Announcements and reminders

- HW 2 due next Friday at 5 PM
- Good progress:
    - 2/4 problems by Sunday night, or maybe a *little* bit more than that…?

      (because 2 weeks to do 4 problems)

Lecture 7: Discrete random variables and their distributions

Previously, on CSCI 3022…

Definition: A **discrete random variable** (r.v.) X is a function that maps the elements of the sample space Ω to a finite number of values $a_1, a_2, \ldots, a_n$ or an infinite number of values $a_1, a_2, \ldots$

Definition: A **probability mass function** (pmf) is the map between the random variable's values and the probabilities of those values

$$f(a) = P(X=a)$$

Definition: A **cumulative distribution function** (cdf) is a function whose value at a point *a* is the cumulative sum of probability masses up until *a*

$$F(a) = P(X \leq a)$$

Warm-up problem

Example: S'pose you roll two fair, six-sided dice. Let X be a random variable representing the maximum of the two dice. Some questions:

Q1: What are the possible values that X can take? ✓ $X \in \{1, 2, 3, 4, 5, 6\}$

Q2: Which elements of the sample space map to which values of X?

Q3: What is the pmf of the random variable X?

ω_1 = roll 1

ω_2 = roll 2

	1	2	3	4	5	6
1						
2					5	
3						
4			4			
5						
6						

Warm-up problem

all 36 possible outcomes in Ω are equally likely

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ω_1

ω_2

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

each has 1/36 probability mass assoc. w/ it

a	1	2	3	4	5	6
f(*a*)	1/36	3/36				

Warm-up problem

Example: S'pose you roll two fair, six-sided dice. Let X be a random variable representing the maximum of the two dice. Some questions:

Q4: What is the probability that X is an even number?

a	1	2	3	4	5	6
f(a)	1/36	3/36	5/36	7/36	9/36	11/36

valid pmf if:
all probs ≥ 0 ✓
& Σ to 1 ✓

$P(X \text{ is even}) = P(X=2 \cup X=4 \cup X=6)$

disjoint

$= P(X=2) + P(X=4) + P(X=6)$

$= 21/36$

if not disjoint ...
we'd have to subtract off the intersection to account for double-counting

Warm-up problem

Example: S'pose you roll two fair, six-sided dice. Let X be a random variable representing the maximum of the two dice. Some questions:

Q5: What is the probability that X is 3 or smaller?

$$= f(1) + f(2) + f(3)$$

$$= F(3)$$

$$= 9/36$$

a	1	2	3	4	5	6
$f(a)$	1/36	3/36	5/36	7/36	9/36	11/36

Warm-up problem

Example: S'pose you roll two fair, six-sided dice. Let X be a random variable representing the maximum of the two dice. Some questions:

Q6: What is the complete cdf of X?

pmf

a	1	2	3	4	5	6
f(*a*)	1/36	3/36	5/36	7/36	9/36	11/36

cdf

a	1	2	3	4	5	6
F(*a*)	1/36	4/36	9/36	16/36	25/36	1

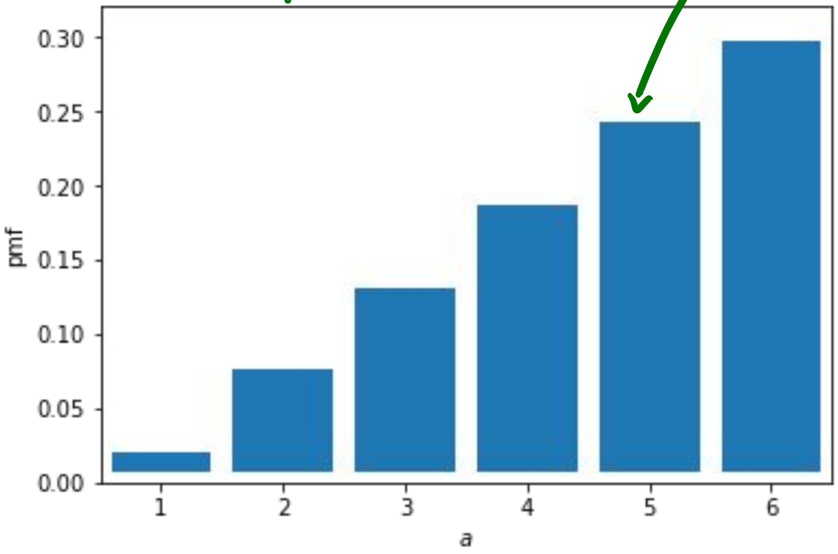
Visualizing pmfs and cdfs

Example: S'pose you roll two fair, six-sided dice. Let X be a random variable representing the maximum of the two dice.

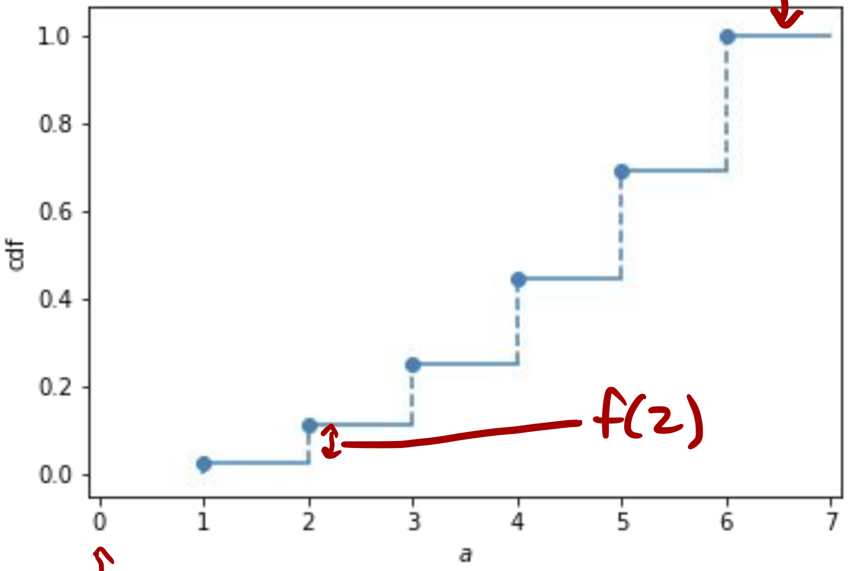
f(5) is hgt.

F(6)=1

PMF

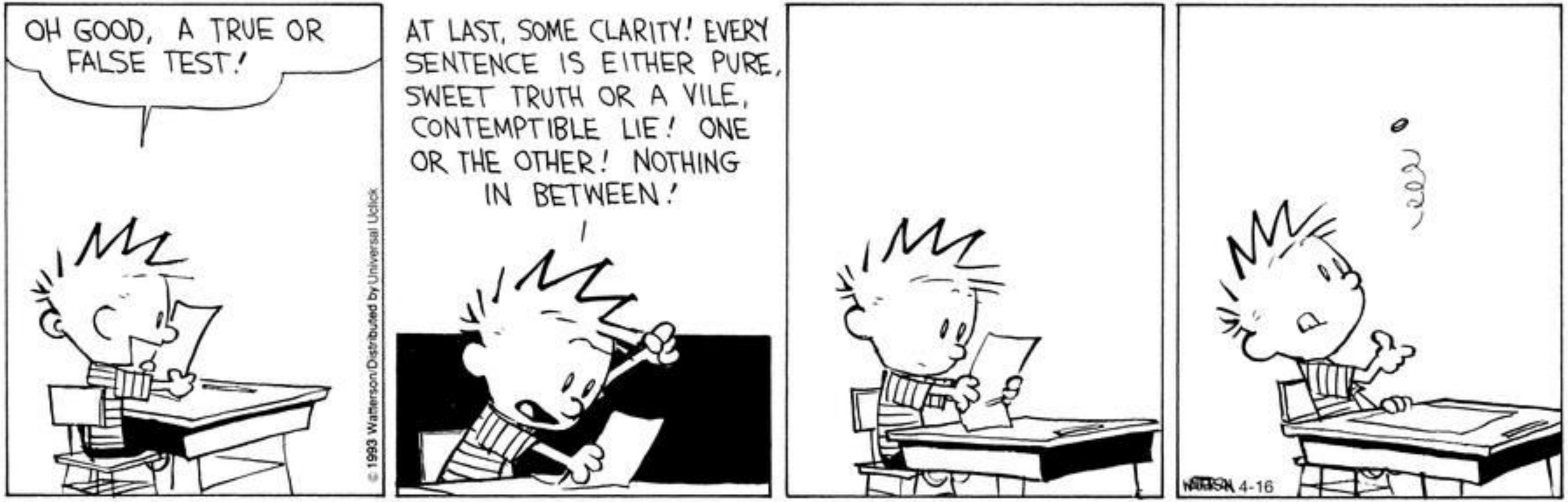


CDF



Common discrete r.v. distributions

Discrete r.v.'s can be categorized into different types or classes that each **model** different real-world situations



The Bernoulli distribution

The Bernoulli distribution is used to model experiments with only two possible outcomes.

Often referred to as “success” and “failure”, and encoded as 1 and 0, respectively.

Definition: A discrete random variable X has a **Bernoulli distribution** with parameter p, where $0 \leq p \leq 1$, if its probability mass function is given by

prob

$$\underbrace{f(1)}_{\text{pmf}} = \underbrace{p_X(1)}_{\text{r.v.}} = P(X=1) = p \quad \textbf{and} \quad p_X(0) = P(X=0) = 1-p = f(0)$$

We denote this distribution by Ber(p)

Distr Name (parameters for that distr.)

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Question: Wouldn’t it be nice if we could describe the pmf with a single equation?

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We denote this distribution by Ber(p)

Question: Wouldn’t it be nice if we could describe the pmf with a single equation?

→ if we have $p_X(1)=p$, and $p_X(0)=1-p$, then for x in {0, 1}, we have: $p_X(x) = p^x (1-p)^{1-x}$

A counting interlude

We'll come back to the Bernoulli distribution in a minute.   First… we ***count!***

Counting comes up all over the place in probability
(and therefore in data science, comp sci, math, physics, etc…)

Some counting is easy: how many integers are there in the interval [0, 9] ?

But we're interested in counting problems that require a bit more thought:

- Dan, Chris, Rhonda and Tony line up at the coffee stand. How many different orders could they stand in?
- If there are 10 problems on an exam, and you need 7 correct to pass, how many different ways are there to pass?

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order matters

- Dan, Chris, Rhonda and Tony line up at the coffee stand. How many different orders could they stand in?
- If there are 10 problems on an exam, and you need 7 correct to pass, how many different ways are there to pass?

order doesn't matter

A counting interlude

We'll talk about two important kinds of counting problems today:

1) Counting **permutations** means counting the number of ways that a set of objects can be ordered (or *permuted!*)

Example: Dan, Chris, Rhonda and Tony line up at the coffee stand. How many different orders could they stand in?

2) Counting **combinations** means counting the number of ways that a set of objects can be combined into subsets

Example: If there are 10 problems on an exam, and you need 7 correct to pass, how many different ways are there to pass?

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2) Counting **combinations** means counting the number of ways that a set of objects can be combined into subsets

Example: If there are 10 problems on an exam, and you need 7 correct to pass, how many different ways are there to pass?

Permutations

Questions:

- How many ways are there to order a set of 1 object? 1
- How many ways are there to order a set of 2 objects? AB or BA $\to$ 2
  3 places to put C   3 places to put C
- How many ways are there to order a set of 3 objects?
  $\hookrightarrow$ $3 \cdot 2$

The Big Question: What is a formula for the number of ways you can order *n* objects?

(# of ways to arrange all n objects) $= n \times$ (# ways to arrange the first n-1 objects)

$\to$ $P(n) = n \cdot P(n-1)$

$P(1) = 1$

$\to$ $\boxed{P(n) = n!}$ $= n \times (n-1) \times \dots \times 1$

Permutations

Question: What if we have *n* objects, but want to count permutations of only *r* of them?

Example: How many 3-character strings can we make if each character is a distinct letter from the English alphabet?

$$\underline{26} \times \underline{25} \times \underline{24} = \frac{26 \times 25 \times 24 \times 23 \times 22 \times \cdots \times 2 \times 1}{23 \times 22 \times \cdots \times 2 \times 1} = \frac{26!}{23!}$$

Question: What is the general formula for *r*-permutations of *n* objects?

$$\boxed{P(n, r) = \frac{n!}{(n-r)!}}$$

larger collection

permutations of r of them

Permutations

Question: What if we have *n* objects, but want to count permutations of only *r* of them?

Example: How many 3-character strings can we make if each character is a distinct letter from the English alphabet?

Answer: $26 \times 25 \times 24$

Question: What is the general formula for *r*-permutations of *n* objects?

Answer: $P(n, r) = \frac{n!}{(n-r)!}$

write our previous perm:

$\underline{P(n, n) = n!}$

Combinations

Counting **combinations** means counting the number of ways a set of objects can be combined into subsets

Key difference: When counting combinations, **order does not matter**.

r $P(n,r)$

Example: How many 3-character **combinations** can we make if each character is a distinct letter from the English alphabet?

permutations: abc, acb, bac, bca, cab, cba

combos: these are all the same. ∴ so # perms counted 6x too many

6 came from # ways to arrange those 3 letters:

$P(3,3) = 3! = r!$

Combinations

Example: How many 3-character **combinations** can we make if each character is a distinct letter from the English alphabet?

→ Start with the number of 3-permutations of 26 letters:

→ But if order doesn't matter, we are counting combinations **multiple times**

$$C(n,r) = \frac{P(n,r)}{r!} \longrightarrow \boxed{C(n,r) = \frac{n!}{(n-r)!\,r!}}$$

$$\binom{n}{r}$$

Combinations

There are many different notations for combinations. You can write the number of ways to choose *r* objects from a set of *n* objects as:

$C(n, r)$ **or** $C_{n,\,r}$ **or** $\binom{n}{r}$ $= \dfrac{n!}{(n-r)!\,r!}$

Example: If there are 10 problems on an exam, and you need 7 correct to pass, how many different ways are there to pass?

$\#\text{ways} = C(10,7) + C(10,8) + C(10,9) + C(10,10)$

Combinations

There are many different notations for combinations. You can write the number of ways to choose *r* objects from a set of *n* objects as:

$$C(n, r) \quad \textbf{or} \quad C_{n,\, r} \quad \textbf{or} \quad \binom{n}{r}$$

Example: If there are 10 problems on an exam, and you need 7 correct to pass, how many different ways are there to pass?

Answer: # ways = C(10, 7) + C(10, 8) + C(10, 9) + C(10, 10) = …

Combinations

Example: A coin is flipped 10 times. How many possible outcomes have exactly 2 Heads?

$C(10, 2)$

#flips

choose 2 to be H

Example: A coin is flipped 10 times. How many possible outcomes have 2 Heads or fewer?

$= C(10, 2) + C(10, 1) + C(10, 0)$

Sum of Bernoulli random variables

Example: S'pose you show up to a quiz completely unprepared. Oh no! The quiz has 5 questions, each with 4 multiple choice options. You decide to guess each answer in a completely random way. What is the probability that you get 3 questions correct?

Bernoulli r.v. for each problem w/ $p = 1/4$

$\hookrightarrow \quad R_i = \begin{cases} 0 & \text{if problem } i \text{ is wrong} \\ 1 & \text{if } \ldots \text{ correct} \end{cases}$

Sum of Bernoulli random variables

Example: S'pose you show up to a quiz completely unprepared. Oh no! The quiz has 5 questions, each with 4 multiple choice options. You decide to guess each answer in a completely random way. What is the probability that you get 3 questions correct?

For $i = 1, 2, 3, 4, 5$ let $$R_i = \begin{cases} 1 & \text{if the } i^{th} \text{ answer is correct} \\ 0 & \text{if the } i^{th} \text{ answer is incorrect} \end{cases}$$

$R_i \sim \text{Ber}(p = 1/4)$

R_i "is distributed" w/ Ber...

LateX: \sim

Question: What can you say about R_i?

$P(3 \text{ of } R_i \text{ are } = 1)$?

Define r.v. $X = R_1 + R_2 + R_3 + R_4 + R_5$

$\hookrightarrow \quad P(X=3)$?

Sum of Bernoulli random variables

HERE

Example: S'pose you show up to a quiz completely unprepared. Oh no! The quiz has 5 questions, each with 4 multiple choice options. You decide to guess each answer in a completely random way. What is the probability that you get 3 questions correct?

→ Let r.v. X = # correct answers. → $X = R_1 + R_2 + R_3 + R_4 + R_5$

Question: What values can X take? $0, 1, \dots, 5$

Question: What is the probability that you get 0 problems correct? :(

$$P(X=0) = P(R_1=0 \cap R_2=0 \cap R_3=\dots \cap R_5=0)$$

if indep

$$= P(R_1=0)\,P(R_2=0)\cdots P(R_5=0)$$

$$= (3/4)^5$$

Sum of Bernoulli random variables *(Example, cont.)*

Question: What is the probability that you get 0 problems correct? :(

Answer:
$$\begin{aligned} P(X=0) &= P(R_1=0, R_2=0, R_3=0, R_4=0, R_5=0) \\ &= P(R_1=0)P(R_2=0)P(R_3=0)P(R_4=0)P(R_5=0) \\ &= \left(\frac{3}{4}\right)^5 \end{aligned}$$

Question: What is the probability that you get exactly 1 problem correct?

Sum of Bernoulli random variables *(Example, cont.)*

Question: What is the probability that you get 0 problems correct? :(

Answer:

$$\begin{aligned} P(X=0) &= P(R_1=0, R_2=0, R_3=0, R_4=0, R_5=0) \\ &= P(R_1=0)P(R_2=0)P(R_3=0)P(R_4=0)P(R_5=0) \\ &= \left(\frac{3}{4}\right)^5 \end{aligned}$$

Question: What is the probability that you get exactly 1 problem correct?

→ P(X=1) = ???

→ Could have gotten Q1 correct → $P(R_1=1, \text{others} = 0) = (1/4)(3/4)^4$

→ Could have gotten Q2 correct → $P(R_2=1, \text{others} = 0) = (3/4)(1/4)(3/4)^3 = (1/4)(3/4)^4$

→ … *and so on* … $\mathbf{P(X=1) = 5 \cdot (1/4) \cdot (3/4)^4}$

Sum of Bernoulli random variables…

Question: What is the probability that you get k problems correct out of n problems total?
(k some ≥ 0, integer)

Answer: $$p_X(k) = \binom{n}{k} p^k \ (1-p)^{n-k}$$

Where the combination (or binomial coefficient) is $$\binom{n}{k} = \frac{n!}{(n-k)!\ k!}$$

Sum of Bernoulli random variables… a Binomial distribution!

Question: What is the probability that you get k problems correct out of n problems total?
(k some ≥ 0, integer)

Answer: $$p_X(k) = \binom{n}{k} p^k \ (1-p)^{n-k}$$

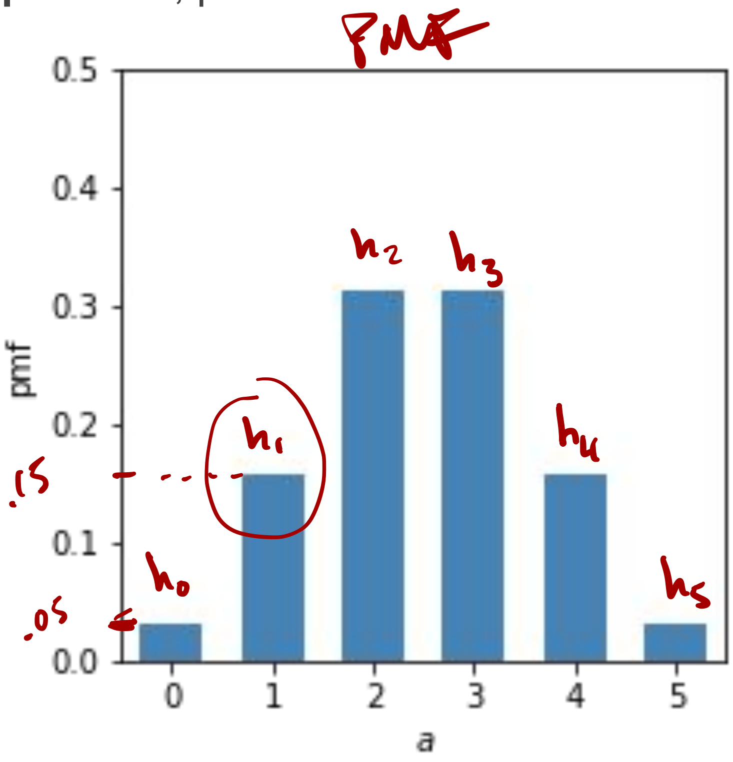
Definition: A discrete r.v. X has a **binomial distribution** with parameters *n* and *p*, where *n* = 1, 2, … and 0 ≤ *p* ≤ 1, if its probability mass function is given by

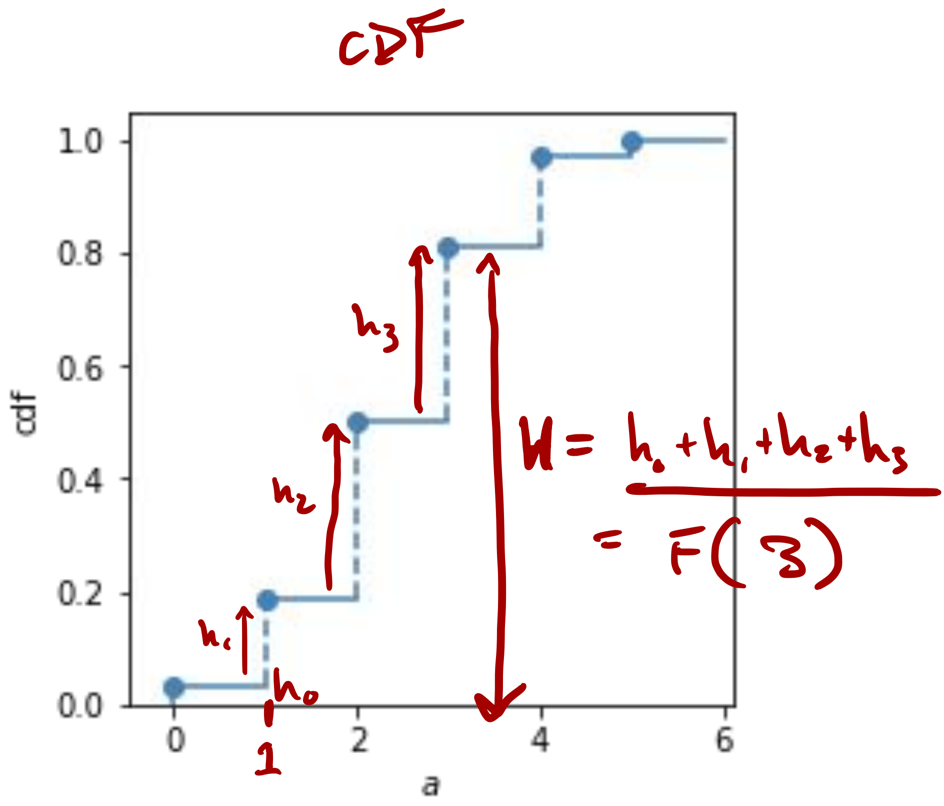
$$p_X(k) = P(X = k) = \binom{n}{k} p^k \ (1-p)^{n-k} \quad \text{for } k = 0, 1, 2, \ldots, n$$

We denote this distribution by Bin(n, p)

The Binomial distribution

Example: n=5, p=0.5



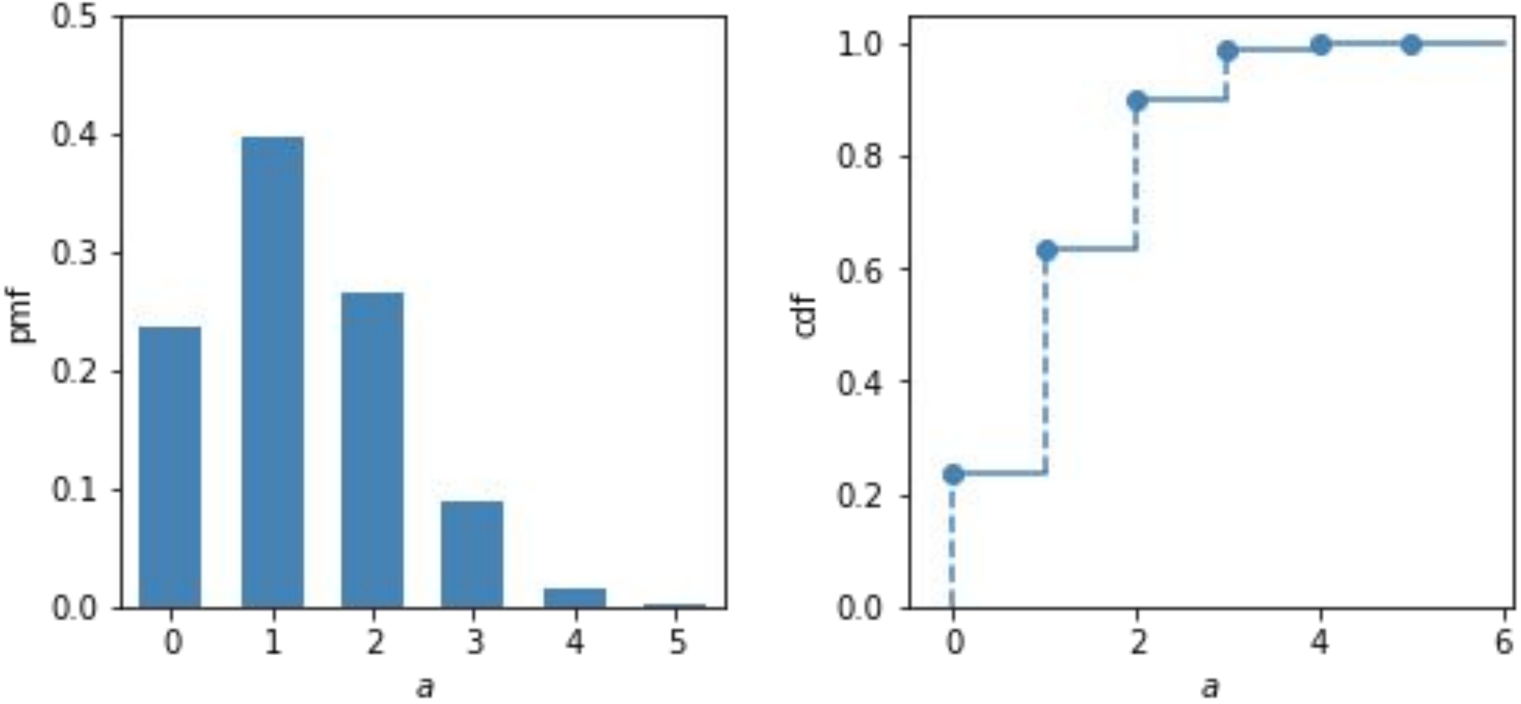


The Binomial distribution

Example: $n=5$, $p=0.25$

The Binomial distribution

Example: n=5, p=0.25

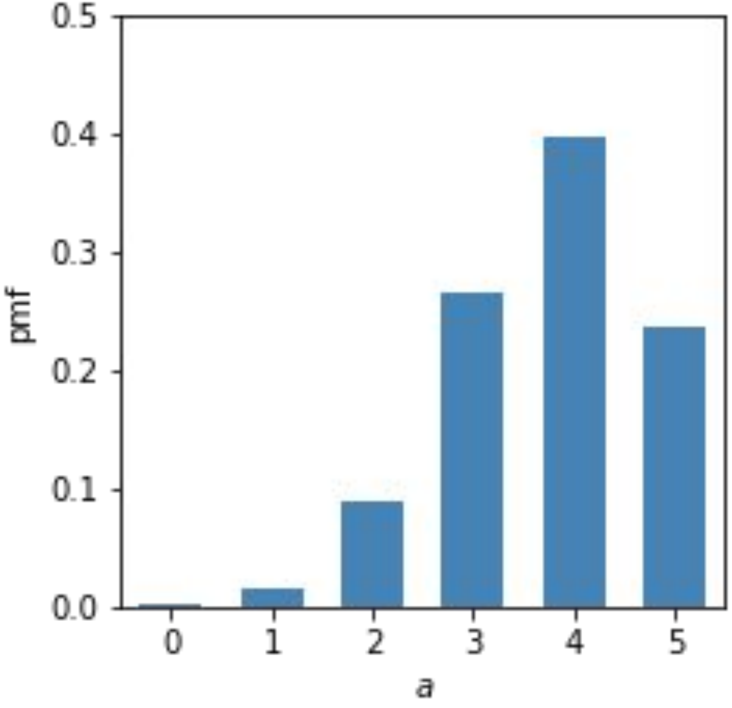


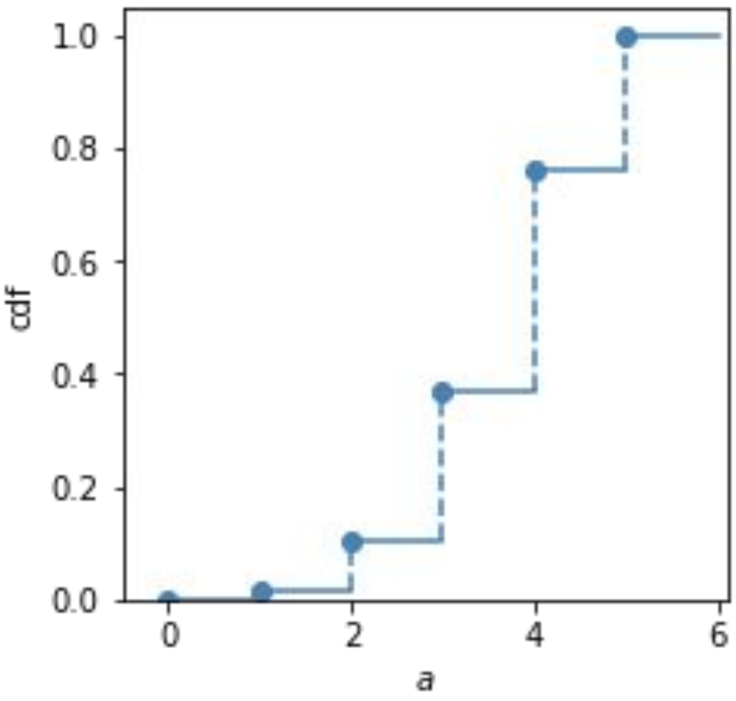
The Binomial distribution

Example: $n=5$, $p=0.75$

The Binomial distribution

Example: n=5, p=0.75





The Binomial distribution

What **assumptions** did we make in going from Ber(p) to Bin(n, p) ?

The Binomial distribution

What **assumptions** did we make in going from Ber(p) to Bin(n, p) ?

- Each of the n Bernoulli trials are independent

- Each of the Bernoulli trials has the same probability of success p

The Most Boring (but Common) Distribution of Them All

What is the distribution of a fair die?

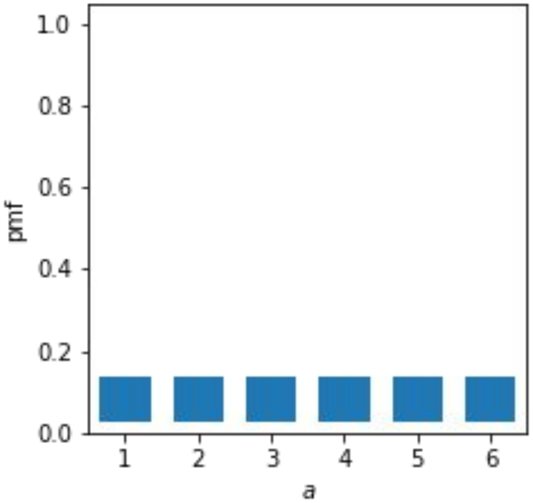
The Most Boring (but Common) Distribution of Them All

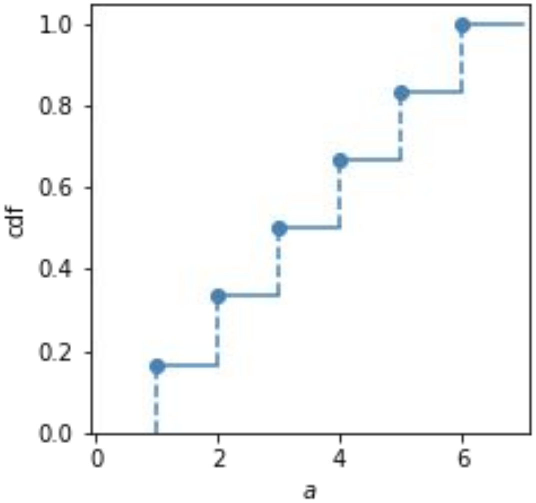
What is the distribution of a fair die?

Definition: A discrete r.v. X has a **discrete uniform distribution** with parameters *a*, *b*, and *n*=*b*-*a*+1 if

$\frac{1}{\text{total \# outcomes}}$

$$p_X(k) = \frac{1}{n} \quad \text{for} \quad k = a, a+1, a+2, \dots, b$$





What just happened?

- We learned about some important **discrete** distributions! (what does "discrete" mean?)
  - **Bernoulli** distribution -- a coin flip // success or failure
  - **Binomial** distribution -- how many successes out of *n* Bernoulli trials

Next time: More **coin flipping**! But, a little bit **different**!