Script "Fundamentals of Laser Physics" – Chapter 3

3. The principle of lasers

The goal of a laser is to emit intense coherent radiation.

In the optical spectral range, thermal sources possess poor spatial and temporal coherence, because their individual emitters radiate independently in all directions of space and their frequency spectrum is stochastic (determined by statistics). However, the spatial and temporal coherence of a light source can be improved through spatial and spectral filtering, respectively. Additionally, a light wave can be amplified by means of stimulated emission. Hence, a straightforward approach to obtain intense coherent radiation is to filter the emission of a thermal source (so that we obtain a low power coherent source) and amplify it by stimulated emission. This approach is, however, very inefficient.

An alternative, more efficient approach follows a concept known from oscillators in radio frequency technology. It can be shown that by applying a feedback to an amplifying element, it is possible to create a system which oscillates by its own. In order to achieve oscillation at a well-defined frequency it is necessary to additionally add a frequency-selecting element. RF technology, therefore, typically uses resonant structures such as oscillating circuits or cavity resonators. The design of an optical resonator will be discussed in a later lecture.

At this point we have to consider that there are competing processes to the stimulated emission during the interaction of light with matter, i.e absorption and spontaneous emission. The one that poses the main challenge is spontaneous emission. In principle, spontaneous emission is a largely useless loss of energy from the active medium adding noise to the emission. Moreover, due to the ν^3 frequency scaling law, the impact of spontaneous emission rapidly increases towards higher (optical) frequencies. Hence, an amplifier based on stimulated emission is much simpler to realize in the microwave spectral region than in the optical spectral range and it becomes practically impossible in the X-ray spectral region.

3.1 Inversion condition / population inversion

The challenge for getting a laser to work is to ensure that stimulated emission is "stronger" than the reversal process (absorption) and the spontaneous emission. In fact, we already know *the condition for optical amplification: the excited state must have a higher population than the ground state*, i.e.:

$$N_2 > N_1 \tag{3.1}$$

But, in thermal equilibrium the population of the different energy states follows the Boltzmann distribution, given by:

$$\left.\frac{N_2}{N_1}\right|_{equilibrium} = \exp\left(-\frac{h\nu}{kT}\right) \tag{3.2}$$

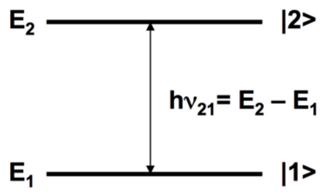
Consequently, even in the best case only an equal distribution of population $N_2 \approx N_1$ is possible; and this only happens if the thermal energy kT is significantly larger than the photon energy $h\nu = E_2 - E_1$. For transitions in the optical spectral range and at room temperature that condition is not fulfilled. Accordingly, the condition given by (3.1) is the inverse of the normal state, and that is why it is called population inversion.

In principle, inequation (3.1) could be fulfilled if a negative temperature is inserted in (3.2). However, particles in thermal equilibrium cannot possess a negative temperature. Please note that this is a rather artificial consideration it is just used herein as an argument to conclude that a medium with population inversion must be a system in disequilibrium.

In nature, all systems tend to their equilibrium state. In this context, all relaxation processes act to reestablish the thermal equilibrium of the system, i.e. they transfer particles in the excited states back to the ground state. In the context of LASERs this relaxation mechanism is the spontaneous emission. Hence, *the realization of a LASER, which demands population inversion, requires a mechanism which maintains the inversion in spite of the relaxation processes*. Such a mechanism is called *the pump process*.

3.1.1 Two-level system / Inversion condition

Let's discuss the pump process in a two-level system:



allowed transitions: $B_{12} \neq 0$

MASER

At first, we consider the special case of excitation in a MASER. For microwave quanta, the photon energy is much smaller than the thermal energy. Thus, following the Boltzmann distribution (3.2.), a nearly equal population in the excited and the ground states is given even in the thermal equilibrium.

$$\frac{N_2}{N_1} \approx 1$$

If we now assume low energy levels, both states of the two-level system will be thermally populated. In particular, it is important to remark that level 2 has a significant population ($N_2 >> 0$).

In principle, in this situation inversion can be achieved simply by removing particles that are in the lower level from the observation volume. In fact, this is exactly the approach of inversion generation in Hydrogen and Ammonia MASERs: the spatial separation of excited and not excited particles by means of spatially inhomogeneous electric or magnetic fields, which create different forces to the different dipole moments of the particles in level 1 and level 2.

Remark: That separation of states might remind you of the Stern-Gerlach-experiment, which is a well-known historic proof of the quantization of the angular momentum.

LASER

For transitions in the optical spectral range and at room temperature the excited state energy level is not/negligibly thermally populated. Hence, a simple separation of the particle as in the case of the MASER does not lead to the desired result (i.e. population inversion).

In this case a population mechanism is required, e.g. by optical excitation. This is considered in the following for a two-level system.

Optical excitation

Let's consider, step by step, which processes populate and de-populate level 2 and recap the equations we know from earlier lectures.

A decrease of population density in level 1 through absorption is described by:

$$\left.\frac{dN_1}{dt}\right|_{Absorption} = -B_{12} \cdot \rho(\nu) \cdot N_1 \quad , \tag{2.3a}$$

which leads to an increase of population density in level 2 through absorption:

$$\left.\frac{dN_2}{dt}\right|_{Absorption} = -\left.\frac{dN_1}{dt}\right|_{Absorption}$$

(Because we only consider two levels, i.e. whatever particle leaves level 1 has to end up in level 2).

Of course, the two emission processes act in the opposite direction (i.e. they depopulate level 2 and populate level 1). Therefore, a decrease of population density in level 2 through spontaneous emission is given by:

$$\left.\frac{dN_2}{dt}\right|_{Spontaneous\ Emission} = -A_{21} \cdot N_2 \quad . \tag{2.3c}$$

And a decrease of population density in level 2 through induced emission is given by:

$$\left.\frac{dN_2}{dt}\right|_{Induced\ Emission} = -B_{21} \cdot \rho(\nu) \cdot N_2 \quad . \tag{2.3b}$$

From these simple equations we can learn that the product of the Einstein B-coefficient times the spectral energy density possesses units of "per seconds" $[B_{12}\cdot\rho(\nu)] = \text{time}^{-1}$.

Thus, this product can be understood as a rate, i.e. as a probability per unit time of a particle making a transition (herein absorption due to B_{12}).

With that knowledge we can define the so-called transition probabilities:

- from |1> to |2> due to absorption:

$$W_{12} = B_{12} \cdot \rho(\nu)$$

- from |2> to |1> due to induced emission:

$$W_{21} = B_{21} \cdot \rho(\nu)$$

- from |2> to |1> due to spontaneous emission:

$$S_{21} = A_{21}$$

In order to gain insight into the evolution of the population densities in the two-level system, we can sum up all the contributions of the three processes, resulting in an overall change of population in level 2 (under the assumption: $B_{12} = B_{21}$):

$$\begin{aligned} \frac{dN_2}{dt} &= B_{12} \cdot \rho(\nu) \cdot (N_1 - N_2) - A_{21} N_2 \\ &= W_{12}(N_1 - N_2) - S_{21} N_2 \end{aligned} \tag{3.3a}$$

This pump process/inversion build-up continues until a steady state is reached, whereby that steady state is, in fact, a dynamic equilibrium (i.e. emission and absorption processes permanently occur) i.e.:

$$\frac{dN_2}{dt} = 0$$

Hence, in steady state, the ratio of the population densities in level 2 and level 1 is given by:

$$\frac{N_2}{N_1} = \frac{W_{12}}{W_{12} + S_{21}} < 1 \quad , \tag{3.3b}$$

which is always smaller than 1, i.e. even for an infinitely strong optical excitation (W_{12} -> infinity), inversion is not observed. This is because, in this extreme case, stimulated emission occurs at the same rate / with the same probability as absorption. Hence, optical excitation is not a suitable pump process to achieve population inversion in a two-level system.

One may ask about other mechanisms to populate level 2, e.g. electron collision excitation. According to quantum mechanics, the transition probability for any processes is identical to its reversal process and, thus, we can write:

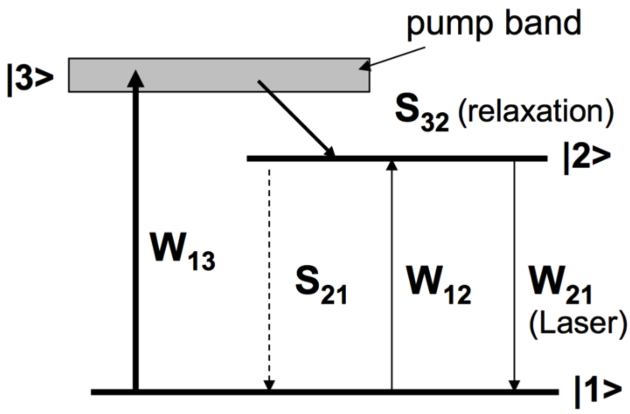
$$W_{ij} = W_{ji} .$$

Hence, inversion generation is simply not possible in a two-level system assuming non-degenerated energy states.

3.1.2 Three-level system

We have just learned that the *creation of population inversion in a two-level system* via absorption (i.e. optical excitation) *fails due to* the presence of a reverse process with the same probability: *stimulated emission*. One possible way of avoiding this reverse process is to empty the upper level as fast as possible, i.e. before the reverse process can take place.

This can be done with a three-level system as illustrated in the figure below. In this scheme, absorption of photons is used to excite the particles and send them to a so-called pump band (level 3) and not to the upper laser level 2. Please note that, in order to do this, the injected photons (so-called pump photons) will have a different wavelength (i.e. higher energy) than the laser photons (i.e. the photons generated by the transitions between level 2 and 1). A fast relaxation from level 3 to level 2 follows that excitation immediately and, therefore, there is no population accumulated in level 3 available for de-excitation back to the ground state via stimulated emission induced by the pump photons. On the contrary, level 2 accumulates a significant population. In this level stimulated emission, spontaneous emission and absorption occur at the signal wavelength, i.e. for the transition between level 2 and level 1 (it will be later termed laser transition). Since, for the time being, there are no photons at the laser wavelength present in the medium (remember that only pump photons at a different wavelength are injected in the medium), stimulated emission cannot take place and inversion population can be obtained. The relevant processes and their corresponding transition probabilities are summarized in the following picture.

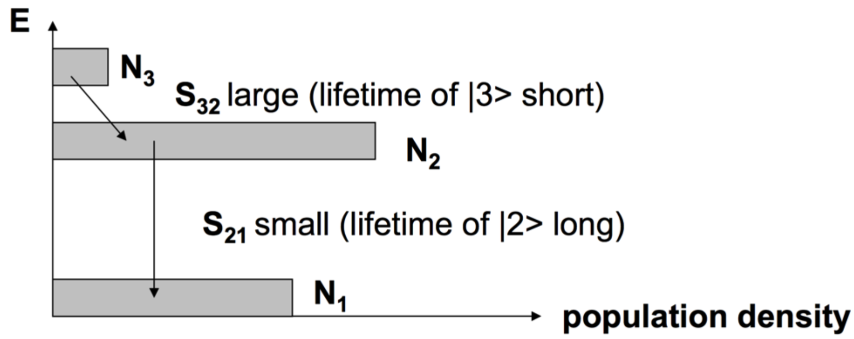


The pump process (absorption of photons with a photon energy corresponding to the energy difference between level 1 and level 3) is characterized by a pump rate and transition probability W_{13}. The fast relaxation from 3 -> 2 (that empties the pump band) is described by the relaxation rate S_{32}. In most cases that relaxation is a non-radiative transition. Excitation and de-excitation between level 2 and level 1 via spontaneous emission, absorption and stimulated emission happen with the rates S_{21}, W_{12} and W_{21}, respectively. The (laser) photon energy corresponds to the energy difference between level 2 and level 1.

The conditions for laser oscillation in a 3-level system can be summarized as follows:

- A fast relaxation 3 -> 2 is needed to avoid the reverse process during pumping.
- In addition, a weak spontaneous emission rate S_{21} is required for the accumulation of population in level 2 and, consequently, for the creation of population inversion between level 2 and level 1.

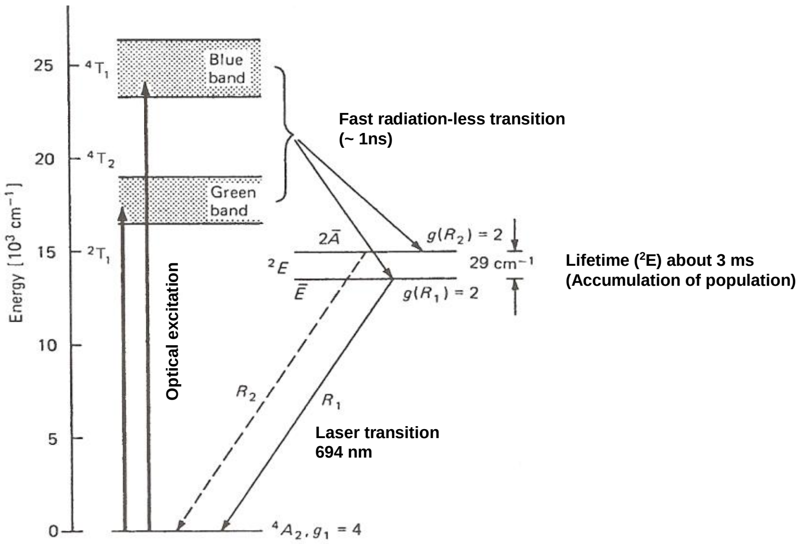
The distribution of population density in a three-level system can be illustrated in a histogram representation, where the length of the bars represents the amount of population in each of the considered levels.



As can be seen, the inversion condition (i.e. that level 2 is stronger populated than level 1) requires, in the best case, that at least 50% of the particles are in the excited states (i.e. level 2 and level 3), but in reality typically more than 50% of the particles need to be in the excited states to obtain inversion. This is the reason why three-level schemes typically have high demands on pumping schemes to achieve inversion or laser emission in general.

It is worth mentioning that the three-level system described above cannot exist in free atoms / particles, as the selection rules for dipole transitions would be disobeyed. An allowed transition requests, e.g. a change of the angular momentum by $\Delta L = \pm 1$. Assuming that the transitions 1 <-> 3 and 2 <-> 3 are allowed, then 2 <-> 1 has to be forbidden. However, the selection rules are only strictly valid for isolated particles. However, in reality particles are in interaction and, hence, the selection rules are relaxed. This implies that a forbidden transition in an isolated particle becomes just a transition with a low likelihood of happening for a particle in interaction with others. This, in turn means in our three-level system that the lifetime of the particles in level 2 is not infinity (as it would be if it were a forbidden transition) but it is just very long. Thus level 2 becomes what it is known as a metastable state. This is exactly the condition we have just identified for the creation of population inversion in a 3-level system and it is an important part of the working principle of a ruby laser:

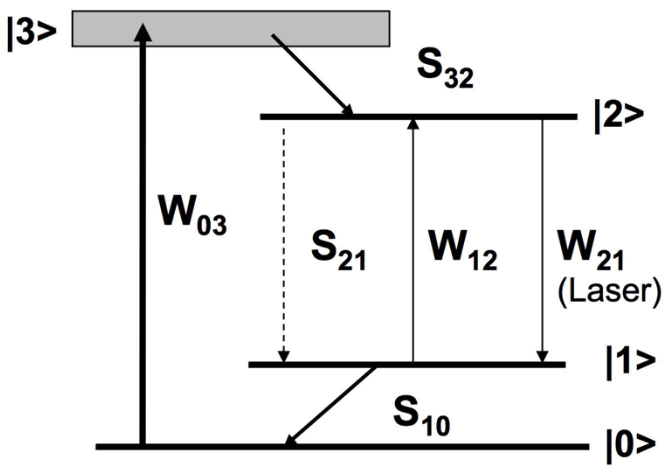
In the melt of Al_2O_3 (known as sapphire), a tiny percentage of Cr_2O_3 is given, resulting in the reddish color of ruby. Below, the energy level diagram with all relevant transitions in a ruby laser is illustrated, showing that it is a three-level system.



As the lower laser level in a three-level system is the ground state, an important feature of 3-level systems is that >50% of the laser active particles have to be excited. This usually requires a significant amount of pump power. It would be more appealing to have an energy level system with a lower laser level which is not thermally populated. Indeed, that is the basic idea of a four-level system.

3.1.3 Four-level system

The main difference to a 3-level system is that the lower laser level here is not (significantly) thermally populated. I.e. the ground state (now level 0) and the lower laser level 1 are separated and they have a sufficiently large energy separation.



Most of the relevant processes are identical to the ones found in 3-level systems and can be summarized as follows:

- The pump rate W_{03} describes the absorption process 0 -> 3.
- A fast decay to level 2 with the rate S_{32} de-populates the pump level 3 before stimulated emission back to level 0 occurs, leading to the population of level 2 (upper laser level).
- Between level 2 and level 1 we can observe the known transitions: spontaneous emission S_{21}, stimulated emission W_{21} and absorption W_{12}.

- Finally, we need to take into account the decay rate S_{10}, which describes the de-population of level 1 back to the ground state.

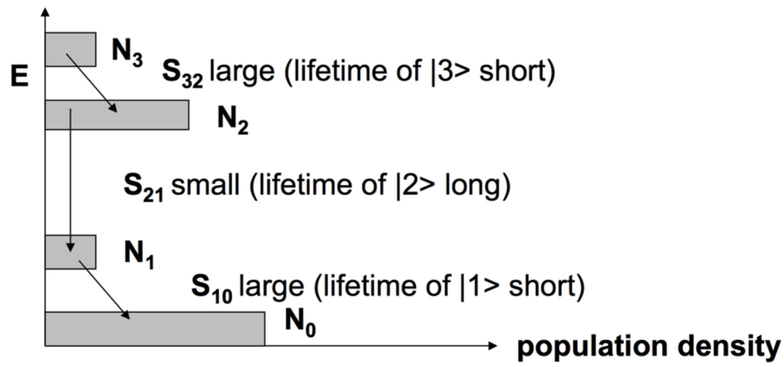
A fast relaxation 3 -> 2 is needed to avoid reversal process but, in contrast to the 3-level scheme, a small decay rate between level 2 and level 1 S_{21} is just a favorable but not a necessary condition.

Another important difference to the 3-level system is the fact that it is not required to pump the upper laser level faster than it decays to achieve inversion. The reason for this advantage is, of course, that the lower laser level 1 is not thermally populated. Hence, a single excited particle in level 2 leads already to inversion between level 2 and level 1.

The bottleneck for the realization of 4-level lasers is the limited decay rate of the lower laser level (S_{10}), i.e. the emptying of the lower laser level. If this is not fast enough, then the accumulation of population in level 1 can annihilate the inversion. Hence, an essential condition on the relaxation rates for achieving continuous inversion (stationary regime) is given by:

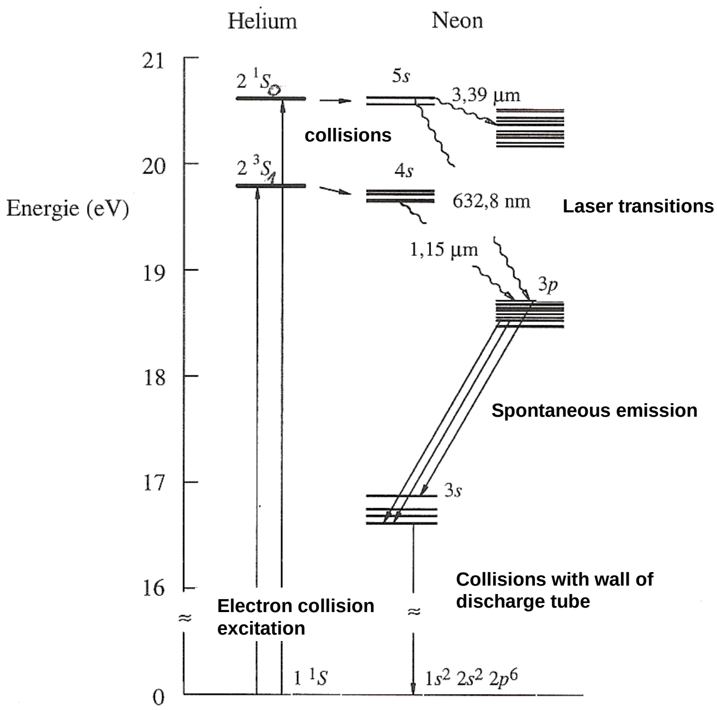
$$S_{10} > S_{21} \tag{3.5}$$

Similar to the 3-level system, the distribution of population densities among the levels can be illustrated in a histogram view:



The most prominent example of a 4-level laser is the Helium-Neon-laser, which simplified energy level diagram is shown below. It works as follows:

- The first step is an electron collision excitation of Helium atoms in a gas discharge tube, resulting in a population of the 2^1S_0 and 2^3S_1 states around 20eV above the ground state (pump process).
- Later on, collisions of the so-called “second kind” transfer this excitation to Neon 5s and 4s states (the ground state electronic configuration is $1s^2\ 2s^2\ 2p^6$, i.e. one of the 2p electrons is excited). With that population inversion with respect to levels 4p and 3p is immediately present.
- Laser transitions at 3.39µm, 633nm and 1.15µm wavelength are possible. The most commonly used one is the 633nm, i.e. the transition between the 5s and 3p states.
- The lower laser level (3p) is de-populated (to 3s) via spontaneous emission.
- Finally, the particles in 3s return to the ground state configuration by collisions with the wall of the discharge tube. This is, in fact, the reason why these tubes are rather thin (diameter <2mm), to provide a large collision surface.



3.2 Laser Amplifier and Amplified Spontaneous Emission

Before we can move on to laser oscillators, we should discuss the principles of laser amplifiers. What we know up to now is that stimulated emission allows for an exponential growth of an incoming light wave. Hence, we can define a gain coefficient g(ν):

$$g(\nu)=\sigma_\nu\cdot(N_2-N_1)=\sigma_\nu\cdot n \tag{3.6}$$

with

$$n=(N_2-N_1)$$

being the inversion density.

Using equation (2.13), the evolution of the intensity in an amplifying medium can be described by

$$\frac{I_\nu(x)}{I_\nu(0)}=\exp(g(\nu)\cdot x) \tag{3.7}$$

The overall gain, also known as gain factor, G(ν) in an amplifier of length l is given by:

$$G(\nu)=\frac{I_\nu(l)}{I_\nu(0)}=\exp(g(\nu)\cdot l)=\exp(\sigma_\nu\cdot n\cdot l) \tag{3.8}$$

Due to the exponential function of the gain factor, the frequency dependence of the gain factor G(ν) and the gain coefficient g(ν) can strongly differ from each other. In particular, in active media with high gain factors:

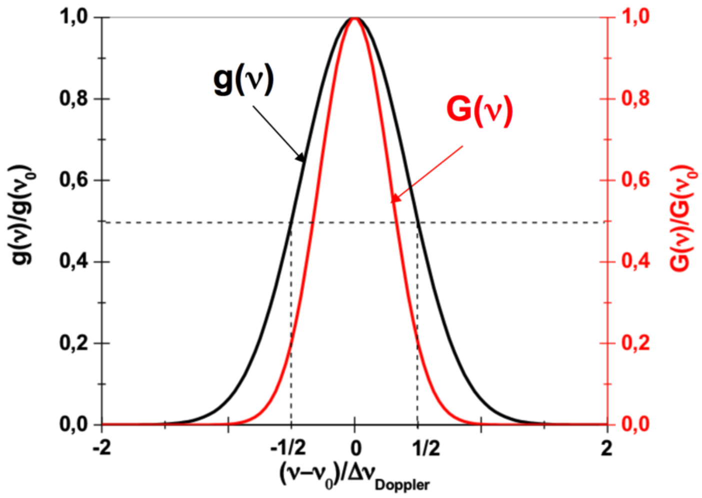
$$g(\nu_0)\cdot l = \sigma(\nu_0)\cdot n \cdot l >> 1$$

the gain factor can shrink towards the resonance frequency. This leads to a spectral narrowing of the output signal, an effect known as gain narrowing.

If the gain coefficient g(ν) possesses a Gaussian distribution, then a simple relation between the gain factor and the gain coefficient can be derived, which is:

$$\frac{\Delta g(\nu)}{\Delta G(\nu)} = \sqrt{\ln(G(\nu_0))}$$

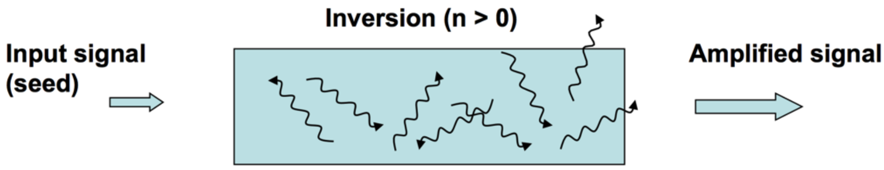
where Δg(ν) is the full-width at half maximum (FWHM) of the gain coefficient g(ν) and ΔG(ν) is the FWHM of the gain factor G(ν). The following plot shows these two functions for a given peak gain factor $G(\nu_0)$ equal to 10.



Remember that the gain coefficient g(ν) has the same frequency dependence as the spontaneous emission. However, as implied by the figure above, the amplified light possesses a narrower spectral distribution than the spontaneously emitted light.

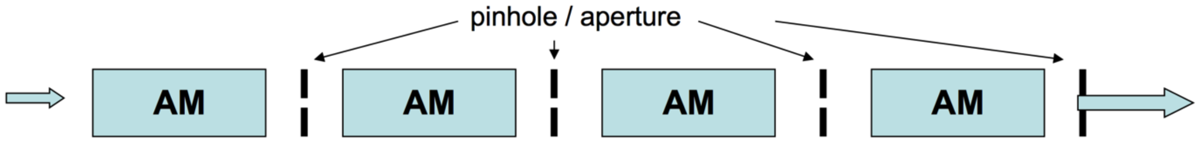
Apart from these considerations, *we also need to consider that an increasing signal will reduce the inversion*, i.e. the population distribution among the relevant energy levels, since more and more particles will get removed from level 2 due to stimulated emission. As a consequence *the gain decreases* (because the inversion decreases), *a scenario known as amplifier saturation or gain saturation.* So far, in our discussions we have disregarded the influence of the signal on the population level by assuming a small signal (even at the end of the amplifier). This case is known as small-signal amplification.

As a matter of fact, spontaneous emission is also present in an amplifier and it is amplified together with the input signal, creating amplifier noise.

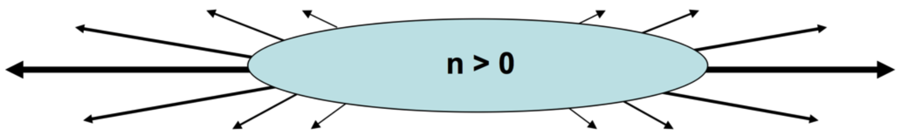


Spontaneous emission is particularly critical at the beginning of the amplifying medium, as it is these spontaneously emitted photons the ones that experience the whole gain of the active medium over its length. This so called amplified spontaneous emission (ASE) is particularly pronounced if the product of the gain coefficient and the amplifier length is large (i.e for large gain factors).

Approaches to reduce ASE are: spectral filtering, temporal gating, polarization filtering and spatial filtering, e.g. by placing apertures between the amplification stages.

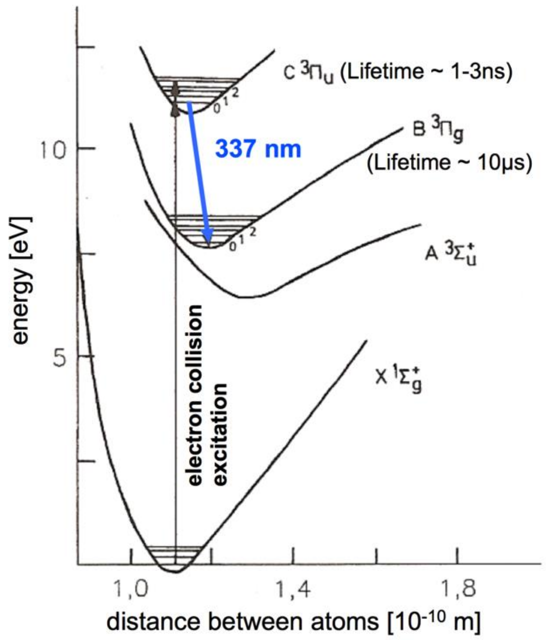


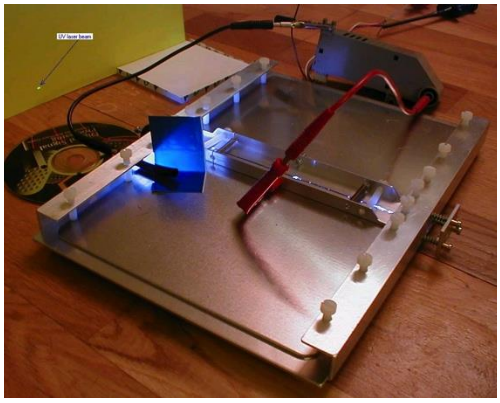
Let's assume that inversion in a certain active volume results in a large gain factor G. We can expect ASE emission even without any input signal. In fact, it is possible that the probability for induced processes grows beyond the probability for spontaneous processes. In that case, ASE dominates over the spontaneous emission, and a spectral narrowing of the emitted spectrum can be observed. Besides, if the inverted volume is not a sphere, its spatial radiation characteristic / pattern will be anisotropic. Such an emitter is called superradiant source.



In summary, the dominance of stimulated emission over spontaneous emission leads to a more intense emission and to a more "monochromatic" / narrower spectral distribution.

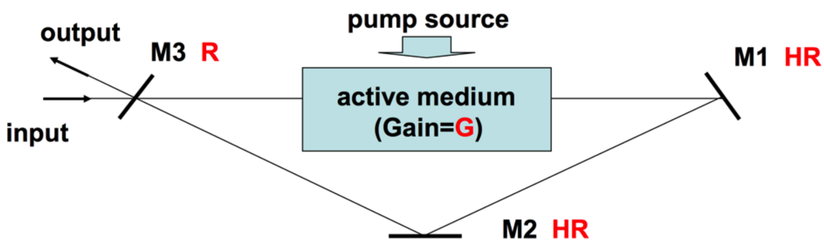
Nitrogen can offer an enormous gain of 10^{10} over 1m in a discharge tube. This allows getting a superradiant source in the UV spectral range (337nm). The excitation process is shown in the energy level diagram below. It is noteworthy that the unfavorable excitation process (compare the lifetimes of the upper and lower laser levels) leads to a self-termination of the inversion and, hence, only pulsed operation is possible.





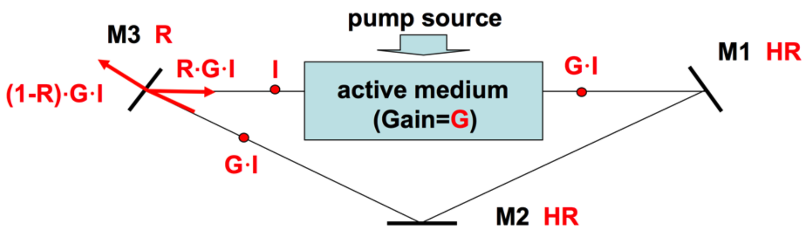
3.3 Laser Amplifier with feedback / Laser oscillators

Not all types of laser amplifiers can offer such enormous gain as the Nitrogen laser does. In reality, one has to deal quite often with gain factors of just 1.1 or even less. There is a way to get around the problem of a limited gain, though. The main idea is to effectively increase the gain and, therefore, extract a considerable power level by using the active medium several times. This can be realized by placing the active medium in a ring cavity, as illustrated in the figure below.



Coupling into and out of that cavity is possible through the partially reflective mirror M3. Mirrors M1 and M2 are high reflective and close the beam path of the ring cavity.
For the time being we will disregard the ability of light to interfere and, hence, we just consider the power or intensity evolution in such a mirror arrangement.

Assuming a radiation field with intensity I in front of the active medium (AM), which offers a gain factor G, the intensity at the output of the AM is G·I. In this case the yield (energy/intensity given by the AM to the light field) is given by (G-1)·I.

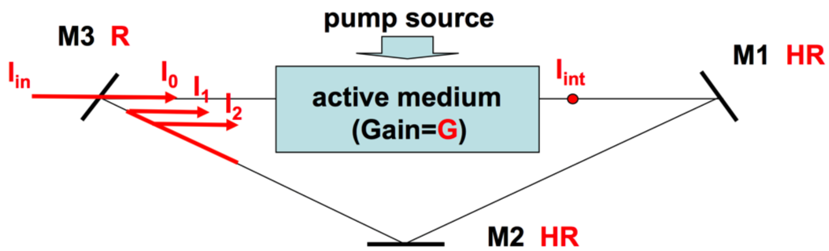


The reflectivity of the out-coupling mirror M3 is R. This means that the intensity that is fed back into the cavity (the so-called regenerated intensity) is R·G·I. Consequently, the intensity leaving the cavity (loss / leakage) is (1-R)·G·I.

If the yield is smaller than the losses, i.e. $(G-1)\cdot I < (1-R)\cdot G\cdot I$ or, in short, if $\underline{R\cdot G < 1}$, then the intensity in the cavity will decay with each round trip.

In the following consideration we assume a constant intensity I_{in} going through mirror M3 into the ring cavity:

$I_0 = (1-R)\cdot I_{in}$



With that, the circulating intensity at the output of the AM is:

$$I_{int} = G\cdot(I_0 + I_1 + I_2 + \ldots), \tag{3.9a}$$

with I_n denoting the intensity contribution after the n-th round trip.

From the figure above we can learn that there is a simple recursion relation between I_n and I_{n-1}, which is given by:

$I_n = R\cdot G\cdot I_{n-1}$.

Hence, equation (3.9a) can be written as a geometric series

$$I_{int} = G\cdot I_0\cdot(1 + R\cdot G + (R\cdot G)^2 + \ldots)\ , \tag{3.9b}$$

which converges to:

$$I_{int} = \frac{G\cdot I_0}{(1-R\cdot G)}\ , \tag{3.9c}$$

if $R\cdot G < 1$.

Thus, a partial feedback of the amplifier output signal back to the amplifier input leads to an enhancement of the amplification by a factor $(1-R\cdot G)^{-1}$. This approach is known as a regenerative amplifier.

If the gain factor G is larger than 1, then, in principle the amplification offered by regenerative amplifiers can approach infinity; one just needs to achieve that $R\cdot G \rightarrow 1$.

Let's assume now the case when

$$R\cdot G = 1, \tag{3.10}$$

i.e. the yield is equal to the losses. In this case an existing intensity will be maintained in the cavity (without the need for any injection of I_0). In reality, however, it is rather unlikely that equation (3.10) is fulfilled exactly. Hence, usually either $R\cdot G < 1$ or $R\cdot G > 1$ is given. The case $R\cdot G < 1$ has already been discussed before, see e.g. equation (3.9c).

So, let's assume now that $R\cdot G > 1$. In this case it can be observed that the intensity increases with each round trip. This also implies that the series (3.9b) diverges, which would lead to a violation of the energy conservation law. As such a violation is not possible, we need to include saturation effects in our considerations as soon as the intensity increases, i.e. the intensity influences the inversion and, consequently, the gain is reduced. Hence, a more accurate starting condition is $R\cdot G_0 > 1$, where G_0 is the small-signal gain factor (low intensity gain factor).

What happens in such case?
Even without an input signal, spontaneous emission will be present in the active medium. Thus, a fraction of that spontaneous emission will be able to pass through the ring resonator several times. Initially the yield is larger than the losses and, hence, a build-up of intensity from noise is observable. However, now we need to consider that the active medium (or more precisely, the gain of the active medium) is saturable:

$$G(I_2) < G(I_1) < G_0 \text{ with } I_2 > I_1 > 0, \tag{3.11}$$

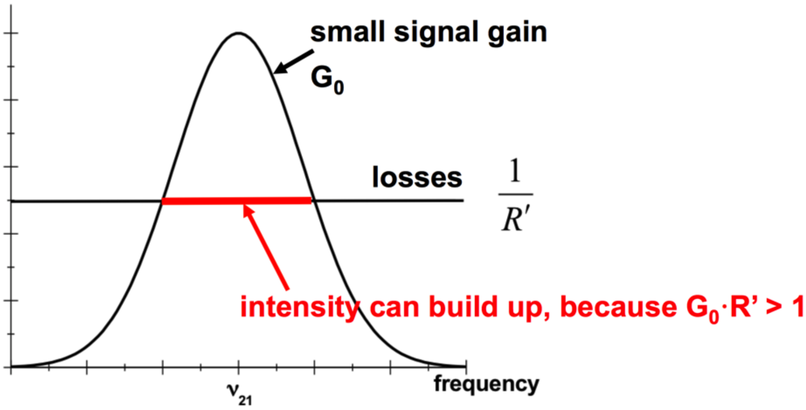
i.e. *the higher the intensity, the lower the gain, because the radiation field depletes the inversion*.

This saturation behavior allows exactly fulfilling the condition $R\cdot G = 1$ (3.10), i.e. that the yield becomes equal to the losses and, therefore, the intensity in the cavity is sustained without any violation of the energy conservation. Hence, the intensity will increase to a value called stationary intensity I_S and it will remain indefinitely at this value given by

$$G(I_S) = R^{-1}.$$

A more refined consideration of the problem includes other loss mechanisms in addition to the out-coupling loss R. These additional loss terms can have their origin in diffraction, scattering and absorption and are all contained in the parameter V (with $0 < V < 1$). Consequently, instead of $R\cdot I_{int}$, now just the intensity $R\cdot V\cdot I_{int}$ is fed back to the active medium. In order to consider that into the discussion, we just need to replace R by $R' = R\cdot V$ in the equations above.

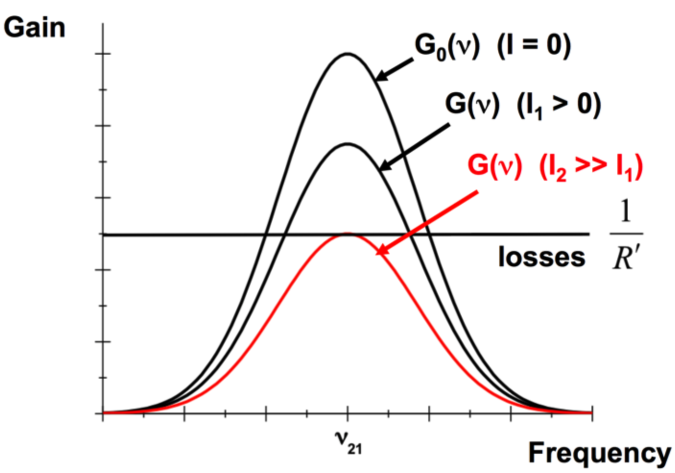
Assuming, that the out-coupling and all the other loss mechanisms have a weak frequency dependence (compared to the gain profile), 1/R' will be represented by a horizontal line in the frequency domain. This is the so-called loss line in the following plot. Within the frequency range marked in red, the initial yield (before saturation takes place in the AM) is larger than all the losses and, thus, the intensity can build up from noise.



To discuss what happens next, we need to distinguish between homogeneous and inhomogeneous broadening of the considered transition.

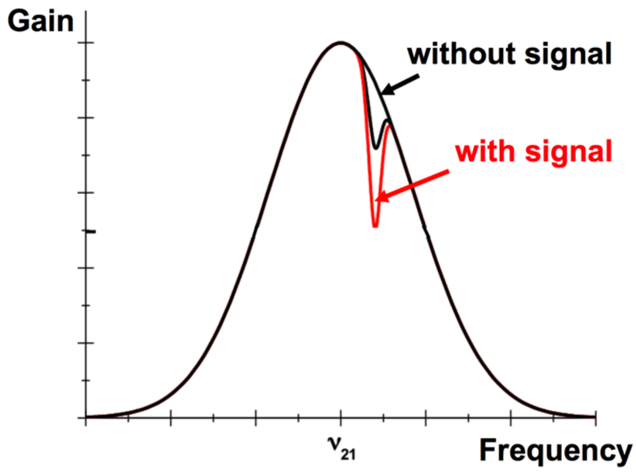
A) Homogeneous broadening

A homogeneous saturation of the gain profile can be observed in this case, because all inverted particles contribute to the amplification in the same manner. As soon as the intensity increases, the gain is reduced and, hence, the spectral region which fulfills $R' \cdot G > 1$ gets narrower and narrower. Actually, at frequencies at which the yield falls below the losses, the intensity decays with time (i.e. round-trips). Ultimately, this behavior results in only one contact point between the loss line and the gain curve, i.e. emission at only one frequency occurs.



B) Inhomogeneous broadening

As we have seen before, inhomogeneous broadening allows defining classes of emitters/absorbers with different transition frequencies. Hence, if the incoming light has a certain frequency, then it will interact only with the class of atoms responsive to that frequency. This way, a selective decrease of the gain near the frequency of the incoming signal can be observed. This is an effect known as *spectral hole burning*. The width of the hole is similar to the homogeneous linewidth (e.g. natural linewidth or pressure width)



If the light emission in such an inhomogeneously broadened system starts from noise, a superposition of homogeneous contributions can be observed, i.e. an emission in multiple spectral lines. On the contrary, within a homogeneous class or, in general, in the case of homogeneous broadening, only one contact point (gain/losses) is expected. This implies that the system adjusts itself so that only oscillation at that particular frequency (within the homogenous band) occurs, i.e. emission of monochromatic radiation at the frequency ν_{opt}.

The stationary intensity of that monochromatic radiation is given by:

$$G(\nu_{opt}, I_S) = \{R(\nu_{opt}) \cdot V(\nu_{opt})\}^{-1}$$

Interference effects

In the discussion made so far we have just added intensities or photon numbers. However, light has the ability to interfere (either constructively or destructively). Hence, if we want to take interference effects into account, it is necessary to consider the electric field strength instead of the intensity. Moreover, the phase of the regenerated light is very important.

Thus, in our formalism, we have to replace the quantities R, G, V (related to intensity) by r, g, v, which are valid for electric fields:

$$r = \sqrt{R}, \quad g = \sqrt{G}, \quad v = \sqrt{V} \; .$$

Furthermore, we have to consider any phase shifts that the electric field might undergo while travelling through the system (e.g. introduced by the active medium or simply by free-space propagation in the cavity). Thus, we define the electric field at the input of the active medium as:

$$E_1 = E_0 \cdot \cos(\omega \cdot t + \psi)$$

where

$$\omega = 2\pi\nu.$$

This way, the electric field after one round-trip (i.e. at the same spatial position) in the ring cavity will be:

$$E_2 = r \cdot g \cdot v \cdot E_0 \cdot \cos(\omega \cdot t + \psi - \phi)$$

with ϕ being the overall phase shift per round trip.

Without any external field (i.e. starting from noise) the stationary operation regime requires self-consistency, which is given by fulfilling the following condition:

$$E_2 = E_1$$

The self-consistency condition is fulfilled for all times, only if the following further two conditions are satisfied:

$$r \cdot g \cdot v = 1$$

corresponding to yield = losses, and

$$\cos(\omega \cdot t + \psi) = \cos(\omega \cdot t + \psi - \phi).$$

This last condition implies constructive interference with the field from the previous round trip. Such condition is fulfilled if the sum of all phase shifts per round trip is a multiple integer of 2π, i.e.

$$\phi = N \cdot 2\pi = \omega \cdot \tau_R + \phi_0 \tag{3.12 a}$$

where τ_R is the round trip time in the resonator, ϕ_0 is the sum of the phase jumps (e.g. at the mirrors) and N is an integer number. However, the main contribution to the phase shift per round trip originates from the term $\omega \cdot \tau_R$, which is directly related to the optical path length in the cavity l.

In a ring cavity the round trip time is:

$$\tau_R = \frac{l}{c}.$$

On the other hand, in a linear cavity it is given by:

$$\tau_R = \frac{2 \cdot l}{c}.$$

Consequently, the product of the angular frequency times the round trip time is given by (in a ring cavity):

$$\omega \cdot \tau_R = \omega \cdot \frac{l}{c} = 2\pi \cdot \frac{l}{\lambda}.$$

Therefore, the change of the phase shift per round trip with the angular frequency ν or the wavelength λ is:

$$d\phi = -\left(2\pi \cdot \frac{l}{\lambda^2}\right) \cdot d\lambda = 2\pi \cdot \left(\frac{l}{\lambda}\right) \frac{d\omega}{\omega}$$

Considering the second part of the equation and taking into account that real cavity dimensions are in the order of 1m, we can derive that, in the visible spectral range, the quotient

$$\frac{l}{\lambda_{VIS}} \approx 10^6$$

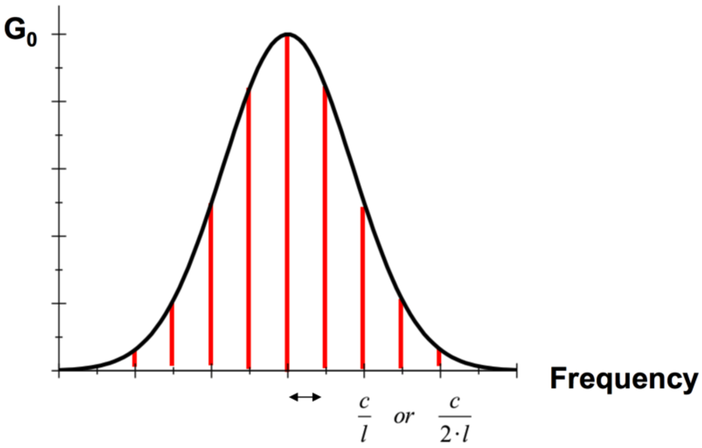
is a rather large quantity. Hence, starting from any arbitrary frequency, a very small relative change in frequency ($d\omega/\omega$) will suffice to satisfy (3.12 a).

Let's assume that we have found a frequency ω_0 that satisfies (3.12 a), then also other frequencies given by:

$$\omega_{\pm N} = \omega_0 \pm 2\pi \cdot N \cdot \frac{c}{l}$$

will comply with the self-consistency condition. These frequencies $\omega_{+/-N}$ are called eigen-frequencies of the cavity or, alternatively, *longitudinal modes* of the cavity.

To conclude, not all frequencies can start to oscillate in a cavity even if the active medium (AM) would provide enough gain for them. Only the longitudinal modes can oscillate, as illustrated in the following figure.



The existence of these eigen-frequencies suggests considering the mirror arrangement as an optical resonator. The damping of that oscillator (due to losses) can be fully compensated by stimulated emission (gain) and, hence, long term oscillation on one or more of its eigen-frequencies is possible.

Actually, this concept leads to the textbook definition of a *LASER*:

An optical resonator, which is de-damped on one or more of its eigen-frequencies by means of stimulated emission. To achieve this, population inversion (i.e. gain) is required, which can be realized by a pump process.