AE353: Midterm

November 13, 2024

You are allowed to use any online or offline notes at your disposal, **but no communication and computational tools**. You must **write a complete solution, showing the process that you took to reach your answer**. The problems are **not** sorted by difficulty.

Problem 1. (25 points)

Consider matrix

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let $N = (MM^T)^{-1}$. Show that N is symmetric, i.e., that $N = N^T$.

(I suggest working smart, not hard.)

Problem 2. (25 points)

This problem had a typo that allowed it to be understood in multiple ways. This document contains the problem and the solution that were originally intended, but a correct solution to any interpretation is equally acceptable.

Consider a linear control system given by $\dot{x} = Ax + Bu$, where $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 2}$. Consider the problem of finding a control signal u which minimizes

$$\int_0^1 (x_1(t) + x_2(t))^2 + (x_1(t) - x_2(t))^2 + \|u(t)\|^2 dt.$$

Convert it (you do not have to solve the problem!) into the LQR problem of minimizing

$$\int_0^1 (x(t))^T Q x(t) + (u(t))^T R u(t) dt,$$

i.e., find positive semidefinite matrices Q and R such that the two problems match.

Problem 3. (25 points)

Consider a two-state linear control system given by

$$\dot{x} = Ax + Bu, \quad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x.$$

Assume that all elements of A are non-zero. Show that this system is detectable and observable.

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Problem 4. (25 points)

Consider a linear control system given by

$$\dot{x} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \quad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x.$$

Assuming that the controller can only read output values $y(t)$, but not full $x(t)$ directly, construct a control signal u which ensures $\lim_{t\to+\infty} x(t) = 0$.

You do not need to write $u(t)$ explicitly for all t, but, if you want, you can write it as an output of an observer system.

Bonus Problem. (25 points)

Consider a single-input, single-state linear control system given by $\dot{x} = x + u$, with $x(0) = 1$. Show that there does not exist a minimum of

$$\int_0^1 x^2(t) - u^2(t)dt,$$

i.e., that one can choose u such that the integral above is negative and arbitrarily large in magnitude.

Solutions

Problem 1.
We first note that M is invertible, and MM^T is thus invertible as well, since $\det(MM^T) = \det(M)\det(M^T) = \det(M)^2 \neq 0$. Then, using X^{-T} to denote the inverse of X^T and, equivalently, the transpose of X^{-1}, we get $N^T = (MM^T)^{-T} = M^{-T}M^{-1} = N$.

Problem 2.
We first note that $(x_1+x_2)^2 + (x_1-x_2)^2 = 2x_1^2 + 2x_2^2 = x^TQx$, and $\|u\|^2 = u_1^2 + u_2^2 = u^TRu$, where

$$Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Both of these matrices are clearly symmetric and positive definite.

Problem 3.
Let us calculate the observability matrix. It equals

$$O = \begin{pmatrix} 1 & 0 \\ a_1 & a_2 \end{pmatrix},$$

where a_1 and a_2 are the elements of the first row of A. Thus, $\det(O) = a_2 \neq 0$. Since matrix O is thus full rank, this system is observable. Since it is observable, it is detectable.

Problem 4.
Let us set matrix $K = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Then, using our usual notation for matrices A, B, and C, matrix $A - BK$ is upper-triangular with -1 and -1 on the diagonal. Hence, $A - BK$ is stable. Let us set $L = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$. Then, $A - LC$ is also upper triangular with -1 and -1 on the diagonal. Hence, $A - LC$ is stable. Thus, as described in class, a control signal $u(t) = -K\hat{x}(t)$, where $\hat{x}(t)$ is the observer generated by $\hat{x}(t) = A\hat{x} + Bu(t) - L(C\hat{x}(t) - y(t))$ and $\hat{x}(t) = 0$, ensures that $x(t) \to 0$ as $t \to +\infty$.

Bonus Problem.
Consider any $n \geq 0$ and a control input u given by $u(t) = nx(t)$. Then, $\dot{x} = (n+1)x$, so $x(t) = e^{(n+1)t}$. The value of the integral in question is thus

$$\int_0^1 (1-n^2)x^2(t)dt = (1-n^2)\int_0^1 e^{2(n+1)t}dt = \frac{(1-n^2)(e^{2(n+1)}-1)}{2(n+1)} = -\frac{(n-1)(e^{2(n+1)}-1)}{2}.$$

As $n \to +\infty$, the integral's value clearly diverges to $-\infty$.