A=\$1,846.47+\$3,000=\$4,846.47
- P=\$4,846.47(P|A 8%,5)=\$4,846.47*3.99271=\$19,350.55

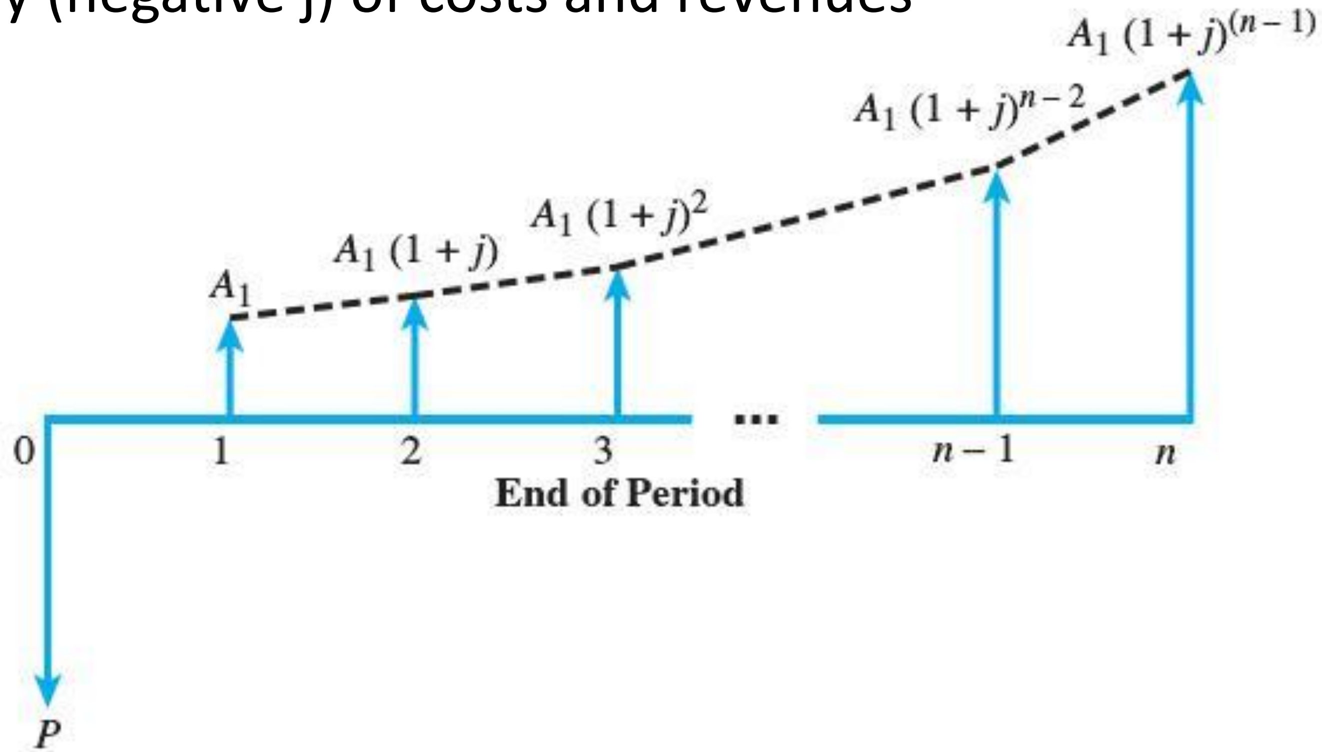
- Solution 2: the NPV worksheet function

- the following entry can be made in any cell: =NPV(8%,3000,4000,5000,6000,7000)
- or, =1000*NPV(8%,3,4,5,6,7)=$19,350.56.

- The uniform series equivalent and future worth equivalent can be obtained using the PMT and FV worksheet functions:
- A =PMT(8%,5,-19350.56)=$4846.47
- F =FV(8%,5,,-19350.56)=$28,432.32

Geometric Series

- When the size of a cash flow increases (decreases) by a fixed percent from one time period to the next.
- Let j denotes the percent change
- $A_t = A_{t-1}(1+j)$ t=2,..., n or, $A_t = A_1(1+j)^{t-1}$ t=1,..., n
- geometric series is used to represent the growth (positive j) or decay (negative j) of costs and revenues



- The present worth $P = A_1(1 + j)^{-1} \sum_{t=1}^{n} (1 + j)^t (1 + i)^{-t}$
- which equivalent to $P = \begin{cases} A_1\left[\dfrac{1 - (1 + j)^n (1 + i)^{-n}}{i - j}\right] & i \neq j \\ nA_1/(1 + i) & i = j \end{cases}$
- or written as $P=A_1(P|A_1\ i\%, j\%, n)$
- $(P|A_1\ i\%, j\%, n)$ is *the geometric series, present worth factor*

Illustration

A company is considering purchasing a new machine tool. In addition to the initial purchase and installation costs, management is concerned about the machine's maintenance costs, which are expected to be $1,000 at the end of the first year of the machine's life and increase 8 percent/year there-after. The machine tool's expected life is 15 years. Company management would like to know the present worth equivalent for expected costs. If the firm's time value of money is 10 percent/year compounded annually, what is the present worth equivalent?

- Using the NPV worksheet function
- entering the following in any cell: =1000*NPV(10%,1,1.08,1.08^2,1.08^3,1.08^4,1.08^5,1.08^6,1.08^7,1.08^8,1.08^9,1.08^10,1.08^11,1.08^12,1.08^13,1.08^14)

- or use spreadsheet (when more than 10 values are included in the range of cash flows, the spreadsheet approach is much preferred)

C19 fx =NPV(10%,C4:C18)

End of Year (n)	Cash Flow (CF)	
0	$0	
1	$1,000	
2	$1,080	
3	$1,166	
4	$1,260	
5	$1,360	
6	$1,469	
7	$1,587	
8	$1,714	
9	$1,851	
10	$1,999	
11	$2,159	
12	$2,332	=C14*1.08
13	$2,518	
14	$2,720	
15	$2,937	
P =	$12,030.40	=NPV(10%,C4:C18)

A uniform series equivalent to the geometric series is obtained by multiplying the value of the geometric series, present worth factor by the value of the $(A|P\ i\%,n)$ factor resulting in

$$A = \begin{bmatrix} A_1\left[\dfrac{1-(1+j)^n(1+i)^{-n}}{i-j}\right]\left[\dfrac{i(1+i)^n}{(1+i)^n-1}\right] & i \neq j \\ nA_1\left(\dfrac{i(1+i)^{n-1}}{(1+i)^n-1}\right) & i = j \end{bmatrix} \tag{2.39}$$

or

$$A = A_1(A|A_1\ i\%,j\%,n) \tag{2.40}$$

- $(A|A_1\ i\%,j\%,n)$ is the *geometric-to-uniform series conversion factor*.

The future worth equivalent of the geometric series is obtained by multiplying the value of the geometric series, present worth factor by the ($F|P\ i\%,n$) factor to obtain

$$F = \begin{cases} A_1\left[\dfrac{(1+i)^n - (1+j)^n}{i-j}\right] & i \neq j \\ nA_1(1+i)^{n-1} & i = j \end{cases} \tag{2.41}$$

or

$$F = A_1(F|A_1\ i\%,j\%,n)$$

- ($F|A_1$ i%,j%,n) is the *geometric series, future worth factor*

Example

Mattie Bookhout receives an annual bonus and deposits it in a savings account that pays 8 percent compounded annually. The size of her bonus increases by 10 percent each year; her initial deposit is $500. Determine how much will be in the fund immediately after her tenth deposit.

Solution: first using NPV worksheet function to find P, then find F

C14 =NPV(8%,C4:C13)

End of Year (n)	Cash Flow (CF)	
0	$0	
1	$500	
2	$550	
3	$605	
4	$666	=C6*1.1
5	$732	
6	$805	
7	$886	
8	$974	
9	$1,072	
10	$1,179	
P =	$5,035.12	=NPV(8%,C4:C13)
F =	$10,870.44	=FV(8%,10,,-C14)

Conclusion: using NPV function for solving the problem of a gradient/geometric series!!

COMPOUNDING FREQUENCY

- So far, when referring to an interest rate, we said that it was x% annual compound interest.
- All engineering economic analyses incorporate annual compounding.
- However in personal financing, compounding typically occurs more frequently than once a year.
- Credit cards charge interest (say, 1½ percent on the unpaid balance of the account) each month.
- Here *monthly compounding is at work*.
- When cash flow frequency and compounding frequency are different, the period interest rate or the effective interest rate must be employed.

Period Interest Rate

- 1½% per month=18% per annum compounded monthly, or 18% per year per month.
- The 18% is known as the nominal annual interest rate (r).
- The 1½% is known as the period interest rate.

$$\text{Period interest rate} = \frac{\text{Nominal annual interest rate}}{\text{Number of interest periods per year}}$$

Let r denotes nominal rate and m is the number of compounding periods in a year

$$i_{per} = r/m$$

Illustration

Two thousand dollars is invested in an account. What is the account balance after 3 years when the interest rate is:

a. 12 percent per year compounded monthly?

b. 12 percent per year compounded semiannually?

c. 12 percent per year compounded quarterly?

Solution: P = \$2,000; nominal interest rate = 12%; duration = 3 years; find period interest rate; number of interest periods; F

a. Period interest rate =(12%/year)/(12 months/year) =1%/month

Number of periods = 3 years(12 months/year) = 36 months

F = FV (1%, 36, -2000)= $2,861.54

b. Period interest rate =(12%/year)/(2 semiannual periods/year) =6%/semiannual period

Number of periods = 3 years(2 semiannual periods/year) = 6 semiannual periods

F = FV(6%, 6, -2000) = $2,837.04

c. Period interest rate =(12%/year)/(4 quarters/year) =3%/quarter

Number of periods = 3 years(4 quarters/year) = 12 quarters

F = FV(3%, 12, -2000) = $2,851.54

When nominal interest rate is fixed, the results of different compounding periods don't differ too much!

Determining Car Payments

Rebecca Carlson purchases a car for $25,000 and finances her purchase by borrowing the money at 8 percent per year compounded monthly; she pays off the loan with equal monthly payments for 5 years. What will be the size of her monthly loan payment?

Solution:

Period interest rate =(8%/year)/(12 months/year)=0.66667%/month

Number of periods=5 years(12 months/year) = 60 months

Using the Excel PMT function

A =PMT(0.066667,60,-25000)=$506.91

Effective Annual Interest Rate

- The annual interest rate equivalence to the period interest rate
- Example: if the interest rate is 12 percent per year compounded quarterly, then the period interest rate is 3 percent per quarter.
- Hence, $1 invested for 1 year at 3 percent per quarter has a future worth of F = FV(3%,4,-1) =$1.12551
- To obtain the same value in 1 year requires an annual compound interest rate of 12.551 percent, or $(1+0.03)^4-1$
- The 12.551% is called the effective annual interest rate (i_{eff})
- $i_{eff}=(1+r/m)^m-1=(1+i_{per})^m-1$, r denotes nominal rate and m is the number of compounding periods in a year
- The Excel EFFECT worksheet function can be used, parameters in order: r, m
- =EFFECT(r,m)

Distinguishing the three interest rates

Nominal annual interest rate, r=12%
Effective annual interest rate, i_{eff}=12.551%
Period interest rate, i_{per}=3%

Usually we know the nominal annual interest rate r,

$i_{per}=r/m$

$i_{eff}=(1+r/m)^m-1$

Note that,

a) r and i_{eff} are annual interest rates, while i_{per} is a period (e.g., monthly, quarterly, semi-annually) interest rate.

b) i_{eff} and i_{per} are actual (precise) interest rates, while r is a nominal (approximate) interest rate.

Illustration

Calculate the effective annual interest rate for each of the following cases:

(a) 12 percent per year compounded quarterly;

(b) 12 percent per year compounded monthly;

(c) and 12 percent per year compounded every minute.

Solution:

a. $i_{eff}= (1 + 0.12/4)^4 - 1 =12.551\%$

or using the Excel EFFECT function, i_{eff}=EFFECT(12%,4)=12.551%

b. $i_{eff}= (1 + 0.12/12)^{12} - 1 =12.683\%$

or using the Excel EFFECT function, i_{eff}=EFFECT(12%,12)=12.683%

a. m=(60 min/hour)(24 hour/day)(365 day/year)=525,600

$i_{eff}= (1 + 0.12/525{,}600)^{525{,}600} -1 =12.750\%$

or using the Excel EFFECT function, i_{eff}=EFFECT(12%, 525,600)=12.750%

When nominal interest rate is fixed, the results of different compounding periods don't differ too much (even when compounding every minute)!