Optical Modeling and Design

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Chapter 1

Maxwell's equations and plane wave solution

In field tracing light is always represented by electromagnetic fields. First we deal with electromagnetic fields in homogeneous media.

1.1 Maxwell's equations in the time domain

- Light is an electromagnetic field whose physical nature is mathematically governed by Maxwell's equations:

$$\nabla \times \bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r}, t) = -\frac{\partial}{\partial t}\bar{\boldsymbol{B}}^{(\mathrm{r})}(\boldsymbol{r}, t)\,, \tag{1.1}$$

$$\nabla \times \bar{\boldsymbol{H}}^{(\mathrm{r})}(\boldsymbol{r}, t) = \bar{\boldsymbol{j}}^{(\mathrm{r})}(\boldsymbol{r}, t) + \frac{\partial}{\partial t}\bar{\boldsymbol{D}}^{(\mathrm{r})}(\boldsymbol{r}, t)\,, \tag{1.2}$$

$$\nabla \cdot \bar{\boldsymbol{D}}^{(\mathrm{r})}(\boldsymbol{r}, t) = \bar{\rho}^{(\mathrm{r})}(\boldsymbol{r}, t)\,, \tag{1.3}$$

$$\nabla \cdot \bar{\boldsymbol{B}}^{(\mathrm{r})}(\boldsymbol{r}, t) = 0 \tag{1.4}$$

  with the position vector $\boldsymbol{r} = (x, y, z)$.

- All vectorial field quantities have three components, for instance $\bar{\boldsymbol{E}}^{(\mathrm{r})} = (\bar{E}_x^{(\mathrm{r})}, \bar{E}_y^{(\mathrm{r})}, \bar{E}_z^{(\mathrm{r})})$.

- We introduce the index r to emphasize, that the field quantities stand for the real optical fields, in contrast to the complex valued fields which are introduced later.

The units of the field quantities are:

- Electric field: $[E] = \mathrm{V/m} = \mathrm{m\ kg/s^3A}$
- Magnetic field: $[H] = \mathrm{A/m}$
- Dielectric displacement: $[D] = \mathrm{C/m^2} = \mathrm{A\ s/m^2}$
- Magnetic induction: $[B] = \mathrm{T} = \mathrm{kg/s^2A}$
- Current density: $[j] = \mathrm{A/m^2}$
- Charge density: $[\rho] = \mathrm{C/m^3} = \mathrm{A\ s/m^3}$

1.2 Maxwell's equations in frequency domain

- The time dependency of all field quantities may be transformed into the frequency domain by the Fourier transformation[1]

$$\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r},\omega) = \mathcal{F}_\omega \bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r},t) \tag{1.5}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r},t)\, \mathrm{e}^{+\mathrm{i}\omega t}\, \mathrm{d}t \tag{1.6}$$

[1]Note, that the field units in frequency domain have an extra time dimension factor.

and vice versa by the inverse transformation

$$\bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r},t) = \mathcal{F}_{\omega}^{-1}\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r},\omega) \tag{1.7}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r},\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega \tag{1.8}$$

at the example of the electric field.

- The same may be done for all other electromagnetic quantities, that is for $\boldsymbol{D}$, $\boldsymbol{H}$, $\boldsymbol{B}$, $\boldsymbol{j}$ and ρ.
- Remarks:(1) Fourier transformation can also be defined by the opposite sign in the integral kernel. (2) In optics we typically use the angular frequency ω, which is related to the temporal frequency ν via $\omega = 2\pi\nu$.
- Because the values of $\bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r},t)$ are real-valued, its Fourier transformation is hermitian, that is

$$\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r},\omega) = (\boldsymbol{E}^{(\mathrm{r})})^{*}(\boldsymbol{r},-\omega) \tag{1.9}$$

with the phase conjugation operation $*$.

- Obviously, the mathematical step of the Fourier transformation formally leads to negative frequencies. Because of (1.9) they do not carry physical information.

- In order to simplify mathematical procedures the negative frequencies are truncated and the complex field vector in frequency domain

$$\boldsymbol{E}(\boldsymbol{r},\omega) = \begin{cases} 2\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r},\omega) & \text{if } \omega \geq 0 \\ 0 & \text{otherwise} \end{cases} . \tag{1.10}$$

is obtained. That yields the complex field vector

$$\bar{\boldsymbol{E}}(\boldsymbol{r},t) = \boldsymbol{\mathcal{F}}_{\omega}^{-1}\boldsymbol{E}(\boldsymbol{r},\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \boldsymbol{E}(\boldsymbol{r},\omega) \exp\left(-\mathrm{i}\omega t\right) \,\mathrm{d}\omega \tag{1.11}$$

in time domain.

- The function $\boldsymbol{E}(\boldsymbol{r},\omega)$ in (1.10) is not hermitian and therefore $\bar{\boldsymbol{E}}(\boldsymbol{r},t)$ is complex valued. That is the reason to call $\bar{\boldsymbol{E}}(\boldsymbol{r},t)$ the complex electric field which is emphasized by skipping the index r.

- The basic relationship between the real electric field and its complex generalization is given by[2]

$$\bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r},t) = \Re\big(\bar{\boldsymbol{E}}(\boldsymbol{r},t)\big)\,. \tag{1.12}$$

- The proof uses Eqs. (1.9)-(1.10).

- In order to formulate the Maxwell's equations in the frequency domain we first replace the real fields, i.e. $\bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r},t)$, by its complex generalization

$$\bar{\boldsymbol{E}}(\boldsymbol{r},t) = \bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r},t) + \mathrm{i}\bar{\boldsymbol{E}}'(\boldsymbol{r},t)\,. \tag{1.13}$$

Obviously any $\bar{\boldsymbol{E}}(\boldsymbol{r},t)$ of Eq.(1.13) satisfies Eq.(1.12) by definition.

- Inserting $\Re\Big(\bar{\boldsymbol{E}}(\boldsymbol{r},t)\Big)$ into Eq.(1.1) - Eq.(1.4) and moving the $\Re$-operator in front

[2] see Mandel/Wolf, Eq. (3.1-8b)

of the terms leads to the equations

$$\Re\Big(\nabla \times \bar{\boldsymbol{E}}(\boldsymbol{r},t) + \frac{\partial}{\partial t}\bar{\boldsymbol{B}}(\boldsymbol{r},t)\Big) = 0\,, \tag{1.14}$$

$$\Re\Big(\nabla \times \bar{\boldsymbol{H}}(\boldsymbol{r},t) - \bar{\boldsymbol{j}}(\boldsymbol{r},t) - \frac{\partial}{\partial t}\bar{\boldsymbol{D}}(\boldsymbol{r},t)\Big) = 0\,, \tag{1.15}$$

$$\Re\Big(\nabla \cdot \bar{\boldsymbol{D}}(\boldsymbol{r},t) - \bar{\rho}(\boldsymbol{r},t)\Big) = 0\,, \tag{1.16}$$

$$\Re\Big(\nabla \cdot \bar{\boldsymbol{B}}(\boldsymbol{r},t)\Big) = 0\,. \tag{1.17}$$

- The move of $\Re$ to the front of the expressions is allowed due to the linear nature of Maxwell's equations.

- Next we can replace all time dependent quantities by its Fourier transformed version according to Eq.(1.11) and obtain

$$\Re\left(\mathcal{F}_{\omega}^{-1}\left[\nabla\times\boldsymbol{E}(\boldsymbol{r},\omega)-\mathrm{i}\omega\boldsymbol{B}(\boldsymbol{r},\omega)\right]\right)=0\,, \tag{1.18}$$

$$\Re\left(\mathcal{F}_{\omega}^{-1}\left[\nabla\times\boldsymbol{H}(\boldsymbol{r},\omega)-\boldsymbol{j}(\boldsymbol{r},\omega)+\mathrm{i}\omega\boldsymbol{D}(\boldsymbol{r},\omega)\right]\right)=0\,, \tag{1.19}$$

$$\Re\left(\mathcal{F}_{\omega}^{-1}\left[\nabla\cdot\boldsymbol{D}(\boldsymbol{r},\omega)-\rho(\boldsymbol{r},\omega)\right]\right)=0\,, \tag{1.20}$$

$$\Re\left(\mathcal{F}_{\omega}^{-1}\left[\nabla\cdot\boldsymbol{B}(\boldsymbol{r},\omega)\right]\right)=0\,. \tag{1.21}$$

- Here we use one big advantage of working in the Fourier domain, that is the replacement of the time derivation by the factor $-\mathrm{i}\omega$. Moreover, we apply the commutativity of the ∇ and $\mathcal{F}_{\omega}^{-1}$ operators.

- From Eq.(1.18)-Eq.(1.21) we conclude: Since a Fourier transformation of a function f can only be zero if $f\equiv 0$ and because of the linear nature of the equations $\bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r},t)$ of Eq.(1.12) satisfies Maxwell's equations if the complex field ($\boldsymbol{E}$ and $\boldsymbol{H}$)

satisfies the equations

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},\omega) = \mathrm{i}\omega\boldsymbol{B}(\boldsymbol{r},\omega) \ , \tag{1.22}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},\omega) = \boldsymbol{j}(\boldsymbol{r},\omega) - \mathrm{i}\omega\boldsymbol{D}(\boldsymbol{r},\omega) \ , \tag{1.23}$$

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r},\omega) = \rho(\boldsymbol{r},\omega) \ , \tag{1.24}$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r},\omega) = 0 \, . \tag{1.25}$$

- Thus in what follows we mainly deal with the complex field expressions and the corresponding field equations Eq.(1.22) - Eq.(1.25).
- However, we like to give the following urgent warning when dealing with the complex field quantities: Detector functions, like the energy, are typically defined by the real fields in time domain. Then, Eq.(1.12) is to be used before the detector function can be evaluated. At the example of the electric field that requires ultimately to apply

$$\bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r},t) = \Re\Big(\mathcal{F}_{\omega}^{-1}\Big[\boldsymbol{E}(\boldsymbol{r},\omega)\Big]\Big) \tag{1.26}$$

before the detector function can be evaluated.

- This warning remains valid for any nonlinear operations with the field quantities.

1.3 Linear matter equations

- In macroscopic media we assume specific relations between $\boldsymbol{E}$ and $\boldsymbol{D}$, between $\boldsymbol{E}$ and $\boldsymbol{j}$ and between $\boldsymbol{H}$ and $\boldsymbol{B}$ in the frequency domain. Those relations are called matter or constitutive equations.[3]

- In what follows we assume linear media and obtain linear dependencies of the form:

$$\boldsymbol{j}(\boldsymbol{r},\omega) = \boldsymbol{\sigma}(\boldsymbol{r},\omega)\,\boldsymbol{E}(\boldsymbol{r},\omega)\,, \tag{1.27}$$

$$\boldsymbol{D}(\boldsymbol{r},\omega) = \boldsymbol{\epsilon}(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega) = \epsilon_0\boldsymbol{\epsilon}_{\mathrm{r}}(\boldsymbol{r},\omega)\,\boldsymbol{E}(\boldsymbol{r},\omega)\,, \tag{1.28}$$

$$\boldsymbol{B}(\boldsymbol{r},\omega) = \boldsymbol{\mu}(\boldsymbol{r},\omega)\,\boldsymbol{H}(\boldsymbol{r},\omega)\,. \tag{1.29}$$

- In general the material quantities are tensors (3×3) and called conductivity $\boldsymbol{\sigma}$, electric permittivity $\boldsymbol{\epsilon}$ and magnetic permeability $\boldsymbol{\mu}$.

- The electric permittivity is often factorized by $\boldsymbol{\epsilon} = \epsilon_0\boldsymbol{\epsilon}_{\mathrm{r}}$[4] with the relative permittivity $\boldsymbol{\epsilon}_{\mathrm{r}}$ and the vacuum electric permittivity

$$\epsilon_0 = 8.8541878176 \times 10^{-12}\,\mathrm{N}^{-1}\,\mathrm{m}^{-2}\,\mathrm{C}^2\,. \tag{1.30}$$

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[3] Often that is also discussed in terms of the material polarization vector $\boldsymbol{P}$. Comprehensive discussion in *Fundamentals* lecture.

[4] Index r in $\boldsymbol{\epsilon}_{\mathrm{r}}$ refers to "‘relative"’ and not "‘real"’!

- Analogously a vacuum magnetic permeability is given by

$$\mu_0 = 1.2566370614 \times 10^{-6}\,\mathrm{m\,kg\,C^{-2}}\,. \tag{1.31}$$

- Both are related via

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.99792458 \times 10^{8}\,\mathrm{m\,s^{-1}}\,. \tag{1.32}$$

  with the speed of light in vacuum c.

- In optics Maxwell's equations are often discussed with at least some of the following restrictions:

  1. In optics free charges are typically of no concern and therefore $\rho = 0$.
  2. Isotropic media: Matter functions are scalar functions instead of tensors.
  3. In optics (from IR to UV) natural media are not magnetic and $\boldsymbol{\mu}(\boldsymbol{r}, \omega) = \mu_0$.
  4. Homogeneous media: Matter functions are not dependent of the location $\boldsymbol{r}$.
  5. Non-dispersive media: Matter functions are not dependent of the frequency ω.

- Application of matter equations under consideration of restrictions 1-3 results in the following set of Maxwell's equations:[5]

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},\omega) = \mathrm{i}\omega\mu_0 \boldsymbol{H}(\boldsymbol{r},\omega) \tag{1.33}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},\omega) = -\mathrm{i}\omega\epsilon_0 \check{\epsilon}_\mathrm{r}(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega) \tag{1.34}$$

$$\nabla \cdot \Big(\check{\epsilon}_\mathrm{r}(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega)\Big) = 0 \tag{1.35}$$

$$\nabla \cdot \boldsymbol{H}(\boldsymbol{r},\omega) = 0 \tag{1.36}$$

- Here we introduced the *generalized permittivity*

$$\check{\epsilon}_\mathrm{r}(\boldsymbol{r},\omega) := \epsilon_\mathrm{r}(\boldsymbol{r},\omega) + \mathrm{i}\frac{\sigma(\boldsymbol{r},\omega)}{\omega\epsilon_0}\,. \tag{1.37}$$

- If we restrict in addition to homogeneous media[6] Maxwell's equations in frequency

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[5] Transformation into time domain results in convolution integrals!

[6] Here we exclude the case $\check{\epsilon}_\mathrm{r}(\omega) = 0$.

domain read

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},\omega) = \mathrm{i}\omega\mu_0\boldsymbol{H}(\boldsymbol{r},\omega) \tag{1.38}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},\omega) = -\mathrm{i}\omega\epsilon_0\check{\epsilon}_\mathrm{r}(\omega)\boldsymbol{E}(\boldsymbol{r},\omega)\,, \tag{1.39}$$

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r},\omega) = 0\,, \tag{1.40}$$

$$\nabla \cdot \boldsymbol{H}(\boldsymbol{r},\omega) = 0\,. \tag{1.41}$$

- From Maxwell's equations in homogeneous media the Helmholtz' equation can be concluded by combination of the $\nabla\times$-equations and application of the vector identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}\,. \tag{1.42}$$

Then we obtain

$$\nabla^2\boldsymbol{E}(\boldsymbol{r},\omega) + k_0^2\check{\epsilon}_\mathrm{r}(\omega)\boldsymbol{E}(\boldsymbol{r},\omega) = 0 \tag{1.43}$$

with the wave number in vacuum

$$k_0 = \frac{\omega}{c} = \frac{2\pi\nu}{c}\,. \tag{1.44}$$

- The same result is valid for the magnetic field. It can be read componentwise. Thus, each field component of $\boldsymbol{H}$ and $\boldsymbol{E}$ must satisfy the Helmholtz equation.

- In non-dispersive media, that is $\check{\epsilon}_\mathrm{r} \neq \check{\epsilon}_\mathrm{r}(\omega)$, the Helmholtz equation can be directly transformed into the time domain (see Eq.(1.11)) and that leads to the wave equation

$$\nabla^2 \bar{\boldsymbol{E}}(\boldsymbol{r}, t) - \frac{\check{\epsilon}_\mathrm{r}}{c^2} \ddot{\bar{\boldsymbol{E}}}(\boldsymbol{r}, t) = 0\,. \tag{1.45}$$

- For non-damping dielectrics we have $\check{\epsilon}_\mathrm{r} = \epsilon_\mathrm{r} > 0$ and then c^2/ϵ_r is identical to the square of the phase velocity v of the wave according to Eq.(1.45). That leads to the definition of the refractive index $n := c/v = \sqrt{\epsilon_\mathrm{r}}$ which expresses the “resistance” of the matter to the wave propagation.

- This definition can be generalized by introducing the complex refractive index by

$$\check{n}(\omega) := \sqrt{\check{\epsilon}_\mathrm{r}(\omega)} = n(\omega) + \mathrm{i}n'(\omega). \tag{1.46}$$

1.4 Harmonic fields

- General fields can be always described as a superposition of fields with a single frequency ω_0, which are called monochromatic or harmonic fields.[7] So in what follows we concentrate on the discussion of harmonic fields.[8]

- There are several ways to introduce harmonic fields mathematically. We define a harmonic field by its complex version in the frequency domain, that is [9]

$$\boldsymbol{E}(\boldsymbol{r},\omega) = \boldsymbol{E}(\boldsymbol{r})\delta(\omega-\omega_0) \tag{1.47}$$

  with the Dirac delta function $\delta(\omega)$ and the *complex amplitude* electric field vector

$$\boldsymbol{E}(\boldsymbol{r}) = \begin{bmatrix} |E_x(\boldsymbol{r})| \exp[\mathrm{i}\varphi_x(\boldsymbol{r})] \\ |E_y(\boldsymbol{r})| \exp[\mathrm{i}\varphi_y(\boldsymbol{r})] \\ |E_z(\boldsymbol{r})| \exp[\mathrm{i}\varphi_z(\boldsymbol{r})] \end{bmatrix} . \tag{1.48}$$

- With the definition of the delta function the inverse Fourier transformation Eq.(1.8) of Eq. (1.47) yields the *complex* electric field vector of a harmonic field in time

[7] General fields are discussed in the lecture OMD Part II (summer period) in detail.
[8] Comment on stationary: Monochromatic fields are stationary. Polychromatic fields can be stationary, e.g. sun light, or non-stationary, e.g. fs pulses.
[9] Though we concentrate on the electric field, the same is valid for the magnetic one.

domain according to

$$\bar{\boldsymbol{E}}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r}) \exp\left(-\mathrm{i}\omega_0 t\right) \ . \tag{1.49}$$

- According to Eq.(1.12) the *real* electric field vector of a harmonic field is obtained by taking the real part of Eq.(1.49) and we obtain

$$\bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r},t) = \Re\big[\boldsymbol{E}(\boldsymbol{r}) \exp(-\mathrm{i}\omega_0 t)\big] = \begin{bmatrix} |E_x(\boldsymbol{r})| \cos[\varphi_x(\boldsymbol{r}) - \omega_0 t] \\ |E_y(\boldsymbol{r})| \cos[\varphi_y(\boldsymbol{r}) - \omega_0 t] \\ |E_z(\boldsymbol{r})| \cos[\varphi_z(\boldsymbol{r}) - \omega_0 t] \end{bmatrix} \ . \tag{1.50}$$

- The analogous result is valid for the magnetic field vector and we obtain explicitly

$$\bar{\boldsymbol{H}}^{(\mathrm{r})}(\boldsymbol{r},t) = \Re\big[\boldsymbol{H}(\boldsymbol{r}) \exp(-\mathrm{i}\omega_0 t)\big] = \begin{bmatrix} |H_x(\boldsymbol{r})| \cos[\phi_x(\boldsymbol{r}) - \omega_0 t] \\ |H_y(\boldsymbol{r})| \cos[\phi_y(\boldsymbol{r}) - \omega_0 t] \\ |H_z(\boldsymbol{r})| \cos[\phi_z(\boldsymbol{r}) - \omega_0 t] \end{bmatrix} \ . \tag{1.51}$$

- Demonstration of Eq.(1.50) by VirtualLab Fusion: (1) Gaussian (1D) $w_0 = 10\ \mu$m ; (2) $z = 0 - 2$ mm in 200 steps (phase enabled!); (3) data array; (4) show real part; (5) add time dependency.

- If we restrict Maxwell's equations to one monochromatic solution (frequency $\omega =$

ω_0) we obtain equations which relate the complex amplitude electric and magnetic field vectors by[10]

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = \mathrm{i}\omega\mu_0 \boldsymbol{H}(\boldsymbol{r}) \tag{1.52}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}) = -\mathrm{i}\omega\epsilon_0\check{\epsilon}_{\mathrm{r}}(\omega)\boldsymbol{E}(\boldsymbol{r}) \,, \tag{1.53}$$

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = 0 \,, \tag{1.54}$$

$$\nabla \cdot \boldsymbol{H}(\boldsymbol{r}) = 0 \,. \tag{1.55}$$

1.5 Plane wave fields

1.5.1 Definitions

- Next we assume electric and magnetic complex amplitude vectors of a harmonic field of the form

$$\boldsymbol{E}(\boldsymbol{r}) = \breve{\boldsymbol{E}} \, \exp(\mathrm{i}\check{\boldsymbol{k}} \cdot \boldsymbol{r}) \tag{1.56}$$

and

$$\boldsymbol{H}(\boldsymbol{r}) = \breve{\boldsymbol{H}} \, \exp(\mathrm{i}\check{\boldsymbol{k}} \cdot \boldsymbol{r}) \,. \tag{1.57}$$

[10] Mathematically we perform an integration over ω on both sides and eliminate the delta distribution by that.

- The vectors $\breve{\boldsymbol{E}}$ and $\breve{\boldsymbol{H}}$ are assumed to be elements of $\mathbb{C}^3$.
- The so-called complex wave vector $\breve{\boldsymbol{k}}$

$$\breve{\boldsymbol{k}} = \boldsymbol{k} + \mathrm{i}\boldsymbol{k}' = k\hat{\boldsymbol{k}} + \mathrm{i}k'\hat{\boldsymbol{k}}' \tag{1.58}$$

possesses the real-valued vectors $\boldsymbol{k} \in \mathbb{R}^3$ and $\boldsymbol{k}' \in \mathbb{R}^3$.

- The lengths of the vectors is defined by

$$k = \|\Re\breve{\boldsymbol{k}}\| = \|\boldsymbol{k}\| = \sqrt{k_x^2 + k_y^2 + k_z^2} \tag{1.59}$$

$$k' = \|\Im\breve{\boldsymbol{k}}\| = \|\boldsymbol{k}'\| = \sqrt{(k_x')^2 + (k_y')^2 + (k_z')^2} \tag{1.60}$$

$$|\breve{k}| = \|\breve{\boldsymbol{k}}\| = \sqrt{\breve{\boldsymbol{k}} \cdot \breve{\boldsymbol{k}}^*} = \sqrt{k^2 + k'^2} \tag{1.61}$$

with the complex wave number $\breve{k} = k + \mathrm{i}k'$ and $k, k' \in \mathbb{R}^+$.

- The direction unit vectors

$$\hat{\boldsymbol{k}} = \boldsymbol{k}/k = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{1.62}$$

$$\hat{\boldsymbol{k}}' = \boldsymbol{k}'/k' = (\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta') \tag{1.63}$$

define the directions of the real and imaginary part of $\check{\boldsymbol{k}}$ respectively.

- The unit direction vectors are fully specified by the spherical angles (θ, ϕ) and (θ', ϕ'). Such angles can be visualized in the Angle Viewer of VirtualLab Fusion.

- Obviously, for the complex wave vector $\check{\boldsymbol{k}}$ a geometric interpretation of the direction is not possible. Instead, as indicated in Eq.(1.62) and Eq.(1.63), the directions of the real and the imaginary parts must be considered separately.

- We distinguish between two basic types of plane waves: (1) Plane waves are called to be *homogeneous* for $k' = 0$ or $\boldsymbol{k} \parallel \boldsymbol{k}'$ and hence $\hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}'$. (2) Plane waves are called to be *inhomogeneous* for $\boldsymbol{k} \nparallel \boldsymbol{k}'$ and hence $\hat{\boldsymbol{k}} \neq \hat{\boldsymbol{k}}'$.

- Hence, homogeneous plane waves have always a well-defined unique direction $\hat{\boldsymbol{s}} := \hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}'$. Together with Eq.(1.58) follows, that homogeneous plane waves have a complex wave vector of the form

$$\check{\boldsymbol{k}} = (k + \mathrm{i}k')\hat{\boldsymbol{s}} = \check{k}\hat{\boldsymbol{s}}\,. \tag{1.64}$$

- Combination of Eq.(1.58) and Eq.(1.56) (analogously for Eq.(1.57)) yields

$$\boldsymbol{E}(\boldsymbol{r}) = \breve{\boldsymbol{E}} \exp(\mathrm{i}k\hat{\boldsymbol{k}} \cdot \boldsymbol{r}) \exp(-k'\hat{\boldsymbol{k}}' \cdot \boldsymbol{r})\,. \tag{1.65}$$

- Because $\hat{\boldsymbol{k}}\cdot\boldsymbol{r} = \text{const}$ and $\hat{\boldsymbol{k}}'\cdot\boldsymbol{r} = \text{const}$ define planes in space, the fields of Eq.(1.56) and Eq.(1.57) are called plane waves. The first factor in Eq.(1.65) governs the planes of constant phase (wavefront) and the second factor does the same for the planes of constant amplitude of the field.

- Due to the interpretation of the plane wavefront and the amplitude term we next define two quantities, that is the wavelength $\lambda(k)$ and the decay distance d_{d}.

- The wavelength $\lambda(k)$ is defined as the perpendicular distance between two planes of constant phase, having a phase difference of 2π. According to Eq.(1.65) $\hat{\boldsymbol{k}}$ is the normal vector on the planes of constant phase and thus we conclude $k\lambda = 2\pi$ and thus

$$\lambda(k) = 2\pi/k = 2\pi/\|\Re\breve{\boldsymbol{k}}\|\,. \tag{1.66}$$

- It is noteworthy, that the concept of the wavelength is per definition restricted to plane waves. This is in contrast to the general concept of frequency ω.

- However, we can define the unique connection

$$\lambda := \lambda(k_0) = 2\pi/k_0 = 2\pi c/\omega = c/\nu \tag{1.67}$$

between the wavelength in vacuum λ and the angular frequency ω. Thus, in optics we often apply the concept of the wavelength instead of the frequency independent of the type of optical field. Nevertheless, strictly speaking, the geometric interpretation of the wavelength is restricted to plane waves.

- Comment: Locally most fields can be considered to be plane.

- It should be emphasized, that in general the wavelength according to Eq.(1.66)is a function of k, that is the length of the real part of the wave vector $\breve{\boldsymbol{k}}$. That is of course not always k_0. This is discussed in more detail below.

- The distance s_{d} is defined as the distance along $\hat{\boldsymbol{k}}'$ in Eq.(1.65) at which the amplitude of $\breve{\boldsymbol{E}}$ is reduced to $1/\,\mathrm{e}$ of its original value. That leads to

$$d = 1/\|\Im\breve{\boldsymbol{k}}\| = 1/k'\,. \tag{1.68}$$

1.5.2 Electromagnetic plane wave field

- Next we answer the questions if and when the ansatz functions of Eq.(1.56) and Eq.(1.57) are solutions of Maxwell's equations. In other words, we aim to formulate conditions for the free parameters $\breve{E}$, $\breve{H}$, and $\check{\boldsymbol{k}}$, which ensures that the ansatz functions represent electromagnetic plane wave fields.

- To this end we insert the ansatz functions into Maxwell's equations Eq.(1.52) to Eq.(1.55) and obtain by the substitution $\nabla \rightarrow \mathrm{i}\check{\boldsymbol{k}}$ the plane wave field equations

$$\check{\boldsymbol{k}} \times \breve{\boldsymbol{E}} = \omega \mu_0 \breve{\boldsymbol{H}} \,, \tag{1.69}$$

$$\check{\boldsymbol{k}} \times \breve{\boldsymbol{H}} = -\omega \epsilon_0 \check{\epsilon}_{\mathrm{r}}(\omega) \breve{\boldsymbol{E}} \,, \tag{1.70}$$

$$\check{\boldsymbol{k}} \cdot \breve{\boldsymbol{E}} = 0 \,, \tag{1.71}$$

$$\check{\boldsymbol{k}} \cdot \breve{\boldsymbol{H}} = 0 \,. \tag{1.72}$$

- Since all three vectors in the equations are complex valued, they have different directions for their real and the imaginary part. Thus, direct conclusions about the directions of the vectors cannot be drawn from the plane field equations in general.

- Obviously, the field amplitude components of $\breve{\boldsymbol{E}}$ and $\breve{\boldsymbol{H}}$ significantly depend on each other. One way to discuss that dependency is as follows: Assume $\breve{E}_x$ and $\breve{E}_y$ would be specified. Then from Eq.(1.71) follows

$$\breve{E}_z = -\frac{\check{k}_x \breve{E}_x + \check{k}_y \breve{E}_y}{\check{k}_z} \,. \tag{1.73}$$

- Now $\breve{\boldsymbol{E}}$ is known and via Eq.(1.69) follows

$$\breve{\boldsymbol{H}} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{\check{\boldsymbol{k}} \times \breve{\boldsymbol{E}}}{k_0} \,. \tag{1.74}$$

- The vector cross product in (1.74) can be written as the determinant

$$\check{\boldsymbol{k}} \times \breve{\boldsymbol{E}} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \check{k}_x & \check{k}_y & \check{k}_z \\ \breve{E}_x & \breve{E}_y & \breve{E}_z \end{vmatrix} \tag{1.75}$$

with the unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ of the coordinate system in use. That gives in detail

$$\check{\boldsymbol{k}} \times \breve{\boldsymbol{E}} = (\check{k}_y\, \breve{E}_z - \check{k}_z\, \breve{E}_y)\hat{\boldsymbol{x}} \\ + (\check{k}_z\, \breve{E}_x - \check{k}_x\, \breve{E}_z)\hat{\boldsymbol{y}} + (\check{k}_x\, \breve{E}_y - \check{k}_y\, \breve{E}_x)\hat{\boldsymbol{z}}\,. \tag{1.76}$$

- Eq.(1.73) and Eq.(1.74) constitute the conditions on the other four components of $\breve{\boldsymbol{E}}$ and $\breve{\boldsymbol{H}}$ for given $\breve{E}_x$ and $\breve{E}_y$, which follows from and are therefore required to satisfy Maxwell's equations (1.69) and (1.71).
- Also Eq.(1.72) is then automatically satisfied. However, Eq.(1.70) yields an additional condition which can be easily obtained by plugging $\breve{\boldsymbol{H}}$ from Eq.(1.69) into Eq.(1.70) and we obtain with Eq.(1.71) and by using Eq.(1.42) the so-called dispersion relation

$$\check{\boldsymbol{k}}(\omega) \cdot \check{\boldsymbol{k}}(\omega) = \|\check{\boldsymbol{k}}(\omega)\|^2 = \check{k}_x^2 + \check{k}_y^2 + \check{k}_z^2 \stackrel{!}{=} k_0^2 \check{n}^2(\omega) = k_0^2 \big(n(\omega) + \mathrm{i}n'(\omega)\big)^2 \tag{1.77}$$

  as the final constraint to our plane wave ansatz functions in order to satisfy Maxwell's equations.
- Because of Eq.(1.77) the three components $(\check{k}_x, \check{k}_y, \check{k}_z)$ are not independent.
- The dispersion relation Eq.(1.77) can also be obtained from the Helmholtz' equation

Eq.(1.43) by inserting the plane wave ansatz function. The situations is summarized in Fig.1.1.

- Next we turn to a more detailed discussion of the dispersion equation (1.77). [11]
- We rewrite the left side of Eq.(1.77) and obtain (skipping ω; j indicates the three components)

$$\check{\boldsymbol{k}}^2 = (\boldsymbol{k} + \mathrm{i}\boldsymbol{k}')^2 = \sum_j (k_j + \mathrm{i}k_j')^2 \tag{1.78}$$
$$= \sum_j (k_j^2 - {k_j'}^2) + 2\mathrm{i}\sum_j k_j k_j'$$
$$= (k^2 - k'^2) + 2\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{k}'$$

using (1.58).

- Expanding the right side of Eq.(1.77) leads to

$$\check{\boldsymbol{k}}^2 = k_0^2(n + \mathrm{i}n')^2 = k_0^2(n^2 - {n'}^2) + 2\mathrm{i}k_0^2 n n'\,. \tag{1.79}$$

- Equalizing Eq.(1.78) and Eq.(1.79) provides the dispersion relation in the more

[11] In *Fundamentals* a comprehensive discussion of models of $\check{n}^2(\omega)$ is given.

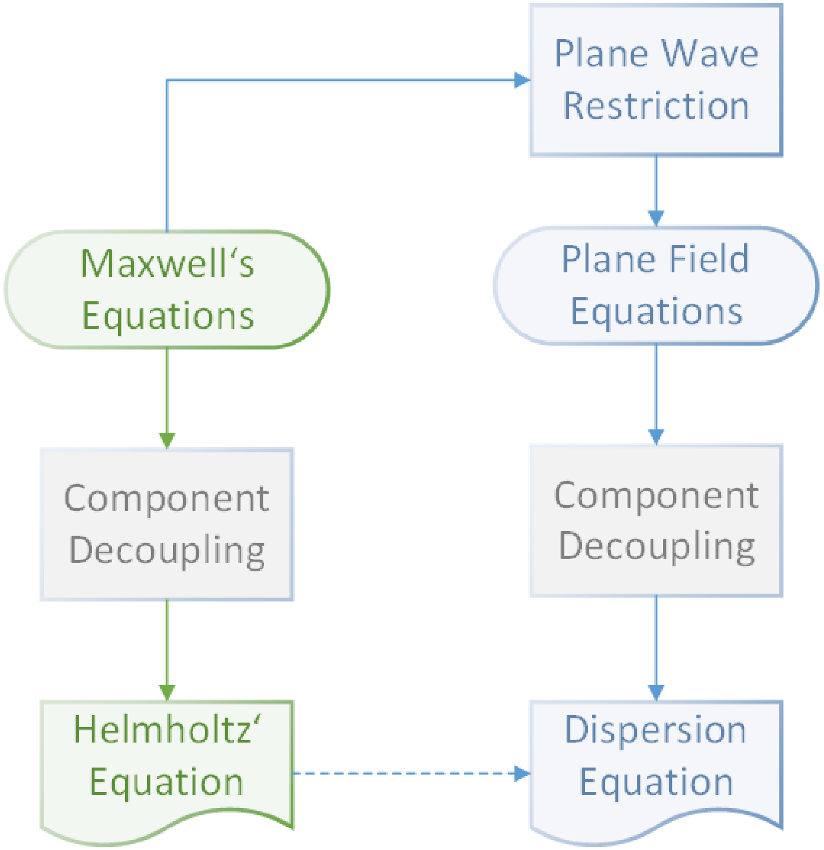


Figure 1.1: For plane waves Maxwell's equations reduce to the plane field equations and Helmholtz' equation finds its counterpart in the dispersion relation.

explicit form

$$(k^2 - k'^2) = k_0^2(n^2 - n'^2) \tag{1.80}$$

and

$$\boldsymbol{k} \cdot \boldsymbol{k}' = k_0^2 n n' \,. \tag{1.81}$$

- From our discussion so far we conclude:
  - Let us assume we specify $\breve{\boldsymbol{E}}_\perp = (\breve{E}_x, \breve{E}_y) \in \mathbb{C}^2$ in the plane wave field ansatz function. The selection of $\breve{\boldsymbol{E}}_\perp$ is *not restricted*, but typically a normalized vector is applied and called Jones vector.
  - Moreover $\breve{\boldsymbol{k}} = \boldsymbol{k} + \mathrm{i}\boldsymbol{k}'$ should be given, which satisfies the dispersion relation Eq.(1.77).
  - Then the missing field components $\breve{E}_z$ and $\breve{\boldsymbol{H}}$ can be calculated by Eq.(1.73) and Eq.(1.74).
- Obviously there is one open task to be solved: How to specify a complex vector $\breve{\boldsymbol{k}}$ which satisfies the dispersion relation Eq.(1.77) or its more explicit versions Eq.(1.80) and Eq.(1.81).

- Two ways to specify the k-vector are discussed next. Both are of great importance in optical modeling.

1.5.3 Specification of wave vector: Type I

- The specification type I is systematized by:
  - The direction vector of the wavefront propagation $\hat{\boldsymbol{k}}$ is given (see Eq.(1.62)).
  - The wavenumber $k = \|\Re\check{\boldsymbol{k}}\|$ is given.
  - The wavelength in vacuum λ and by that k_0 is known.
  - The complex refractive index $\check{n} = n + \mathrm{i}n'$ is specified.
  - From that we like to conclude $\boldsymbol{k}' = k'\hat{\boldsymbol{k}}'$, that is the direction $\hat{\boldsymbol{k}}'$ and length k' of the imaginary part of the complex wavevector.
  - Reminder: According to Eq.(1.68) the larger k' the shorter the decay distance of the field.
- From Eq.(1.80) we can immediately conclude

$$k' = \sqrt{k^2 - k_0^2(n^2 - n'^2)}\,. \tag{1.82}$$

- Since k' is the length of a vector it is positive and thus we use the positive root only.

- Moreover, here we must introduce a first constraint to k, that is $k^2 \geq k_0^2(n^2 - n'^2)$, in order to ensure real valued k'. Obviously, the specification of k is not completely unrestricted.

- Next we turn to the determination of the direction $\hat{\boldsymbol{k}}'$. That can be done via Eq.(1.81). To this end we reformulate it to

$$\Delta\gamma = \arccos \frac{k_0^2 n n'}{k k'} \leq \pi/2 \tag{1.83}$$

  with $\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}' = \cos \Delta\gamma$.

- In fact, we cannot conclude an unambiguous direction $\hat{\boldsymbol{k}}'$, but the angle $\Delta\gamma$ between $\hat{\boldsymbol{k}}'$ and $\hat{\boldsymbol{k}}$ only. All directions which lie on a cone with opening angle $\Delta\gamma$ around $\hat{\boldsymbol{k}}$ satisfy the dispersion relation.

- From Eq.(1.83) we can conclude the constraint $k_0^2 n n'/(kk') \leq 1$. Inserting k' from

Eq.(1.82) into this inequality yields

$$k^4 - k^2 k_0^2 (n^2 - n'^2) - k_0^4 n^2 n'^2 \geq 0\,. \tag{1.84}$$

- The only real-valued and positive solution of this inequality is given by (see Mathematica: 2016-01-25_FW_Plane.Waves_.nb)

$$k \geq k_0 n\,. \tag{1.85}$$

- Obviously, that is even stricter than the constraint $k^2 \geq k_0^2(n^2 - n'^2)$ and thus it is the universal constraint we need to satisfy when selecting a k in type I specification of a wavevector.

- Then we can calculate k' and $\Delta\gamma$, which allows us to construct a vector $\boldsymbol{k}'$, which satisfies together with $\boldsymbol{k}$ the dispersion relation.

- Alternatively to Eq.(1.84) we can obtain

$$k'^4 + k'^2 k_0^2 (n^2 - n'^2) - k_0^4 n^2 n'^2 \geq 0\,. \tag{1.86}$$

Then we conclude the condition

$$k' \geq k_0 n' \,. \tag{1.87}$$

In Sec.1.5.3 we discuss the importance of the special case $k = k_0 n$ and $k' = k_0 n'$.

- Illustration of formulas in Mathematica 2016-01-25_FW_Plane.Waves_.nb
- The dependency $k'(k)$ of Eq.(1.82) is illustrated in Fig.1.2.
- The dependency $Delta\gamma(k)$ of Eq.(1.83) is illustrated in Fig.1.3.
- The Type I specification of a plane wave can be summarized as follows: (1) A $k \geq k_0 n$ and direction $\hat{\boldsymbol{k}}(\theta, \phi)$ is selected. (2) Then k' and $\Delta\gamma$ are calculated via Eq.(1.82) and Eq.(1.83) respectively. (3) From $\hat{\boldsymbol{k}}(\theta, \phi)$ and $\Delta\gamma$ a possible $\hat{\boldsymbol{k}}'(\theta', \phi')$ is derived.
- The last step can be done in the following way:
  - Assume $\hat{\boldsymbol{k}}_{\text{in}} = (0, 0, 1)$ to initialize the construction process.
  - Select the rotation matrix $\mathsf{R}(\theta, \phi)$, which rotates $\hat{\boldsymbol{k}}_{\text{in}}$ into the selected $\hat{\boldsymbol{k}}$.

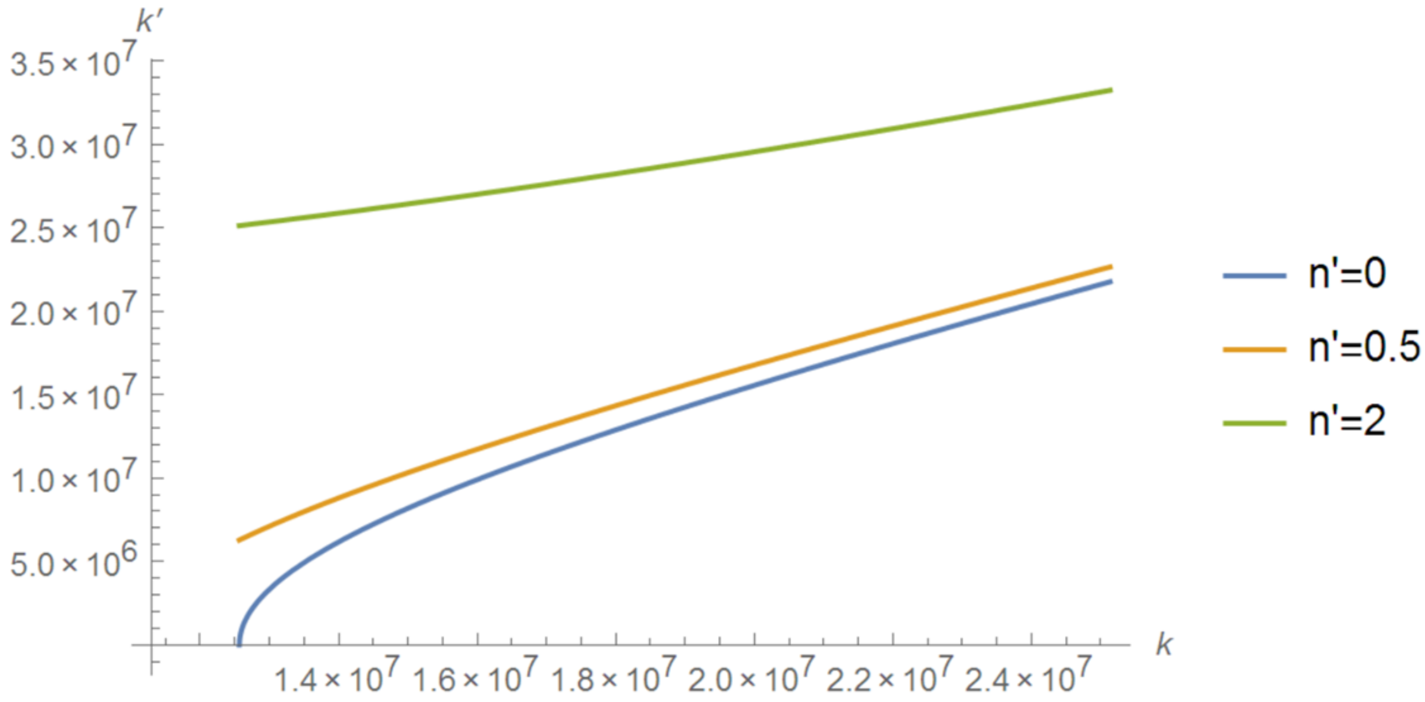


Figure 1.2: The function $k'(k)$ is monotonically increasing starting with $k' = k_0 n'$ at $k = k_0 n$.

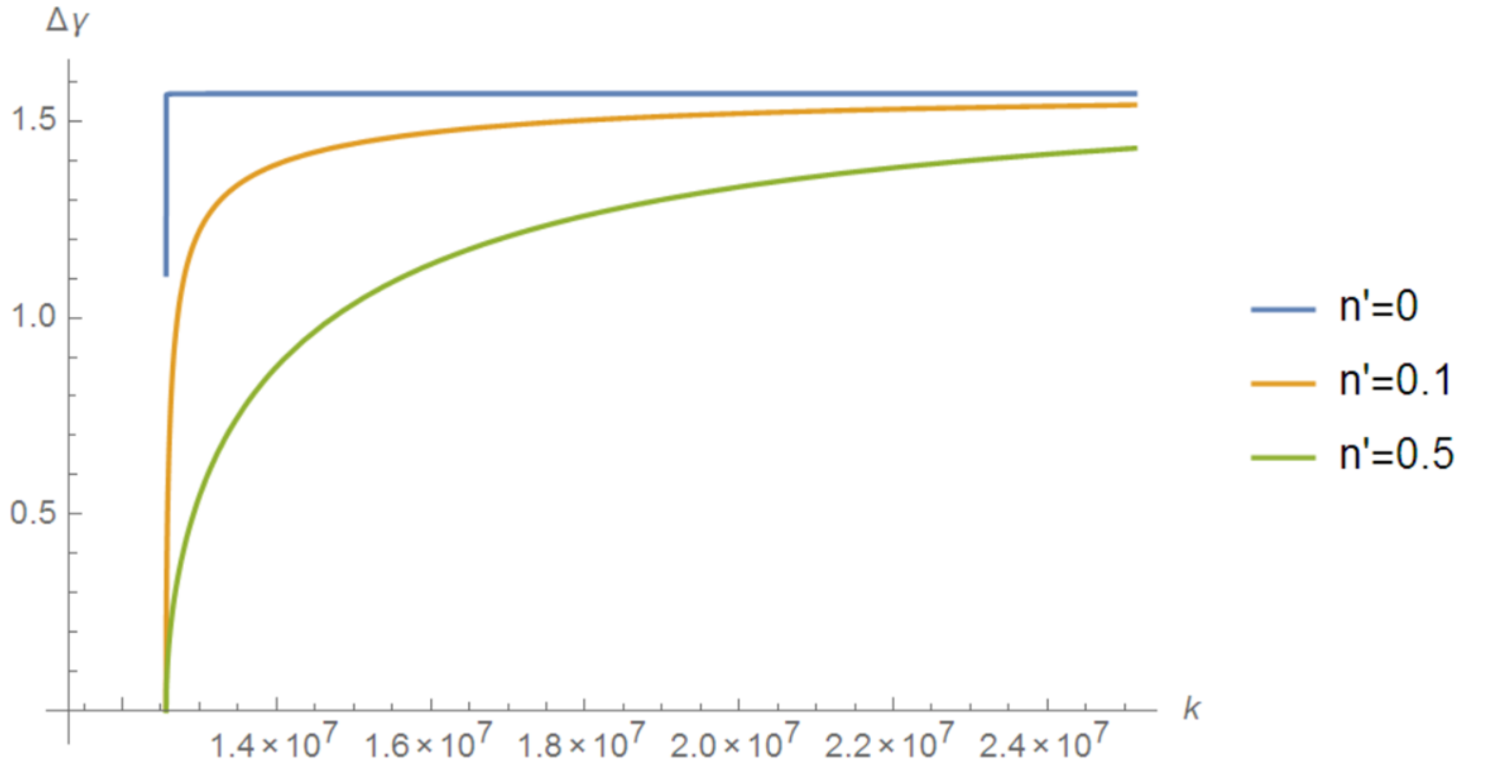


Figure 1.3: The function $\Delta\gamma(k)$ is monotonically increasing starting with $Delta\gamma(k) = 0$ at $k = k_0 n$. In the case $n' = 0$ the function jumps from zero to $\pi/2$ for $k > k_0 n$.

– Select

$$\hat{\boldsymbol{k}}'_{\text{in}} = (\sin\Delta\gamma\cos\phi_{\text{in}}, \sin\Delta\gamma\sin\phi_{\text{in}}, \cos\Delta\gamma) \tag{1.88}$$

with any ϕ_{in}, e.g. $\phi_{\text{in}} = 0$.

– Obviously this construction ensures $\hat{\boldsymbol{k}}_{\text{in}} \cdot \hat{\boldsymbol{k}}'_{\text{in}} = \cos\Delta\gamma$. Thus, in the final step $\hat{\boldsymbol{k}}' = \mathsf{R}(\theta, \phi)\hat{\boldsymbol{k}}'_{\text{in}}$ is performed and the construction of $\hat{\boldsymbol{k}}'$ is completed.

- See also 2016-01-18_CH_RaD_Sources_Module_Generate_PlaneWave.cs.

Homogeneous plane waves

- Obviously the lower limit $k = k_0 n$ is of special concern, also because it is directly related with the minimum $k' = k_0 n'$ according to Eq.(1.82) and Eq.(1.87).

- Minimum k and k' have two consequences according to Eq.(1.66) and Eq.(1.68) respectively: (1) Minimum k results in the maximum wavelength

$$\lambda_{\text{max}} = \lambda/n \tag{1.89}$$

and the maximum decay distance

$$d_{\max} = \lambda/n' \,. \tag{1.90}$$

- Shorter wavelengths are directly related to smaller decay distances and by that only small propagation distances of the plane wave field, before energy is dissipated!
- The general dependency $\lambda(k)$ according to eqReq.plane.wave.lambda.general is shown in Fig.1.4.
- Examples of the decay function $d(k)$ are illustrated in Fig.1.5.
- The case with $k = k_0 n$ and $k' = k_0 n'$ leads according to Eq.(1.83) to $\Delta\gamma = 0$, that is we have a homogeneous plane wave. Since the inequalities Eq.(1.85) and Eq.(1.87) it follows, that this constitutes the only homogeneous plane wave solution.
- Conclusion:
  - Homogeneous plane waves possess the wave vector

$$\check{\boldsymbol{k}} = (k + \mathrm{i}k')\hat{\boldsymbol{s}} = k_0(n + \mathrm{i}n')\hat{\boldsymbol{s}} = k_0\check{n}\,(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)\,. \tag{1.91}$$

  - They are the plane waves with maximum wavelength of Eq.(1.89) and maximum

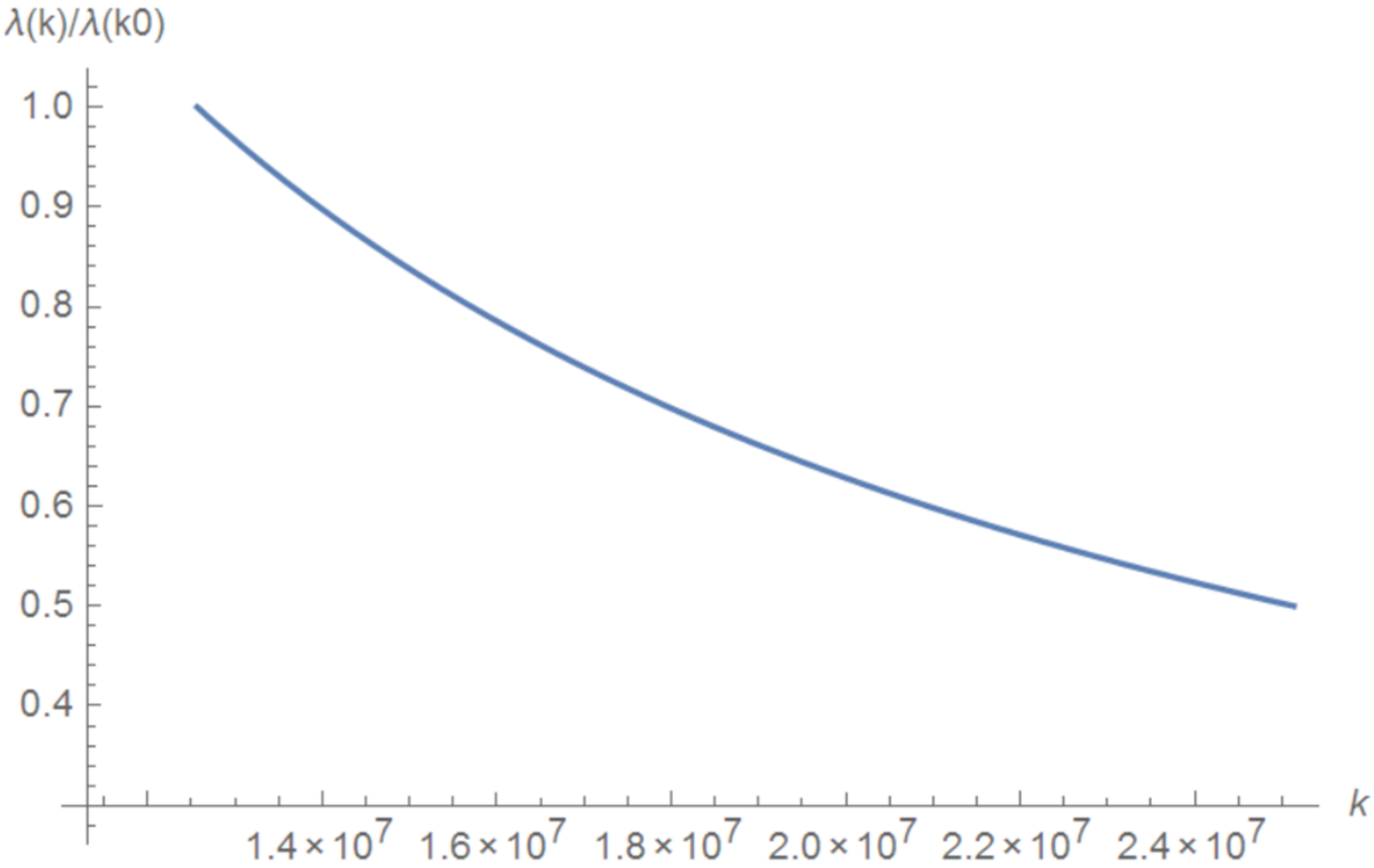


Figure 1.4: The function $\lambda(k)/\lambda(k_0)$ is shown for $n = 1$.

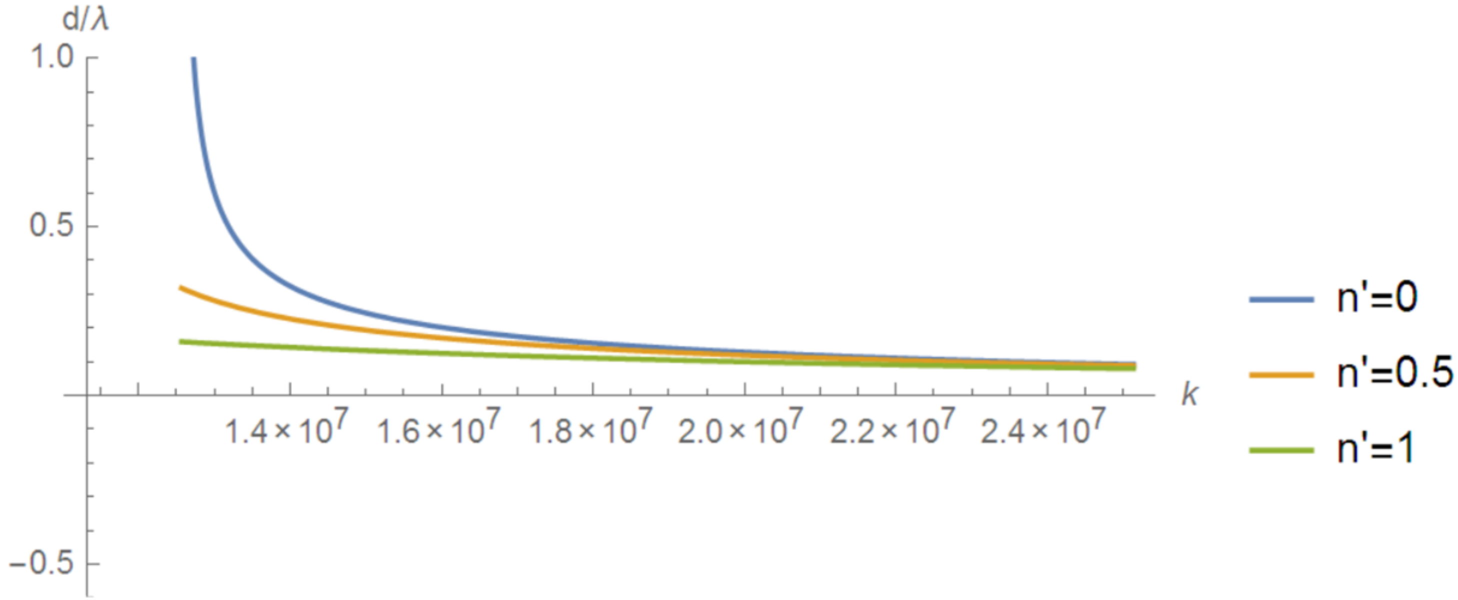


Figure 1.5: Examples of the normalized decay distance $d(k)/\lambda$ for $n' = 0, 0.5, 1$ are shown.

decay distance Eq.(1.90).

  – Smaller wavelengths ultimately leads to inhomogeneous plane waves and by that small decay distance.

- Inserting Eq.(1.91) into Eq.(1.65) yields

$$\boldsymbol{E}(\boldsymbol{r}) = \breve{\boldsymbol{E}} \exp(\mathrm{i}k_0 n \hat{\boldsymbol{s}} \cdot \boldsymbol{r}) \exp(-k_0 n' \hat{\boldsymbol{s}} \cdot \boldsymbol{r}) \,. \tag{1.92}$$

- Eq.(1.73) gets with Eq.(1.91) the form

$$\breve{E}_z = -\frac{\hat{s}_x \breve{E}_x + \hat{s}_y \breve{E}_y}{\hat{s}_z} \,. \tag{1.93}$$

- Moreover, from Eq.(1.74) follows with Eq.(1.91)

$$\breve{\boldsymbol{H}} = \frac{\breve{k}}{k_0} \sqrt{\frac{\varepsilon_0}{\mu_0}} \hat{\boldsymbol{s}} \times \breve{\boldsymbol{E}} \,. \tag{1.94}$$

- These equations are easier to discuss, if we choose $\hat{\boldsymbol{s}} = (0, 0, 1)$ w.l.o.g. Obviously, then we conclude $\breve{E}_z = 0$ from Eq.(1.93).

- For this choice of $\hat{\boldsymbol{s}}$ Eq.(1.94) leads to

$$\breve{\boldsymbol{H}} = \frac{\check{k}}{k_0}\sqrt{\frac{\varepsilon_0}{\mu_0}}\Big(- \breve{E}_y\hat{\boldsymbol{x}} + \breve{E}_x\hat{\boldsymbol{y}}\Big)\,. \tag{1.95}$$

- Thus, homogeneous plane waves are transversal and $\hat{\boldsymbol{s}}$, $\breve{\boldsymbol{E}}$, and $\breve{\boldsymbol{H}}$ are orthogonal to each other according to the the right-hand rule.

Special case $n' = 0$

- For dielectrics without damping we have $n' = 0$ and thus Eq.(1.82) reduces to

$$k' = \sqrt{k^2 - k_0^2 n^2}\,. \tag{1.96}$$

- For $k = k_0 n$ we have $k' = 0$, which is in full accordance with Eq.(1.91) for homogeneous plane waves in case of $n' = 0$.
- From Eq.(1.83) we conclude $\Delta\gamma = \pi/2$ for any $k \geq k_0 n$.
- In summary: In dielectrics without damping we have homogeneous plane waves with $\check{\boldsymbol{k}} = k_0 n \hat{\boldsymbol{s}}$ or inhomogeneous waves with $k > k_0 n$, the orthogonal direction

vectors $\hat{\boldsymbol{k}} \perp \hat{\boldsymbol{k}'}$, and

$$\check{\boldsymbol{k}} = k\hat{\boldsymbol{k}} + \mathrm{i}\sqrt{k^2 - k_0^2 n^2}\,\hat{\boldsymbol{k}'} \tag{1.97}$$

These inhomogeneous plane waves are typically called evanescent plane waves, since they vanish very fast in the direction of $\hat{\boldsymbol{k}'}$ in contrast to the propagating homogeneous waves.

1.5.4 Specification of wave vector: Type II

- In Sec.2.2 and numerous other discussions in optical modeling we use the concept of spatial Fourier transformation of field components, e.g.

$$E_x(x, y, z) = \frac{1}{2\pi} \int\int_{-\infty}^{\infty} \hat{E}_x(k_x, k_y, z)\, \mathrm{e}^{\mathrm{i}(k_x x + k_y y)}\, \mathrm{d}k_x\, \mathrm{d}k_y \tag{1.98}$$

  is a Fourier transform representation of the E_x-component in some plane z.

- We will discuss in Sec.2.2, that (k_x, k_y) represents wave vectors of plane waves. In order to use this wave vector interpretation of k_x and k_y in Eq.(1.98), we must demand for two major constraints on the wave vectors: (1) The x-and y-components of the wave vector must be real valued, that means $k'_x = k'_y = 0$ and (2) $k_x, k_y \in \mathbb{R}$ must be allowed to take any value out of $\mathbb{R}$, because the limits of the integral in Eq.(1.98) are $\pm\infty$.

- The second constraint should be expressed mathematically by introducing $\boldsymbol{\kappa} = \boldsymbol{k}_\perp = (k_x, k_y)$, $\kappa = \|\boldsymbol{\kappa}\| = \sqrt{k_x^2 + k_y^2}$ and the normalized radial spatial frequency

$$f(\kappa) = \kappa / k_0 = \frac{\sqrt{k_x^2 + k_y^2}}{k_0} \,. \tag{1.99}$$

  Obviously constraint (2) requires that f should be allowed to take an arbitrary large value.

- These considerations lead us to the specification of type II for a complex wave vector:
  - The x- and y-components of the complex wave vector are given as real valued but otherwise unrestricted values, that is $\check{k}_x = k_x \in \mathbb{R}$ and $\check{k}_y = k_y \in \mathbb{R}$.
  - The wavelength in vacuum λ and by that k_0 is known.
  - The complex refractive index $\check{n} = n + \mathrm{i}n'$ is specified.

– From that we like to conclude the complex wave vector $\check{\boldsymbol{k}}$.

- The solution of this task is straightforward.
- The condition $k'_x = k'_y = 0$ for the wave vector $\check{\boldsymbol{k}}$ of Eq.(1.58) yields the special form

$$\check{\boldsymbol{k}} = (k_x, k_y, \check{k}_z) = (k_x, k_y, k_z) + \mathrm{i}(0, 0, k'_z) = k\hat{\boldsymbol{k}} + \mathrm{i}\,\mathrm{sgn}(k'_z)\,k'\hat{\boldsymbol{z}}\,. \tag{1.100}$$

- By inserting this $\check{\boldsymbol{k}}$ into Eq.(1.65) we obtain

$$\boldsymbol{E}(\boldsymbol{r}) = \check{\boldsymbol{E}}\,\exp(\mathrm{i}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r})\,\exp(-\mathrm{sgn}(k'_z)\,k'z)\,. \tag{1.101}$$

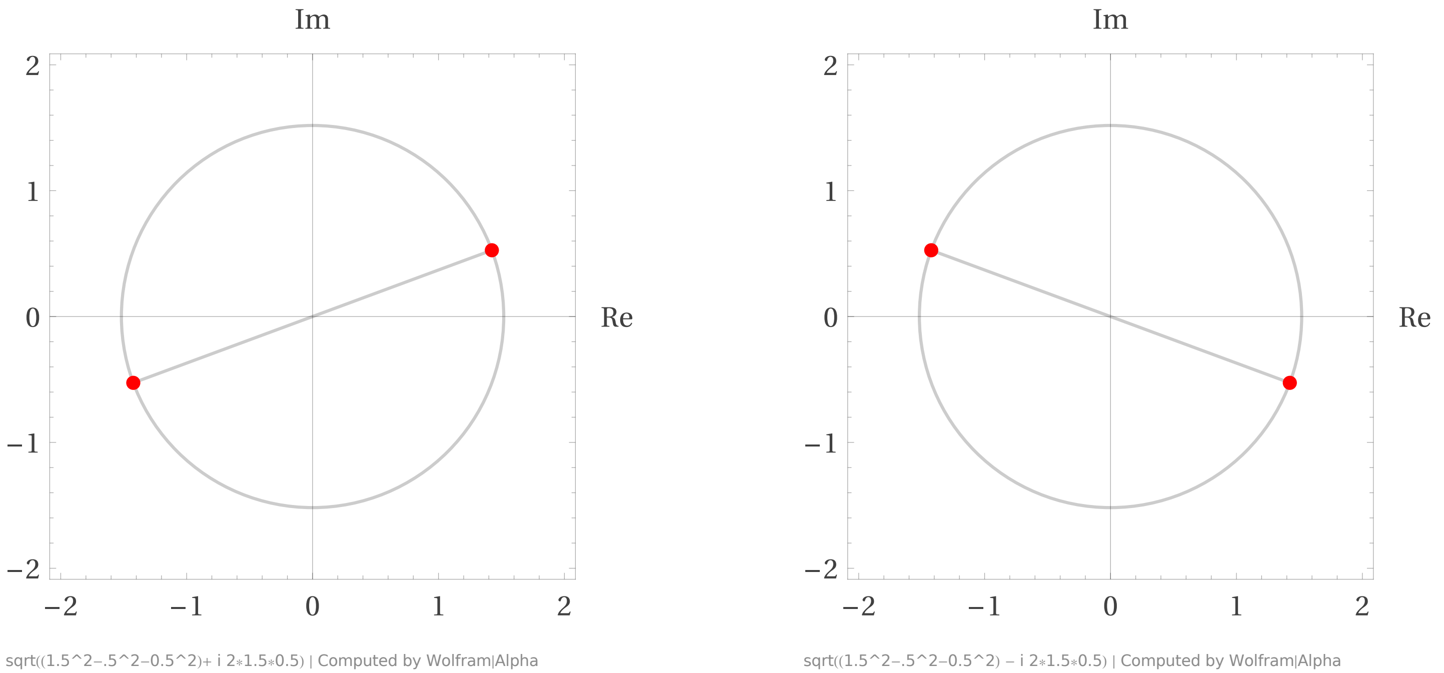
Because of $\hat{\boldsymbol{k}}' = \hat{\boldsymbol{z}}$ the attenuating (amplifying) factor affects the plane wave in z-direction only.

- Resolving the dispersion relation Eq.(1.77) for $\check{k}_z$ results in

$$\begin{aligned} \check{k}_z(\kappa) &= k_0\sqrt{\check{n}^2 - f^2(\kappa)} \\ &= k_z(\kappa) + \mathrm{i}k'_z(\kappa)\,. \end{aligned} \tag{1.102}$$

- The two solutions of $\check{k}_z$ of Eq.(1.102) are complex valued in general. Dependent on the sign of n', the imaginary and real part of $\check{k}_z$ have equal or different sign. That is illustrated in Fig.1.6.

- In optical modeling we have the convention to consider fields which propagate into positive z-direction (with respect to the field coordinate system, which can change after interaction with a component). Hence, the root from Eq.(1.102) with $\Re(\check{k}_z) = k_z > 0$ is used.
- It should be mentioned, that the plane waves which follow from type II specification can of course be also discussed in the context of type I. To this end we calculate for a given $\boldsymbol{\kappa}$ the z-component according to Eq.(1.102) and calculate then $k = (\boldsymbol{k}^2 + k_z^2)^{0.5}$. Then we are in the type I specification with a specially constructed k.
- Of course also this k and related k' satisfy all equations which were discussed in Sec.1.5.3. In particular the universal inequalities for k and k' hold.
- Illustration in Mathematica by 2015-01-16_k_z.Discussion.nb.



(a) Parameters: $f = \kappa/k_0 = 0.5, n = 1.5, n' = 0.5$

(b) Parameters: $f = \kappa/k_0 = 0.5, n = 1.5, n' = -0.5$

Figure 1.6: Illustration of the solutions of Eq.(1.102) for (a) positive (attenuation) and (b) negative (gain) imaginary part n' of the complex refractive index $\check{n}$.

Chapter 2

Solution of Maxwell's equations in homogeneous media

2.1 Optical modeling as a boundary value problem

- Plane waves constitute a special solution of Maxwell's equations in homogeneous media, that are Eq.(1.52) to Eq.(1.55).
- They form a class of possible electromagnetic fields in homogenous media.
- Besides the search for solution classes of Maxwell's equations in optical modeling we typically face another task: We search for specific solution for given boundary conditions.
- This task is illustrated in Fig.2.1 and Fig.2.2.

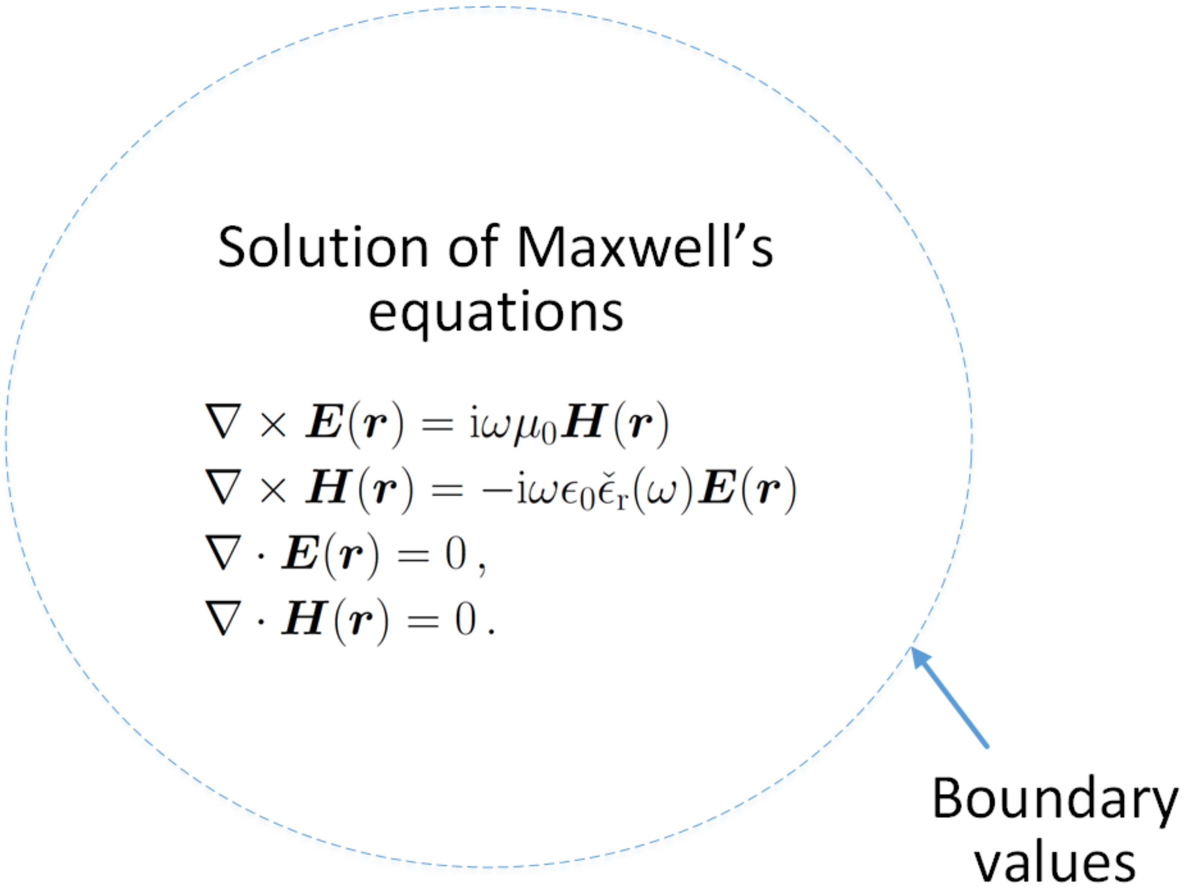


Figure 2.1: Solution of Maxwell's equations with boundary values.

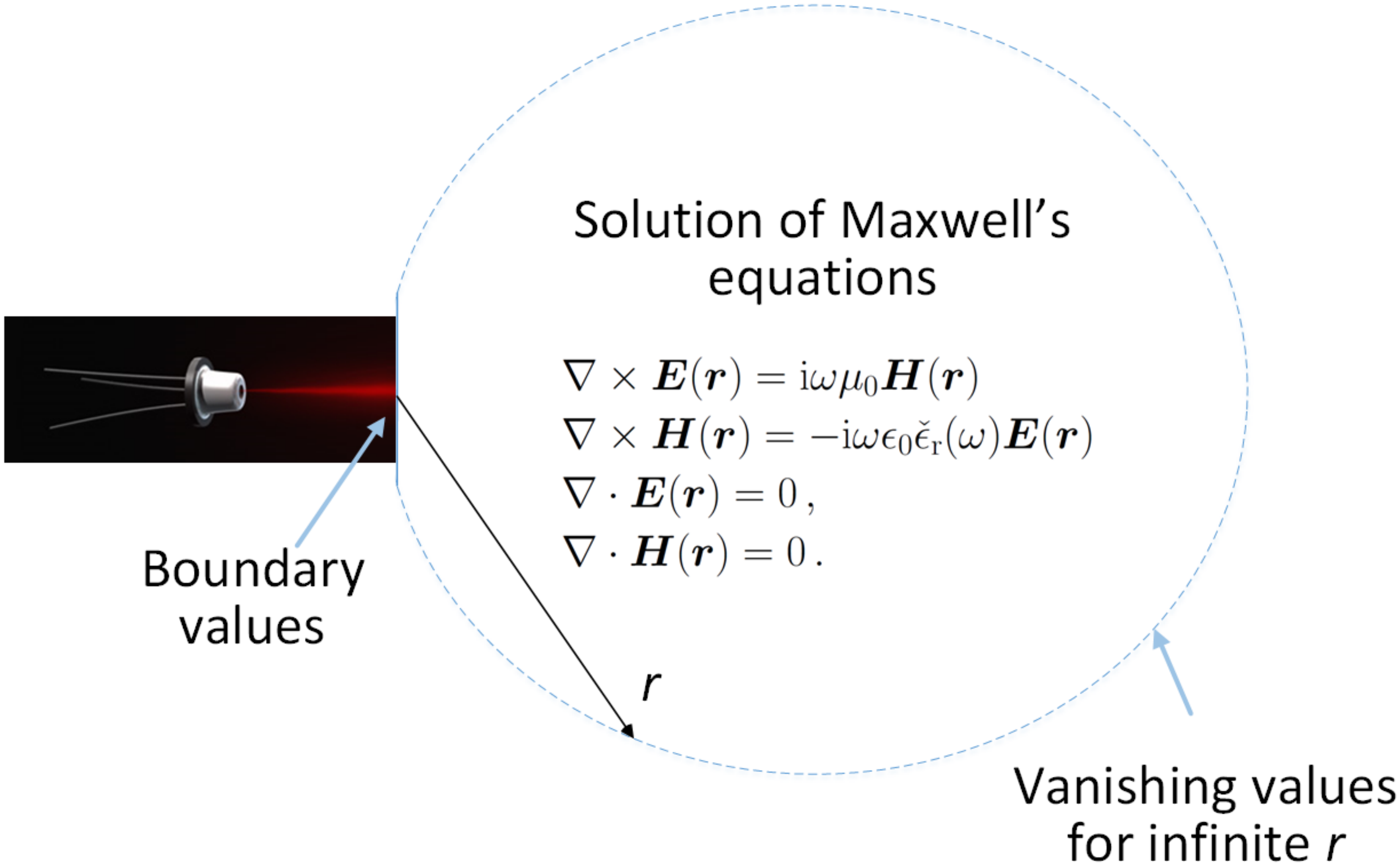


Figure 2.2: In optical modeling a typical boundary problem is formulated by a defined field in a source plane.

- In optical modeling it is also typical, that we are not interested in the field in full space, which is originated by the source field, but often just in another plane, e.g. the detector plane as illustrated in Fig.2.3.
- That leads us to the interpretation of the solution of Maxwell's equation as an propagation problem. This interpretation does not include any approximation, as long as we deal with the full electromagnetic field!
- The propagation problem is illustrated in Fig.2.4.
- In optical modeling this propagation task does not only appear to proapagte from the source plane to the detector or first component, but also between different components, as illustrated in Fig.2.5.
- The classical diffraction problem at an aperture is a special case of the general propagation task. It is illustrated in Fig.2.6.
- Thus, often we refer to propagation operators in integral form as diffraction integrals.
- Conclusion: In order to solve the general propagation problem we need to tackle to subtasks: (1) Electromagnetic specification of an electromagnetic field in the input

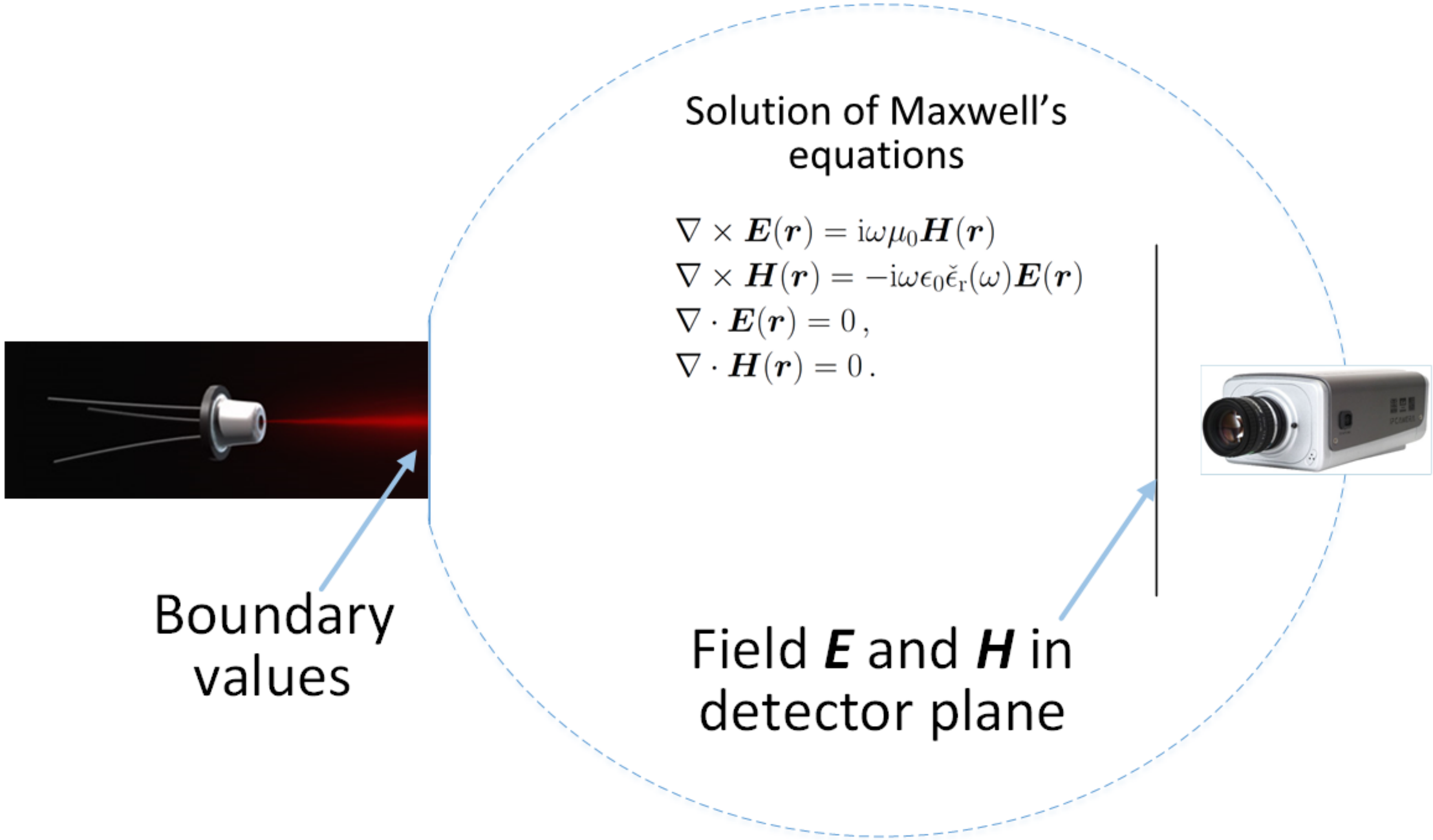


Figure 2.3: Often we are only interested to know the field, which is origoi9nated by the source, in a detector plane.

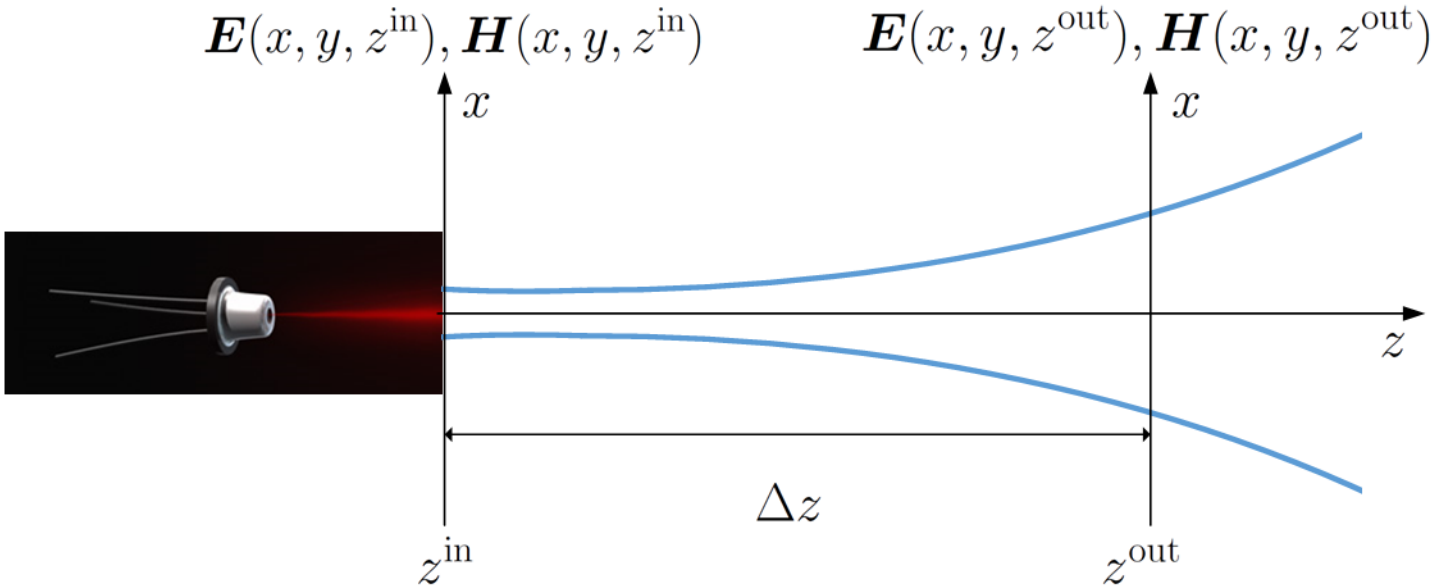


Figure 2.4: Illustration of the typical free-space propagation problem: The electromagnetic field is given in an input plane and should be calculated in another plane some distance Δz away. Here the input plane is the source plane.

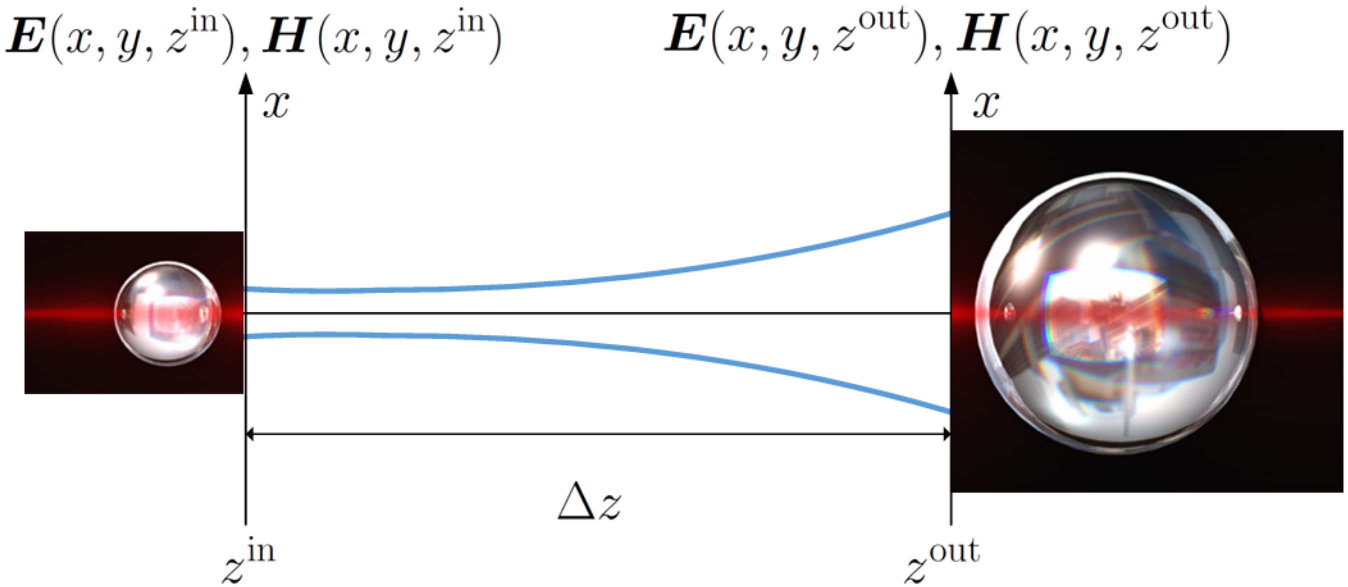


Figure 2.5: Also between components the free-space propagation task is to be solved in optical modeling.

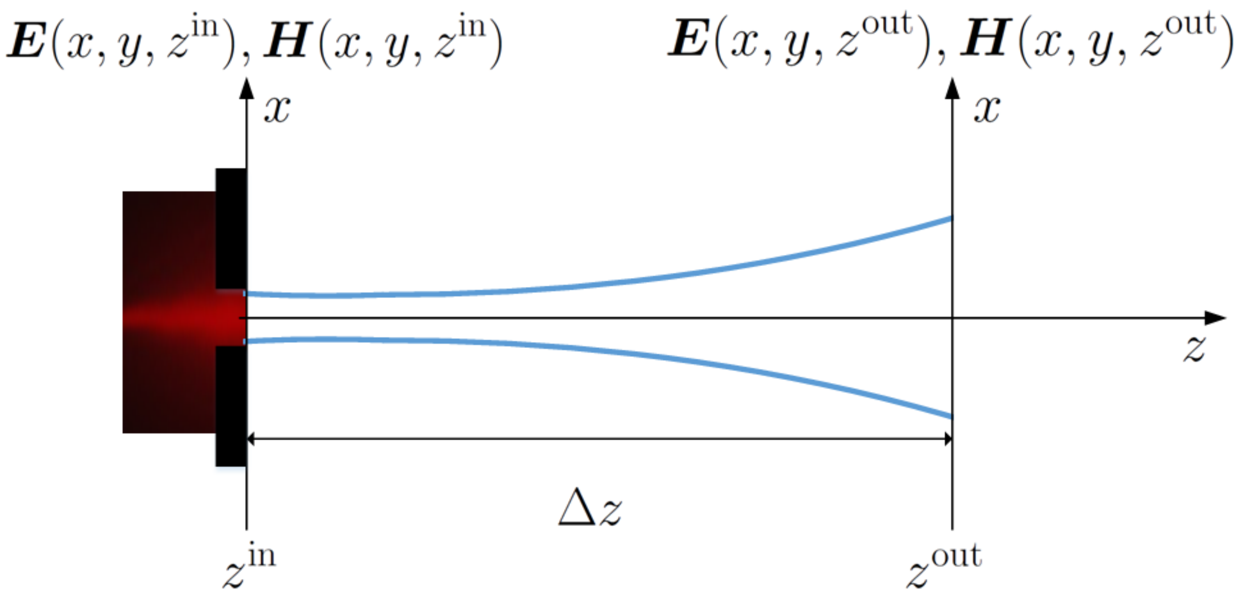


Figure 2.6: Illustration of diffraction at an aperture as a special case of the general propagation problem.

plane z^{in}. (2) Propagation of this electromagnetic field to the plane z^{out}.

- Both tasks are to be discussed next.
- Remark: Here we discuss the propagation between two parallel planes only. The generalization to tilted planes and even general surfaces requires a repeated application of the basic technique, which is very time consuming in practice, or more sophisticated techniques, which are subject of Optical Modeling and Design I.

2.2 Spectrum of plane wave (SPW) decomposition of fields

- To solve the propagation problem the spectrum of plane waves technique is of basic concern. Thus it is introduced next.
- Let us describe an arbitrary component of $\boldsymbol{E}(x, y, z_0)$ and $\boldsymbol{H}(x, y, z_0)$ of a harmonic field in an arbitrary plane z_0 which propagates in $+z$-direction with $V(\boldsymbol{\rho}, z_0)$ and $\boldsymbol{\rho} = (x, y)$.
- The plane z_0 can be the one in z^{in} or z^{out}. The SPW techniques can be applied in any plane and thus the conclusions are valid in any plane.

- Formally we may introduce the **spatial** Fourier transformation of $V(\boldsymbol{\rho}, z_0)$ in form of

$$\hat{V}(\boldsymbol{\kappa}, z_0) = \mathcal{F}_k V(\boldsymbol{\rho}, z_0) = \frac{1}{2\pi} \int\int_{-\infty}^{\infty} V(\boldsymbol{\rho}, z_0)\, \mathrm{e}^{-\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}\, \mathrm{d}^2\rho \tag{2.1}$$

with the **real-valued** $\boldsymbol{\kappa} = (k_x, k_y)$.

- The related inverse Fourier transformation is given by

$$V(\boldsymbol{\rho}, z_0) = \mathcal{F}_k^{-1} \hat{V}(\boldsymbol{\kappa}, z_0) = \frac{1}{2\pi} \int\int_{-\infty}^{\infty} \hat{V}(\boldsymbol{\kappa}, z_0)\, \mathrm{e}^{\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}\, \mathrm{d}^2\kappa\,. \tag{2.2}$$

- Important special case: $\mathcal{F}_k \exp(\mathrm{i}\boldsymbol{\kappa}_0 \cdot \boldsymbol{\rho}) = \delta(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0)$.[1]
- Example of spatial Fourier transformation: Fig.2.7 shows an example of a high-passed spectrum and the resulting picture.
- Next we reformulate Eq.(2.1) and obtain

$$V(\boldsymbol{\rho}, z_0) = \frac{1}{2\pi} \int\int_{-\infty}^{\infty} \breve{V}(\boldsymbol{\kappa}, z_0)\, \mathrm{e}^{\mathrm{i}(k_x x + k_y y + \check{k}_z(\kappa) z_0)}\, \mathrm{d}^2\kappa\,. \tag{2.3}$$

[1] Here and in what follows we use for the effect of $\mathcal{F}$ and other operators on functions $G(\boldsymbol{\rho})$ the short notation $\mathcal{F}G(\boldsymbol{\rho})$ instead of the mathematically more accurate one $\mathcal{F}(\boldsymbol{\rho} \mapsto G(\boldsymbol{\rho}))$. Mathematically an operator applies to the mapping character of a function and not on one value $G(\boldsymbol{\rho})$ of it.

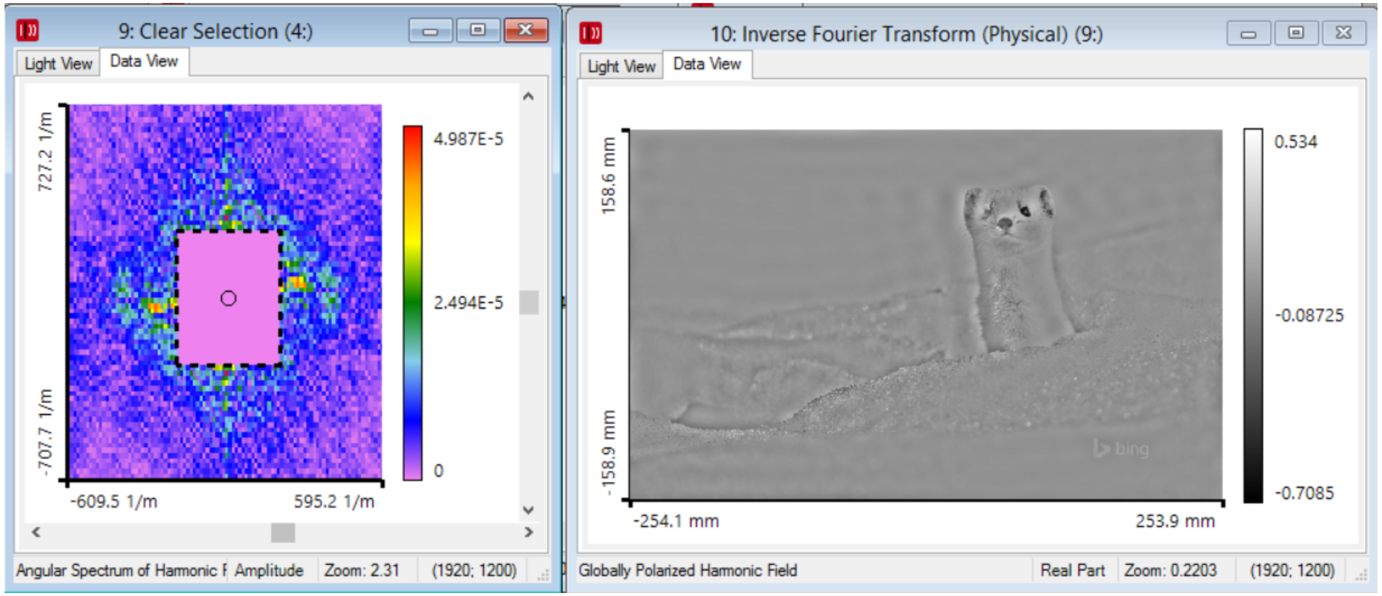


Figure 2.7: Demonstrations of spatial Fourier transformation in VirtualLab.

with

$$\breve{V}(\boldsymbol{\kappa}, z_0) := \hat{V}(\boldsymbol{\kappa}, z_0)\, \mathrm{e}^{-\mathrm{i}\check{k}_z(\kappa) z_0}\,. \tag{2.4}$$

- We may interpret $\check{k}_z$ as the z-component of the complex wave vector of an inhomogeneous plane wave according to Eq.(1.100) and Eq.(1.102). Then Eq.(2.3) can be interpreted as the decomposition of $V(\boldsymbol{\rho}, z_0)$ into plane waves in $z = z_0$ with the amplitudes (see Eq.(1.56) and Eq.(1.57)) $\breve{V}(\boldsymbol{\kappa}, z_0)$ per spatial frequency vector $\boldsymbol{\kappa}$.

- Equation (2.1) (respectively Eq.(2.3)) is typically referred to as spectrum of plane waves (SPW) decomposition and it is valid for all components of $\boldsymbol{E}(\boldsymbol{\rho}, z_0)$ and $\boldsymbol{H}(\boldsymbol{\rho}, z_0)$.

- However we know, that for plane waves the field components are not independent. Thus we expect some consequences for the SPW field representation in Eq.(2.1).

2.3 Electromagnetic fields in a plane

2.3.1 Electric field in a plane

- Next we discuss the SPW representation of the electric field components including the dependency between them.
- Let us consider the complex amplitude components of the electric field in the plane z_0, that is

$$\boldsymbol{E}(\boldsymbol{\rho}, z_0) = \Big(E_x(\boldsymbol{\rho}, z_0), E_y(\boldsymbol{\rho}, z_0), E_z(\boldsymbol{\rho}, z_0)\Big) . \tag{2.5}$$

- For each of them we can calculate

$$\hat{V}_\ell(\boldsymbol{\kappa}, z_0) = \mathcal{F}_k\, V_\ell(\boldsymbol{\rho}, z_0) \tag{2.6}$$

  according to Eq.(2.1) with $\ell = 1, 2, 3$ for the x, y and z components of the electric field in space and k-domain, that are for instance $V_1 = E_x$ and $\hat{V}_2 = \mathcal{F}_k E_y$.
- From $\hat{V}_\ell(\boldsymbol{\kappa}, z_0)$ follows via Eq.(2.4) the plane wave amplitudes $\breve{V}_\ell(\boldsymbol{\kappa}, z_0)$ in Eq.(2.3).
- Let us assume that we know $E_x(\boldsymbol{\rho}, z_0)$ and $E_y(\boldsymbol{\rho}, z_0)$ and via Fourier transformation and Eq.(2.4) $\breve{E}_x(\boldsymbol{\kappa}, z_0)$ and $\breve{E}_y(\boldsymbol{\kappa}, z_0)$, which are interpreted as the amplitudes $\breve{E}_x$

and $\breve{E}_y$ of plane waves with a wave vector $\breve{\boldsymbol{k}}$ which is specified for given $\boldsymbol{\kappa}$ by $\breve{k}_z(\boldsymbol{\kappa})$ of Eq.(1.102).

- Then we know through (1.73) for real valued (k_x, k_y), how the complex amplitude $\breve{E}_z(\boldsymbol{\kappa}, z_0)$ depends directly from the others. The factors $\mathrm{e}^{-\mathrm{i}\breve{k}_z z_0}$ on the left and right side of the equation cancel each other and we find via Eq.(2.4)

$$\hat{E}_z(\boldsymbol{\kappa}, z_0) = -\frac{k_x \hat{E}_x(\boldsymbol{\kappa}, z_0) + k_y \hat{E}_y(\boldsymbol{\kappa}, z_0)}{\breve{k}_z(\boldsymbol{\kappa})} . \tag{2.7}$$

- Inverse Fourier transformation Eq.(2.2) and expression of $\hat{E}_{x,y}(\boldsymbol{\kappa}, z_0)$ by Eq.(2.1) leads to

$$E_z(\boldsymbol{\rho}, z_0) = -\mathcal{F}_k^{-1} \left\{ \frac{k_x \left(\mathcal{F}_k E_x(\boldsymbol{\rho}, z_0)\right) + k_y \left(\mathcal{F}_k E_y(\boldsymbol{\rho}, z_0)\right)}{\breve{k}_z(\boldsymbol{\kappa})} \right\} . \tag{2.8}$$

- Equation (2.8) formulates an algorithm to calculate $E_z(\boldsymbol{\rho}, z_0)$ for given $E_x(\boldsymbol{\rho}, z_0)$ and $E_y(\boldsymbol{\rho}, z_0)$.

- Important conclusion: The electric field is completely specified in a plane P, which is situated in a homogeneous medium by specification of $E_x(\boldsymbol{\rho}, z_0)$ and $E_y(\boldsymbol{\rho}, z_0)$ in this plane. Thus the z-component is redundant in the representation of the electric field and can be calculated on demand!

- Via Eq.(1.102) we see, that the values of E_z in Eq.(2.8) significantly depend on the ratio $f/\sqrt{\check{n}^2 - f^2}$. In particular for $f \approx n$ and $n' \approx 0$ this ratio can become very large. However, for $f \approx 0$, the ratio is almost zero and we obtain very small E_z values. Thus, paraxial fields are almost transversal (we will see the same for H_z).

- VirtualLab Fusion simulations with 2015-08-25_FW_Field.Components.lpd.

2.3.2 Magnetic field in a plane

- Next we consider the complex amplitude of the magnetic field in the plane z_0, that is

$$\boldsymbol{H}(\boldsymbol{\rho}, z_0) = \Big(H_x(\boldsymbol{\rho}, z_0), H_y(\boldsymbol{\rho}, z_0), H_z(\boldsymbol{\rho}, z_0)\Big)\,. \tag{2.9}$$

- We can calculate for each of them the corresponding SPW component via

$$\hat{V}_\ell(\boldsymbol{\kappa}, z_0) = \mathcal{F}\, V_\ell(\boldsymbol{\rho}, z_0) \tag{2.10}$$

with $\ell = 4, 5, 6$ for the x, y and z components of the magnetic field in both domains.

- From $\hat{V}_\ell(\boldsymbol{\kappa}, z_0)$ follows via Eq.(2.4) the plane wave amplitudes $\breve{V}_\ell(\boldsymbol{\kappa}, z_0)$ in Eq.(2.3).

- Analogously to the discussion for electric fields we interpret $\hat{V}_\ell$ for $\ell = 4, 5, 6$ as the magnetic field amplitudes of plane waves and conclude from Eq.(1.74)

$$\hat{\boldsymbol{H}}(\boldsymbol{\kappa}, z_0) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, \frac{\breve{\boldsymbol{k}} \times \hat{\boldsymbol{E}}(\boldsymbol{\kappa}, z_0)}{k_0}\,. \tag{2.11}$$

- Explicitly we find via (1.76)

$$\hat{H}_x = -a_0(\breve{k}_z \hat{E}_y - k_y \hat{E}_z) \tag{2.12}$$

$$\hat{H}_y = +a_0(\breve{k}_z \hat{E}_x - k_x \hat{E}_z) \tag{2.13}$$

$$\hat{H}_z = a_0(k_x \hat{E}_y - k_y \hat{E}_x) \tag{2.14}$$

where we skipped the arguments of the field terms and used

$$a_0 := \frac{1}{k_0}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,. \tag{2.15}$$

- Substituting $\hat{E}_z$ of (2.7) leads to:

$$\hat{H}_x = -\frac{a_0}{\check{k}_z}\Big(k_x k_y \hat{E}_x + (k_y^2 + \check{k}_z^2)\hat{E}_y\Big) \tag{2.16}$$

$$\hat{H}_y = +\frac{a_0}{\check{k}_z}\Big(k_x k_y \hat{E}_y + (k_x^2 + \check{k}_z^2)\hat{E}_x\Big) \tag{2.17}$$

$$\hat{H}_z = a_0\,(k_x\hat{E}_y - k_y\hat{E}_x) \tag{2.18}$$

- Inverse Fourier transformation Eq.(2.2) and expression of $\hat{E}_{x,y}(\boldsymbol{\kappa}, z_0)$ by Eq.(2.1) leads to the magnetic field components

$$H_x(\boldsymbol{\rho}, z_0) = -a_0\,\mathcal{F}_k^{-1}\Big\{\frac{1}{\check{k}_z}\Big(k_x k_y\,\big(\mathcal{F}_k E_x(\boldsymbol{\rho}, z_0)\big) + (k_y^2 + \check{k}_z^2)\big(\mathcal{F}_k E_y(\boldsymbol{\rho}, z_0)\big)\Big)\Big\}\,, \tag{2.19}$$

$$H_y(\boldsymbol{\rho}, z_0) = +a_0\, \mathcal{F}_k^{-1}\left\{ \frac{1}{\check{k}_z} \Big(k_x k_y\, \big(\mathcal{F}_k E_y(\boldsymbol{\rho}, z_0)\big) + (k_x^2 + \check{k}_z^2)\big(\mathcal{F}_k E_x(\boldsymbol{\rho}, z_0)\big)\Big) \right\}, \tag{2.20}$$

and

$$H_z(\boldsymbol{\rho}, z_0) = a_0\, \mathcal{F}_k^{-1}\left\{ k_x\, \big(\mathcal{F}_k E_y(\boldsymbol{\rho}, z_0)\big) - k_y\, \big(\mathcal{F}_k E_x(\boldsymbol{\rho}, z_0)\big) \right\}. \tag{2.21}$$

- It follows, that the magnetic field in a plane P can be completely calculated from given $E_x(\boldsymbol{\rho}, z_0)$ and $E_y(\boldsymbol{\rho}, z_0)$.
- VirtualLab Fusion simulations with 2015-08-25_FW_Field.Components.lpd.
- In conclusion, the harmonic field in a plane P in homogeneous media is completely specified by specification of the x- and y-components of the electric field in that plane.

2.4 Electromagnetic fields in a space

- In Sec.2.2 the representation of an electromagnetic harmonic field in a plane which is situated in a homogeneous and isotropic medium is presented.

- That allows us to specify the field in a plane P, e.g. behind a source, as the boundary conditions for the boundary value problem which we discussed in Sec.2.1.

- In Fig.2.4 the typical scenario in optical modeling is illustrated. From our knowledge of the field in P^{in}, we like to conclude via Maxwell's equations the field in another plane $\mathsf{P}^{\mathrm{out}}$, which is placed at some distance $\Delta z > 0$ from the input plane.

- Since we restrict ourselves in this section Sec.2.4 to parallel planes, we can use z^{in} and z^{out} to refer to the planes.

- As discussed in Sec.2.1, the solution of the boundary value problem can be interpreted as a propagation of the light field from z^{in} to z^{out} with $z^{\mathrm{out}} = z^{\mathrm{in}} + \Delta z$ with $\Delta z \geq 0$, that is we assume without loss of generality, that the field propagates into the positive z-direction.

- The propagation of the field from z^{in} to z^{out} can be formulated by the operator equation

$$V_\ell(\boldsymbol{\rho}, z^{\mathrm{out}}) = \mathcal{P} V_\ell(\boldsymbol{\rho}, z^{\mathrm{in}}) \tag{2.22}$$

with the propagation operator $\mathcal{P}$, to which we often also refer to as free-space propagation operator.[2]

- V_ℓ with $\ell = 1, \ldots, 6$ denotes the six field components, that is $\boldsymbol{V} = (E_x, E_y, E_z, H_x, H_y, H_z)$.

- With Eq.(2.22) we assume, that the propagation operator $\mathcal{P}$ is the same for all field components and we have no crosstalk between the components. Without this assumptions we would need to write Eq.(2.22) in form of a 6×6-matrix.

- Of course we need to justify these assumptions. That is done below. Here we like to emphasize, that those assumptions are reasonable: (1) The Helmholtz' equation Eq.(1.43) is valid for all six field components independently and they express the evolution of the field components in space. (2) The dependency of the components from each other is well known in any plane by the results of Sec.2.3. These dependencies are the same in any z and independent of the propagation.

- In what follows we derive a solution of the propagation problem and give a solid justification of the form in Eq.(2.22).

[2] In literature *free space* is sometimes also used for vacuum only, however we use it also for homogeneous and isotropic media or in other words, in regions without components.

2.4.1 SPW propagation operator

- From Sec.2.2 we know, that the harmonic fields in z^{in} and z^{out} can be decomposed into inhomogeneous plane waves. Thus, the propagated harmonic field in z^{out} can be considered as a superposition of propagated plane waves specified in z^{in}. Hence, in order to solve the propagation problem we first consider propagated plane waves.

- By definition through Eq.(1.56) and Eq.(1.57) plane waves are known in space for a given wave vector $\check{\boldsymbol{k}}$ per definition. Hence, the propagation "problem" for plane waves is solved *a priori*. For plane waves with real-valued $\boldsymbol{\kappa}$ (see Sec.1.5.4) the "propagation" can be stated by

$$\breve{V}_\ell\, \mathrm{e}^{\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}\, \mathrm{e}^{\mathrm{i}\check{k}_z z^{\mathrm{out}}} = \breve{V}_\ell\, \mathrm{e}^{\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}\, \mathrm{e}^{\mathrm{i}\check{k}_z\left(z^{\mathrm{in}}+\Delta z\right)} = \breve{V}_\ell\, \mathrm{e}^{\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}\, \mathrm{e}^{\mathrm{i}\check{k}_z z^{\mathrm{in}}} \times\ \mathrm{e}^{\mathrm{i}\check{k}_z\Delta z} \tag{2.23}$$

  for $\ell = 1, ..., 6$.

- Hence, for a plane wave the propagation operator $\mathcal{P}$ constitutes a simple multiplication with the so-called propagation kernel $\exp(\mathrm{i}\check{k}_z\Delta z)$ with $\check{k}_z$ from Eq.(1.102). Obviously, it is identical for all components and we have no crosstalk.

- This result can be directly applied to the SPW decomposition of Eq.(2.3) with

$\breve{V}(\boldsymbol{\kappa}, z_0)$ from Eq.(2.4) in z^{in}. To this end we identify $\breve{V}(\boldsymbol{\kappa}, z^{\text{in}})$ as the complex amplitudes of plane waves with wave vector $\breve{\boldsymbol{k}} = \boldsymbol{\kappa} + \breve{k}_z \hat{\boldsymbol{z}}$ and apply Eq.(2.23). Then we obtain

$$\begin{aligned} V_\ell(\boldsymbol{\rho}, z^{\text{out}}) &= \frac{1}{2\pi} \int\int_{-\infty}^{\infty} \breve{V}_\ell(\boldsymbol{\kappa}, z^{\text{in}})\, \mathrm{e}^{\mathrm{i}(k_x x + k_y y + \breve{k}_z(\kappa) z^{\text{in}})} \times \mathrm{e}^{\mathrm{i}\breve{k}_z(\kappa)\Delta z}\, \mathrm{d}^2\rho \qquad (2.24) \\ &= \frac{1}{2\pi} \int\int_{-\infty}^{\infty} \hat{V}_\ell(\boldsymbol{\kappa}, z^{\text{in}})\, \mathrm{e}^{\mathrm{i}\breve{k}_z(\kappa)\Delta z}\, \mathrm{e}^{\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}\, \mathrm{d}^2\kappa \end{aligned}$$

by using Eq.(2.4) to replace $\breve{V}_\ell(\boldsymbol{\kappa}, z^{\text{in}})\, \mathrm{e}^{\mathrm{i}\breve{k}_z(\kappa) z^{\text{in}}}$ by $\hat{V}_\ell(\boldsymbol{\kappa}, z^{\text{in}})$.

- Eq.(2.24) formulates the SPW propagation integral and can be represented by Fourier transforms (see Eq.(2.1) and Eq.(2.2)) and we obtain

$$V_\ell(\boldsymbol{\rho}, z^{\text{out}}) = \mathcal{F}_k^{-1}\left[\left(\mathcal{F}_k V_\ell(\boldsymbol{\rho}, z^{\text{in}})\right) \mathrm{e}^{\mathrm{i}\breve{k}_z(\kappa)\Delta z}\right]. \qquad (2.25)$$

- The operator in form of Eq.(2.25) allows the application of the Fast Fourier Transformation (FFT) algorithm for the numerical implementation of the SPW integral operator. For the practice of optical modeling and design this is a significant benefit from mathematics in all simulations in which diffraction (and scattering) must be included.

- From its derivation it is clear, that the propagation operator is to be applied componentwise for $\ell = 1, \ldots, 6$. That finally justifies our way to formulate the propagation problem in Eq.(2.22).

- In practice the propagation it is only done for $\ell = 1, 2$ and the other components are calculated on demand from the propagated two electric field components using the results from Sec.2.3.

- The Fourier transformation formulation in Eq.(2.25) allows an interesting interpretation of the SPW operator: It is a Fourier filtering step with the filter function[3]

$$\hat{G}^{\mathrm{I}}(\boldsymbol{\kappa}) = \mathrm{e}^{\mathrm{i}\check{k}_z(\kappa)\Delta z} = \mathrm{e}^{\mathrm{i}k_z\Delta z}\,\mathrm{e}^{-k_z'\Delta z} \tag{2.26}$$

  with $k_z(\kappa)$ and $k_z'(\kappa)$ from Eq.(1.102).

- The first term is a phase factor and the second term is an amplitude factor which

[3]The superscript I is used to identify this integral kernel as of first kind. In context of the discussion of the Rayleigh integral, which can be derived from the SPW integral, this is of concern.

acts as a low-pass filter in the k-domain. From Eq.(1.102) we conclude

$$|\hat{G}^{\mathrm{I}}(\kappa)| = \mathrm{e}^{-k_z' \Delta z} = \exp[-k_0\,\Delta z\,\Im\sqrt{\check{n}^2 - f^2(\kappa)}]\,. \tag{2.27}$$

- We know, that $|\hat{G}(\kappa)| \to 0$ for the short distances $\Delta z > z_d$ if $\kappa > k_0 n$. But for ultrashort distances the $|\hat{G}(\kappa)|$-filter effect may allow passing of quite high frequencies f.

- However, in most applications $\Delta z \gg z_d$ and the filter effect comes asymptotically like a circ-function truncation in k-domain, i.e. $|\hat{G}(f)| = \mathrm{circ}(f/n)$. Moreover, most fields in practice do not include frequencies f even close to n but much smaller ones only and the $\hat{G}^{\mathrm{I}}$ constitutes a pure phase filter function only. Then the second factor in Eq.(2.26) has no effect.

- Fig.2.8 and Fig.2.9 illustrate the amplitude filtering effect and its strong dependency on the propagation distance. Of course again $f = n$ constitute the essential threshold for the profile of the f-filter.

- In Fig.2.10 four examples of the amplitude $|\hat{G}^{\mathrm{I}}(f)|$ are shown, which were calculated by VirtualLab Fusion.

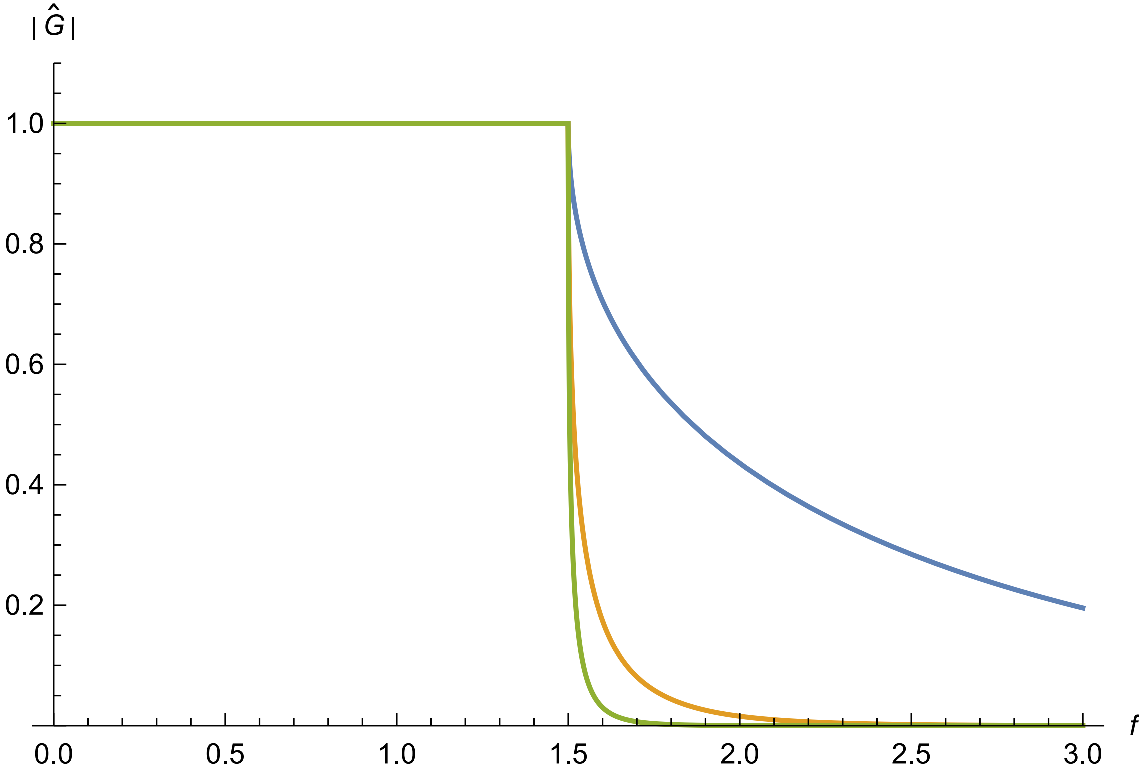


Figure 2.8: The amplitude $|\hat{G}(f)|$ for $n = 1.5$, $n' = 0$, and $\Delta z/\lambda_0 = 0.1, 0.5, 1$ (from blue to green line).

2.4.2 Other propagation operators

- The SPW operator solves the propagation problem rigorously, that means without physical approximations. However, in practice it is often not practical, because of too heavy numerical effort.

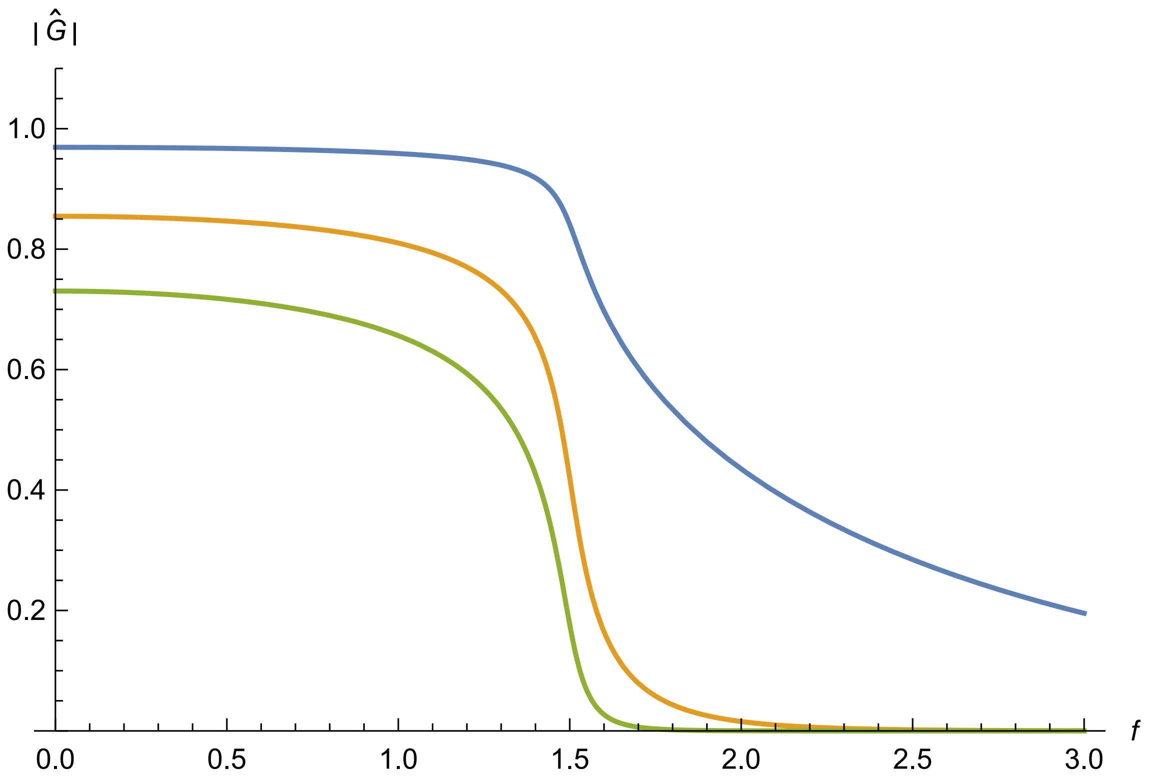


Figure 2.9: The amplitude $|\hat{G}(f)|$ for $n = 1.5$, $n' = 0.05$, and $\Delta z/\lambda_0 = 0.1, 0.5, 1$ (from blue to green line).

- Thus, other operators are derived from SPW operator, in particular the paraxial Fresnel operator and the far field operator.

- VirtualLab selects automatically the one with sufficient accuracy and minimum numerical effort.

- Research is still going on in that field! Efficient and accurate free-space propagation is still an R&D topic!
- See 2016-01-31_FW_RaD_Freespace_Propagation_Automatic_Case.Studies.lpd simulations.

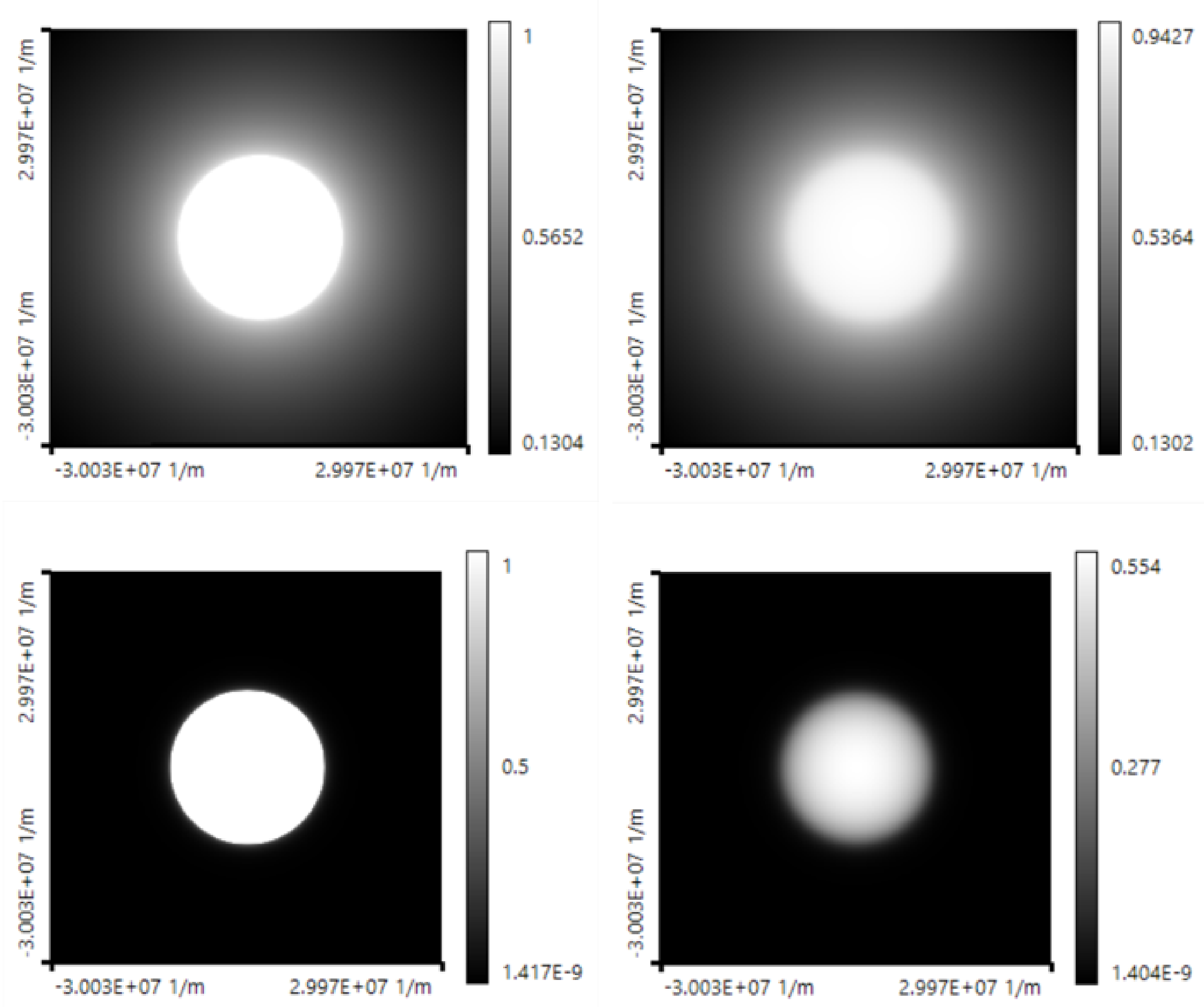


Figure 2.10: Examples of $|\hat{G}(f)|$ for $n = 1.0$, $n' = 0, 0.1$ (left, right column) and $\Delta z/\lambda_0 = 0.05, 0.5$ (upper and lower row

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