Fundamentals of Modern Optics
series 5
13.11.2015

to be returned: 20.11.2015, at the beginning of the lecture

Problem 1 - Normal modes in dielectrics (2+2+2+1+1 points)

For transversal waves in an isotropic, homogeneous, and dielectric medium, the field amplitude $\vec{E}\,(\vec{r},\omega)$ is given by the Helmholtz equation in Fourier domain,

$$\left[\Delta + \frac{\omega^2}{c^2}\epsilon(\omega)\right]\vec{E}\,(\vec{r},\omega) = 0, \quad \text{with} \quad \epsilon(\omega) = \epsilon'(\omega) + i\epsilon''(\omega).$$

Assume an electric field $\vec{E}\,(\vec{r},\omega) = \vec{E}_0(\omega)e^{i\vec{k}\vec{r}}$, describing a plane wave with the complex wave vector $\vec{k} = \vec{k}' + i\vec{k}''$.

a) Derive the set of equations which connects $\vec{k}'$ and $\vec{k}''$ with ϵ' and ϵ''.

b) Discuss the conditions for propagating and evanescent waves with respect to $\epsilon(\omega)$ from that set of equations.

c) Now assume a dielectric function containing one sharp resonance at frequency ω_0,

$$\epsilon(\omega) = 1 + \frac{f_0}{(\omega_0^2 - \omega^2 - i\gamma\omega)}, \quad \text{where} \quad \gamma \ll \omega_0 \text{ (weak damping regime).}$$

with

$$f_0 = 10 \text{ fs}^{-2}\,,\ \omega_0 = 5 \text{ fs}^{-1} \text{ and } \gamma = 10^{-3} \text{ fs}^{-1}.$$

Separate the real and imaginary parts of $\epsilon(\omega)$ to get $\epsilon'(\omega)$ and $\epsilon''(\omega)$ and calculate the ratio $\eta(\omega) \equiv \epsilon''(\omega)/\epsilon'(\omega)$. Sketch/plot this function and identify the frequency regions where $|\eta(\omega)| \ll 1$.

d) Just above the resonance frequency, we have $\epsilon'(\omega) < 0$. Determine the local minimum of $|\eta(\omega)|$ in this regime, assume $\vec{k}' \parallel \vec{k}'' \parallel \vec{e}_z$, and calculate the approximated refractive index $n\,(\omega_1) + i\kappa\,(\omega_1)$ at this point for $\epsilon''\,(\omega_1) \ll |\epsilon'\,(\omega_1)|$. Find the distance $z_0\,(\omega_1)$, in units of the vacuum wavelength λ, where the amplitude of the plane wave is reduced by $1/e$ (the so-called $1/e$ penetration depth).

e) Further away from ω_0, the regime with $0 < \epsilon'(\omega) < 1$ is reached. Calculate the approximated refractive index $n\,(\omega_2) + i\kappa\,(\omega_2)$ for $\vec{k}' \parallel \vec{k}'' \parallel \vec{e}_z$ at $\omega_2 = 6 \text{ fs}^{-1}$ as well as the $1/e$ penetration depth $z_0\,(\omega_2)$ in units of the vacuum wavelength λ. Compare your result with the one obtained in (d).

Problem 2 - Travelling and standing wave (2+2+1+1 points)

The electric-field complex amplitude vector, for the sum of two monochromatic plane waves of wavelength λ_0 travelling in free space in different directions, is $\vec{E}(\vec{r}) = E_0 \sin(\beta y)\, e^{i\beta z}\, \vec{e_x}$.

a) Derive an expression for the magnetic-field complex amplitude vector $\vec{H}(\vec{r})$.

b) Determine the direction of the flow of optical power by calculating the time averaged Poyting Vector.

c) Derive the complex amplitude vector for each plane wave and determine their directions of propagation. Draw a simple sketch of each plane wave.

d) Determine a relation between β and λ_0.

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Problem 3 - Evanescent waves (2+2 points)

Imagine an external source of electric field, with an amplitude distribution $u_0(x, y)$ of a Gaussian beam in Cartesian coordinates in vacuum (neglect the polarization),

$$u_0(x, y) = A \exp\left[-(x^2 + y^2)/W^2\right] \quad \text{with} \quad W > 0.$$

a) Compute its spatial frequency spectrum $U_0(k_x, k_y)$. Grafically explain your result in units of the wavelength and indicate homogenous and evanascent wave regions in the case of vacuum.

b) What is the minimum value for the waist W, to allow at least 90% of the electric field energy of the beam to be able to turn into a propagating field? Assume the electric field energy of this beam to be proportional to $\iint |u_0(x, y)|^2 dxdy$. *Hint*: Maybe Parseval's theorem comes in handy.