C2475 Linear Algebra
Lecture 1: Systems of Linear Equations

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Contents

1 Solving systems of linear equations

1.1 Systems of linear equations

Consider the following chemistry problem (which I have lifted directly from Hefferon's book [?]). I've got x molecules of toluene, C_7H_8, and y molecules of nitric oxide, HNO_3. Putting them together, I can produce trinitrotoluene (TNT), which has the form $C_7H_5O_6N_3$, with some water (H_2O) byproduct. Say I produce z molecules of the first and w of the second. I'd like to balance this equation;

$$xC_7H_8 + yHNO_3 \to zC_7H_5O_6N_3 + wH_2O$$

That is, I'd like to find values of x, y, z, and w so that the number of atoms of each type is the same before and after the reaction. Counting them each individually, this leads to the following system of equations;

$$7x = 7z \quad (1)$$
$$8x + 1y = 5z + 2w \quad (2)$$
$$1y = 3z \quad (3)$$
$$3y = 6z + 1w \quad (4)$$

Of course, we're only interested in nonnegative integer values of x, y, z, w in this situation (since these represent the number of molecules we're dealing with), so we'll have to watch out for that when we're searching for solutions. Equation (1) clearly implies $x = z$, and so subtracting equation (3) from equation (2), we get

$$8z = 2z + 2w \implies 6z = 2w \implies w = 3z$$

So if (x, y, z, w) is any solution, then $x = z$, $y = 3z$, and $w = 3z$. Moreover, it's easy to check that no matter what choice I make for the value of z, then letting $x = z$ and $y, w = 3z$, I get a solution to the above system, and if z is a nonnegative integer then so are x, y, and w. So I've found a solution; in fact, infinitely many!

In general, there is a straightforward process which will tell us all of the solutions to a given system of linear equations, if there are any; and that there aren't any, if there aren't any. This is called Gaussian elimination. To describe it I'll need to formally define some of the concepts we've been using informally up to now.

Definition 1.1. *Let $x_1, \ldots, x_n$ be variables. A linear combination of $x_1, \ldots, x_n$ is an expression of the form $a_1x_1 + \cdots + a_nx_n$, where $a_1, \ldots, a_n \in \mathbb{R}$. $a_1, \ldots, a_n$ are called the coefficients of the combination. A linear equation in variables $x_1, \ldots, x_n$ is an equation that can be transformed to the form $a_1x_1 + \cdots a_nx_n = b$, where $b \in \mathbb{R}$. b is called the constant of the equation.*

A system of linear equations is simply a finite set of linear equations in the same variables, which can be transformed to the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

A tuple $(s_1, \ldots, s_n) \in \mathbb{R}^n$ is a solution to this system of equations if the equations obtained by replacing x_i by s_i for each i, in each equation, results in unanimously true statements. If S is a system of linear equations then we write sol(S) for its set of solutions. So sol$(S) \subseteq \mathbb{R}^n$.

Example 1.1. *Determine whether each of the following systems of equations is linear. If so, put it in standard format.*

(a) $\begin{aligned} x+2 &= y+z \\ 3x-y &= 4 \end{aligned}$ (b) $\begin{aligned} xy+2 &= 1 \\ 2x-6 &= y \end{aligned}$ (c) $\begin{aligned} x+2y &= -2y \\ 2x &= y \\ 2 &= x+y \end{aligned}$

Example 1.2. *Check if* $(3,5,0)$ *is a solution of linear system (a) in Example 1.1. What about* $(2,0,4)$ *and* $(2,2,2)$*?*

1.2 Matrix notation

A system of linear equations can be represented in a so-called matrix form. Let's define what a matrix is:

Definition 1.2. *A matrix is a rectangular array of numbers. If a matrix has m rows and n columns, then the size of the matrix is said to be $m \times n$. If the matrix is $1 \times n$ or $m \times 1$, it is called a vector. If $m = n$, then it is called a square matrix of order n. Finally, the number that occurs in the i-th row and j-th column is called the (i, j)th entry of the matrix. The leading entry of a row is the first nonzero element of that row, counting from left to right. If all entries are zero, the row have no leading entry.*
In many cases, the matrix is denoted by $A = [a_{ij}]_{m \times n}$ whose (i, j)th entry is a_{ij}. If the size is clear from context, we can simply write $A = [a_{ij}]$.

Example 1.3. *Consider the following matrices*

$$1) \begin{bmatrix} 3 & 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 9 \\ 0 & 2 & 0 & 1 & -3 \end{bmatrix} \qquad 2) \begin{bmatrix} 2 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 19 \end{bmatrix}$$

Find

a) the sizes of these matrices. Is there any square matrix?

b) the 2nd row vector of matrix 1) and the 3rd column vector of matrix 2).

c) the largest entry in matrix 1) and its position.

d) all the leading entries in matrix 1).

How can we define the general linear system of equations using matrices and vectors? First, there is the $m \times n$ coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Next, there is the $m \times 1$ right-hand-side column vector of constants

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Finally, stack this coefficient matrix and the vector along side each other to obtain the $m \times (n+1)$ augmented matrix

$$\tilde{A} = [A \,|\, b] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Example 1.4. *Find the associated augmented matrices of the following systems:*

a)

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= \frac{5}{2} \\ x_3 &= 5 \end{aligned}$$

b)

$$\begin{aligned} x_1 \qquad\qquad &= 1 \\ x_2 \qquad &= 0 \\ x_3 &= 5 \end{aligned}$$

c)

$$\begin{array}{rrrrrl} x_1 + & x_2 + & x_3 + & x_4 + & x_5 & = 1 \\ & & x_3 - & x_4 & & = 0 \\ x_1 - & 2x_2 & & + x_4 & & = 5 \\ & x_2 & & - x_4 & + 2x_5 & = 0 \end{array}$$

1.3 Echelon form

Definition 1.3. *When a system of equations is represented in the form of an augmented matrix, we often refer to the* i*th equation down as row* i, *or* ρ_i. *Of course, the order of the equations does not matter when considering solutions; it only matters to our written system. The first entry* a_{ij} *in row* i *such that* $a_{ij} \neq 0$ *is called the leading term of that row, and* a_{ij} *the leading coefficient. Note that a row may not have a leading term, e.g., if all of the coefficients are zero;*

$$0x_1 + 0x_2 + \cdots + 0x_n = b$$

A system and its corresponding augmented matrix are in row echelon form (or just echelon form) if the leading term of each row (except the first) is strictly to the right of all the leading terms of the rows above it, and all of the rows without a leading term are below the ones with one. Visually;

$$\begin{aligned}
a_{1j_1}x_{j_1} + \cdots + a_{1j_2}x_{j_2} + \cdots + a_{1j_i}x_{j_i} + \cdots + a_{1n}x_n &= \quad b_1 \\
a_{2j_2}x_{j_2} + \cdots + a_{2j_i}x_{j_i} + \cdots + a_{2n}x_n &= \quad b_2 \\
&\quad\;\; \vdots \\
a_{ij_i}x_{j_i} + \cdots + a_{in}x_n &= \quad b_i \\
0 &= b_{i+1} \\
&\quad\;\; \vdots \\
0 &= \quad b_m
\end{aligned}$$

or

$$\begin{bmatrix}
a_{1j_1} & \cdots & a_{1j_2} & \cdots & a_{1j_i} & \cdots & a_{1n} & b_1 \\
 & & a_{2j_2} & \cdots & a_{2j_i} & \cdots & a_{2n} & b_2 \\
 & & & & & & & \vdots \\
 & & & & a_{ij_i} & \cdots & a_{in} & b_i \\
 & & & & & & 0 & b_{i+1} \\
 & & & & & & & \vdots \\
 & & & & & & 0 & b_m
\end{bmatrix}$$

Here $j_1, j_2, \ldots,$ *etc are the column indices of the leading terms in rows* $1, 2, \ldots,$ *respectively, and row* i *is the last with a leading term.*

For instance, the following is in echelon form;

$$\begin{aligned}
x - 2y + z &= 0 \\
z &= 2
\end{aligned}$$

together with the corresponding augmented matrix:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

However, the following system is not in echelon form.

$$\begin{aligned}
y + 3z &= 1 \\
-x - y - z &= 0 \\
5y - 2z &= -1
\end{aligned}$$

together with the corresponding augmented matrix:

$$\begin{bmatrix} 0 & 1 & 3 & 1 \\ -1 & -1 & -1 & 0 \\ 0 & 5 & -2 & -1 \end{bmatrix}$$

Definition 1.4. *A system or matrix is called in reduced row echelon form (or just reduced echelon form) if*

- *it is in echelon form.*
- *each leading entry is 1.*
- *the leading entry in each row is the only non-zero entry in its column.*

Example 1.5. *Check if the following systems of equations or augmented matrices are in echelon form? Reduced echelon form?*

a)

$$\begin{aligned} z &= -1 \\ x - y - z &= 1 \\ 2x - y + z &= 0 \end{aligned}$$

b)

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c)

$$\begin{aligned} x - y - z &= 1 \\ -y + z &= 0 \\ z &= -1 \end{aligned}$$

2 Gaussian elimination

2.1 Elementary row operations

The object of Gaussian elimination is to produce, through a series of operations on a given system of linear equations, a final system which is in echelon form. There are three different operations that we use;

- swap: Swapping two of the rows, written E_{ij} (where $i \neq j$).
- scaling: Multiplying some row by a nonzero number; $E_i(\lambda)$, where $\lambda \neq 0$.
- row combination: Replacing the row i by the sum of itself with a multiple of row j; $E_{ij}(\lambda)$. Here λ may be zero but i and j must be different.

These operations are called elementary row operations.

There is a systematic way of applying these operations to get a system in echelon form, but first we need to know that they won't change the solution set we're after. This is proven in the following theorem.

Theorem 2.1. *If S is a system of linear equations and T is the result of applying one of the above operations to S, then S and T have exactly the same set of solutions, i.e., $sol(S) = sol(T)$.*

Proof. For clarity, let's say S is the following system of linear equations;

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

To prove the theorem it suffices to show;

1. If $r \in \mathbb{R}^n$ is a solution to S then r is also a solution to T.
2. There is a row-operation which, when applied to T, produces S.

(1) proves that $\mathrm{sol}(S) \subseteq \mathrm{sol}(T)$; then, (2) and (1) prove together that $\mathrm{sol}(T) \subseteq \mathrm{sol}(S)$.

Let's prove (1) first. Let $(s_1, \dots, s_n) \in \mathbb{R}^n$ be a solution to S. This means we have

$$\begin{aligned} a_{11}s_1 + a_{12}s_2 + \cdots + a_{1n}s_n &= b_1 \\ &\vdots \\ a_{m1}s_1 + a_{m2}s_2 + \cdots + a_{mn}s_n &= b_m \end{aligned}$$

To fully prove (1) we'd have to handle three cases, according to which type of row operation we applied to S to get T; I'll just do the row-combination case, since that's the hardest. So for some $i \neq j$ and some $\lambda \in \mathbb{R}$, T is the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ (a_{i1}x_1 + \cdots a_{in}x_n) + \lambda(a_{j1}x_1 + \cdots a_{jn}x_n) &= b_i + \lambda b_j \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Since $(s_1, \dots, s_n)$ is a solution to S, we have $a_{i1}s_1 + \cdots a_{in}s_n = b_i$ and $a_{j1}s_1 + \cdots a_{jn}s_n = b_j$, so

$$(a_{i1}s_1 + \cdots a_{in}s_n) + \lambda(a_{j1}s_1 + \cdots a_{jn}s_n) = b_i + \lambda b_j$$

The other equations in T are the same as those in S. Hence $(s_1, \dots, s_n)$ is a solution to T.

Now we prove (2). This comes down to another proof by cases, based on which type of row operation we applied to get T from S. I'll write the row operation down along with its reverse.

- To reverse a swap $\rho_i \leftrightarrow \rho_j$, we just apply the same swap again; $\rho_i \leftrightarrow \rho_j$.
- To reverse a row combination $\rho_i \to \rho_i + \lambda\rho_j$ we apply the row combination $\rho_i \to \rho_i - \lambda\rho_j$. (It's implicit that $i \neq j$, here. Why is this important?)
- To reverse a scaling operation $\rho_i \to \lambda\rho_i$, we apply $\rho_i \to \frac{1}{\lambda}\rho_i$. (Here it's implicit that $\lambda \neq 0$.)

This completes the proof of the theorem.

$\square$

2.2 Gaussian elimination

Now I'll describe the Gaussian elimination algorithm. Let S be the following system.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Perform the following operations for each i from 1 to m in that order, or until I tell you to stop. The resulting system will be in echelon form.

1 Find the row, below row i, with the leftmost leading term among those rows. If there is no such thing (ie, if $a_{jk} = 0$ for all $j \geq i$ and $1 \leq k \leq m$), stop. Otherwise, swap this row with row i.

2 Say the leading coefficient in row i is in column j. For each $k > i$, perform the row combination $E_{ki}(-a_{kj}/a_{ij}) : \rho_k \to \rho_k - (a_{kj}/a_{ij})\rho_i$.

3 Repeat.

After the first i steps of this algorithm, we've ensured that the leading terms of the first i rows go from left to right (this is easily proven by induction on i). If we go through every row, then we've ensured that the leading terms are ordered this way throughout the whole matrix. If we stop at row i, then none of the rows below have leading terms. Either way, after finishing, the system is in echelon form. The columns that contain a leading entry are called pivot columns.

After we obtain the echelon form, continue to perform the following operations to obtain the reduced echelon form.

1 Based on the echelon form, identify the last row which has non-zero entries. Perform a scaling operation to make the leading entry in that row to be 1.

2 conduct row combinations to each of the upper rows, until every element above the leading entry equals 0.

3 Moving up the matrix, repeat this process for each row.

Row echelon form pattern

The following are two typical row echelon matrices.

$$\begin{bmatrix} \bullet & * & * & * & * & * & * & * & * \\ 0 & \bullet & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \bullet & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the circled stars $\bullet$ represent arbitrary nonzero numbers, and the stars $*$ represent arbitrary numbers, including zero. The following are two typical reduced row echelon matrices.

$$\begin{bmatrix} 1 & 0 & * & * & 0 & * & 0 & * & * \\ 0 & 1 & * & * & 0 & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & * & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

What are the pivot columns?

For instance, let's conduct Gaussian elimination on the following system to obtain its echelon form

$$\tilde{A} = [A \,|\, \vec{b}] = \begin{bmatrix} 0 & 1 & 9 & 2 & 2 \\ 1 & 0 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

We will conduct the following elementary row operations:

$$i=1: \quad \begin{bmatrix} 0 & 1 & 9 & 2 & 2 \\ 1 & 0 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{E_{12}} \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 9 & 2 & 2 \\ 2 & 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{E_{31}(-2)} \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 9 & 2 & 2 \\ 0 & 1 & -4 & -1 & -5 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$i=2: \quad \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 9 & 2 & 2 \\ 0 & 1 & -4 & -1 & -5 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{E_{32}(-1)} \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 9 & 2 & 2 \\ 0 & 0 & -13 & -3 & -7 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$i=3: \quad \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 9 & 2 & 2 \\ 0 & 0 & -13 & -3 & -7 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{E_{43}(1/13)} \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 9 & 2 & 2 \\ 0 & 0 & -13 & -3 & -7 \\ 0 & 0 & 0 & -3/13 & 6/13 \end{bmatrix}$$

Or we can also do the following:

$$i=1: \quad \begin{bmatrix} 0 & 1 & 9 & 2 & 2 \\ 1 & 0 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{E_{12}} \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 9 & 2 & 2 \\ 2 & 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{E_{31}(-2)} \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 9 & 2 & 2 \\ 0 & 1 & -4 & -1 & -5 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$i=2: \quad \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 9 & 2 & 2 \\ 0 & 1 & -4 & -1 & -5 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{E_{32}(-1)} \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 9 & 2 & 2 \\ 0 & 0 & -13 & -3 & -7 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{E_{34}} \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 9 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -13 & -3 & -7 \end{bmatrix}$$

$$i=3: \quad \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 9 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -13 & -3 & -7 \end{bmatrix} \xrightarrow{E_{43}(13)} \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 9 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -3 & 6 \end{bmatrix}$$

We see that they are both in echelon form. But they are different matrices. The echelon form is not unique.
Now let's try to find the reduced echelon form based on each of the echelon forms we had.

$$i=4: \begin{bmatrix}1&0&3&1&2\\0&1&9&2&2\\0&0&-13&-3&-7\\0&0&0&-3/13&6/13\end{bmatrix} \xrightarrow{E_4(-13/3)} \begin{bmatrix}1&0&3&1&2\\0&1&9&2&2\\0&0&-13&-3&-7\\0&0&0&1&-2\end{bmatrix} \xrightarrow[E_{14}(-1)]{E_{34}(3),E_{24}(-2)} \begin{bmatrix}1&0&3&0&4\\0&1&9&0&6\\0&0&-13&0&-13\\0&0&0&1&-2\end{bmatrix}$$

$$i=5: \begin{bmatrix}1&0&3&0&4\\0&1&9&0&6\\0&0&-13&0&-13\\0&0&0&1&-2\end{bmatrix} \xrightarrow{E_3(-1/13)} \begin{bmatrix}1&0&3&0&4\\0&1&9&0&6\\0&0&1&0&1\\0&0&0&1&-2\end{bmatrix} \xrightarrow[E_{23}(-9)]{E_{14}(-3)} \begin{bmatrix}1&0&0&0&1\\0&1&0&0&-3\\0&0&1&0&1\\0&0&0&1&-2\end{bmatrix}$$

Let's try to find the reduced echolon form based on the other echelon form:

$$i=4: \begin{bmatrix}1&0&3&1&2\\0&1&9&2&2\\0&0&1&0&1\\0&0&0&-3&6\end{bmatrix} \xrightarrow{E_4(-1/3)} \begin{bmatrix}1&0&3&1&2\\0&1&9&2&2\\0&0&1&0&1\\0&0&0&1&-2\end{bmatrix} \xrightarrow[E_{24}(-2)]{E_{14}(-1)} \begin{bmatrix}1&0&3&0&4\\0&1&9&0&6\\0&0&1&0&1\\0&0&0&1&-2\end{bmatrix}$$

$$i=5: \begin{bmatrix}1&0&3&0&4\\0&1&9&0&6\\0&0&1&0&1\\0&0&0&1&-2\end{bmatrix} \xrightarrow[E_{23}(-9)]{E_{14}(-3)} \begin{bmatrix}1&0&0&0&1\\0&1&0&0&-3\\0&0&1&0&1\\0&0&0&1&-2\end{bmatrix}$$

Compare the reduced echelon forms and see what you find!
They are identical!!! In fact, we have the following theorem.

Theorem 2.2. *Every matrix can be reduced using a sequence of elementary row operations to an unique reduced row echelon form.*

Example 2.1. *Find a echelon form of the following system.*

a)

$$\begin{bmatrix} 3 & 0 & 0 & 2 \\ -3 & 1 & 6 & -5 \\ 0 & 0 & 1 & 2 \\ 6 & 0 & 0 & 4 \end{bmatrix}$$

$$\xrightarrow{E_{41}(-2)} \begin{bmatrix} 3 & 0 & 0 & 2 \\ -3 & 1 & 6 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{E_{21}(1)} \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 1 & 6 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{E_{1}(\frac{1}{3})} \begin{bmatrix} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 6 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{E_{23}(-6)} \begin{bmatrix} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & -15 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b)

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 1 \\ x_3 - x_4 &= 0 \\ x_1 - 2x_2 + x_4 &= 5 \\ x_2 - x_4 + 2x_5 &= 0 \end{aligned}$$

$$A \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & -1 & 2 & 0 \end{bmatrix} \xrightarrow{E_{23},\ E_{34}}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{matrix} \overline{E_{21}(-1)} \\ E_{12}(\frac{1}{3}) \\ E_{41}(-1) \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -3 & \cancel{0}\,-1 & 0 & -1 & 4 \\ 0 & 0 & -\frac{1}{3} & 0 & \frac{5}{3} & \frac{5}{3} \end{bmatrix}$$

Example 2.2. *Find the reduced echelon form of the systems in the previous example.*

3 Solving linear system

3.1 consistent and inconsistent systems

Let's consider several examples first.

Example 3.1. *Solve the following system*

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 + x_2 + 5x_3 &= 2 \\ 3x_1 + 3x_2 + 4x_3 &= 1 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & 5 & 2 \\ 3 & 3 & 4 & 1 \end{bmatrix} \xrightarrow[E_{31}(-3)]{E_{21}(-2)} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 7 & 0 \\ 0 & -3 & 7 & -2 \end{bmatrix} \xrightarrow{E_{32}(-1)} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 7 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$\xrightarrow{E_2(-\frac{1}{3})} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -\frac{7}{3} & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow{E_{12}(-2)} \begin{bmatrix} 1 & 0 & \frac{11}{3} & 1 \\ 0 & 1 & -\frac{7}{3} & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow[E_3(-\frac{1}{2})]{E_{13}(\frac{1}{2})} \begin{bmatrix} 1 & 0 & \frac{11}{3} & 0 \\ 0 & 1 & -\frac{7}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In the previous example, we obtain an equation of the form

$$0x_1 + 0x_2 + \cdots + 0x_n = b$$

at the bottom of the echelon form. This equation is either true or false outright, depending on whether $b = 0$ or $b \neq 0$. If $b \neq 0$ then we call the equation inconsistent. If, after performing Gaussian elimination to bring a system into echelon form, we find an inconsistent equation at the bottom, then we've determined that it has no solutions, since no system with an inconsistent equation can have a solution. But what if there are no inconsistent equations at the bottom? Do we have solutions in this case?

Example 3.2. *Solve the following system*

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 2 \\ 3x_1 + 3x_2 + 4x_3 &= 1 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & 4 & 2 \\ 3 & 3 & 4 & 1 \end{bmatrix} \xrightarrow[\substack{E_{21}(-1) \\ E_{31}(-3)}]{} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 6 & 0 \\ 0 & -3 & 7 & -2 \end{bmatrix} \xrightarrow[E_{23}(-1)]{} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & -3 & 7 & -2 \end{bmatrix}$$

$$\xrightarrow[E_{23}]{} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 7 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow[E_{23}(7)]{} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 0 & 12 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow[\substack{E_2(-\frac{1}{4}) \\ E_3(-1)}]{} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow[\substack{E_{12}(-2) \\ E_{13}(1)}]{} \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

In the previous example, we have one unique solution. We have learned two examples: one with no solution, the other with one solution. Are there other cases? Consider the following example.

Example 3.3. *Solve the following system*

$$
\begin{aligned}
x_1 - \ \ x_2 + \ \ x_3 - x_4 &= 2 \\
x_1 - \ \ x_2 + \ \ x_3 + x_4 &= 0 \\
4x_1 - 4x_2 + 4x_3 \qquad &= 4 \\
-2x_1 + 2x_2 - 2x_3 + x_4 &= -3
\end{aligned}
$$

$$
\begin{matrix} 1 & -1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 0 \\ 4 & -4 & 4 & 0 & 4 \\ -2 & 2 & -2 & 1 & -3 \end{matrix}
\xrightarrow[E_4(-\frac{1}{2})]{E_3(\frac{1}{4})}
\begin{matrix} 1 & -1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \\ 1 & -1 & 1 & -\frac{1}{2} & \frac{3}{2} \end{matrix}
\quad
\begin{matrix} E_{21}(-1) \\ E_{31}(-1) \\ E_{41}(-1) \end{matrix}
\longrightarrow
\begin{matrix} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{matrix}
$$

$x_1 - x_2 + x_3 = 1$

$x_4 = -1$

Definition 3.1. *A system of linear equations is consistent if it has at least one solution. Otherwise it is called inconsistent.*

Definition 3.2. *A variable in a consistent linear system is called free if its corresponding column in the coefficient matrix is not a pivot column.*

Hence, The system in Example 3.1 is inconsistent since it has no solution. The system in Example 3.2 is consistent. It has an unique solution but no free variables. The system in Example 3.3 is also consistent. It has infinitely many solutions and it has 2 free variables.

Theorem 3.1. *A linear system is consistent if and only if the row echelon form of its augmented matrix contains no row of the form*

$$\begin{bmatrix} 0 & ,\dots, & 0 & b \end{bmatrix}$$

where $b \neq 0$.
For consistent systems, it has an unique solution if and only if there is no free variables. It has infinitely many solutions if and only if there is at least one free variable.

Example 3.4. *Determine if the following system is consistent? If so, are there any free variables?*

$$\tilde{A} = [A \,|\, \vec{b}\,] = \begin{bmatrix} 0 & 0 & 1 & -1 & 2 & 1 & 0 \\ 3 & 6 & 0 & 3 & -3 & 2 & 7 \\ 1 & 2 & 0 & 1 & -1 & 0 & 1 \\ 2 & 4 & -2 & 4 & -6 & -5 & -4 \end{bmatrix}$$

$$E_{41} \begin{bmatrix} 2 & 4 & -2 & 4 & -6 & -5 & -4 \\ 3 & 6 & 0 & 3 & -3 & 2 & 7 \\ 1 & 2 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 & 0 \end{bmatrix}$$

Example 3.5. *For what values of* c *are the following systems inconsistent, with unique solution or with infinitely many solutions?*

$$\tilde{A} = [A \,|\, \vec{b}\,] = \begin{bmatrix} 1 & 2 & -1 & c \\ 1 & 3 & 1 & 1 \\ 3 & 7 & -1 & 4 \end{bmatrix}$$

$$\begin{matrix} 1 & 2 & -1 & c \\ 1 & 3 & 1 & 1 \\ 3 & 7 & -1 & 4 \end{matrix} \xrightarrow[E_{31}(-3)]{E_{21}(-1)} \begin{matrix} 1 & 2 & -1 & c \\ 0 & 1 & 2 & 1-c \\ 0 & 1 & 2 & 4-3c \end{matrix} \xrightarrow{E_{32}(-1)} \begin{matrix} 1 & 2 & -1 & c \\ 0 & 1 & 2 & 1-c \\ 0 & 0 & 0 & 3-2c \end{matrix}$$

$c \neq \frac{3}{2}$, it will hape no solution

3.2 Rank and nullity of a matrix

Definition 3.3. *The rank of a matrix A is the number of nonzero rows of the row echelon form of A. It is denoted as $\text{rank}(A)$ or rank A.*
The nullity of a matrix A is the number of columns of the row echelon form of A that do not contain a leading entry. It is denoted as $\text{null}(A)$ or null A. The rank of a matrix with all the entries being zero (zero matrix) is 0.

In the case that A is the coefficient matrix of a linear system, we can interpret the nullity of A as the number of free variables of the system.

Example 3.6. *The rank of the following matrices can be determined by inspection. Inspect these matrices and specify their rank.*

(a) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

3 0 3 $\cancel{3}$ $\cancel{2}$

Example 3.7. *Find the rank and nullity of the matrix* $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 5 \\ 3 & 3 & 2 \end{bmatrix}$

$$\xrightarrow[E_{31}(-3)]{E_{21}(-1)} \begin{matrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -4 \end{matrix} \xrightarrow{E_{32}(4)} \begin{matrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}$$

$r = 2$

$n = 1$

column

Example 3.8. *Find the rank and nullity of the matrix* $\begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 4 & 0 & 2 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix}$

$$\xrightarrow[E_{31}(-1)]{E_{21}(-1)} \begin{matrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{matrix} \xrightarrow[E_{24}]{E_{43}(1)} \begin{matrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

$r = 2$

$n = 3$

Theorem 3.2. *Let A be an $m \times n$ matrix. Then*

1) $0 \le rank(A) \le \min\{m, n\}$.
2) $rank(A) + null(A) = n$.

Theorem 3.3 (Consistency in terms of rank)**.** *The general linear system with $m \times n$ coefficient matrix A and right-hand-side vector $\vec{b}$, and the augmented matrix $\tilde{A} = [A\,|\vec{b}\,]$ is consistent if and only if $rank(A) = rank(\tilde{A})$, in which case either*

1) *$rank(A) = n$, in which case the system has a unique solution, or*
2) *$rank(A) < n$, in which case the system has infinitely many solutions.*

Example 3.9. *Determine the rank of the augmented matrix of the following system. Based on that result, determine if it is consistent. If so, how many solutions are there?*

$$\begin{aligned} x_1 - x_2 + x_3 - x_4 &= 2 \\ x_1 - x_2 + x_3 + x_4 &= 0 \\ 4x_1 - 4x_2 + 4x_3 &= 4 \\ -2x_1 + 2x_2 - 2x_3 + x_4 &= -3 \end{aligned}$$

$$\begin{matrix} 1 & -1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 0 \\ 4 & -4 & 4 & 0 & 4 \\ -2 & 2 & -2 & 1 & -3 \end{matrix} \xrightarrow{} \begin{matrix} E_{21}(-1) \\ E_{31}(-4) \\ E_{41}(2) \end{matrix} \begin{matrix} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & -1 & \end{matrix}$$

$$\begin{matrix} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Example 3.10. *Find the unique value of t for which the following system is consistent*

$$\begin{array}{rcrcrcrcl} -x_1 & & & + & x_3 & - & x_4 & = & 3 \\ 2x_1 & + & 2x_2 & - & x_3 & - & 7x_4 & = & 1 \\ 4x_1 & - & x_2 & - & 9x_3 & - & 5x_4 & = & t \\ 3x_1 & - & x_2 & - & 8x_3 & - & 6x_4 & = & 1 \end{array}$$

$$\begin{matrix} -1 & 0 & 1 & -1 & 3 \\ 2 & 2 & -1 & -7 & 1 \\ 4 & -1 & -9 & -5 & t \\ 3 & -1 & -8 & -6 & 1 \end{matrix} \xrightarrow{\begin{matrix} E_{21}(2) \\ E_{31}(-4) \\ E_{41}(3) \end{matrix}} \begin{matrix} -1 & 0 & 1 & -1 & 3 \\ 0 & 2 & 1 & -9 & 7 \\ 0 & -1 & -13 & -1 & t-12 \\ 0 & -1 & -5 & -9 & 10 \end{matrix} \xrightarrow{E_{24}} \begin{matrix} -1 & 0 & 1 & -1 & 3 \\ 0 & -1 & -5 & -9 & 10 \\ 0 & -1 & -13 & -1 & t+12 \\ 0 & 2 & 1 & -9 & 7 \end{matrix}$$

$$\xrightarrow{\begin{matrix} E_{32}(-1) \\ E_{42}(2) \end{matrix}} \begin{matrix} -1 & 0 & 1 & -1 & 3 \\ 0 & -1 & -5 & -9 & 10 \\ 0 & 0 & -8 & 8 & t-22 \\ 0 & 0 & -9 & -27 & -27 \end{matrix} \xrightarrow{\begin{matrix} E_3(-\frac{1}{8}) \\ E_4(-9) \end{matrix}} \begin{matrix} -1 & 0 & 1 & -1 & 3 \\ 0 & -1 & -5 & -9 & 10 \\ 0 & 0 & 1 & -1 & \frac{22-t}{8} \\ 0 & 0 & 1 & 3 & 3 \end{matrix}$$

4 Exercise

1. Determine if each of the following systems of linear equations is linear. If so put it in standard form.

(a) $\begin{aligned} x+2 &= 1 \\ x+3 &= y^2 \end{aligned}$ (b) $\begin{aligned} x+2z &= y \\ 3x-y &= y \end{aligned}$ (c) $\begin{aligned} x+y &= -3y \\ 2x &= xy \end{aligned}$

2. Express the following systems of equations in the notation of the definition of linear systems by specifying the numbers of m, n, a_{ij}, and b_i.

(a) $\begin{aligned} x_1-2x_2+x_3 &= 2 \\ x_2 &= 1 \\ -x_1+x_3 &= 1 \end{aligned}$ (b) $\begin{aligned} x_1-3x_2 &= 1 \\ x_2 &= 5 \end{aligned}$

3. For each of the following matrices, identify the size.

(a) $\begin{bmatrix} 1 & -1 & 2 & 1 \\ -2 & 2 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ 2 & -1 \\ 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

4. Exhibit the augmented matrix of each system and give its size.

(a) $\begin{aligned} 2x+3y &= 7 \\ x+2y &= -2 \end{aligned}$ (b) $\begin{aligned} 3x_1+6x_2-x_3 &= -4 \\ -2x_1-4x_2+x_3 &= 3 \\ x_3 &= 1 \end{aligned}$ (c) $\begin{aligned} x_1+x_2 &= -2 \\ 5x_1+2x_2 &= 5 \\ x_1+2x_2 &= -7 \end{aligned}$

5. Use Gaussian elimination to find the solution of the following systems.

(a) $\begin{aligned} x+3y &= 7 \\ x+2y &= 1 \end{aligned}$ (b) $\begin{aligned} 2x_1+6x_2 &= 2 \\ -2x_1+x_2 &= 1 \end{aligned}$ (c) $\begin{aligned} x_1+x_2 &= 1 \\ 5x_1+2x_2 &= 5 \\ x_1+2x_2 &= -7 \end{aligned}$

6. Use Gaussian elimination to find the solution of the following systems.

(a) $\begin{aligned} x_1+x_2+x_4 &= 1 \\ 2x_1+2x_2+x_3+x_4 &= 1 \\ 2x_1+2x_2+2x_4 &= 2 \end{aligned}$ (b) $\begin{aligned} x_3+x_4 &= 0 \\ -2x_1-4x_2+x_3 &= 0 \\ -x_3+x_4 &= 0 \end{aligned}$ (c) $\begin{aligned} x_1+x_2+3x_3 &= 2 \\ 2x_1+5x_2+9x_3 &= 1 \\ x_1+2x_2+4x_3 &= 1 \end{aligned}$

7. Each of the following matrices results from apply Gaussian elimination to the augmented matrix of a linear system. In each case, write out the general solution to the system or indicate that it is inconsistent.

(a) $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

8. Circle leading entries and determine which of the following matrices are in echelon form or reduced echelon form.

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (h) $[1\ 3]$

9. The rank of the following matrices can be determined by inspection. Inspect these matrices and specify their rank.

(a) $\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

10. Find the rank of the augmented and coefficient matrix of the following linear systems and the solution sets to the following systems. Are these systems equivalent?

(a)
$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 2 \\ 2x_1 + x_2 - 2x_4 &= 1 \\ 2x_1 + 2x_2 + 2x_3 - 2x_4 &= 4 \end{aligned}$$

(b)
$$\begin{aligned} x_3 + x_4 &= 0 \\ -2x_1 - 4x_2 &= 0 \\ 3x_1 + 6x_2 - x_3 + x_4 &= 0 \end{aligned}$$

11. For what values of c are the following systems inconsistent, with unique solution, or with infinitely many solutions?

(a)
$$\begin{aligned} x_2 + cx_3 &= 0 \\ x_1 - cx_2 &= 1 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 2x_2 - x_3 &= c \\ x_1 + 3x_2 + x_3 &= 1 \\ 3x_1 + 7x_2 - x_3 &= 4 \end{aligned}$$

(c)
$$\begin{aligned} cx_1 + x_2 + x_3 &= 2 \\ x_1 + cx_2 + x_3 &= 2 \\ x_1 + x_2 + cx_3 &= 2 \end{aligned}$$