Heat transfer and mass transfer

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Heat transfer and thermodynamics

Thermodynamics enables us to calculate the amount of heat transfer associated with a change from one equilibrium state to another. This is achieved through concepts such as heat capacity, latent heat, enthalpy and entropy. Thermodynamics considers the exchange of energy from one form to another. However, reaching equilibrium, in principle, takes infinite time (or infinite heat transfer area in a continuous system).

Infinite capital cost is not an acceptable choice, hence these lectures

Heat Transfer enables us to calculate the rate at which heat is transferred during a real change in system state, *e.g.* the rate of cooling of a hot object. Heat transfer does not usually consider energy in forms other than heat.

Thermodynamics teaches us that mechanical energy is more valuable than heat so it is always best to say 'heat' if that is correct.

?

Temperature and heat transfer

If two adjacent regions are at different temperatures, heat will flow from hot to cold in an effort to establish thermal equilibrium. Given sufficient time, equilibrium will be established and no further heat transfer will occur.

Conversely, where no temperature difference exists, no heat transfer will take place.

Some applications

Heat transfer equipment (heat exchangers)

Heat dissipation systems (heat sinks in electronics)

Thermal insulation (pipe lagging, heat shields)

This subject is of interest to all branches of engineering and science.

Heat transfer mechanisms

There are three fundamental heat transfer mechanisms

Conduction

The propagation of thermal energy by interactions between adjacent molecules. Does not involve bulk movement of molecules.

Convection

The transfer of thermal energy between a solid surface and the adjacent fluid by bulk movement of the fluid

Radiation

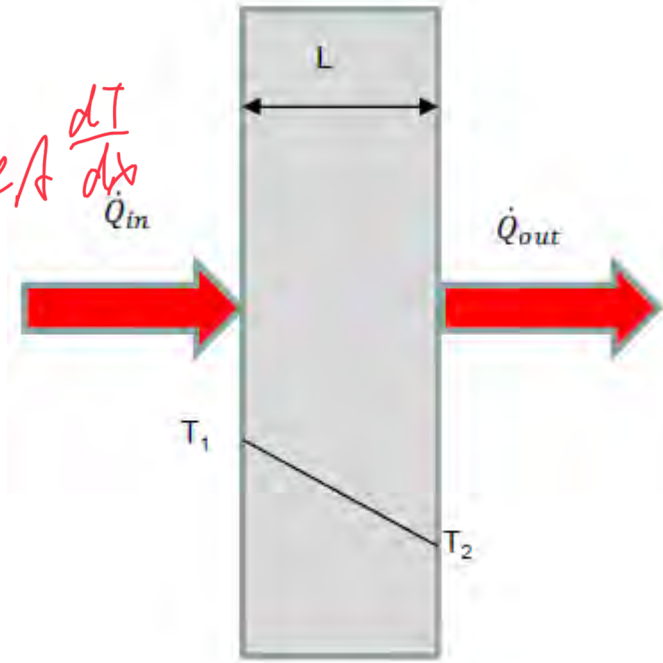
Thermal energy emitted in the form of electron-magnetic waves. All bodies above 0 K emit radiation according to the Stefen-Boltzmann law.

Fourier's law and 1D steady state conduction

The rate of heat transfer by conduction is described by Fourier's law

$$\dot{Q}_{cond} = -kA\frac{dT}{dx}$$

$\dot{Q}_{cond} = -kA\frac{dT}{dx}$



Here k is the thermal conductivity of the material ($W.m^{-1}.K^{-1}$). A is the conductive area (perpendicular to the heat flow, but assume 1-D heat flow)

For steady state, as $\dot{Q}_{in} = \dot{Q}_{out}$ (otherwise the wall temperature will keep changing, i.e. unsteady state), the temperature gradient $\frac{dT}{dx}$ will be linear in a uniform wall. Therefore the equation becomes:

$$\dot{Q}_{cond} = \frac{kA}{L}(T_1 - T_2)$$

In vector notation, Fourier's law is $\vec{q} = -k\,\overrightarrow{grad(T)}$

Where $\vec{q}$ is the heat flux vector (W/m^2), k is the thermal conductivity scalar, and $\overrightarrow{grad(T)}$ is the vector gradient of temperature with respect to distance

Convection

Convection involves bulk movement of fluid.

Forced convection – fluid movement "forced" by mechanical input (fan or pump) or by wind

Natural convection – fluid movement occurs through buoyancy forces (local variation in fluid temperature lead to local variation in fluid density which establish buoyancy forces)

Note: natural convection is only important when forced convection is largely absent.

$\dot{Q}_{conv} = hA(T_s - T_\infty)$

The rate of heat transfer by convection is described by Newton's law of cooling.

$$\dot{\boldsymbol{Q}}_{\boldsymbol{conv}} = \boldsymbol{hA}(\boldsymbol{T_s} - \boldsymbol{T}_\infty)$$

A: surface area of convection, T_s: surface temperature, T_∞: bulk fluid temperature, h: heat transfer coefficient ($W.m^{-2}.K^{-1}$)

Radiation

For an ideal “black body” emitter the heat transfer by radiation is given by

$$\dot{Q}_{emit} = \sigma A T_s^4$$

Where $\sigma = 5.67 \times 10^{-8}\ W.m^{-2}.K^{-4}$ (the Stefen-Boltzmann constant)

T_s is the surface temperature of the black body.

Note: the power of 4 on temperature means that at moderate temperatures radiation is often not significant (negligible) compared to conduction and convection, but at higher temperatures, radiation becomes significant.

Sometimes a combined heat transfer coefficient considering convection and radiation

$$\dot{Q}_{total} = \dot{Q}_{emit} + \dot{Q}_{conv} = h_{combined} A (T_s - T^{\infty})$$

Note2: radiation is the only heat transfer mechanism that does not need any intermedia

Typical values

Typical values for thermal conductivity k (W/mK)			
silver	~400	Concrete	~1.5
copper	~380	Glass	~1
aluminium	~200	Water (~20 ˚C)	~0.6
steel	~50	Expanded polystyrene	~0.03
titanium	~15	Air (~0 ˚C)	~0.02
Typical values for heat transfer coefficient h (W/m^2K)			
Gases in free convection	5-37	Water in free convection	100-1200
Oil under free convection	50-350	Gas flow in tubes	10-350
Water flowing in tubes	500-1200	Oil flow in tubes	300-1700
Molten metal flowing in tubes	2000-45000	Water nucleate boiling	2000-45000
Water film boiling	100-300	Conden. of organic fluid	500-2300

Conduction through composite walls

Very common to have multi-layer (sandwich) wall, e.g. metal wall covered in a layer of insulation.

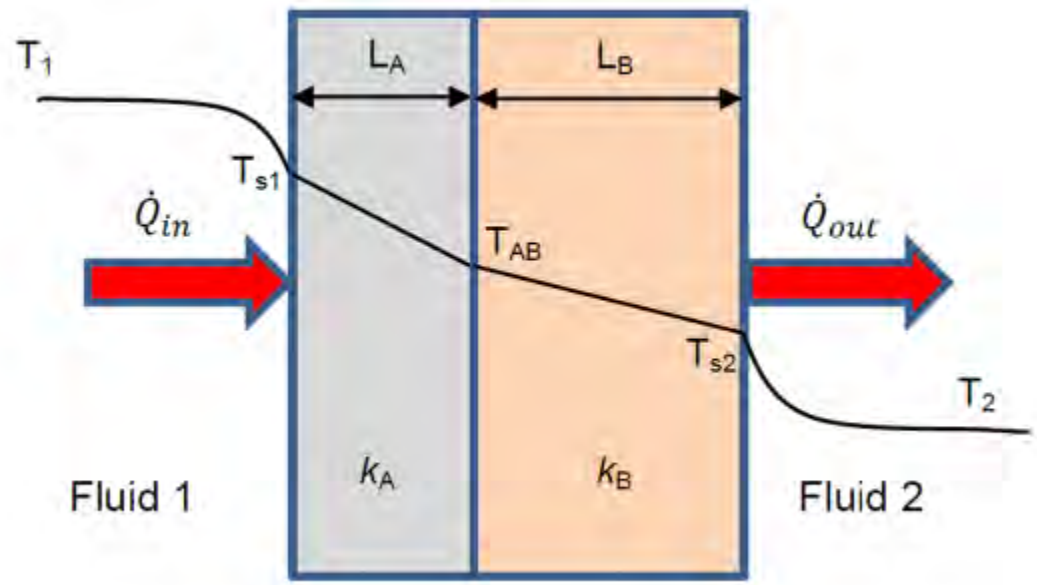
The wall can be treated as thermal resistances (equivalent to electric resistances), temperature equivalent as voltage, heat flow rate as current.

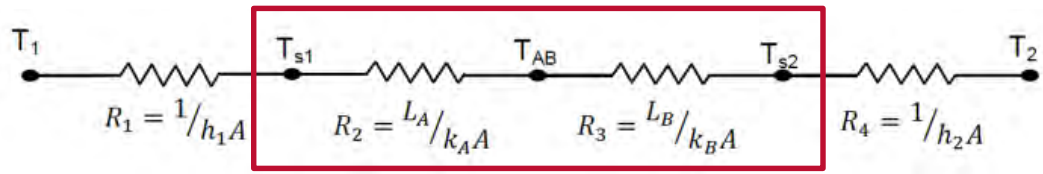
Heat transfer from fluid to wall by convection

Thermal resistance for conduction defined as

$$R = \frac{L}{kA}$$

$$U = IR \leftrightarrow \quad \Delta T = \dot{Q}R$$

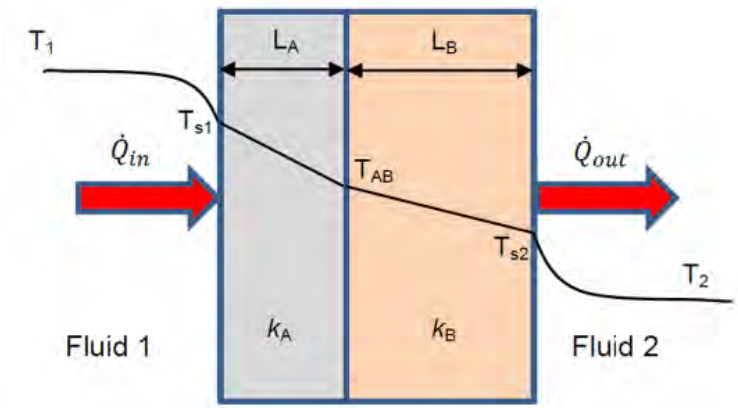




Conduction through composite walls

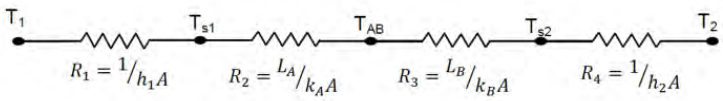
Similarly, thermal resistance for convection can be defined as

$$\boxed{R = \frac{1}{hA}} \qquad U = IR \leftrightarrow \quad \Delta T = \dot{Q}R$$



So the rate of heat flow can be written

$$\dot{Q} = \frac{driving\ force}{resistance} = \frac{T_1 - T_2}{R_{tot}}$$



where $R_{tot} = \frac{1}{h_1 A} + \frac{L_A}{k_A A} + \frac{L_B}{k_B A} + \frac{1}{h_2 A}$

Heat flows through each component is identical

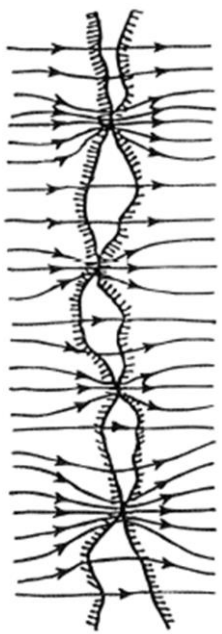
$$\dot{Q} = \frac{T_1 - T_{s1}}{\frac{1}{h_1 A}} = \frac{T_{s1} - T_{AB}}{\frac{L_A}{k_A A}} = \frac{T_{AB} - T_{s2}}{\frac{L_B}{k_B A}} = \frac{T_{s2} - T_2}{\frac{1}{h_2 A}}$$

This approach enables us to solve problems such as finding the interface temperature (e.g. $T_{s1},\ T_{AB}$) or wall thicknesses (e.g. L_A)

Thermal contact resistance

The previous example assumed perfect thermal contact between wall layers. Real surfaces are microscopically rough, and contact only occurs at certain points. The air in between act as an insulator. This create a thermal contact resistance and an accompanying temperature drop across the interface $\Delta T_{interface}$

Typical values for R_c range from 5×10^{-6} to $5 \times 10^{-4}\ K\ m^2/W$



$$\dot{Q}_{cond} = h_c A \Delta T_{interface}$$

$$R_c = \frac{1}{h_c}$$

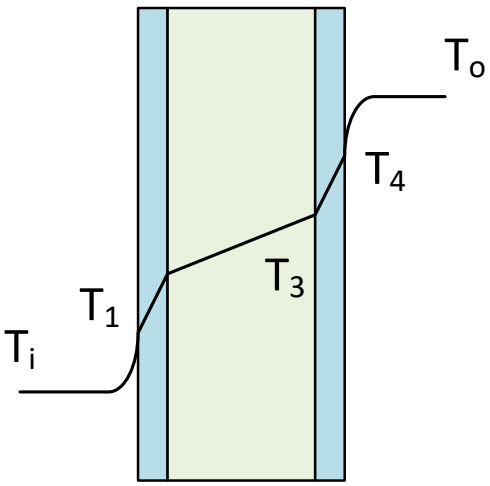
$$\dot{Q}_{cond} = \frac{A \Delta T_{interface}}{R_c}$$

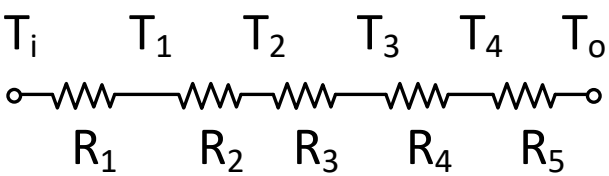
Example

The wall of an industrial furnace is constructed from 0.15 m thick fireclay brick having a thermal conductivity of 1.7 $W.m^{-1}.K^{-1}$. Measurements made during steady-state operation reveal temperatures of 1400 and 1150 K at the inner and outer surfaces respectively. What is the rate of heat loss through a wall that is 0.5 m by 3 m on a side?

$$\dot{Q}_{cond} = kA\frac{\Delta T}{x}$$
$$= 1.7\ W.m^{-1}.K^{-1} \times 0.5\ m \times 3\ m \times \frac{1400\ K - 1150\ K}{0.15\ m}$$
$$= 4250\ W$$

The wall of a refrigerator is constructed of rigid foam insulation ($k = 0.035\ W.m^{-1}.K^{-1}$) sandwiched between two layers of 1 mm thick sheet metal ($k = 15.1\ W.m^{-1}.K^{-1}$). The refrigerated space is maintained at 3°C and the average heat transfer coefficients at the inner and outer surfaces of the wall are 4 $W.m^{-2}.K^{-1}$ and 9 $W.m^{-2}.K^{-1}$ respectively. The kitchen air temperature averages 25°C. It is observed that condensation occurs on the outer surfaces of the refrigerator when the outer wall surface temperature drops to 20°C. Determine the minimum thickness of rigid foam insulation that needs to be used in the wall to avoid condensation.





Solution

Known: $T_o = 25\ °C, T_{4max} = 20°C, T_i = 3°C, h_1 = 4\frac{W}{m^2K}, h_5 = 9\frac{W}{m^2K},$

$$x_2 = x_4 = 1mm, \qquad k_2 = k_4 = 15.1\frac{W}{mK}, k_3 = 0.035\frac{W}{mK}$$

Find: x_3

Consider the worst case, $T_4 = 20°C$

For a unit area:

$$R_1 = \frac{1}{h_1} = 0.25\frac{m^2K}{W}$$

$$R_2 = R_4 = \frac{x_2}{k_2} = \frac{10^{-3}m}{15.1\frac{W}{mK}} = 6.62 \times 10^{-5}\frac{m^2K}{W}$$

$$R_5 = \frac{1}{h_5} = 0.11\frac{m^2K}{W}$$

$$\dot{Q} = \frac{T_o - T_4}{R_5} = 45\ W/m^2$$

Therefore

$$R_{tot(1-4)} = \frac{T_4 - T_i}{\dot{Q}} = \frac{17\ K}{45\ W/m^2} = 0.378\frac{m^2K}{W}$$

$$R_3 = R_{tot(1-4)} - \mathrm{R}_1 - \mathrm{R}_2 - \mathrm{R}_4 = 0.1276\frac{m^2K}{W} = \frac{x_3}{k_3}$$

Hence

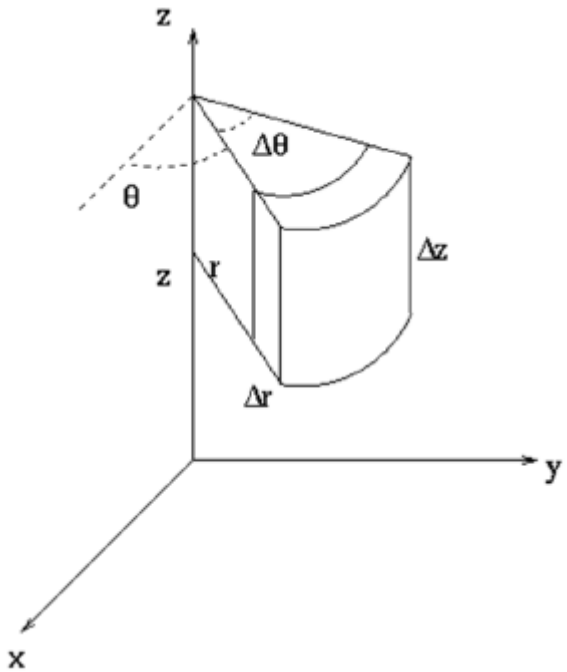
$$x_3 = R_3 \times k_3 = 4.5 \times 10^{-3}m = 4.5\ mm$$

Steady state conduction through cylinders

Fourier's Law for cylinders

Cylindrical geometry is very common in engineering applications, e.g. pipes, rods, tubes etc.

It's more convenient to consider cylindrical coordinates (z, r, θ) rather than Cartesian coordinates (x, y, z)



Cylindrical coordinates are: *axial* (distance along the length of the cylinder, z), *radial* (distance from the centre the cylinder, r) and *tangential* (angle for the rotation in the tangential direction, θ)

In many application, no (little) heat flow in the axial or tangential directions.

$$\dot{Q}_{cond} = -kA\frac{dT}{dr}$$

Where r is the shortest distance from the axis to any point.

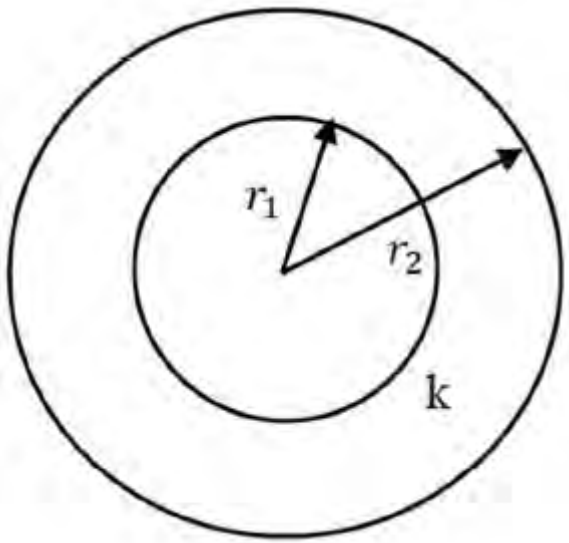
Cylindrical heat transfer

Consider a thick pipe with an inner radius r_1 and outer radius r_2

The heat transfer area, A, is normal to the direction of heat flow which is the increasing or decreasing of r direction.

For a unit length of pipe, the heat transfer area is $A = 2\pi r$, therefore Fourier's law becomes

$$\dot{Q}_{cond} = -2\pi k r \frac{dT}{dr}$$



Integrating the heat flow between the inner and outer walls, where T_1 is at r_1 and T_2 at r_2

$$\dot{Q}_{cond} \frac{dr}{r} = -2\pi k \, dT$$

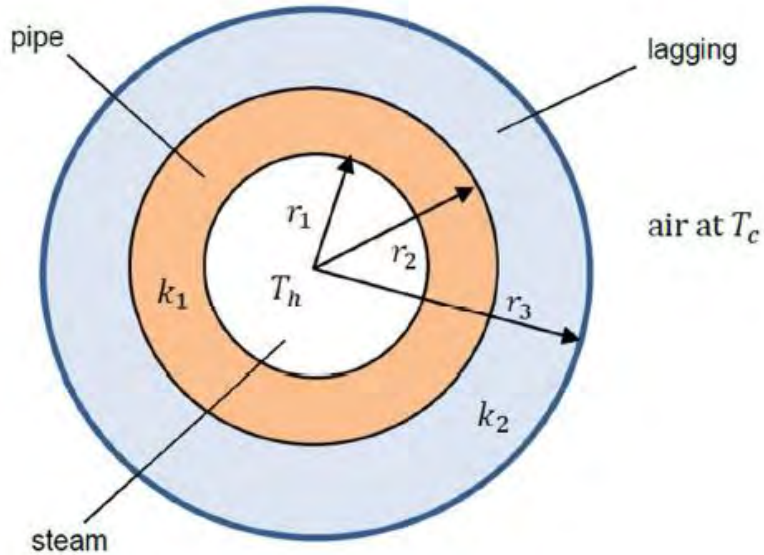
$$\dot{Q}_{cond} \int_{r_1}^{r_2} \frac{dr}{r} = -2\pi k \int_{T_1}^{T_2} dT$$

$$\dot{Q}_{cond} = 2\pi k \frac{T_1 - T_2}{\ln(\frac{r_2}{r_1})}$$

Note the sign, negative eliminated by switching the temperature around

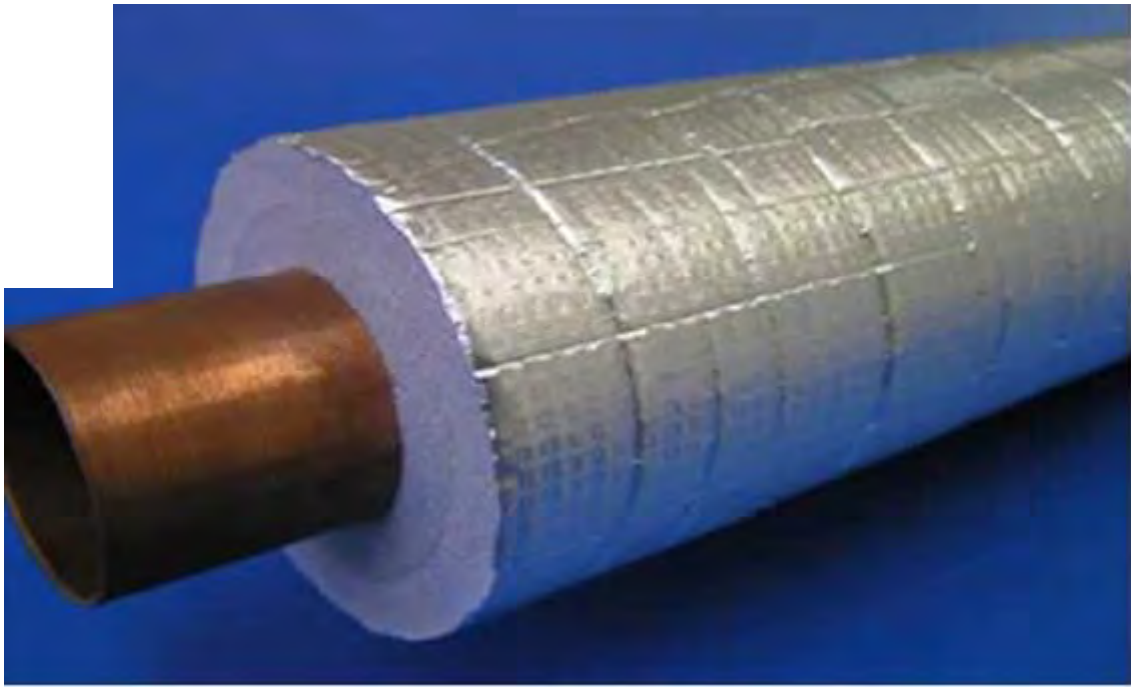
Cylindrical heat transfer

Multi layer pipe



A multi-layer pipe can be modelled as shown in the schematic drawing.

Very common in engineering applications.



Multi layer pipe

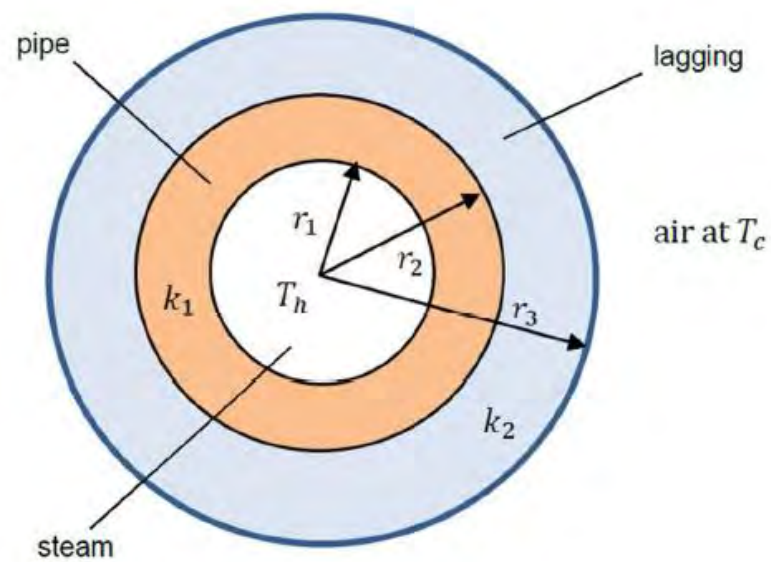
Referring to the familiar concept of

$$\dot{Q}_{cond} = \frac{driving\ force}{resistance} = \frac{T_1 - T_2}{R}$$

The resistance to conduction through a cylindrical layer can be considered as

$$R = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi k}$$

Note this is the resistance for a unit length of pipe

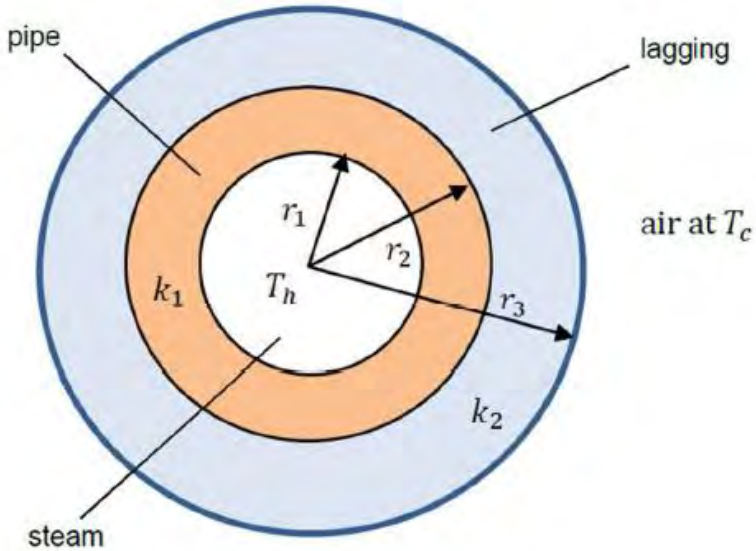


Pipe insulation

It is common for pipes carrying hot fluid to be covered in a layer of insulation or “lagging” to prevent heat loss. The lagging has a low thermal conductivity to give a high thermal resistance.

Cylindrical heat transfer

Multi layer pipes



Usually for a metal pipe, R_{conv1} and R_{wall} are small. So the rate of heat transfer is determined by R_{conv2} which can also be small (depending on the environment) and $R_{logging}$.

The rate of heat loss per unit length is given by

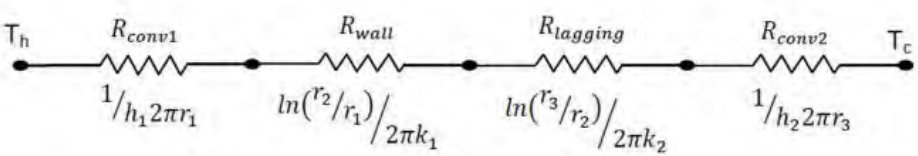
$$\dot{Q}_{cond} = \frac{T_h - T_c}{R_{tot}}$$

where

$$R_{tot} = \frac{1}{h_1 2\pi r_1} + \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi k_1} + \frac{\ln\left(\frac{r_3}{r_2}\right)}{2\pi k_2} + \frac{1}{h_2 2\pi r_3}$$

2π is area, A, for 1 m pipe length

Corresponding thermal resistance circuit, note these are the resistance for 1m of pipe length

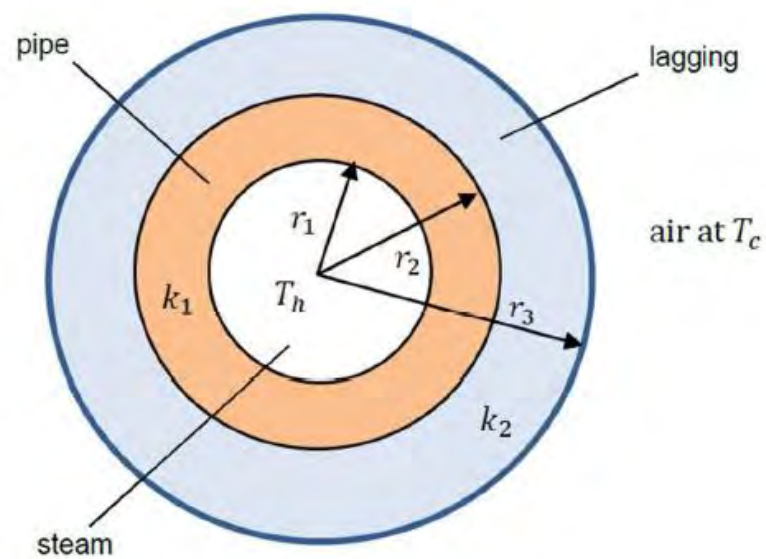


Example

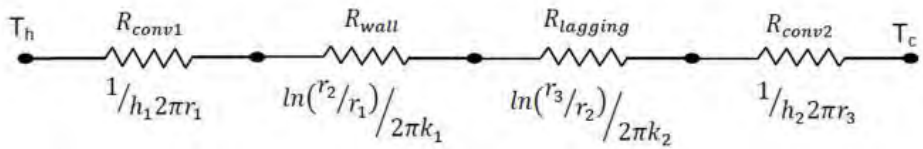
Steam at a temperature of 320°C flows in a cast iron pipe (k = 80 W.m-1.K-1) having an inner and outer diameter of 5 cm and 5.5 cm. The pipe is covered with glass wool insulation (k = 0.05 W.m-1.K-1) to a thickness of 3 cm. Heat is lost from the lagging surface by convection and radiation, with a combined heat transfer coefficient of 18 W.m-2.K-1. The surroundings are at a temperature of 5°C and the internal heat transfer coefficient for heat transfer between the steam and the cast iron pipe is 60 W.m-2.K-1.

Determine the rate of heat loss from the pipe per unit length, and the corresponding temperature drop across the pipe wall and across the lagging.

r_1=0.025 m
r_2=0.0275 m
r_3=0.0575 m
k_1=80 W m^-1. K^-1
k_2=0.05 W m^-1. K^-1
h_1=60 W.m-2.K-1
h_2=18 W.m-2.K-1
T_h=320°C, T_c=5°C



Solution



$$R_{conv1} = \frac{1}{h_1 2\pi r_1} = \frac{1}{60(2\pi)0.0025} = 0.106\ K/W$$

$$R_{pipe} = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi k_1} = \frac{\ln\left(\frac{2.75}{2.5}\right)}{2\pi 80} = 0.0002\ K/W$$

$$R_{lagging} = \frac{\ln\left(\frac{r_3}{r_2}\right)}{2\pi k_2} = \frac{\ln\left(\frac{5.75}{2.75}\right)}{2\pi 0.05} = 2.35\ K/W$$

$$R_{conv2} = \frac{1}{h_2 2\pi r_3} = \frac{1}{18(2\pi)0.0575} = 0.154\ K/W$$

$$R_{tot} = \sum R = 2.61\ K/W$$

$$\dot{Q} = \frac{T_h - T_c}{R_{tot}} = \frac{320 - 5}{2.61} = 120.7W \quad \text{per m of pipe}$$

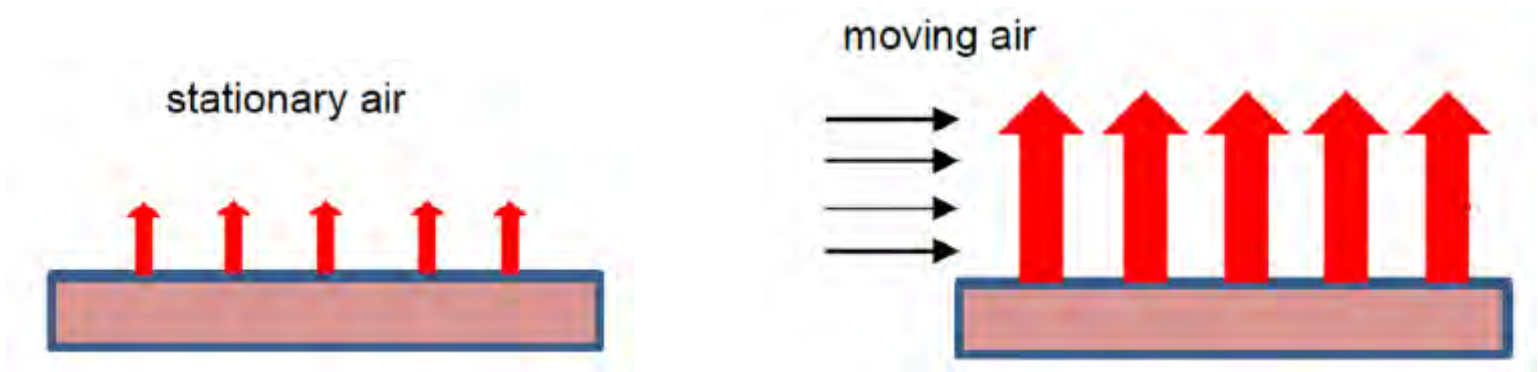
$$\Delta T_{pipe} = \dot{Q} R_{pipe} = 120.7 \times 0.002 = 0.02°\text{C}$$

$$\Delta T_{lagging} = \dot{Q} R_{lagging} = 120.7 \times 2.35 = 274°\text{C}$$

Natural convection is characterised by fluid movement as a result of buoyancy forces.

Forced convection is characterised by fluid movement assisted through the input of mechanical energy.

Consider a hot metal plate in air



The rate of heat flow by convection is given by Newton's law of cooling

$$\dot{q} = \dot{Q}_{conv}/\mathbf{A} = h(T_s - T_\infty)$$

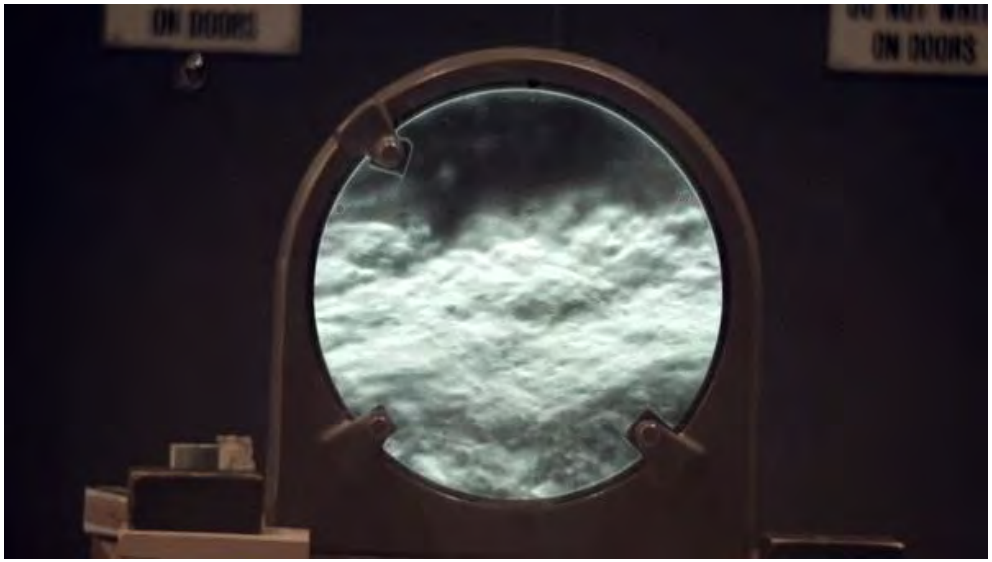
Where h is the heat transfer coefficient $(W.m^{-2}.K^{-1})$

Forced convection

External flow: here the fluid is unconstrained, and therefore the flow characteristics can develop unhindered above heat transfer surface (e.g. flow over a flat plate, aircraft wind or pipe in the open air)

Internal flow: here the fluid flow is constrained by physical boundaries, and therefore the development of flow characteristics is restricted (e.g. fluid flowing in a pipe or air inlet duct)

Usually forced convection has much higher heat transfer coefficient than natural convection



Schlieren photography showing forced convection and natural convection

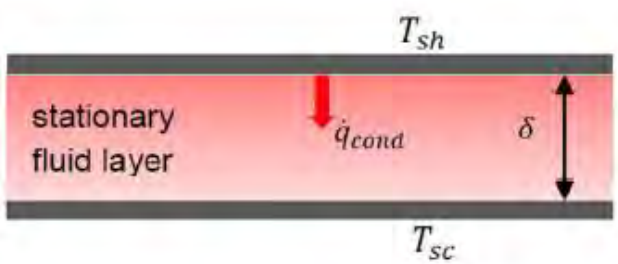
Forced convection

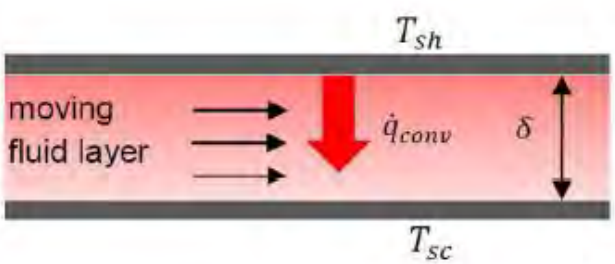
The heat transfer coefficient h depends on the fluid properties, the flow characteristics, and geometry of the system.

People tried to correlate h with dimensionless groups (in thermo-fluid area, dimensionless groups are often used to describe flow characteristics).

For forced convection, the groups are the Nusselt number, the Prandtl number and the Reynolds number

The Nusselt Number (Nu): is the dimensionless group that describes the magnitude to convective heat transfer compared to the heat transfer that would occur through the fluid by conduction only. Consider fluid layer between two flat plates maintained at temperature T_{sh} and T_{sc}





Stationary fluid layer

Heat transfer between the plates by is conduction only, (Fourier's law):

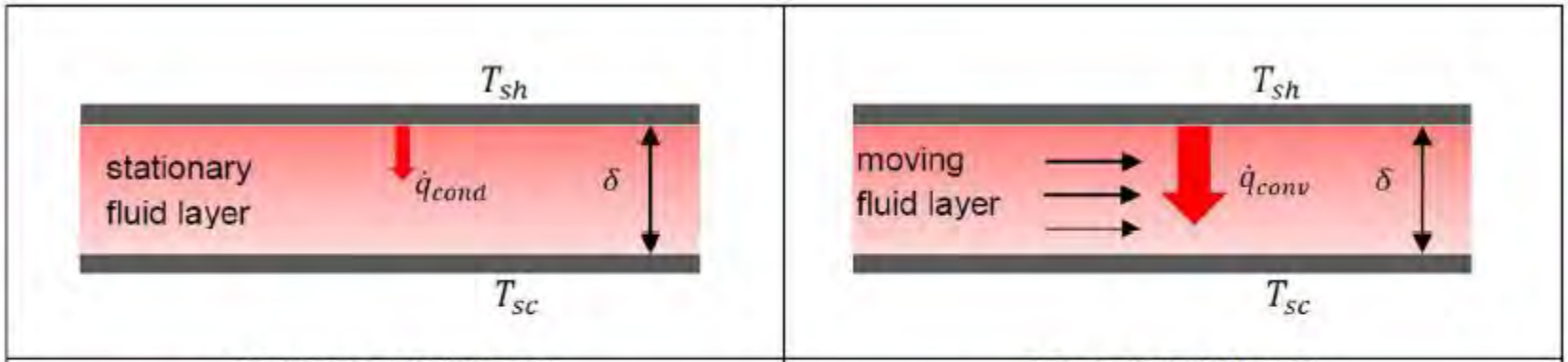
$$\dot{q}_{cond} = \frac{k}{\delta}\Delta T$$

Moving fluid layer

Heat transfer between the plates is by convection, which includes the conduction component, (Newton's law of cooling):

$$\dot{q}_{conv} = h\Delta T$$

Nusselt Number (Nu)



Stationary fluid layer	**Moving fluid layer**
Heat transfer between the plates by is conduction only, (Fourier's law): $\dot{q}_{cond} = \frac{k}{\delta}\Delta T$	Heat transfer between the plates is by convection, which includes the conduction component, (Newton's law of cooling): $\dot{q}_{conv} = h\Delta T$

The Nusselt Number Nu is defined as

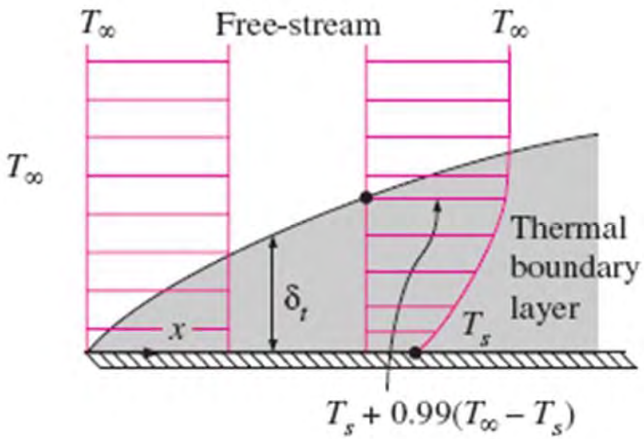
$$\mathrm{Nu} = \frac{\dot{q}_{conv}}{\dot{q}_{cond}} = \frac{h\Delta T}{\frac{k}{\delta}\Delta T} = \frac{h\delta}{k}$$

The Nu is always equal or greater than 1

Prandtl Number (Pr)

Thermal boundary layer

The temperature transition between a surface and a bulk fluid will occur over a region of the flow, known as the thermal boundary layer



Thermal boundary layer on a flat plate (the fluid is hotter than the plate surface).

The relative effectiveness of the velocity and thermal boundary layers to transport momentum and heat depends on the Prandtl number (Pr) which is defined as

$$\mathrm{Pr} = \frac{molecular\ diffuivity\ of\ momentum}{molecular\ diffusivity\ of\ heat} = \frac{\nu}{\alpha}$$

Here ν is the kinematic viscosity ($m^2.s^{-1}$), defined by $\nu = \frac{\mu}{\rho}$ and α is the conductive diffusivity of heat ($m^2.s^{-1}$) defined as $\alpha = k/\rho c_p$. Therefore

$$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{c_p \mu}{k}$$

Prandtl number (Pr)

$$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{c_p \mu}{k}$$

Apparently, Pr number encapsulates important material properties. The range of typical values of Pr are listed below

Liquid metals	$\mathrm{Pr} \ll 1$	Large k, $\alpha \gg \nu$
Gases	$\mathrm{Pr} \cong 1$	$\alpha \cong \nu$
Water	$\mathrm{Pr} \cong 10$	
Oils	$\mathrm{Pr} \gg 1$	Small k, $\alpha \ll \nu$

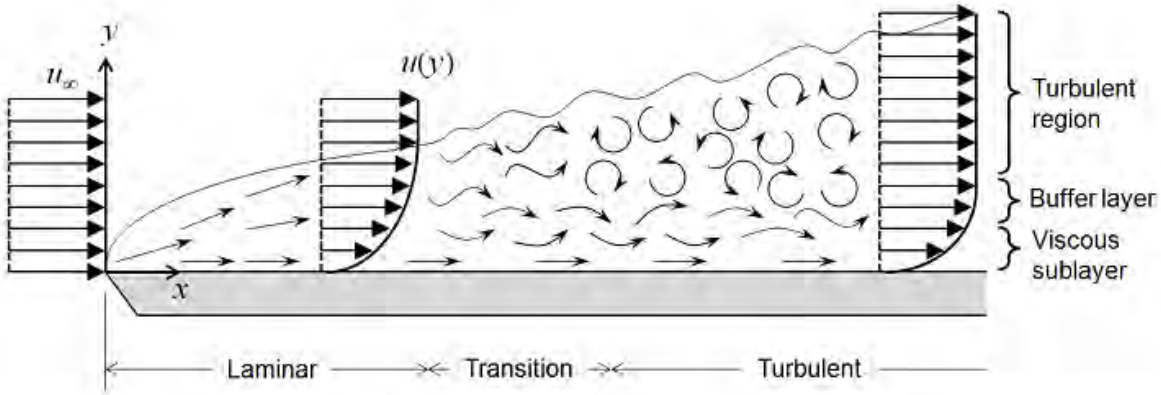
The Reynolds Number Re

The Reynolds number expresses the ratio of **inertial forces** to **viscous forces** in fluid flow. Very widely used to predict the transition between laminar flow and turbulent flow regimes.

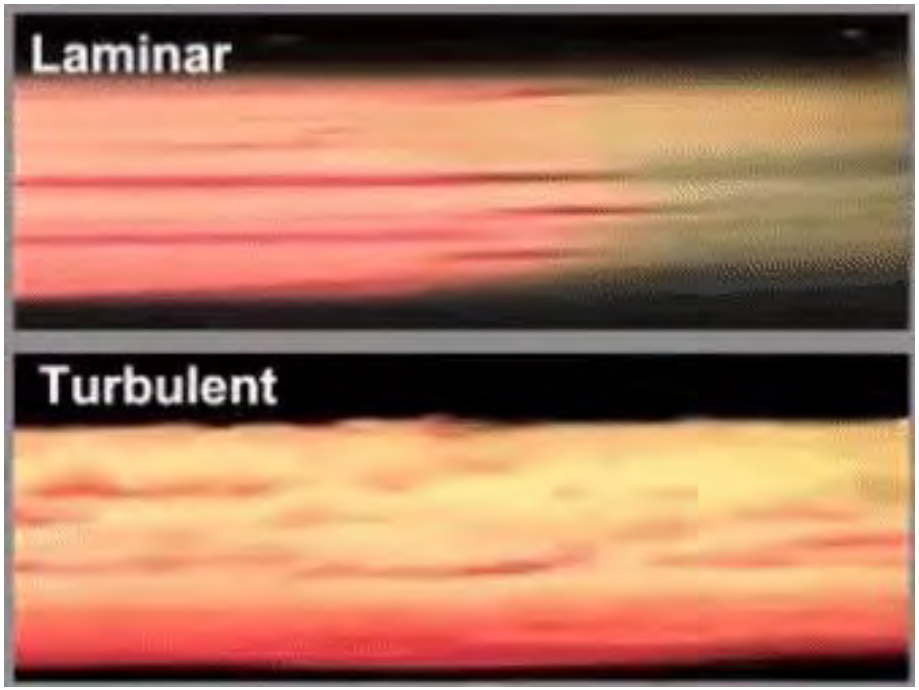
$$\mathrm{Re}_x = \frac{\rho v x}{\mu}$$

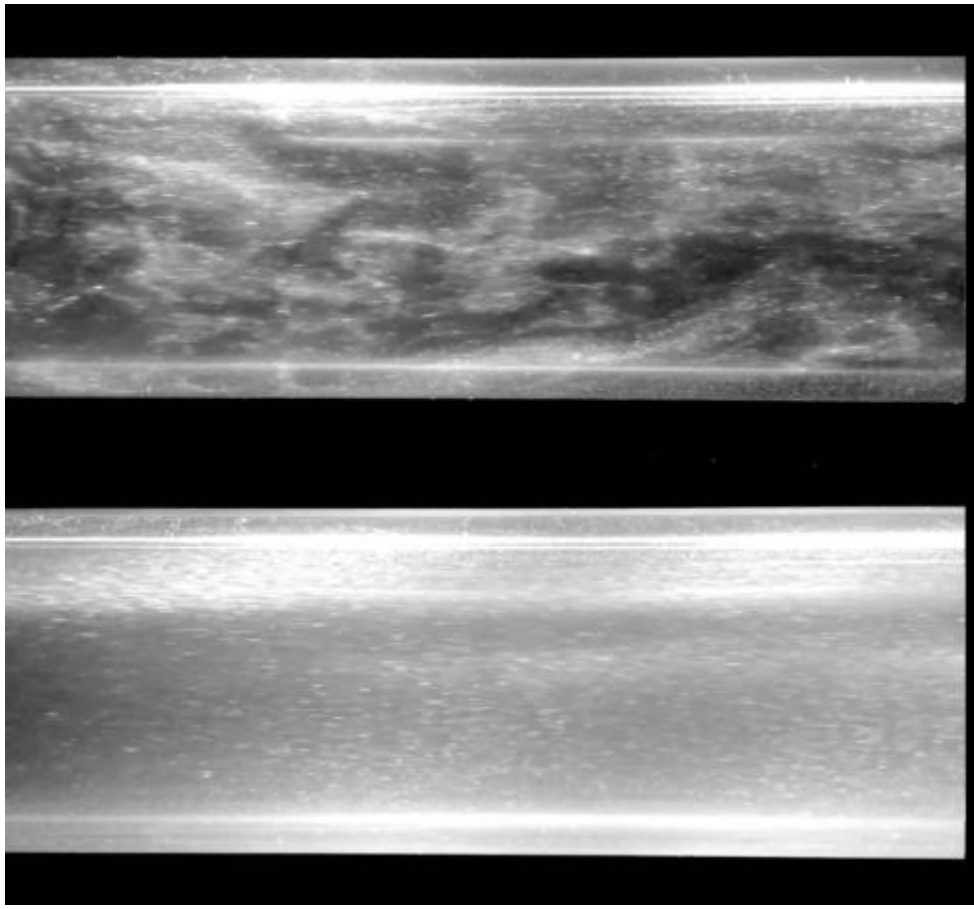
Here v is fluid velocity.

For flow along a flat plate, the flow transition occurs at a value of $\mathrm{Re}_x \cong 5 \times 10^5$. The transition occurs at some distance down the plate, this distance corresponding to the value of $\mathrm{Re}_x \cong 5 \times 10^5$



Laminar flow and turbulent flow





Heat transfer coefficient correlations

The heat transfer coefficient in forced convection depends on a complex set of variables. These have been assembled in the dimensionless groups of Nu, Pr and Re. These groups are often correlated in a generic form as

$$\mathrm{Nu} = C\mathrm{Re}_x^m \mathrm{Pr}^n$$

The specific values of C, m and n depend on the system geometry and flow regime. The same form of correlation is applied for forced convection in internal flow as well as external flow.

For example

External Flow Correlations

Flow over a flat surface

Laminar Flow: $Nu = \frac{hL}{k} = 0.664 Re_L^{0.5} Pr^{1/3}$ $\quad Re_L < 5 \times 10^5$

Turbulent Flow: $Nu = \frac{hL}{k} = 0.037 Re_L^{08} Pr^{1/3}$ $\quad 0.6 \leq Pr \leq 60$

$5 \times 10^5 \leq Re_L \leq 10^7$

Key points

- The transfer of thermal energy in the form of electromagnetic waves.
- Wavelength range for the thermal radiation is 0.1<λ<100 μm
- All bodies above 0 Kelvin emit radiation
- Transmission of radiation does not require a medium. In fact it works better in a vacuum.

Black body

A black body represents an ideal reference case. It can be considered as a body with a surface that emits the maximum possible radiation in a diffuse manner, i.e. the emission does not depend on the direction. Furthermore a black body absorbs all radiation that is incident upon it.

The radiation heat flux emitted from a black body is termed the Blackbody Emissive Power E_b defined as

$$E_b = \sigma T^4$$

Where σ is the Stefan-Boltzmann constant ($= 5.67 \times 10^{-8} W.m^{-2}.K^{-4}$), and T is the absolute temperature (K)

Emissivity

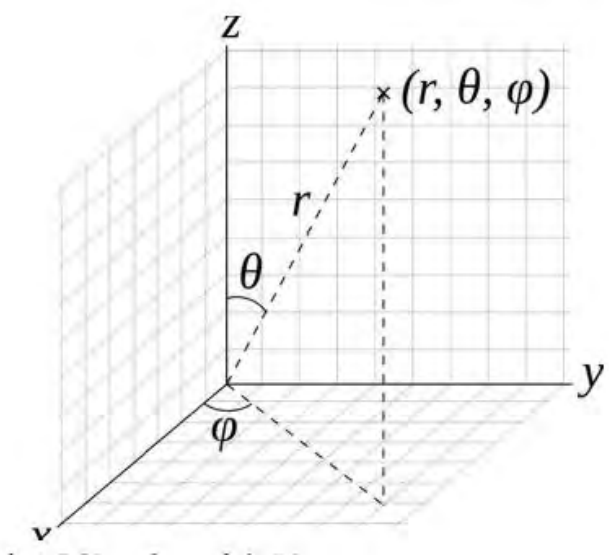
Real body emit radiation at a rate $E(T)$ that is less than the black body $E_b(T)$. The two are related by the emissivity ε as

$$\varepsilon = \frac{emissive\ power\ of\ real\ body\ at\ T}{emissive\ power\ of\ black\ body\ at\ T} = \frac{E(T)}{E_b(T)}$$

In reality ε can be a function of temperature (T), wavelength (λ) and direction (θ). For any given temperature we can define the following

- $\varepsilon(\lambda) = constant$: grey body/surface
- $\varepsilon(\theta) = constant$: diffuse body/surface
- $\varepsilon(\lambda) = \varepsilon(\theta) = constant$: diffuse, grey body/surface

θ is the spherical polar coordinate with z being normal to the surface

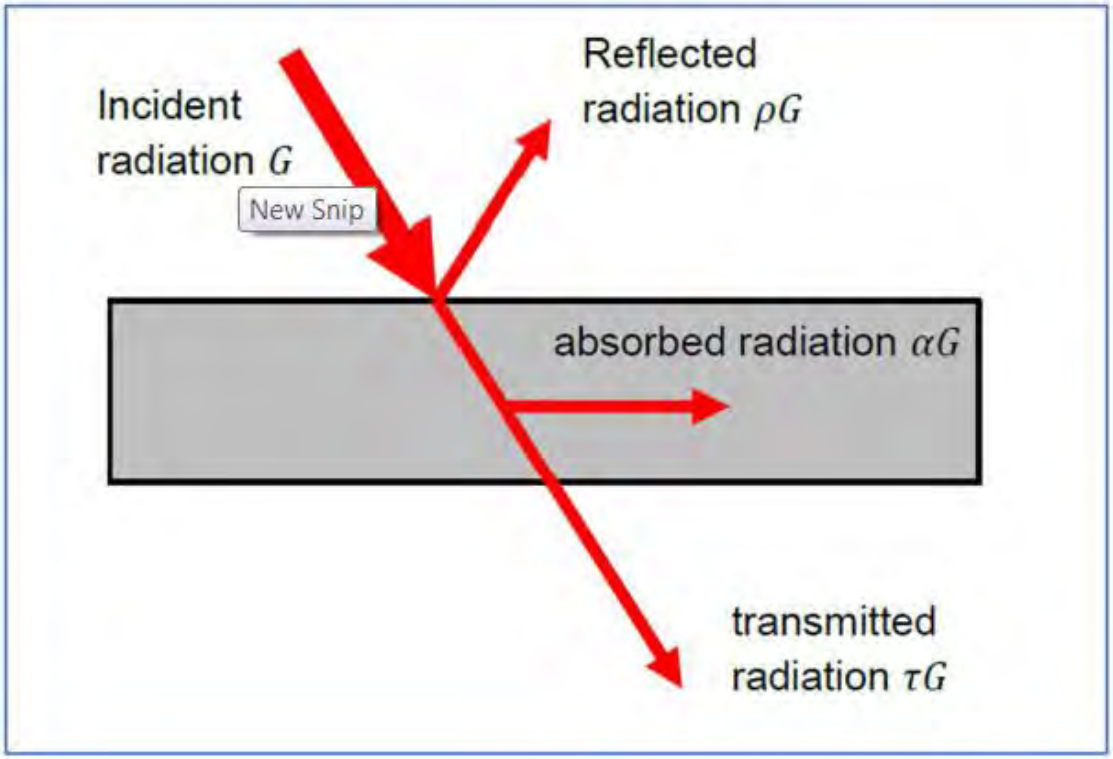


Absorptivity, Reflectivity and Transmissivity

The rate at which incident radiation arrives at a surface per unit area is known as the irradiation G $(\mathrm{W.m^2})$

In order to calculate radiation heat exchange, it is important to know what happens to the irradiation when it arrives a surface. Essentially it has three options, which can be

- Absorbed
- Reflected
- Transmitted



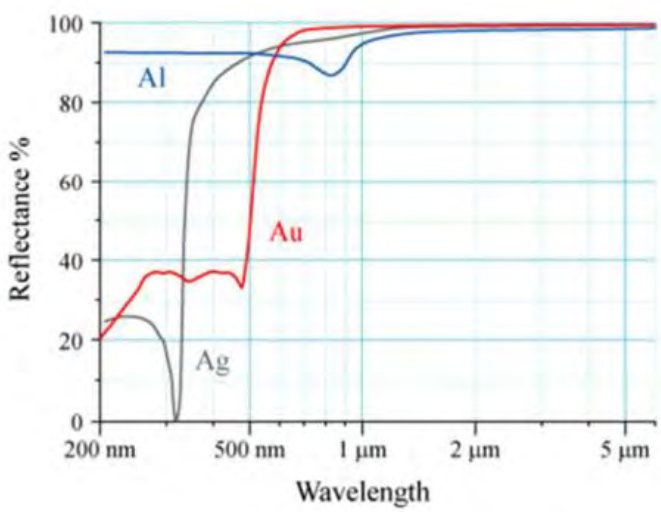
Absorptivity, Reflectivity and Transmissivity

$$\text{Absorptivity } \alpha = \frac{rate\ of\ absorption}{irradiation}$$

$$\text{Reflectivity } \rho = \frac{rate\ of\ reflection}{irradiation}$$

$$\text{Transmissivity } \tau = \frac{rate\ of\ transmission}{irradiation}$$

In general we have $\alpha + \rho + \tau = 1$



For aluminium, $\tau = 0$, therefore $\alpha = 1 - \rho$ and by Kirchhoff's law, $\alpha = \varepsilon$ so this is a graph of $1 - \varepsilon$ versus λ

Kirchoff's Law

Consider a small body of area A emissivity ε and absorptivity α at temperature T inside a large blackbody enclose at the same temperature. The irradiation G received by the small body ($W.m^{-2}$) must be equal to the black body emissive power, i.e.

$$G = E_b(T) = \sigma T^4$$

The radiation absorbed by the small body per unit area must therefore be

$$G_{abs} = \alpha G = \alpha \sigma T^4$$

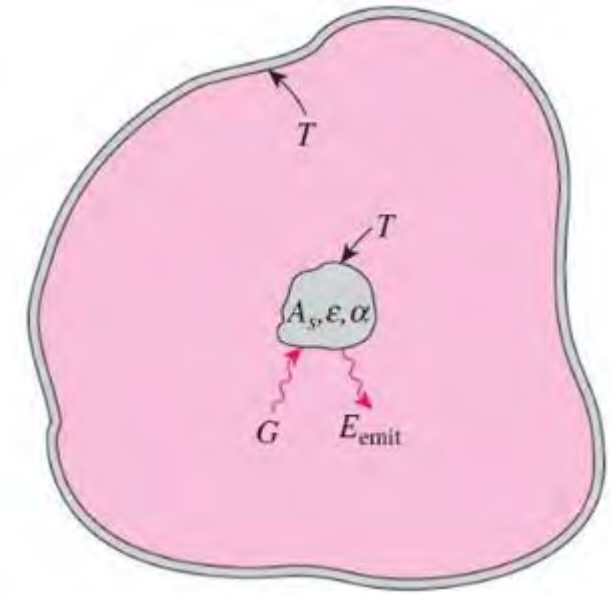
The radiation emitted by the small body per unit area must be

$$E_{emit} = \varepsilon \sigma T^4$$

As the two surfaces are at the same temperature, the net radiation exchange must be zero, hence

$$A\varepsilon\sigma T^4 = A\alpha\sigma T^4$$

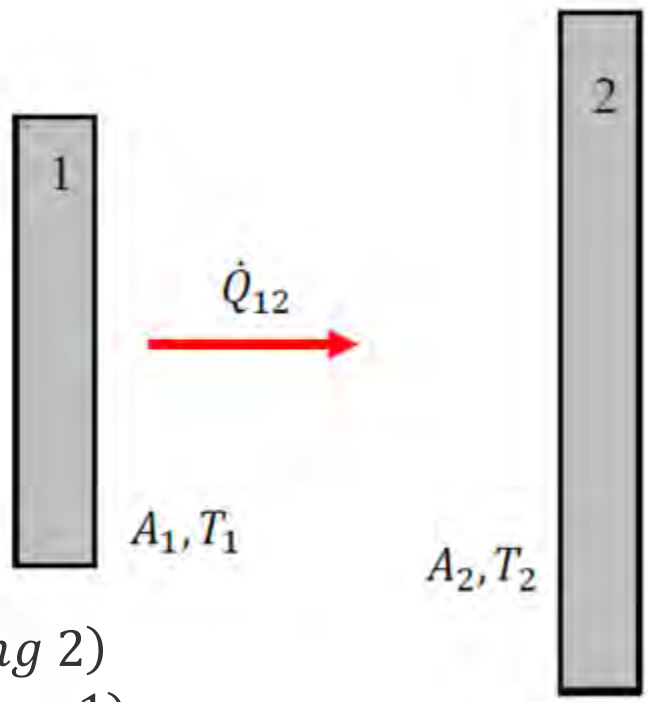
This gives Kirchhoff's Law $\boldsymbol{\varepsilon(T) = \alpha(T)}$



Radiation between black bodies

Consider two opposing black bodies with areas A_1 and A_2 maintained at temperature T_1 and T_2 respectively

The net radiative heat exchange between the two surfaces can be expressed by



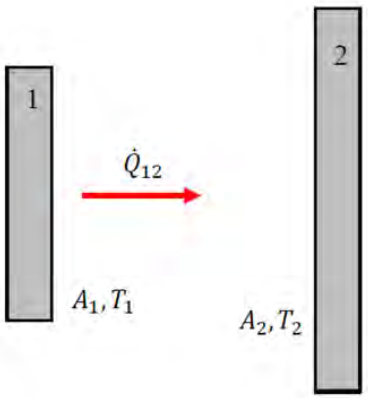
$$\dot{Q}_{12} = (radiation\ leaving\ 1\ and\ striking\ 2) - (radiation\ leaving\ 2\ and\ striking\ 1)$$

Radiation between black bodies

$$\dot{Q}_{12} = (radiation\ leaving\ 1\ and\ striking\ 2) - (radiation\ leaving\ 2\ and\ striking\ 1)$$

Therefore

$$\dot{Q}_{12} = A_1 F_{1-2} E_{b1} - A_2 F_{2-1} E_{b2}$$



F_{1-2} is the view factor which is defined as the fraction of radiation leaving surface 1 and strikes surface 2.

Determine the view factor is complex, even for simple geometries. Values for commonly encountered systems are published in tables and charts.

One useful attribute of the view factor is the reciprocity rule, which states:

$$A_1 F_{1-2} = A_2 F_{2-1}$$

This identity greatly simplifies radiation exchange calculations. The above expression then becomes

$$\dot{Q}_{12} = A_1 F_{1-2} E_{b1} - A_2 F_{2-1} E_{b2} = A_1 F_{1-2}(E_{b1} - E_{b2}) = A_1 F_{1-2} \sigma (T_1^4 - T_2^4)$$

Radiation between grey bodies

Radiation exchange between black bodies is relatively straightforward because all the radiation received gets absorbed. However when dealing with real bodies (including grey bodies) account needs to be taken of the reflected radiation between the two surfaces. The method for this is too involved to be covered in the scale of this module.

The case of grey body radiation between a small object (area A_1, emissivity ε_1 and temperature T_1) contained in a large cavity (area A_2, emissivity ε_2 and temperature T_2) has a relatively simple solution

$$\dot{Q}_{12} = A_1 \varepsilon_1 \sigma (T_1^4 - T_2^4)$$

The simplicity arises because $A_1 \ll A_2$

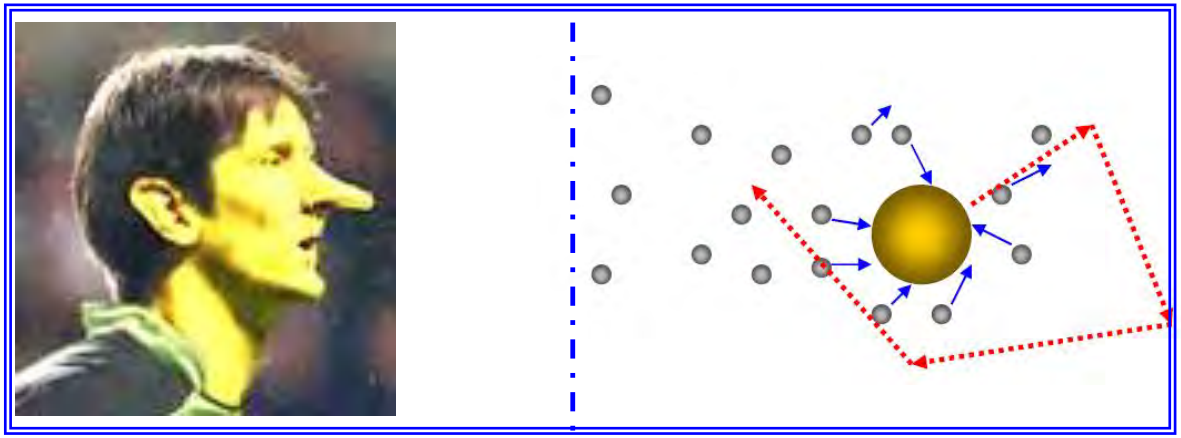
Mass transfer

Imagine opening a bottle of perfume

- Immediate detection of perfume adjacent to the bottle
- Later everywhere in the room
- Perfume molecules (mass) transported from high concentration region to a low concentration region, subject to mass transfer.

Mass may be transported by

- molecular processes -- slow and small spatial scales
- turbulent processes -- depends on the velocity and size of an eddy



Mass transfer phenomenological description

Mass transfer	**vs**	**Heat transfer**
Gradient of concentration		Gradient of temperature
From high concentration low concentration		From high temperature to to low temperature

Mass transfer is caused by concentration gradient: a non-equilibrium and irreversible process

Fick's law of diffusion

For the case of **one-dimensional binary diffusion**

$$\dot{m}_A'' = Y_A(\overset{\dot{m}_A''}{\dot{m}_A''} + \dot{m}_B'') - \rho D_{AB}\frac{dY_A}{dx}$$

Mass flow of species A per unit area	Mass flow of species A associated with bulk flow per unit area	Mass flow of species A associated with molecular diffusion per unit area

Where $\dot{m}_A''$ is the mass flux (mass flow rate per unit area, $\frac{\dot{m}_A}{A}, kg.s^{-1}.m^{-2}$) of species A, Y_A is the mass fraction. D_{AB}, the binary diffusivity, is a property of the mixture ($m^2.s^{-1}$)

Mass transfer rate laws

$$\dot{m}_A'' = Y_A(\dot{m}_A'' + \dot{m}_B'') - \rho D_{AB}\frac{dY_A}{dx}$$

Species A is transported by two means, bulk motion of the fluid (the first term on the right side of the equation) and the diffusion of A superimposed on the bulk flow.

In the absence of diffusion,

$$\dot{m}_A'' = Y_A(\dot{m}_A'' + \dot{m}_B'') = Y_A\dot{m}''$$

Where $\dot{m}''$ is the mixture mass flux.

The diffusional flux adds an additional component

$$-\rho D_{AB}\frac{dY_A}{dx} \equiv \text{Diffusional flux of species A, } \dot{m}''_{A,\text{diff}}$$

Meaning that the diffusional flux of A, is proportional to the gradient of the mass fraction.

Minus sign means it moves from high concentration region to low concentration region.

Fick's law

In a more general expression

$$\dot{\boldsymbol{m}}_{\boldsymbol{A}}'' = Y_A(\dot{\boldsymbol{m}}_{\boldsymbol{A}}'' + \dot{\boldsymbol{m}}_{\boldsymbol{B}}'') - \rho D_{AB} \nabla Y_A$$

All the bold symbols represent vector quantities

Or the molar form

$$\dot{\boldsymbol{n}}_{\boldsymbol{A}}'' = x_A(\dot{\boldsymbol{n}}_{\boldsymbol{A}}'' + \dot{\boldsymbol{n}}_{\boldsymbol{B}}'') - cD_{AB} \nabla x_A$$

Where $\dot{\boldsymbol{n}}_{\boldsymbol{A}}''$ is the molar flux of species A ($\mathrm{kmol}.s^{-1}.m^{-2}$), x is the mole fraction, and c is the mixture molar concentration ($kmol_{mix}.m^{-3}$)

The total mass flux is the sum of mass flux of species A and that of species B

$$\dot{\boldsymbol{m}}'' = \dot{\boldsymbol{m}}_{\boldsymbol{A}}'' + \dot{\boldsymbol{m}}_{\boldsymbol{B}}''$$

Mixture mass flux | Species A mass flux | Species B mass flux

For a one dimensional question, we have

$$\dot{m}'' = \underbrace{\boxed{Y_A \dot{m}'' - \rho D_{AB} \frac{dY_A}{dx}}}_{\dot{\boldsymbol{m}}_{\boldsymbol{A}}''} + \underbrace{\boxed{Y_B \dot{m}'' - \rho D_{BA} \frac{dY_B}{dx}}}_{\dot{\boldsymbol{m}}_{\boldsymbol{B}}''}$$

Or in another form

$$\dot{m}'' = (Y_A + Y_B)\dot{m}'' - \rho D_{AB} \frac{dY_A}{dx}'' - \rho D_{BA} \frac{dY_B}{dx}$$

For a double components system, as $Y_A+Y_B=1$, we have

$$-\rho D_{AB}\frac{dY_A}{dx}-\rho D_{AB}\frac{dY_B}{dx}=0$$

Diffusion of component A

Diffusion of component B

In general, by mass conservation

$$\boxed{\sum \dot{m}''_{i,diff}=0}$$

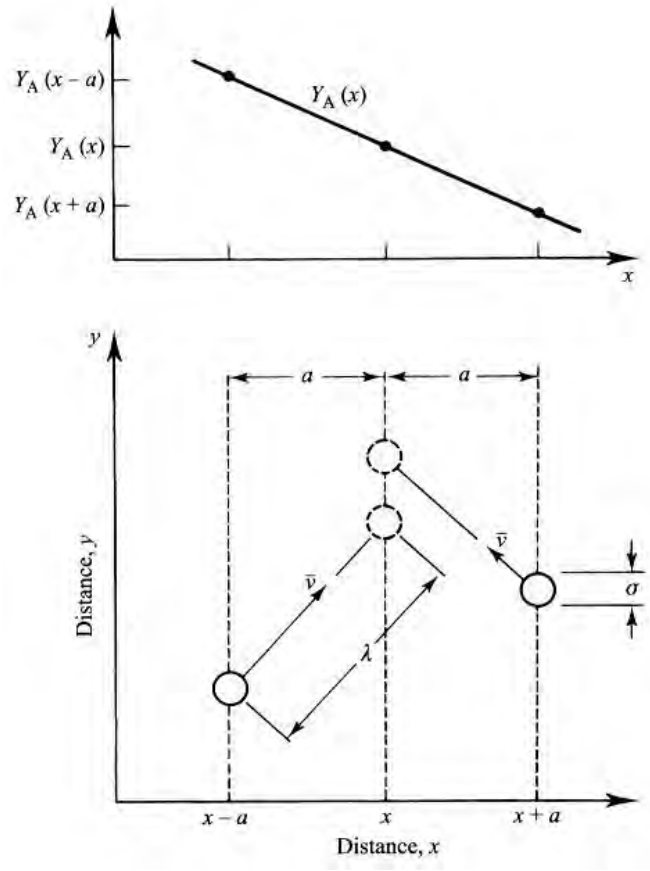
So far we only concern diffusion by concentration gradient, which is termed **ordinary diffusion**. Also gradients of temperature and pressure can produce species diffusion, i.e. the **thermal diffusion** (known as **Soret**) and **pressure diffusion** effects respectively.

Molecular basis of diffusion

Some concepts from kinetic theory of gases are applied.

Consider a stationary (no bulk flow) plane layer of a binary gas mixture consisting of rigid, non-attracting molecules, in which the molecular mass of species A and B is equal.

A concentration gradient exists in the gas layer in the x-direction and sufficiently small that the mass-fraction distribution can be considered linear over a distance of a few molecular mean free path, λ.



From kinetic theory (not included in this course) we define

$$\bar{v} \equiv \frac{\text{mean speed of}}{\text{species A molecules}} = \left(\frac{8k_B T}{\pi m_A}\right)^{1/2} \quad \sqrt{\frac{8k_B T}{\pi m_A}}$$

$$Z''_A \equiv \frac{\text{Wall collision frequency}}{\text{of A molecules per unit area}} = \frac{1}{4}\left(\frac{n_A}{V}\right)\bar{v}$$

$$\lambda \equiv \text{Mean free path} = \frac{1}{\sqrt{2}\pi\left(\frac{n_{tot}}{V}\right)\sigma^2}$$

$$a \equiv \frac{\text{Avearage perpendicular distance from plane of}}{\text{last collision to plane where next collision occurs}} = \frac{2}{3}\lambda$$

Where k_B is the Boltzmann's constant, m_A is the mass of a single A molecule, n_A/V is the number of A molecules per unit volume, n_{tot}/V is the total number of molecules per unit volume, and σ is the diameter of both A and B molecules.

$$pV = \frac{m}{M}R_0 T = NR_0 T = nk_B T$$

Molecular basis of diffusion

Assuming no bulk flow for simplicity, the net flux of A molecules at the x-plane is the difference between positive and negative x-direction flux

$$\dot{m}''_A = \dot{m}''_{A,(+)x-dir} - \dot{m}''_{A,(-)x-dir}$$

By express in terms of collision frequency

$$\dot{m}''_A = (Z''_A)_{x-a} m_A - (Z''_A)_{x+a} m_A$$

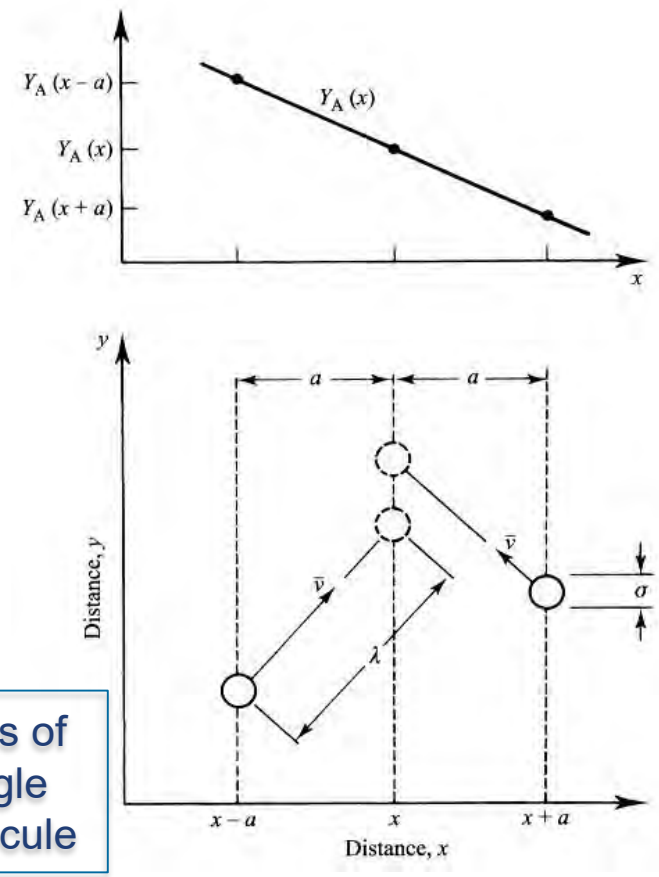
Net mass flux of species A

Number of A molecules crossing plane at x originating from plane at x-a per unit time and area

Mass of single molecule

Number of A molecules crossing plane at x originating from plane at x+a per unit time and area

Mass of single molecule



Molecular basis of diffusion

Use the definition of density ($\rho \equiv m_{tot}/V_{tot}$) to relate Z''_A to the mass fraction of A molecules:

$$Z''_A m_A = \frac{1}{4}\left(\frac{n_A}{V}\right)\bar{v} m_A = \frac{1}{4}\frac{n_A m_A}{m_{tot}}\rho\bar{v} = \frac{1}{4}Y_A\rho\bar{v}$$

Substitute this to the last equation, and treat mixture density and mean molecular speeds as constants, we have

$$\dot{m}''_A = \frac{1}{4}\rho\bar{v}(Y_{A,x-a} - Y_{A,x+a})$$

With the assumption of linear concentration distribution,

$$\frac{dY_A}{dx} = \frac{Y_{A,x+a} - Y_{A,x-a}}{2a} = \frac{Y_{A,x+a} - Y_{A,x-a}}{4\lambda/3}$$

$a = \frac{2}{3}l$

Therefore we have

$$\dot{m}''_A = -\frac{1}{3}\rho\lambda\bar{v}\frac{dY_A}{dx}$$

Remember $\dot{m}_A'' = \cancel{Y_A(\dot{m}_A'' + \dot{m}_B'')} - \rho D_{AB}\frac{dY_A}{dx}$

we have

$$D_{AB} = \frac{1}{3}\lambda\bar{v}$$

Molecular basis of diffusion

Using the definitions of mean molecular speed and mean free path, and the ideal gas equation $pV = nk_BT$, the temperature and pressure dependence of D_{AB} can be determined:

$$D_{AB} = \frac{2}{3}\left(\frac{k_B^3 T}{\pi^3 m_A}\right)^{\frac{1}{2}} \frac{T}{\sigma^2 p}$$

Or

$$D_{AB} \propto T^{\frac{3}{2}} p^{-1}$$

Thus we see diffusivity depends strong on temperature (to the 3/2 power), and inversely proportional to pressure.

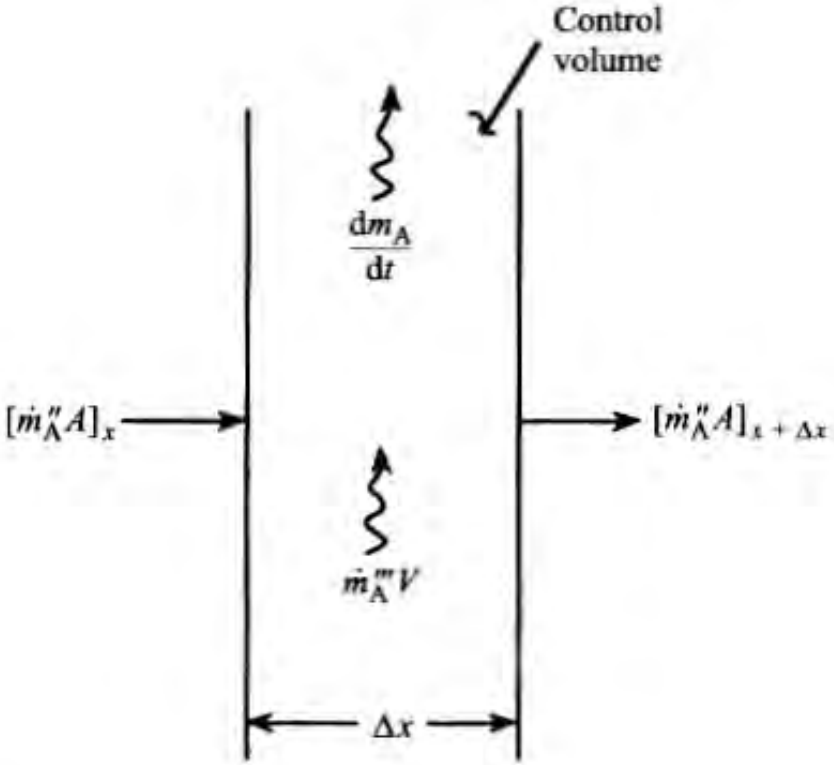
The mass flux of species A, depends on the product of ρD_{AB}, has

$$\rho D_{AB} \propto T^{\frac{1}{2}} p^0 = T^{1/2}$$

In simplified analysis, the weak temperature dependence is neglected and ρD is treated as a constant.

Species conservation

- Consider the one-dimensional control volume, a plane layer Δx thick.
- Species A flows into and out of the volume due to bulk flow and diffusion.
- Within the volume, species A may be created or destroyed as a result to chemical reaction.



The net rate of increase in the mass of A in the volume is

$$\frac{dm_{A,cv}}{dt} = [\dot{m}_A{}''A]_x - [\dot{m}_A{}''A]_{x+\Delta x} + \dot{m}_A'''V$$

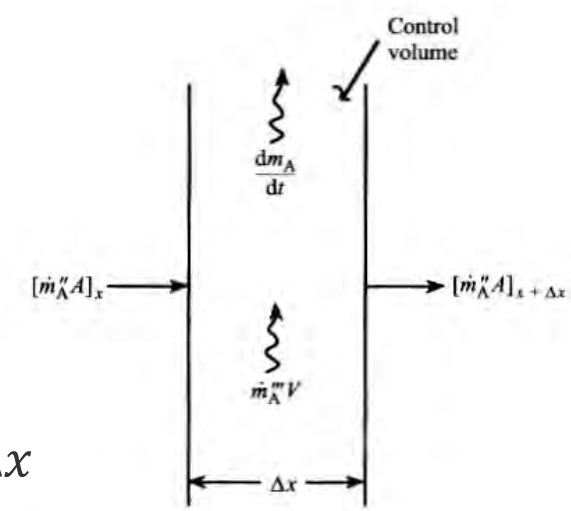
Rate of increase of mass of A within control volume	Mass flow of A into the control volume	Mass flow of A out of the control volume	Mass production rate of species A by chemical reaction

Recognising that the mass of A within the control volume is

$$m_{A,cv} = Y_A m_{cv} = Y_A \rho V_{cv}$$

The above equation can be written as

$$A\Delta x \frac{\partial(\rho Y_A)}{\partial t} = A[Y_A \dot{m}'' - \rho D_{AB} \frac{\partial Y_A}{\partial x}]_x - A[Y_A \dot{m}'' - \rho D_{AB} \frac{\partial Y_A}{\partial x}]_{x+\Delta x} + \dot{m}_A''' A\Delta x$$

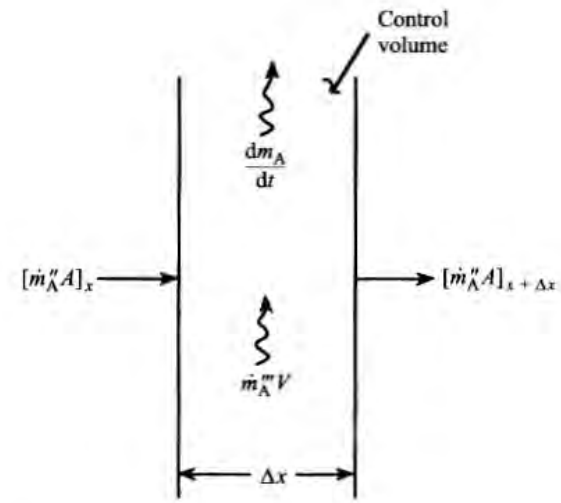


Species conservation

$$A\Delta x\frac{\partial(\rho Y_A)}{\partial t} = A[Y_A\dot{m}'' - \rho D_{AB}\frac{\partial Y_A}{\partial x}]_x - A[Y_A\dot{m}'' - \rho D_{AB}\frac{\partial Y_A}{\partial x}]_{x+\Delta x} + \dot{m}_A'''A\Delta x$$

Dividing through by $A\Delta x$ and taking the limit as $\Delta x \to 0$, it becomes

$$\frac{\partial(\rho Y_A)}{\partial t} = -\frac{\partial[Y_A\dot{m}'' - \rho D_{AB}\frac{\partial Y_A}{\partial x}]}{\partial x} + \dot{m}_A'''$$



Or for steady flow $\frac{\partial(\rho Y_A)}{\partial t} = 0$, therefore we have

$$\dot{m}_A''' - \frac{\partial[Y_A\dot{m}'' - \rho D_{AB}\frac{\partial Y_A}{\partial x}]}{\partial x} = 0$$

For multi-dimensional case,

$$\dot{m}_A''' \quad - \quad \nabla\cdot\dot{m}_A'' = 0$$

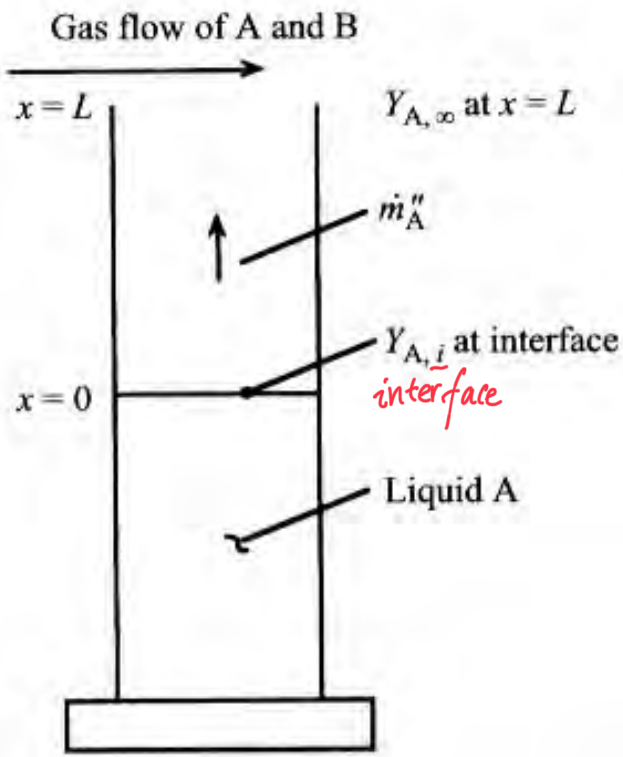
Net rate of production of species A by chemical reaction per unit volume

Net flow of species A out of control volume per unit volume

Some applications

The Stefan problem

- Consider liquid a, maintained at a fixed height in a glass cylinder.
- A mixture of gas A and gas B flow across the top.
- If the concentration of A in the flowing gas is less than the concentration of A at the liquid-gas interface a driving force of mass transfer exists
- A diffuse from liquid-gas interface to the open end of the cylinder.
- Consider a steady state (neglect the interface movement), assume B is insoluble in A, then there is no B transfer



The Stefan problem

Overall mass conservation of the system

$$\dot{m}''(x) = constant = \dot{m}''_A + \dot{m}''_B$$

As $\dot{m}''_B = 0$

$$\dot{m}''(x) = constant = \dot{m}''_A$$

As we have $$\dot{m}''_A = Y_A \dot{m}''_A - \rho D_{AB} \frac{dY_A}{dx}$$

Rearrange $$-\frac{\dot{m}''_A}{\rho D_{AB}} dx = \frac{dY_A}{1-Y_A}$$

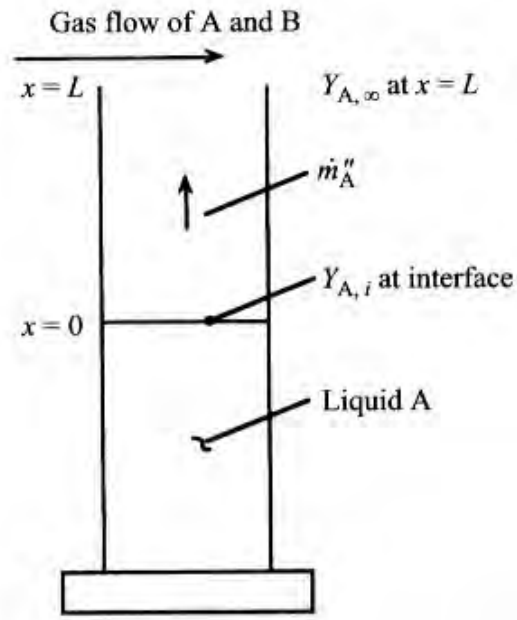
Assume ρD_{AB} to be constant, integration of above gives

$$-\frac{\dot{m}''_A}{\rho D_{AB}} x = -\ln[1 - Y_A] + C$$

C is the constant of integration. With boundary condition

$$Y_A(x = 0) = Y_{A,i}$$

$$C = \ln(1 - Y_{A,i})$$



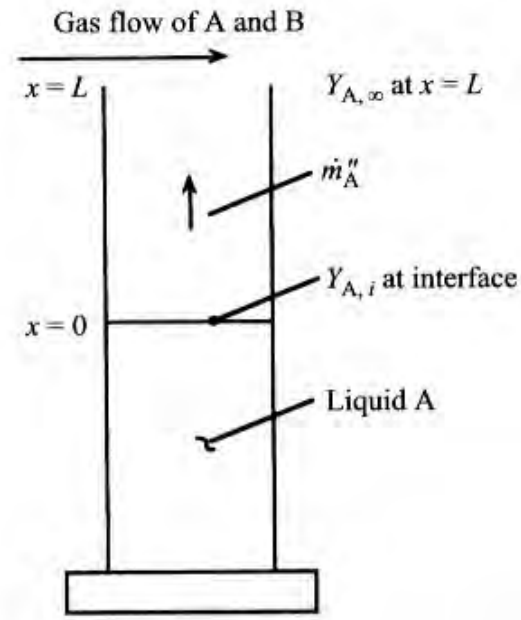
The Stefan problem

Removed the logarithm by exponentiation

$$Y_A(x) = 1 - \left(1 - Y_{A,i}\right) \exp[\frac{\dot{m}_A''}{\rho D_{AB}}]$$

Let $Y_A(x = L) = Y_{A,\infty}$,

$$\dot{m}_A'' = \frac{\rho D_{AB}}{L} \ln\left[\frac{1 - Y_{A,\infty}}{1 - Y_{A,i}}\right]$$

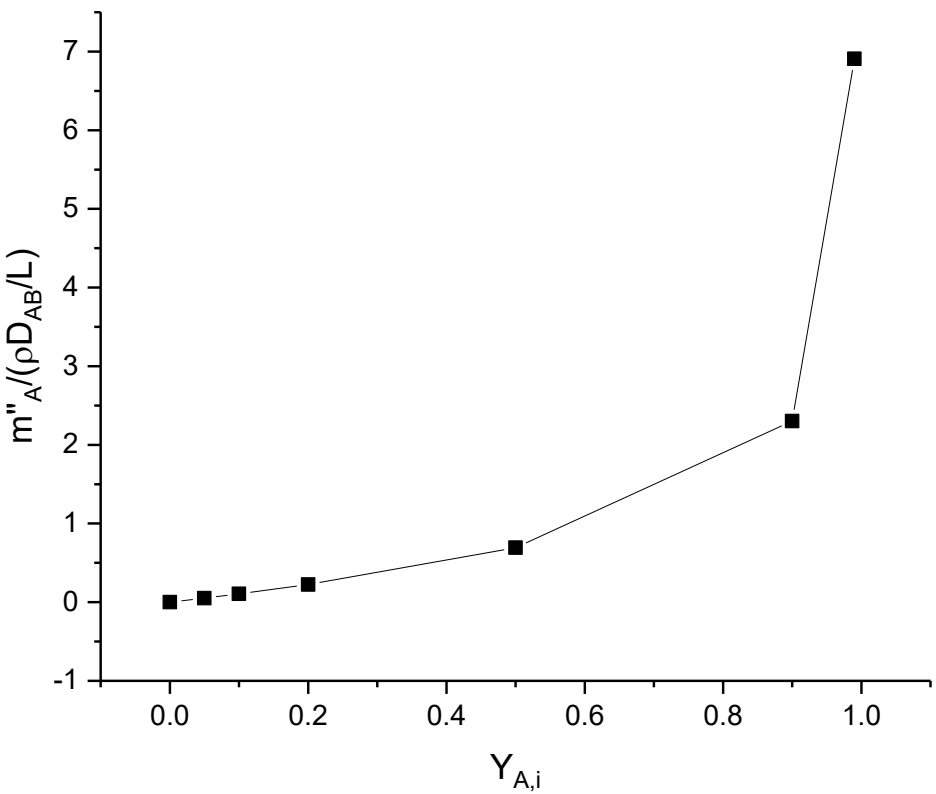


Observations:

- Mass flux directly proportional to density ρ and mass diffusivity D_{AB}
- Inversely proportional to the length L
- The mass fraction of A at the interface $Y_{A,i}$ has a substantial impact on mass flux: for small $Y_{A,i}$, mass flux is proportional to $Y_{A,i}$; for $Y_{A,i}$ greater than 0.5, mass flux increases rapidly

Some applications

The Stefan problem



$Y_{A,i}$	$\dot{m}''_A/(\frac{\rho D_{AB}}{L})$
0	0
0.05	0.0513
0.10	0.1054
0.20	0.2231
0.50	0.6931
0.90	2.303
0.99	6.908

Liquid-vapour interface boundary conditions

How to decide $Y_{A,i}$?

If species A has reached equilibrium at the liquid-vapour interface, the partial pressure of A at the surface is the saturation pressure at that temperature

$$p_{A,i} = p_{sat}(T_{liq,i})$$

Partial pressure $p_{A,i}$ can be related to the mole fraction of A $x_{A,i} = \frac{p_{A,i}}{p_{total}}$, therefore to mass fraction

$$Y_{A,i} = \frac{p_{sat}(T_{liq,i})}{p_{total}} \frac{\boxed{M_A}}{\boxed{M_{total,i}}}$$

Molar mass of A

Molar mass of mixture at interface

In this equation, $M_{total,i}$ depends on $x_{A,i}$, hence on p_{sat}

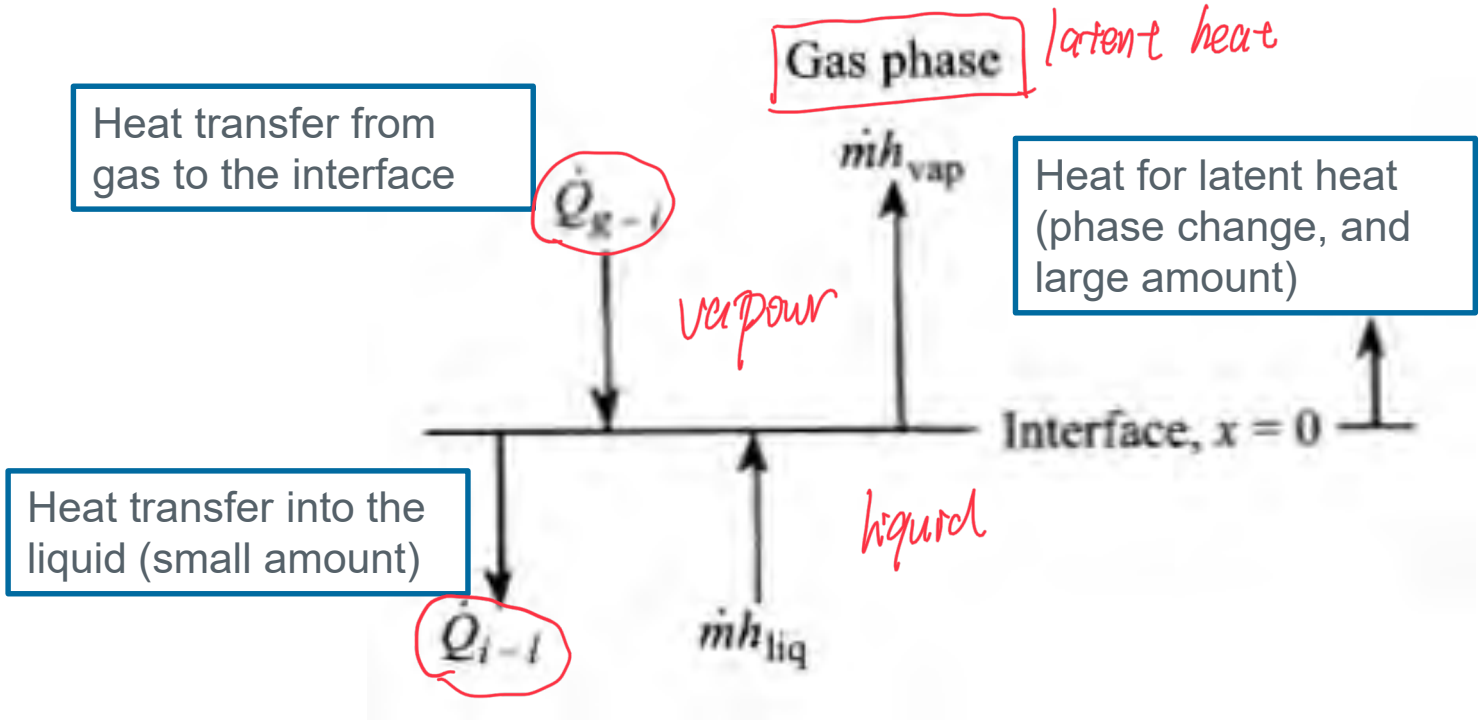
$T_{liq,i} \rightarrow P_{A,i} \rightarrow X_{A,i} \rightarrow M_{total,i}$

Liquid-vapour interface boundary conditions

Now the analysis has transformed the question of finding the vapour mass fraction at the interface to finding the temperature at the interface (not the liquid).

If the temperature is known, problem solved.

But in general, energy balance is needed to work out the temperature.



Liquid-vapour interface boundary conditions

In crossing the boundary, we maintain the continuity of temperature

$$T_{liq}(x = 0^-) = T_{vap}(x = 0^+) = T(0)$$

Due to energy conservation at the interface, we have

$$\dot{Q}_{g-i} - \dot{Q}_{i-l} = \dot{m}(h_{vap} - h_{liq}) = \dot{m}\boxed{h_{fg}}$$

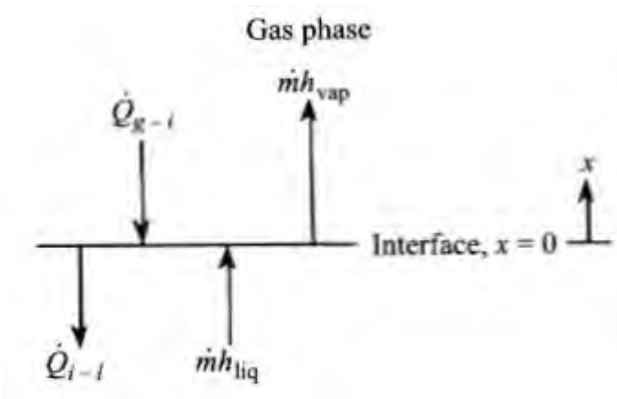
Latent heat

Or

$$\dot{Q}_{net} = \dot{m}h_{fg}$$

$\dot{Q}_{net} \overset{?}{\Longrightarrow} T$

If evaporation rate is known, this equation can be used to calculate the heat transfer, or vice versa, if $\dot{Q}_{net}$ is known, evaporation rate can be calculated.



Example

Liquid benzene (C6H6), at 298, is contained in a 1-cm-diameter glass tube and maintained at a level 10 cm below the top of the tube, which is open to the atmosphere. The following properties of benzene are given.

$$T_{boil} = 353\ K@1atm$$
$$h_{fg} = 393\ kJ/kg@T_{boil}$$
$$M = 78.108\ kg/kmol$$
$$\rho_l = 879\ kg/m^3$$
$$D_{C_6H_6-air} = 0.88 \times 10^{-5}\ m^2/s@298K$$

A. Determine the mass evaporation rate (kg/s) of the benzene.
B. How long does it take to evaporate 1 cm3 of benzene.
C. Compare the evaporation rate of benzene with that of water, assume $D_{H_2O-air} = 2.6 \times 10^{-5}\ m^2/s$

Solution

This question represent Stefan problem, we apply

$$\dot{m}''_{C_6H_6} = \frac{\bar{\rho} D_{C_6H_6-air}}{L} \ln\left[\frac{1 - Y_{C_6H_6,\infty}}{1 - Y_{C_6H_6,i}}\right]$$

In above, we know $D, L, Y_{C_6H_6,\infty}\ (= 0)$. So we need to evaluate the mass fraction of benzene at the interface $Y_{C_6H_6,i}$, and the mean density $\bar{\rho}$.

We know that

$$Y_{C_6H_6,i} = x_{C_6H_6,i} \frac{M_{C_6H_6}}{M_{mix,i}}$$

Where

$$x_{C_6H_6,i} = \frac{p_{sat}(T_{liq,i})}{p_{total}}$$

Solution

In order to evaluate $\frac{p_{sat}}{p}$, we need Clausius-Clapeyron equation

$$\frac{dp}{dT} \approx \frac{h_{fg}}{Tv_{vapor}}$$

$$\frac{dp}{p} = \frac{h_{fg}}{R_0/M_{C_6H_6}} \frac{dT}{T^2}$$

$v = \frac{R_0 T}{Mp}$

Integration the above from reference state (p=1 atm, $T=T_{sat}=T_{boil}$=353 K) to the state at 298 K

$$\frac{p_{sat}}{p(=1\ atm)} = \exp[-\frac{h_{fg}}{\frac{R_0}{M_{C_6H_6}}}(\frac{1}{T} - \frac{1}{T_{boil}})]$$

Substitute the values,

$$\frac{p_{sat}}{p(=1\ atm)} = \exp[-\frac{393{,}000}{\frac{8315}{78.108}}(\frac{1}{298} - \frac{1}{353})] = 0.145$$

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Clausius-Clapeyron equation (equilibrium during phase change)

$$\Delta G = -SdT + Vdp = 0$$

$$dG_{vapor} = dG_{liquid}$$

$$dG_{vapor} = V_{vapor}dp - S_{vapor}dT$$

$$dG_{liquid} = V_{liquid}dp - S_{liquid}dT$$

$$\frac{dp}{dT} = \frac{S_{vapor} - S_{liquid}}{V_{vapor} - V_{liquid}} = \frac{\Delta S}{\Delta V}$$

$$\Delta S = \frac{\Delta H}{T} = \frac{mh_{fg}}{T}$$

Neglect the volume of liquid

$$\frac{dp}{dT} \approx \frac{h_{fg}}{Tv_{vapor}}$$

$$pv_{vapor} = RT$$

Assume vapour is an ideal gas

$$\frac{dp}{p} = \frac{h_{fg}}{R_0/M_{C_6H_6}} \frac{dT_{sat}}{T_{sat}^2}$$

Solution

$P_{sat} \rightarrow X_{C_6H_6} \rightarrow M_{mix} \rightarrow$

Hence, $p_{sat} = x_{C_6H_6} = 0.145$. The molar mass at the interface is (air is considered simplified composition)

$$M_{mix,i} = 0.145(78.108) + (1 - 0.145)28.85 = 35.99\ kg/kmol$$

Mass fraction of benzene at the interface is

$$Y_{C_6H_6,i} = x_{C_6H_6,i}\frac{M_{C_6H_6}}{M_{mix,i}} = 0.145\frac{78.108}{35.99} = 0.3147$$

The mean density is (the mean value of interface and infinity)

$PV = nR_0T = \frac{m}{M}R_0T$

$\Rightarrow \rho = \frac{m}{V} = \frac{P}{R_0T/M}$

$$\bar{\rho} = \frac{p}{\left({}^{R_0}/_{\bar{M}}\right)T} = \frac{101{,}325}{\left({}^{8315}/_{0.5(28.85 + 35.99)}\right)298} = 1.326\ kg/m^3$$

Now we can calculate the mass flux of benzene

Solution

$$\dot{m}''_{C_6H_6} = \frac{1.326(0.88 \times 10^{-5})}{0.1} \ln\left[\frac{1-0}{1-0.3147}\right] = 4.409 \times 10^{-5}\ kg/sm^2$$

Therefore

$$\boxed{\dot{m}_{C_6H_6} = \dot{m}''_{C_6H_6} \frac{\pi D^2}{4} = 4.409 \times 10^{-5} \frac{\pi 0.01^2}{4} = 3.46 \times 10^{-9}\ kg/s}$$

The time needed for 1 cm^3 benzene evaporation

$$t = \frac{m_{evap}}{\dot{m}_{C_6H_6}} = \frac{V\rho_{C_6H_6}}{\dot{m}_{C_6H_6}} = \frac{879 \cdot 10^{-6}}{3.46 \times 10^{-9}} = 2.54 \times 10^5 \text{ s or } 70.6 \text{ hours}$$

If replace benzene with water

From steam table

$$p_{sat,H_2O}(298\ K) = 3.619\ kPa$$

solution

So

$$x_{H_2O,i} = \frac{3.619}{101.325} = 0.03128$$

$$\boxed{x_{C_6H_6,i} = 0.145}$$

And

$$M_{mix,i} = 0.03128(18.016) + (1 - 0.03128)28.85 = 28.51\ kg/kmol$$

Thus

$$Y_{H_2O,i} = x_{H_2O,i}\frac{M_{H_2O}}{M_{mix,i}} = 0.03128\frac{18.016}{28.51} = 0.01977$$

The mean molar mass and mean density are

$$\overline{MW} = \frac{1}{2}(28.51 + 28.85) = 28.68\ kg/kmol$$

$$\overline{MW} = \frac{1}{2}(28.52 + 28.85) = 28.68\ kg/kmol$$

$$\bar{\rho} = \frac{101{,}325}{\left(\frac{8315}{28.86}\right)298} = 1.173\,kg/m^3,$$

Solution

Therefore mass flux is

$$\dot{m}''_{H_2O} = \frac{1.173(2.6 \times 10^{-5})}{0.1} \ln\left[\frac{1-0}{1-0.01977}\right] = 6.09 \times 10^{-6}\ kg/sm^2$$

So

$$\boxed{\frac{\dot{m}_{C_6H_6}}{\dot{m}_{H_2O}} = \frac{4.409 \times 10^{-5}}{6.09 \times 10^{-6}} = 7.2}$$

Comments:

Although water has higher diffusivity, the higher vapour pressure of benzene dominates, thus causing the benzene to evaporate more than seven times faster than water.

Another application

Droplet evaporation

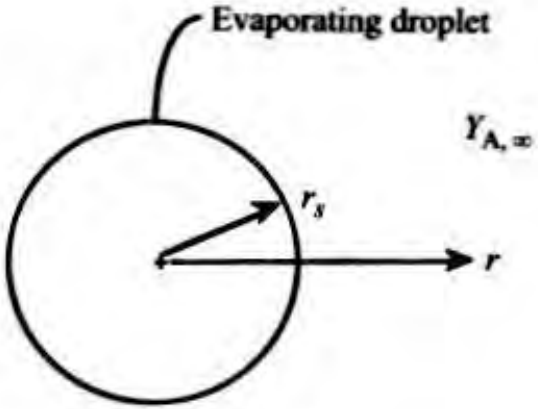
Evaporation of a single liquid droplet in a quiescent environment – Stefan problem in a spherically symmetric coordinate system.

- Radius r is the only coordinate variable.
- Origin at the centre of the droplet
- r_s is the liquid-vapour interface

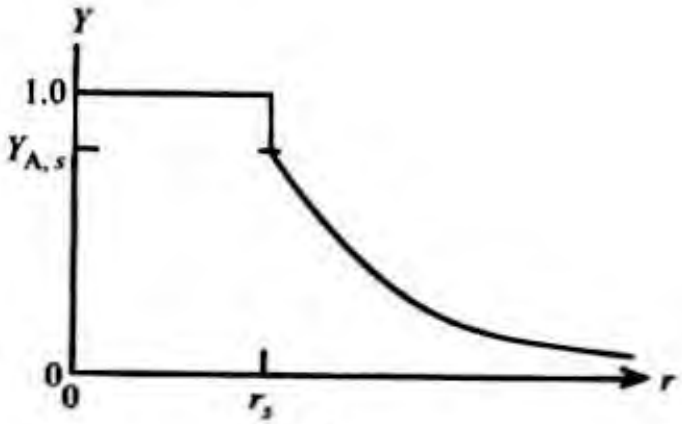
surface

Physically

- Heat transferred from environment for evaporation
- Vapour diffuses to ambient gas
- Mass loss causes droplet shrink until completely evaporate (r_s =0)



(a)



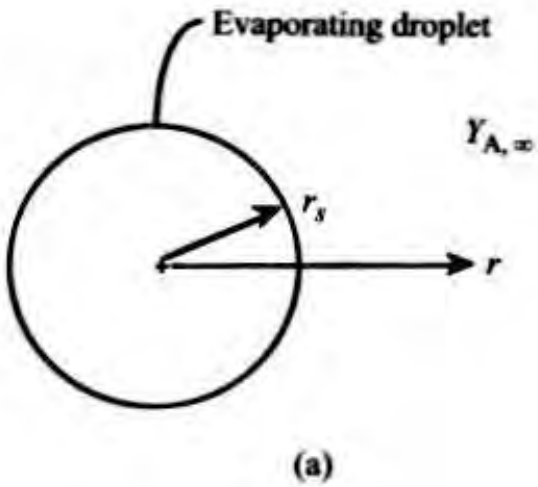
(b)

Another application

Droplet evaporation

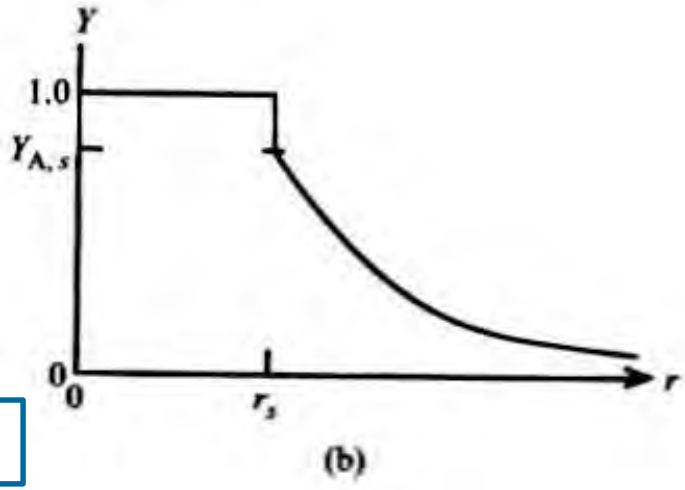
We need to determine:

- Mass flowrate of the vapour from the surface at any time $\dot{m}''_{\alpha}$
- Droplet radius as a function of time $r(t)$
- Droplet lifetime τ



Conservation laws of

- *Droplet*: mass conservation, energy conservation
- *Droplet vapour/ambient gas mixture ($r_s<r<\infty$)*: overall mass conservation, droplet vapour (species) conservation, and energy conservation



FIVE equations!

Droplet evaporation

Assumptions

1. The evaporation process is quasi-steady – at any instant in time, the process can be described as if it were in steady state, in order to eliminate the need of dealing with partial differential equations.

   No

   $r \to 0$

2. Droplet temperature is uniform and a fixed value below boiling point.

3. The mass fraction of vapour at the droplet surface is determined by liquid-vapour equilibrium at the droplet temperature

4. All thermophysical properties – particularly ρD– are constant.

Evaporation rate

The same as Cartesian coordinate Stefan problem, just adapt new coordinate.

For mass conservation

$S=4\pi r^2$

$$① \quad \dot{m}(r) = constant = 4\pi r^2 \dot{m}''$$

Where $\dot{m}'' = \dot{m}_A'' + \dot{m}_B'' = \dot{m}_A''$ ($\dot{m}_B'' = 0$ at the interface). Note it is the mass flow rate not the mass flux that is constant.

Species conservation for the droplet vapour becomes

$$② \quad \dot{m}_A'' = Y_A \dot{m}_A'' - \rho D_{AB} \frac{dY_A}{dr}$$

Substitute and rearrange

$$① + ② \Rightarrow \quad \dot{m} = -4\pi r^2 \frac{\rho D_{AB}}{1 - Y_A} \frac{dY_A}{dr}$$

Integrate the above and apply boundary condition that at the surface, the vapour mass fraction is $Y_{A,s}$ ($Y_A(r - r_s) = Y_{A,s}$), yields

$$Y_A(r) = 1 - \frac{\left(1 - Y_{A,s}\right) \exp[-\dot{m}/(4\pi\rho D_{AB} r)]}{\exp[-\dot{m}/(4\pi\rho D_{AB} r_s)]}$$

Evaporation rate

The evaporation rate can be determined by letting $Y_A = Y_{A,\infty}$ for $r \to \infty$, and solving for $\dot{m}$

$$\dot{m} = 4\pi r_s \rho D_{AB} \ln\left[\frac{1 - Y_{A,\infty}}{1 - Y_{A,s}}\right]$$

$\dot{m}'' = \frac{\dot{m}}{4\pi r^2}$

Recap: for Cartesian coordinate, we have

$$\boxed{\dot{m}_A'' = \frac{\rho D_{AB}}{L} \ln\left[\frac{1 - Y_{A,\infty}}{1 - Y_{A,i}}\right]}$$

We define a dimensionless **transfer number B_Y**

$$1 + B_Y \equiv \frac{1 - Y_{A,\infty}}{1 - Y_{A,s}}$$

or

$$\boxed{B_Y = \frac{Y_{A,s} - Y_{A,\infty}}{1 - Y_{A,s}}}$$

The evaporation rate can be expressed as

$$\dot{m} = 4\pi r_s \rho D_{AB} \ln[1 + B_Y]$$

- Transfer number is zero → evaporation rate is zero
- Transfer number increases → evaporation rate increases

Physical meaning: from the appearance of $(Y_{A,s} - Y_{A,\infty})$ in B_Y definition, B_Y can be considered as a "driving potential" for mass transfer.

Droplet mass conservation

The droplet mass decrease is equal to the rate of evaporation

$$\frac{dm_d}{dt} = -\dot{m}$$

Where the droplet mass m_d is given by

$$m_d = \rho_l V = \rho_l \pi D^3/6$$

Where V and D are the volume and diameter of the droplet, respectively

Substituting gives us

$$d(\rho_l \pi D^3/6)/dt = -4\pi r_s \rho D_{AB} \ln[1 + B_Y]$$

Rearrange (with $D=2r_s$):

$$\frac{dD}{dt} = -\frac{4\rho D_{AB}}{\rho_l D} \ln[1 + B_Y]$$

In combustion context, we usually use D^2 rather than D

$dD^2 = 2D\,dD$

$$\boxed{\frac{dD^2}{dt} = -\frac{8\rho D_{AB}}{\rho_l} \ln[1 + B_Y]}$$

Droplet evaporation

$$\boxed{\frac{dD^2}{dt} = -\frac{8\rho D_{AB}}{\rho_l}\ln[1 + B_Y]}$$

$B_Y = \frac{Y_{A,s} - Y_{A,\infty}}{1 - Y_{A,s}}$

The above equation tells us the time derivative of the droplet diameter squared is constant, or D^2 varies linearly with t with the slope $-\frac{8\rho D_{AB}}{\rho_l}\ln[1 + B_Y]$. This slope is defined as **evaporation constant K**.

$$K = \frac{8\rho D_{AB}}{\rho_l}\ln(1 + B_Y)$$

We can find the time it takes a droplet to evaporate completely, or lifetime t_d

$$\int_{D_0^2}^{0} dD^2 = -\int_0^{t_d} K dt \quad \text{yields:} \quad t_d = D_0^2 / K$$

By change the upper limit of the integration, we have a general relationship

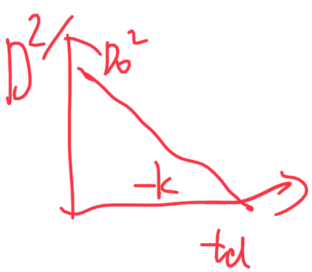
$$D^2(t) = D_0^2 - Kt$$

This is referred to as the **D² Law** for droplet evaporation. From experiments, D2 law holds after a initial transient period, and can also be used to describe burning of fuel droplets.

Example

In mass-diffusion-controlled evaporation of a fuel droplet, the droplet surface temperature is an important parameter. Estimate the droplet lifetime of a 100 μm diameter n-dodecane ($C_{12}H_{26}$) droplet evaporating in dry nitrogen at 1 atm if the droplet temperature is 10 K below the dedecane boiling point. Repeat the calculation for a temperature 20 K below the boiling temperature and compare the results. For simplicity, assume that, in both cases, the mean gas density is that of nitrogen at a mean temperature of 800 K. Use this same temperature to estimate the fuel vapour diffusivity. The density of liquid dodacane is 749 kg/m^3.

Given: n-dodecane droplet



$$D = 100\mu m$$
$$p = 1\ atm$$
$$\rho_l = 749\ kg/m^3$$
$$T_s = T_{boil} - 10\ or\ -20$$
$$\rho = \rho_{N_2}@\bar{T} = 800\ K$$

Find: droplet lifetime

Solution

We can employ D^2 law to evaluate the lifetime. In order to do that, we need to calculate the evaporation constant K.

Properties required:

$T_{boil} = 216.3 + 273.15 = 489.5\ K$(from property table)

$h_{fg} = 256\ kJ/kg$ (from property table)

$M_{C_{12}H_{26}} = 170.337\ kg/kmol$

$D_{AB} = 8.1 \times 10^{-6}\ m^2/s$ @399 K(from property table)

In order to calculate B_Y, we need to know the fuel mass fraction at the surface. As in the last example, we integrate Clausius-Clapeyron equation to find the saturation pressure at the given temperature.

For $T = T_{boil} - 10\ (= 479.5\ K)$

$$\frac{p_{sat}}{p(= 1\ atm)} = \exp[-\frac{h_{fg}}{\frac{R_0}{M_{C_{12}H_{26}}}}(\frac{1}{T} - \frac{1}{T_{boil}})]$$

$$= \exp\left[\frac{-256{,}000}{8315/170.337}(1/479.5 - 1/489.5)\right] = 0.7998$$

Solution

So $p_{sat} = 0.7998$ and $x_A = 0.7998$. Thus we can calculate mass fraction

$$Y_{C_{12}H_{26},s} = x_{A,s}\frac{M_{C_{12}H_{26}}}{M_{mix,s}} = 0.7998\frac{170.337}{0.7998(170.337) + (1 - 0.7998)28.014} = 0.9605$$

Now evaluate the transfer number B_Y

$$B_Y = \frac{Y_{A,s} - Y_{A,\infty}}{1 - Y_{A,s}} = \frac{0.9605 - 0}{1 - 0.9605} = 24.32$$

We also need to estimate ρD_{AB}, which we treat as $\bar{\rho}_{N_2} D_{AB}(\bar{T} = 800\ K)$. As $D_{AB} \propto T^{3/2}$

$$D_{AB}(\bar{T} = 800\ K) = 8.1 \times 10^{-6}\left(\frac{800}{399}\right)^{3/2} = 23.0 \times 10^{-6}\ m^2/s$$

Based on ideal gas law

$$\bar{\rho}_{N_2} = \frac{101{,}325}{\frac{8315}{28.014} \cdot 800} = 0.4267\ kg/m^3$$

Solution

Hence

$$K = \frac{8\bar{\rho} D_{AB}}{\rho_l} \ln(1 + B_Y)$$

$$= \frac{8(0.4267)23.0 \times 10^{-6}}{749} \ln(1 + 24.32)$$

$$= 3.39 \times 10^{-7}\ m^2/s$$

And the droplet lifetime is

$$t_d = D_0^2 / K$$

$$= \frac{(100 \times 10^{-6})^2}{3.39 \times 10^{-7}}$$

$$= 0.030\ s$$

Solution

Repeating the calculation for $T=T_{boil}-20=469.5$ K, we have

ΔT	T	p_{sat}	Y_s	B_Y	$K(10^{-7})$	t_d
10	479.5	0.7998	0.9605	24.32	3.39	0.03
20	469.5	0.6336	0.9132	10.52	2.56	0.039

Comments:

- 2% surface temperature change results in a 30% increase in droplet lifetime
- Evaporation of fuel at high temperature is very important for power generation devices (gas turbine, diesel engine, etc.)
- Forced convection are usually important (not considered in the example) in the evaporation process.

Summary

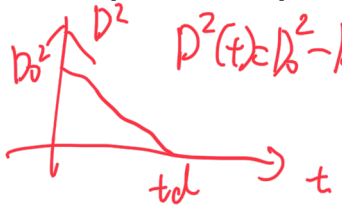
Heat transfer

- Conduction $\dot{Q} = -kA\frac{dT}{dx}$
  - Heat transfer coefficient and correlation
- Convection $\dot{Q} = hA(T_s - T_\infty)$ $Nu = \frac{h\delta}{k}$ $Nu = C Re^m \cdot Pr^n$
  - Conductivity, heat transfer network
- Radiation $\dot{Q} = \sigma T^4$
  - Blackbody, grey body, Kirchhoff's Law

Mass transfer

- Fick's law of diffusion $\dot{m}_A'' = Y_A \dot{m}_f'' - \rho D_{AB} \cdot \frac{dY_A}{dx}$
- Applications
  - Stefen problem $\dot{m}_A'' = \frac{\rho D_{AB}}{L} \ln\left[\frac{1 - Y_{A,\infty}}{1 - Y_{A,i}}\right]$
  - Droplet evaporation $k = \frac{8\rho_{AB}}{\rho_L} \ln[1 + B_M]$

$B_M = \frac{Y_{A,s} - Y_{A,\infty}}{1 - Y_{A,s}}$



$D^2(t) = D_0^2 - kt$