Minimal Surface and Its Application to 3-manifolds II

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Toronto Geometric Analysis Seminar

Some remark for last time

Stability operator and stable minimal surface

Assume $\Sigma^{n-1} \subset M^n$ is minimal. Let F be a normal variation, $F_t^T \equiv 0$. **The second variation** of Σ is defined by

$$\frac{d^2}{dt^2}\bigg|_{t=0} Vol(F(\Sigma, t)) = -\int_\Sigma \langle\, F_t, \mathcal{L}F_t \rangle, \tag{1}$$

where $\mathcal{L}$ is called stability or Jacobi operator.

Assume Σ has trivial normal bundle, then for a normal vector field $X = \nu N$,

$$\mathcal{L}\nu := \Delta_\Sigma \nu + |A|^2 \nu + Ric_M(N, N)\nu \tag{2}$$

A minimal surface Σ is called **stable** if

$$\frac{d^2}{dt^2}\bigg|_{t=0} Vol(F(\Sigma, t)) = -\int_\Sigma \langle\, F_t, \mathcal{L}F_t \rangle \geq 0 \tag{3}$$

Morse index

We call λ an eigenvalue of $\mathcal{L}$ if

$$\mathcal{L}X + \lambda X = 0 \tag{4}$$

The **Morse index** is defined to be the number of negative eigenvalues of $\mathcal{L}$.

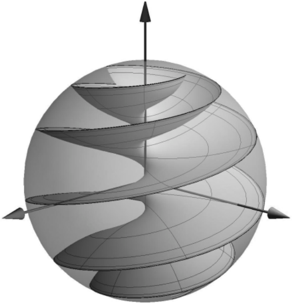
An index 1 minimal surface and an index 5 minimal surface:

Theorem, Urbano '90

Let S be a compact orientable nontotally geodesic minimal surface in S^3. Then $\mathrm{ind}(M) \geq 5$, and the equality holds iff M is a Clifford torus.

Examples of stable minimal surfaces

2-torus and hyperbolic helicoid (Biao Wang '19)



Area minimizing surfaces in a homotopy class or homology class are stable.

Complete stable surface in Euclidean space

Existence of minimal surface with boundary, non-compact minimal surface like helicoid and catenoid. Stable non-compact min surfaces in $\mathbb{H}^3$.

Carmo-Peng '79, Schoen-Colbrie '80 (Bernstein 1917)

A stable complete minimal surface in $\mathbb{R}^3$ is a plane.

From ambient to submanifold and vice versa

Suppose $\Sigma^{n-1} \subset M^n$ is a closed stable minimal hypersurface with trivial normal bundle.

1. Simons '68 : If $\mathrm{Ric}_M \geq 0$, then Σ is totally geodesic and $\mathrm{Ric}_M(N, N) = 0$ on Σ
2. Schoen-Yau '79: If $\mathrm{Scal}_M > 0$ and $n = 3$, then Σ is an S^2 or an $\mathbb{RP}^2$ and
$$\int_\Sigma \mathrm{Scal}_M + |A|^2 \leq 8\pi \tag{5}$$
Proof:

Gao-Yau '86: Every closed M^3 admits a metric with negative Ricci.

Classical and geometric Dehn's Lemma

Dehn's lemma: If a Jordan curve on ∂M^3 compact is homotopically trivial, then it bounds an **embedded** disk.
Announced by Dehn 1910, a gap found Kneser '29, fixed by Papakyriakopoulos '57, simplified by Whitehead and Shapiro '57.

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Geometric Dehn's Lemma, GDL (Meeks-Yau, '82):
M^3 compact with ∂M convex. If Γ is a scc in ∂M homotopically trivial in M, then:

1. $\exists f : D \to M$ of least-energy such that $f|_{\partial D}$ parameterizes Γ.
2. f above is $1-1$ and smooth immersion in D°.
3. f is as regular as Γ along ∂D. Γ $C^2 \implies f$ an immersion.
4. If f_1 and f_2 are two such solutions and $f_1(D^{\circ}) \cap f_2(D^{\circ}) \neq \emptyset$, then $f_2 = f_1$ up to conformal diffeomorphism reparameterization.

Cut and paste for Statement 4

Suppose that D_1, D_2 are two least-area embedded disks in M with $\partial D_1 = \partial D_2 = \Gamma$, which intersect transversely at some interior point of the disks.

Outline for Statement 1 and 3

1. Extend M to a homogeneously regular $\widetilde{M}$ without boundary.

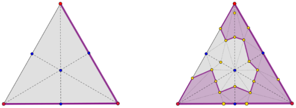
2. Use Morrey '48 to get a least-energy map f.
3. Use Osserman '70 and Gulliver '73 to conclude $f|_{\mathrm{int}(D)}$ is an immersion. Use Lewy '51 and Hildebrandt '69 to conclude the boundary regularity.

Proof of Statement 2

Let M^3 be compact analytic. Suppose D is the closed unit disk in the plane and γ is an analytic curve on ∂M and $f : D \to M$ is a least-area (energy) map with $f(\partial D) = f(D) \cap \partial M = \gamma$. Then f is $1-1$.

1. f is an analytic immersion (Last slide + Morrey '48 for analyticity)
2. f is simplicial w.r.t. fixed triangulations of D and M (by Lojasiewicz '64).
3. two distinct analytic embedded disks (least-area w.r.t. boundary) have no interior intersections.

4. by barycentric subdivision, can assume the simplicial nbhd of $f(D)$ is a regular nbhd.



Refined tower construction using minimal surface

Construction of a tower for $f : D \to M$ to simplify the self-intersection for f:
N_1 a simplicial regular nbhd of $f(D)$. Restrict the range to get $f_1 : D \to N_1$. If N_1 is not simply connected, take universal cover and lift to get f_2.

A combinatorial upper bound on complexity

T is the collection of open simplices and vertices. Let $c = \# T \times T$. Terminates $\leq c$ steps.

Lemma

If $S(f_i) = \{(\sigma, \tau) \in T \times T | \sigma \neq \tau, f(\sigma) = f(\tau)\}$, then $S(f_{i+1})$ is a proper subset of $S(f_i)$. Hence the tower construction terminates at finite steps.

Mostow Rigidity and Effective Geometrization

When M is finite-volume hyperbolic manifold with dim≥ 3, its geometry is determined by its topology.

Norms on the Cohomology

Measuring $H^1(M) \cong H_2(M)$
Thurston norm, L^2-norm, least area norm, L^1-norm, L^∞-norm.

Thurston Norm

For a compact irreducible 3-manifold M, the Thurston norm of $\phi \in H^1(M;\mathbb{Z}) \cong H_2(M;\mathbb{Z})$ is defined by

$$\chi_-(S) = \max\{0, -\chi(S)\}$$
$$\|\phi\|_{Th} = \min\{\chi_-(S) | \text{S is a properly embedded surface dual to } \phi\}$$

When M is closed hyperbolic, it is non-degenerate and a genuine norm.

L^2 and L^1 Norm

$$|\alpha|_{L^2}^2 := \int_M |\alpha(x)|^2 d\mathrm{vol}_g(x) = \int_M \alpha \wedge *\alpha$$

For $\phi \in H^1$, we have $\|\phi\|_{L^2} = \inf\{|\alpha|_{L^2} | \alpha \text{ represents } \phi\}$

$$|\alpha|_{L^1} := \int_M |\alpha(x)| d\mathrm{vol}_g(x)$$

Least Area Norm

For $\phi \in H^1$, let $\mathcal{F}_\phi$ be the collection of smooth maps $f : S \to M$ where S is a closed oriented surface with $f_*([S])$ dual to ϕ. The least area norm of ϕ is

$$\|\phi\|_{LA} = \inf\left\{\mathrm{Area}(f(S)) | f \in \mathcal{F}_\phi\right\}$$

Theorem (Bergeron-Sengün-Venkatesh 2015)

If M is a finite of a fixed closed orientable hyperbolic 3-manifold M_0, then we have

$$\frac{C_1}{\mathrm{vol}(M)}\|\cdot\|_{Th} \leq \|\cdot\|_{L^2} \leq C_2\|\cdot\|_{Th} \text{ on } H^1(M;\mathbb{R})$$

where C_1 and C_2 depend only on M_0. It generalizes Kronheimer-Mrowka.

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Theorem (Brock-Dunfield 2017), (Lin 2017)

For all closed orientable hyperbolic 3-manifolds M one has :

$$\frac{\pi}{\sqrt{\mathrm{vol}(M)}}\|\cdot\|_{Th} \leq \|\cdot\|_{L^2} \leq \frac{10\pi}{\sqrt{\mathrm{inj}}}\|\cdot\|_{Th}, \text{ on } H^1(M;\mathbb{R})$$

(Lin 2017) uses the technique of Kronheimer-Mrowka to prove the LHS.

Theorem (Stern 2019, Bray-Stern 2019)

Generalize the LHS to 3-manifolds with boundary and reducible cases.

A Glance at the Noncompact Case

Finite-volume case also shares the rigidity and it is natural to wonder.

inj$= 0$

One immediate challenge: injectivity radius of M is zero. We have to modify the inequality so that it can produce useful bounds on the 1-form.

Thurston norm degenerate $H_2(M)$

The boundary-parallel torus has Thurston norm 0 and area goes to 0.

Solution is Surprisingly Simple!

Only need to ask for L^2

The L^2 condition works extraordinarily well with Thurston norm and minimal surface theory. Denote $\mathcal{H}^1$ the space of L^2 harmonic 1-form. [Zucker, 1982 and Mazzeo-Phillips 1990] showed that
$\mathcal{H}^1 \cong \mathrm{Im}(H_0^1(M) \to H^1(M)) \cong \mathrm{Im}(H_2(M) \to H_2(M, \partial M))$.

Lemma (H. '20):

1. The Thurston norm is a genuine norm on $\mathcal{H}^1$.
2. The following estimates hold on $\mathcal{H}^1$:

$$\pi\|\cdot\|_{Th} \leq \|\cdot\|_{LA} \leq 2\pi\|\cdot\|_{Th}$$

Existence of LA surface established by Huang-Wang, Collin-Hauswirth-Mazet-Rosenberg is crucial. Apply arguments in Hass or Collin-Hauswirth-Mazet-Rosenberg for the estimates.

Compact Subdomain Suffices for the Inequality

Theorem (H. '20):

For all finite-volume orientable hyperbolic 3-manifolds M one has

$$\frac{\pi}{\sqrt{\mathrm{vol}(M)}}\|\cdot\|_{Th} \leq \|\cdot\|_{L^2} \leq \max\{\frac{10\pi}{\sqrt{\mathrm{inj}(M_\tau)}}, \frac{8\pi}{\mathrm{inj}(M_\tau)}\}\|\cdot\|_{Th} \text{ on } \mathcal{H}^1 \quad (6)$$

τ is a constant explicitly computatble from M.

Proof of Upper Bound

Counting duplicates when M is lifted:

Poincaré duality in closed vs. non-compact

Proof of Upper Bound

For the closed case: Fix a surface S dual to α, and α is the harmonic representative of ϕ. Thus we have

$$\begin{aligned}
\|\alpha\|_{L^2}^2 &= \int_M *\alpha \wedge \alpha = \int_S *\alpha \\
&\leq \int_S |*\alpha| dA \leq \int_S \|\alpha\|_{L^\infty} dA \\
&\leq \|\alpha\|_{L^\infty} Area(S) \leq 2\pi \|\alpha\|_{L^\infty} \|\alpha\|_{Th}
\end{aligned}$$

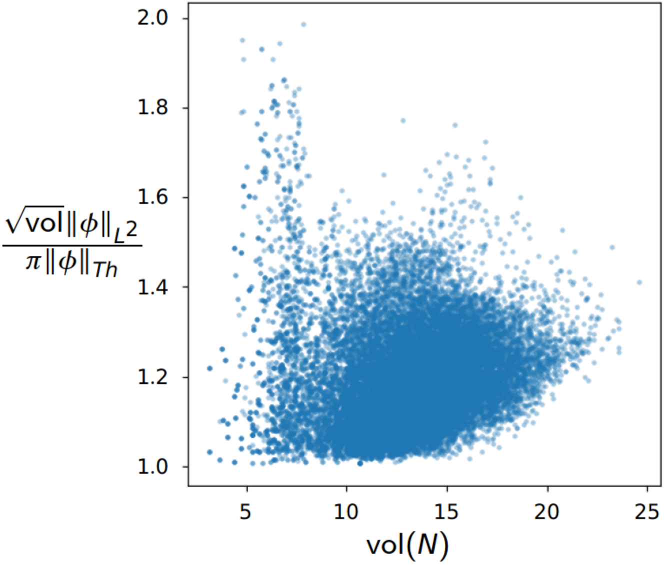

Figure 1: Data from Dunfield-Hirani

Sharpness for M closed

Can it achieve 1?

Proposition (H. '20):

The LHS of the inequality in Brock-Dunfield will never be realized:

$$\frac{\pi}{\sqrt{\mathrm{vol}(M)}}\|\cdot\|_{Th} < \|\cdot\|_{L^2} \tag{7}$$

Proof of non-sharpness for Brock-Dunfield

First Proof by Bochner technique (Inspired by Stern)

Let α be a harmonic 1-form with constant length c, By Bochner formula,

$$|\nabla\alpha|^2 = 2|\alpha|^2 = 2c^2$$

The second fundamental form on S is

$$\sigma_S = \frac{\nabla\alpha}{|\alpha|}\Big|_S$$

which implies that

$$|\sigma_S|^2 = 2$$

Contradicts the property of holomorphic quadratic differential.

Second proof based on Wolf-Wu

Wolf-Wu: Geometric folitation

Locally geometric 1-parameter family of closed minimal surface: if $\exists$ a closed surface S, $\epsilon > 0$, an embedding:

$$h : (-\epsilon, \epsilon) \times S \to M \tag{8}$$

$\forall p \in S$, $f(t,p) := \langle (h_t)_*(\partial_t), \nu \rangle|_{t=0}$ only depends on the principal curvature of S at p.

Geometric foliation and harmonic forms of constant length

1. Harmonic form with constant length gives a foliation by minimal surface (fibered over S^1) with orthogonal geodesic flow. Particular case of Wolf-Wu.
2. Theorem (Zeghib, '83): There is no smooth vector fields on a closed hyperbolic 3-manifold where all flow lines are geodesic.

Non-sharpness in the non-compact case is easier to prove, thanks to Mazzeo-Phillips, on asymptotic of harmonic forms near infinity.

A conjecture from Brock-Dunfield

Let M_j be a sequence of orientable closed hyperbolic 3-manifolds, converging geometrically to M. Then the

$$\sup_{H^1(M_j)} \frac{\|\cdot\|_{L^2}}{\|\cdot\|_{Th}} \sim \mathcal{O}\left(\sqrt{-\log(\mathrm{inj}(M_j))}\right) \tag{9}$$

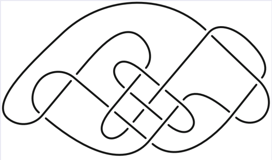

Figure 2: L14n21792

M_j is obtained from M by particular Dehn fillings.

Natural questions: one cusp? More than two cusps?

Theorem, H' 21

Let M_j be a sequence of orientable closed hyperbolic 3-manifolds, converging geometrically to M with one cusp. Then for j large enough,

$$\sup_{H^1(M_j)} \frac{\|\cdot\|_{L^2}}{\|\cdot\|_{Th}} < C \tag{10}$$

where C depends on M.

For n cusps, this is generic. Geometric convergence vs. Dehn fillings:

Outline of proof (for large j)

1. Use Hatcher '82 to conclude $H_2(M_j)$ all come from the closed non-peripheral surfaces from M, Im. Use Agol '01 to control the topological complexity;

2. Use barrier surface arguments to argue no deep disk, as in Wang '12, Huang-Wang '18, Hass '15, Mazet-Rosenberg '20. Use k-bi-Lipshitz diffeomorphism between thick part of M_j and M to argue no deep annulus, by the work of Futer-Purcell-Schleimer '19 (based on Bromberg-Brock 04, Hodgson-Kerckhoff '02) and Huang-Wang '17;

3. Modify Brock-Dunfield, replace inj(M_j) by $\epsilon(M)$.

Thanks to the organizers and the audiences!