Note on the Call/Put replication

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Abstract

We show 1. how to replicate the general cash flow as a function of underlying asset price with calls and puts avilable in the market, 2. how to find the price of the general cash flow with prices of calls and puts, 3. how to recover a risk-neutral density using the second derivative of call/prices with respect to strike price.

1 Derivation

With the fundamental theorem of calculus, we have

$$\begin{aligned} f(S_T) &= f(F_0) + \int_{F_0}^{S_T} f'(x)\,dx && (1) \\ f'(x) &= f'(F_0) + \int_{F_0}^{x} f''(K)\,dK. && (2) \end{aligned}$$

By plugging (2) into (1),

$$\begin{aligned} f(S_T) &= f(F_0) + \int_{F_0}^{S_T} \left(f'(F_0) + \int_{F_0}^{x} f''(K)\,dK \right) dx \\ &= f(F_0) + \int_{F_0}^{S_T} f'(F_0)\,dx + \int_{F_0}^{S_T} \int_{F_0}^{x} f''(K)\,dK dx \\ &= f(F_0) + f'(F_0)(S_T - F_0) + \int_{F_0}^{S_T} \int_{F_0}^{x} f''(K)\,dK dx. && (3) \end{aligned}$$

1.1 $S_T > F_0$

Then, the last component of (3) is can be expressed as follows:

$$\begin{aligned}
\int_{F_0}^{S_T}\int_{F_0}^{x} f''(K)\,dK dx &= \int_{F_0}^{S_T}\int_{K}^{S_T} f''(K)\,dx dK \\
&= \int_{F_0}^{S_T} f''(K)\left(\int_{K}^{S_T} dx\right) dK \\
&= \int_{F_0}^{S_T} f''(K)\,(S_T - K)\,dK \\
&= \int_{F_0}^{\infty} f''(K)\max(S_T - K, 0)\,dK, \qquad (4)
\end{aligned}$$

where first equlity is due to Fubini's Theorem.

1.2 $S_T < F_0$

Then, the last component of (3) is can be expressed as follows:

$$\begin{aligned}
\int_{F_0}^{S_T}\int_{F_0}^{x} f''(K)\,dK dx &= \int_{S_T}^{F_0}\int_{x}^{F_0} f''(K)\,dK dx \\
&= \int_{S_T}^{F_0}\int_{S_T}^{K} f''(K)\,dx dK \\
&= \int_{S_T}^{F_0} f''(K)\left(\int_{S_T}^{K} dx\right) dK \\
&= \int_{S_T}^{F_0} f''(K)\,(K - S_T)\,dK \\
&= \int_{0}^{F_0} f''(K)\max(K - S_T, 0)\,dK, \qquad (5)
\end{aligned}$$

where second equlity is due to Fubini's Theorem.

By combining (3), (4) and (5), we get

$$\begin{aligned} f\left(S_T\right) = f\left(F_0\right)\ &:\text{bond} \\ + f'\left(F_0\right)\left(S_T - F_0\right) &: \text{futures} \\ + \int_{F_0}^{\infty} f''\left(K\right)\max\left(S_T - K, 0\right) dK &: \text{call} \\ + \int_{0}^{F_0} f''\left(K\right)\max\left(K - S_T, 0\right) dK &: \text{put} \end{aligned}$$

2 Pricing

Now, we want to find today's price of $f\left(S_T\right)$. By the fundamental pricing equation,

$$\begin{aligned} e^{-rT}\mathbb{E}^Q\left[f\left(S_T\right)\right] &= e^{-rT}\mathbb{E}^Q\left[f\left(F_0\right)\right] + e^{-rT}\mathbb{E}^Q\left[f'\left(F_0\right)\left(S_T - F_0\right)\right] \\ &\quad + e^{-rT}\mathbb{E}^Q\left[\int_{F_0}^{\infty} f''\left(K\right)\max\left(S_T - K, 0\right) dK\right] \\ &\quad + e^{-rT}\mathbb{E}^Q\left[\int_{0}^{F_0} f''\left(K\right)\max\left(K - S_T, 0\right) dK\right] \\ &= e^{-rT} f\left(F_0\right) + f'\left(F_0\right) e^{-rT}\mathbb{E}^Q\left[S_T - F_0\right] \\ &\quad \int_{F_0}^{\infty} f''\left(K\right) e^{-rT}\mathbb{E}^Q\left[\max\left(S_T - K, 0\right)\right] dK \\ &\quad + \int_{0}^{F_0} f''\left(K\right) e^{-rT}\mathbb{E}^Q\left[\max\left(K - S_T, 0\right)\right] dK \\ &= e^{-rT} f\left(F_0\right) + \int_{F_0}^{\infty} f''\left(K\right) C\left(K\right) dK + \int_{0}^{F_0} f''\left(K\right) P\left(K\right) dK, \end{aligned}$$

where the last equality is due to

$$\begin{aligned} C\left(K\right) &= e^{-rT}\mathbb{E}^Q\left[\max\left(S_T - K, 0\right)\right], \\ P\left(K\right) &= e^{-rT}\mathbb{E}^Q\left[\max\left(K - S_T, 0\right)\right] \end{aligned}$$

and

$$e^{-rT}\mathbb{E}^Q\left[S_T - F_0\right] = e^{-rT}\mathbb{E}^Q\left[S_T\right] - e^{-rT} F_0 = S_0 - e^{-rT} e^{rT} S_0 = 0.$$

3 Recovery of Q-measure

Lastly, we show how to recover Q-measure from the second derivative of call option prices. First, note that the put-call parity implies

$$C''(K) = P''(K)$$

and that simple economic arguments justify

$$P(0) = P'(0) = C(\infty) = C'(\infty) = 0.$$

Next, we will exploit the following equality:

$$g(x)\,h(x) - g(y)\,h(y) = \int_y^x g'(z)\,h(z)\,dz + \int_y^x g(z)\,h'(z)\,dz.$$

Now, we rewrite $\int_{F_0}^{\infty} f''(K)\,C(K)\,dK$ as follows:

$$\begin{aligned}
&\int_{F_0}^{\infty} f''(K)\,C(K)\,dK \\
&= f'(\infty)\,C(\infty) - f'(F_0)\,C(F_0) + \int_{F_0}^{\infty} f'(K)\,C'(K)\,dK \\
&= -f'(F_0)\,C(F_0) + \int_{F_0}^{\infty} f'(K)\,C'(K)\,dK \\
&= -f'(F_0)\,C(F_0) + f(\infty)\,C'(\infty) - f(F_0)\,C'(F_0) + \int_{F_0}^{\infty} f(K)\,C''(K)\,dK \\
&= -f'(F_0)\,C(F_0) - f(F_0)\,C'(F_0) + \int_{F_0}^{\infty} f(K)\,C''(K)\,dK. \qquad (6)
\end{aligned}$$

In a symmetric way, we rewrite $\int_0^{F_0} f''(K)\,P(K)\,dK$ as follows:

$$\begin{aligned}
&\int_0^{F_0} f''(K)\,P(K)\,dK \\
&= f'(F_0)\,P(F_0) - f'(0)\,P(0) + \int_0^{F_0} f'(K)\,P'(K)\,dK \\
&= f'(F_0)\,P(F_0) + \int_0^{F_0} f'(K)\,P'(K)\,dK \\
&= f'(F_0)\,P(F_0) + f(F_0)\,P'(F_0) - f(0)\,P'(0) + \int_0^{F_0} f(K)\,P''(K)\,dK \\
&= f'(F_0)\,P(F_0) + f(F_0)\,P'(F_0) + \int_0^{F_0} f(K)\,P''(K)\,dK. \qquad (7)
\end{aligned}$$

Combining (6) and (7) yields

$$\begin{aligned}
e^{-rT}\mathbb{E}^{Q}\left[f\left(S_T\right)\right] &= e^{-rT}f\left(F_0\right) - f'\left(F_0\right)C\left(F_0\right) - f\left(F_0\right)C'\left(F_0\right) + \int_{F_0}^{\infty} f\left(K\right)C''\left(K\right)dK \\
&+ f'\left(F_0\right)P\left(F_0\right) + f\left(F_0\right)P'\left(F_0\right) + \int_{0}^{F_0} f\left(K\right)P''\left(K\right)dK \\
&= f\left(F_0\right)\left(P'\left(F_0\right) - C'\left(F_0\right) + e^{-rT}\right) + f'\left(F_0\right)\left(P\left(F_0\right) - C\left(F_0\right)\right) \\
&+ \int_{F_0}^{\infty} f\left(K\right)C''\left(K\right)dK + \int_{0}^{F_0} f\left(K\right)P''\left(K\right)dK \\
&= \int_{0}^{\infty} f\left(K\right)C''\left(K\right)dK = \int_{0}^{\infty} f\left(S_T\right)C''\left(S_T\right)dS_T \\
&= e^{-rT}\int_{0}^{\infty} f\left(S_T\right)\underbrace{e^{rT}C''\left(S_T\right)}_{\text{Q-density}}dS_T.
\end{aligned}$$