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WS 2022/2023

Series 1
Fundamentals of Modern Optics

to be returned on **20.10.2022**, at the beginning of the lecture

Initial Remarks

Please read these carefully!

- return the completed assignments by the above mentioned date.
- group work is allowed and encouraged; however each student has to hand in an individual assignment; literal copies will not be accepted.
- problems marked with an asterisk (*) are non-mandatory and can be used to gain extra points.
- assignments will be checked, returned, and discussed in the seminars in the week after the return dates.
- hand in your assignments in hand-writing only; write neatly.
- use a pen, not a pencil.
- note the name of the seminar tutor, whose seminar you are planning to attend, on the assignments you return.
- write down all calculations and derivations in a clear and concise manner.
- some *Hint* sections are there to suggest one way of solving a problem. Feel free to solve it in any other way you know.

Task 1: Fourier transformations (a=2, b=2+2*, c=2* pts.)

Given is the definition of the Fourier transformation and its inverse, which transforms the time-domain representation of a signal $f(t)$ into its frequency-domain representation $\tilde{f}(\omega)$ and vice versa:

$$\tilde{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\exp(\mathrm{i}\omega t)\,\mathrm{d}t$$
$$f(t) = \int_{-\infty}^{\infty} \tilde{f}(\omega)\exp(-\mathrm{i}\omega t)\,\mathrm{d}\omega.$$

Use these definitions to find the frequency-domain representation of the following signals

a) $f(t) = \begin{cases} 0 & t < 0 \\ A\exp[-\gamma t]\cos(\omega_0 t) & t \geq 0 \end{cases}$

b) $f(t) = A\exp[-\frac{1}{2}\frac{t^2}{t_0^2}]$
This problem involves a complex-valued Gaussian integral. You can use its solution without deriving it. However, two bonus points are awarded for correctly solving the integral.

c*) Show for the second function that the product of the square root of the second momentum in time-domain and in frequency-domain is a constant.

Hint: The square root of the second moment $\sqrt{\langle f^2\rangle}$ of a symmetric function is defined as

$$\sqrt{\langle f^2\rangle} = \sqrt{\frac{\int_{-\infty}^{\infty} t^2\,|f(t)|^2\,\mathrm{d}t}{\int_{-\infty}^{\infty} |f(t)|^2\,\mathrm{d}t}}$$

When $f(t)$ is centered around $t = 0$, the above definition is a measure of Δt, the width of $f(t)$.

Task 2: Fourier transform properties (a=2, b=2 pts.)

Assume that a signal $f(t)$ is given and its frequency representation $\tilde{f}(\omega)$ is known. Now calculate the frequency domain representation of

a) $f(t-t_0)$, a signal that is translated by t_0

b) $\frac{\mathrm{d}}{\mathrm{d}t}f(t)$, the temporal derivative of the signal.

Task 3: δ-Functions (a=1,b=1,c=1,d=1*,e=1,f=1* pts.)

Given is a function $\delta(t)$, with the following properties

i) $\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undefined} & t = 0 \end{cases}$

ii) $\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1.$

With this knowledge, calculate the following integrals:

a) $\int_{-\infty}^{\infty} \delta(t) f(t)\mathrm{d}t$

b) $\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\mathrm{d}t$

c) $\int_{-\infty}^{\infty} \delta(at) f(t)\mathrm{d}t$

d*) $\int_{-\infty}^{\infty} \delta(g(t)) f(t)\mathrm{d}t$, where $g(t)$ is an arbitrary analytic function, with $g(t) = 0 \Leftrightarrow t \in \{t_0^i\} \wedge i \in \{1 \ldots N\}$.The roots of $g(t)$ must be simple roots, meaning that $g'(t_0^i) \neq 0$.

Now calculate

e) the Fourier transform of the delta function.

For another bonus point:

f*) Show that the function $f(t) = \lim_{w \to 0} \frac{1}{\sqrt{\pi} w} \exp[-\frac{t^2}{w^2}]$ fulfills the above mentioned properties and is thus a possible representative of the δ function.

Hint: While the solution of the problems b) to f) is possible with a representative function, we suggest to just use the definition i) and ii), in combination with Taylor expansions or change of variables to find solutions.

Task 4: Convolution (a=4,b*=2 pts.)

Given are two functions $f(t)$ and $g(t)$ and their Fourier transformations $\tilde{f}(\omega)$ and $\tilde{g}(\omega)$. The convolution $[f \otimes g](t)$ of both functions is defined as:

$$[f \otimes g](t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\mathrm{d}\tau.$$

The convolution theorem:

$$\mathcal{F}\{f \otimes g\} = 2\pi \mathcal{F}\{f\}\mathcal{F}\{g\} \tag{1}$$

a) Please show the proof of the convolution theorem by calculating the Fourier transform of the convolution of the two functions $f(t)$ and $g(t)$:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} [f \otimes g](t) \exp[i\omega t]\mathrm{d}t.$$

Hint: Replace $f(t)$ and $g(t)$ with their respective Fourier integrals. Reorder the resulting quadruple integral to generate δ-functions that allow you to solve the integrals.

b*) Use the convolution theorem to calculate the Fourier transform of:

$$\Pi(t) cos(\omega_0 t)$$

where:

$$\Pi(t) = \begin{cases} 1 & -t_0/2 \leq t \leq +t_0/2 \\ 0 & else \end{cases}$$

Hint: Use the symmetry of the convolution theorem: $\mathcal{F}\{f(t)\} \otimes \mathcal{F}\{g(t)\} = 2\pi \mathcal{F}\{f(t) g(t)\}$