Name: ____________________ Date of birth: ____________ Student ID No. __________

Midterm Exam
"Fundamentals of modern optics"
WS 2017/18
to be written on December 18, 8:15 - 9:45 am

Problem 1 – Maxwell's Equations **$2+2+3+2+3 = 12$ points**

a) Write down the Maxwell's equations in material in time domain in general form.

b) Write down the material equations relating the vector fields $\boldsymbol{D}, \boldsymbol{E}$, both in time domain and in frequency domain.

c) Derive the wave equation for the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ in a source-free, non-magnetic, isotropic, homogeneous medium with constant real-valued susceptibility χ and constant real-valued conductivity σ (so that the induced electric current density is $\boldsymbol{j}(\boldsymbol{r}, t) = \sigma \boldsymbol{E}(\boldsymbol{r}, t)$).

d) Derive the dispersion relation $k = k(\omega)$ for a plane wave solving the wave equation from part (c) and find how the complex dielectric function $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$ is defined from χ and σ.

e) Derive the wave equation in frequency domain for the electric field $\boldsymbol{E}(\boldsymbol{r}, \omega)$ in source-free, non-magnetic, non-conducting, inhomogeneous medium with spatially-varying dielectric permittivity

$$\varepsilon(\boldsymbol{r}) = \tilde{\varepsilon} + \boldsymbol{\kappa}\boldsymbol{r},$$

where $\tilde{\varepsilon}$ is some constant value and $\boldsymbol{\kappa}$ is a constant vector.

Problem 2 – Normal Modes and Poynting Vector **$2+3+3 = 8$ points**

A plane wave of frequency ω, in a homogeneous, isotropic, lossy dielectric medium which is given by

$$\boldsymbol{E}(\boldsymbol{r}) = E_0 \cos\left[(\beta + i\alpha)x\right] e^{(\mathrm{i}\beta - \alpha)z} \hat{\boldsymbol{y}} \qquad (*)$$

a) Calculate the corresponding magnetic field $\boldsymbol{H}$ from Maxwell's equations.

b) Consider a low loss dielectric ($\varepsilon'' \ll \varepsilon'$) and find an expression for both β and α in terms of ε' and ε'' such that Equ. (*) fulfills the wave equation.
Hint: In your calculation you should only use the first three terms of Taylor expansion.

c) Now neglect the losses in the medium.
Write down the electric field for the lossless case and determine the direction of the flow of the optical power by calculating the time averaged Poynting vector.

Problem 3 – Beam propagation **$1+1+3+2+2+2 = 11$ points**

a) Write down the expression for the general transfer function $H(\alpha, \beta, z)$, which describes the scalar field propagation in free space along z-direction.

b) Specify the conditions of paraxial (Fresnel) approximation in the spatial frequency domain and derive the transfer function in Fresnel approximation $H_F(\alpha, \beta, z)$ from the general transfer function $H(\alpha, \beta, z)$ in part (a).

c) Consider an initial field distribution in the plane $z = 0$ of the form:

$$u_0(x, z = 0) = A + B\cos\left(\frac{2\pi x}{L}\right), \quad L > \lambda_0,$$

where λ_0 is the vacuum wavelength. Calculate the field distribution after propagation for an arbitrary distance $z > 0$ *without* Fresnel approximation.

d) Show that the field periodically reappears upon propagation up to a constant phase factor and calculate the distance along the z-axis until the first reappearance of the initial field.

e) Specify the conditions of applicability of Fresnel approximation for the field distribution from part (c) and calculate the value of the distance of field reappearance from part (d) under these conditions.

f) Show that for certain distances z the transverse *intensity* distribution will be periodic with twice the spatial frequency of the original field and find the distance along z-axis where this happens for the first time.

Problem 4 – Pulses **$2 + 3 + 2 = 7$ points**

a) Describe how the first three coefficients of the Taylor expansion of $k(\omega)$ with respect to ω at the center frequency ω_0 are connected to physical parameters of pulse propagation and explain their physical meaning.

b) A Gaussian pulse with center frequency ω_1 is launched into a material at $t = 0$. A second Gaussian pulse with center frequency ω_2 is launched into the same material at a later time $t = \Delta t$. The refractive index of the material is given by $n(\omega) = A + B\omega^2$ with $A > 0$ and $B > 0$. State under which conditions the second pulse can catch up to the first one. Derive the expression for the time t at which that happens in terms of ω_1 and ω_2.

c) By analogy of diffraction to pulse propagation in dispersive media argue how the temporal intensity profile $|v(t, z)|^2$ of a pulse envelope $v(t, z = 0)$ looks like for very large z when you neglect dispersion terms higher than second order.

Problem 5 – Propagation of Gaussian Beam **$3 + 3 + 1 + 2 = 9$ points**

The evolution of a Gaussian beam in paraxial approximation is given as

$$v(x, y, z) = A_0 \frac{1}{1 + iz/z_0} exp\left(-\frac{x^2 + y^2}{w_0^2 (1 + iz/z_0)}\right).$$

a) Restructure the provided equation and give explicit expressions for the following terms as functions of z:
i) Amplitude evolution
ii) Beam width evolution
iii) Radius of phase curvature
iiii) The Gouy phase

b) Sketch three diagrams for the evolution of the normalized beam intensity, beam width and radius of phase curvature over propagation distance z and indicate the values of the quantities at the propagation distances $z = -z_0$, $z = 0$ and $z = z_0$.

c) Plot the lines of constant phase of a Gaussian beam in a diagram over x and z. The diagram only needs to cover the region $-z_0 \le z \le z_0$.
What happens for $z \gg z_0$?

d) Explain in your own words how to derive the stability condition for the fundamental gaussian mode in a resonator.

Problem 6 – Fraunhofer diffraction **$3 + 3 = 6$ points**

We consider a one-dimensional slit with its center in the origin of the x-axis. The field directly after the slit is:

$$u(x, z = 0) = \begin{cases} 1 & \text{for } |x| < d/2 \\ 0 & \text{otherwise} \end{cases}.$$

a) Calculate the resulting far field. (Hint: We are only interested in the general behaviour, so leave out prefactors.)

In Figure 1, a transmission mask is shown with the dimensions, indicated by d_{big} and d_{small} for the diameters of the holes and l_{vertical} and $l_{\text{horizontal}}$ for the distances between the holes. They obey the following relation: $l_{\text{vertical}} > l_{\text{horizontal}} \gg d_{\text{big}} > d_{\text{small}}$.

b) Which of these dimensions defines the applicability of the Fresnel approximation and the Fraunhofer approximation, respectively? Explain the condition(s) for both approximations.

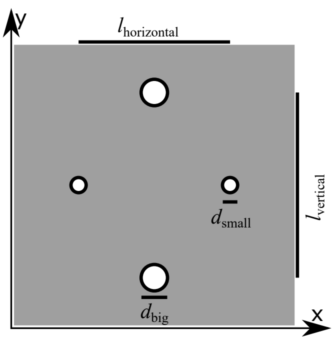

Figure 1: Sketch.