$\mathbb{N} \cup \{0\} = \mathbb{N}_0 = \{0, 1, 2, \ldots\}$

$\mathbb{P} \subseteq \mathbb{N}$: set of prime numbers

$\mathbb{Z}$ - integers

$\mathbb{Q}$ - rationale $\{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{N}\}$ - ordered field, x complete, x closed

$\mathbb{R}$ - reals (Appendix, Chap 1) - ordered field, complete, x closed

$\mathbb{C}$ - complex numbers - algebraically closed field, complete, x ordered

(x form a ordered set)

(x form a field)

$\mathbb{I}$ - irrationals $\mathbb{R} \setminus \mathbb{Q}$

$\mathbb{N} \subseteq \mathbb{N} \cup \{0\} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

$A \subseteq B$ means $A = B$ or A is contained in B

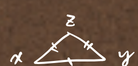
$A \subsetneq B$ means --- properly

In $\mathbb{R}$, we have a metric $d(x, y) \in \mathbb{R}$ for $x, y \in \mathbb{R}$ satisfying

i) $d(x, y) \geq 0 \quad \forall x, y$

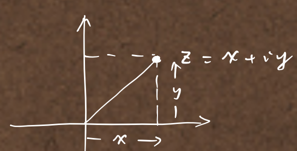
ii) $d(x, y) = 0 \iff x = y$

iii) $d(x, y) = d(y, x) \quad \forall x, y$

iv) $d(x, y) \leq d(x, z) + d(z, y)$ 

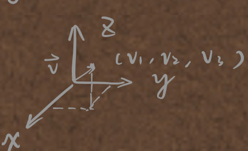
One such metric $d(x, y) = |x - y|$ where $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$\mathbb{C}$: $|z| = |x + iy| = \sqrt{x^2 + y^2}$



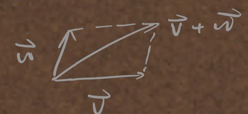
$\mathbb{R}^n$ $\quad +: \mathbb{R}^2 \to \mathbb{R} \quad +(x, y) = x + y$ $\quad$ e.g. $\mathbb{R}^3$: $\vec{v} = (v_1, v_2, v_3)$

$\cdot: \mathbb{R}^2 \to \mathbb{R} \quad \cdot(x, y) = xy$



$\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$

$c\vec{v} = (cv_1, cv_2, cv_3)$



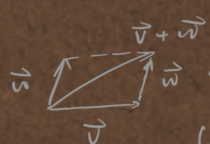
norm $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

i) $\|\vec{v}\| \geq 0$

ii) $\|\vec{v}\| = 0 \iff \vec{v} = \vec{0}$

iii) $\|c\vec{v}\| = |c| \, \|\vec{v}\|$

iv) $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$



Triangle Inequality

(equality $\iff \vec{v} = c\vec{w}$ for some $c \geq 0$)

$\mathbb{C}^n$ $\quad \vec{z} = (z_1, \ldots, z_n) \quad z_j \in \mathbb{C}$

$\|\vec{z}\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ $\quad \lambda \in \mathbb{C}, \ \lambda\vec{z} = (\lambda z_1, \ldots, \lambda z_n)$

$\mathbb{R}^n$ $\quad d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| = \|(x_1 - y_1, \ldots, x_n - y_n)\|$

$\mathbb{R}^n$: $d_2(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\| = \|(x_1-y_1,\dots,x_n-y_n)\|$

$\vec{x} = (x_1,\dots,x_n)$ $\qquad = [(x_1-y_1)^2+\dots+(x_n-y_n)^2]^{1/2}$

$\vec{y} = (y_1,\dots,y_n)$

$\|\vec{x}\| = d(\vec{x},\vec{0})$

Def. of norm isn't unique!

$\mathbb{R}^n$: $\vec{x} = (x_1,\dots,x_n)$

$\|\vec{x}\|_1 = |x_1|+\dots+|x_n|$ (1-norm) satisfies 4 properties

measures the distance along the axises

$d_1(\vec{x},\vec{y}) = |x_1-y_1|+\dots+|x_n-y_n|$

$\mathbb{R}^n$: $\vec{x} = (x_1,\dots,x_n)$ (max-norm)

$\|\vec{x}\| = \max\{|x_j|\}$ measures the longest distance along the axises

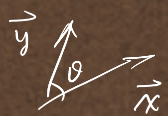
$d(x,y) = \max\{|x_j-y_j|\}$

<u>Later</u> $\|f\| = [\int_I |f|^2]^{1/2}$ $\quad \|f\| = \int_I |f|$ $\quad \|f\| = \max_I |f|$

A variety of measure functions!

<u>Dot</u> / inner product of vectors : $\vec{x},\vec{y} \in \mathbb{R}^n$

$\vec{x}\cdot\vec{y} = \langle\vec{x},\vec{y}\rangle = x_1y_1+\dots+x_ny_n$



$\vec{x}\cdot\vec{y} = \|\vec{x}\|\cdot\|\vec{y}\|\cos\theta$

i) $\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y}+\vec{x}\cdot\vec{z}$

$\vec{x}\perp\vec{y} \iff \vec{x}\cdot\vec{y} = 0$

ii) $\vec{x}\cdot(c\vec{y}) = c(\vec{x}\cdot\vec{y})$

Cauchy Schwartz Ineq. $|\vec{x}\cdot\vec{y}| \le \|\vec{x}\|_2\|\vec{y}\|_2$

$\vec{x}\cdot\vec{x} = x_1^2+\dots+x_n^2 = \|\vec{x}\|_2^2$ $\qquad \langle\vec{x},\vec{x}\rangle = \|\vec{x}\|_2^2$

Trade-off of simpler norms is losing geometry!

<u>Claim</u> If $\vec{x} = (x_1,\dots,x_n), \vec{y} = (y_1,\dots,y_n) \in \mathbb{R}^n$ true for $\mathbb{C}^n$ as well

then $|\vec{x}\cdot\vec{y}| \le \|\vec{x}\|\cdot\|\vec{y}\|$ equiv. $(x_1y_1+\dots+x_ny_n)^2 \le (x_1^2+\dots+x_n^2)(y_1^2+\dots+y_n^2)$

<u>Consider</u> $0 \le (\lambda x_1+y_1)^2+\dots+(\lambda x_n+y_n)^2 \quad \forall \lambda\in\mathbb{R}$

$$0 \le \lambda^2\underbrace{(x_1^2+\dots+x_n^2)}_{a} + 2\lambda\underbrace{(x_1y_1+\dots+x_ny_n)}_{b} + \underbrace{(y_1^2+\dots+y_n^2)}_{c}$$

$0 \le p(\lambda)$ where p is poly. degree 2

$\Rightarrow$ p doesn't have 2 distinct real roots (Discriminent of p is not positive)

$b^2-4ac = (2\vec{x}\cdot\vec{y})^2 - 4\|\vec{x}\|^2\|\vec{y}\|^2 \le 0$

$4\left(\vec{x}\cdot\vec{y} - \|\vec{x}\|^2\|\vec{y}\|^2\right) \le 0 \implies \vec{x}\cdot\vec{y} \le \|\vec{x}\|^2\|\vec{y}\|^2$

If $\mathbb{C}^n$, $\vec{z}=(z_1,\dots,z_n)$ $\vec{w}=(w_1,\dots,w_n)\in\mathbb{C}^n$

$\vec{z}\cdot\vec{w} = z_1\overline{w_1}+\dots+z_n\overline{w_n}$ (inner product on $\mathbb{C}^n$)

$\vec{z}\cdot\vec{z} = z_1\overline{z_1}+\dots+z_n\overline{z_n} = |z_1|^2+\dots+|z_n|^2 = \|\vec{z}\|^2$ others remain the same

($\vec{z}\cdot\vec{w} = \overline{\vec{w}\cdot\vec{z}}$ $\quad \langle f,g\rangle = \int f(x)\overline{g(x)}\,dx$)

Claim If $x^2=2$ and $x\in\mathbb{R}$, then $x\notin\mathbb{Q}$

Proof Suppose $x\in\mathbb{Q}$. Assume $x=\frac{p}{q}$, $p\in\mathbb{Z}$, $q\in\mathbb{N}$, $(p,q)=1$ (relatively prime)

$\left(\frac{p}{q}\right)^2=2$ $\quad p^2=2q^2 \Rightarrow p^2$ is even $\Rightarrow p$ even

Set $p=2m$, $m\in\mathbb{Z}$. Then $(2m)^2=2q^2$ $\quad q^2=2m^2 \Rightarrow q$ even

$(p,q)\ge 2$. contradiction! So $x\notin\mathbb{Q}$

1.17 Definition An *ordered field* is a *field* F which is also an *ordered set*, such that

(i) $x+y<x+z$ if $x,y,z\in F$ and $y<z$,
(ii) $xy>0$ if $x\in F$, $y\in F$, $x>0$, and $y>0$.

If $x>0$, we call x *positive*; if $x<0$, x is *negative*.

For example, Q is an ordered field.

All the familiar rules for working with inequalities apply in every ordered field: Multiplication by positive [negative] quantities preserves [reverses] inequalities, no square is negative, etc. The following proposition lists some of these.

1.8 Definition Suppose S is an ordered set, $E\subset S$, and E is bounded above. Suppose there exists an $\alpha\in S$ with the following properties:

(i) α is an upper bound of E.
(ii) If $\gamma<\alpha$ then γ is not an upper bound of E.

Then α is called the *least upper bound of E* [that there is at most one such α is clear from (ii)] or the *supremum of E*, and we write

$$\alpha = \sup E.$$

The *greatest lower bound*, or *infimum*, of a set E which is bounded below is defined in the same manner: The statement

$$\alpha = \inf E$$

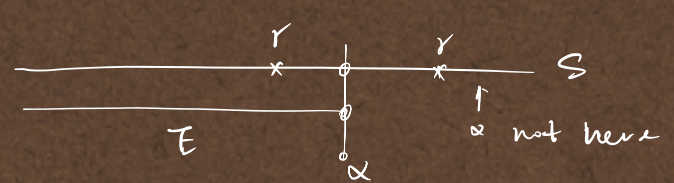
means that α is a lower bound of E and that no β with $\beta>\alpha$ is a lower bound of E.

Suppose S is an ordered set, $E\subseteq S$. Suppose E is bounded above

i.e. $\exists\ \alpha\in S$ s.t. if $x\in E$, then $x\le\alpha$

If $\exists\ \alpha\in S$ satisfies (i) α is an upper bound

(ii) If $r\in S$, with $r<\alpha$, then r isn't an upper bound



Then α is the least upper bound of E

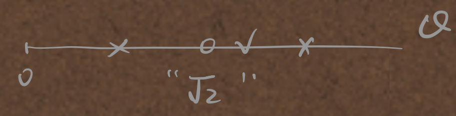
An ordered set S has least upper bound property if

$\forall\ E\subseteq S$ with E bounded above, then $\exists \ge 1$ upper bound of E in S (unique by def)

L.U.B. $(E)=\sup(E)$

e.g. $S = \mathbb{Q}$, then $S(\mathbb{Q})$ doesn't have L.U.B. property

$E = \{ x \in \mathbb{Q} \mid 0 < x, x^2 < 2 \}$ E bounded above



let α be some upper bound of E,

Show α is <u>not</u> a least upper bound

W.T.S. $\exists\, n < \alpha$, n is also an upper bound of E

Take some $\alpha - \frac{1}{q}$ with q sufficiently large $(\alpha - \frac{1}{q}) \in \mathbb{Q}$ ...