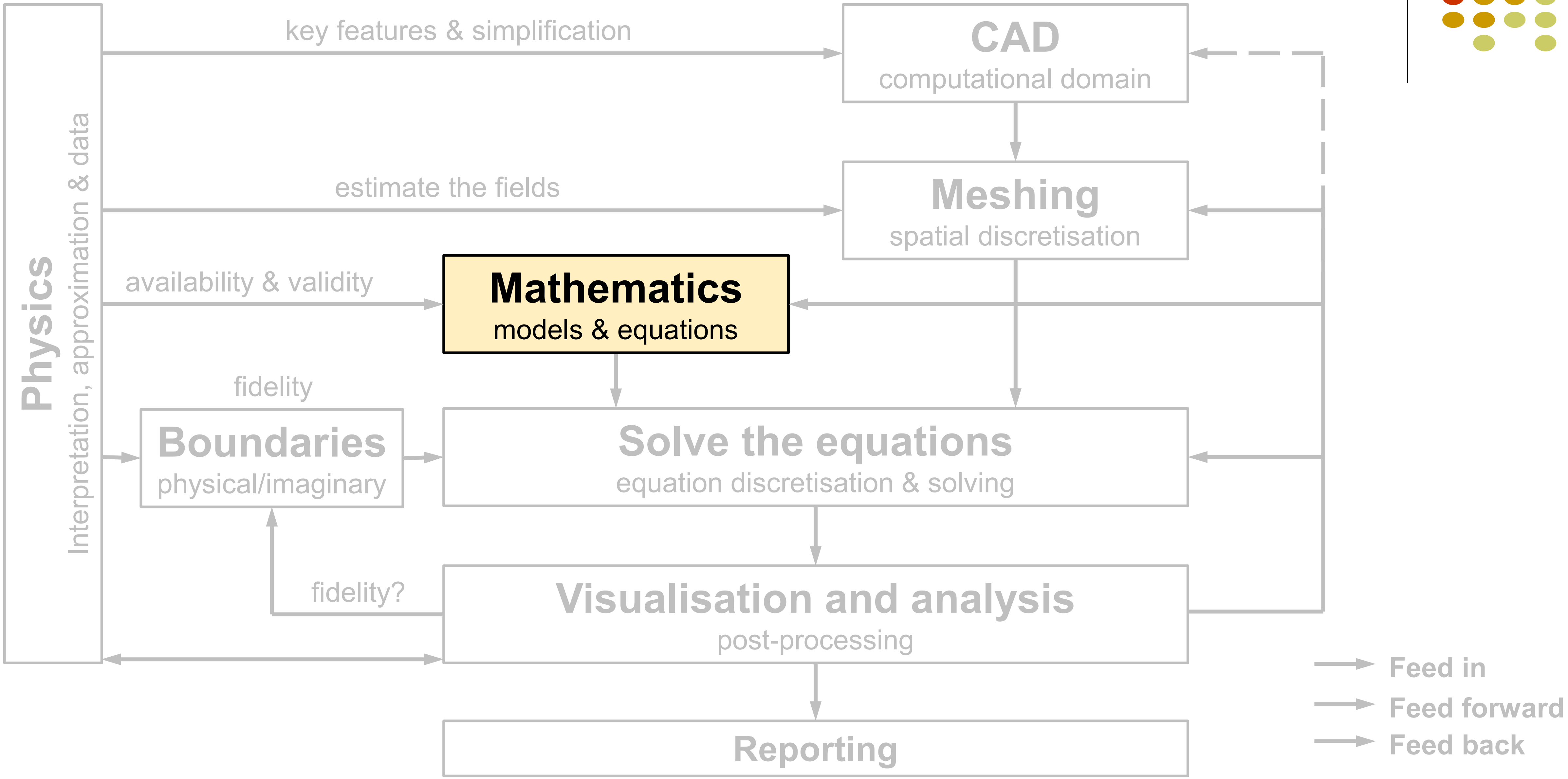
ENIGNEERING COMPUTATIONAL FLUID DYNAMICS (ECFD)

Dr Xiangdong Li

Module 03 – The Governing Equations

CFD workflow



This module

Before we start, you should

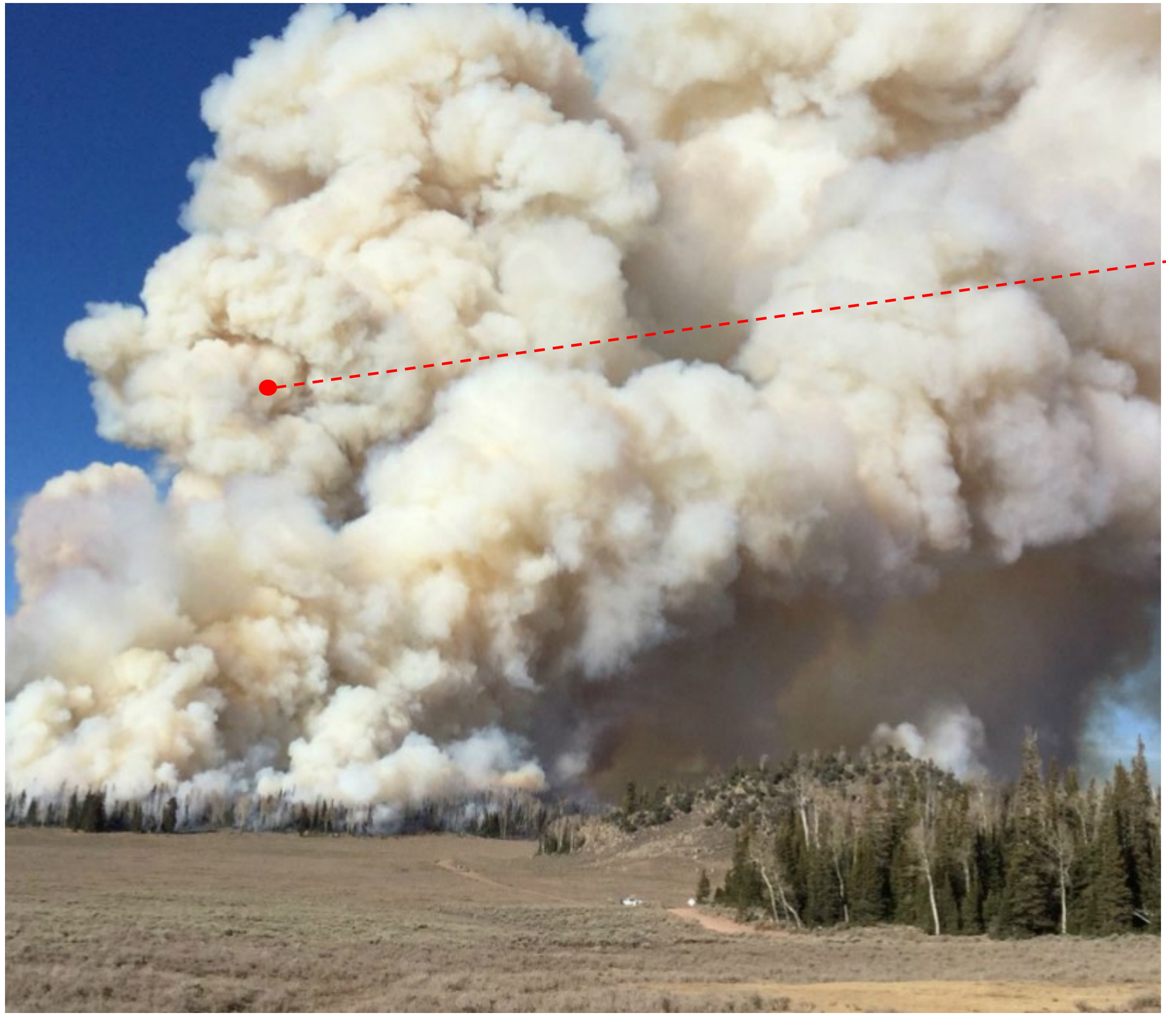
- ❖ Be familiar with concept of partial differentiation
- ❖ Be familiar with 3D coordinate system and vector field
- ❖ Read Chapter 2 (P15 - 43)

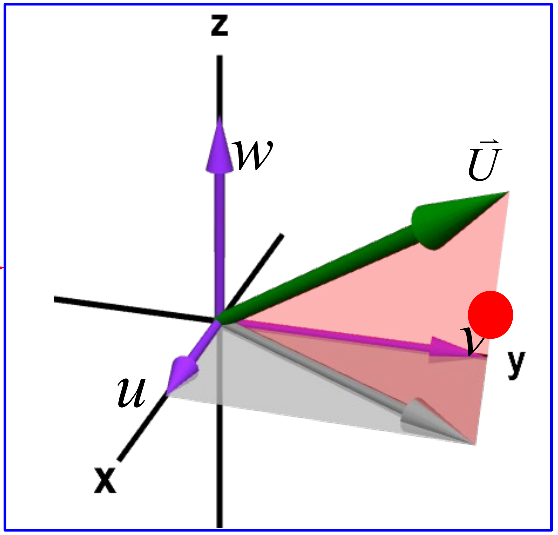
We will be talking about

- ❖ The principles of conservation in fluid dynamics
- ❖ Physical meaning of convection, diffusion and source terms
- ❖ Different forms of the governing equations
- ❖ Applying governing equations for given flow conditions

A FUNDAMENTAL CONCEPT

Think about an infinitesimal volume in a fire field





$$\vec{U} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$u = u(x, y, z, t)$$

$$v = v(x, y, z, t)$$

$$w = w(x, y, z, t)$$

$$\boxed{\rho = \rho(x, y, z, t)} \dashrightarrow \boxed{d\rho = \frac{\partial \rho}{\partial x}dx + \frac{\partial \rho}{\partial y}dy + \frac{\partial \rho}{\partial z}dz + \frac{\partial \rho}{\partial t}dt}$$

Total derivative

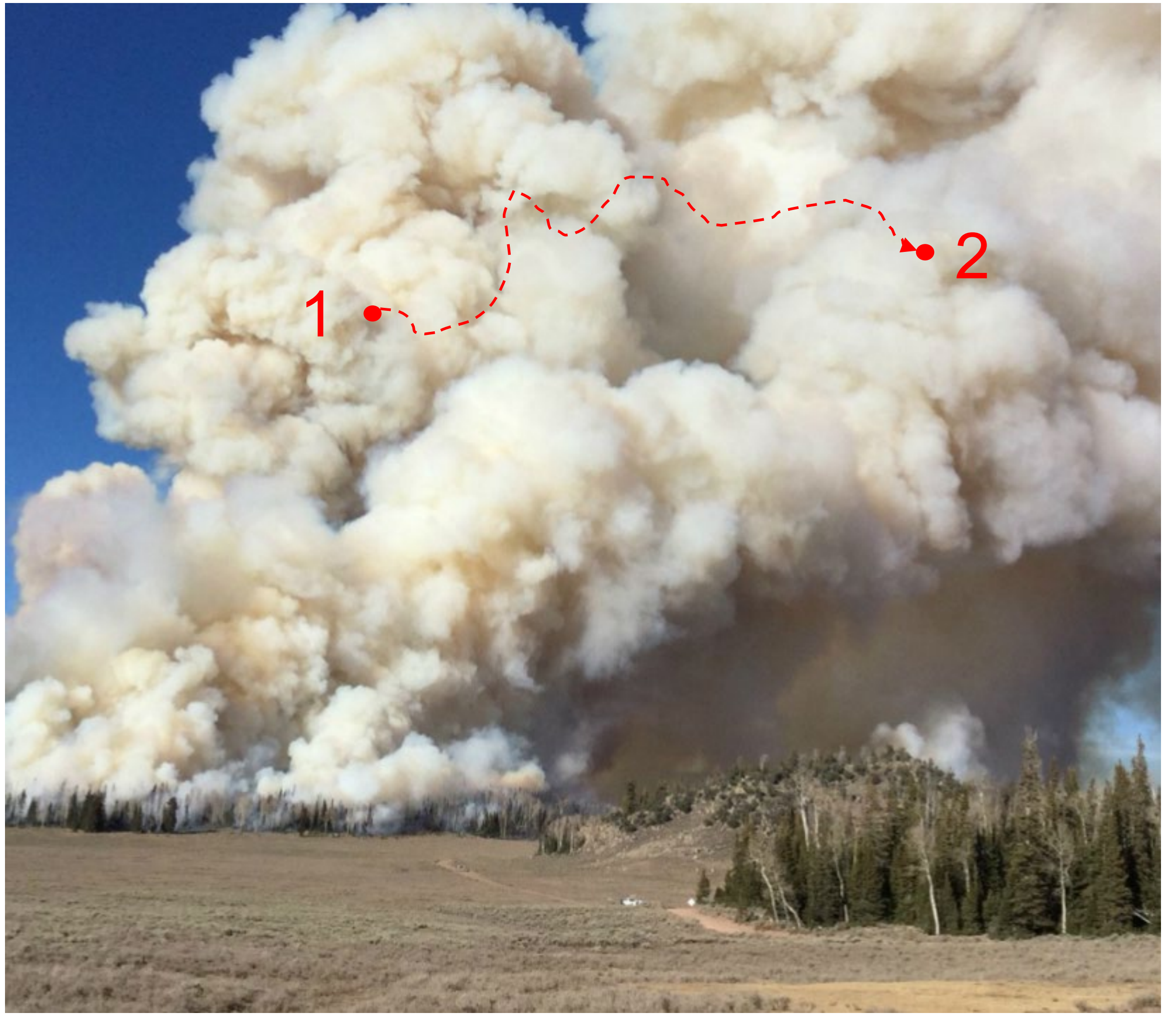
$$p = p(x, y, z, t)$$

$$T = T(x, y, z, t)$$

$$e = e(x, y, z, t)$$

$$\ldots$$

Think about an infinitesimal volume in a fire field



- Moving from Point 1 to Point 2

$$\frac{\rho_2-\rho_1}{t_2-t_1} \cong \left.\frac{\partial\rho}{\partial x}\right|_1 \frac{x_2-x_1}{t_2-t_1} + \left.\frac{\partial\rho}{\partial y}\right|_1 \frac{y_2-y_1}{t_2-t_1} + \left.\frac{\partial\rho}{\partial z}\right|_1 \frac{z_2-z_1}{z_2-z_1} + \left.\frac{\partial\rho}{\partial t}\right|_1$$

- The instantaneous changing rate of density at point 1

$$\lim_{t_2\to t_1}\left(\frac{\rho_2-\rho_1}{t_2-t_1}\right) = \frac{d\rho}{dt}$$

$$\lim_{t_2\to t_1}\left(\frac{x_2-x_1}{t_2-t_1}\right) = u$$

$$\lim_{t_2\to t_1}\left(\frac{y_2-y_1}{t_2-t_1}\right) = v$$

$$\lim_{t_2\to t_1}\left(\frac{z_2-z_1}{t_2-t_1}\right) = v$$

$$\boxed{\frac{d\rho}{dt} = \frac{\partial\rho}{\partial x}u + \frac{\partial\rho}{\partial y}v + \frac{\partial\rho}{\partial z}w + \frac{\partial\rho}{\partial t}}$$

Substantial derivative

The substantial derivative – the total derivative with respective to time

$$\boxed{\frac{D}{Dt} \equiv \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w + \frac{\partial}{\partial t} = \frac{d}{dt}}$$

substantial derivative

$$\boxed{\frac{D}{Dt}} \equiv \boxed{\frac{\partial}{\partial t}} + \boxed{\nabla \cdot \vec{U}}$$

convective derivative

local derivative

Further define the vector operator (del, nabla)

$$\nabla \equiv \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$$

The substantial derivative is applicable to any flow-field parameters.

Physically, the substantial derivative represents the rate of change (of something) in a fluid element as it moves through the flow field. This is different to the local derivative which represents the local rate of change at a fixed point.

The divergence of velocity field

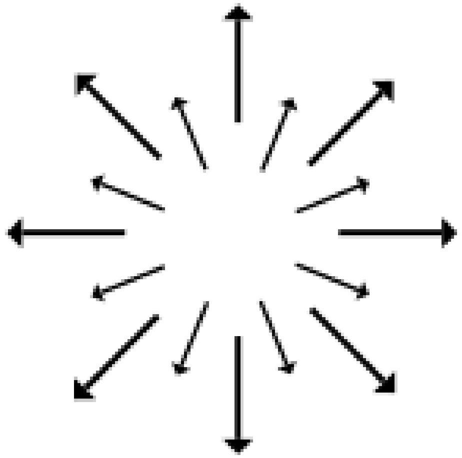
$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boxed{\nabla \cdot \vec{U}}$$

Convective derivative, or

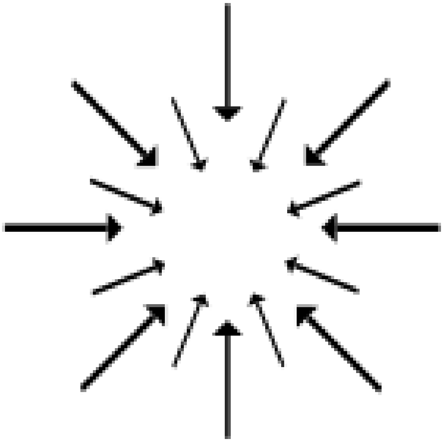
Divergence of velocity field

$\vec{U}$ is a vector and $\nabla \cdot \vec{U}$ is a scalar

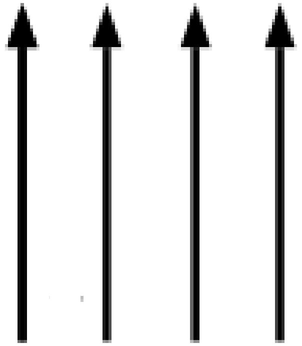
$\nabla \cdot \vec{U}$ *is physically the time rate of change of the volume of a moving fluid element, per unit volume*. Alternatively, divergence is the outflow of flux from a small closed surface area as volume shrink to zero, per unit volume.



$\nabla \cdot \vec{U} > 0$



$\nabla \cdot \vec{U} < 0$



$\nabla \cdot \vec{U} = 0$

Different ways to study flows

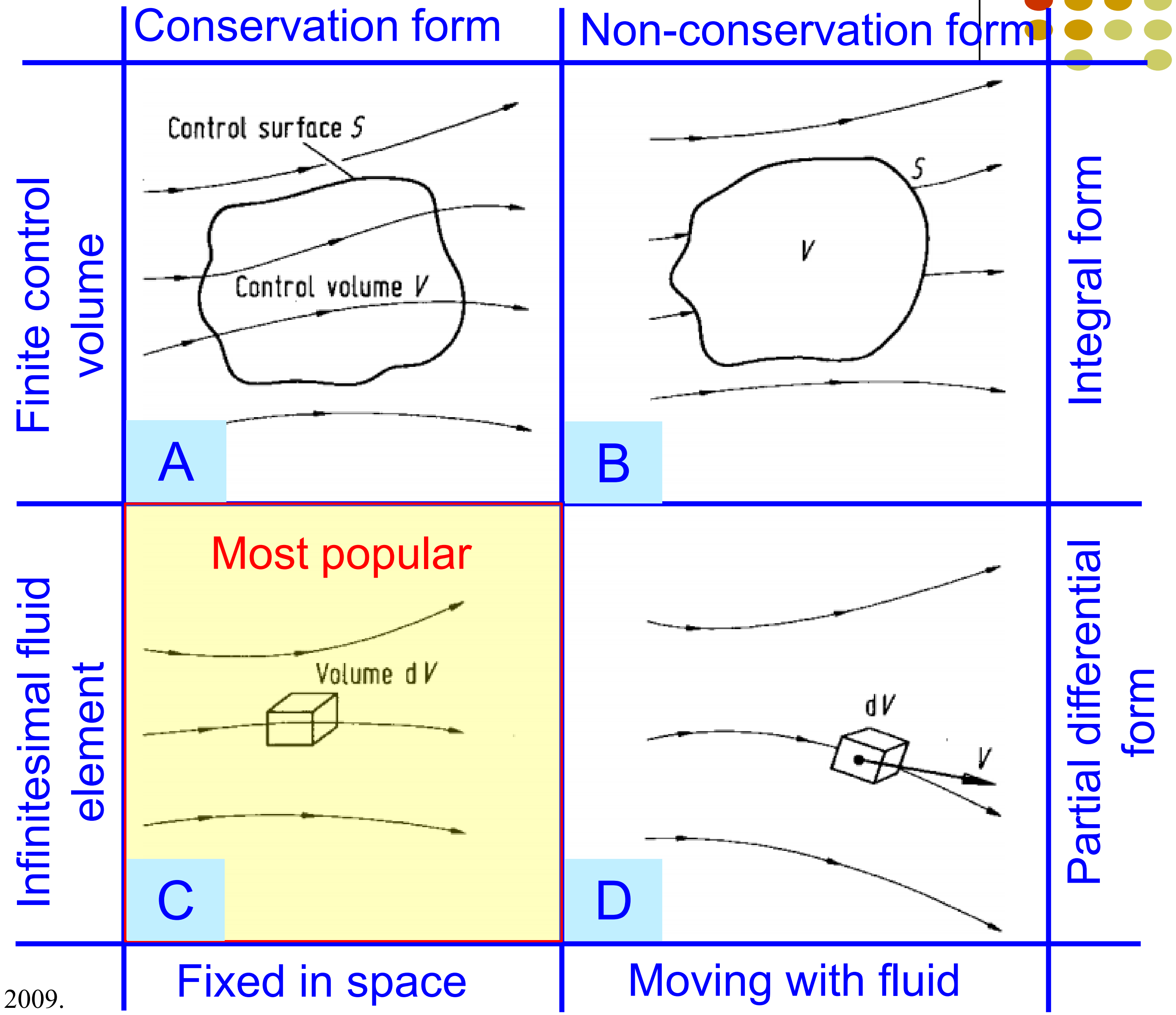
Moving control volume

Fixed control volume

Forms of the governing equations

Depending on the method of derivation, the governing equations have different forms:

- ❖ Integration and partial differential forms
- ❖ Conservation and non-conservation forms
- ❖ Conservation forms are mostly used in CFD
- ❖ Non-conservation forms are easier to derive
- ❖ Different forms are inter-convertible



Modified from: J.D. Aderson Jr. Springer, 2009.

Governing equations for CFD

Mathematical equations governing the conservation of mass, momentum, energy and transportable scalars in fluid flows (hence also called **the conservation equations**), developed based on the conservation relationships over a finite control volume or a infinitesimal fluid element.

The fluid control volume or infinitesimal element is large enough to be regarded as a **continuum**.

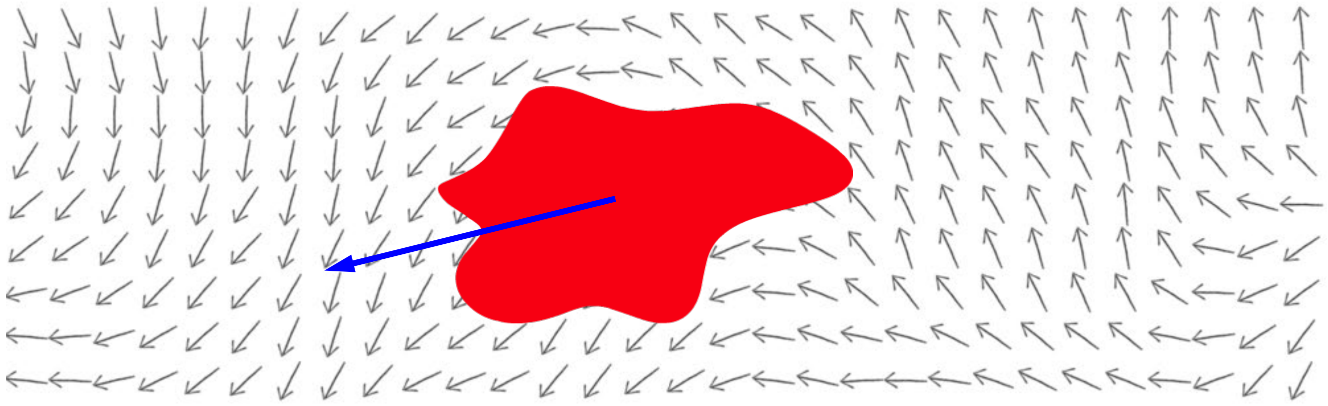
The fundamental principle for developing the governing equations include

- Mass is conserved
- Newton's 2^{nd} law: F = ma
- 1^{st} law of thermodynamics: the rate of energy change equals to the sum of rate of heat addition and work done on the fluid element

Deriving the continuity equation (B)

The non-conservative integration form

$$\boxed{\frac{dm}{dt} = \frac{\partial}{\partial t}\oiiint_V \rho \cdot dV = 0}$$



$$\frac{dm}{dt} = \oiiint_V \frac{d\rho}{dt} dV$$

$$\boxed{d\rho = \frac{\partial\rho}{\partial t}dt + \frac{\partial\rho}{\partial x}dx + \frac{\partial\rho}{\partial y}dy + \frac{\partial\rho}{\partial z}dz}$$

The total derivative

$$\frac{dm}{dt} = \oiiint_V \left(\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x}\frac{dx}{dt} + \frac{\partial\rho}{\partial y}\frac{dy}{dt} + \frac{\partial\rho}{\partial z}\frac{dz}{dt}\right)dV$$

$$= \oiiint_V \left(\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x}u + \frac{\partial\rho}{\partial y}v + \frac{\partial\rho}{\partial z}w\right)dV = 0$$

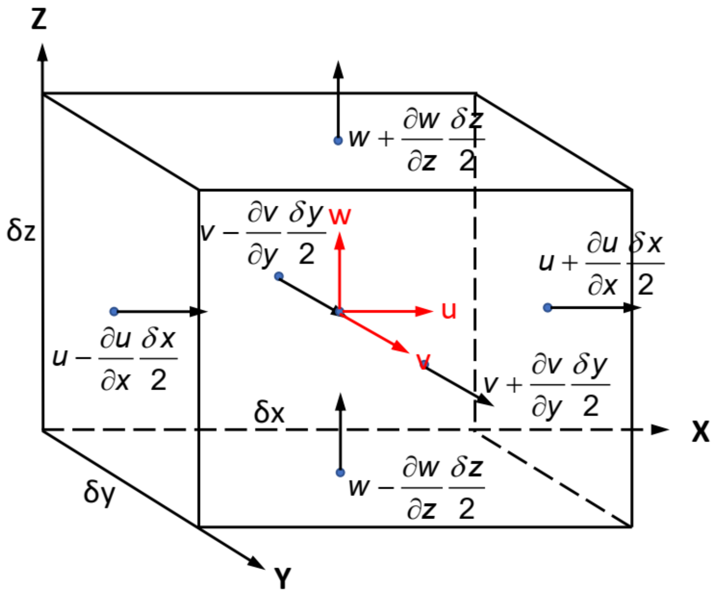
Let's start with a moving control volume (B) with a volume V and surface area S

- Shape changes
- Size changes
- Density changes
- **Mass does not change**

$$\boxed{\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0}$$

Deriving the continuity equation (C)

The conservative partial differential form



$$\frac{\partial \rho}{\partial t}\delta x\delta y\delta z = \left(\rho u - \frac{\partial \rho u}{\partial x}\frac{\delta x}{2}\right)\delta y\delta z - \left(\rho u + \frac{\partial \rho u}{\partial x}\frac{\delta x}{2}\right)\delta y\delta z$$
$$+\left(\rho v - \frac{\partial \rho v}{\partial y}\frac{\delta y}{2}\right)\delta x\delta z - \left(\rho v + \frac{\partial \rho v}{\partial y}\frac{\delta y}{2}\right)\delta x\delta z$$
$$+\left(\rho w - \frac{\partial \rho w}{\partial z}\frac{\delta z}{2}\right)\delta x\delta y - \left(\rho w + \frac{\partial \rho w}{\partial z}\frac{\delta z}{2}\right)\delta x\delta y$$

Let us consider a fixed infinitesimal fluid element (C) with dimensions of δx, δy and δz

- **Mass does not change** (mass increase in the element equals to the mass entering the element across the faces)

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0}$$

The continuity equation

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

$$\frac{\partial \rho}{\partial t}+\vec{U}\cdot\nabla\rho=0$$

$$\underbrace{\boxed{\frac{\partial \rho}{\partial t}}}_{\text{local derivative}}+\underbrace{\boxed{\nabla\cdot\left(\rho\vec{U}\right)}}_{\text{convective derivative}}=0$$

Deriving the momentum equations (D)

Momentum is a vector – let's look at the X direction $ma_x = F_x$

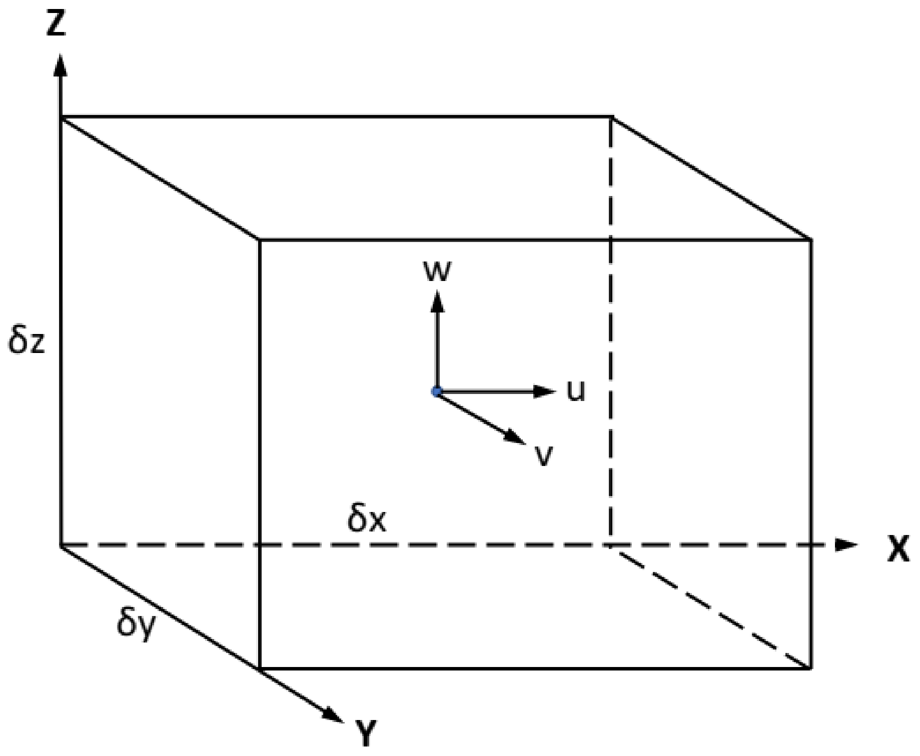
The left-hand side:

$$m = \rho \delta x \delta y \delta z$$

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w$$

$$= \frac{\partial u}{\partial t} + \vec{U} \cdot \nabla u$$

$$= \frac{\partial u}{\partial t} + \nabla \cdot \left(u\vec{U}\right)$$



$$m\vec{a}_x = \left(\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u)}{\partial x}u + \frac{\partial(\rho u)}{\partial y}v + \frac{\partial(\rho u)}{\partial z}w\right)\delta x \delta y \delta z$$

Deriving the momentum equations (D)

Momentum is a vector – let's look at the X direction

$$ma_x = F_x$$

The right-hand side: Forces include

❖ Surface forces

Stress = force / area

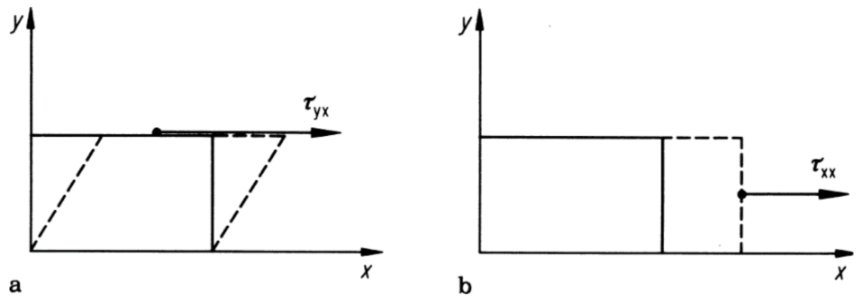
- Pressure
- Viscous forces

❖ Body forces

Force density = force / volume

- Gravity
- Centrifugal force
- Electromagnetic force

The viscous forces



Shear stress Normal stress

Modified from: J.D. Aderson Jr. Springer, 2009.

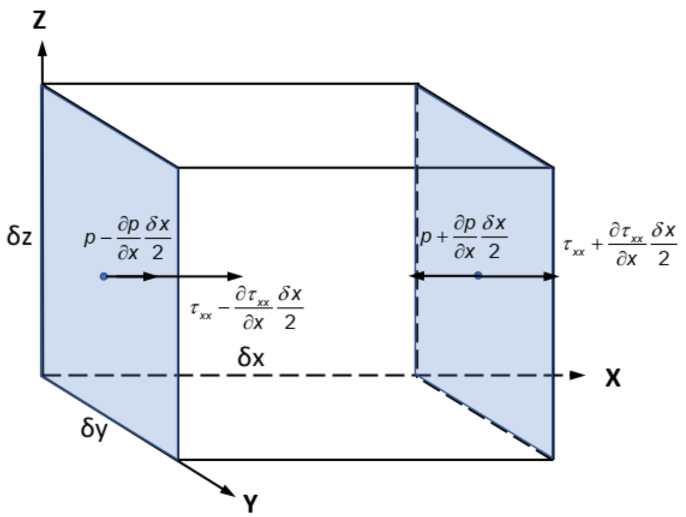
Deriving the momentum equations (D)

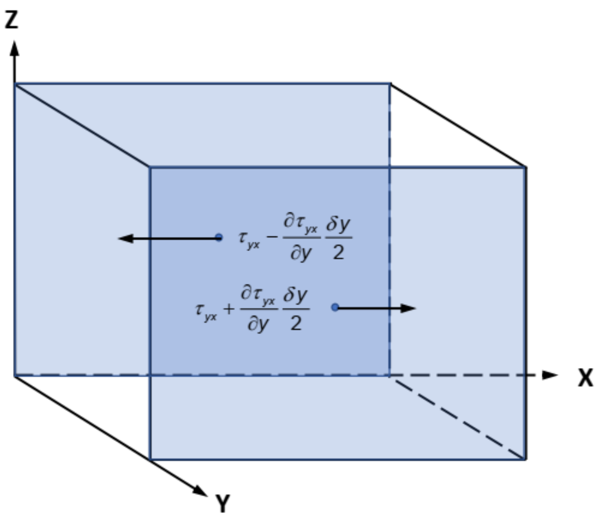
Momentum is a vector – let's look at the X direction $ma_x = F_x$

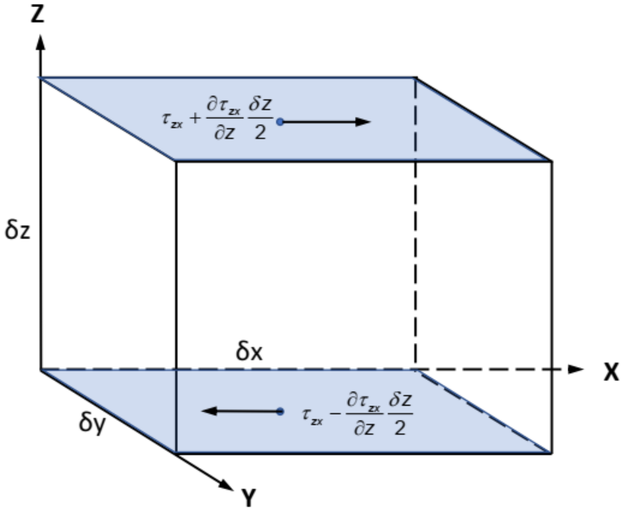
The right-hand side: surface forces include

$$F_{surf,x} = -\frac{\partial p}{\partial x}\delta x\delta y\delta z + \frac{\partial \tau_{xx}}{\partial x}\delta x\delta y\delta z \frac{\partial \tau_{yx}}{\partial y}\delta x\delta y\delta z + \frac{\partial \tau_{zx}}{\partial z}\delta x\delta y\delta z$$

$$F_{surf,x} = \left(-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta x\delta y\delta z$$







Deriving the momentum equations (D)

Momentum is a vector – let's look at the X direction

$$ma_x = F_x$$

$$m\vec{a}_x = \left(\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u)}{\partial x}u + \frac{\partial(\rho u)}{\partial y}v + \frac{\partial(\rho u)}{\partial z}w \right)\delta x \delta y \delta z$$

$$F_{surf,x} = \left(-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right)\delta x \delta y \delta z$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u)}{\partial x}u + \frac{\partial(\rho u)}{\partial y}v + \frac{\partial(\rho u)}{\partial z}w = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

Take the body forces into account

$$\boxed{\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u)}{\partial x}u + \frac{\partial(\rho u)}{\partial y}v + \frac{\partial(\rho u)}{\partial z}w = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_{B,x}}$$

The momentum equations

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u)}{\partial x}u+\frac{\partial(\rho u)}{\partial y}v+\frac{\partial(\rho u)}{\partial z}w=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+F_{B,x}$$

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho v)}{\partial x}u+\frac{\partial(\rho v)}{\partial y}v+\frac{\partial(\rho v)}{\partial z}w=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}+F_{B,y}$$

$$\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho w)}{\partial x}u+\frac{\partial(\rho w)}{\partial y}v+\frac{\partial(\rho w)}{\partial z}w=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}+F_{B,z}$$

or

$$\frac{\partial(\rho\vec{U})}{\partial t}+\nabla\cdot\left(\rho\vec{U}\vec{U}\right)=\overbrace{\boxed{-\nabla p}}^{\text{Pressure}}+\underbrace{\boxed{\nabla\cdot\tau}}_{\text{Viscous forces}}+\boxed{F_B}$$

Body forces

Deriving the energy equation

The first law of thermodynamics

Rate of energy change inside the fluid element	=	Net heat flux into the fluid element	+	Rate of work done on the element due to surface and body forces

Total energy is defined here as: internal energy + kinetic energy

(* Potential energy is included in the source term)

$$\boxed{\begin{aligned} e &= i + \frac{1}{2}\left(u^2 + v^2 + w^2\right) \\ &= C_p\left(T - T_{ref}\right) + h_{fg} + \frac{1}{2}\left(u^2 + v^2 + w^2\right) \end{aligned}}$$

Deriving the energy equation

$$
\begin{aligned}
\frac{\partial(\rho e)}{\partial t}+\frac{\partial(\rho e)}{\partial x}u+\frac{\partial(\rho e)}{\partial y}v+\frac{\partial(\rho e)}{\partial z}w
={}& \boxed{\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)} \rightarrow \text{Thermal conduction}\\
&+\boxed{\begin{aligned}&\frac{\partial(u\tau_{xx})}{\partial x}+\frac{\partial(u\tau_{yx})}{\partial y}+\frac{\partial(u\tau_{zx})}{\partial z}\\
&+\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v\tau_{yy})}{\partial y}+\frac{\partial(v\tau_{zy})}{\partial z}\\
&+\frac{\partial(w\tau_{xz})}{\partial x}+\frac{\partial(w\tau_{yz})}{\partial y}+\frac{\partial(w\tau_{zz})}{\partial z}\end{aligned}} \rightarrow \text{Viscous force work}\\
&-\boxed{\left(\frac{\partial(pu)}{\partial x}+\frac{\partial(pv)}{\partial y}+\frac{\partial(pw)}{\partial z}\right)} \rightarrow \text{Pressure force work}\\
&+\boxed{\rho\dot{q}+S_e} \rightarrow \text{Source terms}
\end{aligned}
$$

The energy equation

$$\frac{\partial(\rho e)}{\partial t}+\nabla\cdot\left(\rho e\vec{U}\right)=\underbrace{-\nabla\cdot\left(p\vec{U}\right)}_{\text{work by pressure}}+\overbrace{\nabla\cdot\left(\lambda\nabla T\right)}^{\text{heat conduction}}+\underbrace{\nabla\cdot\left(\tau\vec{U}\right)}_{\text{work by viscous forces}}+\overbrace{\rho\dot{q}}^{\text{volumetric heating}}+\underbrace{S_e}_{\text{other sources}}$$

$$e=C_p\left(T-T_{ref}\right)+h_{fg}+\frac{1}{2}\left(u^2+v^2+w^2\right)$$

The Euler equations (Momentum equation)

When the viscous stress in the momentum equation effects is neglected, it is know as the Euler equation

$$\frac{\partial\left(\rho\vec{U}\right)}{\partial t}+\nabla\cdot\left(\rho\vec{U}\vec{U}\right)=\underbrace{\boxed{-\nabla p}}_{\text{Pressure}}+\underbrace{\boxed{F_B}}_{\text{Body forces}}$$

However, fluids do have viscosity…

The Navier-Stokes equations

Claude-Louis Navier
(1785-1836)

George Gabriel Stokes
(1819-1903)

- Viscous stress terms exist in the momentum and energy equations
- Navier and Stokes introduced the **viscous stress tensor** to calculate the viscous stresses

$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

- No analytical solution

The Navier-Stokes equations

- The viscous stresses are expressed as functions of the local deformation rate or strain rate, which includes the linear deformation rate and volumetric deformation rate

- The **linear deformation rate** has nine components

3 normal components $s_{xx} = \frac{\partial u}{\partial x}$ $s_{yy} = \frac{\partial v}{\partial y}$ $s_{zz} = \frac{\partial w}{\partial z}$

6 shearing components $s_{xy} = s_{yx} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$ $s_{xz} = s_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$ $s_{yz} = s_{zy} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$

- The **volume deformation rate** is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{U}$$

The Navier-Stokes equations

- In a Newtonian fluid, the viscous stresses are proportional to the rates of deformation (Newton's law of viscosity for compressible flows), through a dynamic viscosity μ related to learning deformation and a second viscosity μ_V related to volumetric deformation.

$$\tau_{xx} = \mu_V \nabla \cdot \vec{U} + 2\mu \frac{\partial u}{\partial x} \qquad \tau_{yy} = \mu_V \nabla \cdot \vec{U} + 2\mu \frac{\partial v}{\partial y} \qquad \tau_{zz} = \mu_V \nabla \cdot \vec{U} + 2\mu \frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

- For gas, the second viscosity μ_V is estimated by $\mu_V = -\frac{2}{3}\mu$
- For liquid, as it is incompressible $\nabla \cdot \vec{U} = 0$

The Navier-Stokes equations

❖ Therefore, the Navier-Stokes (X-momentum) equations take the form

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u)}{\partial x}u+\frac{\partial(\rho u)}{\partial y}v+\frac{\partial(\rho u)}{\partial z}w=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+F_{B,x}$$

$$=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\mu_V\nabla\cdot\vec{U}+2\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial z}\right)\right)+\frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)+F_{B,x}$$

$$=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)$$

$$+\left(\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right)-\frac{\partial}{\partial x}\left(\frac{2}{3}\mu\nabla\cdot\vec{U}\right)\right)+F_{B,x}$$

The Navier-Stokes equations

- The Navier-Stokes equations in all 3 directions

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u)}{\partial x}u+\frac{\partial(\rho u)}{\partial y}v+\frac{\partial(\rho u)}{\partial z}w=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)$$

$$+\left(\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right)-\frac{\partial}{\partial x}\left(\frac{2}{3}\mu\nabla\cdot\vec{U}\right)\right)+F_{B,x}$$

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho v)}{\partial x}u+\frac{\partial(\rho v)}{\partial y}v+\frac{\partial(\rho v)}{\partial z}w=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)$$

$$+\left(\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{2}{3}\mu\nabla\cdot\vec{U}\right)\right)+F_{B,y}$$

$$\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho w)}{\partial x}u+\frac{\partial(\rho w)}{\partial y}v+\frac{\partial(\rho w)}{\partial z}w=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\mu\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right)$$

$$+\left(\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial z}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{2}{3}\mu\nabla\cdot\vec{U}\right)\right)+F_{B,z}$$

The Navier-Stokes equations

$$\frac{\partial}{\partial t}\left(\rho\vec{U}\right)+\nabla\cdot\left(\rho\vec{U}\vec{U}\right)=\underbrace{\boxed{-\nabla p}}_{\text{Pressure}}+\overbrace{\boxed{\nabla\cdot\left(\mu\left(\nabla\vec{U}+\left(\nabla\vec{U}\right)^{T}\right)\right)}}^{\text{Viscous force due to linear deformation}}-\underbrace{\boxed{\nabla\cdot\left(\frac{2}{3}\mu\left(\nabla\cdot\vec{U}\right)\right)}}_{\text{Viscous force due to volumetric deformation}}I+\underbrace{\boxed{S_M}}_{\text{Body forces}}$$

For incompressible fluid, the viscous force due to volumetric deformation is zero

$$\frac{\partial}{\partial t}\left(\rho\vec{U}\right)+\nabla\cdot\left(\rho\vec{U}\vec{U}\right)=-\nabla p+\nabla\cdot\left(\mu\left(\nabla\vec{U}+\left(\nabla\vec{U}\right)^{T}\right)\right)+S_M$$

The Navier-Stokes energy equation

$$\frac{\partial(\rho e)}{\partial t}+\frac{\partial(\rho e)}{\partial x}u+\frac{\partial(\rho e)}{\partial y}v+\frac{\partial(\rho e)}{\partial z}w=\underbrace{\boxed{\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)}}_{\text{Thermal conduction}}$$

$$\underbrace{-\underbrace{\boxed{\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2}}_{\text{normal stress work}}+\underbrace{\boxed{\mu\left(2\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right)+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2\right)}}_{\text{Shear stress work}}}_{\text{Viscous dissipation}}$$

$$-\underbrace{\boxed{p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)}}_{\text{Pressure force work}}$$

$$+\underbrace{\boxed{\rho\dot{q}+S_e}}_{\text{source terms}}$$

The generic convection-diffusion equation

The continuity equation $\frac{\partial \rho}{\partial t}+\nabla\cdot\left(\rho\vec{U}\right)=0$

The momentum equation $\frac{\partial}{\partial t}\left(\rho\vec{U}\right)+\nabla\cdot\left(\rho\vec{U}\vec{U}\right)=-\nabla p+\nabla\cdot\left(\mu\left(\nabla\vec{U}+\left(\nabla\vec{U}\right)^{T}\right)\right)+S_{M}$

The energy $\frac{\partial(\rho e)}{\partial t}+\nabla\cdot\left(\rho e\vec{U}\right)=-\nabla\cdot\left(p\vec{U}\right)+\nabla\cdot(\lambda\nabla T)+\nabla\cdot\left(\tau\vec{U}\right)+\rho\dot{q}+S_{e}$

The generic convection-diffusion equation

(also called **the transport equation** of ϕ)

$$\boxed{\frac{\partial(\rho\phi)}{\partial t}+\nabla\cdot\left(\rho\phi\vec{U}\right)=\nabla\cdot(\Gamma\nabla\phi)+S_{\phi}}$$

Increase rate of ϕ per unit volume	+	Net outflow rate of ϕ per unit volume	=	Increase rate of ϕ due to diffusion	+	Increase rate of ϕ due to source
local term		**convection term**		**diffusion term**		**source term**

The generic convection-diffusion equation

The generic convection-diffusion equation are sometimes written as

$$\boxed{\frac{\partial(\rho\phi)}{\partial t} + \nabla\cdot\left(\rho\phi\vec{U} - \Gamma\nabla\phi\right) = S_\phi}$$

Increase rate of ϕ per unit volume	+	Net outflow rate of ϕ across boundaries per unit volume	=	Increase rate of ϕ due to source per unit volume
local term		**Convection + diffusion term**		**source term**

Substantial derivative and generic equation

$$d(\rho\phi) = \frac{\partial(\rho\phi)}{\partial t}dt + \frac{\partial(\rho\phi)}{\partial x}dx + \frac{\partial(\rho\phi)}{\partial y}dy + \frac{\partial(\rho\phi)}{\partial z}dz$$

→ The total derivative

$$\frac{D(\rho\phi)}{Dt} = \frac{\partial(\rho\phi)}{\partial t} + \frac{\partial(\rho\phi)}{\partial x}\frac{dx}{dt} + \frac{\partial(\rho\phi)}{\partial y}\frac{dy}{dt} + \frac{\partial(\rho\phi)}{\partial z}\frac{dz}{dt}$$

→ The substantial derivative

$$= \frac{\partial(\rho\phi)}{\partial t} + \frac{\partial(\rho\phi)}{\partial x}u + \frac{\partial(\rho\phi)}{\partial y}v + \frac{\partial(\rho\phi)}{\partial z}w$$

$$= \frac{\partial(\rho\phi)}{\partial t} + \nabla\cdot\left(\rho\phi\vec{U}\right)$$

$$= \frac{\partial\phi}{\partial t} + div\left(\rho\phi\vec{U}\right)$$

Increase rate of ϕ in a unit volume of fixed fluid elements

$$\boxed{\frac{D(\rho\phi)}{Dt}} = \boxed{\frac{\partial(\rho\phi)}{\partial t}} + \boxed{\nabla\cdot\left(\rho\phi\vec{U}\right)}$$

Increase rate of ϕ in a unit volume of moving fluid elements

Outflow rate of ϕ from a unit volume of fixed fluid elements

Revisit the concepts

Important concepts

- ❖ Substantial derivative

$$\frac{d\left(\rho\phi\right)}{dt}=\frac{D\left(\rho\phi\right)}{Dt}=\frac{\partial\left(\rho\phi\right)}{\partial t}+\nabla\cdot\left(\rho\phi\vec{U}\right)$$

- ❖ Gradient - vector

$$\nabla p=\vec{i}\,\frac{\partial p}{\partial x}\qquad \nabla p=\vec{i}\,\frac{\partial p}{\partial y}\qquad \nabla p=\vec{i}\,\frac{\partial p}{\partial z}$$

- ❖ Divergence - scalar

$$\nabla\cdot\left(\phi\vec{U}\right)=\mathrm{div}\left(\phi\vec{U}\right)=\frac{\partial\left(\phi u\right)}{\partial x}+\frac{\partial\left(\phi v\right)}{\partial y}+\frac{\partial\left(\phi w\right)}{\partial z}$$

Summary

The governing equations are:

❖ A coupled system of non-linear PDEs, no analytical solution to date.

❖ 5 equations for 6 unknowns: ρ, p, u, v, w, e – closure is needed

❖ Navier-Stokes equations historically mean the momentum equations for viscous flows, but now refer to the entire system of flow equations including the continuity, momentum and energy equations.

Conservation equations in Fluent

- ❖ The continuity and momentum equations are included by default to solve fluid flows. You cannot deselect the continuity and momentum equations.
- ❖ Energy equation can be togged on/off depending on whether the flow is isothermal or non-isothermal.

Q&A