Friedrich Schiller University Jena WS 2022/2023
Institute of Applied Physics
Prof. Dr. Thomas Pertsch
Anastasia Romashkina, Shreyas Ramakrishna, Mostafa Abasifard, Tina (Shiu Hei Lam), Pawan Kumar, Xiao Chen

Series 8
Fundamentals of Modern Optics

to be returned on **05.01.2023**, at the beginning of the lecture

Task 1: Pulsed Beam (a=2, b=2, c=2, d=2 points)

Assume a transform-limited pulsed beam with a Gaussian-shaped intensity in both space and time:

$$I(x,y,t) = I_0 \exp[-2(x^2+y^2)/w_0^2]\exp(-2t^2/\tau_0^2),$$

where the pulse duration is $\tau_0 = 50\,\mathrm{ps}$ and the beam waist is $w_0 = 5\,\mathrm{mm}$. It has a central wavelength of 520 nm and travels in a medium with a group velocity dispersion of $\beta_2 = 0.05\,(\mathrm{ps})^2/\mathrm{m}$. The pulsed beam has a total energy of $E = 10\,\mathrm{mJ}$.

a) Calculate the peak intensity I_0 of the pulsed beam in units of $\mathrm{W/m^2}$.
b) Derive an equation for the decay of the peak intensity $I(z)$ of the pulse due to propagation. Write your answer in terms of the initial peak intensity I_0, the Rayleigh length of the Gaussian beam z_0, and the temporal dispersion length L_D.
c) For the parameters given above, which broadening mechanism is dominant? Spatial broadening due to diffraction or temporal broadening due to dispersion?
d) Approximate the pulse length τ_0 that would lead to equal relative broadening along x, y, and t directions. More precisely, what value of τ_0 results in $\frac{w(z)}{w_0} = \frac{\tau(z)}{\tau_0}$ for all z-values?

Solution Task 1:

a) We have:

$$\begin{aligned}
\text{Energy } E &= \int_{\text{Time}} \text{Power} = \int_{\text{Time}}\int_{\text{Area}} \text{Intensity} \\
&= I_0 \iiint \exp\left[-2\left(\frac{x^2}{w_0^2}+\frac{y^2}{w_0^2}+\frac{t^2}{\tau_0^2}\right)\right]\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&= I_0 \underbrace{\int \exp\left(-2\frac{x^2}{w_0^2}\right)\mathrm{d}x}_{\sqrt{\frac{\pi}{2}}w_0}\underbrace{\int \exp\left(-2\frac{y^2}{w_0^2}\right)\mathrm{d}y}_{\sqrt{\frac{\pi}{2}}w_0}\underbrace{\int \exp\left(-2\frac{t^2}{\tau_0^2}\right)\mathrm{d}t}_{\sqrt{\frac{\pi}{2}}\tau_0}
\end{aligned}$$

Following the above written relation, we obtain the final expression of the peak intensity I_0:

$$I_0 = \frac{E\sqrt{\frac{8}{\pi^3}}}{w_0^2\tau_0} \approx \frac{10\times10^{-3}\mathrm{J}\times 0.5}{25\times10^{-6}\mathrm{m}^2\times 5\cdot10^{-11}\mathrm{s}} \to \boxed{I_0 \approx 4\times10^{12}\,\mathrm{W/m^2} \text{ or } 4\,\mathrm{TW/m^2}}$$

b) From part (a) we notice that the peak intensity $I(z)$ intensity is related to the spatial width $w(z)$ and the temporal width $\tau(z)$:

$$I(z) \sim \frac{1}{w(z)^2\tau(z)}.$$

(notice that the total energy in the pulsed beam in conserved, so at any point in time and space, the term $I(z)w(z)^2\tau(z)$ should remain a constant.) The evolution of the spatial and the temporal width is known from the lecture notes:

$$w(z) = w_0\sqrt{1+\left(\frac{z}{z_0}\right)^2}, \qquad \tau(z) = \tau_0\sqrt{1+\left(\frac{2z}{L_D}\right)^2}.$$

Thus the final expression is obtained by using the relation to I_0:

$$I(z) = \frac{E\sqrt{\frac{8}{\pi^3}}}{w(z)^2\tau(z)} = \frac{I_0 w_0^2 \tau_0}{w(z)^2\tau(z)} \rightarrow \boxed{I(z) = \frac{I_0}{\left(1+\left(\frac{z}{z_0}\right)^2\right)\sqrt{1+\left(\frac{2z}{L_D}\right)^2}}}.$$

c) In order to select the dominant broadening mechanism, we recognize their specific parameters from the equations provided in part (b): z_0 for spatial broadening and L_D for temporal broadening. We can compare the broadening mechanisms by calculating and comparing their values:

$$z_0 = \frac{\pi w_0^2}{\lambda_n} = \frac{\pi}{520}\frac{25\times 10^{-6}}{10^{-9}}\,\mathrm{m} \approx 151\,\mathrm{m},$$

$$L_D = \frac{\tau_0^2}{\beta_2} = \frac{2500\cdot 10^{-24}}{0.05\cdot 10^{-24}}\,\mathrm{m} = 50\,000\,\mathrm{m}.$$

As the broadening is inversely proportional to L_D and z_0, in other words the shorter the distance the more dominant is the mechanism, due to:

$$L_D > z_0.$$

we can conlude that the spatial broadening is the dominant one.

d) For the equal broadening of the spatially symmetric pulse along x, y, and t directions we have to fulfill:

$$\frac{w(z)}{w_0} = \frac{\tau(z)}{\tau_0},$$

which, based on equations in part (b) occurs only if:

$$L_D = 2z_0.$$

We know that $L_D = \frac{\tau_0^2}{\beta_2}$ and we have already calculated $z_0 \approx 151\,\mathrm{m}$ in part (c). This means the new pulse length should be

$$\tau_0 = \sqrt{L_D\beta_2} = \sqrt{2z_0\beta_2} \approx \sqrt{2\times 151\mathrm{m}\times 0.05\frac{(\mathrm{ps}^2)}{\mathrm{m}}} \rightarrow \boxed{\tau_0 \approx 3.88\mathrm{ps}}$$

Task 2: Pulse compression (a=3, b=4, c=4, d=2 points)

A transform-limited Gaussian pulse given by

$$U(t) = B_0 \exp\left(-\frac{t^2}{\tau_0^2}\right),$$

can be compressed by transmitting it first through a quadratic phase modulator (QPM) and then through a chirp filter.

a) Using the QPM the pulse $U(t)$ is multiplied by a quadratic phase factor $\exp(i\zeta t^2)$ resulting in the chirping of the pulse. The resulting chirped pulse can be written as:

$$U_1(t) = B_{10}\exp\left(-(1-iC_1)\frac{t^2}{\tau_1^2}\right).$$

Find the chirp parameter C_1, the amplitude B_{10}, the pulse duration τ_1 and the spectral width ω_1 of the pulse (by spectral width we mean the value ω_1 at which the field envelope is reduced to $1/e$ times its maximum). *Hint:* You can use the fact that FT[$\exp(-\alpha t^2)$]= $\frac{1}{2\sqrt{\pi\alpha}}\exp(-\omega^2/4\alpha)$ for any complex number α with Re[α]> 0.

b) In order to make the chirped pulse transform-limited it is sent through a chirp filter with the envelope transfer function:

$$H_e(\omega) = \exp\left(\frac{-ib\omega^2}{4}\right).$$

The resulting pulse is given as:

$$U_2(t) = B_{20} \exp\left(-(1 - iC_2)\frac{t^2}{\tau_2^2}\right).$$

Find the value of b such that the pulse $U_2(t)$ is transform-limited. *Hint*: The condition for the pulse to be transform-limited is $C_2 = 0$. Also remember that in the Fourier domain $U_2(\omega) = H_e(\omega)U_1(\omega)$.

c) Obtain the pulse duration τ_2, the spectral width ω_2, and the amplitude B_{20} of the pulse after the system, using the expression you found for the parameter b from the previous part.

d) Explain how the comparison of the QPM and the chirp filters can be done via examining their influence on the pulse.

Solution Task 2:

a) The original (transform-limited) pulse is:

$$U(t) = B_0 \exp\left(-\frac{t^2}{\tau_0^2}\right),$$

which after the QPM becomes:

$$U_1(t) = U(t) \cdot \exp(i\zeta t^2) = B_0 \exp(-\frac{t^2}{\tau_0^2} + i\zeta t^2) = B_0 \exp\left(-(1 - i\zeta\tau_0^2)\frac{t^2}{\tau_0^2}\right) \stackrel{!}{=} B_{10} \exp\left(-(1 - iC_1)\frac{t^2}{\tau_1^2}\right)$$

By comparing the last equations, we find:

- $\boxed{C_1 = \zeta\tau_0^2}$, the chirp parameter. The initial transform-limited pulse is now chirped pulse. We will remove it in part (b) with a chirp filter.
- $\boxed{\tau_1 = \tau_0}$, the QPM does not alter the temporal width of the pulse.
- $\boxed{B_{10} = B_0}$, the pulse amplitude remains the same.

To find the spectral width $\Delta\omega_1$ we go to the Fourier domain:

$$\begin{aligned} U_1(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} B_{10} \exp(-(1 - iC_1)t^2/\tau_1^2) \exp(i\omega t)\, \mathrm{d}t \\ &= \frac{B_{10}\tau_1}{2\sqrt{\pi}} \frac{1}{\sqrt{1 - iC_1}} \exp\left(-\frac{\omega^2\tau_1^2}{4(1 - iC_1)}\right) \end{aligned}$$

Here we used the formula for the FT of a Gaussian function, with $\alpha = \frac{1-iC_1}{\tau_1^2}$.
To find $1/e$ of the maximum field envelope, we need to use the exponentially decaying part (real part in exponent). So we have to separate the part in the exponent into its real and imaginary part:

$$\exp\left(-\frac{\omega^2\tau_1^2}{4(1 - iC_1)}\right) = \exp\left(-\frac{\omega^2\tau_1^2(1 + iC_1)}{4(1 + C_1^2)}\right)$$

The decaying part of this exponent then is $\exp\left(-\frac{\omega^2\tau_1^2}{4(1+C_1^2)}\right)$. Hence the spectral width is $\boxed{\omega_1 = \frac{2}{\tau_1}\sqrt{1 + C_1^2}}$.

b) The chirp filter must be evaluated in the Fourier domain.

$$\begin{aligned} U_2(\omega) &= H_e(\omega)U_1(\omega) \\ &= \frac{B_{10}\tau_1}{2\sqrt{\pi}} \frac{1}{\sqrt{1 - iC_1}} \exp\left(-\frac{\omega^2}{4} \frac{\tau_1^2 + ib(1 - iC_1)}{1 - iC_1}\right) \end{aligned}$$

If we calculate $U_2(t)$ by Fourier back transforming:

$$\begin{aligned} U_2(t) &= \int_{-\infty}^{\infty} \frac{B_{10}\tau_1}{2\sqrt{\pi}} \frac{1}{\sqrt{1 - iC_1}} \exp\left(-\frac{\omega^2}{4} \frac{\tau_1^2 + ib(1 - iC_1)}{1 - iC_1}\right) \exp(-i\omega t)\, \mathrm{d}\omega \\ &= \frac{B_{10}\tau_1}{\sqrt{\tau_1^2 + ib(1 - iC_1)}} \exp\left(-t^2 \frac{1 - iC_1}{\tau_1^2 + ib(1 - iC_1)}\right) \\ &\stackrel{!}{=} B_{20} \exp\left(-t^2 \frac{1 - iC_2}{\tau_2^2}\right) \end{aligned}$$

Now we need to find the chirp parameter C_2 from $\exp\left(-t^2 \frac{1-iC_1}{\tau_1^2+ib(1-iC_1)}\right)$. So we need to put $\frac{1-iC_1}{\tau_1^2+ib(1-iC_1)} = \frac{1-iC_2}{\tau_2^2}$. Separating the real and imaginary part gives $\frac{1-iC_1}{\tau_1^2+ib(1-iC_1)} = \frac{1-iC_1}{(\tau_1^2+bC_1)^2+b^2}(\tau_1^2 + bC_1 - ib) = \frac{\tau_1^2}{(\tau_1^2+bC_1)^2+b^2} - i\frac{C_1(\tau_1^2+bC_1)+b}{(\tau_1^2+bC_1)^2+b^2}$. For $C_2 = 0$ we need the imaginary part to be zero, hence

$$C_1(\tau_1^2 + bC_1) + b = 0 \rightarrow \tau_1^2 C_1 + b(C_1^2 + 1) = 0 \rightarrow \boxed{b = -\frac{\tau_1^2 C_1}{C_1^2 + 1}}$$

c) In this task, in order to obtain the amplitude B_{20}, the temporal width τ_2 and the spectral width ω_2 of the pulse, we use our findings from the previous part, plus the value we found for the parameter b:

$$B_{20} = \frac{B_{10}\tau_1}{\sqrt{\tau_1^2 + ib(1-iC_1)}} \rightarrow \boxed{B_{20} = B_{10}\sqrt{1+iC_1}}.$$

Pulse width can also be found from the expansion into the real and imaginary parts that we did in the previous part that gave $\frac{1}{\tau_2^2} = \frac{\tau_1^2}{(\tau_1^2+bC_1)^2+b^2}$:

$$\tau_2^2 = \frac{(\tau_1^2 + bC_1)^2 + b^2}{\tau_1^2} = \frac{\tau_1^2}{1 + C_1^2} \rightarrow \boxed{\tau_2 = \frac{\tau_1}{\sqrt{1 + C_1^2}} \text{ or } \frac{\tau_1}{\sqrt{1 + (\zeta\tau_1^2)^2}}},$$

while the pulse spectral width ω_2 is not changed when it is transmitted through the chirp filter, thus:

$$\boxed{\omega_2 = \omega_1} = \frac{2}{\tau_1}\sqrt{1 + C_1^2},$$

Since $C_1 > 0$, after the QPM and the chirp filter the initial pulse is spectrally broadened and temporally compressed by a factor of $\sqrt{1 + C_1^2}$.

d) The comparison of the QPM and the chirp filters can be done via examining their influence on the pulse:
- Chirp: Both the chirp filter and the QPM introduces chirp to the pulse.
- Temporal width: The QPM preserves the temporal width of the pulse while the chirp filter alters it.
- Spectral width: The QPM changes the spectral width of the pulse while the Chirp filter preserves it.

Task 3: Pulse dispersion (a=2, b=2, c=3, d=4 points)

A pulse with central frequency ω_0 and bandwidth $\Delta\omega$ ($\Delta\omega \ll \omega_0$) is coupled into a dispersive fiber with the frequency-dependent refractive index:

$$n(\omega) = a_1 + a_2\omega^2 + a_3\omega^3 \qquad a_1, a_2, a_3 - \text{real constants}$$

a) While travelling through this fiber, a transform-limited initial pulse will in general undergo temporal broadening. Why?

b) Find the phase velocity v_{ph} at $\omega = \omega_0$ and group velocity v_{g} of the pulse and give short explanations about their physical meaning.

c) Express in terms of a_1, a_2 and a_3, the central frequency ω_0 of a pulse, for which the temporal broadening is minimized.

d) Show that the parabolic approximation for the dispersion relation $k(\omega)$ is justified in this fiber, under the condition that $\frac{1}{\omega_0}$ is negligible compared to $\frac{a_3}{a_2}$.
Hint: Use the Taylor expansion of $k(\omega)$ and see which terms are relevant. Keep in mind that the approximation must be valid for all frequencies of interest, i.e. for $\omega - \omega_0 \leq \Delta\omega$.

Solution Task 3:

a) Pulse broadening occurs as a consequence of group velocity dispersion i.e. the dependence of the group velocity on frequency. Physically, that means that different frequency-components of the pulse will travel at different speeds through the fiber, thus some will lag behind the pulse center and others will pull ahead, thus the pulse appears broadened to a detector at the end of the fiber.

b) The phase velocity describes the velocity of the phase fronts in a plane wave excitation at frequency ω_0 in the medium. It is given by:

$$\frac{1}{v_{\mathrm{ph}}} = \left.\frac{k(\omega)}{\omega}\right|_{\omega=\omega_0} = \frac{a_1}{c} + \frac{a_2}{c}\omega_0^2 + \frac{a_3}{c}\omega_0^3$$

The group velocity describes the velocity of a localized wave packet of central frequency ω_0 in the medium (in simpler terms, the velocity of the center of the pulse). It is given by:

$$\frac{1}{v_{\mathrm{g}}} = \left.\frac{\partial k(\omega)}{\partial \omega}\right|_{\omega=\omega_0} = \frac{a_1}{c} + \frac{3a_2}{c}\omega_0^2 + \frac{4a_3}{c}\omega_0^3$$

c) The dominant contribution to group velocity dispersion comes from the second order term in the expansion of $k(\omega)$ i.e. the one containing the dispersion coefficient D, thus, temporal broadening will be minimized if the dispersion coefficient is zero:

$$D = \left.\frac{\partial^2 k(\omega)}{\partial \omega^2}\right|_{\omega=\omega_0} = 0$$

Now to express the condition in terms of the required coefficients:

$$\begin{aligned}\left.\frac{\partial^2 k(\omega)}{\partial \omega^2}\right|_{\omega=\omega_0} &= \frac{6a_2}{c}\omega_0 + \frac{12a_3}{c}\omega_0^2 \\ &= \frac{12a_3}{c}\omega_0\left(\frac{a_2}{2a_3} + \omega_0\right) = 0\end{aligned}$$

The center frequency of a pulse which undergoes no temporal broadening is:

$$\omega_0 = -\frac{a_2}{2a_3}$$

d) The first 4 terms in the Taylor expansion of $k(\omega)$ are:

$$k(\omega) \approx k(\omega_0) + \partial_\omega k|_{\omega_0}(\omega-\omega_0) + \frac{1}{2}\,\partial_\omega^2 k\Big|_{\omega_0}(\omega-\omega_0)^2 + \frac{1}{6}\,\partial_\omega^3 k\Big|_{\omega_0}(\omega-\omega_0)^3 + \ldots$$

The parabolic approximation means that we're neglecting terms of order 3 and higher, due to them being much smaller than the first and second orders. Thus, in order to prove the validity of the approximation, we need to show that the third-order term (and consequently all higher-order terms) in the above expression is much smaller than the second order term. Written out, this means (the prefactors are irrelevant):

$$\frac{\partial_\omega^3 k\big|_{\omega_0}(\omega-\omega_0)}{\partial_\omega^2 k\big|_{\omega_0}} \ll 1$$

This approximation should be valid for all relevant values of ω, which means for all $\omega \in (\omega_0 - \Delta\omega, \omega_0 + \Delta\omega)$, and for all of them we have $\omega - \omega_0 \leq \Delta\omega$, thus the condition becomes:

$$\frac{\partial_\omega^3 k\big|_{\omega_0}(\omega-\omega_0)}{\partial_\omega^2 k\big|_{\omega_0}} \leq \frac{\partial_\omega^3 k\big|_{\omega_0}}{\partial_\omega^2 k\big|_{\omega_0}}\Delta\omega \ll 1$$

Using the results of the previous section we get:

$$\frac{\partial_\omega^3 k\big|_{\omega_0}}{\partial_\omega^2 k\big|_{\omega_0}}\Delta\omega \propto \Delta\omega\frac{6a_2 + 24a_3\omega_0}{6a_2\omega_0 + 12a_3\omega_0^2} \ll 1$$

We now divide the numerator and denominator by $6\beta\omega_0$ to get terms mentioned in the task description:

$$\Delta\omega\frac{\frac{1}{\omega_0} + 4\frac{\gamma}{a_2}}{1 + 2\frac{a_3}{a_2}\omega_0} \ll 1$$

We can now use the condition from the task description and neglect the first terms in both the numerator and denominator ($\frac{1}{\omega_0} \ll \frac{a_3}{a_2} \Leftrightarrow 1 \ll \frac{a_3}{a_2}\omega_0$), we divide the remaining terms and we're left with:

$$2\frac{\Delta\omega}{\omega_0} \ll 1$$

This is, by the conditions given in the task, **true**. Thus, under the condition that $\frac{1}{\omega_0}$ is negligible compared to $\frac{a_3}{a_2}$, the use of the parabolic approximation in this fiber is justified. QED