

ON THE EQUIDISTRIBUTION OF CLOSED GEODESICS AND GEODESIC NETS

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ABSTRACT. We show that given a closed n -manifold M , for a generic set of Riemannian metrics g on M there exists a sequence of closed geodesics that are equidistributed in M if $n = 2$; and an equidistributed sequence of embedded stationary geodesic nets if $n = 3$. One of the main tools that we use is the Weyl Law for the volume spectrum for 1-cycles, proved in [13] for $n = 2$ and in [10] for $n = 3$. We show that our proof of the equidistribution of stationary geodesic nets can be generalized for any dimension $n \geq 2$ provided the Weyl Law for 1-cycles in n -manifolds holds.

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1. INTRODUCTION

Marques, Neves and Song proved in [16] that for a generic set of Riemannian metrics in a closed manifold M^n , $3 \leq n \leq 7$ there exists a sequence of closed, embedded, connected minimal hypersurfaces which is equidistributed in M . In this paper, we study the equidistribution of closed geodesics and stationary geodesic nets (which are 1-dimensional analogs of minimal hypersurfaces) on a Riemannian manifold (M^n, g) , $n \geq 2$. We prove the following two results, for dimensions 2 and 3 of the ambient manifold respectively:

Theorem 1.1. *Let M be a closed 2-manifold. For a generic set of C^∞ Riemannian metrics g on M , there exists a set of closed geodesics that is equidistributed in M . Specifically, for every g in the generic set, there exists a sequence $\{\gamma_i : S^1 \rightarrow M\}$ of*

closed geodesics in (M, g) , such that for every C^∞ function $f : M \rightarrow \mathbb{R}$ we have

$$\lim_{k \rightarrow \infty} \frac{\sum_{i=1}^k \int_{\gamma_i} f \, dL_g}{\sum_{i=1}^k L_g(\gamma_i)} = \frac{\int_M f \, d\text{Vol}_g}{\text{Vol}(M, g)}$$

Theorem 1.2. *Let M be a closed 3-manifold. For a generic set of C^∞ Riemannian metrics g on M , there exists a set of connected embedded stationary geodesic nets that is equidistributed in M . Specifically, for every g in the generic set, there exists a sequence $\{\gamma_i : \Gamma_i \rightarrow M\}$ of connected embedded stationary geodesic nets in (M, g) , such that for every C^∞ function $f : M \rightarrow \mathbb{R}$ we have*

$$\lim_{k \rightarrow \infty} \frac{\sum_{i=1}^k \int_{\gamma_i} f \, dL_g}{\sum_{i=1}^k L_g(\gamma_i)} = \frac{\int_M f \, d\text{Vol}_g}{\text{Vol}(M, g)}$$

Remark 1.3. We have an equivalent notion of equidistribution for a sequence of closed geodesics or geodesic nets: we say that $\{\gamma_i\}_{i \in \mathbb{N}}$ is equidistributed in (M, g) if for every open subset $U \subseteq M$ it holds

$$\lim_{k \rightarrow \infty} \frac{\sum_{i=1}^k L_g(\gamma_i \cap U)}{\sum_{i=1}^k L_g(\gamma_i)} = \frac{\text{Vol}_g U}{\text{Vol}_g M}$$

Theorem 1.2 is, as far as the authors know, the first result on equidistribution of k -stationary varifolds in Riemannian n -manifolds for $k < n - 1$ (i.e. in codimension greater than 1). Regarding Theorem 1.1, similar equidistribution results for closed geodesics have been proved for compact hyperbolic manifolds in [3] in 1972 and for compact surfaces with constant negative curvature in [20] in 1985. More recently, those results were extended to non-compact manifolds with negative curvature in [21] and to surfaces without conjugate points in [7]. The four previous works have in common that they approach the problem from the dynamical systems point of view. In the present paper, we approach it using Almgren-Pitts min-max theory (as it was done in [16] for minimal hypersurfaces). Additionally, Theorem 1.1 is the first equidistribution result for closed geodesics on closed surfaces that is proved for generic metrics, without any restriction regarding the curvature of the metric or the presence of conjugate points.

Our proof is inspired by the ideas in [16]. There are two key results used in [16] to prove equidistribution of minimal hypersurfaces for generic metrics: the Bumpy Metrics Theorem of Brian White [23] and the Weyl Law for the Volume Spectrum proved by Liokumovich, Marques and Neves in [13]: given a compact Riemannian manifold (M^n, g) with $n \geq 2$ (possibly with boundary), we have

$$\lim_{p \rightarrow \infty} \omega_p^{n-1}(M, g) p^{-\frac{1}{n}} = \alpha(n) \text{Vol}(M, g)^{\frac{n-1}{n}}$$

for some constant $\alpha(n) > 0$. Here, given $1 \leq k \leq n - 1$ we denote by $\omega_p^k(M, g)$ the k -dimensional p -width of M with respect to the metric g (for background on this, see [9], [14], [15] [12]). It was conjectured by Gromov (see [8, section 8.4]) that the Weyl law can be extended to other dimensions and codimensions. In this work, we are interested in the case of 1-dimensional cycles. The following is the most general version of the Weyl law we could expect for 1-cycles.

Conjecture 1.4. *Let (M^n, g) be a closed n -dimensional manifold, $n \geq 2$. Then there exists a constant $\alpha(n, 1) > 0$ such that*

$$\lim_{p \rightarrow \infty} \omega_p^1(M^n, g) p^{-\frac{n-1}{n}} = \alpha(n, 1) \text{Vol}(M^n, g)^{\frac{1}{n}}$$

So far, Conjecture 1.4 has been proved for $n = 2$ as a particular case of [13] and recently for $n = 3$ by Guth and Liokumovich in their work [10]. In this article, we use those two versions of the Weyl law to prove Theorem 1.1 and Theorem 1.2; and we also use the Structure Theorem for Stationary Geodesic Networks proved by Staffa in [22] and the Structure Theorem of White ([23]) for the case of embedded closed geodesics. The work of Chodosh and Mantoulidis in [6] is used to upgrade the equidistribution result for stationary geodesic networks to closed geodesics in dimension 2. The only obstruction to extend our proof of the equidistribution of stationary geodesic nets to arbitrary dimensions of the ambient manifold M is that Conjecture 1.4 has not been proved yet if $n > 3$. As all the rest of our argument works for any dimension $n > 3$, what we do is to prove the following result and then show that it implies Theorem 1.1 and Theorem 1.2.

Theorem 1.5. *Let M^n , $n \geq 2$ be a closed manifold. Assume that the Weyl law for 1-cycles in n -manifolds holds. Then for a generic set of C^∞ Riemannian metrics g on M , there exists a set of connected embedded stationary geodesic nets that is equidistributed in M . Specifically, for every g in the generic set, there exists a sequence $\{\gamma_i : \Gamma_i \rightarrow M\}$ of connected embedded stationary geodesic nets in (M, g) , such that for every C^∞ function $f : M \rightarrow \mathbb{R}$ we have*

$$\lim_{k \rightarrow \infty} \frac{\sum_{i=1}^k \int_{\gamma_i} f \, dL_g}{\sum_{i=1}^k L_g(\gamma_i)} = \frac{\int_M f \, d\text{Vol}_g}{\text{Vol}(M, g)}$$

In order to simplify the exposition, we consider integrals of C^∞ functions instead of the more general traces of 2-tensors discussed in [16]. Next we proceed to describe the intuition behind the proof, the technical issues which appear when one tries to carry on that intuition and how to sort them.

Let g be a Riemannian metric on M . We want to do a very small perturbation of g to obtain a new metric \hat{g} which admits a sequence of equidistributed stationary geodesic networks. Let $f : M \rightarrow \mathbb{R}$ be a smooth function. Consider a conformal

perturbation $\hat{g} : (-1, 1) \rightarrow \mathcal{M}^\infty$ defined as

$$\hat{g}(t) = e^{2tf} g$$

By [14, Lemma 3.4] the normalized p -widths $t \mapsto p^{-\frac{n-1}{n}} \omega_p^1(M, \hat{g}(t))$ are uniformly locally Lipschitz. This combined with the Weyl Law (recall that we assume it holds) implies that the sequence of functions $h_p : (-1, 1) \rightarrow \mathbb{R}$

$$h_p(t) = \frac{p^{-\frac{n-1}{n}} \omega_p^1(M, \hat{g}(t))}{\text{Vol}(M, \hat{g}(t))^{\frac{1}{n}}}$$

converges uniformly to the constant $\alpha(n, 1)$. Considering

$$\tilde{h}_p(t) = \log(h_p(t)) = -\frac{n-1}{n} \log(p) + \log(\omega_p(M, \hat{g}(t))) - \frac{1}{n} \log(\text{Vol}(M, \hat{g}(t)))$$

we have that \tilde{h}_p converges uniformly to the constant $\log(\alpha(n, 1))$. On the other hand, Almgren showed that there is a correspondence between 1-widths and the volumes of stationary varifolds (see [1], [2], [4], [17], [18], [19]) such that for each $p \in \mathbb{N}$ and $t \in (-1, 1)$ there exists a (possibly non unique) stationary geodesic network $\gamma_p(t)$ such that

$$L_{\hat{g}(t)}(\gamma_p(t)) = \omega_p^1(\hat{g}(t))$$

Assume that the $\gamma_p(t)$'s can be chosen so that all of them are parametrized by the same graph Γ and the maps $(-1, 1) \rightarrow \Omega(\Gamma, M)$, $t \mapsto \gamma_p(t)$ are differentiable (this is a very strong assumption and doesn't necessarily hold, as the map $t \mapsto \omega_p^1(\hat{g}(t))$ may not be differentiable; a counterexample is shown below). In that case we can differentiate \tilde{h}_p and obtain

$$\begin{aligned} \frac{d}{dt} \tilde{h}_p(t) &= \frac{1}{\omega_p(M, \hat{g}(t))} \frac{d}{dt} \omega_p^1(M, \hat{g}(t)) - \frac{1}{n \text{Vol}(M, \hat{g}(t))} \frac{d}{dt} \text{Vol}(M, \hat{g}(t)) \\ &= \frac{1}{L_{\hat{g}(t)}(\gamma_p(t))} \int_{\gamma_p(t)} f \, dL_{\hat{g}(t)} - \frac{1}{n \text{Vol}(M, \hat{g}(t))} \int_M n f \, d\text{Vol}_{\hat{g}(t)} \\ &= \int_{\gamma_p(t)} f \, dL_{\hat{g}(t)} - \int_M f \, d\text{Vol}_{\hat{g}(t)} \end{aligned}$$

As $\{\tilde{h}_p\}_p$ converges uniformly to a constant, we could expect that the sequence $\{\tilde{h}'_p(t)\}_p$ converges to 0 for some values of t . If that was the case, the sequence $\{\gamma_p(t)\}_p$ would verify the equidistribution formula for the function f with respect to the metric $\hat{g}(t)$. Nevertheless, this does not have to be true, because of two reasons. The first one is that the uniform convergence of a sequence of functions to a constant does not imply convergence of the derivatives to 0 at any point. Indeed, we can construct a sequence of zigzag functions which converges uniformly to 0 but $\tilde{h}'_p(t)$ does not converge to 0 for any $t \in (-1, 1)$. The second one is that the differentiability

of $t \mapsto \gamma_p(t)$ could fail, a counterexample is shown in the next paragraph. And even if that reasoning was true and such t existed, the sequence $\{\gamma_p(t)\}_p$ constructed would only give an equidistribution formula for the function f (which is used to construct the sequence) instead of for all C^∞ functions at the same time; and with respect to a metric $\hat{g}(t)$ which could also vary with f .

An example when $t \mapsto \omega_p^1(\hat{g}(t))$ is not differentiable is the following. Let us consider a dumbbell metric g on S^2 obtained by constructing a connected sum of two identical round 2-spheres S_1^2 and S_2^2 of radius 1 by a thin neck. Define a 1-parameter family of metrics $\{\hat{g}(t)\}_{t \in (-1,1)}$ such that $\hat{g}(t) = (1+t)^2 g$ along S_1^2 , $\hat{g}(t) = (1-t)^2 g$ along S_2^2 (interpolating along the neck so that it's still very thin). It is clear that for $t \geq 0$, the 1-width is realized by a great circle in S_1^2 with length $1+t$, and for $t \leq 0$ it is realized by a great circle in S_2^2 of length $1-t$. Therefore

$$\omega_1^1(\hat{g}(t)) = \begin{cases} 1-t & \text{if } t \leq 0 \\ 1+t & \text{if } t \geq 0 \end{cases}$$

and hence it is not differentiable at 0.

To fix the previous issue (differentiability of $\omega_p^1(g(t))$), we prove Proposition 3.1 which is a version for stationary geodesic networks of [16, Lemma 2]. Regarding the convergence of $h'_p(t)$ to 0 for certain values of t , we use Lemma 3.4 which is exactly [16, Lemma 3]. To obtain a sequence of stationary geodesic networks that verifies the equidistribution formula for all C^∞ functions (and not only for a particular one as above), we carry on a construction described in Section 4 using certain stationary geodesic nets which realize the p -widths in a similar way of the $\gamma_p(t)$'s above. The key idea here is that the integral of any C^∞ function f over M can be approximated by Riemann sums along small regions with piecewise smooth boundary where f is almost constant. Therefore, if we have an equidistribution formula for the characteristic functions of those regions (or some suitable smooth approximations), then we will be able to deduce it for an arbitrary $f \in C^\infty(M, \mathbb{R})$. The advantage of doing this is that we reduce the problem to a countable family of functions. This argument is also inspired by [16].

The paper is structured as follows. In Section 2, we introduce the set up and necessary preliminaries. In Section 3, the technical propositions necessary to prove Theorem 1.5 are discussed. In Section 4 we prove Theorem 1.5 and get Theorem 1.2 as a corollary using the Weyl law for 1 cycles in 3 manifolds proved in [10]. In Section 5, we use the Weyl law from [13] and the proof of Theorem 1.5 combined with the work of Chodosh and Mantoulidis in [6] (where it is proved that the p -widths on a surface are realized by finite unions of closed geodesics) to prove Theorem 1.1.

Remark 1.6. Rohil Prasad pointed out that an alternative proof of Theorem 1.1 could be obtained using the methods of Irie in [11]. Given a closed Riemannian 2-manifold (M, g) , its unit cotangent bundle U_g^*M is a closed 3-manifold equipped with a natural contact structure induced by the contact form λ_g which is the restriction of the Liouville form λ on T^*M to U_g^*M . It is a well known fact that the Reeb vector field associated to λ_g generates the geodesic flow of (M, g) . Additionally, given a function $f : M \rightarrow \mathbb{R}$, the Riemannian metric $g' = e^f g$ corresponds to the conformal perturbation $\lambda_{g'} = e^{\frac{f \circ \pi}{2}} \lambda_g$ of the contact form in U_g^*M (here $\pi : U_g^*M \rightarrow M$ is the projection map); and both λ_g and $\lambda_{g'}$ are compatible with the same contact structure on U_g^*M . Thus one would like to apply [11, Corollary 1.4] to U_g^*M with the contact structure induced by λ_g . However, that result is about generic perturbations of the contact form of the type $e^{\tilde{f}} \lambda_g$, where $\tilde{f} : U_g^*M \rightarrow \mathbb{R}$ and we only want to consider perturbations $\tilde{f} = f \circ \pi$ which are liftings to U_g^*M of maps $f : M \rightarrow \mathbb{R}$ so some work should be done here in order to apply Irie's result in our setting. This issue was pointed out in [5, Remark 2.3], where a similar problem is studied for Finsler metrics and a solution is given for that class of metrics. Additionally, Irie's theorem would give us an equidistributed sequence for a generic conformal perturbation of each metric g . This immediately implies that for a dense set of Riemannian metrics such an equidistribution result holds, but some additional arguments are needed to prove it for a Baire-generic metric. It is important to point out that the result in [11] uses the ideas of [16] but in the different setting of contact geometry, applying results of Embedded Contact Homology with the purpose of finding closed orbits of the Reeb vector field; while in [16] Almgren-Pitts theory is used to find closed minimal surfaces.

Acknowledgements. We are very grateful to Yevgeny Liokumovich for suggesting this problem and for his valuable guidance and support while we were working on it. We also want to thank Rohil Prasad for his suggestion about using the approach mentioned in Remark 1.6 to get an alternative proof of Theorem 1.1 and for the valuable discussion derived from there. We are thankful to Wenkui Du as well, because of his comments on the preliminary version of this work.

2. PRELIMINARIES

Definition 2.1 (Weighted Multigraph). A weighted multigraph is a graph $\Gamma = (\mathcal{E}, \mathcal{V}, \{n(E)\}_{E \in \mathcal{E}})$ consisting of a set of edges \mathcal{E} , a set of vertices \mathcal{V} and a multiplicity $n(E) \in \mathbb{N}$ assigned to each edge $E \in \mathcal{E}$. A weighted multigraph is good if it is connected and either it is a closed loop with multiplicity or each vertex $v \in \mathcal{V}$ has at least three incoming edges.

Definition 2.2. Given a weighted multigraph $(\mathcal{E}, \mathcal{V}, \{n(E)\}_{E \in \mathcal{E}})$, we identify each edge $E \in \mathcal{E}$ with the interval $[0, 1]$ and we denote $\pi = \pi_E : \{0, 1\} \rightarrow \mathcal{V}$ the map sending $i \in \{0, 1\}$ to the vertex $v \in \mathcal{V}$ under the identification $E \cong [0, 1]$.

Definition 2.3 (Γ -net). A Γ -net f on M is a continuous map $f : \Gamma \rightarrow M$ which is a C^2 -embedding restricted to the edges of Γ . We will denote $\Omega(\Gamma, M)$ the space of Γ -nets on M . We say that two Γ -nets f_1 and f_2 are equivalent if for every edge E of Γ the map $f_1|_E$ is a C^2 reparametrization of $f_2|_E$. This defines an equivalence relation \sim in $\Omega(\Gamma, M)$. We denote $\hat{\Omega}(\Gamma, M) = \Omega(\Gamma, M) / \sim$ the quotient space. Given $f \in \hat{\Omega}(\Gamma, M)$ we will often denote a representative of the equivalence class $[f]$ (which will be an element of $\Omega(\Gamma, M)$) also by f , and regard different representatives as different parametrizations of the geometric object $f \in \hat{\Omega}(\Gamma, M)$.

Notation 2.4. Given $1 \leq q \leq \infty$, let us denote \mathcal{M}^q the set of C^q Riemannian metrics on M .

Definition 2.5. Let $f \in \hat{\Omega}(\Gamma, M)$ and let h be a continuous function defined in $\text{Im}(f) \subseteq M$. Given a metric $g \in \mathcal{M}^q$ we define

$$\int_f h \, dL_g = \sum_{E \in \mathcal{E}} n(E) \int_E h \circ f(u) \sqrt{g_{f(u)}(\dot{f}(u), \dot{f}(u))} du$$

Observe that the right hand side is independent of the parametrization we choose and therefore $\int_f h \, dL_g$ is well defined.

Definition 2.6 (g -Length). Given $g \in \mathcal{M}^q$ and $f \in \hat{\Omega}(\Gamma, M)$, we define the g -length of f by

$$L_g(f) = \int_{\Gamma} 1 \, dL_g = \sum_{E \in \mathcal{E}} n(E) \int_E \sqrt{g_{f(u)}(\dot{f}(u), \dot{f}(u))} du$$

Definition 2.7 (Stationary Geodesic Network). We say that $f \in \Omega(\Gamma, M)$ is a stationary geodesic network with respect to metric $g \in \mathcal{M}^\infty$ if it is a critical point of the length functional $L_g : \Omega(\Gamma, M) \rightarrow \mathbb{R}$. This means that given any one parameter family $\tilde{f} : (-\delta, \delta) \rightarrow \Omega(\Gamma, M)$ with $\tilde{f}(0) = f$ we have

$$\left. \frac{d}{ds} \right|_{s=0} L_g(\tilde{f}(s)) = 0$$

Definition 2.8. We say that $f \in \hat{\Omega}(\Gamma, M)$ is a stationary geodesic network if every representative $\tilde{f} \in \Omega(\Gamma, M)$ of f is a stationary geodesic network.

Remark 2.9. In [14, Lemma 2.5], it was shown that every stationary geodesic network with respect to a metric g can be represented by a map $f : \Gamma \rightarrow M$, where $\Gamma = \bigcup_{i=1}^P \Gamma_i$ is a finite union of the good weighted multigraphs $\{\Gamma_i\}_{1 \leq i \leq P}$ and $f|_{\Gamma_i}$ is an embedded stationary geodesic network for each $1 \leq i \leq P$.

Definition 2.10. Given a stationary geodesic network $f : \Gamma \rightarrow M$, we say that its connected components are nondegenerate if

- (1) We can express $\Gamma = \bigcup_{i=1}^P \Gamma_i$ as a disjoint union of good weighted multigraphs.
- (2) $f|_{\Gamma_i}$ is an embedded nondegenerate stationary geodesic network for every $1 \leq i \leq P$ (for the definition of nondegenerate stationary geodesic network, see [22, Section 1]).

Definition 2.11. An almost embedded closed geodesic in a Riemannian manifold (M, g) is a map $\gamma : S^1 \rightarrow (M, g)$ such that

- (1) γ is geodesic (i.e. $\ddot{\gamma}(t) = 0$ for every $t \in S^1$).
- (2) γ is an immersion (i.e. $\dot{\gamma}(t) \neq 0$ for every $t \in S^1$).
- (3) All self-intersections of γ are transverse, which means that for every $s, t \in S^1$ such that $\gamma(s) = \gamma(t)$ and $s \neq t$, the velocities $\dot{\gamma}(s)$ and $\dot{\gamma}(t)$ are not colinear.

Notation 2.12. Given a symmetric 2-tensor T , a metric $g \in \mathcal{M}^q$, a stationary geodesic network $f : \Gamma \rightarrow M$ on M and $t \in \Gamma$, we denote

$$\text{tr}_{f,g} T(t) = T\left(\frac{\dot{f}(t)}{|\dot{f}(t)|_g}, \frac{\dot{f}(t)}{|\dot{f}(t)|_g}\right)$$

which is the trace of the tensor T along f with respect to the metric g .

Definition 2.13 (Average integral along f). Let Γ be a weighted multigraph. Given $f \in \Omega(\Gamma, M)$, a metric $g \in \mathcal{M}^\infty$ and a continuous function h defined in $\text{Im}(f)$, we define the average integral of h with respect to metric g as

$$\oint_f h \, dL_g := \frac{1}{L_g(f)} \int_f h \, dL_g$$

3. TWO TECHNICAL PROPOSITIONS

Proposition 3.1. Let $g : I^N \rightarrow \mathcal{M}^q$ be a smooth embedding, $N \in \mathbb{N}$, $I = (-1, 1)$. If $q \geq N + 3$, there exists an arbitrarily small perturbation in the C^∞ topology $g' : I^N \rightarrow \mathcal{M}^q$ such that there is a full measure subset $\mathcal{A} \subseteq I^N$ with the following properties: for any $p \in \mathbb{N}$ and any $t \in \mathcal{A}$, the function $s \mapsto \omega_p^1(g'(s))$ is differentiable at t , and there exists a (possibly disconnected) weighted multigraph Γ and a stationary geodesic network $f_p = f_p(t) : \Gamma \rightarrow (M, g'(t))$ such that the following two conditions hold

- (1) $\omega_p^1(g'(t)) = L_{g'(t)}(f_p(t))$.
- (2) $\frac{\partial}{\partial v}(\omega_p^1 \circ g')|_{s=t} = \frac{1}{2} \int_{f_p(t)} \text{tr}_{f_p(t), g'(t)} \frac{\partial g'}{\partial v}(t) \, dL_{g'(t)}$.

To prove the proposition, we will need to have a condition for a sequence of embedded stationary geodesic nets $f_n : \Gamma \rightarrow (M, g_n)$ converging to some $f : \Gamma \rightarrow (M, g)$

that guarantees that f is also embedded. The condition we will work with can be expressed as a collection of lower and upper bounds of certain functionals defined for pairs (g, f) where f is stationary with respect to g . We proceed to describe those functionals.

The first one is

$$F_1(g, f) = \min\{|\dot{f}_E(t)|_g : t \in E, E \in \mathcal{E}\}$$

A lower bound for this functional will imply that the limit net is an immersion along each edge.

Then we have a family of functionals $F_2^{(E_1, i_1), (E_2, i_2)}$ defined for each pair $((E_1, i_1), (E_2, i_2)) \in (\mathcal{E} \times \{0, 1\})^2$ such that $\pi_{E_1}(i_1) = \pi_{E_2}(i_2)$ (see Section 2 for the notation) as follows

$$F_2^{(E_1, i_1), (E_2, i_2)}(g, f) = (-1)^{i_1+i_2} \frac{g(\dot{f}_{E_1}(i_1), \dot{f}_{E_2}(i_2))}{|\dot{f}_{E_1}(i_1)|_g |\dot{f}_{E_2}(i_2)|_g}$$

Notice that $(-1)^{i_j} \frac{\dot{f}_{E_j}(i_j)}{|\dot{f}_{E_j}(i_j)|_g}$ is the unit inward tangent vector to f at $v = \pi_{E_j}(i_j)$ along E_j , $j = 1, 2$ (and observe that in case E is a loop at v , there are two inward tangent vectors to f along E at v represented by the pairs $(E, 0)$ and $(E, 1)$). The condition $F_2^{(E_1, i_1), (E_2, i_2)}(g_n, f_n) \leq 1 - \delta$ for some $\delta > 0$ and for every possible choice $(E_1, i_1) \neq (E_2, i_2)$ with $\pi_{E_1}(i_1) = \pi_{E_2}(i_2)$ implies that the limit (g, f) has the property that given $v \in \mathcal{V}$, there exists an open neighborhood U_v of v in Γ such that $f : U_v \rightarrow f(U_v)$ is a homeomorphism. Explicitly,

$$U_v = \bigcup_{(E, i) : \pi_E(i) = v} \{t \in E : |t - i| < \min\{\frac{\text{inj}(g)}{L_g(f_E)}, \frac{1}{2}\}\}$$

where $\text{inj} : \mathcal{M}^q \rightarrow \mathbb{R} > 0$, $g \mapsto \text{inj}(g)$ is a continuous choice of the injectivity radius for each C^q Riemannian metric g . This is because if we consider U_v as a graph obtained by gluing at v one edge for each pair $(E, i) \in \mathcal{E} \times \{0, 1\}$ such that $\pi_E(i) = v$, this graph is mapped by f into a geodesic ball centered at $f(v)$ of radius $\text{inj}(g)$ and the image of each incoming edge at v has a different inward tangent vector at $f(v)$.

To ensure injectivity along the edges, we define for each edge $E \in \mathcal{E}$ a function

$$d_{(g, f)}^E(t) = \min\{d_g(f(t), f(s)) : s \in E, |t - s| \geq \frac{\text{inj}(g)}{L_g(f_E)}\}$$

In case $\pi_E(0) = \pi_E(1)$, the distance $|s - t|$ between two points $s, t \in E$ is measured with respect of the length of $S^1 = E/0 \sim 1$.

To ensure that the images of different edges under f do not overlap, we define for each pair $E, E' \in \mathcal{E}$, $E \neq E'$ a function $d_{(g,f)}^{E,E'} : E \rightarrow \mathbb{R}_{\geq 0}$ as

$$d_{(g,f)}^{E,E'}(t) = \min\{d_g(f(t), f(s)) : s \in E', |s - i| \geq \frac{\text{inj}(g)}{L_g(f_{E'})} \text{ for each } i \in \{0, 1\} \\ \text{s.t. } \exists j \in \{0, 1\} \text{ with } \pi_{E'}(i) = \pi_E(j) \text{ and } |t - j| \leq \frac{\text{inj}(g)}{L_g(f_E)}\}$$

We will also need the following two lemmas.

Lemma 3.2. *Let $F : \mathbb{R}^n \rightarrow \mathbb{N}$ be a function. Then there exists $m \in \mathbb{N}$ and a basis $\{v_1, \dots, v_n\}$ of \mathbb{R}^n such that $F(v_i) = m$ for all $1 \leq i \leq n$.*

Proof. Observe that $\mathbb{R}^n = \bigcup_{m \in \mathbb{N}} F^{-1}(m)$ and therefore $\mathbb{R}^n = \bigcup_{m \in \mathbb{N}} \langle F^{-1}(m) \rangle$ where given $A \subseteq \mathbb{R}^n$ we denote $\langle A \rangle$ the subspace spanned by A . If $F^{-1}(m)$ did not contain a basis of \mathbb{R}^n for every $m \in \mathbb{N}$, $\langle F^{-1}(m) \rangle$ would be a proper subspace for every m . Therefore, \mathbb{R}^n would be a countable union of closed subspaces with empty interior, which leads to a contradiction due to the Baire Category Theorem. \square

Lemma 3.3. *Let $f : (-1, 1)^N \rightarrow \Omega(\Gamma, M)$ and $g : (-1, 1)^N \rightarrow \mathcal{M}^q$ be smooth maps. Assume that $f(s)$ is stationary with respect to $g(s)$ for every $s \in (-1, 1)^N$. Then for every $t \in (-1, 1)^N$ and every $v \in \mathbb{R}^N$*

$$\frac{\partial}{\partial v} \Big|_{s=t} L(g(s), f(s)) = \frac{1}{2} \int_{f(t)} \text{tr}_{f(t), g(t)} \frac{\partial g}{\partial v}(t) dL_{g(t)}$$

Proof. Using that the length functional is a smooth function $L : \mathcal{M}^q \times \Omega(\Gamma, M) \rightarrow \mathbb{R}$ and the chain rule, we get

$$\begin{aligned} \frac{\partial}{\partial v} \Big|_{s=t} L(g(s), f(s)) &= D L_{(g(t), f(t))}(D(g \times f)_t(v)) \\ &= D L_{(g(t), f(t))}\left(\frac{\partial g}{\partial v}(t), \frac{\partial f}{\partial v}(t)\right) \\ &= D_1 L_{(g(t), f(t))}\left(\frac{\partial g}{\partial v}(t)\right) + D_2 L_{(g(t), f(t))}\left(\frac{\partial f}{\partial v}(t)\right) \\ &= D_1 L_{(g(t), f(t))}\left(\frac{\partial g}{\partial v}(t)\right) \end{aligned}$$

The second term in the penultimate equation vanishes because $f(t)$ is stationary with respect to $g(t)$. Hence

$$\begin{aligned}
\frac{\partial}{\partial v} \Big|_{s=t} L(g(s), f(s)) &= \frac{d}{ds} \Big|_{s=0} L(g(t+sv), f(t)) \\
&= \frac{d}{ds} \Big|_{s=0} \sum_{E \in \mathcal{E}} n(E) \int_E \sqrt{g_{t+sv}(\dot{f}_t(u), \dot{f}_t(u))} du \\
&= \sum_{E \in \mathcal{E}} n(E) \int_E \frac{d}{ds} \Big|_{s=0} \sqrt{g_{t+sv}(\dot{f}_t(u), \dot{f}_t(u))} du \\
&= \sum_{E \in \mathcal{E}} n(E) \int_E \frac{\frac{\partial g}{\partial v}(t)(\dot{f}_t(u), \dot{f}_t(u))}{2\sqrt{g_t(\dot{f}_t(u), \dot{f}_t(u))}} du \\
&= \frac{1}{2} \sum_{E \in \mathcal{E}} n(E) \int_E \frac{\frac{\partial g}{\partial v}(t)(\dot{f}_t(u), \dot{f}_t(u))}{g_t(\dot{f}_t(u), \dot{f}_t(u))} \sqrt{g_t(\dot{f}_t(u), \dot{f}_t(u))} du \\
&= \frac{1}{2} \sum_{E \in \mathcal{E}} n(E) \int_{f(t)_E} \text{tr}_{f(t), g(t)} \frac{\partial g}{\partial v}(t) dL_{g(t)} \\
&= \frac{1}{2} \int_{f(t)} \text{tr}_{f(t), g(t)} \frac{\partial g}{\partial v}(t) dL_{g(t)}
\end{aligned}$$

□

Proof of Proposition 3.1. Notice that it suffices to show that for each $p \in \mathbb{N}$, there exists a full measure subset $\mathcal{A}(p) \subseteq I^N$ where (1) and (2) hold, because in that case $\mathcal{A} = \bigcap_{p \in \mathbb{N}} \mathcal{A}(p)$ will have the desired property. Therefore we will assume $p \in \mathbb{N}$ is fixed.

Let $g : I^N \rightarrow \mathcal{M}^q$ be a smooth embedding. Let $\{\Gamma_i\}_{i \geq 1}$ be a sequence enumerating the countable collection of all good weighted multigraphs. For each $i \geq 1$, let $\mathcal{S}^q(\Gamma_i)$ be the space of pairs $(g, [f])$ where $g \in \mathcal{M}^q$, $f : \Gamma_i \rightarrow (M, g)$ is an embedded stationary geodesic net and $[f]$ denotes its class modulo reparametrization as defined in [22] for connected multigraphs with at least three incoming edges at each vertex and in [23] for embedded closed geodesics. By the structure theorems proved in [22] and [23], each $\mathcal{S}^q(\Gamma_i)$ is a second countable Banach manifold and the projection map $\Pi_i : \mathcal{S}^q(\Gamma_i) \rightarrow \mathcal{M}^q$ mapping $(g, [f]) \mapsto g$ is Fredholm of index 0. A pair $(g, [f]) \in \mathcal{S}_q(\Gamma_i)$ is a critical point of Π_i if and only if f admits a nontrivial Jacobi field with respect to the metric g .

By Smale's transversality theorem, we can perturb $g : I^N \rightarrow \mathcal{M}^q$ slightly in the C^∞ topology to a C^∞ embedding $g' : I^N \rightarrow \mathcal{M}^q$ which is transversal to $\Pi_i : \mathcal{S}^q(\Gamma_i) \rightarrow \mathcal{M}^q$ for every $i \in \mathbb{N}$. Transversality implies that $M_i = \Pi_i^{-1}(g'(I^N))$ is an N -dimensional

embedded submanifold of $\mathcal{S}^q(\Gamma_i)$ for every $i \in \mathbb{N}$. Let $\pi_i = (g')^{-1} \circ \Pi_i|_{M_i} : M_i \rightarrow I^N$. Let $\tilde{\mathcal{A}}_i \subseteq I^N$ be the set of regular values of π_i , which is a set of full measure by Sard's theorem. Let $\tilde{\mathcal{A}}_0 \subseteq I^N$ be the set of points for which the Lipschitz function $s \mapsto \omega_p^1(g'(s))$ is differentiable. Observe that $\tilde{\mathcal{A}}_0$ has full measure by Rademacher's theorem. Therefore, $\tilde{\mathcal{A}} = \bigcap_{i \geq 0} \tilde{\mathcal{A}}_i$ is a full measure subset of I^N . Notice that by transversality, if $t \in \tilde{\mathcal{A}}$ then $g'(t)$ is a bumpy metric, i.e. all embedded stationary geodesic nets with respect to $g'(t)$ and with domain a good weighted multigraph are nondegenerate; and also the map $s \mapsto \omega_p^1(g'(s))$ is differentiable at $s = t$.

Let us fix an auxiliary embedding $\psi : M \rightarrow \mathbb{R}^l$ and identify from now on our manifold M with the submanifold $\psi(M) \subseteq \mathbb{R}^l$. Given a multigraph Γ and a continuous map $f : \Gamma \rightarrow M$ which is C^3 when restricted to each edge, we can consider

$$\|f\|_3 = \|f\|_0 + \|\dot{f}\|_0 + \|\ddot{f}\|_0 + \|\dddot{f}\|_0$$

where given a collection $u = (u_E)_{E \in \mathcal{E}}$ of continuous functions along the edges of Γ , we define

$$\|u\|_0 = \max\{|u_E(t)| : t \in E, E \in \mathcal{E}\}$$

being $|\cdot|$ the Euclidean norm in \mathbb{R}^l .

Given a weighted multigraph $\Gamma = \bigcup_{i=1}^P \Gamma_i$ whose connected components Γ_i are good and a natural number $M \in \mathbb{N}$, we define $\mathcal{B}_{\Gamma, M}$ as the set of all $t \in I^N$ such that there exists a stationary geodesic network $f : \Gamma \rightarrow (M, g'(t))$ verifying

- (1) $f_i = f|_{\Gamma_i}$ is an embedded for each $1 \leq i \leq P$.
- (2) $\|f_i\|_3 \leq M$ for every $1 \leq i \leq P$.
- (3) $F_1(g'(t), f_i) \geq \frac{1}{M}$ for every $1 \leq i \leq P$.
- (4) $F_2^{(E_1, i_1), (E_2, i_2)}(g'(t), f_i) \leq 1 - \frac{1}{M}$ for every $1 \leq i \leq P$ and every pair $(E_1, i_1) \neq (E_2, i_2)$ in $\mathcal{E}_i \times \{0, 1\}$ such that $\pi_{E_1}(i_1) = \pi_{E_2}(i_2)$.
- (5) $d_{(g'(t), f_i)}^E(s) \geq \frac{1}{M}$ for every $1 \leq i \leq P$, $E \in \mathcal{E}_i$ and $s \in E$.
- (6) $d_{(g'(t), f_i)}^{E, E'}(s) \geq \frac{1}{M}$ for every $1 \leq i \leq P$, $E \neq E' \in \mathcal{E}_i$ and $s \in E$.
- (7) $\omega_p^1(g'(t)) = L_{g'(t)}(f)$.

where \mathcal{E}_i denotes the set of edges of Γ_i . Observe that $I^N = \bigcup_{\Gamma, M} \mathcal{B}_{\Gamma, M}$ because of Remark 2.9. We claim that each $\mathcal{B}_{\Gamma, M} \subseteq I^N$ is closed.

Indeed, suppose we have a sequence $\{t_j\}_{j \in \mathbb{N}} \subseteq \mathcal{B}_{\Gamma, M}$ converging to some $t \in I^N$. Let f^j be the stationary geodesic network corresponding to $g'(t_j)$ and verifying properties (1) to (7) above. By property (2) and the Arzela-Ascoli theorem, passing to a subsequence we have that if $f_i^j = f^j|_{\Gamma_i}$ then there exists $f_i : \Gamma_i \rightarrow M$ such that $\lim_{j \rightarrow \infty} f_i^j = f_i$ in $\Omega(\Gamma_i, M)$ for each $1 \leq i \leq P$. By continuity of the map H defined in [22] which is locally defined in $\mathcal{M}^q \times \hat{\Omega}(\Gamma_i, M)$ and vanishes in a pair $(g, [f])$ if and

only if f is stationary with respect to g , we deduce that f is stationary with respect to $g'(t)$. Observe also that if $f = \bigcup_j f_j$

$$L_{g'(t)}(f) = \lim_{j \rightarrow \infty} L_{g'(t_j)}(f^j) = \lim_{j \rightarrow \infty} \omega_p(t_j) = \omega_p(t)$$

Properties (2) to (6) are preserved when we take the limit of the sequence f^j , so it suffices to show that $f|_{\Gamma_i}$ is embedded for each $1 \leq i \leq P$. Fix such i . Properties (3), (4) and (5) imply that f_i is injective along the edges and property (6) combined with property (4) imply that the images of different edges do not intersect (except at the common vertices).

As each $\mathcal{B}_{\Gamma,M}$ is closed, they are measurable and therefore so are the sets $\tilde{\mathcal{A}}_{\Gamma,M} = \tilde{\mathcal{A}} \cap \mathcal{B}_{\Gamma,M}$ (whose union is $\tilde{\mathcal{A}}$). Let $\mathcal{A}_{\Gamma,M}$ be the set of points $t \in \tilde{\mathcal{A}}_{\Gamma,M}$ where the Lebesgue density of $\tilde{\mathcal{A}}_{\Gamma,M}$ at t is 1. By the Lebesgue Differentiation Theorem, $\tilde{\mathcal{A}}_{\Gamma,M} \setminus \mathcal{A}_{\Gamma,M}$ has Lebesgue measure 0 for each pair (Γ, M) . Let us define $\mathcal{A} = \bigcup_{\Gamma,M} \mathcal{A}_{\Gamma,M}$, observe that as $\tilde{\mathcal{A}} \setminus \mathcal{A}$ has measure 0, $\mathcal{A} \subseteq I^N$ has full measure.

Fix $t \in \mathcal{A}$. Let (Γ, M) be such that $t \in \mathcal{A}_{\Gamma,M}$. As the density of $\tilde{\mathcal{A}}_{\Gamma,M}$ at t is 1, given $v \in \mathbb{R}^N$ with $|v| = 1$ we can find a sequence $\{t_m(v)\}_{m \in \mathbb{N}} \subseteq \tilde{\mathcal{A}}_{\Gamma,M}$ such that $\lim_{m \rightarrow \infty} t_m(v) = t$ and $\lim_{m \rightarrow \infty} \frac{t - t_m(v)}{|t - t_m(v)|} = v$. Denoting $\omega_p(s) = \omega_p^1(g'(s))$, using that ω_p is a Lipschitz function we can see that

$$(1) \quad \lim_{m \rightarrow \infty} \frac{\omega_p(t_m(v)) - \omega_p(t)}{|t - t_m|} = \frac{\partial}{\partial v} \omega_p(t)$$

As $t_m(v) \in \tilde{\mathcal{A}}_{\Gamma,M}$, for each $m \in \mathbb{N}$ there exists a stationary geodesic network $f_m : \Gamma \rightarrow M$ with respect to $g'(t_m(v))$ such that

$$\omega_p(t_m(v)) = \omega_p^1(g'(t_m(v))) = L_{g'(t_m(v))}(f_m)$$

and properties (1) to (6) above hold. By the reasoning used to prove that the $\mathcal{B}_{\Gamma,M}$ are closed, we can construct a stationary geodesic net $f : \Gamma \rightarrow (M, g'(t))$ which is embedded when restricted to each connected component Γ_i of Γ , is the limit of (a subsequence of) the f_m 's in the C^2 topology and realizes the width $\omega_p^1(g'(t))$. Hence from (1) we get

$$\frac{\partial}{\partial v} \omega_p(t) = \lim_{m \rightarrow \infty} \frac{L_{g'(t_m)}(f_m) - L_{g'(t)}(f)}{|t - t_m|}$$

As $f|_{\Gamma_i}$ is an embedded stationary geodesic net with respect to $g'(t)$ for each $1 \leq i \leq P$ and $g'(t)$ is bumpy, $\Pi_i : \mathcal{S}^q(\Gamma_i) \rightarrow \mathcal{M}^q$ is a diffeomorphism from a neighborhood U_i of $(g'(t), [f_i])$ to a neighborhood $W_i = \Pi_i(U)$ of $g'(t)$. Denote Ξ_i its inverse. As there exists $m_0 \in \mathbb{N}$ such that $g'(t_m) \in W = \bigcap_{i=1}^P W_i$ and $[f_m|_{\Gamma_i}] \in U_i$ for every $m \geq m_0$, we deduce that $[f_m|_{\Gamma_i}] = \Xi_i(g'(t_m(v)))$ for each $m \geq m_0$ and each

$1 \leq i \leq P$. Let us define $\Xi : W \rightarrow \hat{\Omega}(\Gamma, M)$ as $\Xi(g) = h$ where $h|_{\Gamma_i} = \Xi_i(g)$. Thus by Lemma 3.3

$$\begin{aligned} \frac{\partial}{\partial v} \omega_p(t) &= \lim_{m \rightarrow \infty} \frac{L_{g'(t_m)}(\Xi(g'(t_m))) - L_{g'(t)}(\Xi(g'(t)))}{|t_m - t|} \\ &= \frac{\partial}{\partial v} \Big|_{s=t} L(g'(s), \Xi(g'(s))) \\ &= \frac{1}{2} \int_{f_v} \text{tr}_{f_v, g'(t)} \frac{\partial g'}{\partial v}(t) \, dL_{g'(t)} \end{aligned}$$

Where $f_v = \Xi(g'(t))$ is the one constructed before. Observe that f_v depends on v and that the previous formula holds for each $v \in \mathbb{R}^N$, $|v| = 1$. Notice that each f_v is a stationary geodesic network with respect to $g'(t)$, and as $g'(t)$ is bumpy there are countably many possible f_v 's, say $\{h_j\}_{j \in \mathbb{N}}$. This induces a map $F : \mathbb{R}^N \rightarrow \mathbb{N}$ defined as $F(0) = 1$ and if $w \neq 0$ then $F(w) = j$ where $f_{\frac{w}{|w|}} = h_j$. By Lemma 3.2 we can obtain $m \in \mathbb{N}$ and a basis w_1, \dots, w_N of \mathbb{R}^N with the property $f(w_i) = m$ for every $1 \leq i \leq N$. Therefore if we set $v_i := \frac{w_i}{|w_i|}$, v_1, \dots, v_N is still a basis and by definition $f_{v_i} = h_m$ for every i . By linearity of directional derivatives, denoting $f = h_m$ we deduce that

$$\frac{\partial}{\partial v} \omega_p(t) = \frac{1}{2} \int_f \text{tr}_{f, g'(t)} \frac{\partial g'}{\partial v}(t) \, dL_{g'(t)}$$

for every unit $v \in \mathbb{R}^N$, which completes the proof. \square

Lemma 3.4. *Given $\eta > 0$ and $N \in \mathbb{N}$, there exists $\varepsilon > 0$ depending on η and N such that the following is true: for any Lipschitz function $f : I^N \rightarrow \mathbb{R}$ satisfying*

$$|f(x) - f(y)| \leq 2\varepsilon$$

for every $x, y \in I^N$, and for any subset \mathcal{A} of I^N of full measure, there exist $N + 1$ sequences of points $\{y_{1,m}\}_m, \dots, \{y_{N+1,m}\}_m$ contained in \mathcal{A} and converging to a common limit $y \in (-1, 1)^N$ such that:

- *f is differentiable at each $y_{i,m}$,*
- *the gradients $\nabla f(y_{i,m})$ converge to $N + 1$ vectors v_1, \dots, v_{N+1} with*

$$d_{\mathbb{R}^N}(0, \text{Conv}(v_1, \dots, v_{N+1})) < \eta$$

Proof. See [16, Lemma 3]. \square

4. PROOF OF THE MAIN THEOREM

Fix an n -dimensional closed manifold M . We are going to consider several choices and constructions over M . Let g be a C^∞ Riemannian metric on M . Let $\varepsilon_1 > 0$ be a positive constant such that $\varepsilon_1 < \text{inj}(M, g)$, where $\text{inj}(M, g)$ is the injectivity radius of (M, g) . Let K be an integer and $\hat{B}_1, \dots, \hat{B}_K$ be disjoint domains in M , with piecewise smooth boundary, such that the union of their closures covers M . Let B_1, \dots, B_K be some open neighbourhoods of $\hat{B}_1, \dots, \hat{B}_K$ respectively with the property that each of them is contained in a geodesic ball of radius of ε_1 . Denote \mathcal{M}^q the space of all C^q Riemannian metrics on M . For each $1 \leq k \leq K$, we define a smooth function $0 \leq \phi_k \leq 1$, $\text{spt}(\phi_k) \subseteq B_k$ such that

$$\phi_k = \begin{cases} 1 & \text{on } \hat{B}_k \\ 0 & \text{on } B_k^c \end{cases}.$$

Consider also the partition of unity $\psi_k = \frac{\phi_k}{\sum_{i=1}^K \phi_i}$. We denote

$$\mathcal{C}_{g, \tilde{K}, \varepsilon_1} := \{(K, \{\hat{B}_k\}, \{B_k\}, \{\phi_k\})\}$$

the set of all possible choices as above with $K \geq \tilde{K}$. Notice that $\mathcal{C}_{g, \tilde{K}, \varepsilon_1}$ is non-empty, as we can always find a sufficiently fine triangulation of (M, g) . We claim that the following property holds:

Proposition 4.1. *Assume that the Weyl law for 1-cycles in n -manifolds holds as stated in Conjecture 1.4. Then for any metric $g \in \mathcal{M}^\infty$, for every $\varepsilon_1 > 0$, $\tilde{K} > 0$ and any choice of*

$$S = (K, \{\hat{B}_k\}, \{B_k\}, \{\phi_k\}) \in \mathcal{C}_{g, \tilde{K}, \varepsilon_1},$$

there is a metric $\tilde{g} \in \mathcal{M}^\infty$ arbitrarily close to g in the C^∞ topology such that the following holds: there are stationary geodesic networks $\gamma_1, \dots, \gamma_J$ with respect to \tilde{g} whose connected components are nondegenerate (according to Definition 2.10) and coefficients $\alpha_1, \dots, \alpha_J \in [0, 1]$ with $\sum_{j=1}^J \alpha_j = 1$ satisfying

$$(2) \quad \left| \sum_{j=1}^J \alpha_j \int_{\gamma_j} \psi_k dL_{\tilde{g}} - \int_M \psi_k d\text{Vol}_{\tilde{g}} \right| < \frac{\varepsilon_1}{K}$$

for every $k = 1, \dots, K$.

In the proof, we will need to measure the distance between two rescaled functions. In order to do that, we introduce the following definition.

Definition 4.2. We say that two functions $f, g : (-\delta, \delta)^K \rightarrow \mathbb{R}$ are ε -close if

$$\left\| \frac{1}{\delta} f_\delta - \frac{1}{\delta} g_\delta \right\|_\infty < \varepsilon$$

where $f_\delta, g_\delta : (-1, 1)^K \rightarrow \mathbb{R}$ are given by $f_\delta(s) = f(\delta s)$ and $g_\delta(s) = g(\delta s)$.

Remark 4.3. Observe that $\frac{1}{\delta} f_\delta$ is differentiable at $s \in (-1, 1)^K$ if and only if f is differentiable at $\delta s \in (-\delta, \delta)^K$ and in that case $\nabla(\frac{1}{\delta} f_\delta)(s) = \nabla f(\delta s)$.

Proof of Proposition 4.1. Let $g \in \mathcal{M}^\infty$, $\tilde{K} \in \mathbb{N}$ and $\varepsilon_1 > 0$. Fix $(K, \{\hat{B}_k\}, \{B_k\}, \{\phi_k\}) \in \mathcal{C}_{g, \tilde{K}, \varepsilon_1}$. Let \mathcal{U} be a C^∞ neighborhood of g . Choose $\varepsilon'_0 > 0$ sufficiently small and $q \geq K + 3$ sufficiently large so that if $g' \in \mathcal{M}^\infty$ satisfies $\|g - g'\|_{C^q} < \varepsilon'_0$, then $g' \in \mathcal{U}$. Let $\varepsilon' \leq \varepsilon'_0$ be a positive real number (which we will have to shrink later in the argument). Our goal is to show that there exists $\tilde{g} \in \mathcal{M}^\infty$ such that $\|\tilde{g} - g\|_{C^q} < \varepsilon'_0$ and (2) holds for some stationary geodesic nets $\gamma_1, \dots, \gamma_J$ (whose connected components are nondegenerate) with respect to \tilde{g} and some coefficients $\alpha_1, \dots, \alpha_J$.

Consider the following K -parameter family of metrics. For a $t = (t_1, \dots, t_K) \in (-1, 1)^K$, we define

$$\hat{g}(t) = e^{2 \sum_k t_k \psi_k} g$$

At $t = 0$, for each k , we have

$$\begin{aligned} \left. \frac{\partial}{\partial t_k} \right|_{t=0} \text{Vol}(M, \hat{g}(t)) &= \left. \frac{\partial}{\partial t_k} \right|_{t=0} \int_M (e^{2 \sum_k t_k \psi_k(x)})^{\frac{n}{2}} d\text{Vol}_g \\ &= \int_M n \psi_k(x) d\text{Vol}_g \end{aligned}$$

As t goes to zero, we have the following expansion

$$(3) \quad \text{Vol}(M, \hat{g}(t))^{\frac{1}{n}} = \text{Vol}(M, g)^{\frac{1}{n}} + \sum_{k=1}^K t_k \text{Vol}(M, g)^{-\frac{n-1}{n}} \int_M \psi_k(x) d\text{Vol}_g + R(t)$$

where $|R(t)| \leq C_1 \|t\|^2$ if $t \in (-1, 1)^K$, where C_1 is a constant which depends only on g (this can be checked by computing the second order partial derivatives of $t \mapsto \text{Vol}(M, \hat{g}(t))^{\frac{1}{n}}$ and using the fact that $e^{-n} \text{Vol}(M, g) \leq \text{Vol}(M, \hat{g}(t)) \leq e^n \text{Vol}(M, g)$ as $\text{Vol}(M, \hat{g}(t)) = \int_M e^{n \sum_k t_k \psi_k(x)} d\text{Vol}_g$). Following [16] we can define the following function

$$f_0(t) = \frac{\text{Vol}(M, \hat{g}(t))^{\frac{1}{n}}}{\text{Vol}(M, g)^{\frac{1}{n}}} - \sum_{k=1}^K t_k \int_M \psi_k(x) d\text{Vol}_g$$

Because of (3), $|f_0(t) - 1| = \left| \frac{R(t)}{\text{Vol}(M, g)^{\frac{1}{n}}} \right| \leq C_2 \|t\|^2$ for every $t \in (-1, 1)^K$; where $C_2 = \frac{C_1}{\text{Vol}(M, g)^{\frac{1}{n}}}$ depends only on g (as C_1 and other constants C_i to be defined later).

By the previous, f_0 is $C_2\varepsilon'$ -close to 1 in $(-\delta, \delta)^K$ if $\delta < \varepsilon'$ (see Definition 4.2). Let $\delta < \varepsilon'$ be such that $\hat{g} : (-\delta, \delta)^K \rightarrow \mathcal{M}^q$ is an embedding and $\|\hat{g}(t) - g\|_{C^q} < \frac{\varepsilon'}{2}$ for every $t \in (-\delta, \delta)^K$. We can slightly perturb \hat{g} in the C^∞ topology to another embedding $g' : (-\delta, \delta)^K \rightarrow \mathcal{M}^q$ applying Proposition 3.1. We can assume $\|g'(t) - \hat{g}(t)\|_{C^q} < \frac{\varepsilon'}{2}$ and $\|\frac{\partial g'}{\partial v} - \frac{\partial \hat{g}}{\partial v}\|_{C^q} < \varepsilon'$ for every $t \in (-\delta, \delta)^K$ and $v \in \mathbb{R}^K : |v| = 1$. Consider the function

$$f_1(t) = \frac{\text{Vol}(M, g'(t))^{\frac{1}{n}}}{\text{Vol}(M, g)^{\frac{1}{n}}} - \sum_k t_k \int_M \psi_k(x) d\text{Vol}_g$$

By the properties of g' , there exists $C_3 > 0$ such that f_1 is $C_3\varepsilon'$ -close to the constant function equal to 1 on $(-\delta, \delta)^K$.

Now we will use the assumption that the Weyl law for 1-cycles in n -manifolds holds, which means that

$$(4) \quad \lim_{p \rightarrow \infty} \omega_p^1(M^n, g) p^{-\frac{n-1}{n}} = \alpha(n, 1) \text{Vol}(M^n, g)^{\frac{1}{n}}$$

The normalized p -widths $p^{-\frac{n-1}{n}} \omega_p^1(g'(t))$ are uniformly Lipschitz continuous on $(-\delta, \delta)^K$ by [14, Lemma 2.7]. Hence, by (4) the sequence of functions $t \mapsto p^{-\frac{n-1}{n}} \omega_p^1(M, g'(t))$ converges uniformly to the function $t \mapsto \alpha(n, 1) \text{Vol}(M, g'(t))^{\frac{1}{n}}$. This implies that for the previously defined $\delta > 0$, there exists $p_0 \in \mathbb{N}$ such that $p \geq p_0$ implies

$$|p^{-\frac{n-1}{n}} \omega_p^1(M, g'(t)) - \alpha(n, 1) \text{Vol}(M, g'(t))^{\frac{1}{n}}| < \delta\varepsilon'$$

and hence

$$\left| \frac{\omega_p^1(M, g'(t))}{\alpha(n, 1) p^{\frac{n-1}{n}} \text{Vol}(M, g)^{\frac{1}{n}}} - \frac{\text{Vol}(M, g'(t))^{\frac{1}{n}}}{\text{Vol}(M, g)^{\frac{1}{n}}} \right| < C_4 \delta \varepsilon'$$

for every $t \in (-\delta, \delta)^K$. The previous means that $h(t) = \frac{\omega_p^1(M, g'(t))}{\alpha(n, 1) p^{\frac{n-1}{n}} \text{Vol}(M, g)^{\frac{1}{n}}} - \frac{\text{Vol}(M, g'(t))^{\frac{1}{n}}}{\text{Vol}(M, g)^{\frac{1}{n}}}$ is $C_4\varepsilon'$ -close to 0 in $(-\delta, \delta)^K$ and therefore as f_1 is $C_3\varepsilon'$ -close to 1, by triangle inequality we have that

$$f_2(t) = \frac{\omega_p^1(M, g'(t))}{\alpha(n, 1) p^{\frac{n-1}{n}} \text{Vol}(M, g)^{\frac{1}{n}}} - \sum_{k=1}^K t_k \int_M \psi_k(x) d\text{Vol}_g$$

is $C_5\varepsilon'$ -close to 1 if $p \geq p_0$, for some $C_5 > 0$.

On the other hand, by our choice of g' using Proposition 3.1, there exists a full measure subset $\mathcal{A} \subseteq (-\delta, \delta)^K$ such that for each $t \in \mathcal{A}$ and $p \in \mathbb{N}$ the map $t \mapsto \omega_p^1(g'(t))$ is differentiable at t and there exists a stationary geodesic net $\gamma_p(t)$ with respect to $g'(t)$ so that

$$(1) \quad \omega_p^1(g'(t)) = L_{g'(t)}(\gamma_p(t)).$$

$$(2) \quad \frac{\partial}{\partial v}(\omega_p^1 \circ g'(s))|_{s=t} = \frac{1}{2} \int_{\gamma_p(t)} \text{tr}_{\gamma_p(t), g'(t)} \frac{\partial g'}{\partial v}(t) \, dL_{g'(t)}$$

Define $f_3 : (-1, 1)^K \rightarrow \mathbb{R}$ as $f_3(t) = \frac{1}{\delta} f_2(\delta t)$. We know that $\|f_3 - 1\|_{\infty, (-1, 1)^K} < C_5 \varepsilon'$. Now we want to use Lemma 3.4. In order to do that we will need to impose more restrictions on ε' . Let $\eta > 0$. Let $\varepsilon > 0$ be the one depending on η and $N = K$ according to Lemma 3.4. Choose ε' small enough so that $C_5 \varepsilon' < \varepsilon$, $\varepsilon' < \eta$ and $\varepsilon' \leq \varepsilon'_0$. Observe that this allows us to define $\delta > 0$ and $p_0 \in \mathbb{N}$ with all the properties in the construction above. Then we have

$$|f_3(x) - f_3(y)| \leq 2\varepsilon$$

for every $x, y \in (-1, 1)^K$. As f_3 is Lipschitz, we can apply Lemma 3.4 to f_3 and the full measure subset $\mathcal{A}' = \{\frac{t}{\delta} : t \in \mathcal{A}\}$. After passing to $(-\delta, \delta)^K$ by rescaling and using Remark 4.3, we get $K + 1$ sequences of points $\{s_{1,m}\}_m, \dots, \{s_{K+1,m}\}_{m \in \mathbb{N}}$ contained in \mathcal{A} and converging to a common limit $s \in (-\delta, \delta)^K$ such that:

- (1) f_2 is differentiable at each $s_{j,m}$.
- (2) The gradients $\nabla f_2(s_{j,m})$ converge to $K + 1$ vectors v_1, \dots, v_{K+1} with

$$d_{\mathbb{R}^N}(0, \text{Conv}(v_1, \dots, v_{K+1})) < \eta$$

Let $\alpha_1, \dots, \alpha_{K+1} \in [0, 1]$ be such that $\sum_{j=1}^{K+1} \alpha_j = 1$ and $|\sum_{j=1}^{K+1} \alpha_j v_j| < \eta$. Then if m is sufficiently large,

$$|\sum_{j=1}^{K+1} \alpha_j \nabla f_2(s_{j,m})| < \eta$$

and hence

$$|\sum_{j=1}^{K+1} \alpha_j \frac{\partial f_2}{\partial t_k}(s_{j,m})| < \eta$$

for every $k = 1, \dots, K$. But using the definition of f_2 and denoting $\gamma_{j,m} = \gamma_p(s_{j,m})$,

$$\begin{aligned} \frac{\partial f_2}{\partial t_k}(s_{j,m}) &= \frac{\frac{\partial}{\partial t_k} \omega_p^1(M, g'(s))|_{s=s_{j,m}}}{\alpha(n, 1) \text{Vol}(M, g)^{\frac{1}{n}} p^{\frac{n-1}{n}}} - \int_M \psi_k(x) \, d\text{Vol}_g \\ &= \frac{\int_{\gamma_{j,m}} \text{tr}_{\gamma_{j,m}, g'(s_{j,m})} \frac{\partial g'}{\partial t_k}(s_{j,m}) \, dL_{g'(s_{j,m})}}{2\alpha(n, 1) \text{Vol}(M, g)^{\frac{1}{n}} p^{\frac{n-1}{n}}} - \int_M \psi_k(x) \, d\text{Vol}_g \end{aligned}$$

Passing to a subsequence, we can obtain stationary geodesic networks $\gamma_1, \dots, \gamma_{K+1}$ with respect to $g(s)$, where $\lim_{m \rightarrow \infty} \gamma_{j,m} = \gamma_j$ for every $j = 1, \dots, K + 1$. Hence from the previous,

$$\left| \sum_{j=1}^{K+1} \alpha_j \frac{\int_{\gamma_j} \text{tr}_{\gamma_j, g'(s)} \frac{\partial g'(s)}{\partial t_k} dL_{g'(s)}}{2\alpha(n, 1) \text{Vol}(M, g)^{\frac{1}{n}} p^{\frac{n-1}{n}}} - \int_M \psi_k(x) d\text{Vol}_g \right| \leq \eta$$

for every $k = 1, \dots, K$. Using that $\|\hat{g}(t) - g\|_{C^q} < \frac{\varepsilon'}{2}$, $\|g'(t) - \hat{g}(t)\|_{C^q} < \frac{\varepsilon'}{2}$ and $\|\frac{\partial g'}{\partial v} - \frac{\partial \hat{g}}{\partial v}\|_{C^q} < \varepsilon'$ for every $t \in (-\delta, \delta)^K$ and $v \in \mathbb{R}^K : |v| = 1$; and the fact that $\varepsilon' < \eta$, we can see that there exists a constant $C_6 > 0$ such that

$$\left| \sum_{j=1}^{K+1} \alpha_j \frac{\int_{\gamma_j} \text{tr}_{\gamma_j, \hat{g}(s)} \frac{\partial \hat{g}(s)}{\partial t_k} dL_{\hat{g}(s)}}{2\alpha(n, 1) \text{Vol}(M, g'(s))^{\frac{1}{n}} p^{\frac{n-1}{n}}} - \int_M \psi_k(x) d\text{Vol}_{g'(s)} \right| \leq C_6 \eta$$

By definition of \hat{g} , $\frac{\partial \hat{g}}{\partial t_k}(s) = 2\psi_k \hat{g}(s)$ thus

$$(5) \quad \left| \sum_{j=1}^{K+1} \alpha_j \frac{\int_{\gamma_j} \psi_k dL_{\hat{g}(s)}}{\alpha(n, 1) \text{Vol}(M, g'(s))^{\frac{1}{n}} p^{\frac{n-1}{n}}} - \int_M \psi_k(x) d\text{Vol}_{g'(s)} \right| \leq C_6 \eta$$

Combining (5) with the fact that $\|g'(s) - \hat{g}(s)\|_{C^q} < \frac{\varepsilon'}{2}$,

$$(6) \quad \left| \sum_{j=1}^{K+1} \alpha_j \frac{\int_{\gamma_j} \psi_k dL_{g'(s)}}{\alpha(n, 1) \text{Vol}(M, g'(s))^{\frac{1}{n}} p^{\frac{n-1}{n}}} - \int_M \psi_k(x) d\text{Vol}_{g'(s)} \right| \leq C_7 \eta$$

But we know that $L_{g'(s)}(\gamma_j) = \omega_p^1(g'(s))$ for every $j = 1, \dots, K+1$, so

$$\begin{aligned} & \left| \frac{\int_{\gamma_j} \psi_k dL_{g'(s)}}{\alpha(n, 1) \text{Vol}(M, g'(s))^{\frac{1}{n}} p^{\frac{n-1}{n}}} - \int_{\gamma_j} \psi_k dL_{g'(s)} \right| = \\ & \left| \int_{\gamma_j} \psi_k dL_{g'(s)} \left\| \frac{\omega_p^1(g'(s))}{\alpha(n, 1) \text{Vol}(M, g'(s))^{\frac{1}{n}} p^{\frac{n-1}{n}}} - 1 \right\| \right| \leq \\ & \left| \frac{\omega_p^1(g'(s))}{\alpha(n, 1) \text{Vol}(M, g'(s))^{\frac{1}{n}} p^{\frac{n-1}{n}}} - 1 \right| \leq \eta \end{aligned}$$

if $p \geq p_1$ for some $p_1 \in \mathbb{N}$, because of the Weyl law and the fact $0 \leq \psi_k \leq 1$. Hence from (6),

$$\left| \sum_{j=1}^{K+1} \alpha_j \int_{\gamma_j} \psi_k dL_{g'(s)} - \int_M \psi_k d\text{Vol}_{g'(s)} \right| \leq C_8 \eta$$

for some constant C_8 depending only on g . Let us take $\eta = \frac{\varepsilon_1}{2C_8K}$ and $p \geq \max\{p_0, p_1\}$. Notice that $\|g'(s) - g\|_{C^q} \leq \|g'(s) - \hat{g}(s)\|_{C^q} + \|\hat{g}(s) - g\|_q < \frac{\varepsilon'}{2} + \frac{\varepsilon'}{2} < \varepsilon'_0$. Also, from the proof of Proposition 3.1 the metric $g'(t)$ is bumpy for every $t \in (-\delta, \delta)^K$, and therefore we can represent γ_i as a map $\gamma_i : \Gamma_i \rightarrow M$ where each connected component of the weighted multigraph Γ_i is good and the restrictions of γ_i to those connected

components are embedded and therefore nondegenerate (here we are using Remark 2.9). Therefore, the metric $g'(s)$ has all the properties required by Proposition 4.1 except that it may not be C^∞ (in principle it is only C^q). But applying the Implicit Function Theorem (see [14, Lemma 2.6]), we can find a C^∞ metric \tilde{g} close enough to $g'(s)$ in the C^q topology so that $\|\tilde{g} - g\|_{C^q} < \varepsilon_0$ and it also admits stationary geodesic networks $\tilde{\gamma}_1, \dots, \tilde{\gamma}_{k+1}$ whose connected components are nondegenerate and verifying

$$\left| \sum_{j=1}^{K+1} \alpha_j \int_{\tilde{\gamma}_j} \psi_k dL_{\tilde{g}} - \int_M \psi_k d\text{Vol}_{\tilde{g}} \right| < \frac{\varepsilon_1}{K}$$

for every $k = 1, \dots, K + 1$, which completes the proof. \square

Now we will show that Proposition 4.1 implies Theorem 1.5. Given $g \in \mathcal{M}^\infty$, $\varepsilon_1 > 0$, $\tilde{K} \in \mathbb{N}$ and $S \in \mathcal{C}_{g, \tilde{K}, \varepsilon_1}$ we will denote $\mathcal{M}(g, \varepsilon_1, \tilde{K}, S)$ the set of all metrics $\tilde{g} \in \mathcal{M}^\infty$ such that $\|\tilde{g} - g\|_{C^q} < \varepsilon_1$ (computed with respect to g) and there exist stationary geodesic networks $\gamma_1, \dots, \gamma_J$ with respect to \tilde{g} whose connected components are nondegenerate (according to Definition 2.10) and coefficients $\alpha_1, \dots, \alpha_J \in [0, 1]$ with $\sum_{j=1}^J \alpha_j = 1$ such that (2) holds for every $k = 1, \dots, K$. By the Implicit Function Theorem, $\mathcal{M}(g, \varepsilon_1, \tilde{K}, S)$ is open (see [14, Lemma 2.6]). Therefore given $\varepsilon_1 > 0$ and $\tilde{K} \in \mathbb{N}$ the set

$$\mathcal{M}(\varepsilon_1, \tilde{K}) = \bigcup_{g \in \mathcal{M}^\infty} \bigcup_{S \in \mathcal{C}_{g, \varepsilon_1, \tilde{K}}} \mathcal{M}(g, \varepsilon_1, K, S)$$

is open and by Proposition 4.1 it is also dense in \mathcal{M}^∞ . Define

$$\tilde{\mathcal{M}} = \bigcap_{m \in \mathbb{N}} \mathcal{M}\left(\frac{1}{m}, m\right)$$

which is a generic subset of \mathcal{M}^∞ in the Baire sense. We are going to prove that if $\tilde{g} \in \tilde{\mathcal{M}}$ then there exists a sequence of equidistributed stationary geodesic networks with respect to \tilde{g} .

Let $\tilde{g} \in \tilde{\mathcal{M}}$. Fix $m \in \mathbb{N}$. By definition, there exists $g \in \mathcal{M}^\infty$ such that $\tilde{g} \in \mathcal{M}(g, \frac{1}{m}, m, S)$ for some $S \in \mathcal{C}_{g, \frac{1}{m}, m}$. Therefore, \tilde{g} belongs to a $\frac{1}{m}$ neighborhood of g in the C^K topology; and there exist $J = J_m \in \mathbb{N}$, stationary geodesic networks $\gamma_{m,1}, \dots, \gamma_{m,J_m}$ with respect to \tilde{g} and coefficients $\alpha_1, \dots, \alpha_{m,J_m} \in [0, 1]$ with $\sum_{j=1}^{J_m} \alpha_{m,j} = 1$ satisfying

$$(7) \quad \left| \sum_{j=1}^{J_m} \alpha_{m,j} \int_{\gamma_{m,j}} \psi_k(x) dL_{\tilde{g}} - \int_M \psi_k(x) d\text{Vol}_{\tilde{g}} \right| < \frac{1}{mK}$$

for every $k = 1, \dots, K$. Let $f \in C^\infty(M, \mathbb{R})$. We want to obtain a formula analogous to the previous one but replacing ψ_k by f , which will imply the following proposition.

Proposition 4.4. *For each $m \in \mathbb{N}$, there exists $J = J_m$ depending on m , integers $\{c_{m,j}\}_{1 \leq j \leq J_m}$ and stationary geodesic networks $\{\gamma_{m,j}\}_{1 \leq j \leq J_m}$ such that*

$$\left| \frac{\sum_{j=1}^{J_m} c_{m,j} \int_{\gamma_{m,j}} f dL_{\tilde{g}}}{\sum_{j=1}^{J_m} c_{m,j} L_{\tilde{g}}(\gamma_{m,j})} - \int_M f d\text{Vol}_{\tilde{g}} \right| \leq \frac{D(f)}{m}$$

for every $f \in C^\infty(M, \mathbb{R})$, where $D(f) > 0$ is a constant depending only on f .

Proof. Given $m \in \mathbb{N}$, consider as above $g \in \mathcal{M}^\infty$ and $S \in \mathcal{C}_{g, \frac{1}{m}, m}$ such that $\tilde{g} \in \mathcal{M}(g, \frac{1}{m}, m, S)$. Define $J_m \in \mathbb{N}$, stationary geodesic networks $\gamma_{m,1}, \dots, \gamma_{m,J_m}$ with respect to \tilde{g} and coefficients $\alpha_{m,1}, \dots, \alpha_{m,J_m}$ such that (7) holds. Taking $S = (K, \{\hat{B}_k\}_k, \{B_k\}_k, \{\phi_k\}_k) \in \mathcal{C}_{g, \frac{1}{m}, m}$ into account, let us choose points q_1, \dots, q_K with $q_k \in \hat{B}_k$ for each $k = 1, \dots, K$. The idea will be to approximate the integral of $f(x)$ by the integral of the function $\sum_{k=1}^K f(q_k) \psi_k(x)$. First of all, by using (7) we can see that

$$(8) \quad \left| \sum_{j=1}^{J_m} \alpha_{m,j} \int_{\gamma_{m,j}} \left[\sum_{k=1}^K f(q_k) \psi_k(x) \right] dL_{\tilde{g}} - \int_M \left[\sum_{k=1}^K f(q_k) \psi_k(x) \right] d\text{Vol}_{\tilde{g}} \right| < \frac{D_1}{m}$$

where $D_1 = \|f\|_\infty = \max\{f(x) : x \in M\}$ depends only on f (and not on m , g or S). On the other hand, given $x \in M$

$$\begin{aligned} |f(x) - \sum_{k=1}^K f(q_k) \psi_k(x)| &= |f(x) \sum_{k=1}^K \psi_k(x) - \sum_{k=1}^K f(q_k) \psi_k(x)| \\ &= \left| \sum_{k=1}^K f(x) \psi_k(x) - f(q_k) \psi_k(x) \right| \\ &\leq \sum_{k: x \in B_k} |f(x) - f(q_k)| |\psi_k(x)| \\ &= \sum_{k: x \in B_k} |\nabla_{\tilde{g}} f(c_k)| d_{\tilde{g}}(x, q_k) \psi_k(x) \\ &\leq \frac{2 \|\nabla_{\tilde{g}} f\|_\infty}{m} \sum_{k=1}^K \psi_k(x) \\ &= \frac{2 \|\nabla_{\tilde{g}} f\|_\infty}{m} \end{aligned}$$

We used the Mean Value Theorem and the fact that $\text{supp}(\psi_k) \subseteq B_k$ and $\text{diam}_{\tilde{g}}(B_k) \leq 2 \text{diam}_g(B_k) \leq \frac{2}{m}$ for every i . Combining this and (8) we get

$$(9) \quad \left| \sum_{j=1}^{J_m} \alpha_{m,j} \int_{\gamma_{m,j}} f \, dL_{\tilde{g}} - \int_M f \, d\text{Vol}_{\tilde{g}} \right| < \frac{D_2}{m}$$

where D_2 depends only on f and \tilde{g} . Let us choose integers $c_{m,j}, d_m \in \mathbb{N}$ such that

$$\left| \frac{\alpha_{m,j}}{L_{\tilde{g}}(\gamma_{m,j})} - \frac{c_{m,j}}{d_m} \right| < \frac{1}{m J_m L_{\tilde{g}}(\gamma_{m,j})}$$

Then it holds

$$\begin{aligned} \left| \sum_{j=1}^{J_m} \alpha_{m,j} \int_{\gamma_{m,j}} f \, dL_{\tilde{g}} - \sum_{j=1}^{J_m} \frac{c_{m,j}}{d_m} \int_{\gamma_{m,j}} f \, dL_{\tilde{g}} \right| &\leq \sum_{j=1}^{J_m} \left| \frac{\alpha_{m,j}}{L_{\tilde{g}}(\gamma_{m,j})} - \frac{c_{m,j}}{d_m} \right| \left| \int_{\gamma_{m,j}} f \, dL_{\tilde{g}} \right| \\ &\leq \sum_{j=1}^{J_m} \frac{1}{m J_m L_{\tilde{g}}(\gamma_{m,j})} \|f\|_{\infty} L_{\tilde{g}}(\gamma_{m,j}) \\ &= \frac{D_1}{m} \end{aligned}$$

and hence by (9) and triangle inequality we get

$$\left| \sum_{j=1}^{J_m} \frac{c_{m,j}}{d_m} \int_{\gamma_{m,j}} f \, dL_{\tilde{g}} - \int_M f \, d\text{Vol}_{\tilde{g}} \right| < \frac{D_3}{m}$$

where $D_3 = D_2 + D_1$ depends only on f and \tilde{g} . On the other hand,

$$\begin{aligned} \left| \sum_{j=1}^{J_m} \frac{c_{m,j}}{d_m} \int_{\gamma_{m,j}} f \, dL_{\tilde{g}} - \frac{\sum_{j=1}^{J_m} c_{m,j} \int_{\gamma_{m,j}} f \, dL_{\tilde{g}}}{\sum_{j=1}^{J_m} c_{m,j} L_{\tilde{g}}(\gamma_{m,j})} \right| &\leq \\ \left| \frac{1}{d_m} - \frac{1}{\sum_{j=1}^{J_m} c_{m,j} L_{\tilde{g}}(\gamma_{m,j})} \right| \left| \sum_{j=1}^{J_m} c_{m,j} \int_{\gamma_{m,j}} f \, dL_{\tilde{g}} \right| &\leq \\ \left| \frac{1}{d_m} - \frac{1}{\sum_{j=1}^{J_m} c_{m,j} L_{\tilde{g}}(\gamma_{m,j})} \right| \sum_{j=1}^{J_m} c_{m,j} \|f\|_{\infty} L_{\tilde{g}}(\gamma_{m,j}) &= \\ D_1 \left| \sum_{j=1}^{J_m} \frac{c_{m,j}}{d_m} L_{\tilde{g}}(\gamma_{m,j}) - 1 \right| &\leq \frac{D_1}{m} \end{aligned}$$

because $|\sum_{j=1}^{J_m} \frac{c_{m,j}}{d_m} L_{\tilde{g}}(\gamma_{m,j}) - 1| < \frac{1}{m}$. Hence

$$\left| \frac{\sum_{j=1}^{J_m} c_{m,j} \int_{\gamma_{m,j}} f dL_{\tilde{g}}}{\sum_{j=1}^{J_m} c_{m,j} L_{\tilde{g}}(\gamma_{m,j})} - \int_M f d\text{Vol}_{\tilde{g}} \right| \leq \frac{D_4}{m}$$

for a constant D_4 depending only on f , as desired. \square

Given $\tilde{g} \in \tilde{\mathcal{M}}$, using Proposition 4.4 we can find a sequence of finite lists of connected embedded stationary geodesic nets $\{\beta_{m,1}, \dots, \beta_{m,K_m}\}_{m \in \mathbb{N}}$ with respect to \tilde{g} satisfying the following: given $f \in C^\infty(M, \mathbb{R})$, if we denote $X_{m,j} = \int_{\beta_{m,j}} f dL_{\tilde{g}}$ and $\bar{X}_{m,j} = L_{\tilde{g}}(\beta_{m,j})$, then

$$(10) \quad \left| \frac{\sum_{j=1}^{K_m} X_{m,j}}{\sum_{j=1}^{K_m} \bar{X}_{m,j}} - \alpha \right| \leq \frac{D(f)}{m}$$

where $\alpha = \int_M f d\text{Vol}_{\tilde{g}}$ and $D(f)$ is a constant depending only on f . The lists $\{\beta_{m,j}\}_{1 \leq j \leq K_m}$ are obtained from the lists $\{\gamma_{m,j}\}_{1 \leq j \leq J_m}$ and the coefficients $\{c_{m,j}\}_{1 \leq j \leq J_m}$ from Proposition 4.4 by decomposing each $\gamma_{m,j}$ as a union of embedded stationary geodesic networks whose domain is a good weighted multigraph (see Remark 2.9) and listing each of them $c_{m,j}$ times. From the $X_{m,j}$'s and the $\bar{X}_{m,j}$'s, we want to construct two sequences $\{Y_i\}_{i \in \mathbb{N}}, \{\bar{Y}_i\}_{i \in \mathbb{N}}$ such that

- For all i , there exist integers $m(i), j(i)$ (chosen independently of f) with $Y_i = X_{m(i), j(i)}$ and $\bar{Y}_i = \bar{X}_{m(i), j(i)}$,
- It holds

$$\lim_{k \rightarrow \infty} \frac{\sum_{i=1}^k Y_i}{\sum_{i=1}^k \bar{Y}_i} = \alpha.$$

This can be done as in [16, p. 437-439] and gives us a sequence $\{\gamma_i\}_{i \in \mathbb{N}}$ of connected embedded stationary geodesic networks with respect to \tilde{g} (defined as $\gamma_i = \beta_{m(i), j(i)}$), which is constructed independently of the constant $D(f)$. It holds

$$\lim_{k \rightarrow \infty} \frac{\sum_{i=1}^k \int_{\gamma_i} f dL_{\tilde{g}}}{\sum_{i=1}^k L_{\tilde{g}}(\gamma_i)} = \int_M f d\text{Vol}_{\tilde{g}}$$

for every $f \in C^\infty(M, \mathbb{R})$. This gives us the desired equidistribution result and completes the proof of Theorem 1.5.

Remark 4.5. Observe that combining the Weyl law for 1-cycles in 3-manifolds from [10] with Theorem 1.5 which we just proved, we obtain Theorem 1.2.

5. EQUIDISTRIBUTION OF ALMOST EMBEDDED CLOSED GEODESICS IN 2-MANIFOLDS

In this section we show that the proof of Theorem 1.5 combined with the work of Chodosh and Mantoulidis in [6] (where they show that the p -widths on a surface are realized by collections of almost embedded closed geodesics) imply Theorem 1.1. The strategy to show this result will be to follow the proof of Theorem 1.5 replacing “embedded stationary geodesic network” by “almost embedded closed geodesic”. The main change needed in the proof is the following version of Proposition 3.1:

Proposition 5.1. *Let M be a closed 2-manifold. Let $g : I^N \rightarrow \mathcal{M}^q$ be a smooth embedding, $N \in \mathbb{N}$. If $q \geq N + 3$, there exists an arbitrarily small perturbation in the C^∞ topology $g' : I^N \rightarrow \mathcal{M}^q$ such that there is a full measure subset $\mathcal{A} \subseteq I^N$ with the following property: for any $p \in \mathbb{N}$ and any $t \in \mathcal{A}$, the function $s \mapsto \omega_p^1(g'(s))$ is differentiable at t and there exist almost embedded closed geodesics $\gamma_p^1, \dots, \gamma_p^P : S^1 \rightarrow M$ such that the following two conditions hold*

- (1) $\omega_p^1(g'(t)) = \sum_{i=1}^P L_{g'(t)}(\gamma_p^i(t)).$
- (2) $\frac{\partial}{\partial v}(\omega_p^1 \circ g')|_{s=t} = \frac{1}{2} \sum_{i=1}^P \int_{\gamma_p^i} \text{tr}_{\gamma_p^i, g'(t)} \frac{\partial g'}{\partial v}(t) dL_{g'(t)}.$

Proof. We are going to adapt the proof of Proposition 3.1 by introducing some necessary changes. A priori, the easiest way to do this seems to be substituting “stationary geodesic network” by “finite union of almost embedded closed geodesics” everywhere and use the Bumpy metrics theorem for almost embedded minimal submanifolds proved by Brian White in [24]. Nevertheless, there is not an easy condition (analog to conditions (1) to (7) in the proof of Proposition 3.1) that we can impose on a sequence of almost embedded closed geodesics to converge to another almost embedded closed geodesic without classifying them by their self-intersections and the angles formed there. Therefore, what we will do is to treat the almost embedded closed geodesics as a certain class of stationary geodesic networks, and then proceed as with Proposition 3.1.

To each almost embedded closed geodesic $\gamma : S^1 \rightarrow M$ we can associate a connected graph $\Gamma = S^1 / \sim$ where \sim is the equivalence relation $s \sim t$ if and only if $\gamma(s) = \gamma(t)$. This induces a map $f : \Gamma \rightarrow M$ defined as $f([t]) = \gamma(t)$. Observe that the as the self-intersections of γ are transverse, the vertices of Γ are mapped precisely to those self-intersections and the map $f : \Gamma \rightarrow M$ is injective. Moreover, Γ is a good multigraph and $f : \Gamma \rightarrow (M, g)$ is an embedded stationary geodesic network. We replace the set $\{\Gamma_i\}_{i \in \mathbb{N}}$ which in the proof of Proposition 3.1 is the set of all good connected multigraphs by the countable set of pairs $\mathcal{P} = \{(\Gamma, r)\}$ where Γ is a good multigraph which can be obtained as $\Gamma = S^1 / \sim$ from an almost embedded closed geodesic $\gamma : S^1 \rightarrow (M, g)$ with respect to some metric g as before and r is the set of pairs

$((E_1, i_1), (E_2, i_2))$ such that $\pi_{E_1}(i_1) = \pi_{E_2}(i_2)$ and $(-1)^{i_1+1} \frac{f|_{E_1}(i_1)}{|f|_{E_1}(i_1)|_g} = (-1)^{i_2} \frac{f|_{E_2}(i_2)}{|f|_{E_2}(i_2)|_g}$ (in other words, r contains the necessary information to reparametrize the geodesic net $f : \Gamma \rightarrow M$ to an immersed closed geodesic $\gamma : S^1 \rightarrow (M, g)$). Observe that if $(\Gamma, r) \in \mathcal{P}$ and $f : \Gamma \rightarrow (M, g)$ is an embedded stationary geodesic network verifying $(-1)^{i_1+1} \frac{f|_{E_1}(i_1)}{|f|_{E_1}(i_1)|_g} = (-1)^{i_2} \frac{f|_{E_2}(i_2)}{|f|_{E_2}(i_2)|_g}$ for every $((E_1, i_1), (E_2, i_2)) \in r$ then $f : \Gamma \rightarrow (M, g)$ can be reparametrized as an immersed closed geodesic $\gamma : S^1 \rightarrow (M, g)$ whose self intersections occur precisely at the points $\{f(v) : v \text{ vertex of } \Gamma\}$.

Taking the previous into account, instead of the $\tilde{B}_{\Gamma, M}$ in the proof of Proposition 3.1 we will work with the following. Consider the set of pairs (Γ, r) where Γ is a graph, $\Gamma = \bigcup_{i=1}^P \Gamma_i$ as a union of connected components, $r = (r_i)_{1 \leq i \leq P}$ and $(\Gamma_i, r_i) \in \mathcal{P}$ for every $1 \leq i \leq P$. Given such a pair (Γ, r) and a natural number $M \in \mathbb{N}$ we define $\mathcal{B}_{\Gamma, r, M}$ to be the set of all $t \in (-1, 1)^N$ such that there exists a stationary geodesic network $f : \Gamma \rightarrow (M, g'(t))$ verifying

- (1) For each $1 \leq i \leq P$, $f_i = f|_{\Gamma_i}$ is an embedding and verifies the relations $(-1)^{i_1+1} \frac{f_i|_{E_1}(i_1)}{|f_i|_{E_1}(i_1)|_{g'(t)}} = (-1)^{i_2} \frac{f_i|_{E_2}(i_2)}{|f_i|_{E_2}(i_2)|_{g'(t)}}$ for every $((E_1, i_1), (E_2, i_2)) \in r_i$.
- (2) $\|f_i\|_3 \leq M$ for every $1 \leq i \leq P$.
- (3) $F_1(g'(t), f_i) \geq \frac{1}{M}$ for every $1 \leq i \leq P$.
- (4) $F_2^{(E_1, i_1), (E_2, i_2)}(g'(t), f_i) \leq 1 - \frac{1}{M}$ for every $1 \leq i \leq P$, and every pair $(E_1, i_1) \neq (E_2, i_2) \in \mathcal{E}_i \times \{0, 1\}$ such that $\pi_{E_1}(i_1) = \pi_{E_2}(i_2)$.
- (5) $d_{(g'(t), f_i)}^E(s) \geq \frac{1}{M}$ for every $1 \leq i \leq P$, $E \in \mathcal{E}_i$ and $s \in E$.
- (6) $d_{(g'(t), f_i)}^{E, E'}(s) \geq \frac{1}{M}$ for every $1 \leq i \leq P$, $E \neq E' \in \mathcal{E}_i$ and $s \in E$.
- (7) $\omega_p^1(g'(t)) = L_{g'(t)}(f)$.

Therefore, same as in Proposition 3.1 we have $I^N = \bigcup_{\Gamma, r, M} \mathcal{B}_{\Gamma, r, M}$ because of the fact showed in [6] that the p -widths on surfaces are realized by unions of almost embedded closed geodesics; and each $\mathcal{B}_{\Gamma, r, M}$ is closed. The rest of the proof follows exactly as in Proposition 3.1 if we replace the pairs (Γ, M) by the triples (Γ, r, M) . \square

The rest of the proof follows that of Theorem 1.5 word to word.

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