Name: ______________________ Date of birth: ______________ Student ID No.: ______________

Friedrich Schiller University Jena
Institute of Applied Physics
Prof. Dr. Thomas Pertsch

WS 2022/2023

Midterm Exam

Fundamentals of Modern Optics

to be written on December 12, 2022, 8:15 am – 9:45 am

Problem 1: Maxwell's Equations — 4 + 3 + 3 = 10 points

a) Write down the macroscopic Maxwell's equations in the time domain and in the frequency domain for a linear, isotropic, dispersive, non-magnetizable, and inhomogeneous dielectric medium without any free charges, i.e., current density $\mathbf{j}(\mathbf{r},t) = 0$ and charge density $\rho(\mathbf{r},t) = 0$.

b) Write down the relation between $\mathbf{D}$ and $\mathbf{E}$ in the time and frequency domain in this medium in terms of the response function $R(\mathbf{r},t)$ and the dielectric function $\varepsilon(\mathbf{r},\omega)$. Write down the conversion between the response function and the dielectric function in both directions.

c) For the above case, derive the wave equation for the field $\overline{\mathbf{H}}(\mathbf{r},\omega)$.

Problem 2: Poynting Vector — 2 + 3 + 3 = 8 points

a) Write down the formula of the time-averaged Poynting vector $\langle \mathbf{S}(\mathbf{r},t)\rangle$ for a monochromatic complex electromagnetic field. Write down the relation between the real-valued electric field $\mathbf{E}_r(\mathbf{r},t)$ and the corresponding complex form $\mathbf{E}_c(\mathbf{r})$.

Now, consider a monochromatic plane wave of frequency ω, propagating in a homogeneous, isotropic, and lossy dielectric medium. Its electric field has the form $\mathbf{E}_r(\mathbf{r},t) = E_0\, \mathbf{e}_x\, e^{-k''z} \cos(k'z - \omega t)$ where the wave vector is $k = k' + ik''$.

b) Find the corresponding real-valued magnetic field $\mathbf{H}_r(\mathbf{r},t)$.

c) Find the time-averaged Poynting vector $\langle \mathbf{S}(\mathbf{r},t)\rangle$.

Problem 3: Normal modes in a medium — 2 + 2 + 3 + 2 = 9 points

Consider the complex representation of a plane wave: $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \exp\left[i(\mathbf{k}\cdot\mathbf{r} - \omega t)\right]$, where we can split the wavevector into real and imaginary parts as $\mathbf{k} = \mathbf{k}' + i\mathbf{k}''$. The media is described by the complex dielectric function $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$.

a) Describe how homogeneous, inhomogeneous, evanescent waves are defined in terms of $\mathbf{k}'$ and $\mathbf{k}''$.

b) If the wavevector is $\mathbf{k} = a\mathbf{e}_x + ib\mathbf{e}_z$, where $a, b \in \mathbb{R}$ and $b > 0$, to which case from a) does the wave belong to? Derive the expressions that define the planes of constant amplitude and constant phase for this particular wave.

c) If $\mathbf{k}'$ and $\mathbf{k}''$ are almost parallel, we can introduce the complex refractive index $\hat{n}(\omega) = n(\omega) + i\kappa(\omega)$, where $\mathbf{k}^2 = \frac{\omega^2}{c^2}\hat{n}^2$. Find $\varepsilon'(\omega)$ and $\varepsilon''(\omega)$ as the functions of n and κ.

d) The reflectivity of a metal surface (the ratio of the reflected intensity of light to the incoming light from air) under normal incidence reads as

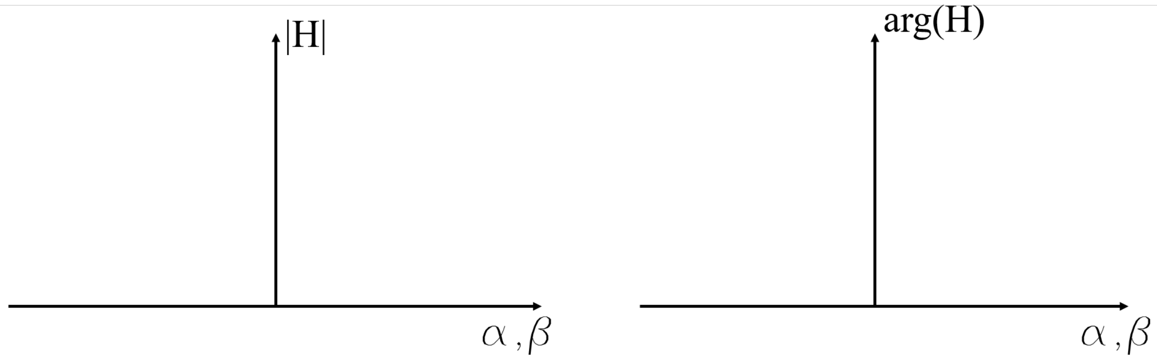
$$\rho = \left|\frac{n + i\kappa - 1}{n + i\kappa + 1}\right|^2 .$$

Find ρ for the case of a lossless metal.

Problem 4: Diffraction and Fresnel approximation **4 + 2 + 3 = 9 points**

a) State the general transfer function of the diffraction theory. What are the necessary conditions for the Fresnel approximation? Further, apply the approximation to derive the transfer function under Fresnel approximation in the spatial frequency domain. *Hint: Use Taylor expansion.*

b) Sketch the amplitude and argument of the general transfer function for a given propagation distance z on the below graphs. And then, sketch the argument of H for the transfer function under Fresnel approximation for the whole range of α and β and confirm the validity of the approximation conditions.



c) Consider a monochromatic light beam of $\lambda = 1$ μm which contains feature sizes of 50 μm and 20 nm. Which of the two features can be described after the propagation under the Fresnel approximation condition? Give reasoning for your answer. Based on this, comment on the physics of evanescent waves in Fresnel approximation.

Problem 5: Gaussian Beams **2 + 1 + 1 + 3 = 7 points**

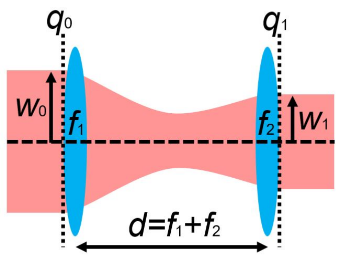
Consider a Gaussian beam propagating in z direction with the beam waist w_0, beam width $W(z)$, the Rayleigh length z_0, and phase curvature $R(z)$:

a) Sketch $W(z)$, the beam width as a function of z, make sure to include the focal depth, the waist w_0, the Rayleigh length z_0 and its corresponding value $W(z = z_0)$.

b) How do $W(z)$ and $R(z)$ evolve in the limit of large z? How does the Rayleigh length z_0 change if a longer wavelength is used?

c) Write the general expression of the q-parameter, which describes the Gaussian beam propagation by its physical parameters.

A collimated Gaussian beam of width w_0 propagates first through a thin lens with a focal distance f_1 and then through a thin lens with a focal distance f_2. The two lenses are separated by a distance $d = f_1 + f_2$ (see figure).



d) Calculate the ABCD matrix of the whole system between q_0 and q_1. The general form of the ABCD matrix of a thin lens is:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}.$$

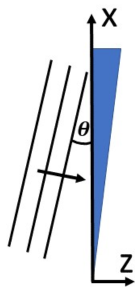
Problem 6: Beam Propagation — 3 + 2 + 2 + 1 = 8 points

Consider the propagation of a monochromatic, scalar field $u(x,y,z)$ at a wavelength λ along the z-direction starting from a given initial field distribution $u(x,y,z=0) = u_0(x,y)$.

a) Describe the steps necessary to calculate the field distribution $u(x,y,z)$ for any $z > 0$ by using the free space transfer function. Name and define all functions and quantities that you use and give the necessary formulas for this calculation.

Now, we consider a 2 dimensional space for the sake of simplicity. As shown in the figure, an incoming plane wave described by its field distribution $u(x,z)$

$$u(x, z \le 0) = Ae^{i(-xk_0 \sin\theta + zk_0 \cos\theta)},$$



is incident on a thin wedge of a transparent optical material. The effect of the wedge can be mathematically described by a phase mask function $t(x) = e^{i\sigma x}$, such that the field just after the wedge is given by $u(x,z)t(x)$.

b) Calculate the Fourier transform $U_0(\alpha)$ of the field right after the optical wedge.

c) Using $U_0(\alpha)$ calculate the field $u(x,z)$ for arbitrary $z > 0$.

d) What should be the value of σ so that the outgoing wavefronts become parallel to the x-axis?

Problem 7: Pulse Propagation — 2 + 3 + 3 = 8 points

a) Describe the physical meanings of the phase velocity, the group velocity, and the group velocity dispersion in one sentence each. Write down the defining formulas as functions of the wave number in free space k_0, the angular frequency ω, and the frequency dependent refractive index of the medium $n(\omega)$.

b) There are striking similarities between the free-space propagated total field equations that describe the paraxial beam diffraction and the pulse dispersion after propagation of z–distance. Fill in the blank spaces (marked with underlines) in the following table by drawing analogies between the beam diffraction and the pulse dispersion.

Beam diffraction	**Pulse dispersion**
______	τ
$(\alpha,\ \beta)$	______
$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$	______
Width: W_0 at $z=0$, $W(z)$	______
______	D
Diffraction length: $L_B = 2z_0 = \frac{2\pi}{\lambda} W_0^2$	Dispersion length: $L_D = 2\lvert z_0\rvert =$ ______
broadening in ______ domain	broadening in **temporal** domain
Paraxial wave equation for a beam: $i\frac{\partial \tilde{v}(x,y,z)}{\partial z} + \frac{1}{2k_0}\Delta^2\,\tilde{v}(x,y,z) = 0$	Paraxial wave equation for a pulse: ____________

c) With the knowledge above in mind, write down the total field equation of an unchirped Gaussian pulse after a propagation of z–distance. Remember that when we propagated a Gaussian beam by a z–distance, we have obtained the total field as

$$v(x,y,z) = A_0 \frac{1}{\sqrt{1+\left(\frac{z}{z_0}\right)^2}} \exp\left(-\frac{x^2+y^2}{W_0^2\left(1+\left(\frac{z}{z_0}\right)^2\right)}\right) \exp\left(i\frac{k}{2}\frac{x^2+y^2}{z\left(1+\left(\frac{z_0}{z}\right)^2\right)}\right) \exp(i\varphi(z)).$$

Hint: The phase curvature of a beam is analogous to the chirp parameter $C(z) = -\frac{z}{z_0\left(1+\frac{z^2}{z_0^2}\right)}$ that also describes the phase curvature in time.