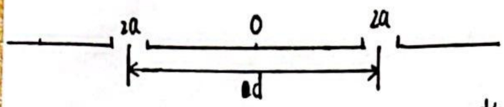
Task 1 Jinsong Liu

a)

Solution:

$$U_0(x, z=0) = \begin{cases} 1, & \text{for } |x \pm d/2| \le a \\ 0, & \text{elsewhere} \end{cases}$$



$$U_+\left(k\frac{x}{z_B}\right) = \int_{-d/2-a}^{-d/2+a} \exp\left(-i\frac{kx}{z_B}x'\right) dx' + \int_{d/2-a}^{d/2+a} \exp\left(-i\frac{kx}{z_B}x'\right) dx'$$

$$= \frac{z_B}{kx}\left\{\sin\left[\frac{kx}{z_B}(-d/2+a)\right] - \sin\left[\frac{kx}{z_B}(-d/2-a)\right] + i\cos\left[\frac{kx}{z_B}(-d/2+a)\right] - i\cos\left[\frac{kx}{z_B}(-d/2-a)\right]\right\}$$

$$+ \frac{z_B}{kx}\left\{\sin\left[\frac{kx}{z_B}(d/2+a)\right] - \sin\left[\frac{kx}{z_B}(d/2-a)\right] + i\cos\left[\frac{kx}{z_B}(d/2+a)\right] - i\cos\left[\frac{kx}{z_B}(d/2-a)\right]\right\}$$

$$= \frac{z_B}{kx}\left[2\cos\left(-\frac{kxd}{2z_B}\right)\sin\left(\frac{kxa}{z_B}\right) - 2i\sin\left(-\frac{kxd}{2z_B}\right)\sin\left(\frac{kxa}{z_B}\right)\right] + \frac{z_B}{kx}\left[2\cos\left(\frac{kxd}{2z_B}\right)\sin\left(\frac{kxa}{z_B}\right) - 2i\sin\left(\frac{kxd}{2z_B}\right)\sin\left(\frac{kxa}{z_B}\right)\right]$$

$$= \frac{2\sin\left(\frac{kxa}{z_B}\right)}{\frac{kx}{z_B}} \exp\left(-i\frac{kxd}{2z_B}\right) + \frac{2\sin\left(\frac{kxa}{z_B}\right)}{\frac{kx}{z_B}} \exp\left(i\frac{kxd}{2z_B}\right)$$

$$= 4a\frac{\sin\left(\frac{kxa}{z_B}\right)}{\frac{kxa}{z_B}} \cos\left(\frac{kxd}{2z_B}\right)$$

$$= 4a\,\mathrm{Sinc}(\alpha a)\cos\left(\frac{kxd}{2z_B}\right)$$

3

(Let $\alpha = \frac{kx}{z_B}$)

$$\therefore I = \frac{1}{(\lambda z_B)^2} 16a^2 \,\mathrm{sinc}^2(\alpha a)\cos^2\left(\frac{kxd}{2z_B}\right)$$

b)

Solution:

0.5

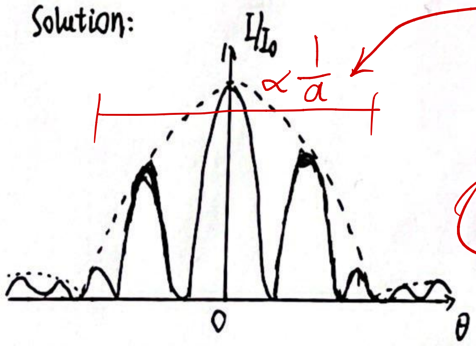
$$\frac{(2a)^2}{\lambda z_B} < 0.1$$

$$N_f = \frac{d^2}{\lambda z_B} << 1$$

also $2a >> \lambda$

c)

Solution:



2

d can decide the [redacted] number of peak values which are included in the envelope.

a can decide the width of the fringes

the width of the sinc envelope

Task 2 Jinsong Liu

a)

3+1.5

Solution:

$$t(x) = \sum_{l=0}^{N-1} \tilde{f}(x - lD)$$

$$\alpha = \frac{kx}{z_B} \quad \text{let } X' = x' - lD$$

$$\therefore T(\alpha) = \sum_{l=0}^{N-1} \int_{-\infty}^{\infty} \tilde{f}(x' - lD)\exp(-i\alpha x')\,dx'$$

$$= \sum_{l=0}^{N-1} \int_{-\infty}^{\infty} \tilde{f}(x')\exp(-i\alpha x')\exp(-i\alpha lD)\,dx'$$

3

$$= \tilde{F}(\alpha) \sum_{l=0}^{N-1} \exp(-i\alpha D l)$$

$$= \tilde{F}(\alpha) \frac{1 - \exp[-iN\alpha D]}{1 - \exp(-i\alpha D)}$$

$$= \tilde{F}(\alpha) \frac{\exp(-iN\alpha D/2)\left[\exp(iN\alpha D/2) - \exp(-iN\alpha D/2)\right]}{\exp(-i\frac{\alpha D}{2})\left[\exp(i\frac{\alpha D}{2}) - \exp(-i\frac{\alpha D}{2})\right]}$$

$$= \tilde{F}(\alpha) \frac{\sin(N\alpha D/2)}{\sin(\alpha D/2)} e^{i(1-N)\alpha D/2}$$

b)

Solution:

$$F\left[\sum_{l=-\infty}^{\infty} \tilde{f}(x - lD)\right] = F\left[\sum_{n=-\infty}^{\infty} F_n e^{in w_0 t}\right] = 2\pi \sum_{n=-\infty}^{\infty} F_n \delta(\alpha - \frac{2n\pi}{D})$$

$$w_0 = \frac{2\pi}{D}$$

1.5, You have to show all the steps.

$$F_n = \frac{1}{D} \tilde{F}(\alpha)$$

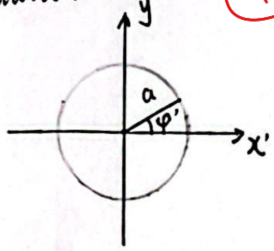
$$\therefore T(\alpha) = \frac{2\pi}{D} \tilde{F}(\alpha) \sum_{n=-\infty}^{\infty} \delta(\alpha - \frac{2n\pi}{D})$$

Task 3 Jinsong Liu

a)

solution:

1.5



Solution:

coordinates:

$$\begin{cases} x' = r'\cos\varphi' \\ y' = r'\sin\varphi' \end{cases}, \quad \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

$$dx'dy' = r'dr'd\varphi'$$

$$U_+(k\tfrac{x}{z_B}, k\tfrac{y}{z_B}) = \iint_{-\infty}^{\infty} \exp\left[-i\left(\tfrac{kx}{z_B}x' + \tfrac{ky}{z_B}y'\right)\right] dx'dy'$$

$$= \int_0^{2\pi}\int_0^{a} \exp\left[-ik\tfrac{rr'}{z_B}(\cos\varphi'\cos\varphi + \sin\varphi'\sin\varphi)\right] r'dr'd\varphi'$$

Let $r/z_B = \theta$

$$U_+(\theta,\varphi) = \int_0^{2\pi}\int_0^{a} \exp\left[-ik\theta r'\cos(\varphi'-\varphi)\right] r'dr'd\varphi'$$

$$\because \int_0^{2\pi} \exp[-ik\theta r'\cos(\varphi'-\varphi)]d\varphi' = \underline{2\pi J_0(kr'\theta)} \text{ (Bessel function)}$$

$$\therefore U_+(\theta,\varphi) = 2\pi\int_0^{k\theta a} (kr'\theta) J_0(kr'\theta)\, d(kr'\theta) \cdot \frac{1}{(k\theta)^2}$$

$$\because \frac{d}{dx}\left[x^{n+1}J_{n+1}(x)\right] = x^{n+1}J_n(x) \qquad \therefore \int_0^t xJ_0(x)dx = tJ_1(t)$$

$$\therefore \int_0^{k\theta a}(kr'\theta)J_0(kr'\theta)\,d(kr'\theta) = \int_0^{k\theta a} xJ_0(x)dx = ka\theta J_1(ka\theta)$$

$$\therefore U_+(\theta,\varphi) = \frac{2\pi}{(k\theta)^2}ka\theta J_1(ka\theta) = \frac{2\pi a J_1(ka\theta)}{k\theta}$$

$$I = \frac{4\pi^2a^2J_1^2(ka\theta)}{k^2\theta^2}$$

$$I_{FR} = |u_{FR}|^2$$

$$u_{FR} = -2\pi i\frac{k}{z}\exp(ikz)\exp\left(i\frac{k}{2z}\rho^2\right)U_0\left(\rho_K = \frac{k_0}{z}\rho\right)$$

↳ $I(x,y,z)$?

b)

Solution: 1

From problem a)

$$U_t\left(k\frac{x}{z_B}, k\frac{y}{z_B}\right) = \int_0^{2\pi}\int_{a_1}^{a_2} \exp\left[-ik\frac{rr'}{z_B}(\cos\varphi'\cos\varphi + \sin\varphi'\sin\varphi)\right] r'dr'd\varphi'$$

Let $r/z_B = \theta$

$$U_t(\theta, \varphi) = \int_0^{2\pi}\int_{a_1}^{a_2} \exp\left[-ik\theta r'\cos(\varphi'-\varphi)\right] r'dr'd\varphi'$$

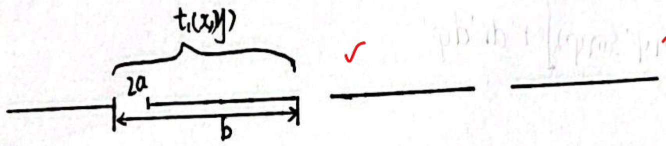
$$= 2\pi\int_{a_1k\theta}^{a_2k\theta} (kr'\theta) J_0(kr'\theta)\, d(kr'\theta) \frac{1}{(k\theta)^2}$$

$$= \frac{2\pi a_2 J_1(ka_2\theta)}{k\theta} - \frac{2\pi a_1 J_1(ka_1\theta)}{k\theta}$$

$$I = \left[\frac{2\pi a_2 J_1(ka_2\theta)}{k\theta} - \frac{2\pi a_1 J_1(ka_1\theta)}{k\theta}\right]^2$$ ✓ wrong factors

c)

Solution: ✓ 1



$$\cancel{t(x) = \sum_{n=0}^{N-1} t_1(x-nb)}$$

$$t(x) = \sum_{n=0}^{N-1} t_1(x-nb, y) \quad \text{with } t_1(x,y) = \begin{cases} 1 & \text{for } x^2+y^2 \leq a^2 \\ 0 & \text{elsewhere} \end{cases}$$

$$T\left(k\frac{x}{z_B}\right) = \sum_{n=0}^{N-1}\int_{-\infty}^{\infty} t_1(x'-nb, y) \exp\left[-i\left(k\frac{x}{z_B}x' + k\frac{y}{z_B}y'\right)\right] dx'dy'$$

(let $x' = x - nb$)

$$= \sum_{n=0}^{N-1}\int_{-a}^{a} t_s(x', y') \exp\left[-i\left(k\frac{x}{z_B}x' + k\frac{y}{z_B}y'\right)\right] \exp\left(-ik\frac{x}{z_B}nb\right) dx'dy'$$

$$= T_s\left(k\frac{x}{z_B}, k\frac{y}{z_B}\right) \sum_{n=0}^{N-1} \exp\left(-ik\frac{x}{z_B}nb\right)$$

From Problem a)

we know that $T_s\left(k\frac{x}{z_B}, k\frac{y}{z_B}\right) = \dfrac{2\pi a J_1(ka\theta)}{k\theta}$

From lecture script

$$\left|\sum_{n=0}^{N-1} \exp(-i\delta n)\right| = \left|\frac{\sin(N\frac{\delta}{2})}{\sin(\frac{\delta}{2})}\right|$$ ✓

$$\therefore T = \frac{2\pi a J_1(ka\theta)}{k\theta}\left|\frac{\sin(N\frac{kxb}{2z_B})}{\sin(\frac{kxb}{2z_B})}\right|$$ → Not exactly!

$$I = \frac{4\pi a^2 J_1^2(ka\theta)}{k^2\theta^2}\left|\frac{\sin(N\frac{kxb}{2z_B})}{\sin(\frac{kxb}{2z_B})}\right|^2$$ ✓

wrong factors