PROVING THE NON-EXISTENCE OF ELEMENTARY ANTI-DERIVATIVES FOR CERTAIN ELEMENTARY FUNCTIONS

VIRAAJ JAYARAM

Abstract. This paper addresses the question of when functions have anti-derivatives that can be expressed in a satisfactory form. We first pose this question in a mathematically precise and algebraic sense by defining the notion of an elementary function and then prove and utilise Liouville's Theorem on elementary anti-derivatives to show that certain elementary functions do not possess elementary anti-derivatives.

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1. Introduction

When one first learns how to compute integrals and anti-derivatives, one may hear that it is 'impossible' to find an anti-derivative of functions like $\sin(x^2)$ or e^{-x^2} or $\frac{1}{\ln(x)}$. This seems to be a rather sad fact of life as the integrals of these functions play important roles in optics, statistics and number theory, respectively. However, of course it is not impossible to find an anti-derivative of these functions in an abstract sense.The question is whether we can find a 'nice' expression for the anti-derivative. The purpose of this paper is firstly to illustrate how one can pose this question in a precise mathematical sense and then to prove that functions like the three mentioned above do not, in fact, have 'nice' anti-derivatives.

2. Differential Fields and Elementary Extensions

We shall see that in order to determine what we mean by 'nice' anti-derivatives, it is helpful to create algebraically abstract notions of the analytic concepts of differentiation, exponentiation and the logarithm. We do this via the following definitions.

Definition 2.1. We say a function f on a ring R satisfies the *sum rule* if for all $a, b \in R$, $f(a+b) = f(a) + f(b)$ and the *product rule* if, for all $a, b \in R$, $f(ab) = f(a) \cdot b + f(b) \cdot a$.

Definition 2.2. A *differential field* is a field F paired with a function $'$ from F to F satisfying the sum and product rules. We shall refer to this function $'$ as the *derivation* on F.

Definition 2.3. Let a and b belong to some differential field F and $b' = a$. We then say that a is the *derivative* of b and that b is an *anti-derivative* of a.

Definition 2.4. An element c in a differential field F is said to be a *constant* if $c' = 0$.

At this point we can notice several results.

Proposition 2.5. *The constants of a differential field form a subfield*

Proof. Constants are clearly closed under addition and multiplication. We then note that $1' = (1 \cdot 1)' = 1' \cdot 1 + 1 \cdot 1' = 2 \cdot 1'$ and thus $1' = 0$.The same proof works for 0. We then have that $0 = 1' = (-1 \cdot -1)' = -2 \cdot (-1)'$ and thus $(-1)'$ is also 0. Thus if c is a constant, so is $-c$. Regarding c^{-1}, note that $(c \cdot c^{-1})' = c' \cdot (c^{-1}) + c \cdot (c^{-1})' = 0$ and thus $(c^{-1})' = \frac{-c'}{c^2} = 0$.

□

Proposition 2.6. *The usual quotient and power rules hold in a differential field.*

Proof. From the proof above, we see that $(\frac{a}{b})' = -a \cdot \frac{b'}{b^2} + \frac{a'}{b} = \frac{a' \cdot b - b' \cdot a}{b^2}$. For the power rule, we want to show that $(a^n)' = n \cdot a^{n-1} \cdot a'$. We can do this by induction. The $n = 1$ case is trivial. So assuming the result holds for $n = k$, we have that $(a^{k+1})' = a' \cdot a^k + a \cdot k \cdot a^{k-1} \cdot a' = (k+1) \cdot a^k \cdot a'$. Thus the rule holds for all positive integers. It shall also be in our interest to consider the case for negative integers and so for the sake of completeness, we shall address that case here as well. We want to prove that $(a^{-n})' = -n \cdot a' \cdot a^{-(n+1)}$. We have seen that the result holds for $n = 1$. Assuming the result holds for $n = k$, we have that $(a^{-(k+1)})' = -k \cdot a' \cdot a^{-(k+1)} \cdot a^{-1} + -a' \cdot a^{-2} \cdot a^{-k} = -(k+1) \cdot a' \cdot a^{-(k+2)}$ and hence the result holds for negative integers as well.

□

We can now begin to talk about exponentials and logarithms of elements of differential fields.

Definition 2.7. Let a and b belong to a differential field F, where b is non-zero. We say the b is an *exponential* of a if $\frac{b'}{b} = a'$. We shall use the notation $b = e^a$ to denote the fact that b is an exponential of a.

Definition 2.8. Let a and b belong to a differential field F, where a is non-zero. We say that b is a *logarithm* of a if $b' = \frac{a'}{a}$. We shall use the notation $b = \ln(a)$ to denote the fact that b is a logarithm of a.

Definition 2.9. Let F be a differential field. We say that adjoining an element t_1 to create the field $F(t_1)$ is an *elementary extension* of F if t_1 is either the exponential of an element of F, the logarithm of an element of F or is algebraic over F.

Definition 2.10. We say the field $F(t_1, ..., t_n)$ is an *elementary extension field* of F if adjoining t_i is an elementary extension of $F(t_1, ..., t_{i-1})$ for each i.

The following proposition is an essential feature of elementary extension fields that we shall make use of time and time again.

Proposition 2.11. *Let F be a differential field of characteristic 0 and $F(f_1, ..., f_n)$ be an elementary extension field of F. Then the derivation on F can be extended to a derivation on $F(f_1, ..., f_n)$ and this extension is unique.*

Proof. We shall prove this result by induction. To this end, note that it suffices to show that if F is a differential field of characteristic 0 and the extension $F(t)/F$ is an elementary extension, then the derivation on F can be extended uniquely to a derivation on $F(t)$. This is because we can then apply this result to the extension $F(f_1, ...f_i)/F(f_1, ..., f_{i-1})$ by replacing F with $F(f_1, ..., f_{i-1})$ and t with f_i.

For the case where t is algebraic, we first show that if such a structure were to exist, it is unique. Let $h(x) = \sum_{i=0}^{j} a_i \cdot x^i$ be the minimal polynomial of t in $F[x]$. We know that for any derivation $'$, $(h(t))' = 0$ and thus

$$\sum_{i=0}^{j} a_j' \cdot (t)^i + t' \cdot \sum_{i=0}^{j} i \cdot a_i \cdot (t)^{i-1} = 0.$$

As the minimal polynomial is unique and $\sum_{i=0}^{j} i \cdot a_i \cdot (t)^{i-1}$ is the evaluation of a polynomial of degree less than that of the minimal polynomial, it does not equal 0 and hence $t' = \frac{-\sum_{i=0}^{j} a_j' \cdot (t)^i}{\sum_{i=0}^{j} i \cdot a_i \cdot (t)^{i-1}}$ and so is uniquely determined. Once we have that t' is uniquely determined, this then uniquely determines the derivative of every other element in $F(t)$ as any element can be written as $\sum_{i=0}^{m} a_i \cdot t^i$, where $a_i \in F$ and $m < j$, and thus the sum and product rules determine its derivative. We now need to show that we can indeed construct a derivation on $F(t)$

To construct this derivation, we shall first seek to construct a function that satisfies the sum and product rules on the polynomial ring $F[x]$ and then use this to induce a derivation on $F(t)$. To this end, we first define D_0 from $F[x]$ to $F[x]$ such that

$$D_0(\sum_i a_i \cdot x^i) = \sum_i a_i' \cdot x^i$$

and D_1 from $F[x]$ to $F[x]$ such that

$$D_1(\sum_i a_i \cdot x^i) = \sum_i i \cdot a_i \cdot x^{i-1}.$$

We can then define a function D from $F[x]$ to $F[x]$ such that

$$D(p(x)) = D_0(p(x)) + g \cdot D_1(p(x))$$

for some fixed $g \in F[x]$. Showing that D satisfies the sum and product rules is equivalent to showing that D_0 and D_1 satisfy the sum and product rules. In verifying this, the sum cases are immediate. For the product cases, if $p = \sum_n a_n \cdot x^n$ and $q = \sum_m b_m \cdot x^m$, we have that the coefficient of the x^k term in their product is equal to $\sum_{j=0}^{k} a_j \cdot b_{k-j}$. Applying D_0, this becomes $\sum_{j=0}^{k} a_j' \cdot b_{k-j} + \sum_{j=0}^{k} a_j \cdot b_{k-j}'$, which is exactly the coefficient of the x^k term of in $D_0(p) \cdot q + p \cdot D_0(q)$. Applying D_1 instead, the coefficient of the x^k term in $D_1(p) \cdot q + p \cdot D_1(q)$ is

$$\sum_{i=0}^{k} (i+1) \cdot a_{i+1} \cdot b_{k-1} + \sum_{i=0}^{k} a_i \cdot (k-i+1) \cdot b_{k-i+1}$$

This can be rewritten as

$$\sum_{i=0}^{k} (i \cdot a_{i+1} \cdot b_{k-i} - i \cdot a_i \cdot b_{k-i+1}) + (k+1) \cdot \sum_{i=0}^{k} a_i \cdot b_{k-i+1} + \sum_{i=0}^{k} a_{i+1} \cdot b_{k-i}.$$

Recognizing the telescoping manner of the first summation, one can rewrite this as

$$k \cdot a_{k+1} \cdot b_0 + (k+1) \cdot \sum_{i=0}^{k} a_i \cdot b_{k-i+1} + \sum_{i=0}^{k} a_{i+1} \cdot b_{k-i} - \sum_{i=1}^{k} a_i \cdot b_{k+1-i}.$$

The two summations on the right also form a telescoping series, simplifying to the single term $a_{k+1} \cdot b_0$. Hence we get that the coefficient of the x^k term is $(k+1) \cdot \sum_{i=0}^{k+1} a_i \cdot b_{k+1-i}$, which is exactly the coefficient on the x^k term of $D_1(p \cdot q)$. Thus D_0 and D_1 both satisfy the sum and product rules and hence so does D.

Given that t is algebraic over F, we can let ϕ be the homomorphism sending $p(x)$ to $p(t)$ and we know that the image of this homomorphism is $F(t)$. We can thus define $p(t)'$ to be $\phi(D(p(x))$ and can see that this $'$ operation will also satisfy the sum and product rules in $F(t)$, as $(p(t)+q(t))' = \phi(D(p+q)) = \phi(D(p))+\phi(D(q)) = p(t)' + q(t)'$ and $(p(t) \cdot q(t))' = (\phi(p \cdot q))' = \phi(D(p \cdot q)) = \phi(D(p) \cdot q + p \cdot D(q)) = p(t)' \cdot q(t) + p(t) \cdot q(t)'$. However, we need to check that $'$ is well defined. Thus we need to check that if $\phi(p) = \phi(q)$ then $\phi(D(p)) = \phi(D(q))$. Therefore we need that if $p-q$ is in the kernel of ϕ then $D(p-q)$ must also be in the kernel and hence that D maps the kernel of ϕ to itself. Once again letting h be the minimal polynomial of t, we know that the the ideal generated by h is the kernel of ϕ. Hence $'$ is well defined if $\phi(D(h)) = 0$, as then $\phi(D(h \cdot p)) = \phi(D(h)) \cdot \phi(p) + \phi(h) \cdot \phi(D(p)) = 0$.

For $\phi(D(h))$ to be 0, we need $\phi(D_0(h)) + \phi(g) \cdot \phi(D_1(h))$ to be 0. Since h is the minimal polynomial of t and $D_1(h)$ is a polynomial of degree less than that of h, and since F has characteristic 0, we know that $\phi(D_1(h))$ is not 0. Hence $\frac{-\phi(D_0(h))}{\phi(D_1(h))}$ is an element of $F(t)$ and thus there does exist a polynomial g in $F[x]$ such that $\phi(g) = \frac{-\phi(D_0(h))}{\phi(D_1(h))}$. We see that for such a g, $'$ is well-defined. Also note that if a is in F, then $D(a) = a'$ and since ϕ fixes F, $'$ extends the derivation on F. Thus such a structure does indeed exist, and noting that $\phi(g)$ is t', we see that this does in fact recreate the structure we expected from noting what t' had to be when proving uniqueness.

For the exponential and logarithm cases, assuming now that in each case t is transcendental over F, we know that any element in $F(t)$ can be written as $\frac{p(t)}{q(t)}$ where $p, q \in F[x]$.

So now to address the exponential case, let $t = e^g$ for some $g \in F$. Then by definition, we have that t' must be $t \cdot g'$, which is clearly in $F(t)$. It is also uniquely determined as g' is uniquely determined. Since any element can be written as $\frac{p(t)}{q(t)}$, and we know what t' has to be, should a differential structure exist, the sum, product and quotient rules dictate what the derivative of every element in the field should be. We thus proceed by defining a possible derivation this way and verifying that it does indeed provide a derivation on $F(t)$. Writing $p(t)$ as $\sum_i a_i \cdot (t)^i$, we can define D_0 and D_1 in practically identical fashion as before, where $D_0(p(t)) = \sum_i a_i' \cdot (t)^i$ and $D_1(p(t)) = \sum_i a_i \cdot i \cdot (t)^{i-1}$. We can then define $\bar{D}(p(t))$ as $D_0(p(t)) + t' \cdot D_1(p(t))$. Given that D_0 and D_1 clearly still satisfy the sum and product rules, we know $\bar{D}$ does as well. Thus we can define $(\frac{p(t)}{q(t)})'$ as $\frac{\bar{D}(p(t)) \cdot q(t) - p(t) \cdot \bar{D}(q(t))}{q(t)^2}$. As t' is in $F(t)$, $F(t)$ is closed under $'$. It is straightforward to check that this $'$ does satisfy the sum and product rules and thus is a derivation on $F(t)$. Since D preserves derivatives in F, it is clear that the $'$ in $F(t)$ does as well. For uniqueness, as we said earlier, it is also clear that any other supposed derivation would act on $p(f_n)$ in the exact same way and so one would recover $\bar{D}$ from the quotient rule. Lastly, given that t' is unique, it is clear that $(\frac{p(t)}{q(t)})'$ is also uniquely determined.

For the logarithm case, let $t = \ln(g)$ for some $g \in F$. Then $t' = g' \cdot \frac{1}{g} =$ which is again clearly in $F(t)$. Therefore we can define the exact same $\bar{D}$ as in the exponential case and make the exact same conclusions.

Thus in all three cases, the derivation can be extended uniquely and therefore by induction, we can conclude that if $F(f_1, ..., f_n)$ is an an elementary extension field of F, it is also a differential field with a unique derivation that extends the derivation on F.

$\square$

Note that we have only shown that if F is a differential field of characteristic 0, and L is an elementary extension field of F, then *given a generating set* for L, there exists a unique way to extend the derivation on F to a derivation on L. However, in all following proofs, any elementary extension field shall be presented with a generating set and the uniqueness of the extension is not important. The key takeaway is that there exists a derivation on any elementary extension field of F, where this derivation extends the derivation on F.

We shall now see how these definitions help us formulate our question.

3. Elementary Extension Fields of $\mathbb{C}(x)$ and Elementary Functions

We are particularly interested in using our algebraic definitions and results to study $\mathbb{C}(x)$, the field of rational polynomial functions from $\mathbb{C}$ to $\mathbb{C}$ of the variable x, with complex coefficients. In this section we shall first see that we can apply our previous ideas of differential fields and elementary extensions to this field and discuss how to interpret elements in the extension fields as holomorphic functions on subsets of $\mathbb{C}$. This is not essential in terms of understanding the main proofs in this paper (although we do use these ideas in an important remark after Theorem 4.1), but does hopefully serve to help the reader understand how to relate the algebraic concepts we have discussed so far with the analytic concepts that we are ultimately interested in. We shall then see how the kinds of functions one might consider as 'nice' functions can be thought of as elements of these fields.

To begin, it is clear that the analytic differentiation with respect to the variable x provides a derivation on this field and thus $\mathbb{C}(x)$ is a differential field. Note that we can also consider this derivation in a purely algebraic sense, by defining $(\frac{p(x)}{q(x)})' = \frac{D_1(p(x)) \cdot q(x) - p(x) \cdot D_1(q(x))}{(q(x))^2}$, where $p(x), q(x) \in \mathbb{C}[x]$. We can also note that if g is in $\mathbb{C}(x)$ then $\frac{d}{dx} \cdot e^{g(x)} = g'(x) \cdot e^{g(x)}$ and $\frac{d}{dx} \cdot \ln(g(x)) = \frac{g(x)}{g'(x)}$ and hence any branch of the analytic logarithm and the analytic exponential satisfy the same differential relations as the algebraic logarithm and exponential. Thus we can talk about elementary extension fields of $\mathbb{C}(x)$ as a field $\mathbb{C}(x, f_1, ..., f_n)$ where each f_i is a function that is either the analytic exponential or a logarithm-after choosing a branch-of an element (function) in $\mathbb{C}(x, f_1, ...f_{n-1})$ or is algebraic over $\mathbb{C}(x, f_1, ...f_{n-1})$, i.e. implicitly defined as the solution to a polynomial with coefficients in $\mathbb{C}(x, f_1, ...f_{n-1})$.

Observe that the derivation we constructed in any elementary extension field (generated by $f_1, ..., f_n$ over $\mathbb{C}(x)$) also agrees with the analytic differentiation, as whether f_n is an exponential or logarithm of an element of $\mathbb{C}(x, f_1, ..., f_{n-1})$ or is algebraic over $\mathbb{C}(x, f_1, ..., f_{n-1})$, its derivative via the extension outlined in the proof of Proposition 2.11 agrees with its analytic derivative. We just saw that this is true for the exponential and logarithmic cases and as we know that the analytic differentiation satisfies the sum and product rules, we know that the derivative of an algebraic f_n must also have the unique form established in Proposition 2.11, which is also what one would obtain through implicit differentiation of the minimal polynomial of f_n.

Definition 3.1. We call a field *elementary* if it is an elementary extension field of $\mathbb{C}(x)$.

Definition 3.2. A function whose domain is some open subset of $\mathbb{C}$ and whose image lies in $\mathbb{C}$ is said to be *elementary* if it is an element of some elementary field.

Now, one must be careful when considering the well-defined-ness and differentiability of elements of these fields as functions. They will generally not be differentiable or even well-defined on all of $\mathbb{C}$. However, the Implicit Function Theorem and other results in complex analysis ensure that every elementary function is well-defined and holomorphic on some open subset of $\mathbb{C}$. We omit a detailed discussion of these results as they are tangential to the main interests of this paper. Nonetheless, for a reader unfamiliar with these concepts, we provide one example.

Example 3.3. We consider the well-defined-ness and differentiability of the function $\sqrt{x}$ as a function from $\mathbb{C}$ to $\mathbb{C}$. $\sqrt{x}$ is algebraic over $\mathbb{C}(x)$ as it is the solution to the polynomial $t^2 - x$ in $\mathbb{C}(x)[t]$. As a function from $\mathbb{C}$ to $\mathbb{C}$, we can write z in polar form as $r \cdot e^{i \cdot \theta}$ where $0 \le \theta < 2\pi$ and $0 \le r$ and thus define $\sqrt{z}$ as $\sqrt{r} \cdot e^{i \cdot \frac{\theta}{2}}$. However, in this case we see that as we traverse the unit circle with θ increasing as we go anti-clockwise, we start with $\sqrt{1} = 1$ but upon returning to 1 we have that $\sqrt{1} = -1$. However, we can fix this problem by removing the non-positive real numbers from $\mathbb{C}$, in what is known as a *branch cut*. We can now relabel z as $r \cdot e^{i\theta}$ where $r > 0$ and $-\pi < \theta < \pi$ and define $\sqrt{z}$ as $\sqrt{r} \cdot e^{i \cdot \frac{\theta}{2}}$. It is clear that the function is well-defined on this new domain and with a little complex analysis it is easy to see that the function is also differentiable on this domain.

Note that when we consider elements in elementary fields as functions, we implicitly associate with each function a domain that is some open, connected subset of $\mathbb{C}$ upon which the function is well-defined and holomorphic. So in particular when we have an identical equality between two elements in an elementary field, or have an expression identically equal to 0, the equation must hold on an open, connected subset of $\mathbb{C}$.

As you may have guessed, elementary functions shall be what we consider to be 'nice' functions. Our question then becomes whether elementary functions always have elementary anti-derivatives, i.e. given an elementary function $f(x)$, can we find another elementary function $g(x)$ (perhaps in a different elementary field) such that $g'(x) = f(x)$, on some open, connected subset of $\mathbb{C}$. Now, one might only be concerned about answering this question where the domain is an open subset of $\mathbb{R}$ and it turns out that essentially the same results in complex analysis tell us that we can indeed view every elementary function as a well-defined and differentiable function from some open subset of $\mathbb{R}$ to $\mathbb{C}$.

It is not necessarily obvious, however, that all the functions we would consider as satisfactory anti-derivatives are, in fact, elementary functions. Hopefully the following results assuage some likely concerns.

Example 3.4. Trigonometric Functions and Their Inverses are Elementary Functions.

We know that $i \cdot x \in \mathbb{C}(x)$. Therefore, $\mathbb{C}(x, e^{ix})$ is an elementary field. Given Euler's Identity $e^{ix} = \cos(x) + i\sin(x)$, one has that $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$ and that $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$. Thus $\sin(x)$ and $\cos(x)$ belong to the elementary field $\mathbb{C}(x, e^{ix})$, as does their quotient and hence $\sin(x)$, $\cos(x)$ and $\tan(x)$ are elementary functions.

As for their inverses, let us consider $\arccos(x)$. To find an expression for it in elementary terms, we want to solve for y in terms of x where $x = \frac{e^{iy} + e^{-iy}}{2}$. Letting $e^{iy} := k$, we have that $2x = k + \frac{1}{k}$. Multiplying through by k and using the quadratic formula gives us that $k = e^{iy} = x \pm \sqrt{x^2 - 1}$ and thus that $y = \arccos(x) = -i \cdot \ln(x + \sqrt{x^2 - 1})$ (where we choose the positive branch). This is elementary as $x + \sqrt{x^2 - 1}$ is algebraic over $\mathbb{C}(x)$ and thus $\mathbb{C}(x, \sqrt{x^2 - 1}, \ln(x + \sqrt{x^2 - 1})$ is an elementary field containing $\arccos(x)$. The other inverse trigonometric functions can also be expressed similarly.

Another class of functions that one might wonder about are functions of the form $f(x)^{g(x)}$, where both $f(x)$ and $g(x)$ are themselves elementary.

Proposition 3.5. *If $f(x)$ and $g(x)$ are elementary, then $f(x)^{g(x)}$ is also elementary.*

Proof. The function $f(x)^{g(x)}$ is defined as $e^{\ln(f(x)) \cdot g(x)}$, which is clearly elementary. □

Lastly, one might be curious about arbitrary composition.

Proposition 3.6. *If $f(x)$ and $g(x)$ are elementary then $f(g(x))$ is also elementary.*

Proof. Given the elementary field $\mathbb{C}(x, ..., g(x))$ and given that $f(x)$ is also elementary, we know that by applying to the element x some string of the operations of exponentiation, taking a logarithm and solving an algebraic equation, we can create the elementary field $\mathbb{C}(x, ..., g(x), ..., f(x))$. However, since $g(x)$ is also in the first field, we can apply the exact same string to $g(x)$ and thus create the elementary field $\mathbb{C}(x, ..., g(x), ..., f(g(x)))$ instead. □

Hopefully the reader now agrees that the notion of an elementary function does indeed encapsulate all functions that one would be satisfied with as 'nice' or 'closed form' expressions for anti-derivatives.

4. The Restrictions Imposed by Having an Elementary Anti-Derivative

To address the question of when elementary functions have elementary anti-derivatives, we shall first consider the more algebraically abstract question of when an element of a differential field F has an anti-derivative in some elementary extension field of F. The main purpose of this section is to prove Liouville's Theorem on elementary anti-derivatives, which tells us that $\alpha \in F$ has an anti-derivative in some elementary extension field of F if and only if α can be written in a particular form.

Theorem 4.1. *(Liouville) Let F be a differential field of characteristic* 0 *and $\alpha \in F$. If the equation $y' = \alpha$ holds for y in some elementary extension field of F, having the same subfield of constants as F, then there exist constants $c_1, ..., c_n \in F$ and elements $u_1, ..., u_n, v \in F$ such that*

$$\alpha = \sum_{i=1}^{n} (c_i \cdot \frac{u_i'}{u_i}) + v' \tag{4.2}$$

Proof. We assume that there exists $y \in F(f_1, ..., f_N)$ such that $y' = \alpha$, where $F \subset F(f_1) ... \subset F(f_1, ..., f_N)$, the adjoining of each f_i has been an elementary extension of the preceding field and $F(f_1, ..., f_N)$ has the same subfield of constants as F. We shall proceed by induction on the number of adjoined elements. The base case is when y belongs to a field where zero elements have been adjoined, i.e. $y \in F$. In this case we can just let all the $c_i = 0$ and let $y = v$ and the theorem holds. We now assume for induction that the theorem holds for $N - 1$ extensions, meaning that if K is a characteristic 0 differential field and $\alpha \in K$ has an anti-derivative in an elementary extension field of K, obtained via $N - 1$ successive elementary extensions and with the same subfield of constants, then (4.2) holds for u_i and v in K. Letting $K = F(f_1)$, this implies that

$$\alpha = \sum_{i=1}^{m} (k_i \cdot \frac{g_i'}{g_i}) + h' \tag{4.3}$$

where each k_i a constant and g_i, h belong to $F(f_1)$. We must now find an equivalent expression

$$\alpha = \sum_{i=1}^{n} (c_i \cdot \frac{u_i'}{u_i}) + v'$$

where each c_i is a constant and each u_i and v are in F. We shall consider separately the cases where f_1 is algebraic over F, f_1 is a logarithm of an element of F or f_1 is an exponential of element of F and show that in each case we can indeed find the desired equivalent expression.

We first consider the algebraic case. If f_1 is algebraic over F, then we know each element of $F(f_1)$ can be written as $p(f_1)$ for some polynomial $p \in F[x]$. Thus, there exist polynomials G_i and $H \in F[x]$ such that $G_i(f_1) = g_i$ and $H(f_1) = h$. So now if we let L be the splitting field of the minimum polynomial of f_1 and let

$f_1 = \tau_1, ..., \tau_s \in L$ be the roots of this minimal polynomial, we have that

$$\alpha = \sum_{i=1}^{m} (k_i \cdot \frac{(G_i(\tau_j))'}{G_i(\tau_j)}) + (H(\tau_j))'$$

for each τ_j (where $1 \leq j \leq s$) as $\alpha \in F$ and is thus invariant under any automorphism in $\mathrm{Gal}(L/F)$. We can then sum this expression for all s different roots and divide by s to get that

$$\alpha = \sum_{i=1}^{m} \sum_{j=1}^{s} (\frac{k_i}{s} \cdot \frac{(G_i(\tau_j))'}{G_i(\tau_j)}) + \frac{(H(\tau_1))' + ... + (H(\tau_s))'}{s}.$$

Given the general formula that

$$\sum_{i=1}^{n} \frac{(q_i)'}{q_i} = \frac{(q_1 \cdot ... \cdot q_n)'}{q_1 \cdot ... \cdot q_n} \tag{4.4}$$

(which can be proven immediately by induction), we have that

$$\alpha = \sum_{i=1}^{m} \frac{k_i}{s} \cdot \frac{(G_i(\tau_1) \cdot ... \cdot G_i(\tau_s))'}{(G_i(\tau_1) \cdot ... \cdot G_i(\tau_s))} + (\frac{H(\tau_1) + ... + H(\tau_s)}{s})'.$$

We can see that for each G_i, $G_i(\tau_1) \cdot ... \cdot G_i(\tau_s)$ and $H(\tau_1) + ... + H(\tau_s)$ are invariant under any automorphism in $\mathrm{Gal}(L/F)$ and thus are actually in F. Thus we can let, $\frac{k_i}{s}$ be c_i, $(G_i(\tau_1) \cdot ... \cdot G_i(\tau_s))$ be u_i and $\frac{H(\tau_1)+...+H(\tau_s)}{s}$ be v and we get our desired expression.

Now we shall tackle the other two cases and assume that f_1 is transcendental over F. In this case, each g_i from (4.3) can be written as $\frac{q_{i1}(f_1)}{q_{i2}(f_1)}$ where each $q_{ij} \in F[x]$. We can hence see that

$$\frac{(g_i)'}{g_i} = \frac{(q_{i1}(f_1))' \cdot q_{i2}(f_1) - q_{i1}(f_1) \cdot (q_{i2}(f_1))'}{(q_{i2}(f_1))^2} \cdot \frac{q_{i2}(f_1)}{q_{i1}(f_1)} = \frac{(q_{i1}(f_1))'}{(q_{i1}(f_1))} - \frac{(q_{i2}(f_1))'}{q_{i2}(f_1)}.$$

Thus we can rewrite $\sum_{i=1}^{m} k_i \cdot \frac{(g_i)'}{g_i}$ as $\sum_{i=1}^{2m} c_i \cdot \frac{(q_i)'}{q_i}$, where each q_i is a polynomial in $F[f_1]$. Factoring each polynomial and using (4.4), we can then further rewrite this as $\sum_{i=1}^{n} c_i \cdot \frac{(p_i)'}{p_i}$ where each p_i is a monic irreducible polynomial in $F[f_1]$ or is a member of F. We can also assume without loss of generality that each c_i is non-zero and no p_i is repeated in the summation. We therefore have that

$$\alpha = \sum_{i=1}^{n} c_i \cdot \frac{p_i'}{p_i} + h'. \tag{4.5}$$

We can also decompose h into partial fractions, thereby expressing h as the sum of an element from $F[f_1]$ plus various terms of the form $\frac{a(f_1)}{(b(f_1))^r}$, where $b(f_1)$ is a monic irreducible polynomial in $F[f_1]$, r is a positive integer and $a(f_1)$ is a non-zero element of $F[f_1]$ of degree less than that of $b(f_1)$.

Now assume that f_1 is a logarithm of an element of F, so $f_1' = \frac{t'}{t}$ for some $t \in F$. Considering $p(f_1)$ as any monic irreducible in $F[f_1]$, we know that $(p(f_1))' \in F[f_1]$ (as $(f_1)' \in F$) and has degree less than $p(f_1)$, so $\frac{(p(f_1))'}{p(f_1)}$ is reduced. If $\frac{a(f_1)}{(b(f_1))^r}$ occurs in the partial fraction expansion of h, considering the term with the maximal value of r for that particular $b(f_1)$, h' will have terms where $b(f_1)$ appears in the denominator with an exponent of at most r, plus one term of the form $-r \cdot a(f_1) \cdot \frac{(b(f_1))'}{(b(f_1))^{r+1}}$. Since $b(f_1)$ does not divide $(b(f_1))'$ nor $a(f_1)$, the exponent of the denominator must remain $r+1$. However, since the entire expression is equal to α, which is in F, we know that all terms containing f_1 must cancel. However, we see that this term cannot be cancelled out by any term in $\sum_{i=1}^{n} c_i \cdot \frac{(p_i(f_1))'}{p_i(f_1)}$ as even if one of the p_i were b, it would occur only once in the denominator. Hence no monic irreducible can occur as a denominator in the partial fraction expansion of h, meaning that $h \in F[f_1]$. Similarly, if any p_i is a monic irreducible in $F[f_1]$, we know that since it occurs exactly once in the denominator of all terms being summed, it cannot be cancelled out. Therefore each p_i must actually belong to F. Hence $\alpha - \sum_{i=1}^{n} c_i \cdot \frac{(p_i)'}{p_i} = h'$ must belong in F. We claim that this forces h to be linear in f_1.

To see this, assume that

$$\sum_{j=1}^{k} (a_j \cdot (f_1)^j) = h.$$

$$\Rightarrow \sum_{j=1}^{k} (a_j \cdot (f_1)^j)' = h' \in F$$

.

$$\Rightarrow \sum_{j=1}^{k} (a_j)' \cdot (f_1)^j + j \cdot a_j \cdot (f_1)^{j-1} \cdot (f_1)' = h'.$$

As h' is in F, all terms containing any non-zero power of f_1 in the summation must cancel. Since f_j is transcendental, a_j must be 0 for $j > 1$. We also see that $(a_1)'$ must be 0 as otherwise f_1 satisfies a linear polynomial in $F[f_1]$. Therefore $h = a_1 \cdot f_1 + a_0$, where a_1 is a constant. Hence

$$h' = a_1 \cdot f_1' + a_0' = a_1 \cdot \frac{t'}{t} + a_0'.$$

Thus,

$$\alpha = \sum_{i=1}^{n} c_i \cdot \frac{p_i'}{p_i} + a_1 \cdot \frac{t'}{t} + a_0',$$

which is in the form we want as a_1 is a constant and t, $a_0 \in F$.

We now must only deal with the exponential case. In this case, $f_1 = e^t$ for some $t \in F$ and thus $\frac{f_1'}{f_1} = t' \in F$. We would wish to make similar arguments about (4.5) as before and conclude that the p_i have to be in F. However we cannot quite make that claim. Instead, we first claim that if $b(f_1)$ is an irreducible monic in $F[f_1]$ and is not identically equal to f_1, then $(b(f_1))' \in F[f_1]$ and $b(f_1)$ does not divide $(b(f_1))'$. To see this, again we write $b(f_1)$ as $\sum_{j=1}^{k} a_j \cdot (f_1)^j$ and so

$$(b(f_1))' = \sum_{j=1}^{k} a_j' \cdot (f_1)^j + a_j \cdot j \cdot (f_1)^{j-1} \cdot (f_1)'.$$

Since $f_1' = t' \cdot f_1$, we get that

$$(b(f_1))' = \sum_{j=1}^{k} (a_j' + a_j \cdot j \cdot t') \cdot (f_1)^j \in F[f_1].$$

We can also see that $(b(f_1))'$ has the same degree as $b(f_1)$ as if the leading term (or any term for that matter) vanishes, then we have that $(a_j \cdot (f_1)^j)' = 0$ meaning that $a_j \cdot (f_1)^j$ is a constant, contradicting the transcendence of f_1. We therefore have that $b(f_1)$ divides $(b(f_1))'$ if and only if $(b(f_1))' = k \cdot b(f_1)$ for some $k \in F$. Now since we are assuming $b(f_1)$ is not a monomial, we can find distinct a_i and a_j such that

$$\frac{a_i + i \cdot a_i \cdot t'}{a_i} = k = \frac{a_j + j \cdot a_j \cdot t'}{a_j}$$

$$\Rightarrow \frac{a_i'}{a_i} + i \cdot \frac{f_1'}{f_1} - (\frac{a_j'}{a_j} + j \cdot \frac{f_1'}{f_1}) = 0$$

$$\Rightarrow (\frac{a_i \cdot (f_1)^i}{a_j \cdot (f_1)^j})' = 0$$

as

$$(\frac{a_i \cdot (f_1)^i}{a_j \cdot (f_1)^j})' = (\frac{a_i \cdot t^i}{a_j \cdot t^j}) \cdot (\frac{a_i'}{a_i} + i \cdot \frac{f_1'}{f_1} - (\frac{a_j'}{a_j} + j \cdot \frac{f_1'}{f_1}))$$

which again contradicts the transcendence of f_1. So now, for the same reasons as in the logarithm case, any $b(f_1)$ appearing in the denominator of a term in partial fraction decomposition of h cannot be cancelled out unless it is a monomial. Thus monomials are the only terms that can appear as denominators in the partial fraction decomposition of h and hence h can be written as $\sum_i a_i \cdot (f_1)^i$ (where i ranges over finitely many positive or negative or 0 integers).

For the same cancellation reasons, we see that all the p_i in $\sum_{i=1}^n \frac{p_i'}{p_i}$ must be in F except for one term which could be $\frac{f_1'}{f_1}$. Even in this case, however, $\frac{f_1'}{f_1} \in F$ and hence all terms in the sum are in F. Therefore again h' must also be in F. Given that $h = \sum_i a_i \cdot (f_1)^i$, we have seen that h' will have terms with the same powers of f_1 and so for h' to be in F, h must also be in F. So now we have that (4.5) holds with h in F and all p_i in F except perhaps one term, which could equal $\frac{f_1'}{f_1}$.

To address this case, assume without loss of generality that $p_1 = f_1$. Then $c_1 \cdot \frac{(p_1)'}{p_1} = c_1 \cdot t'$. Combining this term with the h' term, we can define $v := c_1 \cdot t + h \in F$ and thus see that $\alpha = \sum_{i=2}^n c_i \cdot \frac{p_i'}{p_i} + v'$ is an expression in the form that we want. Thus for all three possibilities of f_1 the theorem still holds.

To summarize, we have now shown that if the theorem holds whenever one adjoins $N-1$ elements, it also holds whenever one adjoins N elements. We did this inductively by assuming that if α is in any characteristic 0 differential field K and has an anti-derivative in some extension field of K, obtained through $N-1$ successive elementary extensions and with the same subfield of constants, one can write α as $\sum_{i=1}^m k_i \cdot \frac{g_i'}{g} + h'$ where the g_i and h are in K. This implied that if α is in F and has an anti-derivative in the elementary extension field $F(f_1, ..., f_N)$, one can write α as $\sum_{i=1}^m k_i \cdot \frac{g_i'}{g} + h'$ where the g_i and h are in $F(f_1)$. We then showed that regardless of whether f_1 was algebraic over F, a logarithm of an element of f or an exponential of an element of F, one can convert the form $\sum_{i=1}^m k_i \cdot \frac{g_i'}{g} + h'$ into the form $\sum_{i=1}^n k_i \cdot \frac{u_i'}{u_i} + v'$, where the u_i and v are in F. We already saw that the base case where y belongs to F is trivial and thus the theorem is proved.

$\square$

Note that since the derivative of a complex-valued function being 0 on an open, connected subset of $\mathbb{C}$ (or $\mathbb{R}$) implies that the function only takes one value on that domain, we have that the only constants in elementary fields are elements of $\mathbb{C}$. We thus can apply Liouville's Theorem to elementary fields and we have that if an elementary function f, belonging to an elementary field F, has an elementary anti-derivative, then f can be written in the form of equation (4.2) where the c_i are in $\mathbb{C}$ (and the u_i and v are in F). Note that in this case we also get an easy converse to Liouville's Theorem:

Proposition 4.6. *If α belonging to some elementary field F can be written in the form of equation (4.2), then α has an anti-derivative in an elementary extension field of F, where the elementary extension field has the same subfield of constants.*

Proof. $F(\ln(u_1), ..., \ln(u_n))$ is an elementary field having the same subfield of constants as F. Letting

$$y = \sum_{i=1}^{n} (c_i \cdot \ln(u_i)) + v$$

we see that y is an elementary anti-derivative of α. □

5. Deriving a Useful Corollary to Liouville's Theorem

Following in Liouville's own footsteps, we shall turn our attention to finding elementary anti-derivatives of functions of the form $f \cdot e^g$, where $f, g \in \mathbb{C}(x)$, f is not zero and g is not constant. In this section we shall show that Liouville's Theorem implies the following corollary:

Corollary 5.1. *If f, $g \in \mathbb{C}(x)$ with f not zero and g non-constant, the function $f(x) \cdot e^{g(x)}$ has an elementary anti-derivative if and only if there exists a rational function $R \in \mathbb{C}(x)$ such that $R'(x) + g'(x) \cdot R(x) = f(x)$.*

Proof. Note that if such an R exists, then

$$\begin{aligned}(R(x) \cdot e^{g(x)})' &= R'(x) \cdot e^{g(x)} + g'(x) \cdot R(x) \cdot e^{g(x)} \\ &= (R'(x) + g'(x) \cdot R(x))e^{g(x)} \\ &= f(x) \cdot e^{g(x)}\end{aligned}$$

and thus $f(x) \cdot e^{g(x)}$ does have an elementary anti-derivative. To prove the converse, we shall first need the following result:

Lemma 5.2. *If g is as in the statement of Corollary 5.1, e^g is transcendental over $\mathbb{C}(x)$.*

Proof. Assume, by contradiction, that e^g satisfies some minimal polynomial

$$e^{ng} + \sum_{j=0}^{n-1} a_j \cdot e^{jg} = 0.$$

Differentiating this, we get that

$$ng' \cdot e^{ng} + \sum_{j=0}^{n-1} (a_j' + a_j \cdot jg') \cdot e^{jg} = 0$$

and so

$$e^{ng} + \sum_{j=0}^{n-1} (\frac{a_j' + a_j \cdot jg'}{ng'}) \cdot e^{jg} = 0.$$

As the minimal polynomial is unique, we know that $\frac{a_j'+a_j \cdot jg'}{ng'} = a_j$ and thus that $a_j \cdot (n-j)g' = a_j'$ for all $0 \le j \le n-1$. As e^{ng} is not 0, we know some a_j is non zero. Thus we have $(n-j) \cdot g' = \frac{a_j'}{a_j}$. We also know a_j is not constant as then we have $g' = 0$, contradicting the assumption that g is not constant. If we write a_j as $\frac{p(x)}{q(x)}$, where $p(x)$, $q(x) \in \mathbb{C}[x]$, we know that $\frac{a_j'}{a_j} = \frac{p(x)'}{p(x)} - \frac{q(x)'}{q(x)}$. We know that p and q are not both constants so consider a non-zero term of the form $\frac{p(x)'}{p(x)}$. Fully factoring $p(x)$ as $\prod_k (x - r_k)^{e_k}$ and using the identity (4.4), we see that

$$\frac{p(x)'}{p(x)} = \sum_k \frac{e_k}{x - r_k}$$

and so similarly

$$\frac{a_j'}{a_j} = \sum_m \frac{e_m}{x - r_m}.$$

This implies that g' and thus g must also have $x - r_m$ as a factor in its denominator for any m. Hence we can we write $g(x)$ as $h(x) \cdot (x - r_m)^{-k}$, where $x - r_m$ does not divide $h(x)$ and $k > 0$. Then $g'(x) = -k \cdot h(x) \cdot (x - r_m)^{-(k+1)} + h'(x) \cdot (x - r_m)^{-k}$. We can see that this expression cannot have $\frac{e_m}{x - r_m}$ as the only term in its partial fraction expansion with the denominator being a power of $(x - r_m)$, and hence cannot equal $\frac{a_j'}{a_j}$, a contradiction. $\square$

We can now complete the proof of the corollary. If $f \cdot e^g$ has an elementary anti-derivative, then, since $f \cdot e^g$ is in the elementary field $\mathbb{C}(x, e^g)$, we can write that

$$f \cdot e^g = \sum_{i=1}^{n} c_i \cdot \frac{p_i'}{p_i} + h'$$

where all p_i are in $\mathbb{C}(x)$ or are irreducible monics in $\mathbb{C}(x)[e^g]$ and h is in $\mathbb{C}(x, e^g)$. Since e^g is transcendental, we can use the same arguments that we made about the transcendental exponential when proving Liouville's Theorem. These arguments showed us that if an irreducible monic was one of the p_i terms or in the partial fraction expansion of h, then it could not be cancelled by any other terms and would remain as a denominator, unless the monic was just e^g. Since $f \cdot e^g$ does not have any monic irreducibles as a denominator, we know that the p_i have to be in $\mathbb{C}(x)$, except for one which could equal e^g, and the only possible denominators in the partial fraction expansion of h are powers of e^g. Hence we can write h as $\sum_j a_j \cdot e^{jg}$ where $a_j \in \mathbb{C}(x)$ and j ranges over a finite set of integer values. As $\sum_i c_i \cdot \frac{p_i'}{p_i}$ is in $\mathbb{C}(x)$, the term linear with e^g in h' must be equal to $f \cdot e^g$ and all other terms must cancel. We can see that the linear term is $(a_1' + a_1 \cdot g') \cdot e^g$ and thus $a_1' + a_1 \cdot g' = f$. Letting a_1 be $R(x)$, we have the desired result.

$\square$

6. Tackling Specific Examples

We are now ready to prove that the three functions mentioned in the introduction: e^{-x^2}, $\frac{1}{\ln(x)}$ and $\sin(x^2)$, do not possess elementary anti-derivatives.

Example 6.1. e^{-x^2} does not have an elementary anti-derivative.

Proof. We can immediately apply the corollary to see that if e^{-x^2} has an elementary anti-derivative, there exists an element $R(x) \in \mathbb{C}(x)$ such that

$$R'(x) - 2x \cdot R(x) = 1.$$

First note that $R(x)$ is clearly not constant and cannot even be a polynomial as if $R(x)$ is a polynomial of degree n then $R'(x) - 2x \cdot R(x)$ is a polynomial of degree $n + 1$. So we can write $R(x)$ as $\frac{p(x)}{q(x)}$ where q is non-constant and $p(x)$ is relatively prime to $q(x)$. As $q(x)$ is non-constant, it has a factor $(x - c)^k$ for $c \in \mathbb{C}$ and $k \geq 1$. Once again, upon considering the partial fraction expansion for $R(x) = \frac{p(x)}{q(x)}$, we see that $R'(x)$ will contribute a term with $(x - c)^{k+1}$ in the denominator and that this term cannot be cancelled by $2x \cdot R(x)$ and hence this expression cannot be identically equal to 1. $\square$

Example 6.2. $\frac{1}{\ln(x)}$ does not have an elementary anti-derivative.

Proof. It is not immediately apparent how we can apply the corollary to this case. However, we can note that if there exists an elementary g such that $g' = \frac{1}{\ln(x)}$, then the derivative of the elementary function $g(e^x)$ is equal to $\frac{1}{\ln(e^x)} \cdot e^x = \frac{e^x}{x}$. Thus it suffices to show that $\frac{e^x}{x}$ does not have an elementary anti-derivative. We can now apply the corollary to see that if $\frac{e^x}{x}$ has an elementary anti-derivative then there exists $R(x) \in \mathbb{C}(x)$ such that $R'(x) + R(x) = \frac{1}{x}$. Again we can note that $R(x)$ is neither constant nor a polynomial and thus we can again write it as $\frac{p(x)}{q(x)}$ where q is non-constant and thus see that again, $R'(x)$ will have a term of the form $(x-c)^{k+1}$ in its denominator for $k \geq 1$. Even if c is 0, the partial fraction expansions still cannot agree and so this equation has no solutions. $\square$

Example 6.3. $\sin(x^2)$ does not have an elementary anti-derivative.

Proof. $\sin(x^2)$ belongs to the elementary field $\mathbb{C}(x, e^{ix^2})$. If it has an elementary anti-derivative then again we can write it in the form of 4.6 and use the exact same arguments about the denominator as we did in the proof of Corollary 5.1 to say that the only monic irreducible that can occur in the partial fraction expansion of h is e^{ix^2}. The e^{ix^2} and e^{-ix^2} terms in this expansion must add to $\sin(x^2)$ and the rest, combined with the terms in $\sum_{i=1}^{n} c_i \cdot \frac{p_i'}{p_i}$ must be zero or cancel out. Considering the e^{ix^2} and e^{-ix^2} terms, we get the two equations

$$a_1 + 2ix \cdot a_1' = \frac{1}{2i}$$

$$a_{-1} - 2ix \cdot a_{-1}' = \frac{-1}{2i}.$$

Using the exact same arguments as used in Example 6.1, we see that neither of these equations can have a solution. $\square$

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