Chapter 4 Hybrid Evolutionary Algorithm: A Case Study on Graph Coloring

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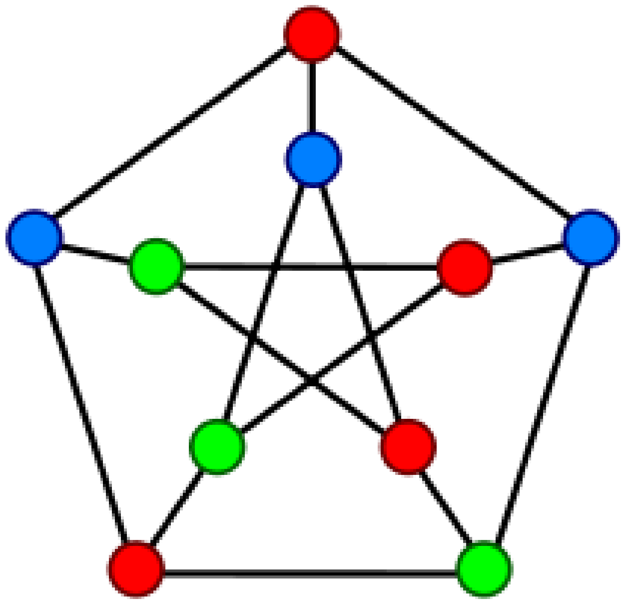
Graph Coloring

- Given an undirected graph G=(V,E), the graph coloring problem (GCP) consists of assigning a color $c_i (1 \leq c_i \leq k)$ to each vertex such that adjacent vertices receive different colors and the number of colors used k is minimized.



Optimization or Decision?

* GCP—**Optimization** Problem (NP Hard):
  * To find the smallest number of colors k.

* ***k***-Coloring—**Decision** Problem (NP Complete):
  * Given a ***k***, we are asked whether there exists a coloring such that all the adjacent coloring constraints are satisfied.

* The **Optimization** version of GCP can be solved by tackling a series of the **Decision** version of GCP problem with a gradually decreasing ***k***.

* Thus, these two versions are equivalent to each other.

Solution Procedure

* We starts from an initial *k* and solve the *k*-coloring problem. As soon as the *k*-coloring problem is solved, we decrease *k* by setting *k* to *k*-1 and solve again the *k*-coloring problem.

* This process is repeated until no legal *k*-coloring can be found.

* Smaller *k* → harder *k*-coloring problem. Thus, the solution approach just described solves thus a series of *k*-coloring problems of increasing difficulty.

* We only consider the K-coloring problem in this presentation.

ILP Formulation

$$x_{v,c} = \begin{cases} 1 & \text{if vertex } v \text{ is coloured with colour } c \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{c=1}^{k} x_{v,c} = 1 \quad \forall \text{ vertices } v \in V$$

$$x_{u,c} + x_{v,c} \leq 1 \quad \forall \text{ colours } c \in K \quad \forall \text{ edges } \{u, v\} \in E$$

- Given ***k*** colors, $x_{v,c}$ is the decision variables.
- The first constraint requires that each vertex receives only one color.
- The second constraint denotes that adjacent vertices should receive different colors.

Assignment Representation

- A solution of k-coloring problem can be represented as a series of colors that each vertex receives:

- $S = \{c_1, c_2, \dots, c_n\}$ where c_v denotes the color of vertex *v*. It is required that for any $(u, v) \in E,\ c_u \neq c_v$.

- This representation is **natural**, but not **intuitive** and **essential**.

Grouping Representation

* The feasible solution of k-coloring problem can also be presented as a set of independent sets, where an independent set is a set of non-adjacent vertices.

* $S = I_1 \cup I_2 \cup \cdots \cup I_k$, where
  1. $I_j \cap I_l = \emptyset$ for any j and l
  2. $I_1 \cup I_2 \cup \cdots \cup I_k = V$
  3. I_j is an independent set for any j.

* Thus, the k-coloring problem becomes to partition the N vertices into k independent sets.

Applications

* Mobile radio frequency assignment
* Timetabling: Education, transportation, sports
* Register allocation
* Crew scheduling
* Printed circuit testing
* Air traffic flow management
* Satellite range scheduling
* Routing and wavelength assignment in WDM networks

Literature Review (1)

Constructive Greedy Algorithms

* The first heuristic approaches to solving the graph coloring problem, which color the vertices of the graph one by one guided by a greedy function.

* They are very fast by nature but their quality is unsatisfactory.

* The best known algorithms in this class are:
    * the *largest saturation degree* heuristic (DSATUR) (D. Brelaz, 1979)
    * *recursive largest first* heuristic (RLF) (F.T. Leighton, 1979)

* Often used to generate initial solutions for advanced algorithms

Literature Review (2)

Local Search Algorithms

* One representative example is the so-called *Tabucol* algorithm which is the first application of Tabu Search to graph coloring (Hertz, de Werra, 1987)

* Other local search metaheuristic methods include:
  * Simulated Annealing (Johnson et al, 1991)
  * Iterated Local Search (Chiarandini and Stutzle, 2002)
  * Reactive Partial Tabu Search (Blochliger, Zufferey, 2008)
  * GRASP (Laguna, Marti, 2001)
  * Variable Neighborhood Search (Avanthay et al, 2003)
  * Variable Space Search (Hertz et al, 2008)
  * *Clustering*-Guided Tabu Search (Porumbel, Hao, 2009)

* Interested readers are referred to [Galinier, Hertz, 2006] for a comprehensive survey of the local search approaches.

* I will describe the famous *Tabucol* algorithm in detail.

Literature Review (3)

Hybrid Evolutionary Algorithms

* One of the most recent and very promising approaches is based upon hybridization that embeds a **local search** algorithm into the framework of an **evolutionary algorithm** in order to achieve a better tradeoff between intensification and diversification, see for examples:
  * Dorne, Hao, 1998
  * Galinier, Hao, 1999
  * Galinier, Hertz, Zufferey, 2008
  * Malaguti, Monaci, Toth, 2008
  * Porumbel, Hao, Kuntz, 2009

Initial Solution——DSATUR

* The heuristic of DSATUR(Degree of Saturation) is to sequentially color vertices according to a DANGER-based heuristic. The main idea of DSATUR is based on least saturation degree.

* At each phase, it consists of two steps: The first is to choose a vertex to color and the other is to choose a color for the chosen vertex.

Initial Solution——DSATUR

* DSATUR starts by assigning color 1 to a vertex of maximal degree.
* Suppose F is a partial coloring of the vertices of G. The degree of saturation of a vertex x, degs(x), is the number of available colors that vertex x can use.
* The vertex to be colored next in the sequential coloring procedure of DSATUR is a vertex x with smallest degs(x), breaking ties by favoring vertex with larger uncolored degree.
* When deciding a color for a chosen vertex the color that is least likely to be required by neighboring vertices is selected.

DSATUR

1. Initialization:

* Color[N] = -1
* Vetex_Color_Avail[N][K]=1
* Num_Avail_Colors[N]=K
* Vertex_Uncolored_Degree[N]=Degree[K]

2. Choose the vertex v1 with the largest degree, color v1 with color 1

```
Color[v1] = 1 ;
for v1's adjacent vertices vj
    Vetex_Color_Avail[vj][1] = 0 ;
    Num_Avail_Colors[vj] -- ;
    Vertex_Uncolored_Degree[vj] -- ;
```

DSATUR

3. for(i = 2 ; i <= N ; i ++)

* 3.1 choose a vertex vi according to:
* < Num_Avail_Colors[vi] (small), Vertex_Uncolored_Degree[vi](large)>
* 3.2 choose a color ki for vertex vi:
* for each available color j for vertex vi
* calculate the number of uncolored vertices for which color j is available
* choose color ki with the smallest value of this number
* 3.3 color vertex vi with color ki and updating:
* Color[vi] = ki ;

  for vi's adjacent vertices vj

  Vertex_Uncolored_Degree[vj] -- ;

  if(Vetex_Color_Avail[vj][ki] == 1)

  Num_Avail_Colors[vj] -- ;

  Vetex_Color_Avail[vj][ki] = 0 ;

Improvements for DSATUR

* It is possible to improve DSATUR heuristic by considering more sophisticated information.

* For example, in case of choosing a color for the vertex, it would be better to consider the number of available colors.

* What else heuristics can be inspired in choosing vertex and choosing color?

Local Search

TabuCol

Search Space

- In this paper, we adapt the ***k-fixed penalty strategy*** which is also used by many coloring algorithms.
- For a given graph $G = (V; E)$, the number k of colors is fixed and the search space contains all possible (legal and illegal) k-colorings.
- A k-coloring is represented by $S = \{V1, \ldots, Vk\}$ such that Vi is the set of vertices receiving color i.
- Thus, if for all *Vi are independent sets*, then S is a legal k-coloring. Otherwise, S is an illegal (or conflicting) k-coloring.

Evaluation Function

- The optimization objective is then to minimize the number of conflicting edges (referred to *confict number* hereafter) and find a legal *k*-coloring in the search space.

- Given a *k*-coloring $S = \{V1, \ldots, Vk\}$, the evaluation function *f* counts the conflict number induced by S such that

$$f(S) = \sum_{\{u,v\} \in E} \delta_{uv}$$

where

$$\delta_{uv} = \begin{cases} 1, \text{ if } u \in V_i,\ v \in V_j \text{ and } i = j, \\ 0, \text{ otherwise.} \end{cases}$$

Initial Coloring

* The initial solution of our algorithm is randomly generated, i.e., each vertex in the graph is randomly assigned a color from 1 to *k*.
* Other greedy constructive heuristics are possible, like DSATUR, RLF, DANGER, etc.
* However, we observe that strong local search algorithms are not sensitive to the initial solutions.

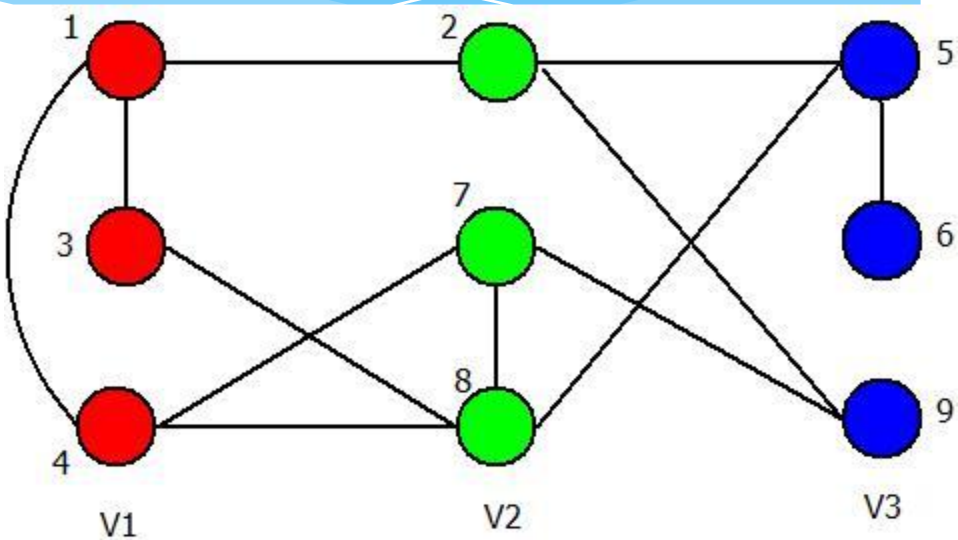
Neighborhood Moves

- A neighborhood of a given k-coloring is obtained by moving a ***conflicting*** vertex u from its original color class V_i to another color class V_j (denoted by $<u, i, j>$), called "critical one-move" neighborhood.

- Therefore, for a k-coloring S with cost $f(S)$, the size of this neighborhood is bounded by $O(f(S) \times k)$.

An Example

* f = 4
* Conflicting pairs: (1,3), (1,4), (7,8), (5,6)
* Critical One-Move: Only considers vertices 1, 3, 4, 7, 8, 5, 6. Totally 7*2=14 moves.



Neighborhood Evaluation

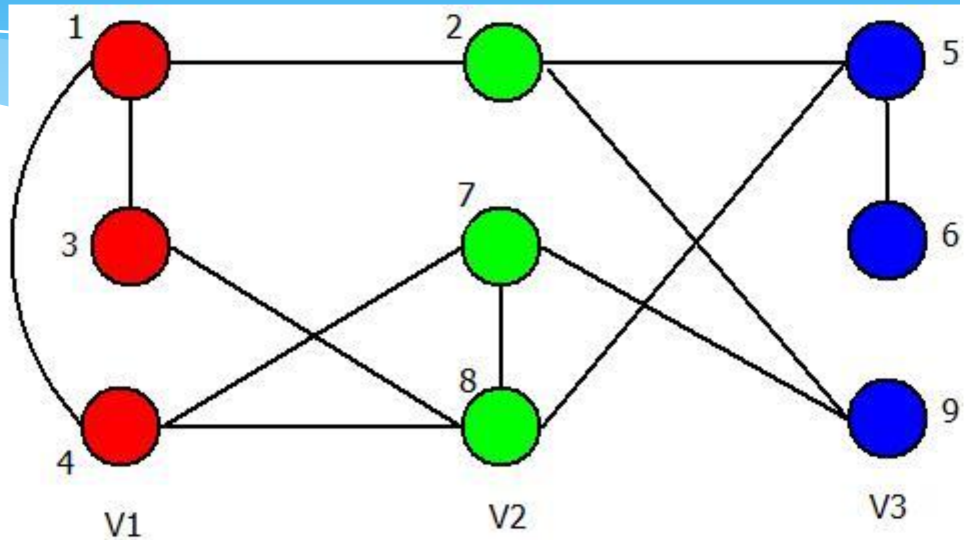
* In order to evaluate the neighborhood efficiently, we employ an ***incremental evaluation*** technique.

* The effect of each move on the objective function can be quickly calculated by a special data structure.

* Each time a move is carried out, only the move values affected by this move are updated accordingly.

Adjacent-Color Table

Vertex	Red V1	Green V2	Blue V3
1	**2**	1	0
2	1	**0**	1
3	**1**	1	0
4	**1**	2	0
5	0	2	1
6	0	0	**1**
7	1	**1**	1
8	2	**1**	1
9	0	2	**0**

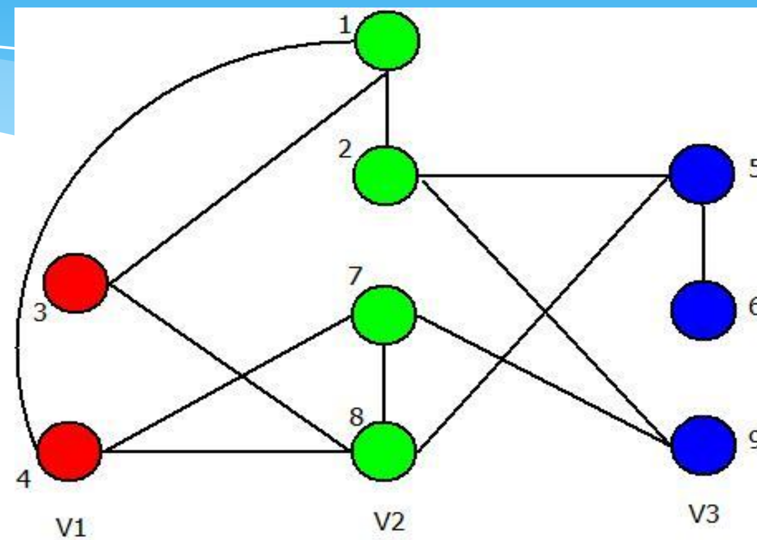


- This matrix M[u][i] (N*k) measures the number of adjacent vertices if vertex ***u*** receives color ***i***.

- Thus, the incremental move value of a move <u, i, j> can be quickly calculated as:
$\Delta(u,i,j) = M[u][j] - M[u][i]$

Updating of Adjacent-Color Table

Vertex	Red V1	Green V2	Blue V3
1	2	1	0
2	1-1=0	**0+1=1**	1
3	**1-1=0**	1+1=2	0
4	**1-1=0**	2+1=3	0
5	0	2	1
6	0	0	**1**
7	1	**1**	1
8	2	**1**	1
9	0	2	**0**



* Move (1, v1, v2):
* Only its adjacent vertices 2, 3 and 4 are affected, and only the v1 and v2 columns need to be updated.
* All old color (v1) columns decrease by 1.
* All new color (v2) columns increase by 1.

Simple Local Search

1. Generate initial solution S, Calculate f(S)
2. Initialize the adjacent-color table M.
3. While {there exist improving moves}

3.1 Construct the neighborhood of S, denoted by N(S)

3.2 Calculate the Δ values of all critical one-moves

3.3 Find the best move with the least Δ value

3.4 Perform the best move: $f' = f + \Delta_{best}$

3.5 Update the adjacent-color table M

End

Tabu Search
Escaping from Local Optimum

* Tabu Search incorporates a tabu list as a “recency-based” memory structure to assure that solutions visited within a certain span of iterations, called tabu tenure, will not be revisited.

* TS then restricts consideration to moves not forbidden by the tabu list, and selects a move that produces the best move value to perform.

Tabu Search

- 1. What?
- 2. Tenure?
- 3. How to judge if a move is forbidden?

Attributes or Solution?

- It should be noted that we generally forbid **attributes** of solutions, but not the **solutions** themselves, since it is too expensive to forbid **solutions**.

- For the tabu list, once move <*u*, *i*, *j*> is performed, vertex *u* is forbidden to move back to color class *Vi* for the next *tt* iterations.

Tabu Tenure

* For the tabu list, once move $<u, i, j>$ is performed, vertex u is forbidden to move back to color class Vi for the next tt iterations.

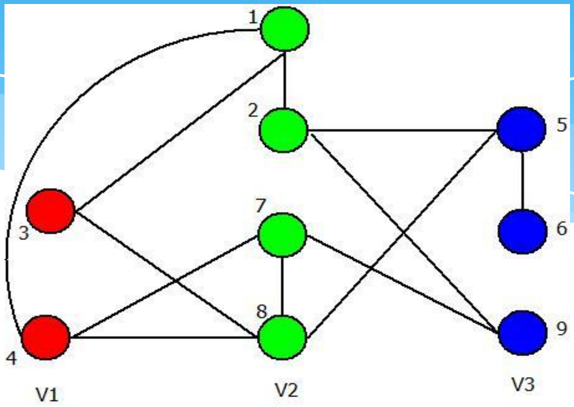
* Here, the tabu tenure tt is dynamically determined by

$$tt = f(S) + r(10)$$

where $r(10)$ takes a random number in $\{1,\ldots,10\}$.

TabuTenure Table

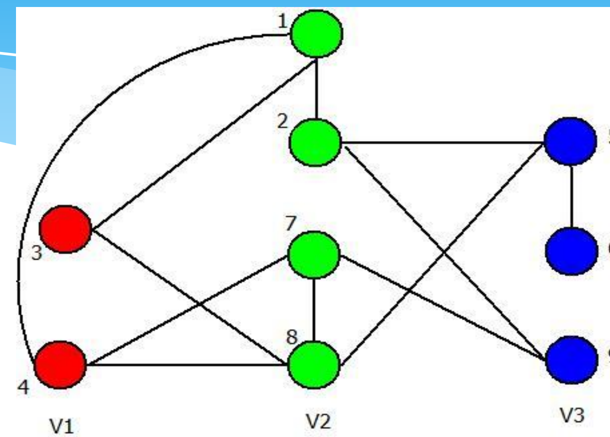
Vertex	Red V1	Green V2	Blue V3
1	9	0	0
2	0	10	0
3	0	0	0
4	0	0	0
5	0	0	0
6	0	0	0
7	0	0	0
8	0	0	0
9	0	0	0



- At the begining of the search, the TabuTenure table is initialized to be zero.
- once move <*u*, *i*, *j*> is performed, the value of Table[u][i]=TabuTenure.
- Once the search progresses, the non-zero value of the table is decreased by one at each time.
- In the following search, we can decide if a move <*u*, *i*, *j*> is tabu by checking if Table[u][j]>0.

Fast Implementation of TabuTenure Table

Vertex	Red V1	Green V2	Blue V3
1	1+10	0	0
2	0	2+15	0
3	0	0	0
4	0	0	3+12
5	0	0	0
6	0	0	0
7	0	4+10	0
8	0	0	0
9	0	0	5+12



* The above Table[u][i] records the relative tabu tenure length. Why not record the absolute tabu tenure length?
* Once move <*u*, *i*, *j*> is performed, the value of Table[u][i] = TabuTenure+Iter.
* In the following search, we can decide if a move <*u*, *i*, *j*> is tabu by checking if Table[u][j]>Iter.
* In this way, the tabu tenure table can be updated in O(1).

Aspiration

* If one move can override the best found solution found so far, it is accepted even if it is in tabu status.

* This is because only the **attributes** but not **solutions** themselves are stored in the tabu table.

TS Algorithm

1. Generate initial solution S, Calculate f(S)
2. Initialize the adjacent-color table M.
3. While {stop condition is not met}

3.1 Construct the neighborhood of S, denoted by N(S)

3.2 Calculate the Δ values of all critical one-moves

3.3 Find the best tabu and non-tabu moves with the least Δ value

3.4 If {the aspiration condition is satisfied}

perform the best tabu move,

else

perform the best non-tabu move

3.5 Update *f* and the adjacent-color table M

End

TS Algorithm

* Data Structures: Sol[N], f, BestSol[N], Best_f, TabuTenure[N][K], Adjacent_Color_Table[N][K]
* Subfunctions:
  * Initialization(): Initialize the values of the data structures.
  * FindMove(u,vi,vj,delt): find the best non-tabu or tabu move.
  * MakeMove(u,vi,vj,delt): update the correponding values.
* TabuSearch()

```
{ int u, vi, vj, iter = 0;
  Initialization() ;
  while( iter < MaxIter) {
       FindMove(u,vi,vj,delt) ;
       MakeMove(u,vi,vj,delt) ; }
}
```

Find Move

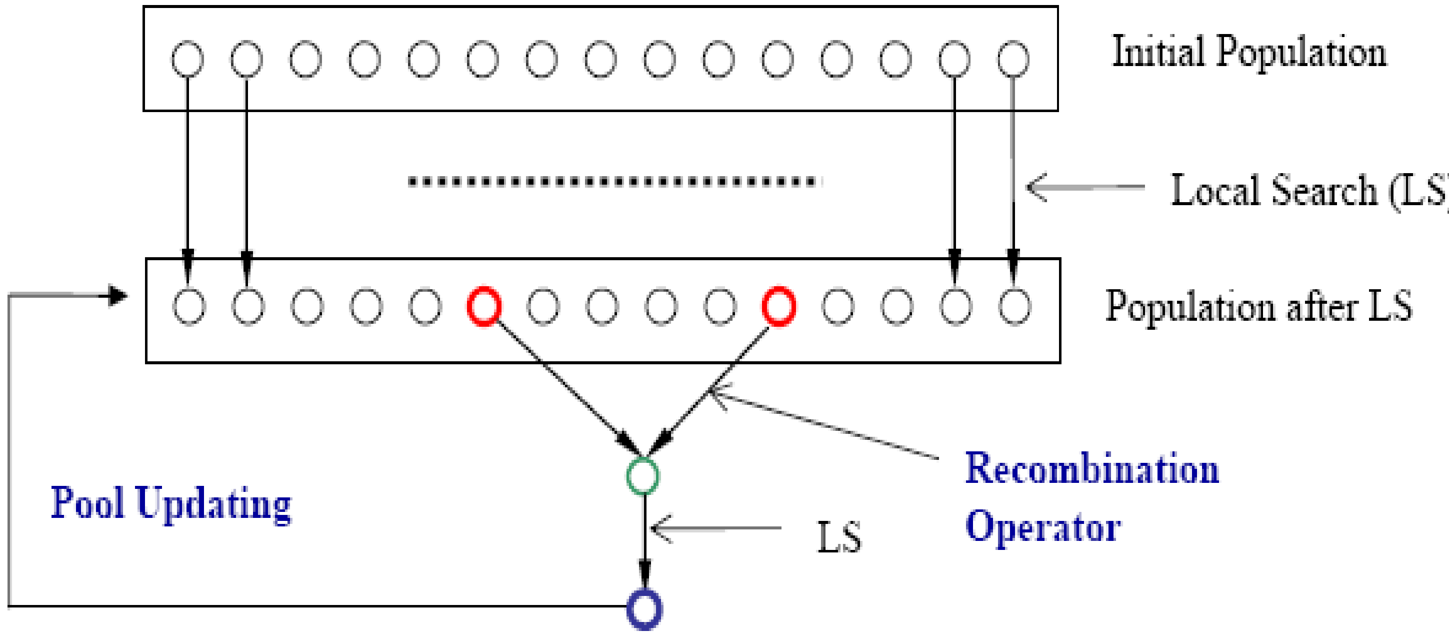
```
FindMove(u,vi,vj,delt)
{
  for(i=1:N)
   if(Adjacent_Color_Table[ i ][ Sol[i] ] > 0) {
    for( k = 1: K)
     if( k != Sol[i] ) {
        calculate delt value of the move <i, Sol[i], k>
        {
          if (Table[i][k] < iter)    update the tabu best move;
          else   update the non-tabu best move;
     } } }
 if(the tabu best move satisfies the tabu aspiration criterion)
        <u, vi, vj, delt> = the tabu best move;
 else     <u, vi, vj,delt> = the non-tabu best move;
}
```

Make Move

* MakeMove(u,vi,vj, delt)
* {
*   Sol[u] = vj ;
*   f = f + delt ;
*   Table[u][vi] = iter + f + rand()%10;
*   Update the Adjacent_Color_Table;
* }

Hybrid Evolutionary Algorithm

Main Scheme



Hybrid Evolutionary Algorithm (LS+EA)

Hybrid Evolutionary Algorithm

The hybrid coloring algorithm

Data : *graph* $G = (V, E)$, *integer k*
Result : *the best configuration found*
begin
    P=*InitPopulation*($|P|$)
    while *not Stop-Condition ()* **do**
        (s1,s2)=*ChooseParents*(P)
        s=*Crossover*(s1,s2)
        s=*LocalSearch*(s, L)
        P=*UpdatePopulation*(P,s)
end

Crossover Operator (1)

Table 2. The crossover algorithm: an example.

parent s_1 →	A B C	$\boxed{\text{D E F G}}$	H I J
parent s_2	C <u>D E G</u>	A <u>F</u> I	B H J
offspring s			

$V_1 := \{D, E, F, G\}$
remove D,E,F and G

A B C		H I J
C	A I	B H J
D E F G		

parent s_1	A <u>B</u> C		<u>H</u> I <u>J</u>
parent s_2 →	C	A I	$\boxed{\text{B H J}}$
offspring s	D E F G		

$V_2 := \{B, H, J\}$
remove B,H and J

A C		I
C	A I	
D E F G	B H J	

parent s_1 →	$\boxed{\text{A C}}$		I
parent s_2	<u>C</u>	<u>A</u> I	
offspring s	D E F G	B H J	

$V_3 := \{A, C\}$
remove A and C

		I
	I	
D E F G	B H J	A C

Crossover Operator (2)

- A legal k-coloring is a collection of k independent sets.
- With this point of view, if we could maximize the size of the independent sets by a crossover operator as far as possible, it will in turn help to push those left vertices into independent sets.
- In other words, the more vertices are transmitted from parent individuals to the offspring within k steps, the less vertices are left unassigned.
- In this way, the obtained offspring individual has more possibility to become a legal coloring.

Crossover Operator (3)

The GPX crossover algorithm

Data : *configurations* $s_1 = \{V_1^1, \ldots, V_k^1\}$ *and* $s_2 = \{V_1^2, \ldots, V_k^2\}$
Result : *configuration* $s = \{V_1, \ldots, V_k\}$
begin
  for $l (1 \leq l \leq k)$ **do**
    if l is odd, then $A := 1$, else $A := 2$
    choose i such that V_i^A has a maximum cardinality
    $V_l := V_i^A$
    remove the vertices of V_l from s_1 and s_2
  Assign randomly the vertices of $V - (V_1 \cup \cdots \cup V_k)$
end

HEA Algorithm Scheme

Algorithm 1 Pseudcode of the Hybrid Evolutionary Algorithm for k-Coloring

Input: Graph G
Output: The best solution S^* found so far
$\{S_1, \dots, S_p\} \leftarrow$ Initial Population
for $i = \{1, \dots, p\}$ **do**
    $S_i \leftarrow$ Tabu Search(S_i)
end for
$S^* = arg\ min\{f(S_i), i = 1, \dots, p\}$
repeat
    Randomly choose two parent solutions $\{S_{i1}, S_{i2}\}$
    $S_0 \leftarrow$ Crossover_Operator (S_{i1}, S_{i2})
    $S_0 \leftarrow$ Tabu Search(S_0)
    if $f(S_0) < f(S^*)$ **then**
        $S^* = S_0$
    end if
    $\{S_1, \dots, S_p\} \leftarrow$ Pool Updating$(S_0, S_1, \dots, S_p)$
until Stop condition met

Thank You!