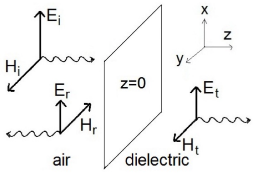
Normal Modes and Poynting Vector

A semi-infinite block of some dielectric (with relative permittivity of $\epsilon' + i\epsilon''$) is illuminated perpendicularly on its surface with a plane monochromatic wave of frequency ω from air. The electric field of the incoming and the reflected and the transmitted waves have the form:

$$\mathbf{E}_i = E_i e^{i(k_0 z)}\hat{\mathbf{x}} \quad , \quad \mathbf{E}_r = E_r e^{i(-k_0 z)}\hat{\mathbf{x}} \quad , \quad \mathbf{E}_t = E_t e^{i(k_1 z)}\hat{\mathbf{x}}$$

respectively, where $k_0 = \omega/c$ and $k_1 = \frac{\omega}{c}\sqrt{\epsilon' + i\epsilon''} = \frac{\omega}{c}(n + i\kappa)$.



a) Find out the three corresponding magnetic fields using the Maxwell's equations.

b) Use the continuity of the tangential components of the electric and magnetic field at the interface between the two media to find E_t as a function of E_i.

c) Calculate the time averaged Poynting vector in the dielectric medium (transmitted power).
Hint: This energy flux in the dielectric medium should be a function of z.
Hint: If you failed to find E_t as a function of E_i from part b, you can write the transmitted Poynting vector as function of E_t.

(a) $\nabla\times\vec{E}(\vec r,t) = -\frac{\partial B(\vec r,t)}{\partial t} \Rightarrow \nabla\times\vec{E}(z) = i\omega\mu_0\vec{H}(z) \Rightarrow \vec{H}(z)\ \frac{1}{i\omega\mu_0}\nabla\times\vec{E}(z)$

$\nabla\times\vec{E}(z) = \begin{vmatrix} \hat x & \hat y & \hat z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \hat x & \hat y & \hat z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & & \end{vmatrix} = \frac{\partial}{\partial z}E_x\hat y - \frac{\partial}{\partial y}E_x\hat z = \frac{\partial}{\partial z}E_x\hat y \Rightarrow H(z) = \frac{1}{i\omega\mu_0}\frac{\partial}{\partial z}E_x\hat y$

$E_x = \vec{E}_i = E_i e^{ik_0 z}\hat x \Rightarrow \vec{H}_i(z) = \frac{1}{i\omega\mu_0} ik_0 E_i e^{ik_0 z}\hat y = \frac{k_0}{\omega\mu_0}E_i e^{ik_0 z}\hat y$

$E_x = \vec{E}_r = E_r e^{i(-k_0 z)}\hat x \Rightarrow \vec{H}_r(z) = -\frac{k_0}{\omega\mu_0}E_r e^{-ik_0 z}\hat y;\quad E_x = \vec{E}_t = E_t e^{ik_1 z}\hat x \Rightarrow \vec{H}_t(z) = \frac{k_1}{\omega\mu_0}E_t e^{ik_1 z}\hat y$

(b) Continuity: $\hat n_{12}\times(\vec E_2 - \vec E_1) = 0 \quad (\vec D_2 - \vec D_1)\cdot\hat n_2 = \rho_s \quad (\vec B_2 - \vec B_1)\cdot\hat n_2 = 0 \quad \hat n_{12}\times(\vec H_2 - \vec H_1) = \vec j$

$\vec E_t\ \vec E_i \Rightarrow E_i e^{ik_0 z} = E_t e^{ik_1 z} \Rightarrow \underline{E_t = E_i e^{i(k_0 - k_1)z}}$

$\langle S(\vec r,t)\rangle = \frac{1}{2}Re[\vec E(\vec r)\times H(\vec r)]$

(c) Time averaged Poynting vector $\langle S(\vec r,t)\rangle = \frac{1}{2}Re[\vec E(z)\times H^*(z)]$

$\vec E_t(z)\times H_t^*(z) = \begin{vmatrix} \hat x & \hat y & \hat z \\ E_x & E_y & E_z \\ H_x^* & H_y^* & H_z^* \end{vmatrix} = \begin{vmatrix} \hat x & \hat y & \hat z \\ E_x & 0 & 0 \\ 0 & H_y^* & 0 \end{vmatrix} \Rightarrow E_x\cdot H_y^*\hat z = E_t e^{ik_1 z}\cdot\frac{k_1}{\omega\mu_0}E_t e^{-ik_1 z}\hat z$

$= \frac{k_1}{\omega\mu_0}E_t^2 = \frac{k_1 E_i^2}{\omega\mu_0}$

Consider a monochromatic plane wave of frequency ω, propagating in a homogeneous isotropic lossy dispersion-less dielectric medium of relative permittivity $\epsilon = \epsilon' + \epsilon''$ (where $\epsilon', \epsilon'' > 0$ and $\epsilon' >> \epsilon''$). Its electric field has the form $\mathbf{E}_r(\mathbf{r},t) = E_0\,\mathbf{e}_x\,e^{-k''z}\cos(k'z - \omega t + \phi)$, where the subscript r is used for the real valued fields.

a) Express k' and k'' (approximately) with respect to ω, ϵ', and ϵ''.

b) Find the real valued magnetic field $\mathbf{H}_r(\mathbf{r},t)$.

c) Write down the formula for the instantaneous Poynting vector $\mathbf{S}_r(\mathbf{r},t)$.

d) Find the time averaged Poynting vector using the formula $\langle\mathbf{S}_r(\mathbf{r},t)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\mathbf{S}_r(\mathbf{r},t)dt$. You also can directly use the formula for time averaged Poynting vector, which uses the complex amplitudes. Your answer should be as simplified as possible.

$E_r(\vec r,t) = \frac{1}{2}[\vec E(\vec r,t) + \vec E^*(\vec r,t)]$

$S(\vec r,t) = E_r(\vec r,t)\times H_r(\vec r,t)$

$\underline{Re[\vec E(\vec r,t)] = E_r(\vec r,t)}$

(a) $k^2 = \frac{\omega^2}{c^2}\varepsilon \Rightarrow (k' + ik'')^2 = \frac{\omega^2}{c^2}(\varepsilon' + i\varepsilon'') \Rightarrow k'^2 - k''^2 + 2ik'k'' = \frac{\omega^2}{c^2}(\varepsilon' + i\varepsilon'')$

$\Rightarrow k'^2 - k''^2 = \frac{\omega^2}{c^2}\varepsilon'$ and $2k'k'' = \frac{\omega^2}{c^2}\varepsilon''$

(b) $E(\vec r,t) = E_0 e^{i(kz - \omega t + \phi)}\vec e_x \qquad \nabla\times E(\vec r,t) = \begin{vmatrix} \vec e_x & \vec e_y & \vec e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \frac{\partial}{\partial z}E_x\vec e_y - \frac{\partial}{\partial y}E_x\vec e_z = \frac{\partial}{\partial z}E_x\vec e_y$

$\Rightarrow \nabla\times E(\vec r,t) = \frac{\partial}{\partial z}E_x\vec e_y = ikE_0 e^{i(kz - \omega t + \phi)}\vec e_y$

$\frac{\partial\vec H(\vec r,t)}{\partial t} = -i\omega\vec H(\vec r,t) \qquad \nabla\times\vec E(\vec r,t) = -\mu_0\frac{\partial\vec H(\vec r,t)}{\partial t} = i\omega\mu_0\vec H(\vec r,t) \Rightarrow \vec H(\vec r,t) = \frac{k}{\omega\mu_0}E_0 e^{i(kz - \omega t + \phi)}$

$\vec H_r(\vec r,t) = \frac{kE_0}{\omega\mu_0}\cos(kz - \omega t + \phi)\,\vec e_y$

$S(\vec r,t) = \vec E_r(\vec r,t)\times\vec H(\vec r,t) = \begin{vmatrix} \hat x & \hat y & \hat z \\ E_x & E_y & E_z \\ H_x & H_y & H_z \end{vmatrix} = \begin{vmatrix} \hat x & \hat y & \hat z \\ E_x & 0 & 0 \\ 0 & H_y & 0 \end{vmatrix} = E_x\cdot H_y\cdot\hat z$

$\Rightarrow S(\vec r,t) = \frac{kE_0^2}{\omega\mu_0}\cos^2(kz - \omega t + \phi)\hat z$

$$\langle S(\vec r,t)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} S(\vec r,t)\,dt$$

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$$\int_{-T}^{T} S(\vec r,t)\,dt = \frac{kE_0^2}{w\mu_0}\int_{-T}^{T}\cos^2(kz-wt+\phi)\,dt = \frac{kE_0^2}{2w\mu_0}\int_{-T}^{T}(\cos[2(kz-wt+\phi)]+1)\,dt$$

$$=\frac{kE_0^2}{2w\mu_0}\left(\int_{-T}^{T}\cos[2(kz-wt+\phi)]dt+\int_{-T}^{T}dt\right)=\frac{kE_0^2}{2w\mu_0}\left(-\frac{\sin[2(kz-wt+\phi)]}{2w}\Big|_{-T}^{T}+2T\right)$$

$$=\frac{kE_0^2}{2w\mu_0}\left\{\left[-\frac{1}{2w}\left(\sin2(kz-wT+\phi)-\sin2(kz+wT+\phi)\right)\right]+2T\right\}$$

$$=\frac{kE_0^2}{2w\mu_0}\left\{\left[2T-\frac{1}{w}\sin(2kz+2\phi)\cos(2wT)\right]\right\}$$

$$\langle S(\vec r,t)\rangle=\frac{kE_0^2}{2w\mu_0}\lim_{T\to\infty}\left[1-\frac{\sin(2kz+2\phi)\cos(2wT)}{2wT}\right]=\frac{kE_0^2}{2w\mu_0}$$

A plane electromagnetic wave has $\mathbf{B}$ given by

$$\mathbf{B}(x,y,z,t)=B_0 sin\left[(x+y)\frac{k}{\sqrt2}-\omega t\right]\hat{z}$$

where k is the wave number and $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the cartesian unit vectors in x, y and z directions respectively.

a) At a given location, how many times $\mathbf{B}(x,y,z,t)$ become zero in one second for $\omega=10^{14} rad/sec$?

$1\ rad=\left(\frac{180}{\pi}\right)^\circ$

$w=2\pi\nu$

(a) Periods $T=\frac{wt}{2\pi}$ Times B is at zero: $n=2T=\frac{wt}{\pi}=\frac{10^{14}}{3.14}\approx 31847133757961$ times

Consider a transverse monochromatic plane wave of frequency ω, propagating in a homogeneous isotropic medium, that is an extremely good conductor with conductivity $\sigma >> \omega\varepsilon_0$. The complex representation of the electric field has the form $\mathbf{E}(\mathbf{r},\omega)=E_0(-\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}})e^{i\beta(1+i)(x+y)}$, where β is a real number and $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors in x and y-direction.

a) Specify the wave-vector $\boldsymbol{k}$ for this plane wave (with its complex amplitude and direction). What is the dispersion relation that this wave-vector satisfies? Find σ as a function of β from this dispersion relation. (You can still fully solve part b and c if you do not manage to solve part a.)

b) Find the magnetic field $\mathbf{H}(\mathbf{r},\omega)$.

c) Write down the formula for the time-averaged Poynting vector $\langle\mathbf{S}(\mathbf{r},\mathbf{t})\rangle$, based on the complex representations of the electric and magnetic fields. Find $\langle\mathbf{S}(\mathbf{r},\mathbf{t})\rangle$ for the field given above.

d) Find the divergence of the time-averaged Poynting vector $\nabla\cdot\langle\mathbf{S}(\mathbf{r},\mathbf{t})\rangle$ and express it as a function of only the absolute value of the electric field $|\mathbf{E}(\mathbf{r},\omega)|$ and conductivity σ.

$\varepsilon(w)=\varepsilon'+i\varepsilon''$

$=1+\chi(w)+\frac{i}{w\varepsilon_0}\sigma(w)$

general complex dielectric function

(a) $\Delta\vec E(\vec r,w)=-k^2$ $\Delta\vec E=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_0(-\hat x+\hat y)e^{i\beta(1+i)(x+y)}=-\beta^2(1+i)^2\cdot\vec E(\vec r,w)$

$\Rightarrow k^2=\beta^2(1+i)^2 \Rightarrow k=\beta(1+i)=\beta+i\beta$

$k^2=\frac{w^2}{c^2}\varepsilon \Rightarrow (\beta+i\beta)^2=\frac{w^2}{c^2}(\varepsilon'+i\varepsilon'') \Rightarrow 2i\beta^2=\frac{w^2}{c^2}\varepsilon'+i\frac{w^2}{c^2}\varepsilon'' \Rightarrow \varepsilon''=\frac{2c^2\beta^2}{w^2}$

$\nabla\times\vec H(\vec r,w)=\vec j(\vec r,w)-iw\vec D(\vec r,t)$ $\quad\nabla\times\vec E(\vec r,w)=iw\vec B(\vec r,w)=iw\mu_0\vec H(\vec r,w)$

$\nabla\times\nabla\times\vec E=iw\mu_0\nabla\times\vec H(\vec r,w)=iw\mu_0\vec j(\vec r,w)+w^2\mu_0\vec D(\vec r,w)$ $\quad\vec D=\varepsilon_0\vec E(\vec r,w)+\varepsilon_0\chi(w)\vec E$ $\quad\vec j=\sigma\vec E(\vec r,w)$

$\nabla(\nabla\cdot\vec E)-\Delta\vec E=iw\mu_0\sigma\vec E+w^2\mu_0\varepsilon_0\vec E+w^2\mu_0\varepsilon_0\chi(w)\vec E$

$k^2=\frac{w^2}{c^2}[1+\chi(w)]-iw\mu_0\sigma(w)=\frac{w^2}{c^2}\left[1+\chi(w)+\frac{i}{w\varepsilon_0}\sigma(w)\right]=\frac{w^2}{c^2}(\varepsilon'+i\varepsilon'')$

$\Rightarrow\frac{2c^2\beta^2}{w^2}=\frac{\sigma(w)}{w\varepsilon_0}\Rightarrow\sigma(w)=\frac{2c^2\beta^2\varepsilon_0}{w}$

$m\ddot s=qE-g\dot s m-mw_0^2 s$

(travel)

v of e^-

$\frac{\partial^2}{\partial t^2}S(\vec r,t)+g\frac{\partial}{\partial t}S(\vec r,t)+w_0^2S(\vec r,t)=\frac{q\vec E(\vec r,t)}{m}$ (oscillate)

$\frac{\partial^2}{\partial t^2}S(\vec r,t)+g\frac{\partial}{\partial t}S(\vec r,t)=\frac{q\vec E(\vec r,t)}{m}$ $\quad j=Nq\frac{\partial}{\partial t}S(\vec r,t)$

$\vec P=Nqs(\vec r,t)\Rightarrow\frac{\partial^2}{\partial t^2}\vec P+g\frac{\partial}{\partial t}\vec P+w_0^2\vec P=\frac{Nq^2}{m}\vec E(\vec r,t)$

$\frac{\partial}{\partial t}\vec j+g\vec j=\frac{Nq^2\vec E}{m}\Rightarrow -iw\sigma\vec E+g\sigma\vec E=\frac{Nq^2}{m}\vec E$

$-w^2\vec P-iwg\vec P+w_0^2\vec P=\frac{Nq^2}{m}\vec E(\vec r,t)$

$\Rightarrow\sigma=\frac{Nq^2}{m}\frac{1}{g-iw}$ $\quad w_p^2=f=\frac{Nq^2}{\varepsilon_0 m}$

$\frac{\vec P}{\vec E}=\frac{Nq^2}{m}\frac{1}{w_0^2-w^2-iwg}$

$\Rightarrow\sigma=\frac{\varepsilon_0 w_p^2}{g-iw}=\frac{-iww_p^2\varepsilon_0}{-w^2-iwg}$

$\chi(w)=\frac{Nq^2}{\varepsilon_0 m}\frac{1}{w_0^2-w^2-iwg}$ $\quad f=\frac{Nq^2}{\varepsilon_0 m}$

(b) $\vec{E}(\vec{r},\omega) = -E_0 \exp[i\beta(1+i)(x+y)]\hat{x} + E_0 \exp[i\beta(1+i)(x+y)]\hat{y}$

$$\nabla\times\vec{E}(\vec{r},\omega) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & 0 \end{vmatrix} = -\frac{\partial}{\partial z}E_y\hat{x} + \frac{\partial}{\partial z}E_x\hat{y} + (\partial/\partial x E_y - \partial/\partial y E_x)\hat{z} = \left(\frac{\partial}{\partial x}E_y - \frac{\partial}{\partial y}E_x\right)\hat{z}$$

$$\Rightarrow \nabla\times\vec{E}(\vec{r},\omega) = iE_0\beta(1+i)\exp[i\beta(1+i)(x+y)]\hat{z} + iE_0\beta(1+i)\exp[i\beta(1+i)(x+y)]\hat{z}$$

$$= 2iE_0\beta(1+i)\exp[i\beta(1+i)(x+y)]\hat{z}$$

$$\nabla\times\vec{E}(\vec{r},\omega) = i\omega\mu_0\vec{H}(\vec{r},\omega) \Rightarrow \vec{H}(\vec{r},\omega) = \frac{2E_0\beta(1+i)}{\omega\mu_0}\exp[i\beta(1+i)(x+y)]\hat{z}$$

$$\langle\vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}[\vec{E}(\vec{r})\times H^*(\vec{r})]$$

$$\vec{E}(\vec{r})\times\vec{H}^*(\vec{r}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_x & E_y & 0 \\ 0 & 0 & H_z^* \end{vmatrix} = E_y H_z^*\hat{x} - E_x H_z^*\hat{y} = \frac{2E_0^2\beta(1+i)}{\omega\mu_0}\hat{x} + \frac{2E_0^2\beta(1+i)}{\omega\mu_0}\hat{y}$$

$$\Rightarrow \langle\vec{S}(\vec{r},t)\rangle = \frac{E_0^2\beta}{\omega\mu_0}\hat{x} + \frac{E_0^2\beta}{\omega\mu_0}\hat{y}$$

$i\omega\varepsilon_0\omega_p^2(\omega^2 - ig\omega)$

$g\omega^2\varepsilon_0\omega_p^2$

For plasma and metal

$\varepsilon = \varepsilon'(\omega)$ $\quad \varepsilon = \frac{-i\omega\varepsilon_0\omega_p^2}{-\omega^2 - i\omega g} = \frac{}{\omega^4 + \omega^2 g^2}$

(c) $\underline{\nabla\cdot\langle\vec{S}(\vec{r},t)\rangle = -\frac{1}{2}\omega_0\varepsilon_0\varepsilon''(\omega)\vec{E}(\vec{r})\vec{E}^*(\vec{r})}$

$$= -\frac{1}{2}\omega_0\varepsilon_0\varepsilon''(\omega)E_0^2 = -\frac{1}{2}\omega_0\varepsilon_0\frac{\sigma}{\omega\varepsilon_0}E_0^2 = -\frac{1}{2}\sigma|\vec{E}(\vec{r},\omega)|^2$$

A plane wave of frequency ω, in a homogeneous, isotropic, lossy dielectric medium which is given by

$$\boldsymbol{E}(\boldsymbol{r}) = E_0\cos[(\beta+i\alpha)x]\,e^{(i\beta-\alpha)z}\,\hat{\boldsymbol{y}} \qquad (*)$$

a) Calculate the corresponding magnetic field $\boldsymbol{H}$ from Maxwell's equations.

b) Consider a low loss dielectric ($\varepsilon'' \ll \varepsilon'$) and find an expression for both β and α in terms of ε' and ε'' such that Equ. (*) fulfills the wave equation.
Hint: In your calculation you should only use the first three terms of Taylor expansion.

c) Now neglect the losses in the medium.
Write down the electric field for the lossless case and determine the direction of the flow of the optical power by calculating the time averaged Poynting vector.

$$\nabla\times\vec{E}(r) = i\omega\mu_0\vec{H}(\vec{r}) \qquad \nabla\times\vec{E}(r) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & E_y & 0 \end{vmatrix} = -\frac{\partial}{\partial z}E_y\hat{x} + \frac{\partial}{\partial x}E_y\hat{z}$$

$$\Rightarrow E(r) = \frac{E_0}{2}\left[e^{i(\beta+i\alpha)x} + e^{-i(\beta+i\alpha)x}\right]e^{(i\beta-\alpha)z}\hat{y} = \frac{E_0}{2}\left[e^{i(\beta+i\alpha)(x+z)} + e^{-i(\beta+i\alpha)(x-z)}\right]\hat{y}$$

$$\Rightarrow \nabla\times\vec{E}(\vec{r}) = -\frac{E_0}{2}i(\beta+i\alpha)\left[e^{i(\beta+i\alpha)(x+z)} + e^{-i(\beta+i\alpha)(x-z)}\right]\hat{x} + \frac{E_0}{2}i(\beta+i\alpha)\left[e^{i(\beta+i\alpha)(x+z)} - e^{-i(\beta+i\alpha)(x-z)}\right]\hat{z}$$

$$\Rightarrow \vec{H}(\vec{r}) = -\frac{E_0(\beta+i\alpha)}{2\omega\mu_0}\left[e^{i(\beta+i\alpha)(x+z)} + e^{-i(\beta+i\alpha)(x-z)}\right]\hat{x} + \frac{E_0(\beta+i\alpha)}{2\omega\mu_0}\left[e^{i(\beta+i\alpha)(x+z)} - e^{-i(\beta+i\alpha)(x-z)}\right]\hat{z}$$

$$-k^2 = [i(\beta+i\alpha)]^2 = -2(\beta+i\alpha)^2 \Rightarrow k = (\beta+i\alpha)\sqrt{2} \quad k^2 = 2(\beta^2-\alpha^2+2i\alpha\beta) = \frac{\omega^2}{c^2}(\varepsilon'+i\varepsilon'') \Rightarrow \begin{cases} 2(\beta^2-\alpha^2) = \frac{\omega^2}{c^2}\varepsilon' \\ 4\alpha\beta = \frac{\omega^2}{c^2}\varepsilon'' \end{cases}$$

$$\Rightarrow 2\beta^2 - 2\left(\frac{\omega^2\varepsilon''}{4c^2\beta}\right)^2 = \frac{\omega^2}{c^2}\varepsilon' \Rightarrow \beta^4 - \frac{\omega^2\varepsilon'}{2c^2}\beta^2 - \frac{\omega^4\varepsilon''^2}{16c^4} = 0 \Rightarrow \beta^2 = \frac{\frac{\omega^2\varepsilon'}{2c^2} \pm \sqrt{\frac{\omega^4\varepsilon'^2}{4c^4} + \frac{\omega^4\varepsilon''^2}{4c^4}}}{2} \Rightarrow \beta = \frac{\omega}{2c}\sqrt{\varepsilon' + \sqrt{\varepsilon'^2+\varepsilon''^2}}$$

$$\Rightarrow 2\left(\frac{\omega^2\varepsilon''}{4c^2\alpha}\right)^2 - 2\alpha^2 = \frac{\omega^2}{c^2}\varepsilon' \Rightarrow \alpha^4 + \frac{\omega^2\varepsilon'}{2c^2}\alpha^2 - \frac{\omega^4\varepsilon''^2}{16c^4} \Rightarrow \alpha^2 = \frac{1}{2}\left(-\frac{\omega^2\varepsilon'}{2c^2} \pm \sqrt{\frac{\omega^4\varepsilon'^2}{4c^4} + \frac{\omega^4\varepsilon''^2}{4c^4}}\right) \Rightarrow \alpha = \frac{\omega}{2c}\left(\sqrt{\sqrt{\varepsilon'^2+\varepsilon''^2} - \varepsilon'}\right)$$

$$\underline{\beta = \frac{\omega}{2c}\left[\varepsilon' + \varepsilon'\left(1+\frac{\varepsilon''^2}{2\varepsilon'^2}\right)\right]^{\frac{1}{2}} = \frac{\omega}{2c}\left[2\varepsilon' + \frac{\varepsilon''^2}{2\varepsilon'}\right]^{\frac{1}{2}} \approx \frac{\omega}{\sqrt{2}c}\sqrt{\varepsilon'}}$$

$$\underline{\alpha \approx \frac{\omega}{2c}\left[\varepsilon'\left(1+\frac{\varepsilon''^2}{2\varepsilon'^2}\right) - \varepsilon'\right]^{\frac{1}{2}} = \frac{\sqrt{2}\omega\varepsilon''}{4c\sqrt{\varepsilon'}}}$$

(c) lossless $\to \alpha = 0 \Rightarrow \vec{E}(\vec{r}) = \frac{E_0}{2}[e^{i\beta(x+z)} + e^{-i\beta(x-z)}]\hat{y}$

$$\vec{H}(\vec{r}) = -\frac{E_0\beta}{2\omega\mu_0}[e^{i\beta(x+z)} + e^{-i\beta(x-z)}]\hat{x} + \frac{E_0\beta}{2\omega\mu_0}[e^{i\beta(x+z)} - e^{-i\beta(x-z)}]\hat{z}$$

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}[\vec{E}(\vec{r}) \times \vec{H}^*(\vec{r})]$$

$$\vec{E}(\vec{r}) \times H^*(\vec{r}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & E_y & 0 \\ H_x^* & 0 & H_z^* \end{vmatrix} = E_y H_z^* \hat{y} - E_y H_x^* \hat{z}$$

$$E_y \cdot H_z^* = \frac{E_0^2\beta}{4\omega\mu_0}[e^{i\beta(x+z)} + e^{-i\beta(x-z)}][e^{-i\beta(x+z)} - e^{i\beta(x-z)}] = e^{-i2\beta x} - e^{i2\beta x} = -\frac{iE_0^2\beta}{8\omega\mu_0}\sin 2\beta x\,\hat{y}$$

$$-E_y \cdot H_x^* = \frac{E_0^2\beta}{4\omega\mu_0}[e^{i\beta(x+z)} + e^{-i\beta(x-z)}][e^{-i\beta(x+z)} + e^{i\beta(x-z)}] = \frac{E_0^2\beta}{4\omega\mu_0}[e^{i2\beta x} + e^{-i2\beta x}) = \frac{E_0^2\beta}{8\omega\mu_0}\cos(2\beta x)\,\hat{z}$$

$$\frac{1}{2}\mathrm{Re}[E(\vec{r}) \times H^*(\vec{r})] = \frac{E_0^2\beta}{16\omega\mu_0}\cos 2\beta x\,\hat{z}$$

a) What is the general connection between the Poynting vector $\mathbf{S}(\mathbf{r},t)$ and the optical intensity $I(\mathbf{r})$?

$$I(\vec{r}) = |\langle \vec{S}(\vec{r},t)\rangle|$$

c) Describe the general relation between medium properties and $\nabla\cdot\langle\mathbf{S}(\mathbf{r},t)\rangle$ in three cases: $\nabla\cdot\langle\mathbf{S}(\mathbf{r},t)\rangle$ is positive, negative, and zero.

$\nabla \cdot \langle \vec{S}(\vec{r},t)\rangle = -\frac{1}{2}\varepsilon_0\omega_0\varepsilon''(\omega)E(\vec{r})E^*(\vec{r})$ Vacuum energy density $u = \frac{1}{2}\varepsilon_0 E r^2 + \frac{1}{2}\mu_0 H r^2$

$\nabla \cdot \langle \vec{S}(\vec{r},t)\rangle > 0 \Rightarrow \varepsilon''(\omega) < 0$, energy is added to the field

$\nabla \cdot \langle \vec{S}(\vec{r},t)\rangle = 0 \Rightarrow \varepsilon''(\omega) = 0$ energy is conserved

$\nabla \cdot \langle \vec{S}(\vec{r},t)\rangle < 0 \Rightarrow \varepsilon''(\omega) >$ Energy of the field is absorbed.

With a good approximation, a medium can be modeled by an ensemble of damped harmonic oscillators, known as the Lorentz model. The response function of this medium is given as:

$$\hat{R}_{mn}(\mathbf{r},t) = \delta_{mn}R(t) \qquad R(t) = \begin{cases} 0 & \text{for } t \le 0 \\ \frac{f}{\Omega}e^{-\gamma t}\sin\Omega t & \text{for } t > 0 \end{cases}, \qquad \Omega = \sqrt{\omega_0^2 - \gamma^2}.$$

a) Based on the given response function, specify the type of the medium by ticking in the table below.

Inhomogeneous	Homogeneous
Anisotropic	Isotropic
Dispersive	Non-dispersive

b) Write down the relation between the polarization $\mathbf{P}(\mathbf{r},t)$, the response function $R(t)$, and the electric field $\mathbf{E}(\mathbf{r},t)$.

c) Calculate the susceptibility $\chi(\omega)$ of the medium.

d) Compute the polarization $\mathbf{P}(\mathbf{r},t)$ for the above medium with an electric field excitation of

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r})\exp(-i\omega_{cw}t).$$

Explain how the complex susceptibility influences on the relation between the polarization $\mathbf{P}(\mathbf{r},t)$ and the electric field $\mathbf{E}(\mathbf{r},t)$. What happens if the damping factor $\gamma = 0$?

e) Compute the polarization $\mathbf{P}(\mathbf{r},t)$ for the above medium with an electric field excitation of

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r})\delta(t - t_0).$$

and explain the result.

(a) Homogeneous, Isotropic, Dispersive

(b) $\vec{P}(\vec{r},t) = \varepsilon_0\int_{-\infty}^{t} R(t-t')E(t')\,dt'$

(c) $\chi(\omega) = \int_{-\infty}^{\infty} R(t)e^{i\omega t}\,dt$

$$= \frac{f}{\Omega}\int_0^\infty e^{-\gamma t}\sin\Omega t\, e^{i\omega t}\,dt$$

$$= \frac{f}{2i\Omega}\int_0^\infty e^{-\gamma t}(e^{i\Omega t} - e^{-i\Omega t})e^{i\omega t}\,dt$$

$$\Rightarrow \chi(\omega) = \frac{f}{2i\Omega}\left[\int_0^\infty e^{(i\omega + i\Omega - \gamma)t}dt - \int_0^\infty e^{(i\omega - i\Omega - \gamma)t}dt\right] = \frac{f}{2i\Omega}\left(\frac{e^{(i\omega+i\Omega)t}e^{-\gamma t}\big|_0^\infty}{i\omega + i\Omega - \gamma} - \frac{e^{i(\omega-\Omega)t}e^{-\gamma t}\big|_0^\infty}{i\omega - i\Omega - \gamma}\right)$$

$$= \frac{f}{2i\Omega}\left(\frac{-1}{i\omega - \gamma + i\Omega} + \frac{1}{(i\omega - \gamma) - i\Omega}\right) = \frac{f}{2i\Omega}\frac{2i\Omega}{(i\omega-\gamma)^2 + \Omega^2} = \frac{f}{\omega_0^2 - \omega^2 - 2i\omega\gamma}$$

(d) $E(r,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\vec{E}(r)e^{-i\omega_{cw}t}e^{i\omega t}\,dt = \frac{\vec{E}(\vec{r})}{2\pi}\int_{-\infty}^{\infty}e^{i(\omega - \omega_{cw})t}\,dt = \vec{E}(\vec{r})\delta(\omega - \omega_{cw})$

$$\vec{P}(\vec{r},\omega) = \varepsilon_0\chi(\omega)E(\vec{r},\omega) = \varepsilon_0\vec{E}(\vec{r})\chi(\omega)\delta(\omega - \omega_{cw})$$

$$\vec{P}(\vec{r},t) = \int_{-\infty}^{\infty}\varepsilon_0\vec{E}(\vec{r})\delta(\omega - \omega_{cw})\chi(\omega)e^{-i\omega t}\,d\omega = \varepsilon_0\vec{E}(\vec{r})\frac{f e^{-i\omega_{cw}t}}{\omega_0^2 - \omega_{cw}^2 - 2i\omega_{cw}\gamma}$$

when $\chi(\omega) < 0$ $\vec{P}(\vec{r},t)$ has opposite direction compared to $\vec{E}(\vec{r},t)$. $\chi(\omega) > 0$, $\vec{P}(\vec{r},t)$ has the same direction compared to $\vec{E}(\vec{r},t)$

if $\gamma=0$, No energy will be absorbed by medium.

(e) $\vec{P}(\vec{r},t)=\varepsilon_0\int_{-\infty}^{t}R(t-t')\vec{E}(t')dt' = \varepsilon_0\int_{-\infty}^{\infty}R(t')\vec{E}(t-t')dt' = \varepsilon_0\int_{-\infty}^{\infty}\frac{f}{\Omega}e^{-\gamma t'}\sin\Omega t'\,\vec{E}(\vec{r})\,\delta(t-t'-t_0)dt'$

$= \frac{\varepsilon_0 f\vec{E}(\vec{r})}{\Omega}\int_{-\infty}^{\infty}\delta[t'-(t-t_0)]\,e^{-\gamma t'}\sin\Omega t'dt' = \frac{\varepsilon_0 f}{\Omega}\vec{E}(\vec{r})\,e^{-\gamma(t-t_0)}\sin\Omega(t-t_0)$

$= \frac{Nq^2}{m\Omega}\vec{E}(\vec{r})\,e^{-\gamma(t-t_0)}\sin\Omega(t-t_0)$

With a good approximation, a medium can be modeled by an ensemble of damped harmonic oscillators, known as the Lorentz model. The response function of this medium is given as:

$$\hat{R}_{mn}(\mathbf{r},t)=\delta_{mn}R(t) \qquad R(t)=\begin{cases}0 & \text{for } t\le 0\\ \frac{f}{\Omega}e^{-\Gamma t}\sin\Omega t & \text{for } t>0\end{cases}, \qquad \Omega=\sqrt{\omega_0^2-\Gamma^2}.$$

a) Based on the given response function, specify the type of the medium by choosing one or more items from the following list: inhomogeneous, homogeneous, anisotropic, isotropic, dispersive, non-dispersive.
b) Calculate the susceptibility $\chi(\omega)$ and the relative permittivity $\varepsilon_r(\omega)$ of the medium. Find its real part $\varepsilon'(\omega)$ and its imaginary part $\varepsilon''(\omega)$.

Consider a superposition of two monochromatic plane waves at frequency ω propagating in this medium with the electric field expression of

$$\mathbf{E}(\mathbf{r})=E_0\cos[(g+ih)x]\,e^{(ig-h)z}\hat{\mathbf{y}} \quad ,$$

where g and h are real values with the unit 1/m.

c) Derive the two wave vectors $\mathbf{k}_1=\mathbf{k}_1'+i\mathbf{k}_1''$ and $\mathbf{k}_2=\mathbf{k}_2'+i\mathbf{k}_2''$ of the two plane waves from the electric field expression. Find the relations which connect g and h with the real and imaginary parts of $\varepsilon_r(\omega)$.

(a) Homogeneous, isotropic, dispersive

(b) $\chi(\omega)=\int_{-\infty}^{\infty}\frac{f}{\Omega}e^{-\Gamma t}\sin\Omega t\, e^{i\omega t}dt$

$= \frac{f}{2i\Omega}\int_0^{\infty}e^{-\Gamma t}(e^{i\Omega t}-e^{-i\Omega t})e^{i\omega t}dt$

$= \frac{f}{2i\Omega}\int_0^{\infty}e^{(i\omega+i\Omega-\Gamma)t}-e^{i\omega t-i\Omega t-\Gamma t}dt$

$= \frac{f}{2i\Omega}\left(\frac{-1}{(i\omega-\Gamma)+i\Omega}+\frac{1}{(i\omega-\Gamma)-i\Omega}\right)$

$= \frac{f}{\omega_0^2-\omega^2-2i\omega\Gamma}$

(b) $\varepsilon_r(\omega)=1+\chi(\omega)=1+\frac{f}{\omega_0^2-\omega^2-2i\omega\Gamma}$

$=1+\frac{f(\omega_0^2-\omega^2)+2if\omega\Gamma}{(\omega_0^2-\omega^2)^2+4\omega^2\Gamma}=1+\frac{f(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+4\omega^2\Gamma^2}+i\frac{2f\omega\Gamma}{(\omega_0^2-\omega^2)^2+4\omega^2\Gamma^2}$

$\Rightarrow \varepsilon'(\omega)=1+\frac{f(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+4\omega^2\Gamma^2} \qquad \varepsilon''(\omega)=\frac{2f\omega\Gamma}{(\omega_0^2-\omega^2)^2+4\omega^2\Gamma^2}$

(c) $\vec{E}(\vec{r})=\frac{E_0}{2}[e^{i(g+ih)x}+e^{-i(g+ih)x}]e^{(ig-h)z}=\frac{E_0}{2}e^{i(g+ih)(x+z)}+\frac{E_0}{2}e^{i(g+ih)(z-x)}$

~~$k_1=k_1'+ik_1''=g+ih \qquad k_2=k_2'+ik_2''=g+ih$~~

$k_1^2=\frac{\omega^2}{c^2}\varepsilon_r(\omega)\Rightarrow 2(g^2-h^2+2igh)=\frac{\omega^2}{c^2}\varepsilon_r(\omega) \qquad k_2^2=2(g^2-h^2+2igh)=\frac{\omega^2}{c^2}\varepsilon_r(\omega)$

$k_{x_1}=g+ih \qquad k_{z_1}=g+ih \qquad k_1=\sqrt{k_{x_1}^2+k_{z_1}^2}=\sqrt{2}(g+ih)$

$k_{x_2}=-(g+ih) \qquad k_{z_2}=g+ih \qquad k_2=\sqrt{k_{x_2}^2+k_{z_2}^2}=\sqrt{2}(g+ih)$

Consider the complex representation of a plane wave: $\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$, where we can split the wavevector into real and imaginary parts as $\mathbf{k}=\mathbf{k}'+i\mathbf{k}''$. The media is described by the complex dielectric function $\varepsilon(\omega)=\varepsilon'(\omega)+i\varepsilon''(\omega)$.

a) Describe how homogeneous, inhomogeneous, evanescent waves are defined in terms of $\mathbf{k}'$ and $\mathbf{k}''$.
b) If the wavevector is $\mathbf{k}=a\mathbf{e}_x+ib\mathbf{e}_z$, where $a,b\in\mathbb{R}$ and $b>0$, to which case from a) does the wave belong to? Derive the expressions that define the planes of constant amplitude and constant phase for this particular wave.
c) If $\mathbf{k}'$ and $\mathbf{k}''$ are almost parallel, we can introduce the complex refractive index $\hat{n}(\omega)=n(\omega)+i\kappa(\omega)$, where $\mathbf{k}^2=\frac{\omega^2}{c^2}\hat{n}^2$. Find $\varepsilon'(\omega)$ and $\varepsilon''(\omega)$ as the functions of n and κ.
d) The reflectivity of a metal surface (the ratio of the reflected intensity of light to the incoming light from air) under normal incidence reads as

$$\rho=\left|\frac{n+i\kappa-1}{n+i\kappa+1}\right|^2.$$

Find ρ for the case of a lossless metal.

(a) Homogeneous: $(k'+ik'')^2=\frac{\omega^2}{c^2}\varepsilon(\omega) \quad (k'\parallel k'')$

$k'^2-k''^2+2ik'k''=\frac{\omega^2}{c^2}\varepsilon(\omega)$

~~Inhomogeneous: $(k'+ik'')^2=\frac{\omega^2}{c^2}\varepsilon(r,\omega)$~~

~~$k'^2-k''^2+2ik'k''=\frac{\omega^2}{c^2}\varepsilon(r,\omega)$~~

Evanescent $k'\perp k''$ $(k'+ik'')^2=k'^2-k''^2=\frac{\omega^2}{c^2}\varepsilon(\omega)$

Homogeneous: $k'\parallel k''$ Evanescent $k'\perp k''$

Inhomogeneous otherwise.

(b) Evanescent wave

$\vec{E}(\vec{r},t)=E_0\exp(-\vec{k}''\cdot\vec{r})\exp[i(\vec{k}'\cdot\vec{r}-\omega t)]$

constant amplitude $-\vec{k}''\cdot\vec{r}=$ constant $\qquad$ constant phase: $\vec{k}'\cdot\vec{r}-\omega t=$ constant

(c) $(k'+ik'')^2=\frac{\omega^2}{c^2}\varepsilon(\omega) \qquad k'+ik''=\frac{\omega}{c}[n(\omega)+i\kappa(\omega)]$

$(k'+ik'')^2=\frac{\omega^2}{c^2}[n(\omega)+i\kappa(\omega)]^2=\frac{\omega^2}{c^2}(\varepsilon'+i\varepsilon'')\Rightarrow\begin{cases}n^2(\omega)-\kappa^2(\omega)=\varepsilon'\\ 2n(\omega)\kappa(\omega)=\varepsilon''\end{cases}$

(d) lossless $\Rightarrow k''=0 \Rightarrow K(\omega)=0 \Rightarrow \rho = \left|\frac{n+iK-1}{n+iK+1}\right|^2 = \left(\frac{n-1}{n+1}\right)^2$

A plane electromagnetic wave in a homogeneous, linear, isotropic, and non-magnetic medium without external charges and currents is **H** given as

$$\mathbf{H}(x,y,z,t) = H_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} - \omega t\right]\hat{\mathbf{z}},$$

where k is the wave number and $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are the unit vectors in the Cartesian coordinate system.

a) Calculate the electric field $\mathbf{E}(x,y,z,t)$ corresponding to the above magnetic field.

b) Write down the formula for the time averaged Poynting vector $\langle \mathbf{S}(\mathbf{r},t)\rangle$.

c) Find the time averaged Poynting vector for this electromagnetic wave.

(a) $\vec{H}(x,y,z,t) = H_0 \cos\left[(x+y)\frac{k}{\sqrt{2}} - \omega t + \frac{\pi}{2}\right]\hat{z} = H_0 \exp\left[i(x+y)\frac{k}{\sqrt{2}}\right]e^{-i\omega t}e^{i\frac{\pi}{2}} = iH_0 \exp\left[i(x+y)\frac{k}{\sqrt{2}}\right]e^{-i\omega t}$

$\Rightarrow \vec{H}(x,y,z) = iH_0 \exp\left[i(x+y)\frac{k}{\sqrt{2}}\right]\hat{z}$ $\qquad \vec{D}(\vec{r},\omega) = \varepsilon_0\varepsilon(\omega)\vec{E}(\vec{r},\omega)$

$\nabla\times\vec{H}(x,y,z) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & H_z \end{vmatrix} = \frac{\partial}{\partial y}H_z\hat{x} - \frac{\partial}{\partial x}H_z\hat{y} = -\frac{kH_0}{\sqrt{2}}\exp\left[i(x+y)\frac{k}{\sqrt{2}}\right]\hat{x} + \frac{kH_0}{\sqrt{2}}\exp\left[i(x+y)\frac{k}{\sqrt{2}}\right]\hat{y}$

$\vec{D}(\vec{r},\omega) = \varepsilon_0\varepsilon(\omega)\vec{E}(\vec{r},\omega) \Rightarrow \vec{D}(\vec{r}) = \varepsilon_0\varepsilon(\omega)\vec{E}(\vec{r})$

$\nabla\times\vec{H}(\vec{r},\omega) = -i\omega\varepsilon_0\varepsilon(\omega)\vec{E}(\vec{r}) \Rightarrow \vec{E}(\vec{r}) = \frac{i}{\omega\varepsilon_0\varepsilon(\omega)}\nabla\times\vec{H}$

$\Rightarrow \vec{E}(x,y,z) = \frac{-ikH_0}{\sqrt{2}\omega\varepsilon_0\varepsilon(\omega)}\exp\left[i(x+y)\frac{k}{\sqrt{2}}\right]\hat{x} + \frac{ikH_0}{\sqrt{2}\omega\varepsilon_0\varepsilon(\omega)}\exp\left[i(x+y)\frac{k}{\sqrt{2}}\right]\hat{y}$

$\vec{E}(x,y,z,t) = \frac{-ikH_0}{\sqrt{2}\omega\varepsilon_0\varepsilon(\omega)}\exp\left[i(x+y)\frac{k}{\sqrt{2}} - i\omega t\right]\hat{x} + \frac{ikH_0}{\sqrt{2}\omega\varepsilon_0\varepsilon(\omega)}\exp\left[i(x+y)\frac{k}{\sqrt{2}} - i\omega t\right]\hat{y}$

(b) $\langle\vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}\left[\vec{E}(\vec{r})\times\vec{H}^*(\vec{r})\right]$

(c) $\vec{E}(x,y,z)\times\vec{H}^*(x,y,z) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_x & E_y & 0 \\ 0 & 0 & H_z^* \end{vmatrix} = E_y H_z^*\hat{x} - H_z^* E_x\hat{y}$

$E_y H_z^* = \frac{ikH_0}{\sqrt{2}\omega\varepsilon_0\varepsilon(\omega)}\exp\left[i(x+y)\frac{k}{\sqrt{2}}\right]\cdot(-i)H_0\exp\left[-i(x+y)\frac{k}{\sqrt{2}}\right] = \frac{kH_0^2}{\sqrt{2}\omega\varepsilon_0\varepsilon(\omega)}$

$H_z^* E_x = -iH_0\exp\left[-i(x+y)\frac{k}{\sqrt{2}}\right]\cdot\frac{-ikH_0}{\sqrt{2}\omega\varepsilon_0\varepsilon(\omega)}\exp\left[i(x+y)\frac{k}{\sqrt{2}}\right] = -\frac{H_0^2 k}{\sqrt{2}\omega\varepsilon_0\varepsilon(\omega)}$

$\langle\vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}\left[\vec{E}(x,y,z)\times\vec{H}^*(x,y,z)\right] = \frac{\sqrt{2}kH_0^2}{4\omega\varepsilon_0\varepsilon(\omega)}\hat{x} + \frac{\sqrt{2}H_0^2 k}{4\omega\varepsilon_0\varepsilon(\omega)}\hat{y}$