Institute of Applied Physics
Friedrich-Schiller-Universität Jena

Laser Physics problem sheet 6

Summer semester 2023

Problem 1 (4 points)

The rate equation for the upper energy level N_2 of a non-degenerated two-level system can be expressed as:

$$\frac{dN_2}{dt} = -W_{21} \cdot (N_2 - N_1) - S_{21} N_2,$$

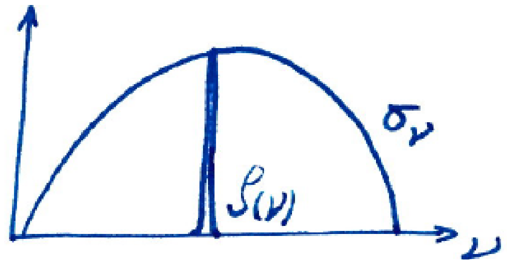
where W_{21} is the probability for stimulated emission or absorption $W_{12} = W_{21}$ and S_{21} the spontaneous decay rate of the upper level, respectively. The general expression for the probability for stimulated emission and absorption W_i is:

$$W_i = \int_0^\infty \frac{\rho(\nu)c}{h\nu} \sigma_\nu d\nu.$$

Here, ν is the frequency, $\rho(\nu)$ is the spectral energy density in the Volume V, c is the speed of light and σ_ν is the cross section.

a) Express W_i as a function of the photon flux Φ for monochromatic light. 1 nts)

real laser:



Intensity I: the energy flux density of the radiation field, i.e. the energy per unit time (= power) through a surface element dA perpendicular to the pointing direction

$$[I] = \frac{\text{W}}{\text{m}^2}$$

Spectral intensity I_ν : the intensity per frequency interval at frequency ν

$$I = \int_{-\infty}^{\infty} I_\nu \, d\nu \qquad [I_\nu] = \frac{\text{W}}{\text{m}^2 \cdot \text{Hz}}$$

Spectral energy density ρ_ν : energy content per frequency interval and unit volume

$$\rho(\nu) = \frac{I_\nu}{c} \quad (2.9a) \qquad [\rho_\nu] = \frac{\text{J}}{\text{m}^3 \cdot \text{Hz}}$$

Photon flux Φ_ν: number of photons per frequency interval, per unit time and surface

$$\Phi_\nu = \frac{I_\nu}{h\nu} \quad (2.9b) \qquad [\Phi_\nu] = \frac{\text{Photons}}{\text{s} \cdot \text{m}^2 \cdot \text{Hz}}$$

$$\Longrightarrow \Phi_\nu = \frac{\rho(\nu) \cdot c}{h\nu}$$

Monochromatic light: $\boldsymbol{\delta(\nu - \nu_0)}$

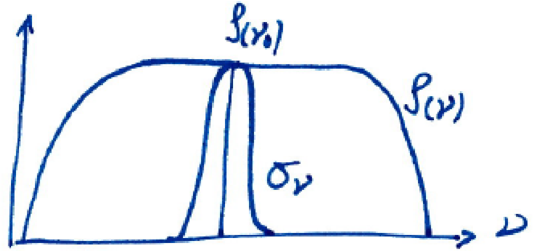
$$W_i = \int_0^\infty \frac{\rho(\nu) \cdot c}{h\nu} \cdot \sigma_\nu \cdot \delta(\nu - \nu_0) d\nu$$

$$= \int_0^\infty \Phi(\nu) \cdot \sigma_\nu \cdot \delta(\nu - \nu_0) d\nu$$

$$\boxed{\int_{-\infty}^{\infty} \mathrm{d}x \, \delta(x - a) \, f(x) = f(a)}$$

$$\boxed{W_i = \Phi(\nu_0) \cdot \sigma_{\nu_0}}$$

b) Express W_i as a function of the upper level lifetime τ_2 for polychromatic light with a broad, smooth spectrum. Assume σ_ν to be much narrower than $\rho(\nu)/(h\nu)$. Use this to extract the expression for the Einstein coefficient B_i. (2 points)



$$W_i = \int_0^\infty \frac{\rho(\nu)\cdot c}{\mathrm{h}\nu}\cdot \sigma_\nu d\nu$$

$$\approx \frac{\rho(\nu_0)\cdot c}{\mathrm{h}\nu}\int_0^\infty \sigma_\nu d\nu$$

$$= \frac{\rho(\nu)\cdot c}{\mathrm{h}\nu}\cdot \sigma$$

$$= \frac{\rho(\nu)\cdot c}{\mathrm{h}\nu}\cdot \frac{\lambda^2}{8\pi\tau_{21}}$$

$$\frac{A_{21}}{B_{12}} = \frac{8\pi h\nu^3}{c^3}$$

$$\sigma = \int_0^\infty \sigma_\nu d\nu = B_{12}\cdot\frac{\mathrm{h}\nu_0}{c}$$

$$A_{21} = {}^1/_{\tau_{21}}$$

$$\lambda = \frac{c}{\nu}$$

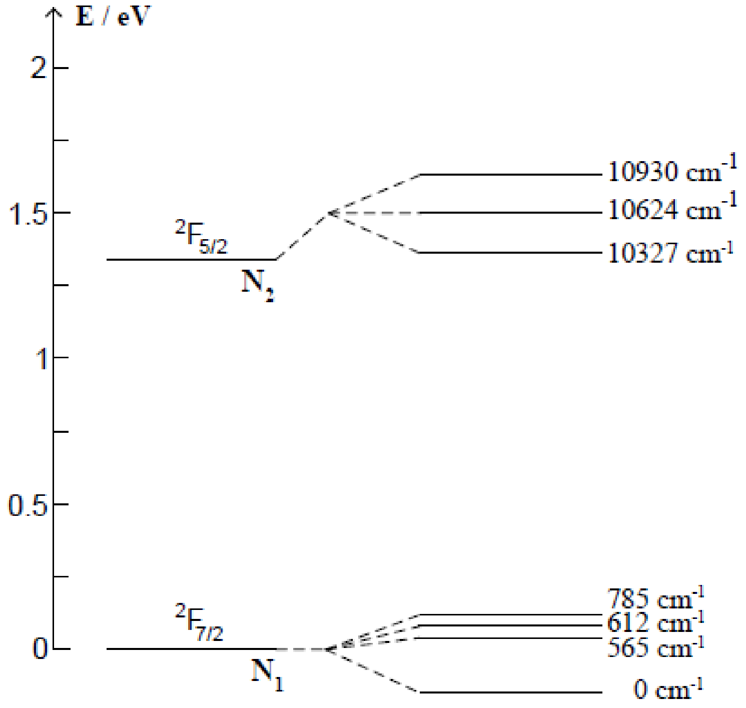
$$\Longrightarrow \sigma = \frac{c^3 A_{21}}{8\pi h\nu^3}\cdot\frac{\mathrm{h}\nu_0}{c} = \frac{A_{21}}{8\pi\left(\frac{\nu}{c}\right)^2} = \frac{\lambda^2}{8\pi\tau_{21}}$$

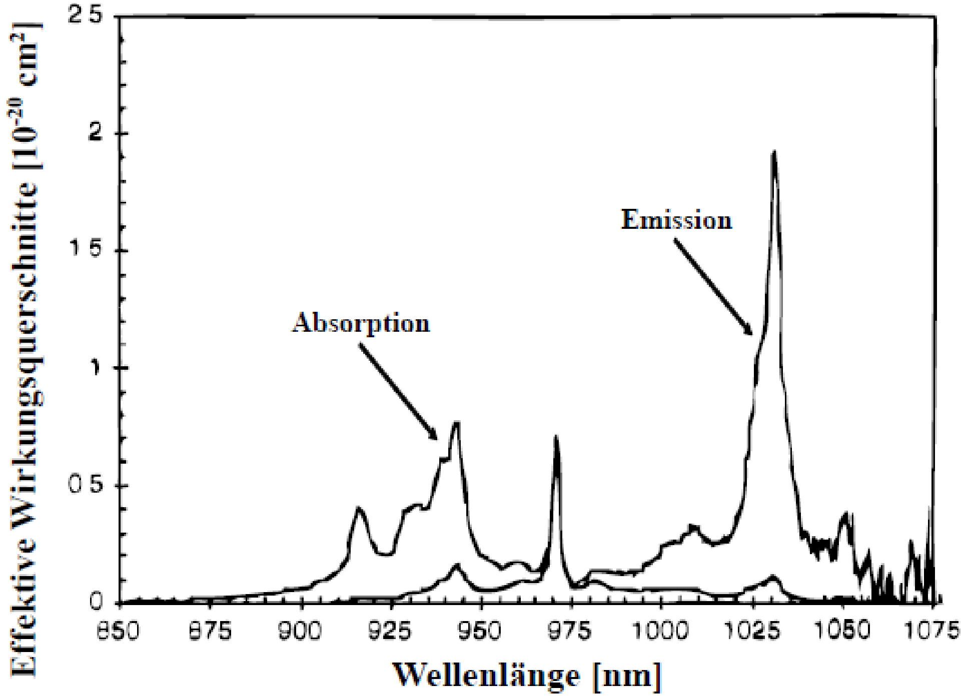
$$\because \quad W_i = B_i\cdot\rho(\nu)$$

$$B_i = \frac{c}{\mathrm{h}\nu}\cdot\frac{\lambda^2}{8\pi\tau_{21}} = \frac{\lambda^3}{8\pi h\tau_{21}}$$

Problem 3 (8 Points)

The figures below show the energy levels diagram (left) and the absorption and emission cross-sections of Yb:YAG. Without pump energy, all the particles populate the lower multiplet (splitted in four sub-levels due to the Stark effect). These particles can be excited to the higher Multiplet through pumping. There is only laser emission for transitions involving energy levels in different multiplets. Within a multiplet, the population of the sub-levels can be approximately described by a Boltzmann distribution.





a) Calculate the emission wavelength for all the possible laser transitions. (2 point)

Calculate the Emission Wavelength via relation to Wavenumber $\Delta\tilde{v}$ in cm^{-1} :

Energy level **E' have the unit** $[\frac{1}{cm}]$ **E have the unit** $[J] = [kg * \frac{m^2}{s^2}]$

$$\mathrm{E[J]} = \mathrm{E'}[\frac{1}{\mathrm{cm}}] * 100 \ * \mathrm{c}\left[\frac{\mathrm{m}}{\mathrm{s}}\right] * h[\frac{m^2}{kg * s}]$$

$$\Delta E = (E_1' - E_2')100ch$$

$$\lambda = \frac{ch}{E} = \frac{1}{100E'}$$

$$\Delta\lambda = \frac{1}{100\Delta E'[\frac{1}{cm}]}$$

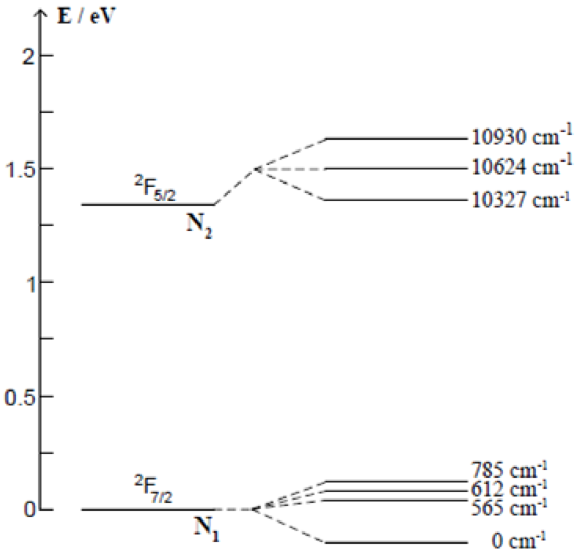
$$\lambda[m] = \frac{1}{\Delta\tilde{v}[cm^{-1}] \cdot 100}$$

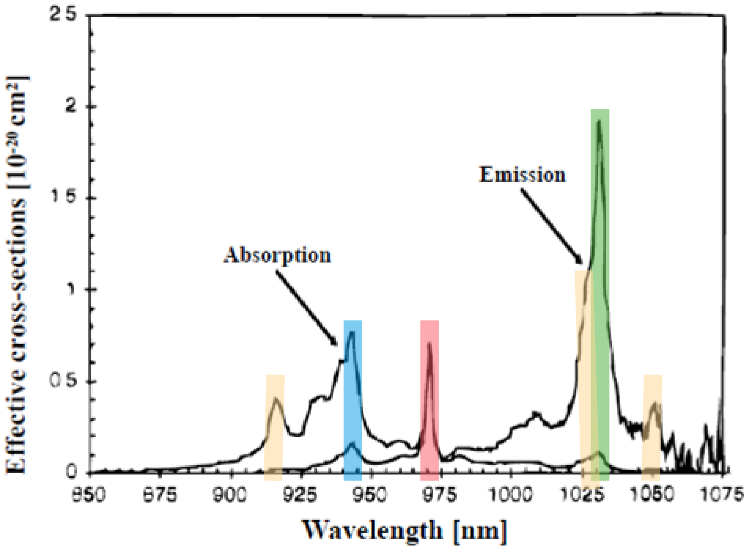
a) Calculate the emission wavelength for all the possible laser transitions. (2 point)

Calculate the Emission Wavelength via relation to Wavenumber $\Delta\tilde{\upsilon}$ in cm^{-1} :

$$\lambda[m] = \frac{1}{\Delta\tilde{\upsilon}[cm^{-1}] \cdot 100}$$

Level 1 / cm-1	Level 2 / cm-1	λ / nm
0	10327	968,3
0	10624	941,3
0	10930	914,9
565	10327	1024,4
565	10624	994,1
565	10930	964,8
612	10327	1029,3
612	10624	998,8
612	10930	969,2
785	10327	1048,0
785	10624	1016,4
785	10930	985,7



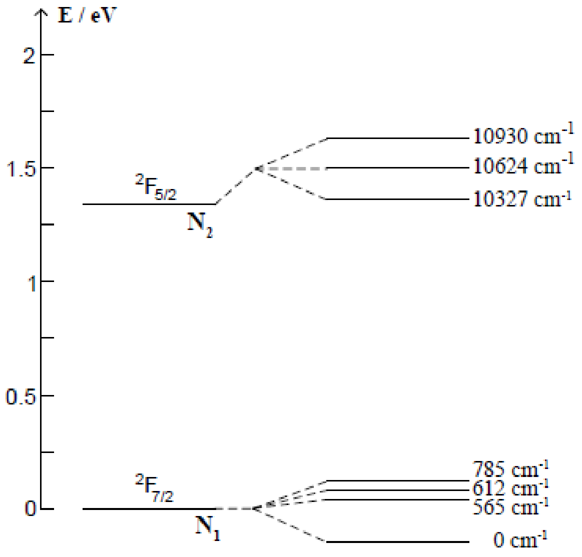


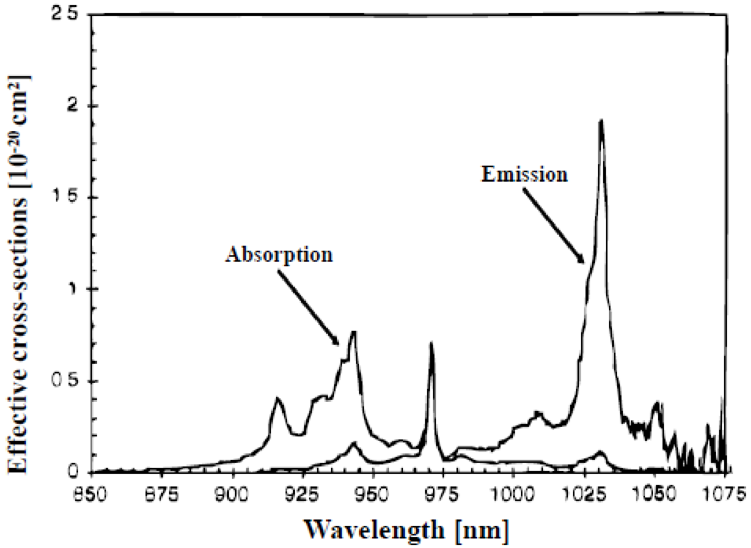
b) Explain why the absorption and emission cross-sections do not show isolated lines corresponding to the different laser transitions. (1 point)

Broadening Mechanisms!

- <u>**Main effect: Stark-effect**</u>

  - **Stark effect = shifting and splitting of spectral lines of atoms & molecules due to presence of external electric field**
  - **Electric field of atoms is created by their neighborhood ⇨ changes from particle to particle**





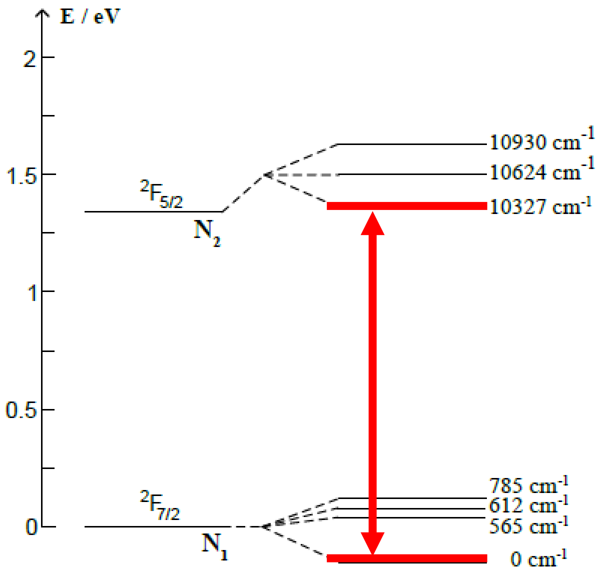
- <u>**Secondary effect: Natural broadening**</u>

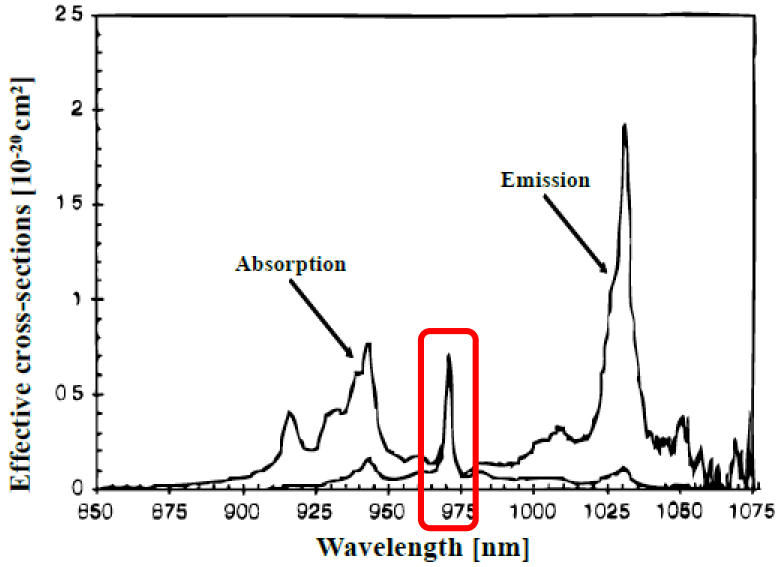
  Each peak is composed of several transitions which will have some broadening and spectral overlap

c) Why are the absorption and emission cross-sections equal in value for the so-called „zero phonon line“ (i.e. the laser transition between the lowest energy level of each multiplet)?. (1 point)

“Zero phonon line”
This is a transition between two manifolds which involves no phonon (there is no non-radiative portion)

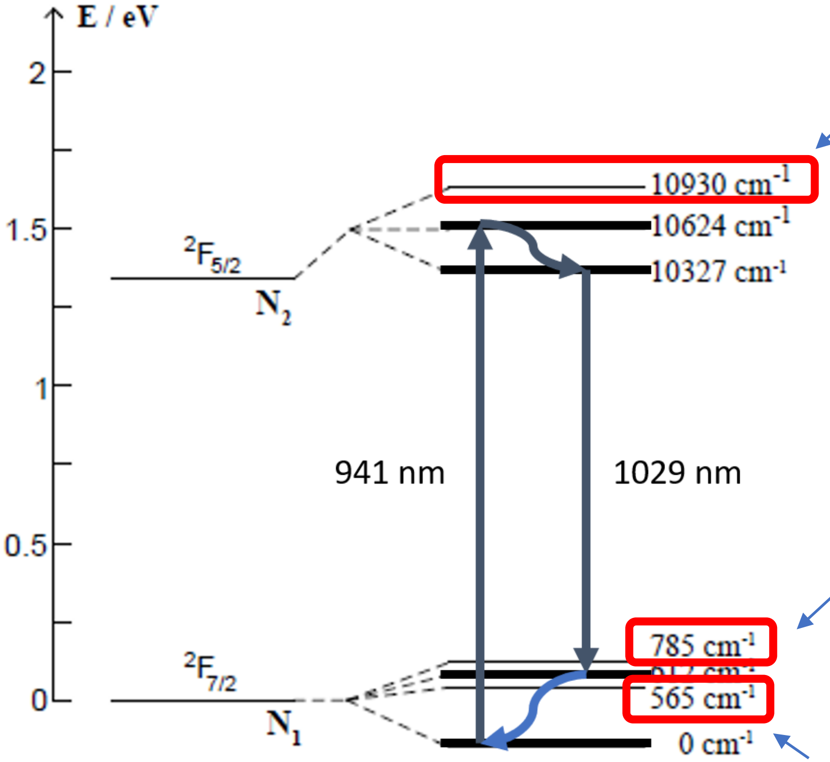
This transition forms an effective two-level system
With a long enough lifetime of the $^2F_{5/2}$ level, or equivalently with a high pump flux, emission or absorbtion at this transition will be about equal.





d) Between which energy levels is it possible to (ideally) obtain a four-level system? Draw its energy level diagram. (1 point)

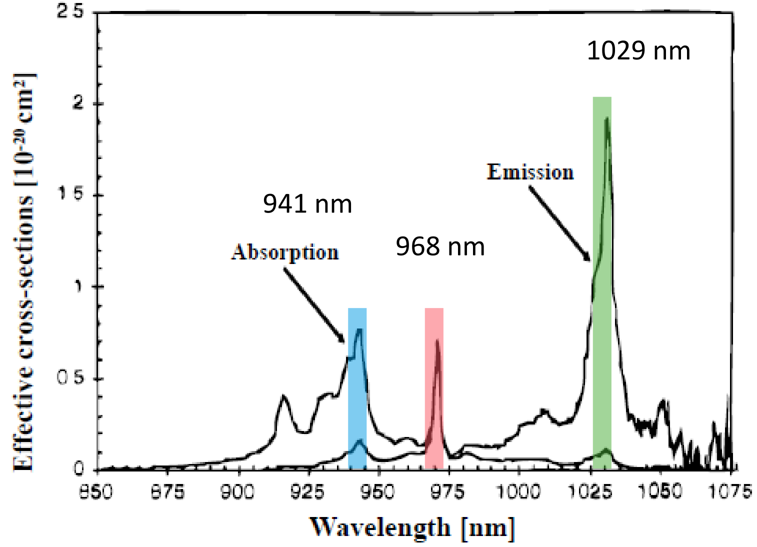
Level 1 / cm^{-1}	Level 2 / cm^{-1}	λ / nm
0	10327	968,3
0	10624	941,3
612	10327	1029,3



Why would this not be a good choice for the pump level? (915 nm)

Why would this not be a good choice for the lower laser level? (1048 nm)

Why would this not be a good choice for the lower pump level? (pump at 964 nm or 994 nm)

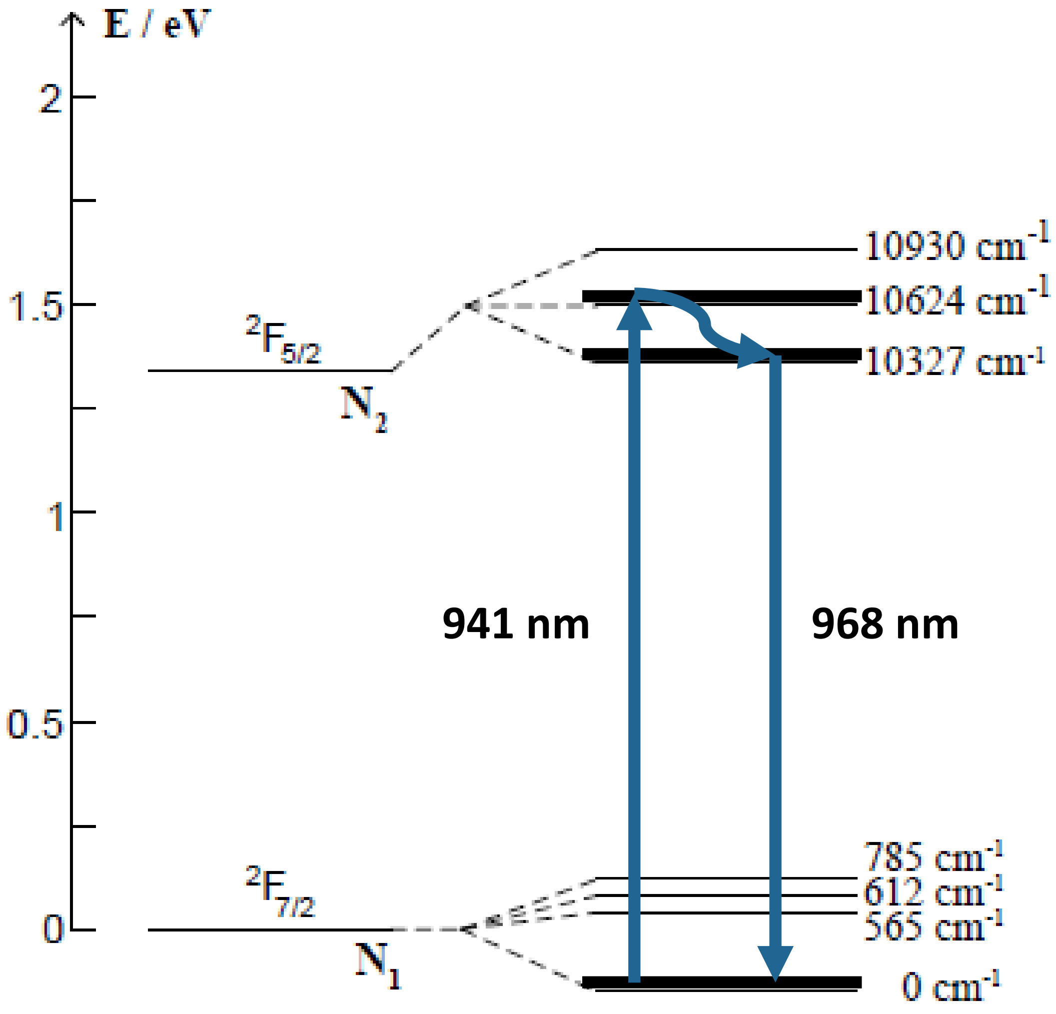


When deciding on pump/laser wavelengths you must consider:

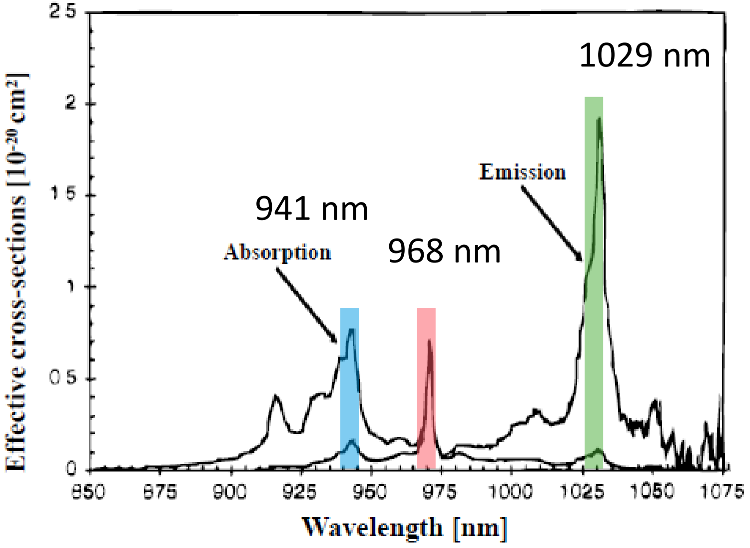
- **Cross-section**
- **Lifetime**
- **Quantum defect**

And many other things…

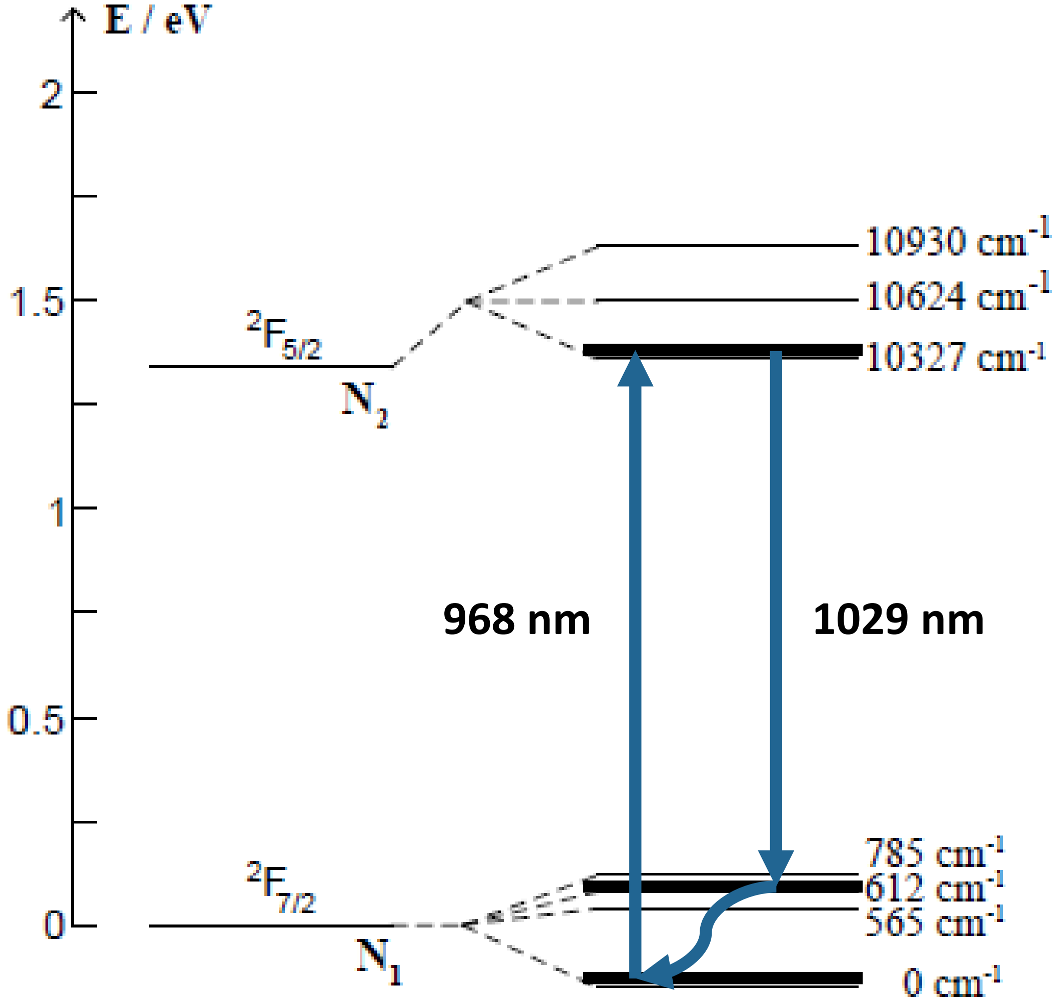
e) Between which energy levels is it possible to (ideally) obtain a three-level system? Draw its energy level diagram. (1 point)



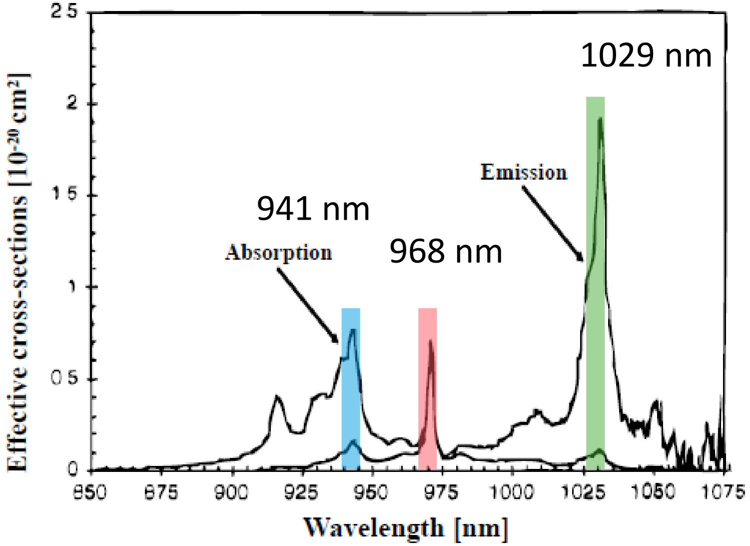
Level 1 / cm^{-1}	Level 2 / cm^{-1}	λ / nm
0	10327	968,3
0	10624	941,3
612	10327	1029,3



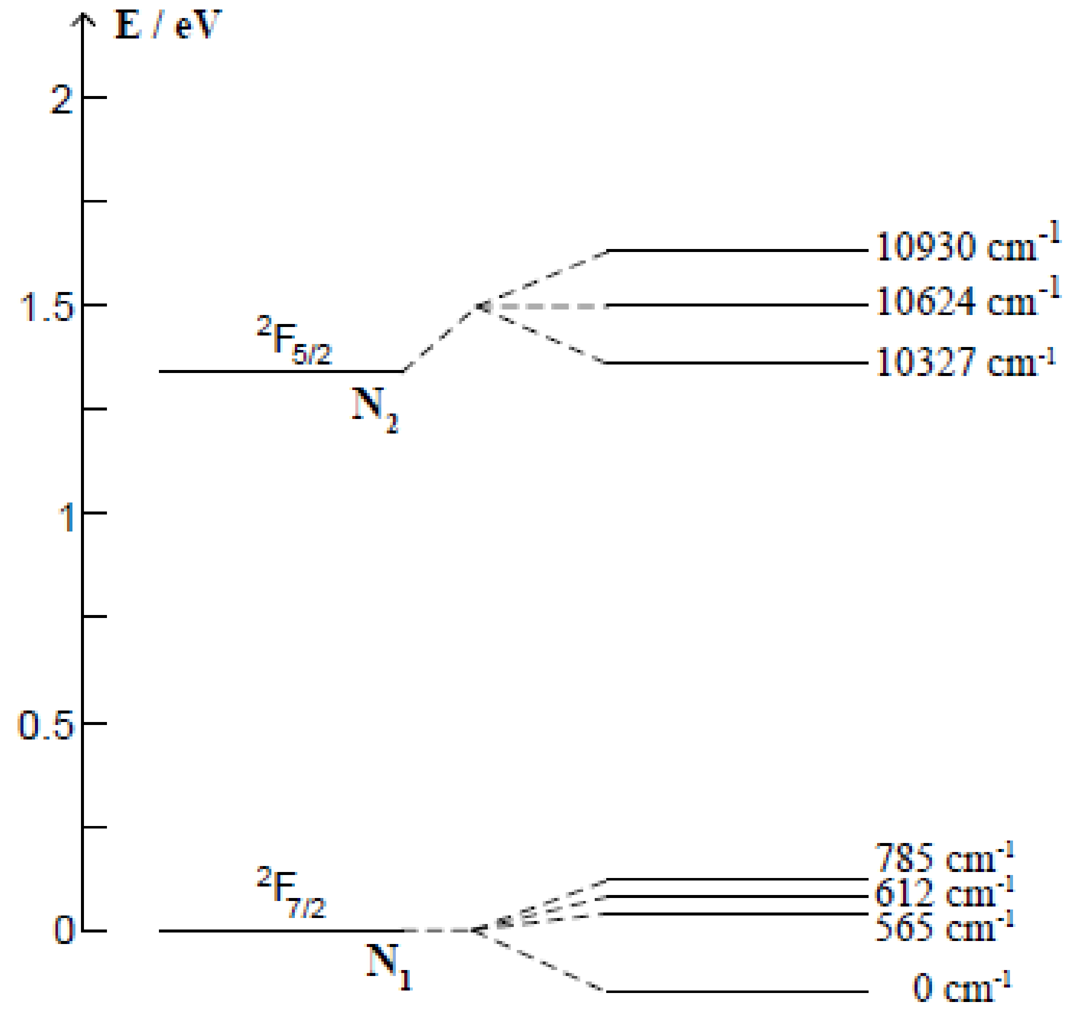
f) Between which energy levels is it possible to (ideally) obtain an inverse three-level system? Draw its energy level diagram. (1 point)



Level 1 / cm^{-1}	Level 2 / cm^{-1}	λ / nm
0	10327	968,3
0	10624	941,3
612	10327	1029,3



g) Which problems might arise if the energy levels lay too close to one another (within a multiplet)?. (1 point)



When energy level lay too close to each other

Thermal excitation usually occurs!

Thermal excitation:
a process where lattice vibrations can provide enough energy to transfer electrons to a higher energy sublevel.

Reduced inversion and hence reduced laser efficiency

Problem 3 (8 points)

a) What is gain saturation? What is its physical origin? (1 point)

b) The basic differential equation governing the growth rate of the signal intensity along an amplifier is:

$$\frac{dI(z)}{dz} = \frac{g_o}{1 + \frac{I(z)}{I_{sat}}} I(z) = g_{sat} I(z)$$

where P is the power, g_o is the small-signal gain coefficient, I_{sat} is the saturation intensity and g_{sat} is the saturated gain coefficient. Using this equation demonstrate that the gain G can be calculated by:

$$G = \frac{I_{out}}{I_{in}} = G_o e^{-\frac{I_{out} - I_{in}}{I_{sat}}}$$

where G_o is the small-signal gain of the amplifier ($G_o = e^{g_o L}$, where L is the length of the active medium). ***Hint:*** Assume that $I(z = 0) = I_{in}$ and that $I(z = L) = I_{out}$. (1 point)

a) Gain satuation is a phenomenon that the gain of an amplifer is reducing when the input power is increasing.
Its physic origin is that the gain medium has finite number of particles and stronger radiation field depletes more inversion.

b) **The equation can be reorganized as**

$$\frac{dI(z)}{I(z)} + \frac{dI(z)}{I_{sat}} = g_0 dz$$

b) The basic differential equation governing the growth rate of the signal intensity along an amplifier is:

$$\frac{dI(z)}{dz} = \frac{g_0}{1+\frac{I(z)}{I_{\text{sat}}}} I(z)$$

$$\frac{dI(z)}{I(z)} + \frac{dI(z)}{I_{sat}} = g_0 dz$$

where g_0 is the small-signal gain coefficient and $G_0 = e^{g_0 L}$ is the small-signal gain of an amplifier of length L. Using this equation demonstrate that the gain is:

$$G = \frac{I_{\text{out}}}{I_{\text{in}}} = G_0 \ \exp\left(-\frac{I_{\text{out}} - I_{\text{in}}}{I_{\text{sat}}}\right)$$

Hint: Assumme that $I(z=0) = I_{\text{in}}$ and that $I(z=L) = I_{\text{out}}$

<u>Integral from 0 to L, where L is the length of the amplifier</u>

$$\left.\begin{aligned} &\int_0^L \frac{dI(z)}{I(z)} + \frac{dI(z)}{I_{sat}} = \ln\big(I(L)\big) - \ln\big(I(0)\big) + \frac{I(L)-I(0)}{I_{sat}} = \ln\left(\frac{I_{out}}{I_{in}}\right) + \frac{I_{out}-I_{in}}{I_{sat}} \\ &\int_0^L g_0 dz = g_0 L \end{aligned}\right\} \ln\left(\frac{I_{out}}{I_{in}}\right) + \frac{I_{out}-I_{in}}{I_{sat}} = g_0 L$$

$$\Rightarrow \quad G = \frac{I_{out}}{I_{in}} = \exp\left(g_0 L - \frac{I_{out}-I_{in}}{I_{sat}}\right) = G_0 \exp\left(-\frac{I_{out}-I_{in}}{I_{sat}}\right)$$

c) We have a $1m$ fiber amplifier. When $I_{in} = 1W/cm^2$, the gain of the amplifier is $10dB$. if we further increase the I_{in} to $2W/cm^2$, the gain decreases to $9dB$, what is the value of the saturation intensity I_{sat} and of the small signal gain G_o? (2 points)

Rewrite the result from b)

$$G = G_0 \exp\left(-\frac{I_{out} - I_{in}}{I_{sat}}\right) = G_0 \exp\left(-(G-1)\frac{I_{in}}{I_{sat}}\right)$$

Pay attention to the unit transfer from dB to unitless

$$P(dB) = 10\log_{10} P$$

$$10\ dB \Rightarrow 10$$

$$9\ dB \Rightarrow 10^{\frac{9}{10}} \approx 7.943$$

Plug the known parameters in

$$\begin{cases} 10 = G_0 \exp\left(-(10-1)\dfrac{1}{I_{sat}}\right) \\ 7.943 = G_0 \exp\left(-(7.943-1)\dfrac{2}{I_{sat}}\right) \end{cases} \Rightarrow \quad I_{sat} = 21.22\ \mathrm{W/cm^2}, \quad G_0 = 15.28$$

d) The extracted intensity in an amplifier is defined as $I_{extr} = I_{out} - I_{in}$. Calculate the extracted intensity in the two cases proposed in c). For which input intensity is it possible to extract more intensity? What is the physical reason behind this? (1 point)

Rewrite the result from b)

$$I_{extr} = I_{out} - I_{in} = \ln\left(\frac{G_0}{G}\right) I_{sat}$$

When input intensity is $1W/cm^2$: $I_{extr} = \ln\left(\frac{15.28}{10}\right) \times 21.22\ \mathrm{W/cm^2} = 8.996\ \mathrm{W/cm^2}$

When input intensity is $2W/cm^2$: $I_{extr} = \ln\left(\frac{15.28}{9}\right) \times 21.22\ \mathrm{W/cm^2} = 11.232\ \mathrm{W/cm^2}$

Input signal with higher intensity depletes more inversion, i.e., more energy is extracted.

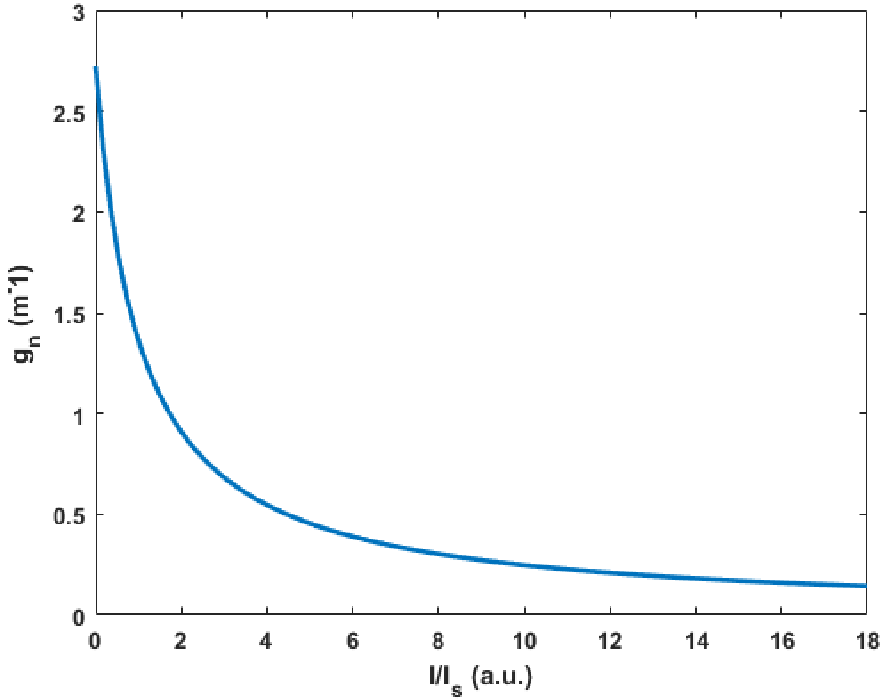
e) Rewrite the differential equation in b) considering a constant loss coefficient α of the amplifier. (1 point)

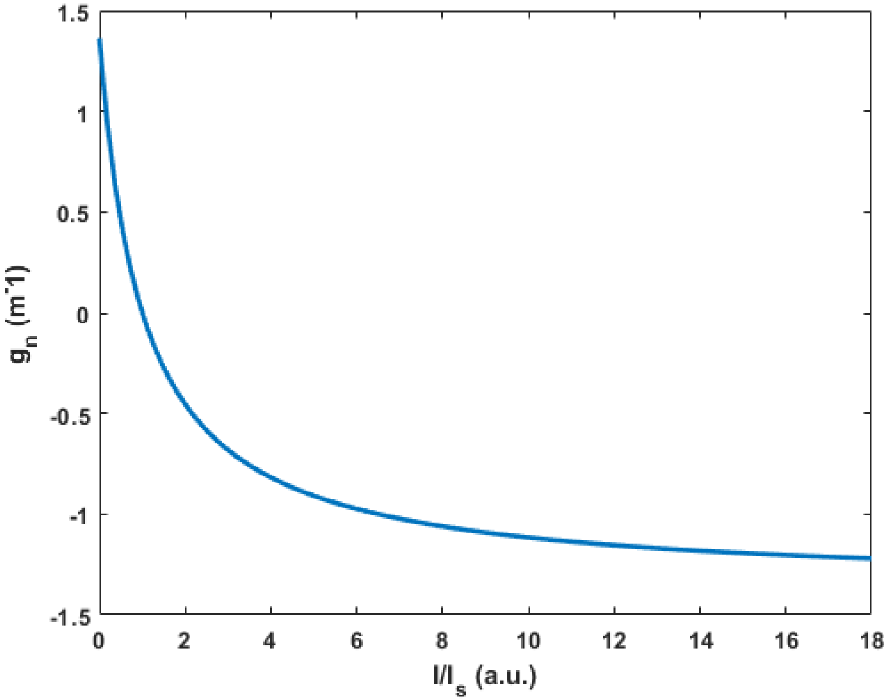
$$\frac{dI(z)}{I(z)dz} = \frac{g_0}{1 + \frac{I(z)}{I_{sat}}} - \alpha$$

f) We have an $8m$ long fiber amplifier. We assume that this amplifier has the same saturated gain coefficient $g_{sat}(z)$ as the amplifier in c) . Now we consider this amplifier under two circumstances: the first one is when there are no extra losses in the system and the second one is when there is a loss coefficient α that is half of the small signal gain coefficient (of the lossless case). Plot how the net gain coefficient (i.e. the saturated gain coefficient $g_{sat}(z)$ minus the loss coefficient α) evolves with the normalized intensity (I/I_{sat}). Additionally, assuming that the input intensity is $I_{in} = I_{sat}/2$ plot the evolution of the normalized intensity with the propagation distance z. (2 points)

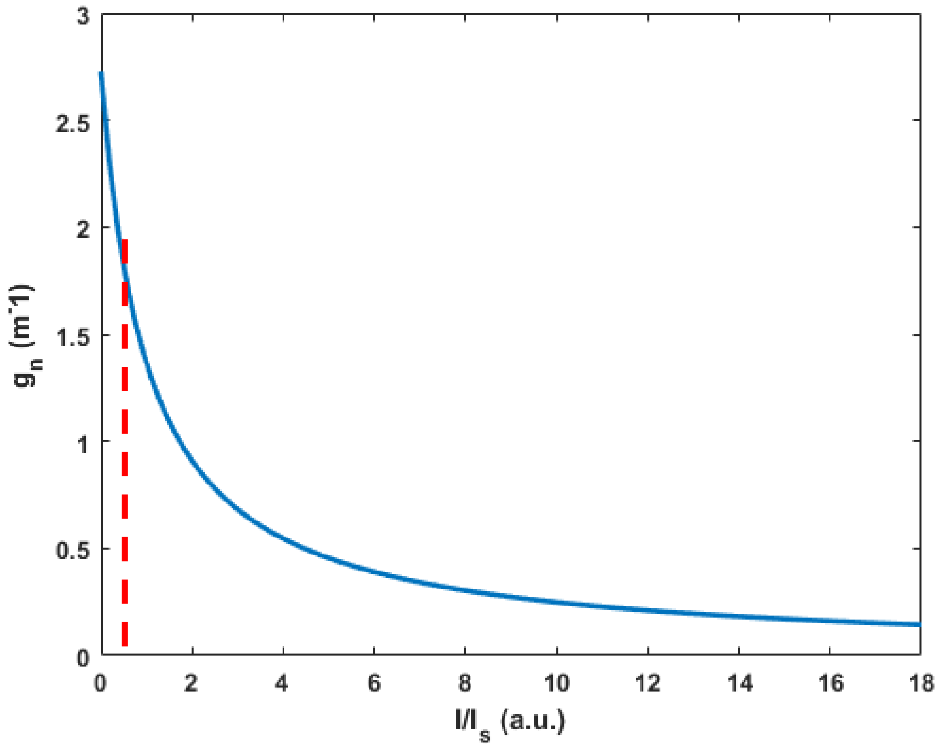
From c)

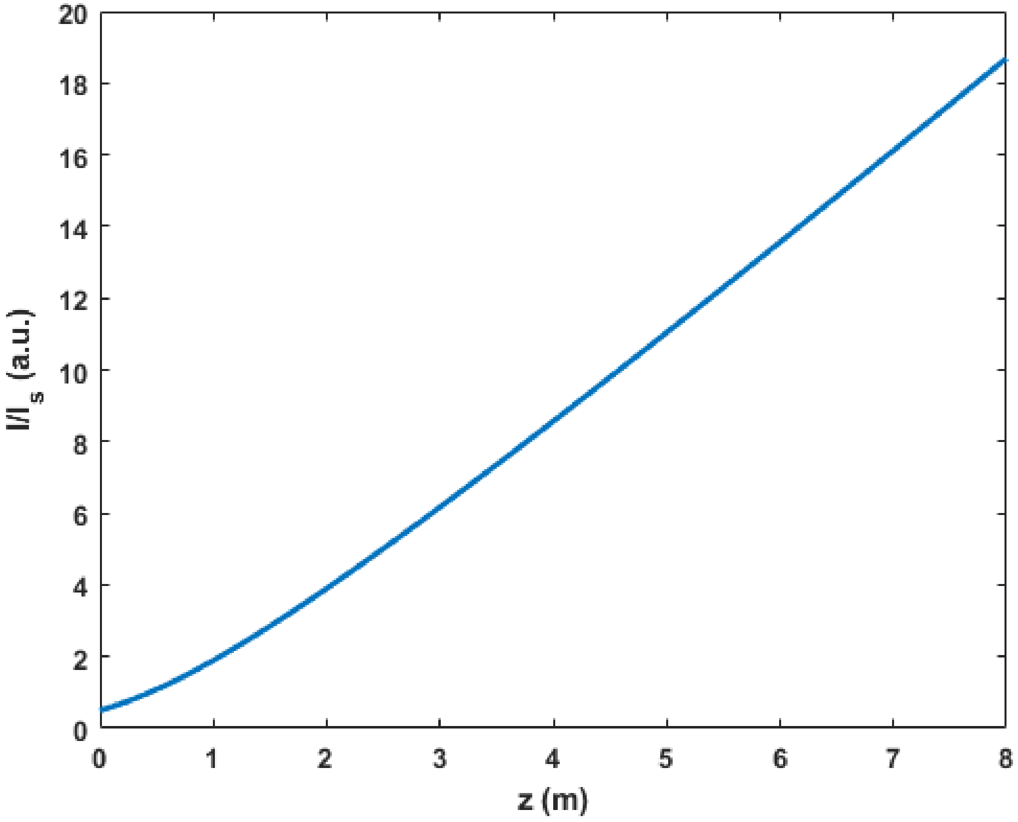
$$G_0 = 15.28 \qquad L = 1\text{ m} \quad \Rightarrow \quad g_0 = \frac{\ln(G_0)}{L} = 2.727\text{ m}^{-1} \qquad \alpha = \frac{g_0}{2} = 1.363\text{ m}^{-1}$$





For the lossless case



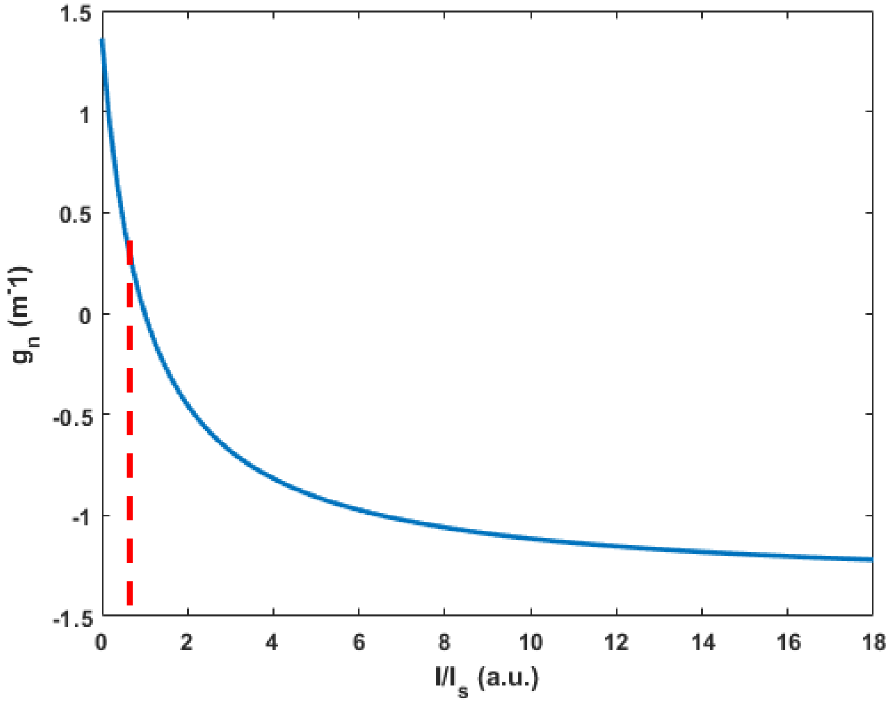


$$g_{net} = \frac{g_0}{1 + I/I_{sat}} = 1.818m^{-1}$$

$$G = \exp(g_{net}\Delta z) = 2.482$$

$$\frac{I}{I_{sat}}(0.5) = G\frac{I}{I_{sat}}(0) = 1.241$$

For the lossless case



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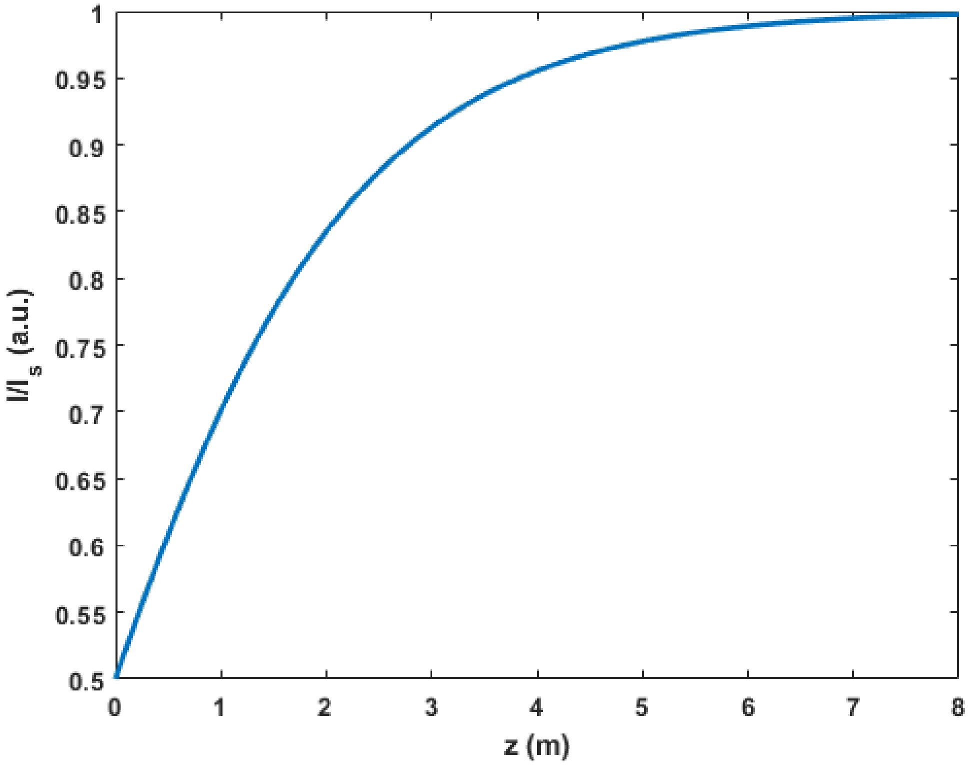

$$g_{net} = \frac{g_0}{1 + I/I_{sat}} - \alpha = 0.455m^{-1}$$

$$G = \exp(g_{net}\Delta z) = 1.256$$

$$\frac{I}{I_{sat}}(0.5) = G\,\frac{I}{I_{sat}}(0) = 0.628$$