AE353: Final Exam

December 18, 2024

You are allowed to use any online or offline notes at your disposal, **but no communication**. You must **write a complete solution, showing the process that you took to reach your answer**.

The entire exam lasts 180 minutes (unless you have been alloted extra time), including all the questions on paper *and* those on PrairieLearn. Computational tools are *not* allowed for the paper element of the exam, but are allowed for PrairieLearn.

Each problem on paper is worth 25 points. Each problem on PrairieLearn is worth 20 points. Consequently, the nominal maximal mark on this exam is 300 points, but it is possible to collect 350. In other words, the “bonus” amount available on this exam is $16.66\dot{6}\%$.

The problems are *not* ranked in order of difficulty.

Problem 1.

Consider a controlled dynamical system with state $q \in \mathbb{R}$ and input $p \in \mathbb{R}$ given by the equation

$$\dddot{q} = 8q^2\ddot{q}p + \sin(p).$$

By adding “dummy states”, convert it into a first-order dynamical system described by equations $\dot{w} = f(w, p)$, where w is the extended system state. *Write the definition of w, and carefully write every component of f in terms of w and p.*

Problem 2.

Consider a nonlinear control system with state $w \in \mathbb{R}^2$ and input $p \in \mathbb{R}^2$ given by

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} e^{w_1} + w_2^2 + p_1p_2 + 1 \\ e^{w_1} + w_1w_2 + p_2 \end{pmatrix},$$

with output $z \in \mathbb{R}$ given by $z = e^{w_1}p_1p_2$. Find the equilibrium (w_e, p_e) of this system such that $w_e = 0$. Linearize this system around this equilibrium. Express the linearized system in the form $\dot{x} = Ax + Bu$, $y = Cx + Du$, and make sure to write down the relationships between x and w, u and p, and y and z.

Problem 3.

Consider the linear control system

$$\dot{x}(t) = \begin{pmatrix} 2 & 7 \\ -5 & 5 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 5 \end{pmatrix} u(t),$$

with $x(0) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$. Let

$$u(t) = \begin{pmatrix} 1 & -1 \end{pmatrix} x(t)$$

for all t. Find $x(5)$.

Optional suggestion: determine x_2 before x_1.

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Problem 4.

Consider the linear control system

$$\dot{x}(t) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} u(t).$$

(a) Determine whether this system is controllable.

(b) Determine whether this system is stabilizable.

(c) Find a control signal u which ensures $x_3(t) \to 0$ regardless of $x_3(0)$.

Optional suggestion for (c): just consider the dynamics of x_3 and design u accordingly.

Problem 5.

Consider a controlled dynamical system with state $z \in \mathbb{R}$ and input $\tau \in \mathbb{R}$ described by

$$\dot{z} = e^{z\tau}.$$

Let $w = e^z$. Describe the dynamics of w, i.e., find a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\dot{w} = f(w, \tau)$.

Problem 6.

Consider matrix

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Show that M can be written as a sum of a *symmetric matrix* P and an *antisymmetric matrix* Q, i.e., that $M = P + Q$, where $P = P^T$ and $Q = -Q^T$.

Problem 7.

Consider a linear control system given by $\dot{x} = Ax + Bu$, where $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 3}$ and $A^T B = 0$. Consider the problem of finding a control signal u which minimizes

$$\int_0^1 \|\dot{x}(t)\|^2 dt.$$

Convert it (you do not have to solve the problem!) into the LQR problem of minimizing

$$\int_0^1 (x(t))^T Q x(t) + (u(t))^T R u(t) dt,$$

i.e., find positive semidefinite matrices Q and R such that the two problems match. If you wish, you can express matrices Q and R in terms of A and B.

(Any matrix M of the form $M = N^T N$ is automatically positive semidefinite; you do not have to verify this fact.)

Problem 8.

Consider a single-state linear control system given by

$$\dot{x} = Ax + Bu, \quad y = Cx.$$

Assume that $B \neq 0$ and $C \neq 0$. Show that, regardless of B and C, this system is detectable, observable, controllable and stabilizable.

Emphasis: the system is single-state, not necessarily single-input or single-output!

Problem 9.

Consider a linear control system given by

$$\dot{x} = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 18 \end{pmatrix} u, \quad y = \begin{pmatrix} 27 & 0 \end{pmatrix} x.$$

As we did in class, construct an asymptotically correct observer $\hat{x}$, i.e., a function $\hat{x}$ such that

$$\lim_{t \to +\infty} \hat{x}(t) - x(t) = 0.$$

You can describe $\hat{x}$ through an ordinary differential equation *and* an initial value.

Problem 10.

Consider a single-input, single-state linear control system given by $\dot{x} = 2x - u$, with $x(0) = 1$. Show that there does not exist a minimum of

$$\int_0^1 x^2(t) - u^2(t)dt,$$

i.e., that one can choose u such that the integral above is negative and arbitrarily large in magnitude.

Solutions

Problem 1.
By taking $(w_1, w_2, w_3) = (q, \dot{q}, \ddot{q})$, we obtain $\dot{w} = f(w, p)$, where

$$f(w,p) = \begin{pmatrix} w_2 \\ w_3 \\ 8w_1^2 w_3 p + \sin(p) \end{pmatrix}.$$

Problem 2.
Let us first find the appropriate equilibrium. If we refer to the right hand side of the control system equation by $f(w,p)$, we obtain that $f(w_e, p_e) = 0$ if and only if $2 + p_{1e}p_{2e} = 0$ and $1 + p_{2e} = 0$. Hence, $p_e = (2, -1)$.

Linearizing the system in the usual way by taking partial derivatives, we obtain

$$A = \begin{pmatrix} e^{w_{1e}} & 2w_{2e} \\ e^{w_{1e}} + w_{2e} & w_{1e} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} p_{2e} & p_{1e} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}.$$

To obtain the linearized output equation, we again take the partial derivatives to get

$$C = \begin{pmatrix} e^{w_{1e}} p_{1e} p_{2e} & 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} e^{w_{1e}} p_{2e} & e^{w_{1e}} p_{1e} \end{pmatrix} = \begin{pmatrix} -1 & 2 \end{pmatrix}.$$

The relationship between states and inputs of the linearized system and those of the original system is given by $x = w - w_e = w$, $u = p - p_e = p - (2, -1)$, and $y = z - e^{w_{2e}} p_{1e} p_{2e} = z + 2$.

Problem 3.
Plugging in u into the equation for system dynamics gives us

$$\dot{x} = Mx = \begin{pmatrix} 3 & 6 \\ 0 & 0 \end{pmatrix} x.$$

The dynamics of x_2 are thus given by $\dot{x}_2 = 0$, so $x_2(5) = x_2(0) = 0$. The dynamics of x_1 are given by $\dot{x}_1 = 3x_1 + 6x_2$, and as $x_2 \equiv 0$, we thus have $\dot{x}_1 = 3x_1$. Hence, $x_1(5) = e^{15} x_1(0) = e^{15}$. Hence, $x(5) = (e^{15}, 0)$

Problem 4.

(a) We compute the controllability matrix of the system by

$$W = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{pmatrix} 1 & 1 & 9 \\ 0 & 4 & 24 \\ 1 & 3 & 5 \end{pmatrix}.$$

This matrix has full rank: its determinant is $\det W = 20 - 72 + 24 - 36 = -64 \neq 0$.

(b) Since the system is controllable, it is stabilizable.

(c) Consider $K = \begin{pmatrix} 2 & 0 & 2 \end{pmatrix}$, and $u = -Kx$. Then, we obtain $\dot{x}_3 = 2x_1 + x_3 + u = 2x_1 + x_3 - 2x_1 - 2x_3 = -x_3$. Thus, $x_3(t) \to 0$ regardless of $x_3(0)$.

Problem 5.
Taking the derivative of w, we obtain $\dot{w} = e^z \dot{z} = e^z e^{z\tau} = e^{z(\tau+1)} = w^{\tau+1}$.

Problem 6.
Indeed, every matrix can be writen in this way by taking $P = (M + M^T)/2$ and $Q = (M - M^T)/2$. Specifically, in this case

$$M = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}.$$

Problem 7.
We first note that $\|\dot{x}\|^2 = \|Ax + Bu\|^2 = (Ax + Bu)^T(Ax + Bu) = (x^T A^T + u^T B^T)(Ax + Bu) = x^T A^T Ax + x^T A^T Bu + u^T B^T Ax + u^T B^T Bu$. Since $A^T B = 0$ and $B^T A = (A^T B)^T = 0$, we get $\|\dot{x}\|^2 = x^T A^T Ax + u^T B^T Bu$. Thus, $Q = A^T A$ and $R = B^T B$.

Problem 8.
Because the system is single-state, its controllability matrix equals $W = B \neq 0$. Its observability matrix equals $O = C \neq 0$. Both of these matrices are full-rank, so the system is controllable and observable, and thus it is also stabilizable and detectable.

Problem 9.
Using the usual notation for matrices A, B, and C, as in class, we define the observer $\hat{x}$ by $\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y)$, and we take the initial value $\hat{x}(0) = 0$. From the discussion in class, this observer will be asymptotically correct if we choose a matrix

$$L = \begin{pmatrix} l_1 & l_2 \end{pmatrix}^T$$

such that $A - LC$ has all eigenvalues with negative real parts. Matrix $A - LC$ equals

$$\begin{pmatrix} 2 - 27l_1 & 3 \\ -27l_2 & -1 \end{pmatrix}.$$

If we choose $l_1 = 1$ and $l_2 = 0$, $A - LC$ is triangular and we obtain that the eigenvalues of $A - LC$ equal -25 and -1. Thus, such L is appropriate for our construction.

Problem 10.
Consider any $n \geq 0$ and a control input u given by $u(t) = -nx(t)$. Then, $\dot{x} = (n+2)x$, so $x(t) = e^{(n+2)t}$. The value of the integral in question is thus

$$\int_0^1 (1 - n^2)x^2(t)dt = (1 - n^2)\int_0^1 e^{2(n+2)t}dt = \frac{(1 - n^2)(e^{2(n+2)} - 1)}{2(n+2)} = -\frac{(n-1)(n+1)(e^{2(n+2)} - 1)}{2(n+2)}.$$

As $n \to +\infty$, the integral's value clearly diverges to $-\infty$.