$\nabla \times H = -i\omega\varepsilon_0 E$

Midterm Exam
"Fundamentals of modern optics"
WS 2014/15
to be written on December 15, 8:15 - 9:45 am

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Problem 1 – Maxwell's equations
$3+2+3+1=9$ points

a) Write down Maxwell's equations in time domain, in its general form. Furthermore, write down the constitutive equations for auxiliary fields $\mathbf{D}$ and $\mathbf{H}$, in time domain (material is dispersive, linear, isotropic, and non-magnetic). inhomogeneous

↳ linear relation btwn D&E

b) Write down Maxwell's equations in frequency domain in a linear, homogeneous and isotropic dielectric medium in absence of free charges and current density ($\rho = 0$ and $\mathbf{j} = 0$).

c) Derive the wave equation in the frequency domain for the electric field in a linear, homogeneous and isotropic dielectric medium in absence of free charges and current density ($\rho = 0$ and $\mathbf{j} = 0$).

d) Give the formula of the time averaged Poynting vector for monochromatic fields.

4.5

Problem 2 – Poynting Vector and Normal Mode
$2+2+1+3=8$ points

Consider a monochromatic plane wave of frequency ω, propagating in a homogeneous isotropic lossy dispersion-less dielectric medium of relative permittivity $\epsilon = \epsilon' + i\epsilon''$ (where $\epsilon', \epsilon'' > 0$ and $\epsilon' >> \epsilon''$). Its electric field has the form $\mathbf{E}_r(\mathbf{r}, t) = E_0\, \mathbf{e}_x\, e^{-k''z} \cos(k'z - \omega t + \phi)$, where the subscript r is used for the real valued fields.

a) Express k' and k'' (approximately) with respect to ω, ϵ', and ϵ''.

b) Find the real valued magnetic field $\mathbf{H}_r(\mathbf{r}, t)$.

c) Write down the formula for the instantaneous Poynting vector $\mathbf{S}_r(\mathbf{r}, t)$.

d) Find the time averaged Poynting vector using the formula $\langle \mathbf{S}_r(\mathbf{r}, t)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} \mathbf{S}_r(\mathbf{r}, t)dt$. You also can directly use the formula for time averaged Poynting vector, which uses the complex amplitudes. Your answer should be as simplified as possible.

Hint: You may, in all the steps of your calculations, use the complex representation as a mean to simplify your calculations. However, the final answers have to be real-valued physical quantities.

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Problem 3 – Beam propagation
$3+3+3=9$ points

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Given is the field directly behind a two dimensional phase mask

$$u_0(x, y, z=0) = A\exp\left(-i\pi\frac{x^2+y^2}{\lambda\, f}\right),$$

where $f > 0$. The field is propagating through vacuum.

a) Calculate the spatial frequency spectrum $U_0(\alpha, \beta; z = 0)$.

b) By introducing the paraxial approximation, derive the free space transfer function ($H_F(\alpha, \beta; z)$). Indicate propagating and evanescent wave regions.

c) Calculate the field $u(x, y, z = f)$.

$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\omega t}\, dt = \frac{1}{2\pi}$

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$

2+2+2=6

6 **Problem 4 - Gaussian beam** 2 + 2 + 2 = 6 points

A lens of focal length f_1 is placed at a distance $d = f_1$ from the waist of a Gaussian beam.

a) Use the ABCD formalism to find the position of the waist and the Rayleigh range of the gaussian beam after the lens.

A second lens of focal length f_2 is placed after the first one at a distance $d_2 = f_1 + f_2$.

b) calculate the position of the waist of the Gaussian beam after the second lens.

c) calculate the waist radius after the second lens as a function of the waist radius W_0 of the initial beam and the focal lengths f_1 and f_2.

4 **Problem 5 – Pulse propagation** 2 + 3 + 2 = 7 points

A gaussian pulse travels through a $L = 20$ meters long medium whose dispersive refractive index is defined as:

$$n(\omega) = B + C\omega^2$$

where $B = 2$ and $C = 10^{-32}\mathrm{s}^2$. Before entering the medium, the pulse is transform limited (has a flat phase) and has a bandwidth of $\omega_s = 10^{12}$Hz and is centered around the carrier frequency $\omega_0 = 2 \times 10^{15}$Hz.

a) What are the phase and group velocities of the ω_0-frequency-component of the pulse? You may leave your answers in terms of the velocity of light c_0.

b) Calculate the pulse width after propagating through $z = L$. (If you cannot remember the exact formulas for the propagation of a gaussian pulse, try to make simple approximations to get a rough number. *Hint*: It is the difference in group velocity at different frequencies that makes a pulse disperse.)

c) Another pulse was simultaneously launched in a different medium whose $n(\omega)$ is the same as before with a small difference that $C = 0$ now. Calculate the difference between the time it takes for the two pulses to reach $z = L$.

3 **Problem 6 – Fraunhofer diffraction** 2 + 2 = 4 points

a) Name two different approximations that can be made in diffraction theory and shortly explain their meaning in two sentences.

b) Calculate the intensity of the diffracted field pattern $I(x, z_B) = |u(x, z_B)|^2$ at $z = z_B$ in paraxial Fraunhofer approximation for one slit illuminated with a normally incident plane wave (just the diffraction pattern, no prefactors). The width of the slit is a ($a > \lambda$):

$$u_0(x, z = 0) = \begin{cases} 1, & \text{for } |x| \le a/2 \\ 0, & \text{elsewhere.} \end{cases}$$

$V_{ph} \times n = c$

— **Useful formulas** —

$\nabla \times \nabla \times \mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) - \Delta \mathbf{a}$

$\nabla \cdot (\nabla \times \mathbf{a}) = 0$

$\nabla \cdot (c\mathbf{a}) = \mathbf{a} \cdot \nabla c + c\, \nabla \cdot \mathbf{a}$

Gaussian q-parameter transform law:

$$q' = \frac{Aq + B}{Cq + D}$$

ABCD matrix for a thin lens:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

$\frac{1}{V_g} = \frac{n_g}{c}$

$\frac{-i(2\pi)^2}{\lambda z_B} e^{ikz} U_t\left(\frac{kx}{z_B}, \frac{ky}{z_B}\right) e^{\frac{ik}{2z_B}(x^2+y^2)}$

$V_g = V_{ph} \frac{n}{n_g}$