Formalization of Ricci Flow in Lean 4

RicciFlow Project

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Chapter 1

Introduction

This project aims to formalize the theory of Ricci Flow in Lean 4, building upon the Mathlib library. The Ricci Flow is a powerful tool in differential geometry, introduced by Richard Hamilton and famously used by Grigori Perelman in his proof of the Poincaré conjecture.

The formalization is organized into several modules, each handling different aspects of the theory:

- Basic definitions and topological foundations
- Riemannian manifolds and metric tensors
- Ricci curvature tensors
- The Ricci Flow equation and existence theorems

Chapter 2

Basic Definitions

Definition 2.1 (Manifold Type)**.** `RicciFlow.Basic` ✓ *We work with a type M equipped with a topological space structure and a charted space structure modeled on $\mathbb{R}$.*

Lemma 2.2 (Positive multiplication)**.** `RicciFlow.pos_mul_pos` ✓ *The product of two positive real numbers is positive.*

Lemma 2.3 (Square positivity)**.** `RicciFlow.square_pos_of_ne_zero` ✓ *The square of a nonzero real number is strictly positive.*

Lemma 2.4 (Existence of positive real)**.** `RicciFlow.exists_pos_real` ✓ *There exists a positive real number.*

Lemma 2.5 (Inverse positivity)**.** `RicciFlow.inv_pos_of_pos` ✓ *The inverse of a positive real number is positive.*

Lemma 2.6 (Continuous at characterization)**.** `RicciFlow.continuousAt_iff_continuousWithinAt` ✓ *A function is continuous at a point if and only if it is continuous within the universal set at that point.*

Chapter 3

Riemannian Manifolds

Definition 3.1 (Riemannian Metric)**.** `RicciFlow.RiemannianMetric` *A Riemannian metric on a manifold M is a smooth, symmetric, positive-definite $(0,2)$-tensor field.*

In the current simplified implementation, we use type parameters M (the manifold) and V (representing the abstract tangent space). The metric at each point $x \in M$ gives a bilinear form $g_x : V \times V \to \mathbb{R}$ satisfying:

1. ***Symmetry**: $g_x(v, w) = g_x(w, v)$ for all $v, w \in V$*
2. ***Positive-definiteness**: $g_x(v, v) > 0$ for all $v \neq 0$*

Remark 3.2. *The current implementation provides the structure definition with type parameter V for the tangent space. Future phases will:*

- *Phase 2: Use dependent types for concrete tangent spaces*
- *Phase 3: Implement as smooth tensor fields* $\mathrm{SmoothSection}(\mathrm{Sym}^2 T^* M)$

Definition 3.3 (Inner Product)**.** `RicciFlow.innerProduct` *The inner product $\langle v, w \rangle_x$ at point $x \in M$ is given by the metric: $\langle v, w \rangle_x = g_x(v, w)$.*

Definition 3.4 (Norm squared)**.** `RicciFlow.normSq` *The squared norm of a vector v at point x is $\|v\|_x^2 = g_x(v, v) = \langle v, v \rangle_x$.*

Lemma 3.5 (Inner product symmetry)**.** `RicciFlow.innerProduct_symm` ✓ *The inner product is symmetric: $\langle v, w \rangle_x = \langle w, v \rangle_x$.*

Lemma 3.6 (Norm positivity)**.** `RicciFlow.normSq_pos` ✓ *For any nonzero vector $v \in V$, we have $\|v\|_x^2 > 0$.*

Chapter 4

Ricci Curvature

Definition 4.1 (Ricci Tensor)**.** *RicciFlow.RicciTensor The Ricci curvature tensor* Ric *is obtained by contracting the Riemann curvature tensor:*

$$\mathrm{Ric}_{ij} = R^k_{ikj}$$

where R^l_{ijk} are the components of the Riemann curvature tensor.

Remark 4.2. *The current simplified implementation stores only the trace value (scalar curvature). A complete implementation should store all components* Ric_{ij} *of the Ricci tensor.*

Definition 4.3 (Scalar Curvature)**.** *RicciFlow.scalarCurvature The scalar curvature R is the trace of the Ricci tensor with respect to the metric:*

$$R = g^{ij}\,\mathrm{Ric}_{ij}$$

where g^{ij} denotes the inverse metric tensor.

Geometric interpretation*:*

- $R > 0$*: The manifold curves inward (like a sphere)*
- $R < 0$*: The manifold curves outward (like hyperbolic space)*
- $R = 0$*: The manifold is locally flat*

Lemma 4.4 (Scalar curvature extraction)**.** *RicciFlow.scalarCurvature_eq_traceValue*
✓ *In the current implementation, the scalar curvature equals the stored trace value.*

Chapter 5

Ricci Flow

Definition 5.1 (Ricci Flow Equation). *RicciFlow.ricci_flow_equation The Ricci Flow is the geometric evolution equation:*

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g)$$

where $g(t)$ is a time-dependent family of Riemannian metrics and $\mathrm{Ric}(g)$ *is the Ricci curvature tensor of g.*

Theorem 5.2 (Short-Time Existence). *RicciFlow.short_time_existence Let (M, g_0) be a compact Riemannian manifold. Then there exists a time $T > 0$ and a smooth family of metrics $g(t)$ for $t \in [0, T)$ such that:*

1. *$g(0) = g_0$ (initial condition)*
2. *$g(t)$ satisfies the Ricci Flow equation for all $t \in (0, T)$*

Proof. This is a deep result in geometric analysis, relying on the theory of parabolic PDEs. The proof involves:

- Showing that the Ricci Flow equation is a weakly parabolic system
- Applying short-time existence theory for parabolic systems
- Establishing appropriate estimates to ensure smoothness

In the Lean formalization, this theorem is currently stated with `sorry`. A complete formalization would require developing substantial PDE theory in Lean. □

Chapter 6

Future Directions

6.1 Immediate Goals

- Complete the definition of Riemannian metrics with smoothness and positive-definiteness conditions
- Implement the Ricci tensor using Mathlib's connection and curvature theory
- Prove basic properties of scalar curvature

6.2 Long-Term Goals

- Formalize Hamilton's maximum principle for the Ricci Flow
- Prove convergence results for special cases (e.g., spheres)
- Develop the theory of Ricci solitons