

Structure of matter: Homework to exercise 5

Electrical and optical properties of continuous media

Due on November 7th 2023 at noon

Please indicate your name on the solution sheets and send it to your seminar leader!

- Multiple-choice test: Please tick the box(es) with the correct answer(s)!
(correctly ticked box: +1/2 point; wrongly ticked box: -1/2 point)

Anomalous dispersion is	impossible	
	typically observed in spectral regions where no absorption is present	
	typically observed in spectral regions where strong absorption is present	✓
Metals	are good electrical conductors	✓
	are usually good light reflectors in the VIS	✓
	may have refractive indices n that are smaller than 1	✓

- True or wrong? Make your decision!

(2 points): 1 point per correct decision, 0 points per wrong or no decision

Assertion	$n = \sqrt{\epsilon_{\text{real}}}$	$K = \sqrt{\epsilon_{\text{imag}}}$	true	wrong
The refractive index n may never be smaller than the extinction coefficient K		$\sqrt{\epsilon_{\text{real}}} > \sqrt{\epsilon_{\text{imag}}}$		✓
For $\omega \rightarrow \infty \Rightarrow \text{Im } \epsilon \rightarrow 1$	$\text{Im} \epsilon \rightarrow 1$	$\frac{1}{1 + \omega^2 C^2}$		✓

- Imagine a fictive isotropic and non-magnetic material with a complex dielectric function ϵ . At a certain frequency you observe that the real part of the dielectric function is zero, while the imaginary is equal to $\text{Im } \epsilon = 2$. Indicate real and imaginary part of the complex index of refraction at that frequency! (4 points)
- From $\hat{n} = 0.1 + 5i$, calculate $\epsilon = -24.99 + i$ and vice versa. (4 points)
- In the EUV and X-ray spectral regions, the complex index of refraction is usually almost real and close to one. Therefore, it may be expressed through the small parameters δ and β (typical EUV-terminology, do not confuse with the polarizability!!) via: $n + iK = 1 - \delta + i\beta$ with $\delta, \beta \ll 1$. Assuming an EUV wavelength of 13.5nm, calculate the penetration depth into silicon when $\beta \approx 0.002$. (2points)
- Find explicit expressions for the so-called dielectric loss function assuming

- a Drude metal and
- a dielectric material described in terms of the single oscillator model
The loss function is defined as $-\text{Im}(1/\epsilon)$. Also, indicate expressions for the resonance angular frequency of the loss function in both situations! (6 points)

$$3. \hat{n}(\omega) = \sqrt{\epsilon(\omega)} = n(\omega) + i/k(\omega) = \text{Re}\sqrt{\epsilon(\omega)} + i\text{Im}\sqrt{\epsilon(\omega)}$$

$$\Sigma = a + bi \quad \text{Re}\Sigma = 0 \quad \text{Im}\Sigma = 2 \Rightarrow \Sigma = 2i \Rightarrow \sqrt{\Sigma} = \sqrt{2}i$$

$$e^{i\chi} = \cos\chi + i\sin\chi \quad \times 2kx + \frac{\lambda}{2} \Rightarrow i = e^{i(2kx + \frac{\lambda}{2})} \Rightarrow \sqrt{\Sigma} = \sqrt{2} e^{i(kx + \frac{\lambda}{4})}$$

$$\Rightarrow \sqrt{\Sigma} = \sqrt{2}i = \sqrt{2} e^{i((kx + \frac{\lambda}{4}))} = \sqrt{2} \cos(kx + \frac{\lambda}{2}) + i\sqrt{2} \sin(kx + \frac{\lambda}{4})$$

$$\Rightarrow \hat{n}(\omega) = \sqrt{2} \cos(kx + \frac{\lambda}{4}) + i\sqrt{2} \sin(kx + \frac{\lambda}{4}) \quad \text{and} \quad n(\omega) = \sqrt{2} \cos(kx + \frac{\lambda}{2}) \quad k(\omega) = \sqrt{2} \sin(kx + \frac{\lambda}{4})$$

$$\text{if } k=2n \text{ and } n \text{ is a Integer} \quad \hat{n}(\omega) = \sqrt{2} \cos(2nx + \frac{\lambda}{4}) + i\sqrt{2} \sin(2nx + \frac{\lambda}{4}) = 1+i \quad \underline{n(\omega)=1 \quad k(\omega)=1}$$

$$\text{if } k=2n+1 \text{ and } n \text{ is a Integer} \quad \hat{n}(\omega) = \sqrt{2} \cos(2nx + \frac{5}{4}\lambda) + i\sqrt{2} \sin(2nx + \frac{5}{4}\lambda) = -1-i \quad \underline{n(\omega)=-1 \quad k(\omega)=-1}$$

case $k=2n+1$ is not physical

$$(4) \hat{n} = 0.1 + 5i = n(\omega) + i k(\omega) = \text{Re}\sqrt{\epsilon} + i\text{Im}\sqrt{\epsilon} \Rightarrow \text{Re}\sqrt{\epsilon} = 0.1 \quad \text{Im}\sqrt{\epsilon} = 5$$

$$\Rightarrow \sqrt{\epsilon} = 0.1 + 5i \quad \Sigma = (0.1 + 5i)^2 = 0.01 - 25 + i = -24.99 + i$$

$$\Sigma = -24.99 + i \Rightarrow \sqrt{\Sigma} = \sqrt{-24.99 + i} = \sqrt{(0.1)^2 + (5i)^2 + i} = \sqrt{(0.1 + 5i)^2} = 0.1 + 5i$$

$$(5) \vec{D} = \epsilon_0 \epsilon(\omega) \vec{E} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \quad \nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} = \epsilon_0 \epsilon(\omega) \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times (\nabla \times \vec{E}) = -\mu_0 \frac{\partial \nabla \cdot \vec{H}}{\partial t} = \mu_0 \epsilon_0 \epsilon(\omega) \frac{\partial^2 \vec{E}}{\partial t^2} \Rightarrow \nabla(\nabla \cdot \vec{E}) - \Delta \vec{E} = -\frac{\epsilon(\omega)}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \Rightarrow \nabla^2 \vec{E} - \frac{\epsilon(\omega)}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\text{Then in frequency domain} \Rightarrow \nabla^2 \vec{E} + \frac{\omega^2}{c^2} \epsilon(\omega) \vec{E} = 0$$

$$\vec{E} = \vec{E}_0 e^{-i(\omega t - \vec{k} \cdot \vec{r})} \quad \nabla^2 \vec{E} = (ik)^2 \vec{E}_0 e^{-i(\omega t - \vec{k} \cdot \vec{r})} = -k^2 \vec{E}$$

$$\Rightarrow -k^2 \vec{E} + \frac{\omega^2}{c^2} \epsilon(\omega) \vec{E} = 0 \Rightarrow k = \pm \frac{\omega}{c} \sqrt{\epsilon(\omega)}$$

$$k = \frac{\omega}{c} \sqrt{\epsilon(\omega)} \quad \vec{E} = \vec{E}_0 \exp[-i(\omega t - \frac{\omega}{c} \sqrt{\epsilon(\omega)} \vec{r})]$$

$$\sqrt{\epsilon(\omega)} = \text{Re}\sqrt{\epsilon(\omega)} + i\text{Im}\sqrt{\epsilon(\omega)} = n + ik \Rightarrow \vec{E} = \vec{E}_0 \exp[-i(\omega t - \frac{\omega}{c} (n + ik) \vec{r})] = \vec{E}_0 \exp[-i(\omega t - \frac{\omega}{c} nr - i \frac{\omega}{c} kr)]$$

$$= \vec{E}_0 e^{-i(\omega t - \frac{\omega}{c} kr)}$$

$$I = (\vec{E}_0 e^{-i(\omega t - \frac{\omega}{c} kr)})^2 = \vec{E}_0^2 e^{-\frac{2\omega}{c} kr} \quad \text{The absorption coefficient} \alpha = \frac{2\omega}{c} k$$

$$k = \beta = 0.002 \quad \omega = \frac{2\pi}{T} = \frac{2\pi c}{\lambda} \Rightarrow \alpha = \frac{4\pi}{\lambda} \beta = \frac{4\pi \times 0.002}{13.5 \times 10^{-7} \text{ cm}} = \frac{13.5 \times 10^{-7} \text{ cm}}{4\pi \times 0.002} \approx 5.37 \times 10^{-5} \text{ cm}^{-1}$$

$$\text{Penetration depth} \quad L = \frac{1}{\alpha} = \frac{13.5 \times 10^{-7} \text{ cm}}{4\pi \times 0.002} \approx 5.37 \times 10^{-5} \text{ cm}$$

$$b. \text{ dielectric loss function: } -\operatorname{Im}\left(\frac{1}{\epsilon}\right) \quad \epsilon(w) = \operatorname{Re}\epsilon(w) + i\operatorname{Im}\epsilon(w)$$

$$\text{Drude model: } \epsilon(w) = 1 - \frac{w_p^2}{w^2 + 2i\gamma w} = \frac{w^2 - w_p^2 + 2i\gamma w}{w^2 + 2i\gamma w}$$

$$\frac{1}{\epsilon(w)} = \frac{w^2 + 2i\gamma w}{w^2 - w_p^2 + 2i\gamma w} = \frac{(w^2 + 2i\gamma w)(w^2 - w_p^2 - 2i\gamma w)}{(w^2 - w_p^2)^2 + 4\gamma^2 w^2} = \frac{w^4 - w_p^2 w^2 + 2i\gamma w^3 - 2i\gamma w p^2 w + 4\gamma^2 w^2}{(w^2 - w_p^2)^2 + 4\gamma^2 w^2}$$

$$= \frac{w^4 - w_p^2 w^2 + 4\gamma^2 w^2}{(w^2 - w_p^2)^2 + 4\gamma^2 w^2} + i \frac{-2\gamma w p^2 w}{(w^2 - w_p^2)^2 + 4\gamma^2 w^2}$$

$$\Rightarrow -\operatorname{Im}\left(\frac{1}{\epsilon}\right) = \frac{2\gamma w p^2 w}{(w^2 - w_p^2)^2 + 4\gamma^2 w^2} \quad \text{resonance frequency } w_0 = w_p$$

$$[-\operatorname{Im}\left(\frac{1}{\epsilon}\right)]' = \frac{2\gamma w p^2 [(w^2 - w_p^2)^2 + 4\gamma^2 w^2] - 2\gamma w p^2 w [4w(w^2 - w_p^2) + 8\gamma^2 w]}{[(w^2 - w_p^2)^2 + 4\gamma^2 w^2]^2} = \frac{2\gamma w p^2 [(w^2 - w_p^2)^2 - 4w^2(w^2 - w_p^2) - 4\gamma^2 w^2]}{[(w^2 - w_p^2)^2 + 4\gamma^2 w^2]^2}$$

$$[-\operatorname{Im}\left(\frac{1}{\epsilon}\right)]' = 0 \Rightarrow (w^2 - w_p^2)^2 - 4w^2(w^2 - w_p^2) - 4\gamma^2 w^2 = 0 \Rightarrow (3w^2 + w_p^2)(w^2 - w_p^2) + 4\gamma^2 w^2 = 0$$

$$\Rightarrow 3w^4 + (4\gamma^2 - 2w_p^2)w^2 - w_p^4 = 0 \Rightarrow w^2 = \frac{2w_p^2 - 4\gamma^2 \pm \sqrt{(4\gamma^2 - 2w_p^2)^2 + 16w_p^4}}{6} = \frac{w_p^2 - 2\gamma^2 \pm 2\sqrt{\gamma^4 - \gamma^2 w_p^2 + w_p^4}}{3}$$

$$w^2 > 0 \text{ thus resonance frequency } w = \frac{1}{3}(w_p^2 - 2\gamma^2 + 2\sqrt{\gamma^4 - \gamma^2 w_p^2 + w_p^4}) \Rightarrow w = \sqrt{\frac{1}{3}(w_p^2 - 2\gamma^2 + 2\sqrt{\gamma^4 - \gamma^2 w_p^2 + w_p^4})}$$

$$\text{Dielectrical material: } \epsilon(w) = 1 + \frac{w_p^2}{\tilde{w}_0^2 - w^2 - 2i\gamma w} = \frac{w_p^2 + \tilde{w}_0^2 - w^2 - 2i\gamma w}{\tilde{w}_0^2 - w^2 - 2i\gamma w}$$

$$\frac{1}{\epsilon(w)} = \frac{\tilde{w}_0^2 - w^2 - 2i\gamma w}{w_p^2 + \tilde{w}_0^2 - w^2 - 2i\gamma w} = \frac{(\tilde{w}_0^2 - w^2 - 2i\gamma w)(w_p^2 + \tilde{w}_0^2 - w^2 + 2i\gamma w)}{(w_p^2 + \tilde{w}_0^2 - w^2)^2 + 4\gamma^2 w^2}$$

$$= \frac{(\tilde{w}_0^2 - w^2)(w_p^2 + \tilde{w}_0^2 - w^2) + 2i\gamma w(\tilde{w}_0^2 - w^2) - 2i\gamma w(w_p^2 + \tilde{w}_0^2 - w^2) + 4\gamma^2 w^2}{(w_p^2 + \tilde{w}_0^2 - w^2)^2 + 4\gamma^2 w^2}$$

$$= \frac{(\tilde{w}_0^2 - w^2)(w_p^2 + \tilde{w}_0^2 - w^2) + 4\gamma^2 w^2}{(w_p^2 + \tilde{w}_0^2 - w^2)^2 + 4\gamma^2 w^2} + i \frac{-2\gamma w w_p^2}{(w_p^2 + \tilde{w}_0^2 - w^2)^2 + 4\gamma^2 w^2}$$

$$\Rightarrow -\operatorname{Im}\left(\frac{1}{\epsilon}\right) = \frac{2\gamma w w_p^2}{(w_p^2 + \tilde{w}_0^2 - w^2)^2 + 4\gamma^2 w^2} \quad \text{resonance frequency } \tilde{w}_0^2 = w_p^2 - \frac{w_p^2}{3}$$

$$[-\operatorname{Im}\left(\frac{1}{\epsilon}\right)]' = \frac{2\gamma w p^2 [(w_p^2 + \tilde{w}_0^2 - w^2)^2 + 4\gamma^2 w^2] - 2\gamma w p^2 w [-4w(w_p^2 + \tilde{w}_0^2 - w^2) + 8\gamma^2 w]}{(w_p^2 + \tilde{w}_0^2 - w^2)^2 + 4\gamma^2 w^2}$$

$$= \frac{2\gamma w p^2 [(w_p^2 + \tilde{w}_0^2 - w^2)^2 + 4w^2(w_p^2 + \tilde{w}_0^2 - w^2) - 4\gamma^2 w^2]}{(w_p^2 + \tilde{w}_0^2 - w^2)^2 + 4\gamma^2 w^2} = 0 \Rightarrow (w_p^2 + \tilde{w}_0^2 + 3w^2)(w_p^2 + \tilde{w}_0^2 - w^2) - 4\gamma^2 w^2 = 0$$

$$\Rightarrow 2w^2(w_p^2 + \tilde{w}_0^2) - 3w^4 + (w_p^2 + \tilde{w}_0^2)^2 - 4\gamma^2 w^2 = 0 \Rightarrow 3w^4 + [4\gamma^2 - 2(w_p^2 + \tilde{w}_0^2)]w^2 - (w_p^2 + \tilde{w}_0^2)^2 = 0$$

$$\Rightarrow w = \sqrt{\frac{1}{3}(w_p^2 + \tilde{w}_0^2 - 2\gamma^2 + 2\sqrt{\gamma^4 - \gamma^2(w_p^2 + \tilde{w}_0^2) - (w_p^2 + \tilde{w}_0^2)^2})}$$

- a Drude metal and

- a dielectric material described in terms of the single oscillator model

The loss function is defined as $-\text{Im}(1/\varepsilon)$. Also, indicate expressions for the resonance angular frequency of the loss function in both situations! (**6 points**)