to be returned: 30.01.2015, at the beginning of the lecture

## Problem 1 – Double Refraction   2+4+4+1 points

Let us consider an uniaxial crystal with $\varepsilon_1 = \varepsilon_2 \neq \varepsilon_3$.

a) Derive the expressions for the refractive indices of the normal modes (ordinary and extraordinary wave) $n_{a,b}(\varphi)$ given in the lecture (see figure below), where $\varphi$ is the angle between the crystal axis and the $k$ vector of the wave inside the crystal.

b) Show that the Poynting vector $\boldsymbol{S}$ is perpendicular to the surfaces (see figure 1).

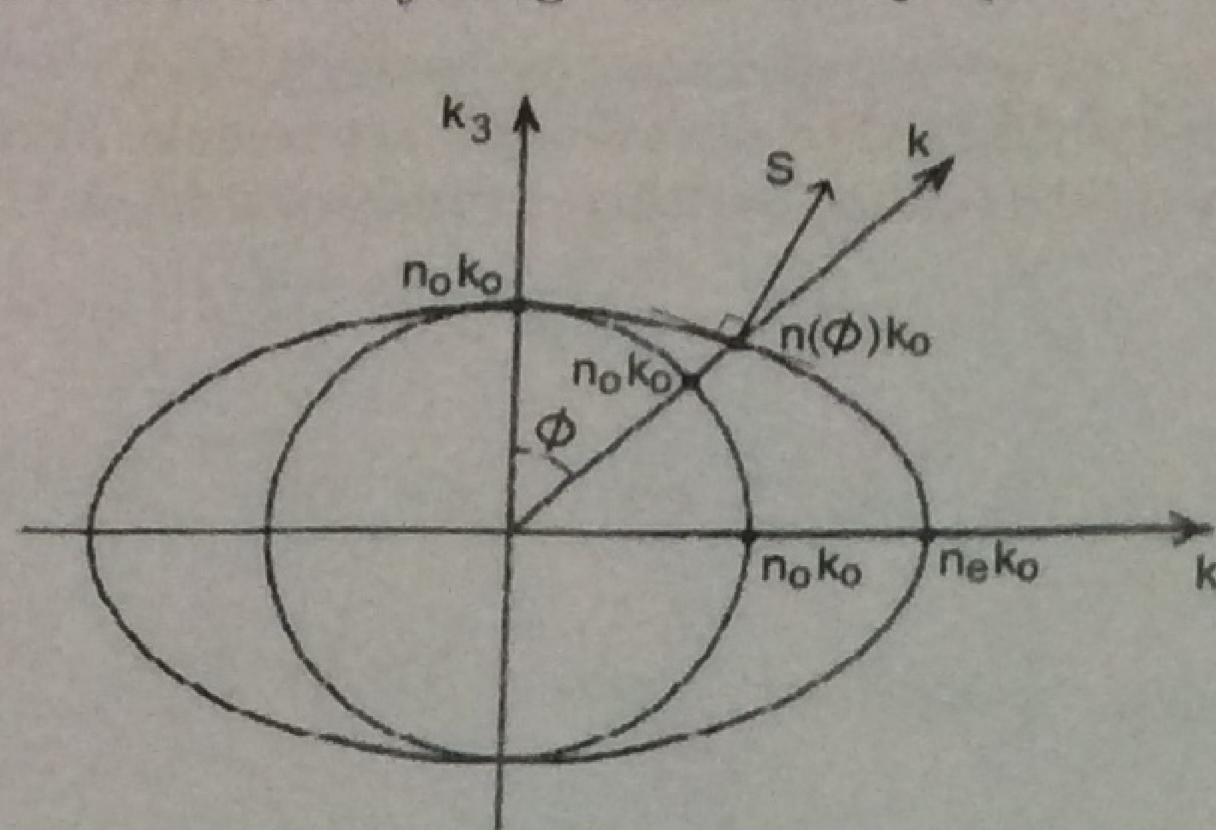



Figure 1

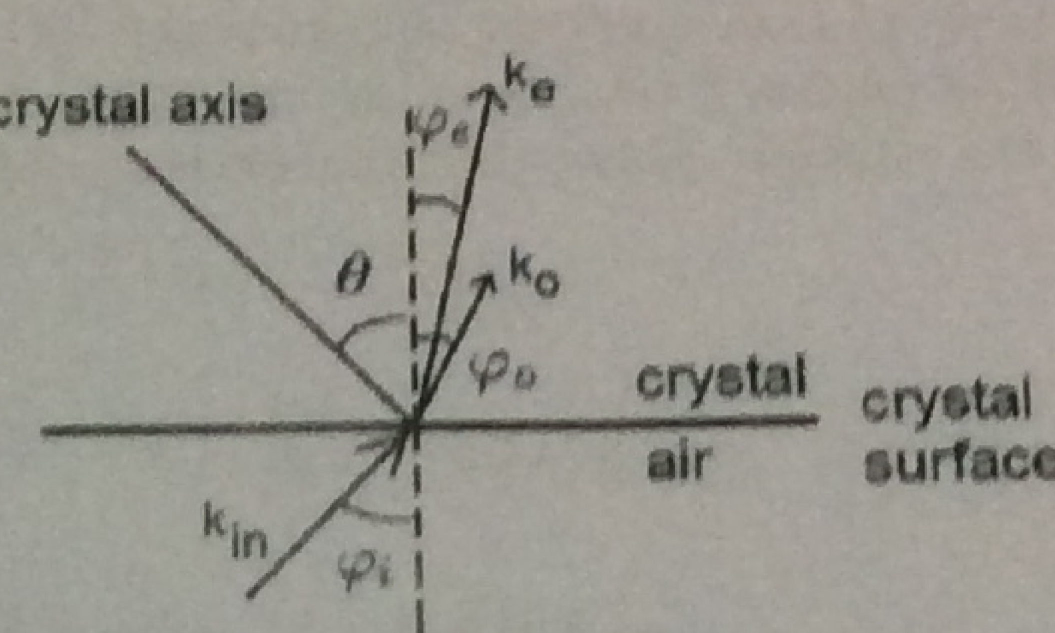



Figure 2

c) Consider the interface between an isotropic medium (refractive index $n_i$) and an uniaxial crystal ($n_o, n_e$) and light impinging at an angle $\varphi_i$ to the surface normal. The optical axis and the surface normal form an angle $\theta$ (see figure 2). Show that there will be two refracted rays with two different directions of wavevectors, $\boldsymbol{k}_o$ and $\boldsymbol{k}_e$ :

I. the *ordinary* one, which obeys ordinary Snell's law

$$n_i \sin \varphi_i = n_o \sin \varphi_o.$$

II. the *extraordinary* one, which obeys

$$n_i \sin \varphi_i = n(\theta + \varphi_e) \sin \varphi_e,$$

where $n(\varphi)$ is the refractive index of the extraordinary ray traveling at an angle $\varphi$ to the optical axis of the crystal (i.e. the formula you derive in part a).
*Hint:* Use the fact that the tangential component of the wavevector k needs to be continuous at the interface. The construction of Ewald's circle and a rotated index ellipse will be 'extraordinary' helpful ...

d) Show the angle between the Poynting vectors of the two refracted rays in part c, on the Ewald's circle.

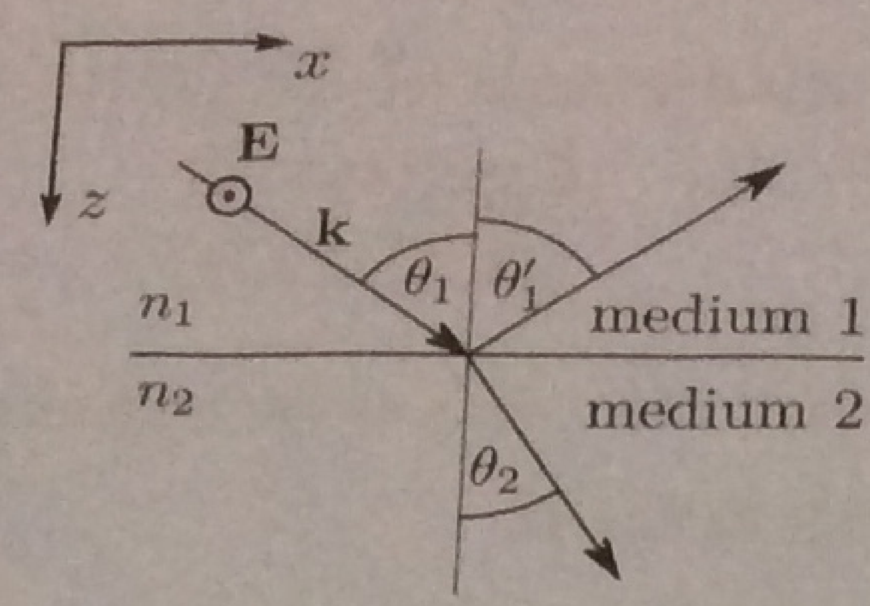
## Problem 2 – Surfaces   2+2+2 points

a) Proof mathematically that Maxwell's equations imply the continuity of the tangential **E** and **H** fields and **k** components at the interface between 2 different homogeneous, isotropic media.

b) Proof that the laws of reflection and refraction

$$\theta_1 = \theta_1', \qquad n_1 \sin \theta_1 = n_2 \sin \theta_2$$

are a direct consequence of this continuity.




c) Use this to derive the reflection and transmission coefficient of a plane wave in TE polarization incident from a medium with refractive index $n_1$ on the surface of a material with refractive index $n_2$ under and angle $\theta$.

## Problem 3 – Total Internal Reflection — 3 points

A plane wave is incident on the planar surface to a *lower* refractive index medium under a fixed angle $\theta > \theta_c$, so that it undergoes total internal reflection (i.e. no light is transmitted to the lower refractive index medium). A solar application engineer tries to make the light pass through by adding multilayer stacks of arbitrary dielectric materials to the surface. Show that his effort is hopeless.

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Problem 1 Double Refraction

9.5

1) From the lecture, we know for uniaxial crystal, we have

$$\frac{u_1^2}{n^2-\varepsilon_{or}} + \frac{u_2^2}{n^2-\varepsilon_{or}} + \frac{u_3^2}{n^2-\varepsilon_e} = \frac{1}{n^2} \quad ①$$

where $\varepsilon_{or} = \varepsilon_1 = \varepsilon_2$, $\varepsilon_e = \varepsilon_3$, $\vec{u} = (u_1, u_2, u_3) = \frac{\vec{k}}{|\vec{k}|}$

Thus $n^2(n^2-\varepsilon_e)(n^2-\varepsilon_{or})(u_1^2+u_2^2) + n^2(n^2-\varepsilon_{or})^2 u_3^2 = (n^2-\varepsilon_e)(n^2-\varepsilon_{or})^2$ ②

a) $n_a^2 = \varepsilon_{or} \Rightarrow n_a = \sqrt{\varepsilon_{or}} = n_o \Rightarrow k_a^2 = \frac{w^2}{c^2} n_a^2 = k_o^2 \varepsilon_{or}$

That means we can find a n in value of $\sqrt{\varepsilon_{or}}$ regardless of the direction, and we can find an ordinary wave, which is independent of direction.

b) $n^2(n^2-\varepsilon_e)(n^2-\varepsilon_{or})(u_1^2+u_2^2) + n^2(n^2-\varepsilon_{or})^2 u_3^2 = (n^2-\varepsilon_e)(n^2-\varepsilon_{or})^2$ ②

$\Rightarrow$ for $n_b^2 \neq \varepsilon_{or}$, $n^2(n^2-\varepsilon_e)(u_1^2+u_2^2) + n^2(n^2-\varepsilon_{or})u_3^2 = (n^2-\varepsilon_e)(n^2-\varepsilon_{or})$ ③

From figure 1, we know $u_1^2+u_2^2+u_3^2 = 1$ and $u_1 = 0$, $u_2 = \sin\varphi$, $u_3 = \cos\varphi$

Thus from ③ $(n^4 - n^2\varepsilon_e)(u_1^2+u_2^2) + (n^4 - n^2\varepsilon_{or})u_3^2 = n^4 - n^2(\varepsilon_e+\varepsilon_{or}) + \varepsilon_e\varepsilon_{or}$

$$n^4(u_1^2+u_2^2+u_3^2) - n^2\varepsilon_e(u_1^2+u_2^2) - n^2\varepsilon_{or}u_3^2 = n^4 - n^2(\varepsilon_e+\varepsilon_{or}) + \varepsilon_e\varepsilon_{or}$$

$$n^2(\varepsilon_e+\varepsilon_{or}) - n^2\varepsilon_e(u_1^2+u_2^2) - n^2\varepsilon_{or}u_3^2 = \varepsilon_e\varepsilon_{or}$$

$$n^2\varepsilon_e u_3^2 + n^2\varepsilon_{or}(u_1^2+u_2^2) = \varepsilon_e\varepsilon_{or}$$

$$\Rightarrow \frac{u_1^2+u_2^2}{\varepsilon_e} + \frac{u_3^2}{\varepsilon_{or}} = \frac{1}{n_b^2(\varphi)}$$

$$\Rightarrow \frac{\sin^2\varphi}{\varepsilon_e} + \frac{\cos^2\varphi}{\varepsilon_{or}} = \frac{1}{n_b^2(\varphi)}$$

$$\Rightarrow n_b = \sqrt{\frac{\varepsilon_e\varepsilon_{or}}{\sin^2\varphi\,\varepsilon_{or} + \cos^2\varphi\,\varepsilon_e}} = \frac{n_o n_e}{\sqrt{(n_o\sin\varphi)^2 + (n_e\cos\varphi)^2}}$$ with $n_o^2 = \varepsilon_{or}$, $n_e^2 = \varepsilon_e$

$$\Rightarrow k_b^2 = \frac{w^2}{c^2} n_b^2(\varphi)$$

That means we get a solution dependent on direction, and we can call it extraordinary wave.

Good

2) We can consider the situation only in $y$ and $z$ direction.

The equation of the ellipse $\frac{k_2^2}{\varepsilon_e k_0^2} + \frac{k_3^2}{\varepsilon_{or} k_0^2} = 1$ ① with $k_i^2 = k_0^2 n^2 u_i^2$

take differential of ① : $\frac{2k_2}{\varepsilon_e k_0^2} dk_2 + \frac{2k_3}{\varepsilon_{or} k_0^2} dk_3 = 0$

That's the ellipse tangential equation.

so the direction of normal $n$ is $\left(\frac{2k_2}{\varepsilon_e k_0^2}, \frac{2k_3}{\varepsilon_{or} k_0^2}\right)$, namely $\left(\frac{k_2}{\varepsilon_e}, \frac{k_3}{\varepsilon_{or}}\right)$

From the lecture, we know $E_2 : E_3 = \frac{k_2}{\frac{w^2}{c^2}\varepsilon_2 - k^2} : \frac{k_3}{\frac{w^2}{c^2}\varepsilon_3 - k^2} = \frac{k_2}{k_0^2 \varepsilon_{or} - k^2} : \frac{k_3}{k_0^2 \varepsilon_e - k^2}$

thus the direction of $E$ is $\left(\frac{k_2}{k_0^2 \varepsilon_{or} - k^2} : \frac{k_3}{k_0^2 \varepsilon_e - k^2}\right)$

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so for the direction of normal $n$ and direction $E$, we have

$$\vec{n} \cdot \vec{E} = \left(\frac{k_2}{\varepsilon_e}, \frac{k_3}{\varepsilon_{or}}\right) \cdot \left(\frac{k_2}{k_0^2 \varepsilon_{or} - k^2}, \frac{k_3}{k_0^2 \varepsilon_e - k^2}\right)$$

$$= \frac{k_2^2}{\varepsilon_e \varepsilon_{or} k_0^2 - \varepsilon_e k^2} + \frac{k_3^2}{\varepsilon_e \varepsilon_{or} k_0^2 - \varepsilon_{or} k^2}$$

$$= \frac{k_2^2(\varepsilon_e \varepsilon_{or} k_0^2 - \varepsilon_{or} k^2) + k_3^2(\varepsilon_e \varepsilon_{or} k_0^2 - \varepsilon_e k^2)}{(\varepsilon_e \varepsilon_{or} k_0^2 - \varepsilon_e k^2)(\varepsilon_e \varepsilon_{or} k_0^2 - \varepsilon_{or} k^2)}$$

as the numerator $k_2^2(\varepsilon_e \varepsilon_{or} k_0^2 - \varepsilon_{or} k^2) + k_3^2(\varepsilon_e \varepsilon_{or} k_0^2 - \varepsilon_e k^2)$

$$= (k_2^2 + k_3^2)\varepsilon_e \varepsilon_{or} k_0^2 - \varepsilon_{or} k^2 k_2^2 - \varepsilon_e k^2 k_3^2$$

($k_2^2 + k_3^2 = k^2$, from ① we have $\varepsilon_{or} k_2^2 + \varepsilon_e k_3^2 = k_0^2 \varepsilon_e \varepsilon_{or}$

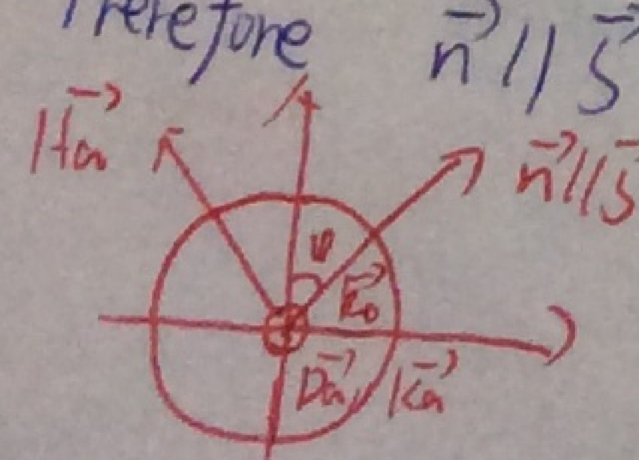
$$= k^2 \varepsilon_e \varepsilon_{or} k_0^2 - k_0^2 \varepsilon_e \varepsilon_{or} k^2$$

$$= 0$$

Thus $\vec{n} \cdot \vec{E} = 0 \Rightarrow \vec{n} \perp \vec{E}$

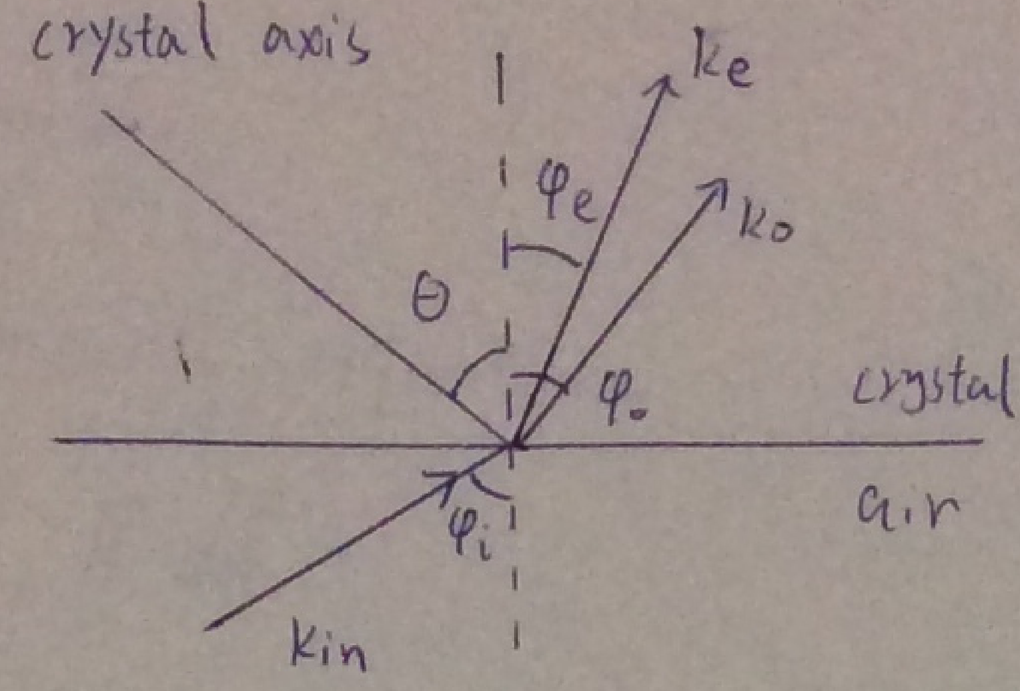
Obviously $\vec{n} \perp \vec{H}$, and $\vec{S} = \vec{E} \times \vec{H}$

Therefore $\vec{n} \parallel \vec{S}$, namely Poynting vector $\vec{S}$ is perpendicular to the surface




$\vec{n} \parallel \vec{S} \rightarrow \vec{S} \perp$ surface

ORDINARY CASE?




we use the fact that the tangential component of the wavevector $\vec{k}$ needs to be continuous at the interface. That is:

$$k_{in} \sin\varphi_i = k' \sin\varphi$$

① for the ordinary wave, $k'$ is indenpendent of the direction and angle $\varphi'$

so $k' = k_{or} = \frac{w}{c} n_o$

so $\frac{w}{c} n_i \sin\varphi_i = \frac{w}{c} n_o \sin\varphi_o \Rightarrow n_i \sin\varphi_i = n_o \sin\varphi_o$

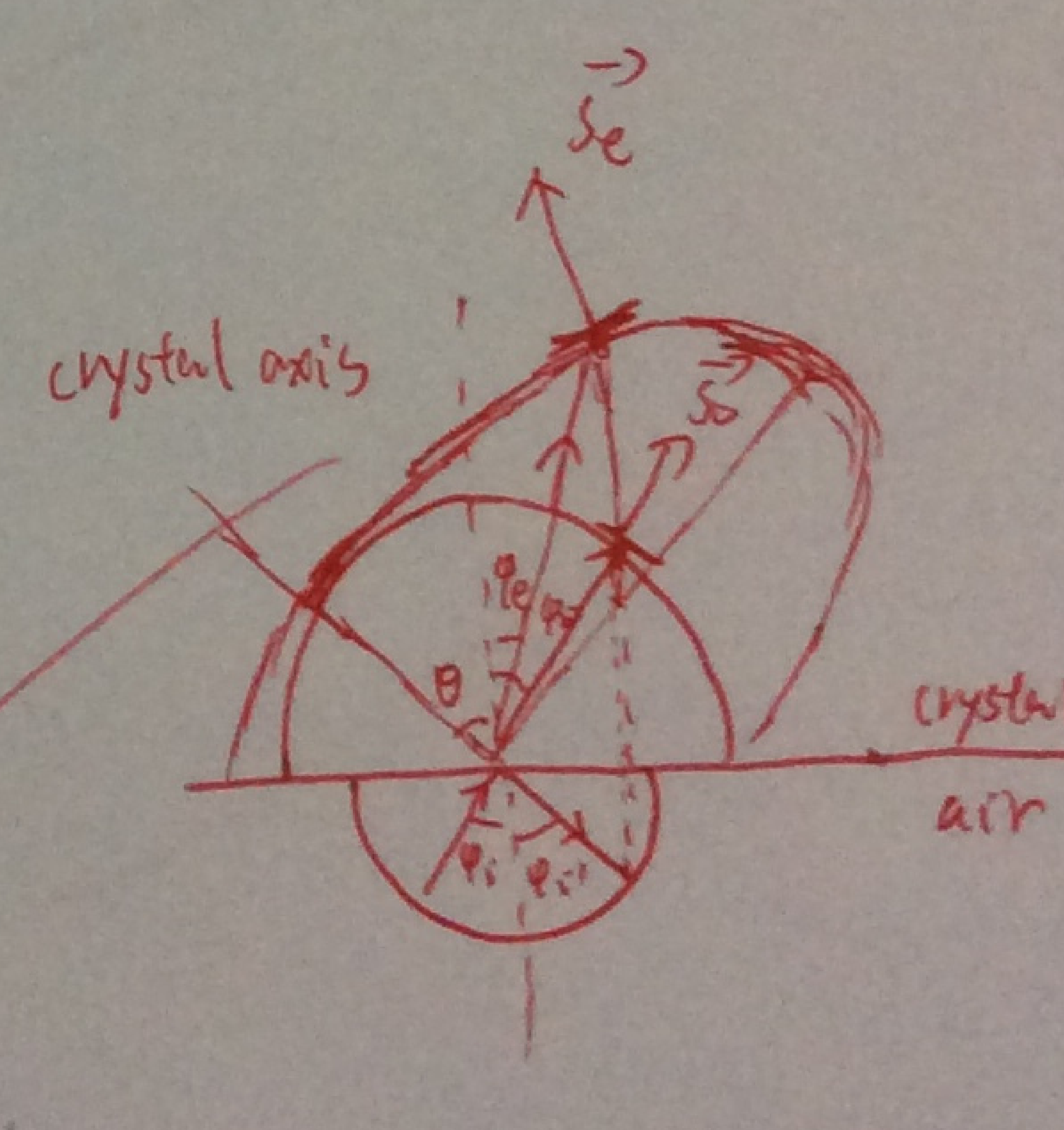
② for the extraordinary wave, $k'$ is dependent on the angle $\varphi'$, we know that the angle between crystal axis and $\vec{ke}$ $\varphi = \theta + \varphi_e$.

From part 1), we know $n_b(\varphi) = n(\theta + \varphi_e) = \frac{n_o n_e}{\sqrt{(n_o \sin\varphi)^2 + (n_e \cos\varphi)^2}}$

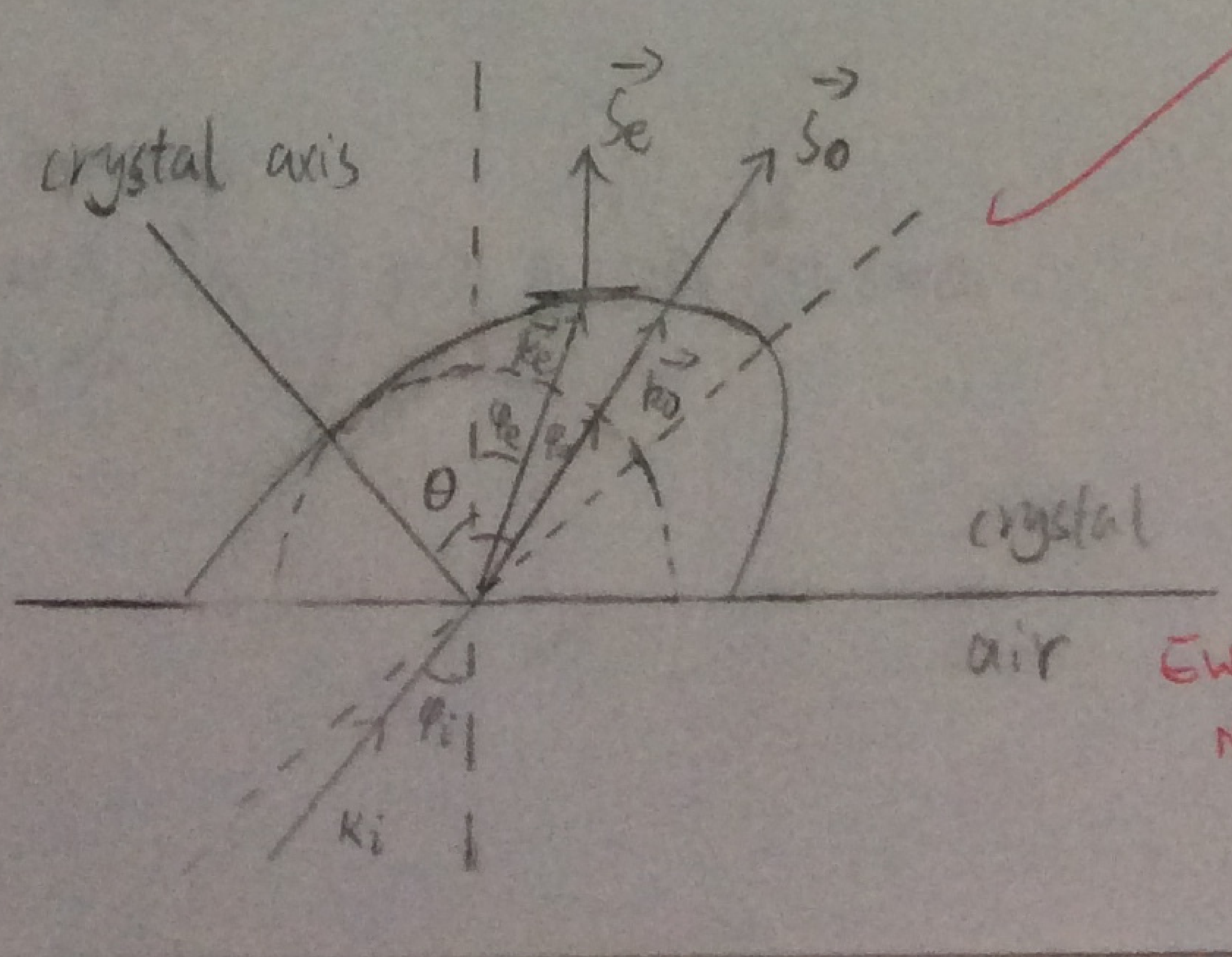
so $k' = n(\theta + \varphi_e) \cdot \frac{w}{c}$

$$\Rightarrow \frac{w}{c} n_i \sin\varphi_i = \frac{w}{c} \cdot n(\theta + \varphi_e) \cdot \sin\varphi_e$$

$$\Rightarrow n_i \sin\varphi_i = n(\theta + \varphi_e) \sin\varphi_e$$




4)




EWALD SPHERE NEEDS WORK. REVIEW FOR EXAM

Problem 2 - Surfaces

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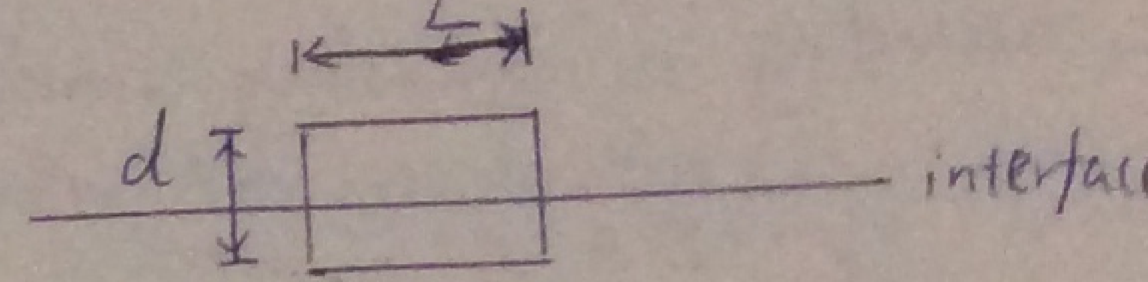



a) According to Maxwell equations:

$$\begin{cases} \oint_L \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint_s \vec{B}\cdot d\vec{s} \\ \oint_L \vec{H}\cdot d\vec{l} = I_f + \frac{d}{dt}\iint_s \vec{B}\cdot d\vec{s} \\ \oint_s \vec{D}\cdot d\vec{s} = Q_f \\ \oint_s \vec{B}\cdot d\vec{s} = 0 \end{cases}$$

when $d \to 0$ $\quad \iint_s \vec{B}\cdot d\vec{s} \to 0$

$$I_f + \frac{d}{dt}\iint_s \vec{B}\cdot d\vec{s} \to 0$$

Let $L$ small enough

$$\oint_L \vec{E}\cdot d\vec{l} = E_{1x}L - E_{2x}L = 0 \Rightarrow E_{1x} = E_{2x}$$

$$\oint_L \vec{H}\cdot d\vec{l} = H_{1x}L - H_{2x}L = 0 \Rightarrow H_{1x} = H_{2x}$$

Similarly, $E_{1y} = E_{2y}$, $H_{1y} = H_{2y}$ where $\vec{e_x}$, $\vec{e_y}$ are parallel to interface.

Therefore, we have the continuity of the tangential $\vec{E}$ and $\vec{H}$

AND $\vec{k}$ CONTINUITY??

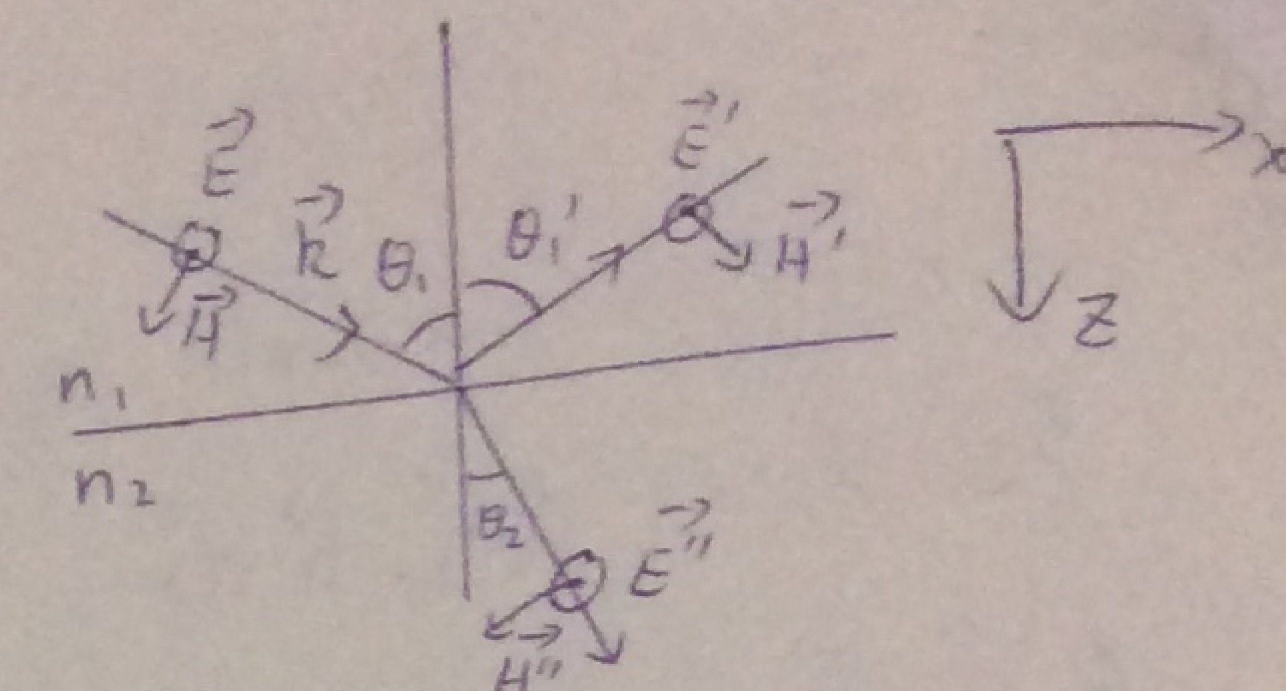
b) Since $k$ is continous tangential.

$$k_z^{in} = k_z^{refl} = k_z^{refr}$$

$$\left.\begin{array}{l} k_z^{in} = k_0 n_1 \sin\theta_1 \\ k_z^{refl} = k_0 n_1 \sin\theta_1' \\ k_z^{in} = k_z^{refl} \end{array}\right\} \Rightarrow \theta_1 = \theta_1'$$

$$\left.\begin{array}{l} k_z^{in} = k_0 n_1 \sin\theta_1 \\ k_z^{refr} = k_0 n_2 \sin\theta_2 \\ k_z^{in} = k_z^{refr} \end{array}\right\} \Rightarrow n_1 \sin\theta_1 = n_2 \sin\theta_2$$

c) when $\vec{E} \parallel$ surface $\vec{E} = \begin{pmatrix} 0 \\ E_y \\ 0 \end{pmatrix}$ $\vec{H} = \begin{pmatrix} H_x \\ 0 \\ H_z \end{pmatrix}$




according to boundary conditions.

$$\begin{cases} E + E' = E'' & ① \\ -H\cos\theta_1 + H'\cos\theta_1' = -H''\cos\theta_2 & ② \end{cases}$$

since $\nabla \times \vec{E} = i\omega\mu_0 \vec{H} \Rightarrow i\vec{k} \times \vec{E} = i\omega\mu_0 \vec{H}$

$$\Rightarrow H = \frac{k}{\omega\mu_0} E = \sqrt{\frac{\varepsilon}{\mu_0}} E$$

from ②, we have $\sqrt{\varepsilon_1}(E' - E)\cos\theta_1 = -\sqrt{\varepsilon_2} E''\cos\theta_2$ $(\theta_1 = \theta_1')$

$$\Rightarrow n_1(E' - E)\cos\theta_1 = -n_2 E''\cos\theta_2 \qquad ③$$

from ① and ③

$$\frac{E'}{E} = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2} = \frac{\frac{\sin\theta_2}{\sin\theta_1}\cos\theta_1 - \cos\theta_2}{\frac{\sin\theta_2}{\sin\theta_1}\cos\theta_1 + \cos\theta_2} = \frac{\sin(\theta_2 - \theta_1)}{\sin(\theta_2 + \theta_1)}$$

$$\frac{E''}{E} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2} = \frac{2\frac{\sin\theta_2}{\sin\theta_1}\cos\theta_1}{\frac{\sin\theta_2}{\sin\theta_1}\cos\theta_1 + \cos\theta_2} = \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_2 + \theta_1)}$$

And $S_z = (\vec{E} \times \vec{H})_z \sim E^2 n\cos\theta$

so reflection coefficient: $R = \frac{(E')^2 n_1\cos\theta_1'}{(E)^2 n_1\cos\theta_1} = \frac{\sin^2(\theta_2 - \theta_1)}{\sin^2(\theta_2 + \theta_1)}$

transmition coefficient $T = \frac{(E'')^2 n_2\cos\theta_2}{(E)^2 n_1\cos\theta_1} = \frac{(2\sin\theta_2\cos\theta_1)^2}{\sin^2(\theta_2 + \theta_1)} \cdot \frac{\cos\theta_2}{\cos\theta_1} \cdot \frac{\sin\theta_1}{\sin\theta_2}$

$$= \frac{4\sin\theta_2\cos\theta_1\cos\theta_2\sin\theta_1}{\sin^2(\theta_2 + \theta_1)}$$

$$= \frac{\sin 2\theta_1 \sin 2\theta_2}{\sin^2(\theta_2 + \theta_1)}$$

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DERIVE FOR $\theta_1$ ONLY

$$R + T = \frac{\sin^2(\theta_2 - \theta_1) + \sin 2\theta_1 \sin 2\theta_2}{\sin^2(\theta_2 + \theta_1)} = \frac{\sin^2(\theta_2 + \theta_1)}{\sin^2(\theta_2 + \theta_1)} = 1$$

(X, z=0)

d, z=0) = blem 3 Total Internal Reflection

3

During the propagation through multilayer system $k_z$ components should conserve because of the boundary condition in previous problem

$$k_z^{in} = k_z^{out} = k_z$$

The angle of total reflection is defined as $\theta_c = \arcsin(\frac{n_2}{n_1})$

$n_1 > n_2$, $k_z^{in} = k_z = k_0 n_1 \sin\theta$, $\theta > \theta_c$

$$\Rightarrow k_x^{out} = \sqrt{(k_{out})^2 - (k_z^{out})^2}$$

$$= \sqrt{(k_0 n_2 \sin\theta_2)^2 - (k_0 n_1 \sin\theta)^2}$$

$$= k_0 \sqrt{n_2^2 \sin^2\theta_2 - n_1^2 \sin^2\theta}$$

but $\sin\theta > \sin\theta_c = \frac{n_2}{n_1}$

$\Rightarrow k_x^{out}$ is not a real number $\Rightarrow$ exponential decay $e^{\sim k_x^{out} x}$

$k_x^{out} = i k_0 \sqrt{n_1^2 \sin^2\theta - n_2^2 \sin\theta_2}$ that is evanescent wave

And at this time, total internal reflection still takes place.