FRIEDRICH SCHILLER UNIVERSITY JENA
Institute of Applied Physics
Prof. Dr. Thomas Pertsch
Anastasia Romashkina, Shreyas Ramakrishna, Mostafa Abasifard, Tina (Shiu Hei Lam), Xiao Chen

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Series 9

FUNDAMENTALS OF MODERN OPTICS

to be returned on **12.01.2023**, at the beginning of the lecture

Task 1: Fraunhofer diffraction (3+1+2 points)

a) Calculate the intensity of the diffracted monochromatic (with the wavelength λ) field pattern $I(x,z_B) = |u(x,z_B)|^2$ in paraxial Fraunhofer approximation for two slits illuminated with a normally incident plane wave (prefactors are not important, the functional dependencies are important). The width of each slit is $2a$ and they are separated by a distance d ($d >> 2a$):

$$u_0(x,z=0) = \begin{cases} 1, & \text{for } |x \pm d/2| \le a \\ 0, & \text{elsewhere.} \end{cases}$$

b) What conditions should the parameters of the initial field satisfy, for the paraxial Fraunhofer approximation to be valid?

c) Try to roughly sketch the shape of the intensity distribution, and explain how parameters a and d influence the main features of the intensity distribution.

Hint: The Fourier transform of a single slit of width $2a$ is $\propto \mathrm{sinc}(\alpha a)$.

Solution Task 1:

a) The field in Fraunhofer approximation is proportional to the Fourier transform of the initial field

$$u(x,z_B) \propto U_0\left(\frac{kx}{z_B}\right).$$

We know the Fourier transform of a single slit centered at the origin:

$$FT[\Theta(a+x)\cdot\Theta(a-x)] = \mathrm{sinc}\,(\alpha a)$$

For the 2 spatially shifted slits we can make use of the shift theorem and get:

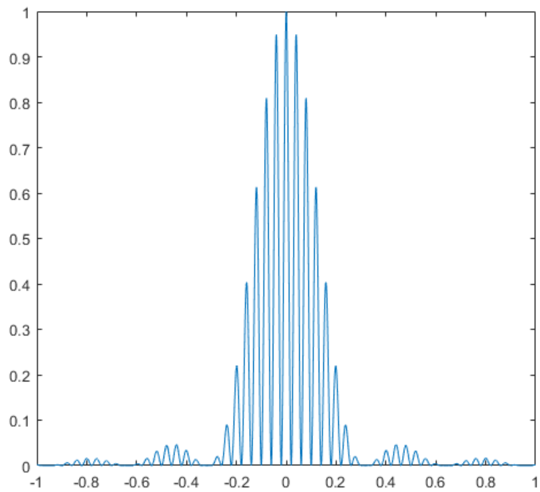
$$\begin{aligned} U_0(\alpha) &= \mathrm{sinc}\,(\alpha a)\cdot\exp[i\alpha d/2] + \mathrm{sinc}\,(\alpha a)\cdot\exp[-i\alpha d/2] \\ &= 2\cdot\mathrm{sinc}\,(\alpha a)\cdot\cos\,(\alpha d/2). \end{aligned}$$

So we get

$$I(x,z_B) = |u(x,z_B)|^2 \propto \mathrm{sinc}^2\left(\frac{kx}{z_B}a\right)\cdot\cos^2\left(\frac{kx}{z_B}\frac{d}{2}\right)$$

b) For paraxial approximation we need $2a >> \lambda$ and for Fraunhofer we need $\frac{d^2}{\lambda z_B} < 0.1$ or $<< 1$.

c) Figure of sinc as envelope and faster oscillation of the cos-term. Increasing a makes the big sinc envelope narrower and increasing d makes the periods of the fast oscillation smaller.



Task 2: Fourier transform of gratings (3+3 points)

a) A finite periodic one-dimensional grating, with period D, has N illuminated periods, so that the transmission function of the whole grating is given by

$$t(x) = \sum_{l=0}^{N-1} \tilde{f}(x - lD),$$

where $\tilde{f}(x)$ is the grating function, which is only nonzero in the range $0 \leq x < D$.
Prove that the spatial spectrum is given by

$$T(\alpha) = \tilde{F}(\alpha)\frac{\sin(N\alpha D/2)}{\sin(\alpha D/2)} e^{i(1-N)\alpha D/2},$$

where $\tilde{F}(\alpha)$ is the Fourier transform of $\tilde{f}(x)$.

Hint: Make use of the Fourier shifting theorem.

b) Now consider an infinitely extended grating, with the transmission function:

$$t(x) = \sum_{l=-\infty}^{+\infty} \tilde{f}(x - lD).$$

Prove that the spatial spectrum is given by

$$T(\alpha) = \tilde{F}(\alpha)\frac{2\pi}{D}\sum_{n=-\infty}^{\infty} \delta\left(\alpha - \frac{2\pi n}{D}\right).$$

Hint: Make use of the fact that an infinitely extended periodic function has a Fourier series expansion.

Solution Task 2:

a) The Fourier transform of $t(x)$ reads as:

$$\begin{aligned} T(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} t(x)\,\exp(-\mathrm{i}\alpha x)\mathrm{d}x && (1)\\ &= \sum_{n=0}^{N-1}\frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{f}(x-nD)\,\exp(-\mathrm{i}\alpha x)\mathrm{d}x. && (2)\end{aligned}$$

We now make use of the Fourier shifting theorem $\mathrm{FT}[f(x-x_0)] = F(\alpha)\exp(-\mathrm{i}\alpha x_0)$ to find

$$T(\alpha) = \sum_{n=0}^{N-1}[\exp(-\mathrm{i}\alpha D)]^n \times \frac{1}{2\pi}\int_0^D \tilde{f}(x)\,\exp(-\mathrm{i}\alpha x)\mathrm{d}x = \sum_{n=0}^{N-1}[\exp(-\mathrm{i}\alpha D)]^n \times \tilde{F}(\alpha), \qquad (3)$$

where we used that $\tilde{f}(x)$ is zero outside of $[0, D)$. This shows that we actually only have to calculate the Fourier transform of one period. Now we need the value of the series, which corresponds to a partial sum of the complex geometric series for which we have:

$$\sum_{n=0}^{N-1} q^n = \frac{1-q^N}{1-q}. \qquad (4)$$

We thus find

$$\begin{aligned} T(\alpha) &= \tilde{F}(\alpha)\frac{1-\mathrm{e}^{-\mathrm{i}N\alpha Dx}}{1-\mathrm{e}^{-\mathrm{i}\alpha Dx}}\\ &= \tilde{F}(\alpha)\left[\frac{\mathrm{e}^{\mathrm{i}N\alpha Dx/2}-\mathrm{e}^{-\mathrm{i}N\alpha Dx/2}}{\mathrm{e}^{\mathrm{i}\alpha Dx/2}-\mathrm{e}^{-\mathrm{i}\alpha Dx/2}}\right]\frac{\mathrm{e}^{-\mathrm{i}N\alpha Dx/2}}{\mathrm{e}^{-\mathrm{i}\alpha Dx/2}}\\ &= \tilde{F}(\alpha)\frac{\sin(N\alpha D/2)}{\sin(\alpha D/2)}\exp\left(\mathrm{i}\alpha D\frac{1-N}{2}\right). && (5)\end{aligned}$$

b) We use the fact that any infinitely periodic function $t(x) = t(x+D)$ has a Fourier series expansion:

$$t(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{\mathrm{i}\,2\pi n x}{D}\right) \quad \text{with} \quad c_n = \frac{1}{D}\int_0^D t(x)\exp\left(-\frac{\mathrm{i}\,2\pi n x}{D}\right)\mathrm{d}x.$$

Take the Fourier transform from the Fourier series expansion of $t(x)$, which gives:

$$T(\alpha) = \sum_{n=-\infty}^{\infty} c_n \delta\left(\alpha - \frac{2\pi n}{D}\right),$$

with

$$c_n = \frac{1}{D}\int_0^D \tilde{f}(x)\,\exp\left(-\mathrm{i}\,\frac{2\pi n}{D}x\right) = \frac{2\pi}{D}\frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{f}(x)\,\exp\left(-\mathrm{i}\,\frac{2\pi n}{D}x\right) = \frac{2\pi}{D}\tilde{F}(\alpha = \frac{2\pi n}{D}).$$

Since the delta function samples any function multiplied by it, we can rewrite the result as

$$T(\alpha) = \tilde{F}(\alpha)\frac{2\pi}{D}\sum_{n=-\infty}^{\infty} \delta\left(\alpha - \frac{2\pi n}{D}\right).$$

Task 3: Fraunhofer diffraction by multiple holes (3+2+2 points)

Calculate the diffraction pattern in Fraunhofer approximation for:

a) A pinhole with radius a.
Hint: Use polar coordinates for $\mathbf{k}$ and $\mathbf{r}$ to solve the Fourier transform, which in polar coordinates looks like

$$U_0(\rho_k, \varphi_k) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^a e^{-i\rho_k\rho\cos(\varphi-\varphi_k)}\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi.$$

b) A ring-shaped aperture bounded by two circles of radius a_1 and a_2 with $a_2 > a_1$.

c) A sequence of N pinholes with radius a placed along the x-axis with distances of $b > 2a$.

Useful formulas are:

$$\frac{\mathrm{i}^{-n}}{2\pi}\int_0^{2\pi} \exp(\mathrm{i}x\cos\alpha)\exp(\mathrm{i}n\alpha)\,\mathrm{d}\alpha = J_n(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{n+1}J_{n+1}(x)\right] = x^{n+1}J_n(x)$$

where J_i are the Bessel functions of first kind.

Solution Task 3:

The far-field in Fraunhofer approximation in polar coordinates:

$$\boxed{u_{\mathrm{FR}}(\rho, z) = -2\pi i\frac{k_0}{z}\exp(ik_0 z)\exp\left(i\frac{k_0}{2z}\rho^2\right)U_0\left(\rho_k = \frac{k_0}{z}\rho\right)}. \tag{6}$$

For the diffraction pattern (= intensity), it is thus sufficient to find the angular spectrum of the field distribution in the plane of the diffracting aperture

$$\boxed{U_0(\alpha, \beta) = \frac{1}{(2\pi)^2}\iint u_0(x, y, z=0)e^{-i\mathbf{k}_\perp\mathbf{r}_\perp}\,\mathrm{d}^2\mathbf{r}_\perp}. \tag{7}$$

a) We should use polar coordinates in both domains (polar coordinate in k domain is named ρ_k, φ_k, in the spatial domain is ρ, φ), so we have

$$\alpha x + \beta y = \mathbf{k}_\perp\mathbf{r}_\perp = \rho_k\rho\cos(\varphi - \varphi_k)$$

and thus

$$\begin{aligned}
U_0(\rho_k, \varphi_k) &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^a e^{-i\rho_k\rho\cos(\varphi-\varphi_k)}\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi \\
&= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^a e^{-i\rho_k\rho\cos(\varphi')}\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi' \\
&= \frac{1}{(2\pi)}\int_0^a J_0(-\rho_k\rho)\rho\,\mathrm{d}\rho = \frac{1}{(2\pi)}\int_0^a J_0(\rho_k\rho)\rho\,\mathrm{d}\rho = \frac{1}{(2\pi)}\int_0^{a\rho_k}\frac{J_0(x)}{\rho_k^2}x\,\mathrm{d}x \\
&= \frac{1}{(2\pi)}a\rho_k\frac{J_1(a\rho_k) - J_1(0)}{\rho_k^2} = \frac{1}{(2\pi)}a\frac{J_1(a\rho_k)}{\rho_k} \\
&= \frac{1}{2\pi}\frac{a}{\rho_k}J_1(a\rho_k) = \left[\frac{a}{2\pi}\frac{1}{\sqrt{\alpha^2+\beta^2}}J_1(a\sqrt{\alpha^2+\beta^2})\right].
\end{aligned}$$

According to (6), we find

$$u_{\text{FR}}(\rho,z) = -i\frac{a}{\rho}J_1\left(\frac{k_0 a}{z}\rho\right)\exp(ik_0 z)\exp\left(i\frac{k_0}{2z}\rho^2\right)$$

and for the diffraction pattern (intensity $I = |u|^2$)

$$I_0(\rho,z) = \frac{a^2}{\rho^2}J_1^2\left(\frac{k_0 a}{z}\rho\right) \qquad \text{(Airy disk)} \tag{8}$$

b) Similar to the calculations in the previous part we obtain:

$$\begin{aligned} U_0(\rho_k,\varphi_k) &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_{a_1}^{a_2} e^{-i\rho_k\rho\cos(\varphi-\varphi_k)}\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi \\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_{a_1}^{a_2} e^{-i\rho_k\rho\cos(\varphi')}\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi' \\ &= \frac{1}{2\pi\rho_k}(a_2J_1(a_2\rho_k) - a_1J_1(a_1\rho_k)) = \frac{1}{2\pi}\frac{1}{\sqrt{\alpha^2+\beta^2}}(a_2J_1(a_2\sqrt{\alpha^2+\beta^2}) - a_1J_1(a_1\sqrt{\alpha^2+\beta^2})\,. \end{aligned}$$

According to (6), we find

$$u_{\text{FR}}(\rho,z) = \frac{-i}{\rho}\exp(ik_0z)\exp\left(i\frac{k_0}{2z}\rho^2\right)\left(a_2J_1\left(\frac{k_0a_2}{z}\rho\right) - a_1J_1\left(\frac{k_0a_1}{z}\rho\right)\right).$$

and for the diffraction pattern (intensity $I = |u|^2$)

$$I(\rho,z) = \frac{1}{\rho^2}\left(a_2J_1\left(\frac{k_0a_2}{z}\rho\right) - a_1J_1\left(\frac{k_0a_1}{z}\rho\right)\right)^2.$$

c) We can immediately use the result of part (a) and the already known fact that the interference of N pinholes will result in an extra factor $\frac{\sin^2\left(\frac{k_0Nbx}{2z}\right)}{\sin^2\left(\frac{k_0bx}{2z}\right)}$ to find the diffraction pattern as

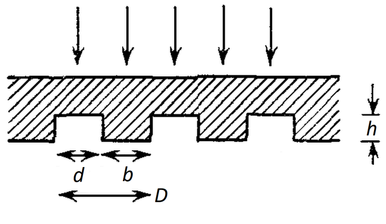
$$I(x,y,z) = \frac{a^2}{x^2+y^2}\left[J_1\left(\frac{k_0a}{z}\sqrt{x^2+y^2}\right)\right]^2\frac{\sin^2\left(\frac{k_0Nbx}{2z}\right)}{\sin^2\left(\frac{k_0bx}{2z}\right)}$$

Task 4: Finite grating with step phase profile (3+2+2 points)

a) Consider that we have a periodic one-dimensional phase grating with the step profile as shown in the figure with N illuminated periods. Assume that the refractive index of the material of the grating is n. We can treat the grating as a phase mask with $\tilde{f}(x) = \exp(\mathrm{i}k_0hn(x))$ within $[0, D = d+b)$, with $k_0 = 2\pi/\lambda$, where:

$$n(x) = \begin{cases} 1 & \text{for} \quad 0 \le x \le d \\ n & \text{for} \quad d < x < d+b \end{cases}$$

Calculate the intensity of the diffraction pattern in the paraxial Fraunhofer approximation using the result of Task 3.



b) Find the field amplitudes of the zeroth and first order diffraction peaks, which appear at $x_0 = 0$ and $x_1 = \frac{\lambda z}{D}$ respectively.

c) Find the values of ridge heights h_0 and h_1 that maximize the amplitudes of zeroth and first order diffraction peaks, respectively.

Solution Task 4:

a) Using the result of task 2(a) we just need to calculate the one dimensional Fourier transform of $\tilde{f}(x)$:

$$
\begin{aligned}
\tilde{F}(\alpha) &= \frac{1}{2\pi}\int_0^D \mathrm{e}^{\mathrm{i}k_0 hn(x)}\mathrm{e}^{-\mathrm{i}\alpha x}dx \\
&= \frac{1}{2\pi}\left(\mathrm{e}^{\mathrm{i}k_0 h}\int_0^d e^{-\mathrm{i}\alpha x}\mathrm{d}x + \mathrm{e}^{\mathrm{i}k_0 nh}\int_d^{d+b} \mathrm{e}^{-\mathrm{i}\alpha x}\mathrm{d}x\right) \\
&= \frac{-1}{2\pi\alpha}\left(2\mathrm{e}^{\mathrm{i}k_0 h}\mathrm{e}^{-\frac{\mathrm{i}\alpha d}{2}}\sin(\frac{\alpha d}{2}) + 2\mathrm{e}^{\mathrm{i}k_0 nh}\mathrm{e}^{-\frac{\mathrm{i}\alpha(2d+b)}{2}}\sin(\frac{\alpha b}{2})\right) \\
&= \frac{-1}{\pi\alpha}\mathrm{e}^{\mathrm{i}k_0 h}\mathrm{e}^{-\frac{\mathrm{i}\alpha d}{2}}\left(\sin(\frac{\alpha d}{2}) + \sin(\frac{\alpha b}{2})\mathrm{e}^{\mathrm{i}k_0(n-1)h}\mathrm{e}^{-\frac{\mathrm{i}\alpha(d+b)}{2}}\right).
\end{aligned}
$$

By combining this with that of part 2(a), and also substituting $\alpha = \frac{k_0 x}{z}$, we find $I \propto |u|^2$ to be proportional to:

$$
I(x,z) \propto \left(\frac{1}{x}\right)^2 \left\{\sin^2\left(\frac{k_0 dx}{2z}\right) + \sin^2\left(\frac{k_0 bx}{2z}\right) + 2\sin\left(\frac{k_0 dx}{2z}\right)\sin\left(\frac{k_0 bx}{2z}\right)\cos\left[k_0(n-1)h - \frac{k_0 Dx}{2z}\right]\right\}\frac{\sin^2(k_0 NDx/2z)}{\sin^2(k_0 Dx/2z)}.
$$

b) The zeroth order corresponds to $l = 0$ (the trivial case $x_0 = 0$). Since $\lim_{x\to 0}\sin(ax)/x = a$ we get for its intensity:

$$
I(x_0,z) \propto 4\left[\left(\frac{k_0 d}{2z}\right)^2 + \left(\frac{k_0 b}{2z}\right)^2 + 2\left(\frac{k_0}{2z}\right)^2 db\cos(k_0(n-1)h)\right]N^2.
$$

For the position of the first order (x_1) with $l = 1$ we get:

$$
x_1 = \frac{2\pi z}{k_0(d+b)} = \frac{\lambda z}{D},
$$

There is also a -1st order, which can be found at $x_{-1} = -x_1$ and therefore is distinct from the first order. However, since $\sin(x)^2$ is symmetric, and we have $\cos(x \pm \pi) = -\cos(x)$, we get the same intensity for both orders:

$$
I(x_{\pm 1},z) \propto \left(\frac{k_0(d+b)}{z\pi}\right)^2 \left\{\sin^2\left(\frac{\pi d}{d+b}\right) + \sin^2\left(\frac{\pi b}{d+b}\right) - 2\sin\left(\frac{\pi d}{d+b}\right)\sin\left(\frac{\pi b}{d+b}\right)\cos\left[k_0(n-1)h\right]\right\}N^2.
$$

c) The amplitude of zeroth order diffraction peaks is maximal if:

$$
\cos(k_0(n-1)h) = 1,
$$

which gives:

$$
h_0 = \frac{\lambda m}{(n-1)},
$$

where m is positive integer or 0.

For the maximum of first order peaks we get the following result:

$$
\cos(k_0(n-1)h) = -1,
$$

and finally obtain:

$$
h_1 = \frac{(2m-1)\lambda}{2(n-1)},
$$

where m is a positive integer.