B0684
Economic Engineering Analysis

Time Value of Money

Learning Objective

- Cash Flow Diagram (CFD)
- Single cash flow calculation
  - Future worth
  - Present worth
- Multiple cash flow calculation
  - Irregular cash flow
  - Uniform series
  - Gradient series
  - Geometric series
- Compounding frequency
  - Period interest rate
  - Effective annual interest rate

This chapter is very important!!

It serves as a foundation for the remainder of the module.

CASH FLOW DIAGRAMS

- A diagram depicting the magnitude and timing of cash flowing in and out of the investment alternative.

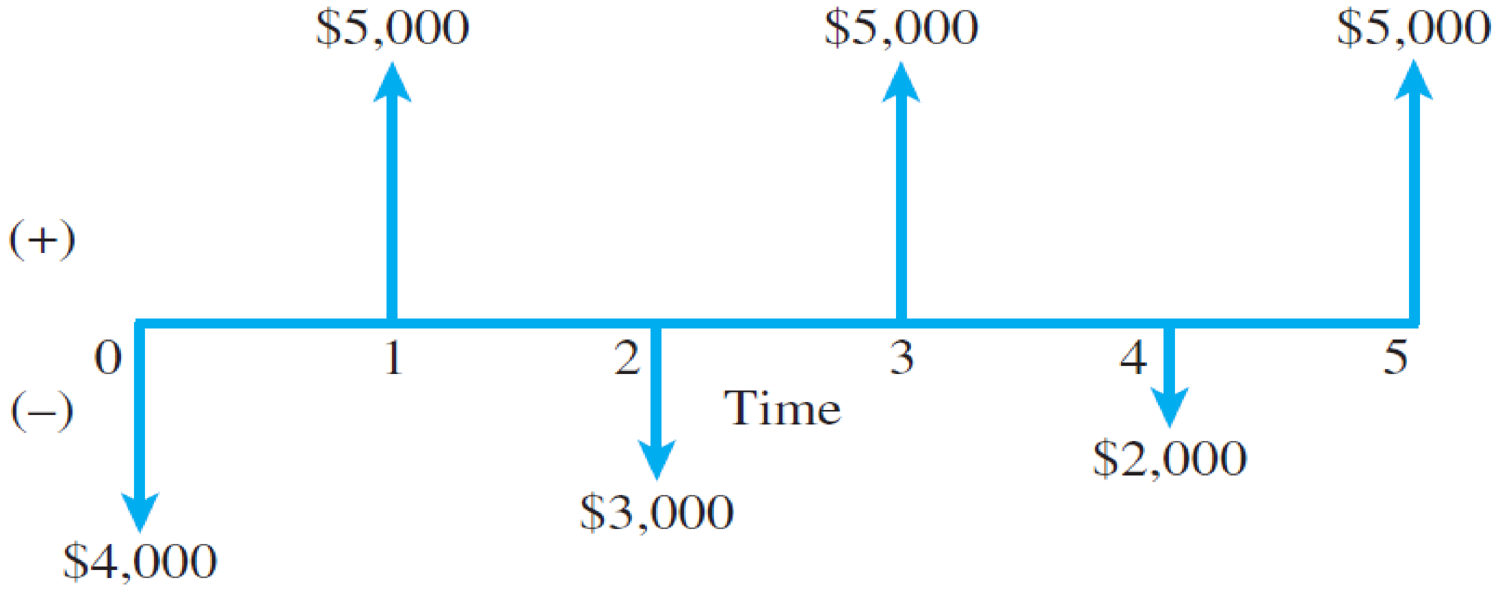
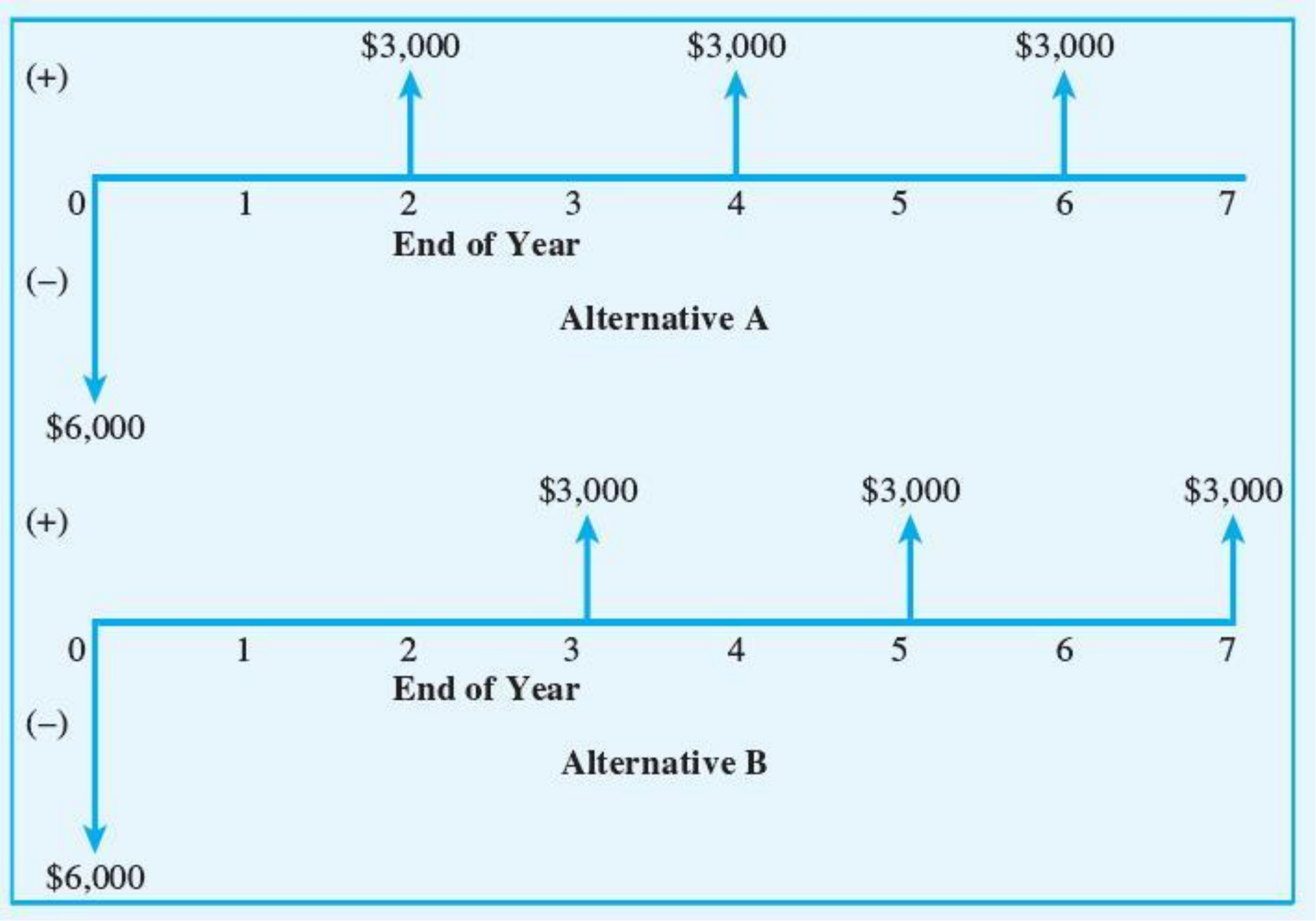


FIGURE 2.1 A Cash Flow Diagram (CFD)

- A horizontal line as a time scale
- Vertical arrows indicating cash flows
  - An upward arrow indicates a cash inflow or positive-valued cash flow (receipts)
  - A downward arrow indicates a cash outflow or negative-valued cash flow (expenditures)
- The lengths of the arrows suggest the magnitudes of cash flows (but not precisely)

Which investment alternative will you choose?



Recall TVOM, when receiving a given sum of money, we prefer to receive it sooner!

Alternative A is preferred to Alternative B

- End-of-period cash flows, end-of-year cash flows, end-of-period compounding, are assumed unless otherwise noted
- The end of period t is the beginning of period t+1
- The role of CFDs:
  - Clearly communicate cash flows (than words do)
  - Identify cash flow patterns (e.g., a uniform/gradient/geometric series)

SINGLE CASH FLOWS

The simplest scenario: there is only one cash flow in the planning horizon.

Compound interest is used in almost all business and lending situations:

- Interest should be charged (or earned) against both the principal and accumulated interest to date.

Future Worth

$$F_n = F_{n-1}(1+i)$$

F_0=P, where P is the present value of a single sum of money; F_n is the accumulated value of P over n years; i – interest rate

$$I_n = \sum_{t=1}^{n} iF_{t-1}$$

In – the accumulated (total) interest over n years

Illustration

Suppose you loan $10,000 for 1 year to an individual who agrees to pay you interest at a compound rate of 10 percent/year. At the end of 1 year, the individual asks to extend the loan period an additional year. The borrower repeats the process several more times. Five years after loaning the person the $10,000, how much would the individual owe you?

Year	Unpaid Balance at the Beginning of the Year	Annual Interest	Payment	Unpaid Balance at the End of the Year
1	$10,000.00	$1,000.00	$0.00	$11,000.00
2	$11,000.00	$1,100.00	$0.00	$12,100.00
3	$12,100.00	$1,210.00	$0.00	$13,310.00
4	$13,310.00	$1,331.00	$0.00	$14,641.00
5	$14,641.00	$1,464.10	$16,105.10	$0.00

$$F=P(1+i)^n$$

P = present worth.

F = future worth. F occurs n periods after P.

i = the interest rate, expressed as a decimal or percentage.

n = the number of interest periods.

- $(1+i)^n$ is referred to as *the single sum, future worth factor*.
- It is denoted (F|P i%, n), and reads "**the F, given P factor at i% for n periods**" Thus,

$$F=P(F|P\ i\%,\ n)$$

Excel financial function FV

- Parameters in order: interest rate (i), number of periods (n), equal-sized cash flow per period (A), present amount (P), and type [either end-of-period cash flows (0 or omitted) or beginning-of-period cash flows (1)].
- Entering the following in any cell in an Excel spreadsheet: =FV(i,n,,-P).
- A negative value is entered for P, since the sign of the value obtained for F by using the FV function will be opposite the sign used for P.
- The FV function was developed for a loan situation where $P are loaned (negative cash flow) in order to receive $F (positive cash flow)
- If you enter P rather than -P, F comes out negative. Remember to change the sign!

Use the Excel function FV to calculate the illustration example.

=FV(0.1,5,,-10000)

C12

	A	B	C
1			
2		$16,105.10	=FV(10%,5,,-10000)
3			
4		$16,105.10	=FV(0.1,5,,-10000)
5			
6		-$16,105.10	=FV(10%,5,,10000)
7			
8		-$16,105.10	=FV(0.1,5,,10000)

Example

Dia St. John borrows $1,000 at 12 percent compounded annually. The loan is to be paid back after 5 years. How much should she repay?

- Solution 1: Using the compound interest tables in Appendix A for 12% and 5 periods, the value of *the single sum, future worth factor* (F|P 12%,5) is shown to be 1.76234. Thus, F = P(F|P 12%,5) = $1,000*1.76234 = $1,762.34
- Solution 2*: Using the Excel FV worksheet function, F = FV(12%,5,,-1000)=1762.34

Question: how long does it take for an investment to double?

if it earns (a) 2 percent, (b) 4 percent, or (c) 12 percent annual compound interest?

- Solution 1: Rule of 72

The quotient of 72 and the interest rate provides a reasonably good approximation of the number of interest periods required to double the value of an investment.

a. $n \approx 72/2 = 36$ yrs

b. $n \approx 72/4 = 18$ yrs

c. $n \approx 72/12 = 6$ yrs

Question: how long does it take for an investment to double?

if it earns (a) 2 percent, (b) 4 percent, or (c) 12 percent annual compound interest?

- Solution 2: mathematics

solve mathematically for n such that $(1 + i)^n = 2$

gives $n = \log 2/\log(1 + i)$. Therefore,

a. $n = \log 2/\log 1.02 = 35.003$ yrs;

b. $n = \log 2/\log 1.04 = 17.673$ yrs;

c. $n = \log 2/\log 1.12 = 6.116$ yrs.

Question: how long does it take for an investment to double?

if it earns (a) 2 percent, (b) 4 percent, or (c) 12 percent annual compound interest?

- Solution 3*: Excel NPER (Number of periods) function

parameters in order: interest rate, equal-sized cash flow per period, present amount, future amount, and type.

Letting F equal 2 and P equal −1, the NPER function yields

a. n = NPER(2%,, -1,2) = 35.003 yrs

b. n = NPER(4%,, -1,2) = 17.673 yrs

c. n = NPER(12%,, -1,2) = 6.116 yrs

Present Worth

As $F=P(1+i)^n$,
$P=F(1+i)^{-n}$
or, $P=F(P|F\ i\%,n)$

- $(1 + i)^{-n}$ and $(P|F\ i\%,n)$ are referred to as *the single sum, present worth factor*.
- Using Excel financial function PV (present value).
- The parameters in order: interest rate (i), number of periods (n), equal-sized cash flow per period (A), future amount (F), and type.
- To solve for P when given i, n, and F, the following can be entered in any cell in an Excel spreadsheet:

=PV(i,n,,-F)

Illustration

Suppose you wish to accumulate $10,000 in a savings account 4 years from now, and the account pays interest at a rate of 5 percent compounded annually. How much must be deposited today?

- using the Excel PV worksheet function,
- P =PV(5%,4,,-10000) = $8227.02

MULTIPLE CASH FLOWS

- We have considered single cash flow.
- We will extend our analysis to multiple cash flows.
- Begin with multiple cash flows that do not exhibit a pattern.
- Follow with cash flow series that form a pattern(the uniform/gradient /geometric series), allowing the use of shortcuts in determining PV and FV.

Irregular Cash Flows

- When there're more than one cash flow,
  - the present worth can be determined by adding the present worth of the individual cash flows.
  - the future worth can be determined by adding the future worth of the individual cash flows.
- let A_t denote a cash flow at the end of time period t, the present worth,

$$P = A_1(1+i)^{-1} + A_2(1+i)^{-2} + A_3(1+i)^{-3} + \cdots + A_{n-1}(1+i)^{-(n-1)} + A_n(1+i)^{-n} \quad (2.8)$$

- or, $P = \sum_{t=1}^{n} A_t(1+i)^{-t}$
- or, $P = \sum_{t=1}^{n} A_t(P|F\ i\%,t)$

Illustration

Consider the following CFD. Using an interest rate of 6 percent per interest period, what is the present worth equivalent of cash flows?