Linear Algebra Exercise #1

Fall 2020

Name:

Instructions:

Please include essential steps in your solution. For most of the problems, answers without essential steps may receive a score of 0.

1. Find the minor, cofactor, and adjoint matrix of the matrix. Compute $A\ adj(A)$.

$$\begin{bmatrix} 1 & 1-i \\ 0 & 1 \end{bmatrix}$$

Solution: The minors are

$$M_{11} = 1,\ M_{12} = 0,\ M_{21} = 1-i\ M_{22} = 1$$

The cofactors are

$$A_{11} = 1,\ A_{12} = 0,\ A_{21} = -1+i\ A_{22} = 1$$

Hence the minor matrix is

$$M(A) = [M_{ij}(A)] = \begin{bmatrix} 1 & 0 \\ 1-i & 1 \end{bmatrix}.$$

The cofactor matrix is

$$A_{cof} = [A_{ij}] = \begin{bmatrix} 1 & 0 \\ -1+i & 1 \end{bmatrix}$$

The adjoint matrix of A is

$$adj(A) = A_{cof}^T = \begin{bmatrix} 1 & -1+i \\ 0 & 1 \end{bmatrix}.$$

$$A\ adj(A) = \begin{bmatrix} 1 & 1-i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1+i \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. Compute the determinant, minors, cofactors, and adjoint matrices for $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}$ and compute $A\ adj(A)$.
Solution

$$det(A) = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} = 2$$

$$M(A) = \begin{bmatrix} \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & -1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 2 \\ -2 & -1 & 0 \end{bmatrix}.$$

To get the matrix of cofactors, simply overlay $M(A)$ with the following "checkerboard" of $+/-$'s $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ to obtain the matrix $A_{\text{cof}} = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & -2 \\ -2 & 1 & 0 \end{bmatrix}$.
Now transpose A_{cof} to obtain

$$\text{adj } A = \begin{bmatrix} 2 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix}.$$

We check that

$$A \,\text{adj } A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = (\det A) I_3. \qquad \square$$

3. Compute these determinants. Which of the matrices represented are invertible?

$$(a) \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{vmatrix}; \quad (b) \begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix}; \quad (c) \begin{vmatrix} 1 & -1 & 4 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 7 \\ -2 & 3 & 4 & 6 \end{vmatrix}$$

Solution:
All except (a) are invertible. (a) 0, (b) 4, (c)-70.

4. Find conditions on the parameters in these matrices under which the matrices

are invertible.

$$(a)\begin{vmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & a \\ 0 & 0 & -a & b \end{vmatrix}; \quad (b)\lambda I_2 - \begin{bmatrix} 0 & 1 \\ -c_0 & -c_1 \end{bmatrix}$$

Solution:
(a) $a \neq 0$. (b)$\lambda^2 + c_1\lambda + c_0 \neq 0$.

5. Find the inverse of following matrices by adjoints

$$(a)\ \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}; \ (b)\begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -1 \\ 1 & -3 & 1 \end{bmatrix}.$$

Solution:
(a) $\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$ (b)$\begin{bmatrix} -1 & -4 & -2 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$.

6. Use Cramer's Rule to solve the following systems.

$$\begin{aligned} x + \ \ y + \ \ z &= 4 \\ 2x + 2y + 5z &= 11 \\ 4x + 6y + 8z &= 24 \end{aligned}$$

Solution: $x = 1, \ y = 2, \ z = 1$.

7. Show that the determinant of the general Vandermonde matrix

$$V_n = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}$$

is a product of factors $(x_j - x_k)$ with $j > k$.

Solution:

This is true for $n = 2$ since $\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = (x_2 - x_1)$.. Inductively, suppose it is true for any Vandermonde matrix of size $n-1$ and subtract row 1 from subsequent rows to obtain

$$\det V_n = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 0 & x_1 - x_0 & x_1^2 - x_0^2 & \cdots & x_1^n - x_0^n \\ \vdots & \vdots & & & \vdots \\ 0 & x_n - x_0 & x_n^2 - x_0^2 & \cdots & x_n^n - x_0^n \end{vmatrix}.$$

Now subtract x_0 times the kth column from the $(k+1)$th, $k = 1, \dots n-1$, to obtain

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & x_1 - x_0 & (x_1 - x_0)\,x_1 & \cdots & (x_1 - x_0)\,x_1^{n-1} \\ \vdots & \vdots & & & \vdots \\ 0 & x_n - x_0 & (x_n - x_0)\,x_n & \cdots & (x_n - x_0)\,x_n^{n-1} \end{vmatrix} = \prod_{j=1}^{n} (x_j - x_0) \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}$$

so by induction we obtain

$$\det V_n = \prod (x_j - x_k)_{0 \le k < j \le n}.$$

8. Find the LU factorization of the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 3 & -2 \\ 4 & 2 & -2 \end{bmatrix}$ and use the LU factorization to solve $A\vec{x} = \vec{b}$, where $\vec{b} = \begin{bmatrix} 6 \\ -8 \\ -4 \end{bmatrix}$.

Solution:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 4 & -3 \\ 0 & 0 & -1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$