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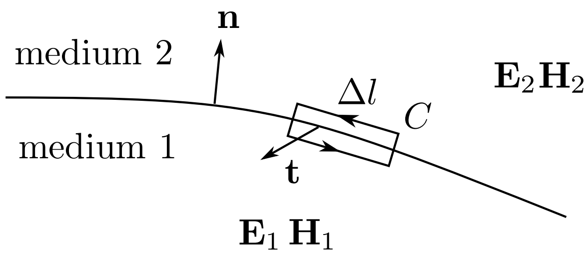
Series 3
Fundamentals of Modern Optics

to be returned on **10.11.2022**, at the beginning of the lecture

Task 1: Interface conditions (a=2, b=2 points)

Consider an interface $\mathcal{S}$ between two homogeneous and isotropic media (refractive indices n_1 and n_2). Do not consider surface current or surface charges.



a) Starting from the differential form of Maxwell's equations, derive the conditions at such an interface for the tangential electric and magnetic field $\mathbf{n}\times\mathbf{E}(\mathbf{r},t)$ and $\mathbf{n}\times\mathbf{H}(\mathbf{r},t)$, respectively. $\mathbf{n}$ is the normal vector to the surface. *Hint:* consider an infinitesimal, closed loop, which contains the surface and use Stokes' theorem to write the integral form of Maxwell's equations (see figure above). $\mathbf{t}$ is the tangential vector to the surface between medium 1 and 2, and is normal to the surface of the infinitesimal closed loop).

b) Similarly, derive the interface conditions for the normal components of the electric flux density $\mathbf{D}$ and the magnetic flux density $\mathbf{B}$. *Hint:* use an infinitesimal volume and Gauss' theorem.

Task 2: Potential Formulation (a=3, b=1, c=1, d=2, e=1 points)

Consider Maxwell's equations (MwEs) in vacuum but with charges $\rho(\mathbf{r},t)\neq 0$ and currents $\mathbf{j}(\mathbf{r},t)\neq 0$. Then the electrodynamic potentials, namely the electric potential $\varphi(\mathbf{r},t)$ and the magnetic potential $\mathbf{A}(\mathbf{r},t)$ can be introduced by the following defining equations:

$$\mathbf{H}(\mathbf{r},t) = \frac{1}{\mu_0}\nabla\times\mathbf{A}(\mathbf{r},t),$$
$$\mathbf{E}(\mathbf{r},t) = -\nabla\varphi(\mathbf{r},t) - \partial_t\mathbf{A}(\mathbf{r},t)$$

a) Show that these definitions are consistent with the MwEs, and find the wave equations for these potentials. *Hint*: Putting these definitions into two of the MwEs results in showing that these potential definitions are consistent with the MwEs. Putting these definitions into the other two of the MwEs will result in two coupled wave equations for $\varphi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$.

b) Show that Maxwell's equations are invariant under the transformations

$$\varphi'(\mathbf{r},t) = \varphi(\mathbf{r},t) + \partial_t\lambda(\mathbf{r},t) \quad \text{and} \quad \mathbf{A}'(\mathbf{r},t) = \mathbf{A}(\mathbf{r},t) - \nabla\lambda(\mathbf{r},t),$$

where $\lambda(\mathbf{r},t)$ is an arbitrary scalar, twice-differentiable function of space and time.

c) Reason if either φ or $\mathbf{A}$ can be measured directly in an experiment.

d) The ambiguity of φ and $\mathbf{A}$ can be fixed by choosing a *gauge*. One choice is the Lorenz gauge

$$\nabla\cdot\mathbf{A}(\mathbf{r},t) = -\frac{1}{c^2}\partial_t\varphi(\mathbf{r},t).$$

Find the wave equations for $\varphi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$, that you found in part (a), under this specific fixed gauge.

e) Find the condition so that the transformation of the electric potential $\varphi(\mathbf{r},t)$ and the magnetic potential $\mathbf{A}(\mathbf{r},t)$ in part (b) always preserve the Lorenz gauge.

Task 3: Lorentz model (a=1, b=2, c=2, d=2* points)

With a good approximation, a dielectric medium can be modeled by an ensemble of damped harmonic oscillators, known as the Lorentz model. In the case of a homogeneous, isotropic medium, the response function reads as

$$\hat{R}_{mn}(\mathbf{r},t) = \delta_{mn} R(t) \qquad R(t) = \begin{cases} 0 & \text{for } t \le 0 \\ \dfrac{f}{\Omega}\, e^{-\gamma t} \sin \Omega t & \text{for } t > 0 \end{cases}, \qquad \Omega = \sqrt{\omega_0^2 - \gamma^2}.$$

a) Calculate the susceptibility $\chi(\omega)$ of the medium. Notice how $\chi(\omega)$ is the Fourier transform of $R(t)$, but with a different normalization convention than our usual definition of the Fourier transform.

b) Sketch the real and imaginary part of the dielectric function $\varepsilon(\omega) = 1 + \chi(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$ for this typical insulator and mark the areas of normal ($\mathrm{d}\varepsilon'(\omega)/\mathrm{d}\omega > 0$) and anomalous ($\mathrm{d}\varepsilon'(\omega)/\mathrm{d}\omega < 0$) dispersion. Where does strong absorption occur?

c) Compute the polarization $\mathbf{P}(\mathbf{r},t)$ for the dielectric medium above with an electric field excitation of

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r}) \cos(\omega_{\mathrm{cw}} t).$$

d*) As you may have noticed, finding $\chi(\omega)$ from $R(t)$ is easier than finding $R(t)$ from $\chi(\omega)$. The former is a simple integral, but the latter requires a complex integral. Use the residue theorem to calculate the inverse Fourier transform of $\chi(\omega)$ to obtain the $R(t)$ from part a.

Task 4: Poynting Vector for Plane Waves (2+2+3+3 points)

Given is a monochromatic plane wave of frequency ω, with the complex representation $\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 \exp[i\mathbf{k}\cdot\mathbf{r}]$. The wave is propagating through a linear, isotropic, homogeneous, non-magnetic medium with a complex relative permittivity $\varepsilon(\omega) = \varepsilon' + i\varepsilon''$.

a) Find the dispersion relation $\omega = \omega(\mathbf{k})$ and identify the refractive index n as a function of ε. Do this by placing the plane wave expression into the wave equation.

b) Calculate both the $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$-fields as functions of z for all $z > 0$. For this assume a specific case of a plane wave with a linear polarization along the y-direction and a $\mathbf{k}$-vector which is pointing in the z-direction.

c) Calculate the time averaged Poynting vector $\langle \mathbf{S}(\mathbf{r},t)\rangle = \frac{1}{2}\mathrm{Re}[\mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})^*]$ and its divergence $\nabla \cdot \langle \mathbf{S} \rangle$ for the plane wave described in (b).

d) Identify ranges of ε which correspond to the cases: (i) propagating waves without loss, (ii) propagating waves with loss, (iii) nonpropagating waves without loss.

Hints: Remember that the complex representation of the field for a monochromatic wave is connected to the real time varying field via the relation $\mathbf{E}_r(\mathbf{r},t) = \frac{1}{2}\left[\mathbf{E}(\mathbf{r})\, e^{-i\omega t} + \mathbf{E}(\mathbf{r})^*\, e^{i\omega t}\right]$. It can be shown that the Maxwell's equations governing the complex representations have the same form as the Maxwell's equations in the frequency domain. Consequently, you can treat the complex field amplitudes as if you are treating the fields in the frequency domain.