AE353: Midterm

October 7, 2024

You are allowed to use any online or offline notes at your disposal, **but no communication and computational tools**. You must **write a complete solution, showing the process that you took to reach your answer**. **"Bonus" problem is not necessarily the hardest.**

Problem 1. (25 points)

Consider a controlled dynamical system with state $q \in \mathbb{R}$ and input $p \in \mathbb{R}$ given by the equation

$$\dddot{q} = 2\dot{q}\ddot{q}p^2 - \cos(p^2).$$

By adding "dummy states", convert it into a first-order dynamical system described by equations $\dot{w} = f(w, p)$, where w is the extended system state. *Write the definition of w, and carefully write every component of f in terms of w and p.*

Problem 2. (25 points)

Consider a nonlinear control system with state $w \in \mathbb{R}^2$ and input $p \in \mathbb{R}$ given by

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} e^{w_2} + w_1 p^2 - 1 \\ w_1 w_2 \end{pmatrix},$$

with output $z \in \mathbb{R}$ given by $z = e^{w_2}$. Linearize this system around an equilibrium (w_e, p_e) such that $p_e = 1$. Express the linearized system in the form $\dot{x} = Ax + Bu$, $y = Cx + Du$, and make sure to write down the relationships between x and w, u and p, and y and z.

Problem 3. (25 points)

Consider the linear control system

$$\dot{x}(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t),$$

with $x(0) = \begin{pmatrix} 2 & 3 \end{pmatrix}^T$. Let

$$u(t) = \begin{pmatrix} 0 & -1 \end{pmatrix} x(t)$$

for all t. Determine, *without use of computational tools*, $x(5)$.

Problem 4. (25 points)

Consider the linear control system

$$\dot{x}(t) = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t).$$

(a) Determine whether this system is controllable.

(b) Determine whether this system is stabilizable.

(c) Determine whether it is possible to design a control signal which ensures $x_2(t) \to 1$.

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Bonus Problem. (25 points)

Consider a controlled dynamical system with state $z \in \mathbb{R}$ and input $\tau \in \mathbb{R}$ described by

$$\dot{z} = 7z^{12} + 5z + \tau^2.$$

Let $w = z^3$. Describe the dynamics of w, i.e., find a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\dot{w} = f(w, \tau)$.

Solutions

Problem 1.
By taking $(w_1, w_2, w_3) = (q, \dot{q}, \ddot{q})$, we obtain $\dot{w} = f(w, p)$, where

$$f(w,p) = \begin{pmatrix} w_2 \\ w_3 \\ 2w_2w_3p^2 - \cos(p^2) \end{pmatrix}.$$

Problem 2.
Let us first find the appropriate equilibrium. If we refer to the right hand side of the control system equation by $f(w, p)$, we obtain that $f(w_e, p_e) = 0$ if and only if $w_{1e}w_{2e} = 0$ and $e^{w_{2e}} + w_{1e} - 1 = 0$. The former equation gives us two options: either $w_{1e} = 0$ or $w_{2e} = 0$. In the first case, from the second equation we get $w_{2e} = 0$, and in the second case the second equation gives $w_{1e} = 0$. Hence, $w_e = (0, 0)$.

Linearizing the system in the usual way by taking partial derivatives, we obtain

$$A = \begin{pmatrix} p_e^2 & e^{w_{2e}} \\ w_{2e} & w_{1e} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2w_{1e}p_e \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

To obtain the linearized output equation, we again take the partial derivatives to get

$$C = \begin{pmatrix} 0 & e^{w_{2e}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad D = 0.$$

The relationship between states and inputs of the linearized system and those of the original system is given by $x = w - w_e = w$, $u = p - p_e = p - 1$, and $y = z - e^{w_{2e}} = z - 1$.

Problem 3.
Plugging in u into the equation for system dynamics gives us

$$\dot{x} = Mx = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x.$$

Matrix M is diagonal, so the solution to this ODE is simply $x_1(t) = e^t x_1(0) = 2e^t$ and $x_2(t) = x_2(0) = 3$. Hence, $x_1(5) = 2e^5$, $x_2(5) = 3$.

Problem 4.

(a) We compute the controllability matrix of the system by

$$W = \begin{bmatrix} B & AB \end{bmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

This matrix clearly does not have full rank, so the system is not controllable.

(b) The system is stabilizable. Consider $u(t) = -\begin{pmatrix} 0 & 3 \end{pmatrix} x$. Plugging u into the equation for system dynamics we obtain $\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} x = -x$, which is clearly stable, as both eigenvalues of the relevant matrix are -1.

(c) Indeed, such a control signal exists: consider matrices A and B as in the usual notation, $K = \begin{pmatrix} 0 & 3 \end{pmatrix}$ as above, $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$ and $k_{ref} = -1/(C(A - BK)^{-1}B)$. Since $(A - BK)^{-1} = -I$, we obtain $k_{ref} = 1/(CB) = 1$. Now, from the theory of reference tracking, we know that the control signal $u(t) = -Kx(t) + k_{ref} = -3x_2(t) + 1$ will lead to $x_2(t) = Cx(t) \to 1$.

Bonus Problem.

Taking the derivative of w, we obtain $\dot{w} = 3z^2\dot{z} = 3z^2(7z^{12} + 5z + \tau^2)$, and noting that $z = \sqrt[3]{w}$, we get $\dot{w} = 3\sqrt[3]{w^2}(7w^4 + 5\sqrt[3]{w} + \tau^2)$.