Institute of Applied Physics
Friedrich-Schiller-Universität Jena

Optical System Design Fundamentals

Lecture 2: The ideal image

2024 / 05 / 14

Vladan Blahnik

Preliminary Schedule - OSDF 2024

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

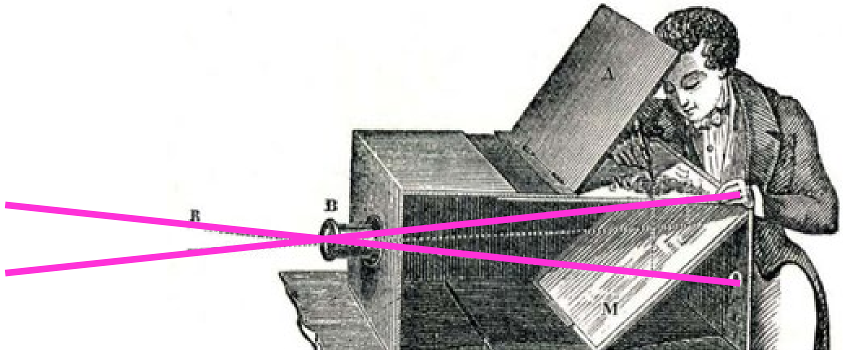
1	07.05.2024	Optical Imaging Fundamentals	Fermat principle, modeling light propagation with ray and wave model; diffraction; Snell´s law, total internal reflection, light propagation in optical system, raytracing; ray and wave propagation in optical systems; pinhole camera; central perspective projection; étendue, f-number and numerical aperture, focal length; ideal projection laws; fish,eye imaging; anamorphic projection; perspective distortion, Panini projection; massive ray-tracing based imaging model	(S) (optic
2	14.05.2024	The ideal image	Paraxial approximation, principal planes, entrance and exit pupil, paraxial imaging equations; Scheimpflug condition; paraxial matrix formulation; Delano diagram; Aplanatic imaging: Abbe sine condition, numerical aperture, sine scaled pupil for microscope imaging	(S)
3	21.05.2024	First order layout, system structures and properties	application of imaging equations; examples: layout of visual imaging system; layout of microscope lens and illumination system; telecentricity, "4f-setups"; tele and reversed tele (retrofocus) system; depth of field, bokeh; equivalence to multi-perspective 3D image acquisition; focusing methods of optical systems; afocal systems; connecting optical (sub-)systems: eye to afocal system; illumination system and its matching to projection optics); zoom systems; first order aberrations (focus and magnification error) and their compensation: lens breathing, aperture ramping; scaling laws of optical imaging	S
4	28.05.2024	Imaging model including diffraction	Gabor analytical signal; coherence function; degree of coherence; Rayleigh-Sommerfeld diffraction; optical system transfer as complex, linear operator; Fraunhofer-, Fresnel-approximation and high-NA field transfer wave-optical imaging equations for non-coherent, coherent and partially coherent imaging; isoplanatic patches, Hopkins Transfer Function for partially coherent imaging, optical transfer functions for non-coherent and coherent imaging; ideal wave-optical imaging equations; Fourier-Optics; spatial filtering; Abbe theory of image formation in microscope, phase contrast microscopy	S
5	04.06.2024	Wavefront deformation, Optical Transfer Function, Point Spread Function and performance criteria	wave aberrations, Zernike polynomials, measurement of system quality; point spread function (PSF), optical transfer function (OTF) and modulation transfer function (MTF); optical system performance criteria based on PSF and MTF: Strehl definition, Rayleigh and Marechal criteria, 2-point resolution, MTF-based criteria, coupling efficiency / insertion loss at photonics interface	S
6	11.06.2024	Aberrations: Classification, diagrams and identification in real images	measurement of wave front deformation: Shack-Hartmann sensor, Fizeau interferometer Symmetry considerations as basis for aberration theory; Seidel polynomial, chromatic aberrations; surface contributions and Pegel diagram, stop-shift-equation; Longitudinal and transverse aberrations, spot diagram; computing aberrations with Hamilton Eikonal Function; examples: defocus, plane parallel plate longitudinal and lateral chromatic aberration, Abbe diagram, dispersion fit; achromat and apochromat aberration in real images of (extended) objects	no
7	18.06.2024	Optimization process and correction principles	pre- and post-computer era correction methodology, merit function, system specification, initial setups, opto-mechanical constraints, damped-least-square optimization; symmetry principles, lens bending, aplanatic surface insertation, lens splitting, aspheres and freeforms	S
8	25.06.2024	Aberration correction and optical system structure	Petzval theorem, field flattening, achromat and apochromat, sensitivity analysis, diffractive elements; system structure of optical systems, e.g. photographic lenses, microscopic objectives, lithographic systems, eyepieces, scan systems, telescopes, endoscopes	(S)
9	02.07.2024	Tolerancing and straylight analysis	optical technology constraints, sensitivity analysis, statistical fundamentals, compensators, Monte-Carlo analysis, Design-To-Cost; false light; ghost analysis in optical systems; AR coating; opto-mechanical straylight prevention	S

Outlook on this lecture „The ideal image“

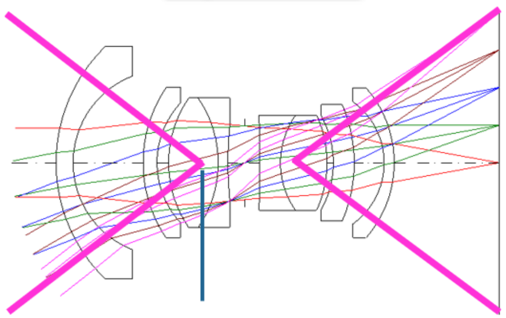
In lecture 1 we looked on different imaging setups (rectilinear, fish-eye, Panini, anamorphic) and realization of these projections as 3D (object) → 2D (image surface) by pinhole setups.

The transfer of the concept of center of perspective of imaging from the hole of a pinhole camera to “entrance pupil” of complex optical systems has been purely heuristic so far, as well as the introduction of focal length and numerical aperture.

Now we will put those terms on a solid base by defining the corresponding quantities in terms of actual optical system construction data (lens radii, thickness / relative distance, refractive indices etc.) and proving the mentioned properties.

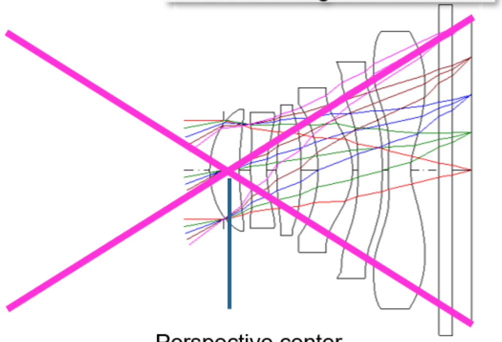


Biogon 2.8/28mm



Crossing point towards object space =
Perspective center
= Entrance Pupil

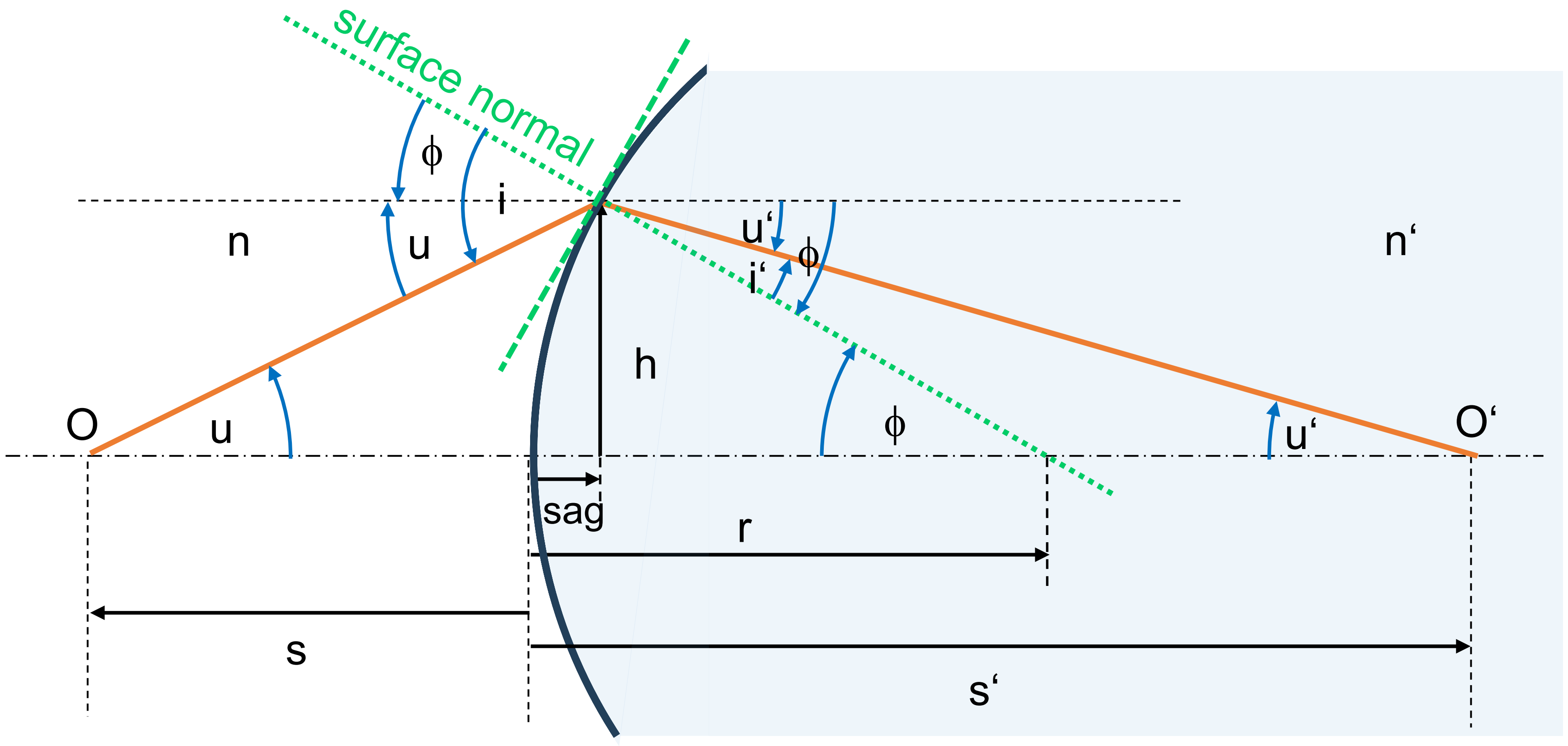
SPC Wide-Angle 2.1/28mm



Perspective center

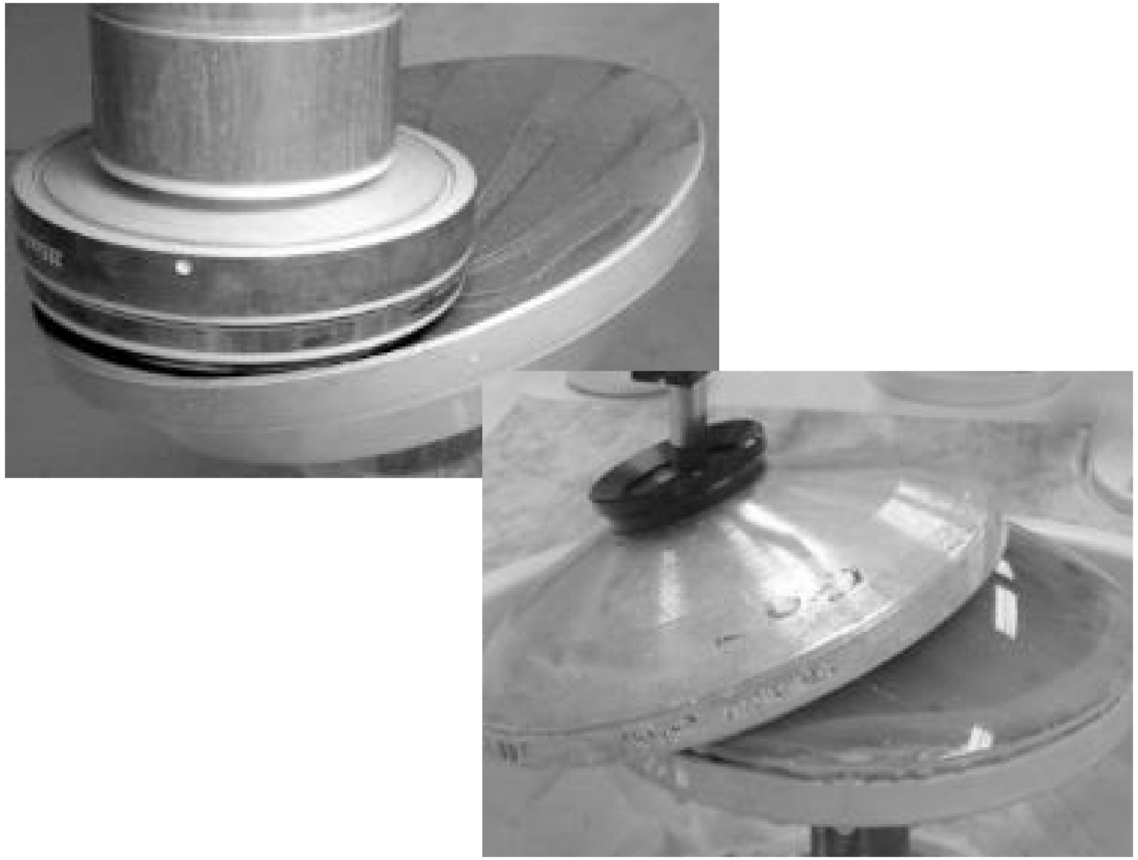
- Imaging equation / invariant law for single refracting surface
- **Image location determination** by paraxial ray tracing (analytical expression of system parameters n_j, r_j, d_j)
- System characteristics **focal length**, **lateral** & **depth magnification** (interpretation and expression of n_j, r_j, d_j)
- Imaging equations of complete optical system as „**black box**“ via pair of conjugate planes: namely **principal planes** or another arbitrary one, via **entrance** and **exit pupil**
- Proof that paraxial imaging defines an **ideal projection between planes**

Paraxial imaging equation of a spherical surface

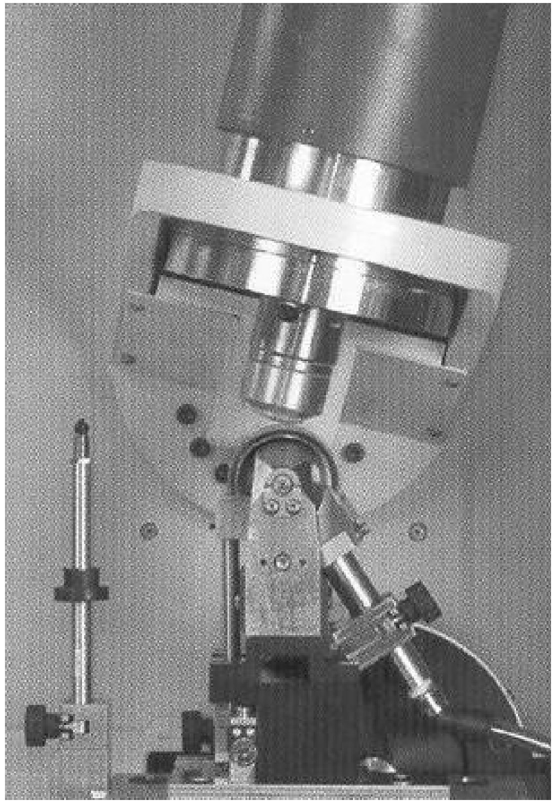


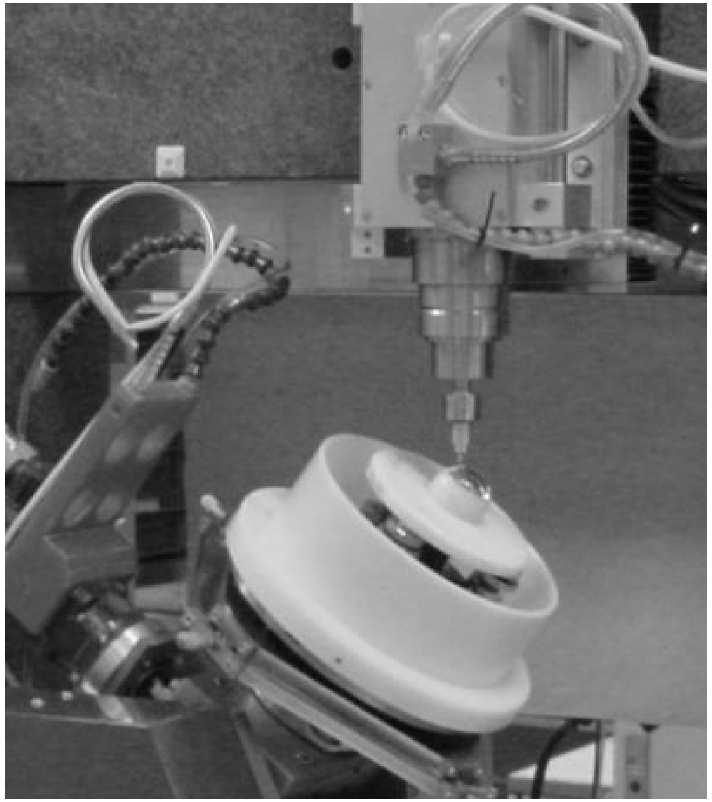
Spherical surface as basis for paraxial imaging.

Lens polishing technology: spherical and aspherical lenses



Large polishing tools for spherical surface precision correction. In 1995 manual process by experts!





Computer controlled polishing with small tools: required for aspherical surfaces

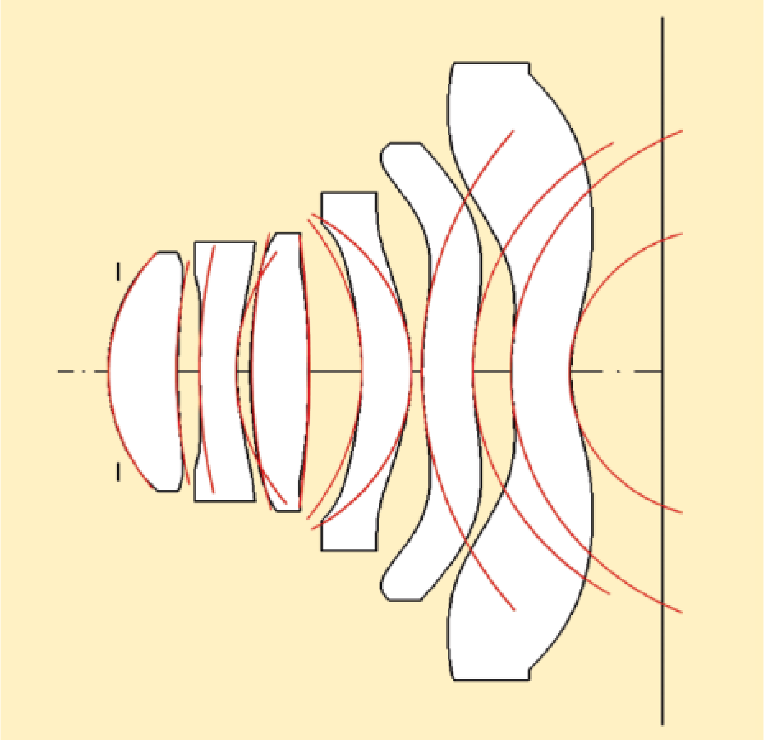
Source: Carl Zeiss, Volkmar Giggel

Paraxial imaging equation widely applicable to determine position of image

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

Smartphone Camera Lens Design

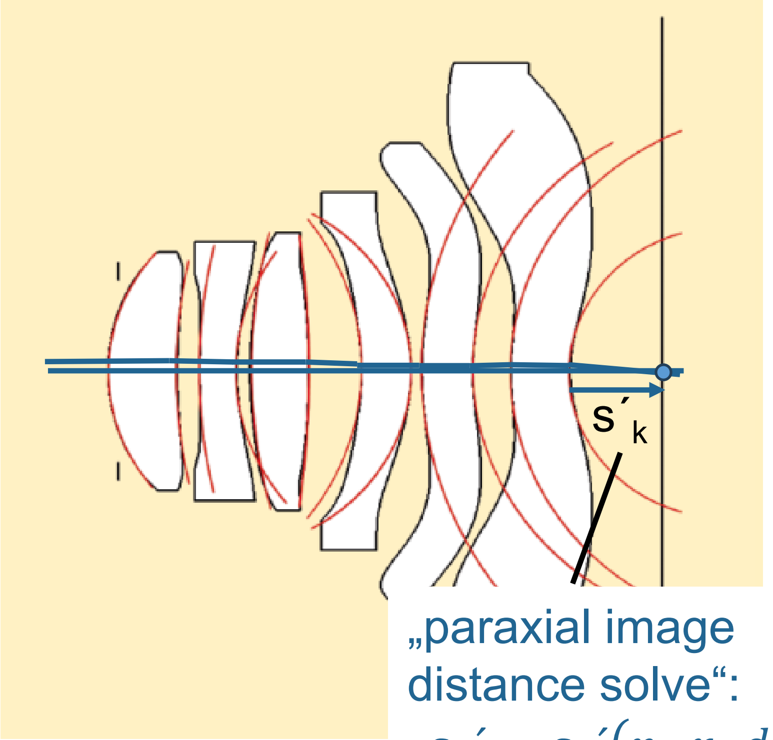
Molded plastic lens elements allow for extreme **aspheres**.



In contrast to ground glass aspheres, a spherical basic shape (shown red for illustration) is irrelevant here.

Paraxial imaging equation widely applicable to determine position of image

Although paraxial theory is based on spherical surfaces the paraxial imaging theory presented in this lecture is also applicable for a wide range of systems with aspherical or freeform surfaces, as the location of the image position is **exactly correct** for a very small aperture (always contained in the actual aperture) in the center of field (always contained in the complete field-of-view)



„paraxial image distance solve“:

$$s_k' = s_k'(n_j, r_j, d_j)$$

Optical system for smartphone camera:
All lens elements are aspheres.
The spherical approximation on axis is drawn also.
Obviously for the image-near surfaces the spherical approximation fails completely.

Anyway, the **position of the image position can be well predicted by paraxial imaging**, which is an analytical expression based on the spherical (or parabolic) part of the lens radii r_j, and n_j and d_j.

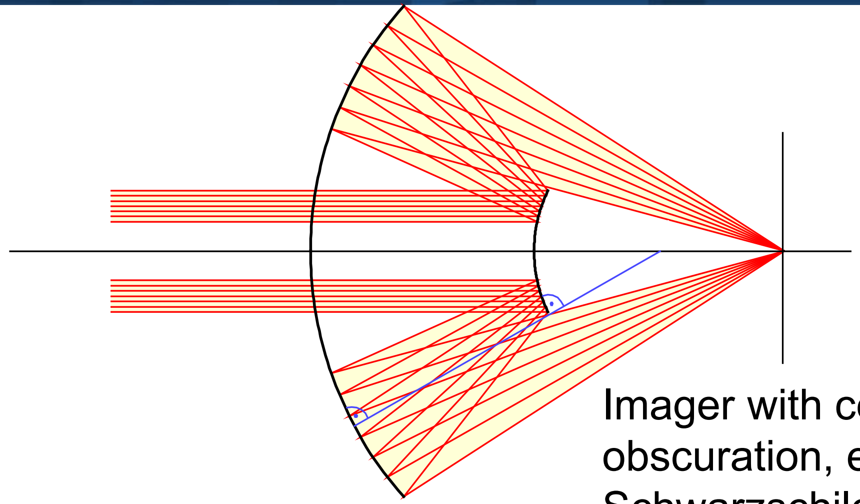
„Non-paraxial“ optical systems

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

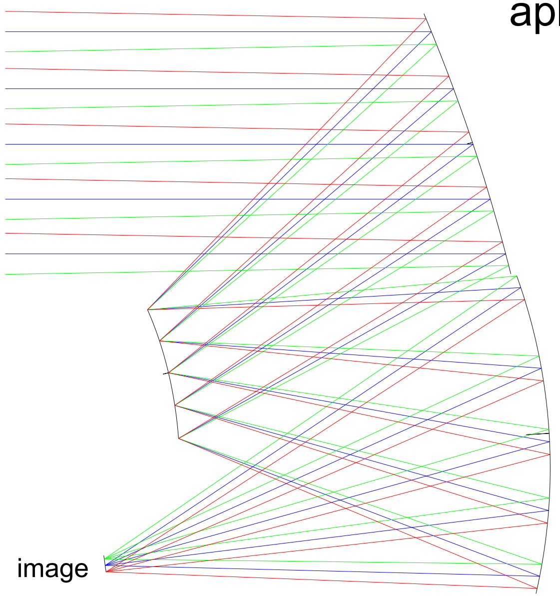
Most optical systems contain a „paraxial region“, that is rays near the optical axis.

There are exceptions.
Sometimes the system can be extrapolated, e.g. mirror systems with central obscuration and typical mirror shapes (conics or similar) may be formally computed along the axis, although the chief ray is blocked.

Off-axis systems may contain no small angle refraction/reflection. More general ray aiming and tracing schemes are necessary then.



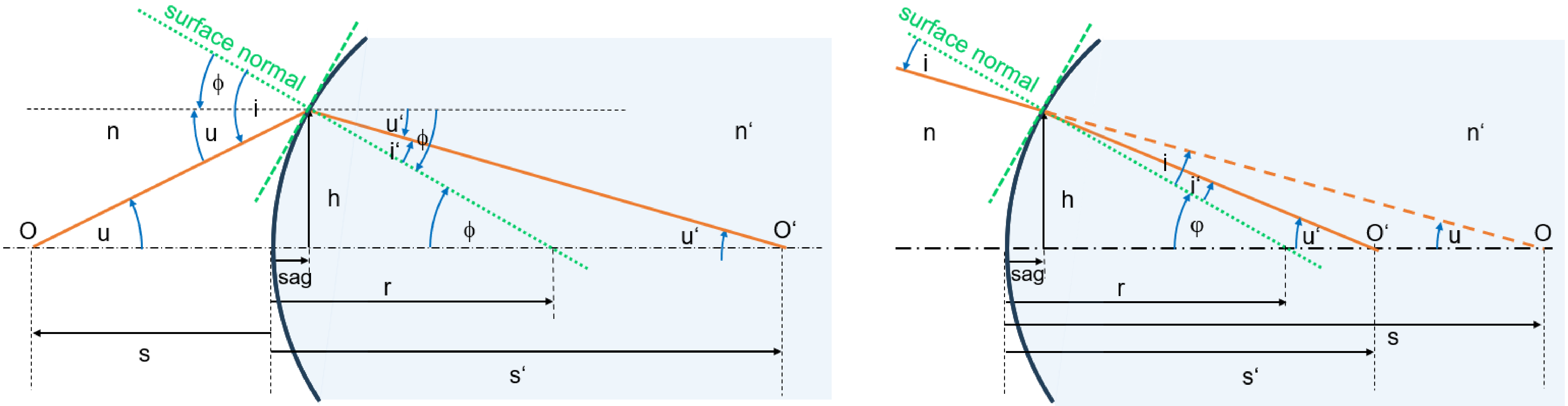
Imager with central obscuration, e.g., Schwarzschild 2-mirror aplanat



Oblique mirror system, e.g., three-mirror anastigmat (TMA)

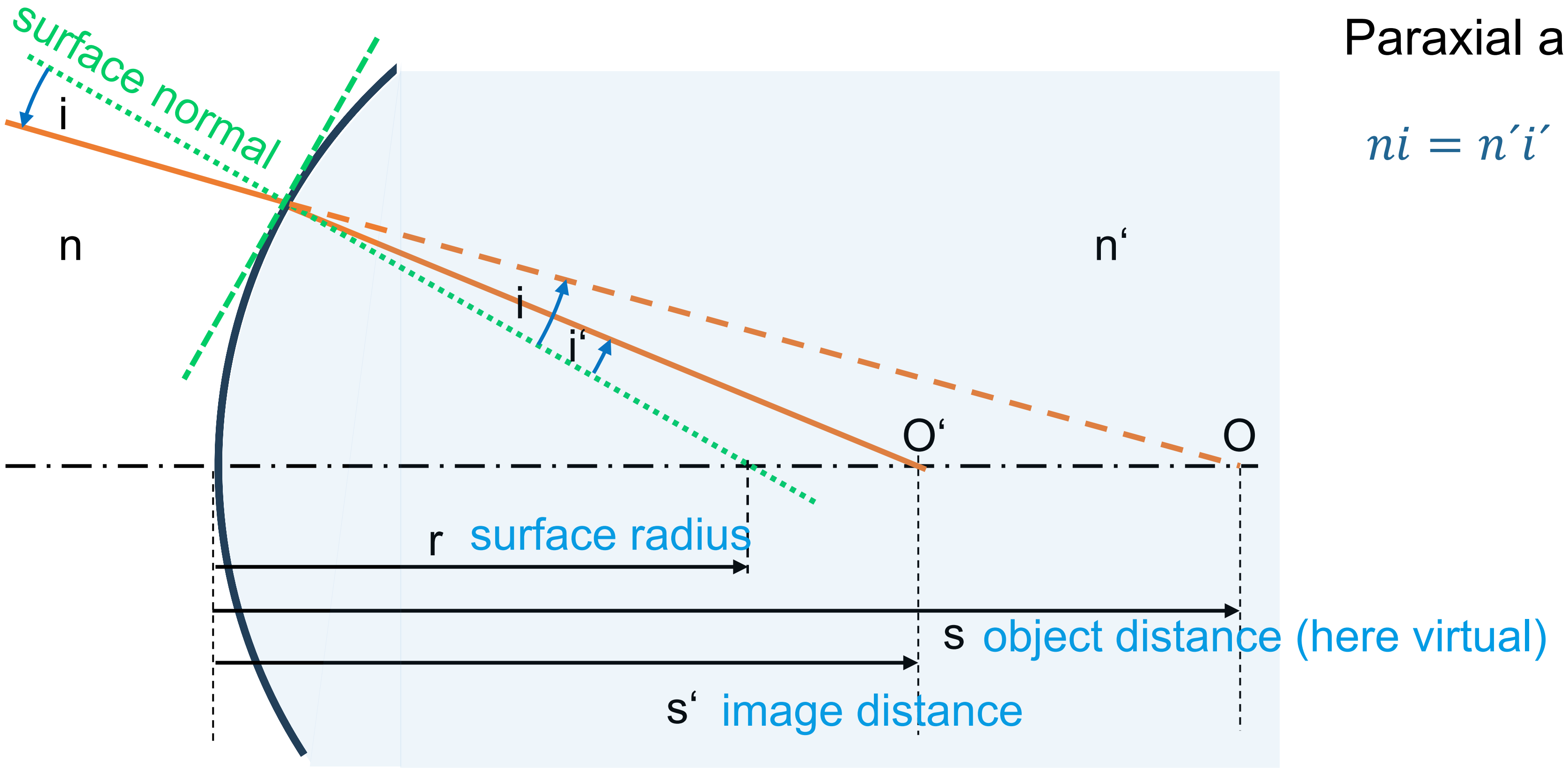
Paraxial imaging equation of a spherical surface

It is convenient to use the special setup on the right-hand side to derive paraxial imaging equations (virtual object position).



Sign convention must be chosen consistently to clearly distinguish all possible cases (object, image orientation, real or virtual locations).

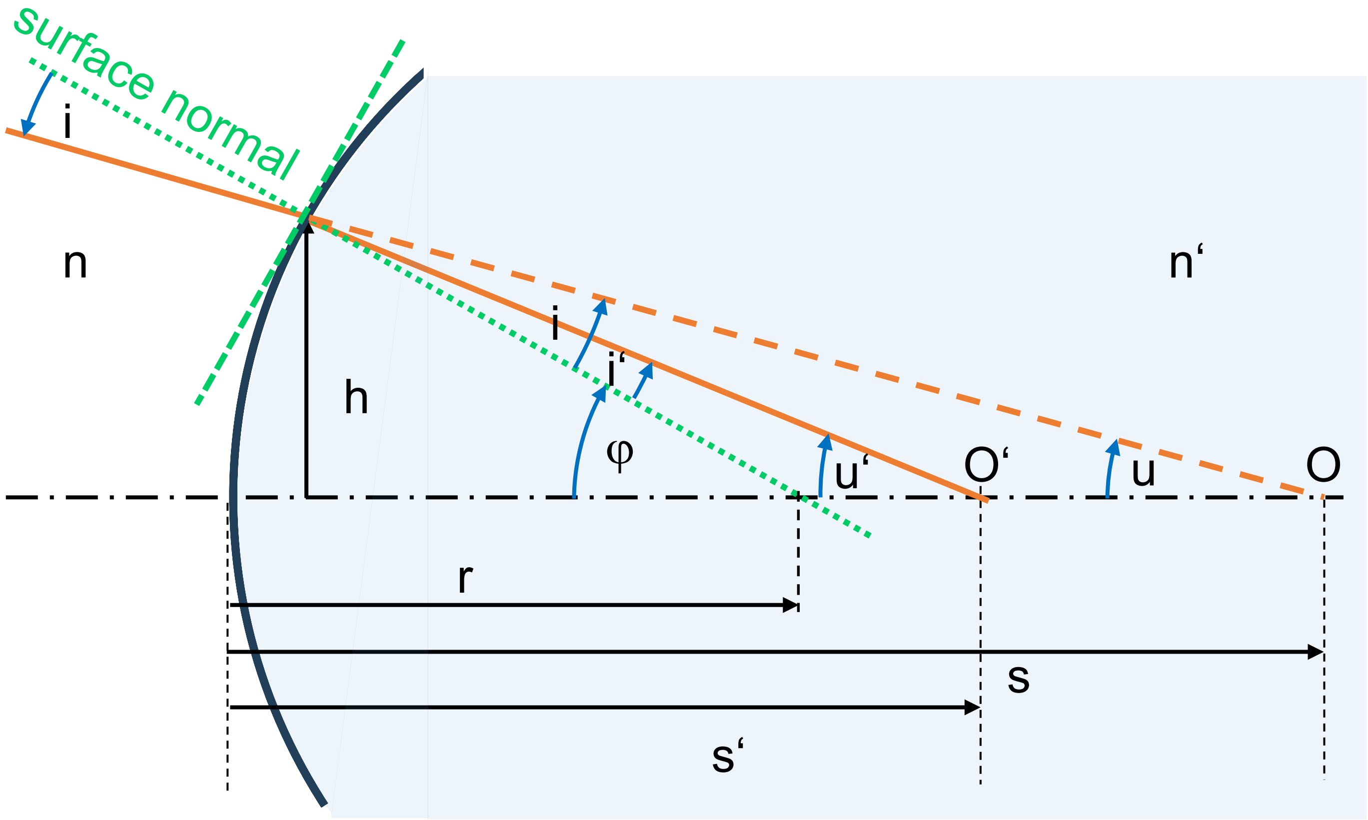
Paraxial imaging equation of a spherical surface



Paraxial approximations:

$ni = n'i'$ [SL,appr] (approximated Snell's law $n \sin i = n' \sin i'$)

Paraxial imaging equation of a spherical surface



Paraxial approximations:

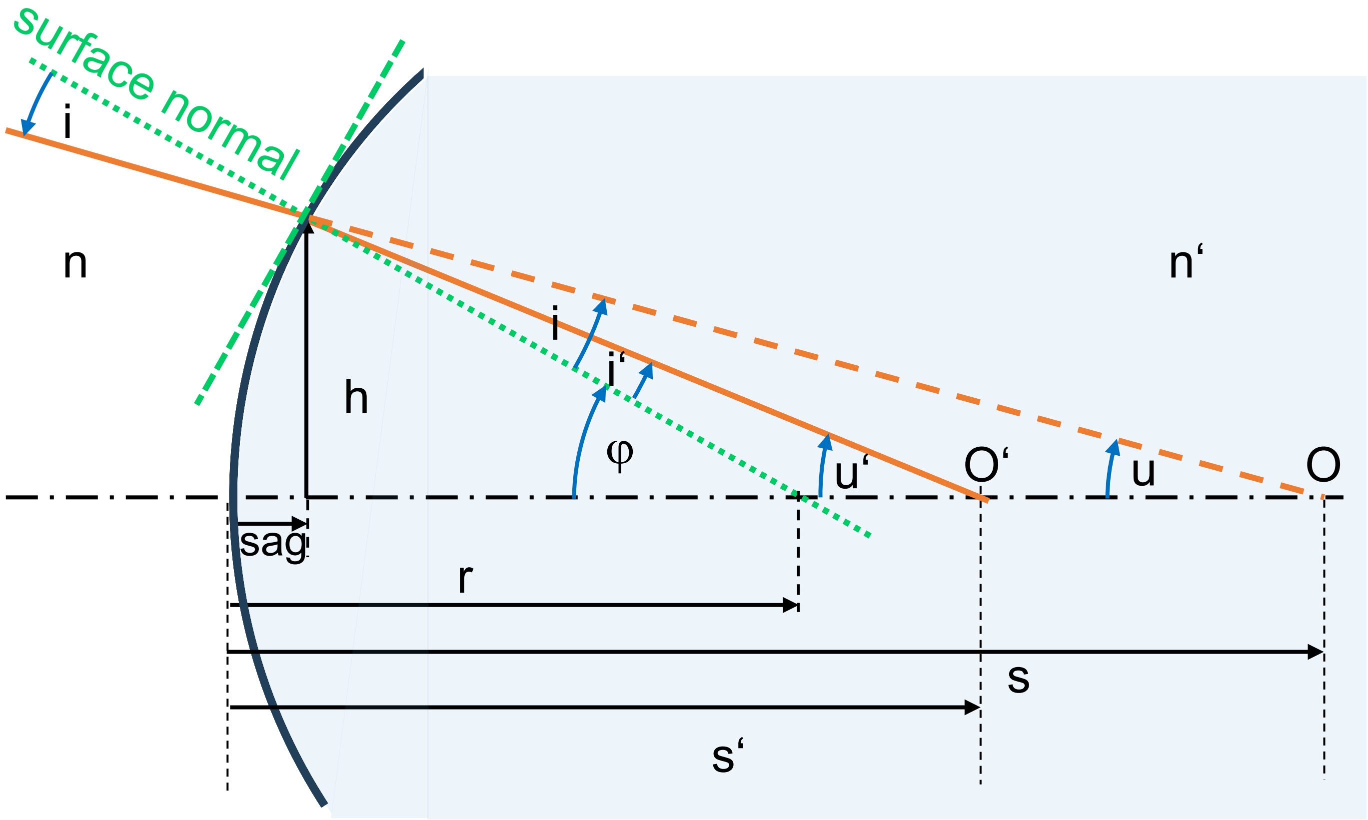
$ni = n'i'$ [SL,appr] (approximated Snell´s law $n \sin i = n' \sin i'$)

$u = \frac{h}{s}, \quad u' = \frac{h}{s'}, \quad \varphi = \frac{h}{r}.$ [NA,appr] (small h, $\tan u \approx u \ldots$)

Paraxial aperture angles u, u´ and angle of ray intersection to center of curvature φ.

Surface height h is measured perpendicular to optical axis.

Paraxial imaging equation of a spherical surface



Paraxial approximations:

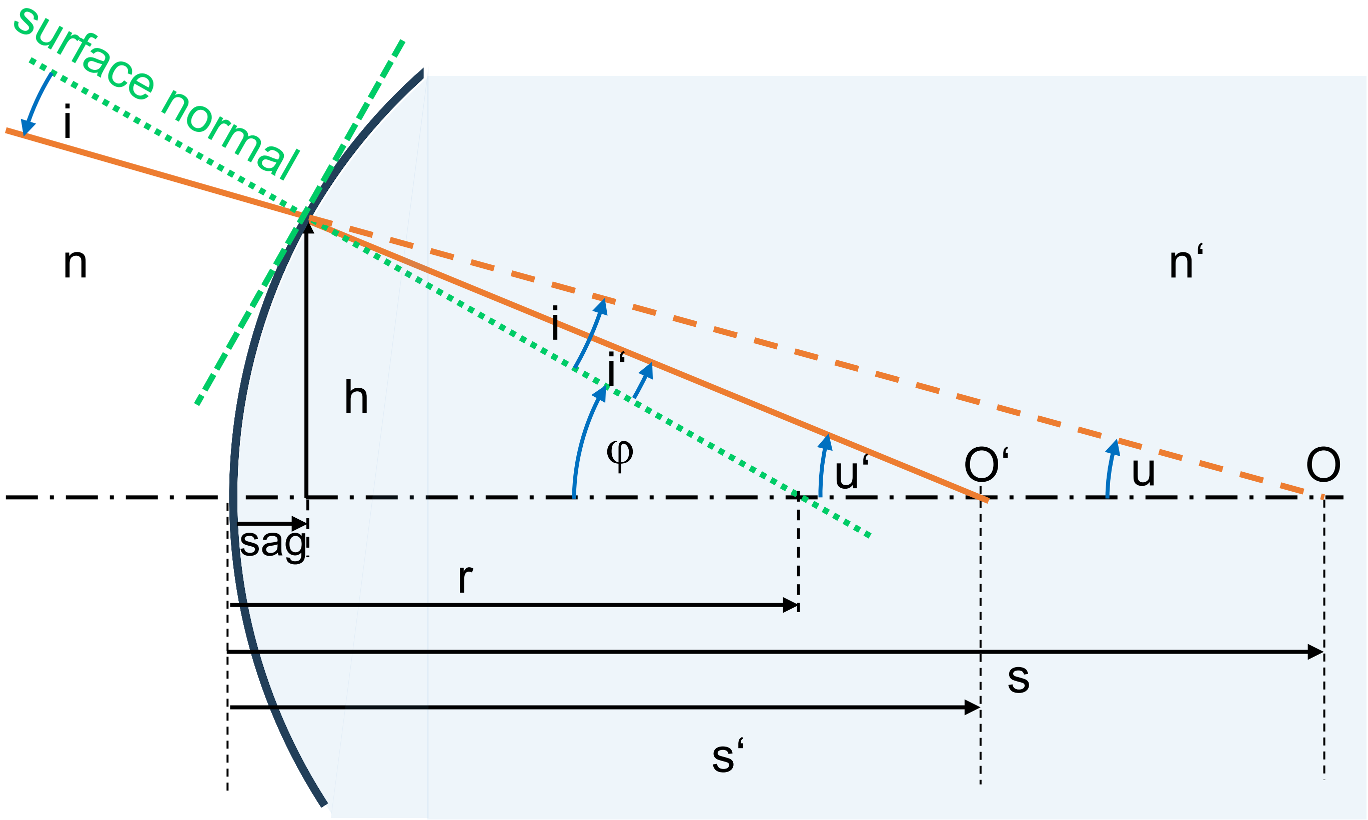
$ni = n'i'$ [SL,appr] (approximated Snell´s law $n \sin i = n' \sin i'$)

$u = \frac{h}{s}, \quad u' = \frac{h}{s'}, \quad \varphi = \frac{h}{r}.$ [NA,appr] (small h, $\tan u \approx u \ldots$)

$sag = 0$ [SAG,appr] consequently, as sag scales quadratically with surface height h:

$$sag = r - \sqrt{r^2 - h^2} = r\left(1 - \sqrt{1 - \frac{h^2}{r^2}}\right) \approx r\left(\frac{h^2}{2r^2} + \cdots\right) \approx \frac{h^2}{2r}$$

Paraxial imaging equation of a spherical surface



Paraxial approximations:

$ni = n'i'$ [SL,appr] (approximated Snell´s law $n \sin i = n' \sin i'$)

$u = \frac{h}{s}, \quad u' = \frac{h}{s'}, \quad \varphi = \frac{h}{r}.$ [NA,appr] (small h, $\tan u \approx u \ldots$)

$sag = 0$ [SAG,appr] consequently, as sag scales quadratically with surface height h:

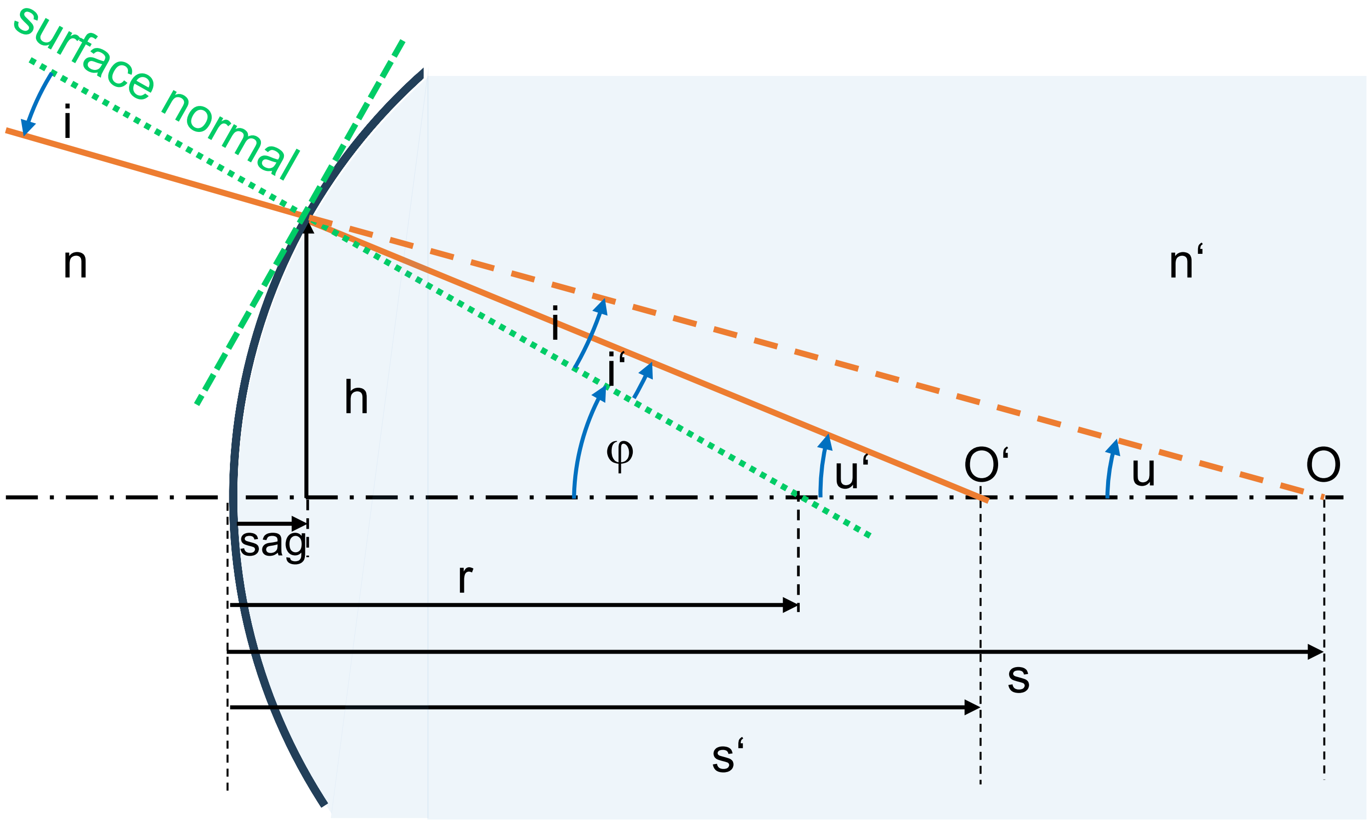
$$sag = r - \sqrt{r^2 - h^2} = r\left(1 - \sqrt{1 - \frac{h^2}{r^2}}\right) \approx r\left(\frac{h^2}{2r^2} + \cdots\right) \approx \frac{h^2}{2r}$$

Geometry: $i = \varphi - u,$ [G,ob]

$i' = \varphi - u'$ [G,im]

Paraxial imaging equation of a spherical surface

Institute of Applied Physics
Friedrich-Schiller-Universität Jena



Paraxial approximations:

$ni = n'i'$ [SL,appr] (approximated Snell's law $n \sin i = n' \sin i'$)

$u = \frac{h}{s}, \quad u' = \frac{h}{s'}, \quad \varphi = \frac{h}{r}.$ [NA,appr] (small h, $\tan u \approx u$ ...)

$sag = 0$ [SAG,appr] consequently, as sag scales quadratically with surface height h:

$$sag = r - \sqrt{r^2 - h^2} = r\left(1 - \sqrt{1 - \frac{h^2}{r^2}}\right) \approx r\left(\frac{h^2}{2r^2} + \cdots\right) \approx \frac{h^2}{2r}$$

Geometry: $i = \varphi - u,$ [G,ob]

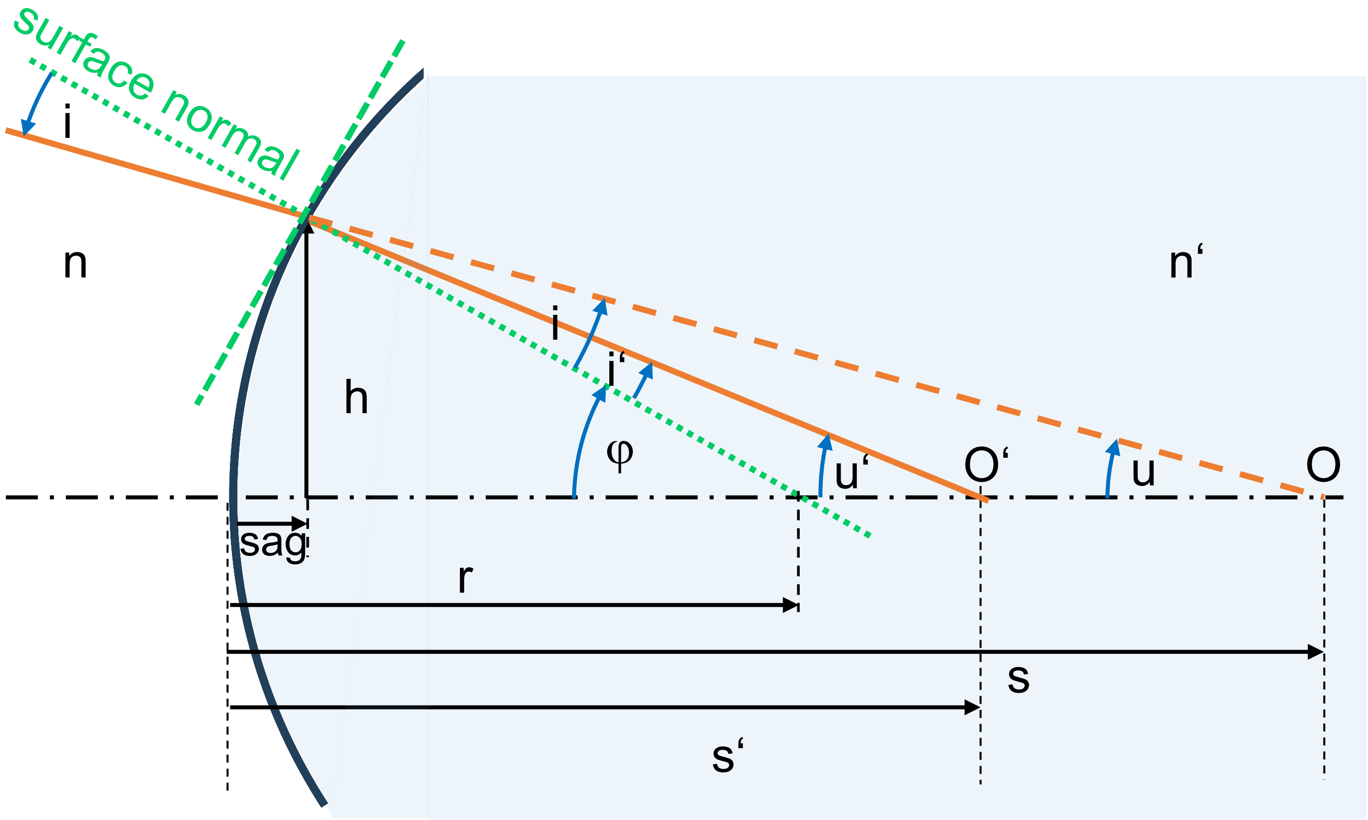
$i' = \varphi - u'$ [G,im]

From the geometrical angle relations, we obtain in paraxial approximation:

$$ni \overset{[G,ob]}{=} n(\varphi - u) \overset{[AP,appr]}{=} n\underline{\left(\frac{h}{r} - \frac{h}{s}\right)} \overset{[SL,appr]}{=} n'i' \overset{[G,im]}{=} n'(\varphi - u') \overset{[AP,appr]}{=} n'\underline{\left(\frac{h}{r} - \frac{h}{s'}\right)}$$

Paraxial imaging equation of a spherical surface

Institute of Applied Physics
Friedrich-Schiller-Universität Jena



From the geometrical angle relations, we obtain in paraxial approximation:

$$ni \overset{[G,ob]}{=} n(\varphi - u) \overset{[AP,appr]}{=} n\left(\frac{h}{r} - \frac{h}{s}\right) \overset{[SL,appr]}{=} n'i' \overset{[G,im]}{=} n'(\varphi - u') \overset{[AP,appr]}{=} n'\left(\frac{h}{r} - \frac{h}{s'}\right)$$

Paraxial approximations:

$ni = n'i'$ [SL,appr] (approximated Snell's law $n \sin i = n' \sin i'$)

$u = \frac{h}{s}, \quad u' = \frac{h}{s'}, \quad \varphi = \frac{h}{r}.$ [NA,appr] (small h, $\tan u \approx u \ldots$)

$sag = 0$ [SAG,appr] consequently, as sag scales quadratically with surface height h:

$$sag = r - \sqrt{r^2 - h^2} = r\left(1 - \sqrt{1 - \frac{h^2}{r^2}}\right) \approx r\left(\frac{h^2}{2r^2} + \cdots\right) \approx \frac{h^2}{2r}$$

Geometry: $i = \varphi - u,$ [G,ob]

$i' = \varphi - u'$ [G,im]

The result is independent of heights h:

$$n\left(\frac{1}{r} - \frac{1}{s}\right) = n'\left(\frac{1}{r} - \frac{1}{s'}\right)$$ **Abbe invariant**

$$-\frac{n}{s} + \frac{n'}{s'} = \frac{n' - n}{r}$$ **Imaging equation**

Ideal Imaging between planes

Berek (1930), Grundlagen der Praktischen Optik [Fundamentals of Practical Optics]; p. 4

For an axis-perpendicular plane to be ideally imaged, the following conditions must be met:

1. every point on the plane must be imaged stigmatically;

2. the entirety of the image points must again fill a plane perpendicular to the axis;

3. the ratio of the distance between any two image points to the distance of the associated object points, the magnification, must be constant within the entire image plane.

These are essential properties of the paraxial imaging equations!
Many modern textbooks fail to emphasize this.

Ideal Imaging between planes

Berek (1930), Grundlagen der Praktischen Optik [Fundamentals of Practical Optics]; p. 4

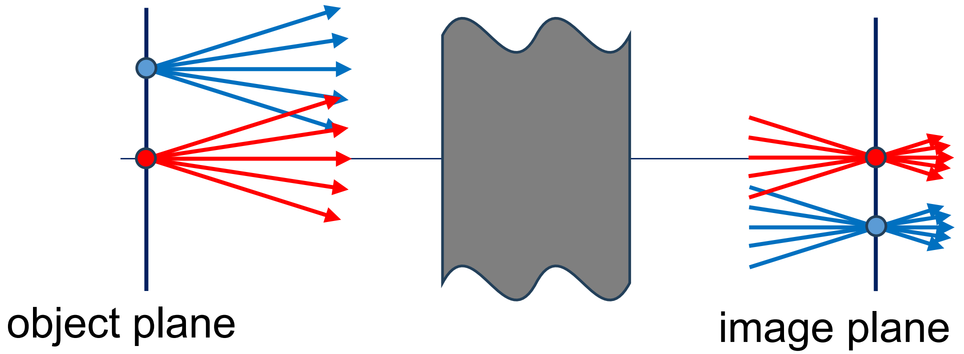
For an axis-perpendicular plane to be ideally imaged, the following conditions must be met:

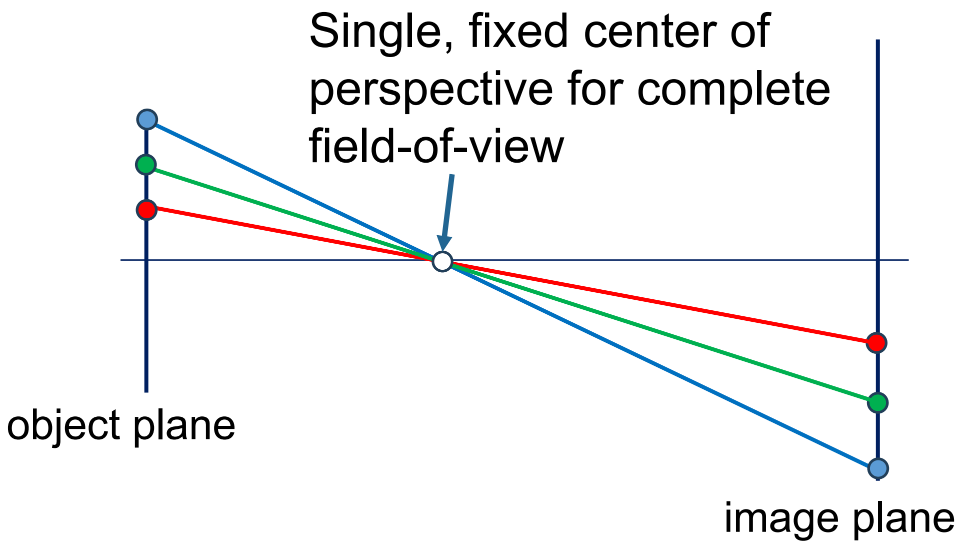
1. **every point** on the plane must be imaged **stigmatically**;

2. the **entirety** of the image points must again **fill a plane** perpendicular to the axis;

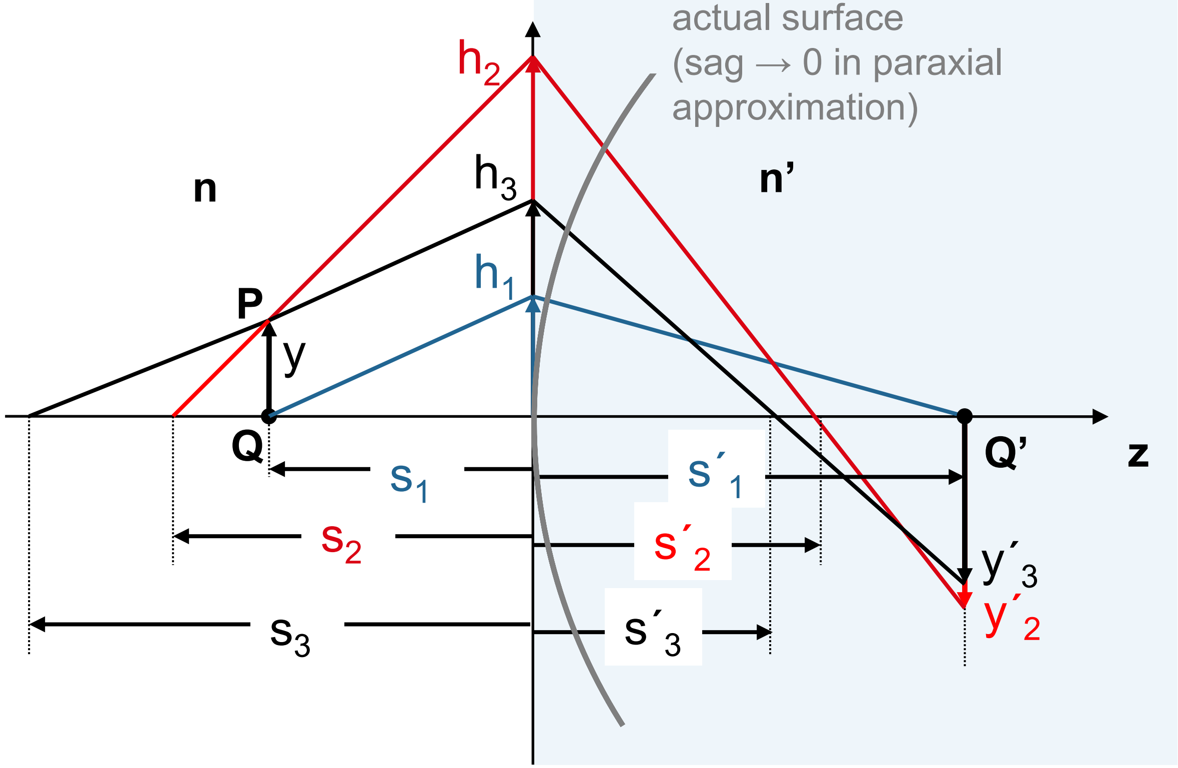
3. the ratio of the distance between any two image points to the distance of the associated object points, the **magnification**, must be **constant** within the entire image plane.

“stigmatic imaging” means “a point object is imaged again perfectly onto a point” for any aperture





The paraxial imaging equations fulfill the ideal imaging conditions



We **proof that the paraxial imaging equations are ideal** with following construction:
The paraxial ray from axial point Q at distance s_1 from the surface intersects the optical axis in Q´ at distance s'_1 in image space.
Now we consider two other rays indexed 2 and 3 which intersect in the same object plane as Q in point P.

The intersection distances with the optical axis s_2, s_3 as well as the intersection heights h_2, h_3 are different.
They cross the plane which is perpendicular to the axis at Q´ at y'_2 and y'_3.
Are y'_2 and y'_3 different or same?

From triangle proportion we have in object space:

$$\frac{h_2}{y} = \frac{s_2}{s_2 - s_1} \text{ and } \frac{h_3}{y} = \frac{s_3}{s_3 - s_1} \qquad [1]$$

and in image space $\frac{-y'_2}{s'_1 - s'_2} = \frac{h_2}{s'_2}$ and $\frac{-y'_3}{s'_1 - s'_3} = \frac{h_3}{s'_3}$. [2]

Solving [1] for y and setting the solved equations equal:

$$\frac{h_2(s_2 - s_1)}{s_2} = \frac{h_3(s_3 - s_1)}{s_3} \text{ or } h_2\left(\frac{1}{s_1} - \frac{1}{s_2}\right) = h_3\left(\frac{1}{s_1} - \frac{1}{s_3}\right), \qquad [3]$$

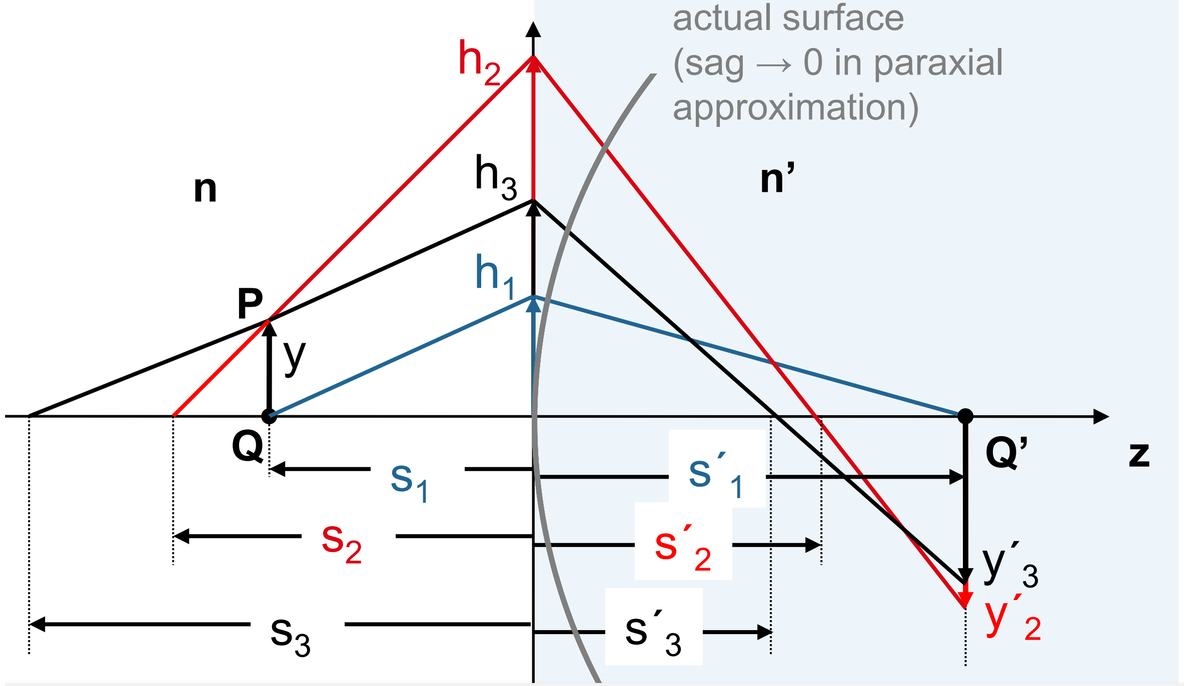
and analogously for [2] by subtracting both equations:

$$y'_2 - y'_3 = \frac{h_2(s'_2 - s'_1)}{s_2'} - \frac{h_3(s'_3 - s'_1)}{s_3'} = h_2\left(\frac{1}{s'_1} - \frac{1}{s'_2}\right)s'_1 - h_3\left(\frac{1}{s'_1} - \frac{1}{s'_3}\right)s'_1. \qquad [4]$$

With the imaging equation $-\frac{n}{s} + \frac{n´}{s´} = \frac{n´}{r}$ we have for j=1,2,3:

$\frac{1}{s'_j} = \left(\frac{n}{n'}\right)\frac{1}{s_j} + \left(\frac{n' - n}{n'}\right)\frac{1}{r}$ and by subtracting relevant index pairs: $\frac{1}{s'_1} - \frac{1}{s'_2} = \frac{n}{n'}\left(\frac{1}{s_1} - \frac{1}{s_2}\right)$ and $\frac{1}{s'_1} - \frac{1}{s'_3} = \frac{n}{n'}\left(\frac{1}{s_1} - \frac{1}{s_3}\right)$ [5]

The paraxial imaging equations fulfill the ideal imaging conditions



For an axis-perpendicular plane to be ideally imaged, the following conditions must be met:

1. every point on the plane must be imaged stigmatically;
2. the entirety of the image points must again fill a plane perpendicular to the axis;
3. the ratio of the distance between any two image points to the distance of the associated object points, the magnification, must be constant within the entire image plane.

Insertion in [4] we obtain with [3]:

$$y_2' - y_3' = \left[h_2\left(\frac{1}{s_1}-\frac{1}{s_2}\right) - h_3\left(\frac{1}{s_1}-\frac{1}{s_3}\right)\right]\left(\frac{n}{n'}\right)s_1' = 0. \quad \text{QED 1., 2.}$$

The intersection of the rays lies in the plane perpendicular to the axis through the point Q and its height $y' = y_2' = y_3'$ above the base point results from [1] and [2] to

$$y' = \left(\frac{s_2'-s_1'}{s_2'}\right)\left(\frac{s_2}{s_2-s_1}\right)y = \left(\frac{\frac{1}{s_1'}-\frac{1}{s_2'}}{\frac{1}{s_1}-\frac{1}{s_2}}\right)\left(\frac{s_1'}{s_1}\right)y \qquad [6]$$

With [5] and the solved imaging equation $s'_j = \frac{n'_j}{\frac{n'_j-n_j}{r_j}+\frac{n_j}{s_j}}$ we finally obtain the expression

$$y' = \left(\frac{n}{n'}\right)\frac{1}{1-\left(\frac{n-n'}{n\,r}\right)s_1}\left(\frac{n'}{n}\right)y = \frac{1}{1-\left(\frac{n-n'}{n\,r}\right)s_1}y.$$

Obviously, the magnification $m = \frac{y'}{y} = \frac{1}{1-\left(\frac{n-n'}{n\,r}\right)s_1}$

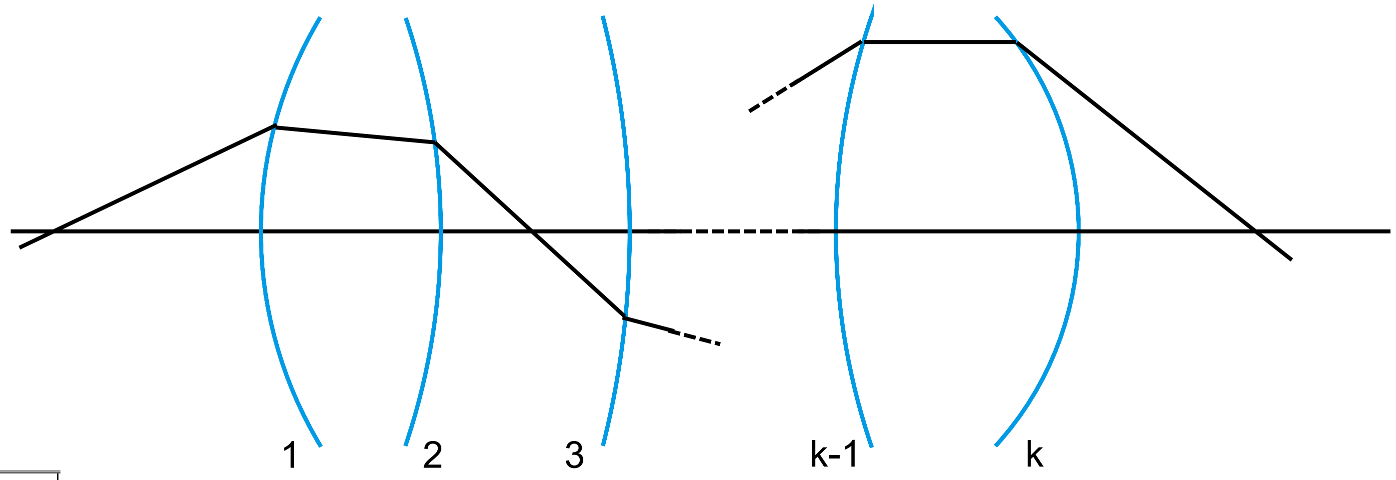
only depends on n, n´, r and the object distance s_1 and is therefore constant in the image plane.

QED 3.

The paraxial image can be calculated by successively applying the paraxial imaging equation to each surface of the system.

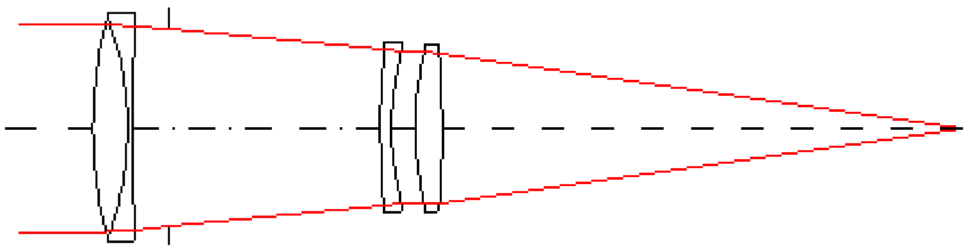
Lens subscription data:

- s_1 (object distance to first surface)
- r_j , j=1…k (surface radii)
- d_j , j=1…k (distances between surfaces)
- n_j , j=1…k (refractive indices)



Example: Data of a Petzval Lens

Surface #	Surface Name	Surface Type	Y Radius	Thickness	Glass	Refract Mode	Y Full Aperture
Object		Sphere	Infinity	Infinity		Refract	
1		Sphere	0.5590	0.0470	517000.650000	Refract	0.2827
2		Sphere	-0.4370	0.0080	575000.575493	Refract	0.2818
3		Sphere	4.6040	0.0470		Refract	0.2769
Stop		Sphere	Infinity	0.2890		Refract	0.2683
5		Sphere	1.1060	0.0150	575000.575493	Refract	0.2114
6		Sphere	0.3890	0.0330		Refract	0.2068
7		Sphere	0.4800	0.0360	517000.650000	Refract	0.2072
8		Sphere	-1.5780			Refract	0.2041
Image		Sphere	Infinity			Refract	0.0003

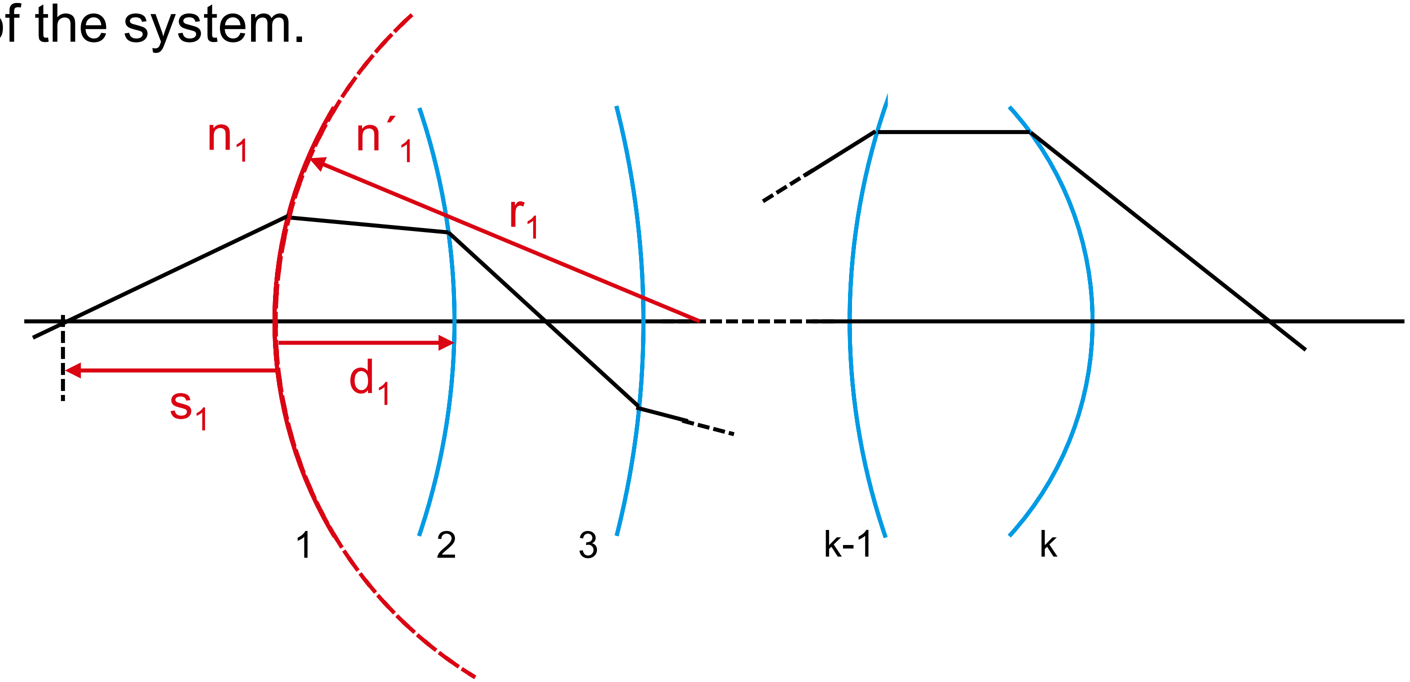


Paraxial ray tracing

The paraxial image can be calculated by successively applying the paraxial imaging equation to each surface of the system.

Lens subscription data:

- s_1 (object distance to first surface)
- r_j , j=1…k (surface radii)
- d_j , j=1…k (distances between surfaces)
- n_j , j=1…k (refractive indices)



The paraxial image can be calculated by successively applying the paraxial imaging equation to each surface of the system.

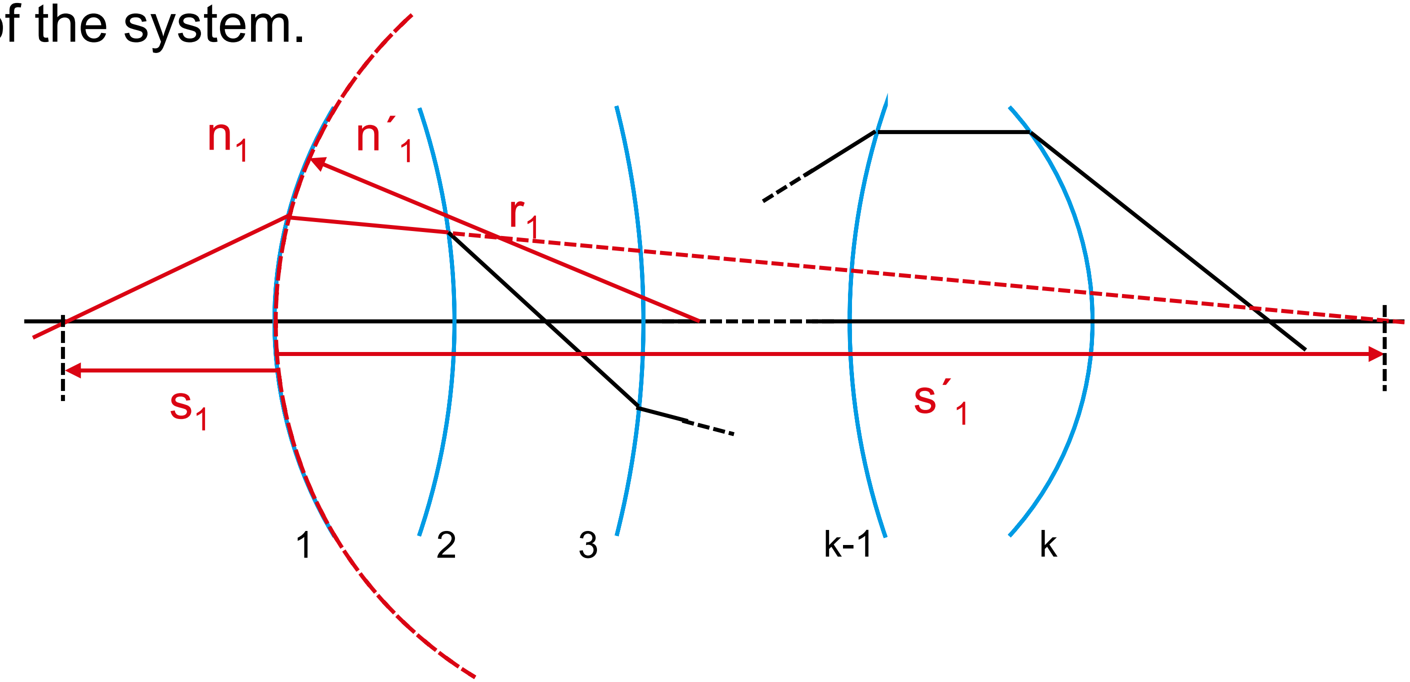
Paraxial ray tracing:

$$-\frac{n_j}{s_j}+\frac{n_j{}'}{s_j{}'}=\frac{n_j{}'-n_j}{r_j}$$

Focusing condition at single surface

Solved for image distance s'_j

$$s_j{}'=\frac{n_j{}'}{\dfrac{n_j{}'-n_j}{r_j}+\dfrac{n_j}{s_j}}$$

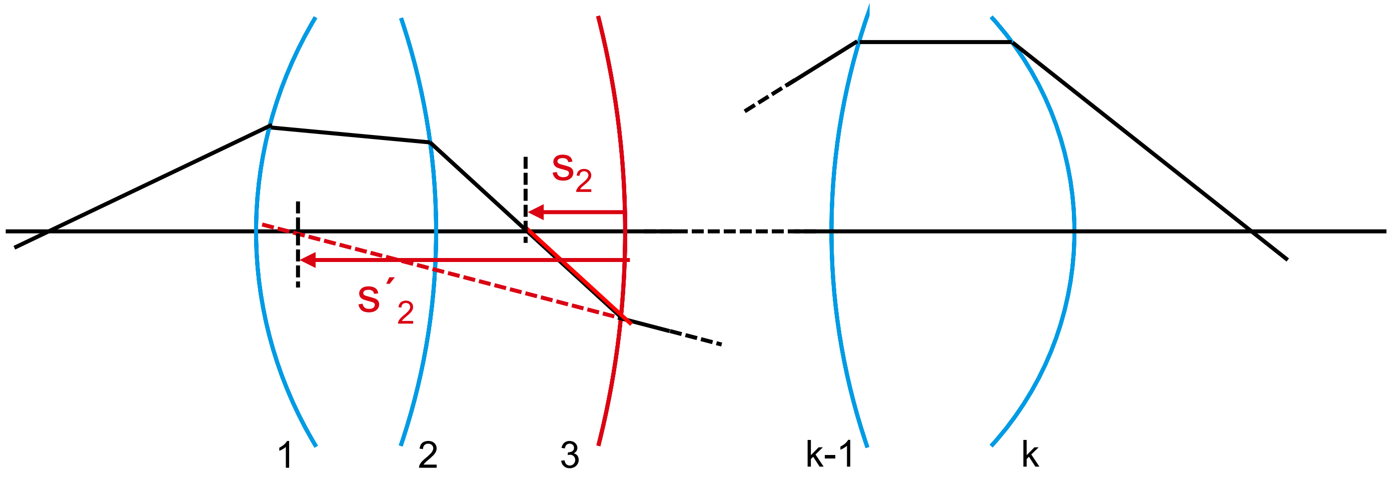


The paraxial image can be calculated by successively applying the paraxial imaging equation to each surface of the system.

Paraxial ray tracing:

$$-\frac{n_j}{s_j}+\frac{n_j'}{s_j'}=\frac{n_j'-n_j}{r_j}$$

Focusing condition at single surface



Solved for image distance s'_j

$$s_j'=\frac{n_j'}{\frac{n_j'-n_j}{r_j}+\frac{n_j}{s_j}}$$

In case of **virtual** imaging both s_j and s'_j are both on the same side of the surface.

Paraxial ray tracing

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

The paraxial image can be calculated by successively applying the paraxial imaging equation to each surface of the system.

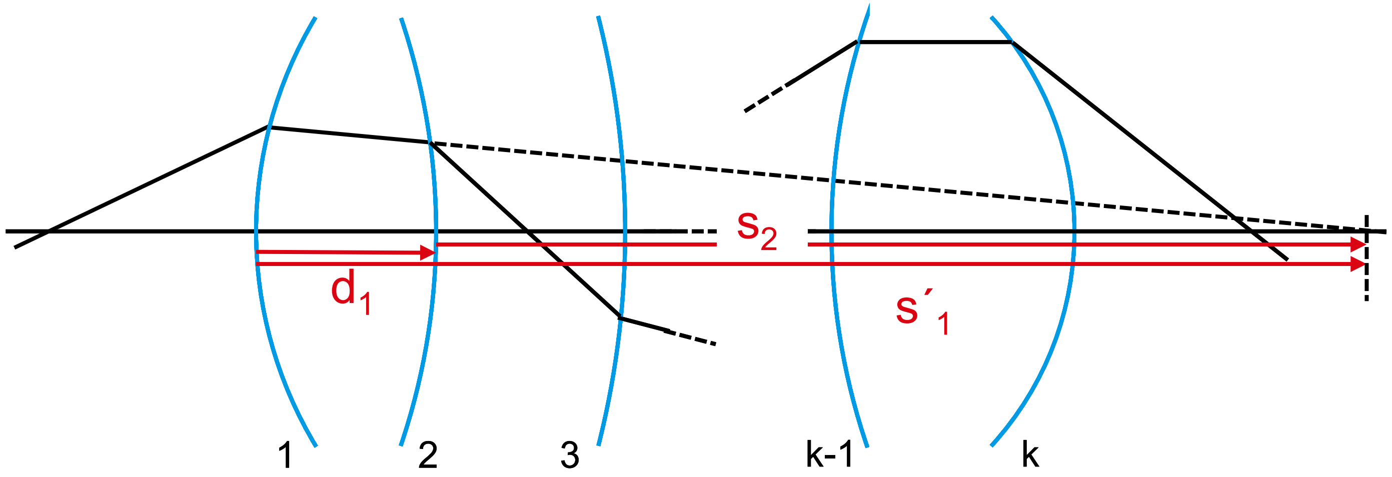
Paraxial ray tracing:

$$-\frac{n_j}{s_j}+\frac{n_j'}{s_j'}=\frac{n_j'-n_j}{r_j}$$

Focusing condition at single surface

$$s_{j+1}=s'_j-d_j$$

Transfer to next surface at distance d_j



Obtaining the next object distance s_{j+1} from the previous image distance s'_j and surface distance d_j.

Paraxial ray tracing

The paraxial image can be calculated by successively applying the paraxial imaging equation to each surface of the system.

Paraxial ray tracing:

$$-\frac{n_j}{s_j}+\frac{n_j{}'}{s_j{}'}=\frac{n_j{}'-n_j}{r_j}$$

Focusing condition at single surface

$$s_{j+1} = s'_j - d_j$$

Transfer to next surface at distance d_j

$$n_{j+1} = n'_j$$

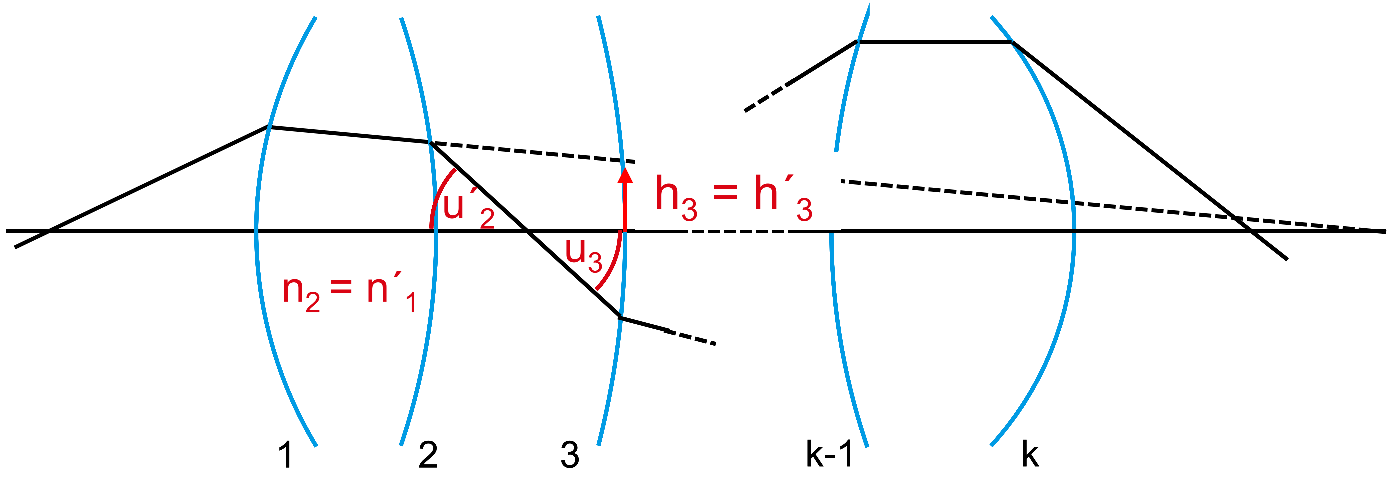
Transfer to next surface for refractive index n_j,

$$u_{j+1} = u'_j$$

aperture angle u_j,

$$h_j = h'_j$$

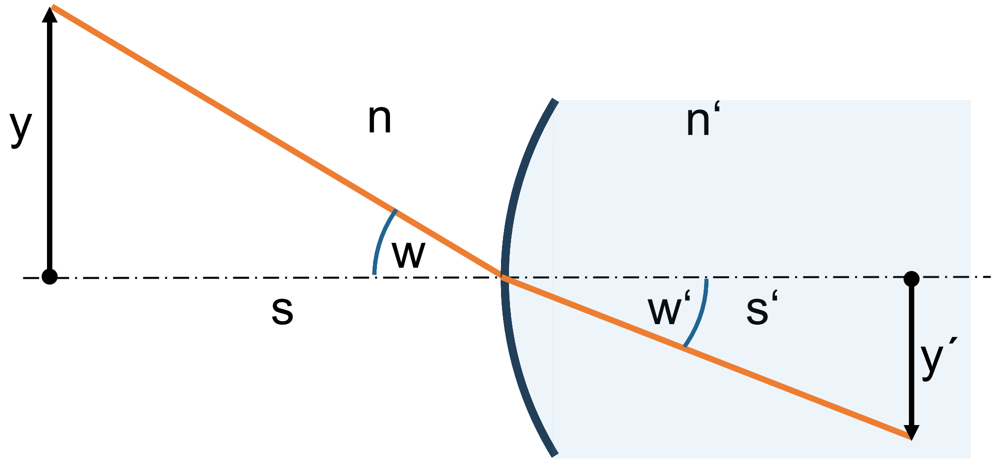
and intersection height h_j.

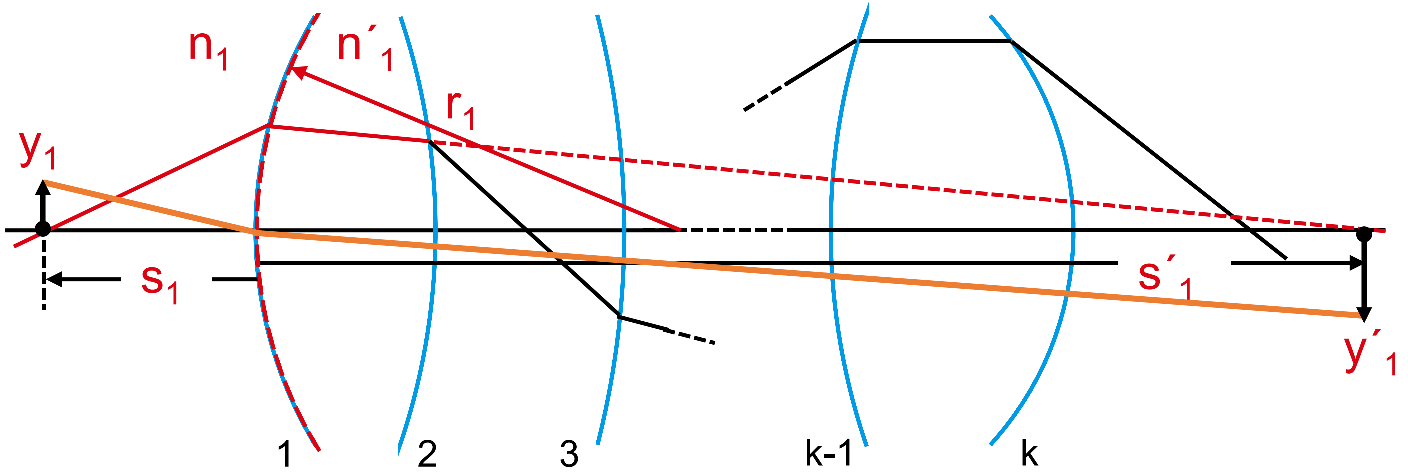


Institute of Applied Physics
Friedrich-Schiller-Universität Jena

Equation [6], $\frac{y'}{y} = \left(\frac{\frac{1}{s_1'}-\frac{1}{s_2'}}{\frac{1}{s_1}-\frac{1}{s_2}}\right)\left(\frac{s_1'}{s_1}\right)$, using the imaging equation $-\frac{n}{s}+\frac{n'}{s'}=\frac{n'}{r}$ or $\frac{1}{s'}=\frac{1}{n'}\left(\frac{n'}{r}+\frac{n}{s}\right)$ the indexed distances in the first factor on the right-hand side cancel out and we have another expression for the magnification:

$$\boxed{m=\frac{y'}{y}=\frac{n}{n'}\frac{s'}{s}}$$

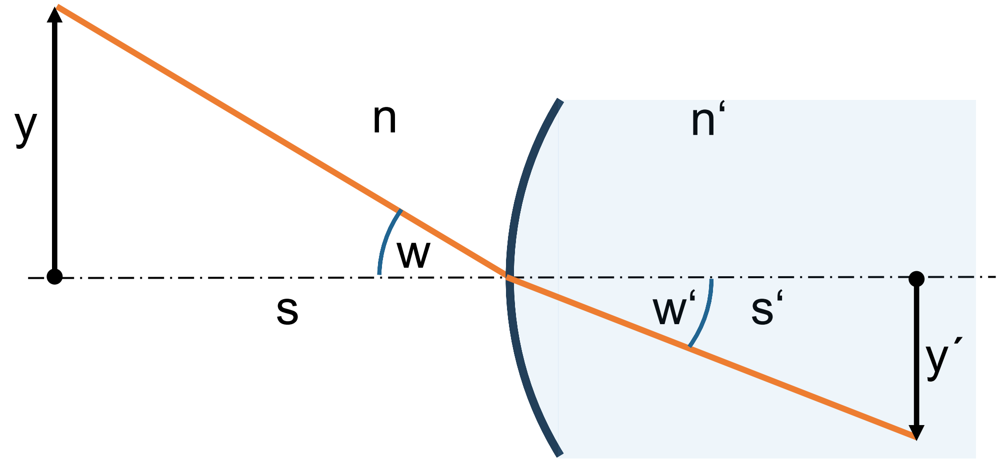




Magnification

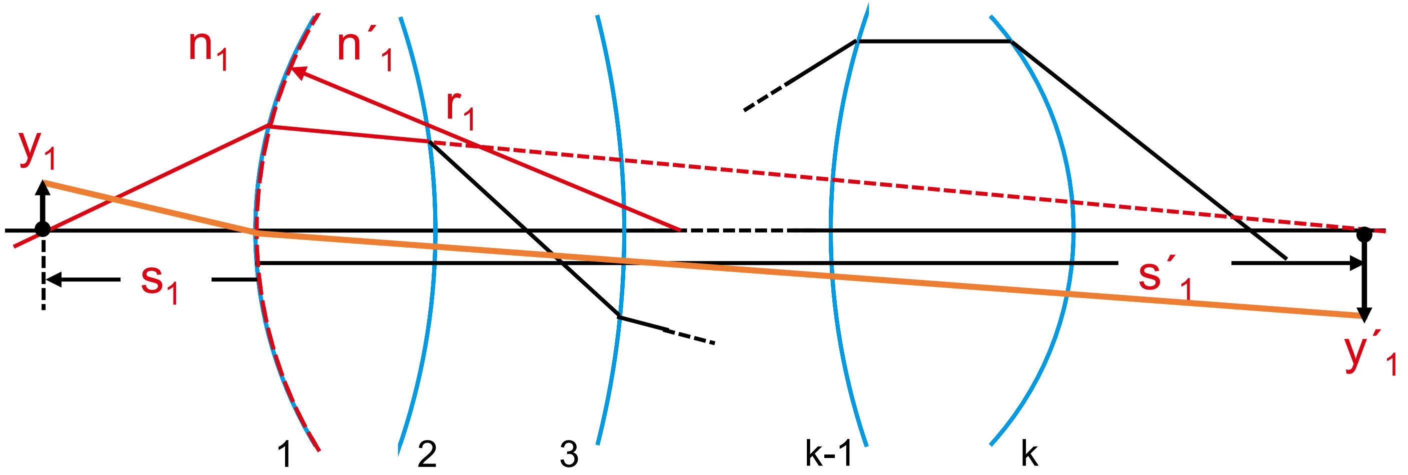
Equation [6], $\frac{y'}{y} = \left(\frac{\frac{1}{s_1'} - \frac{1}{s_2'}}{\frac{1}{s_1} - \frac{1}{s_2}}\right)\left(\frac{s_1'}{s_1}\right)$, using the imaging equation $-\frac{n}{s} + \frac{n'}{s'} = \frac{n'}{r}$ or $\frac{1}{s'} = \frac{1}{n'}\left(\frac{n'}{r} + \frac{n}{s}\right)$ the indexed distances in the first factor on the right-hand side cancel out and we have another expression for the magnification:

$$\boxed{m = \frac{y'}{y} = \frac{n}{n'}\frac{s'}{s}}$$



Applying this for a sequence of surfaces ($n_{j+1} = n'_j$):

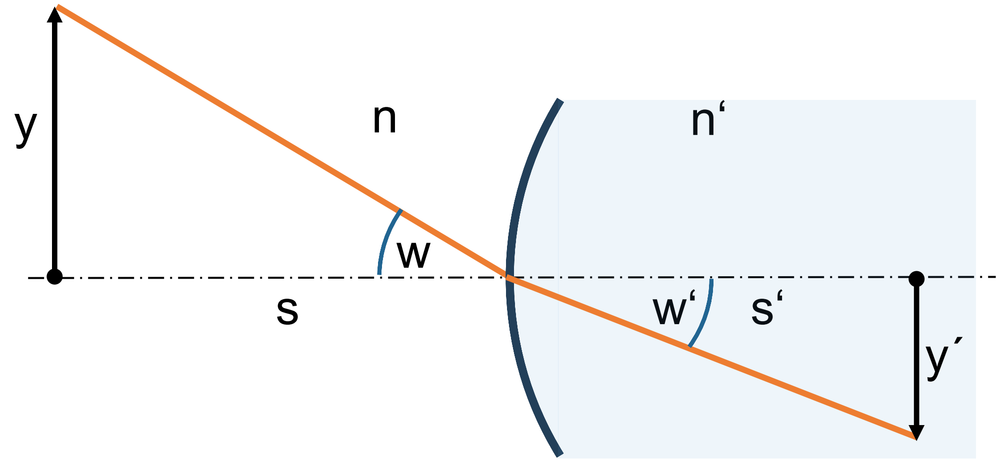
$$\boxed{m = \prod_{j=1}^{k} m_j = \prod_{j=1}^{k} \frac{n_j s_j'}{n_j' s_j} = \frac{n_1}{n_k'} \prod_{j=1}^{k} \frac{s_j'}{s_j}}$$



Magnification

Equation [6], $\frac{y'}{y} = \left(\frac{\frac{1}{s'_1}-\frac{1}{s'_2}}{\frac{1}{s_1}-\frac{1}{s_2}}\right)\left(\frac{s'_1}{s_1}\right)$, using the imaging equation $-\frac{n}{s}+\frac{n´}{s´}=\frac{n´}{r}$ or $\frac{1}{s´}=\frac{1}{n´}\left(\frac{n´}{r}+\frac{n}{s}\right)$ the indexed distances in the first factor on the right-hand side cancel out and we have another expression for the magnification:

$$\boxed{m=\frac{y'}{y}=\frac{n}{n´}\frac{s´}{s}}$$



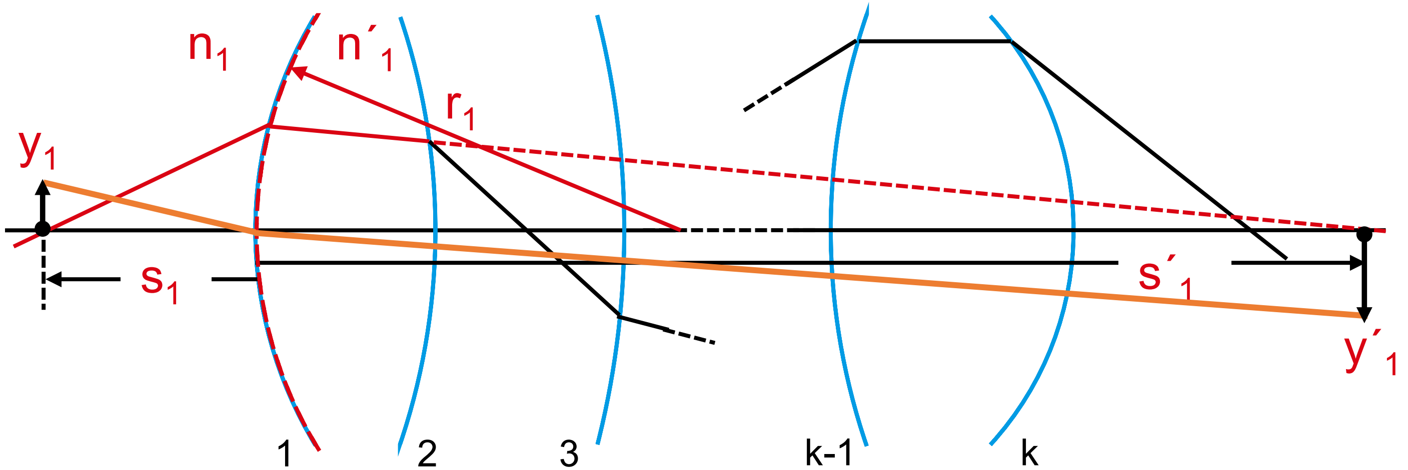
Applying this for a sequence surfaces ($n_{j+1} = n´_j$):

$$\boxed{m=\prod_{j=1}^{k} m_j = \prod_{j=1}^{k}\frac{n_j s'_j}{n'_j s_j} = \frac{n_1}{n'_k}\prod_{j=1}^{k}\frac{s'_j}{s_j}}$$

With $u_j = \frac{h_j}{s_j}$, $u_j = \frac{h_j}{s_j}$, $u_{j+1} = u´_j$, $h_j = h´_j$:

$$\boxed{m=\frac{n_1}{n'_k}\prod_{j=1}^{k}\frac{h'_j u'_j}{h_j u_j} = \frac{n_1 u_1}{n'_k u'_k}}$$

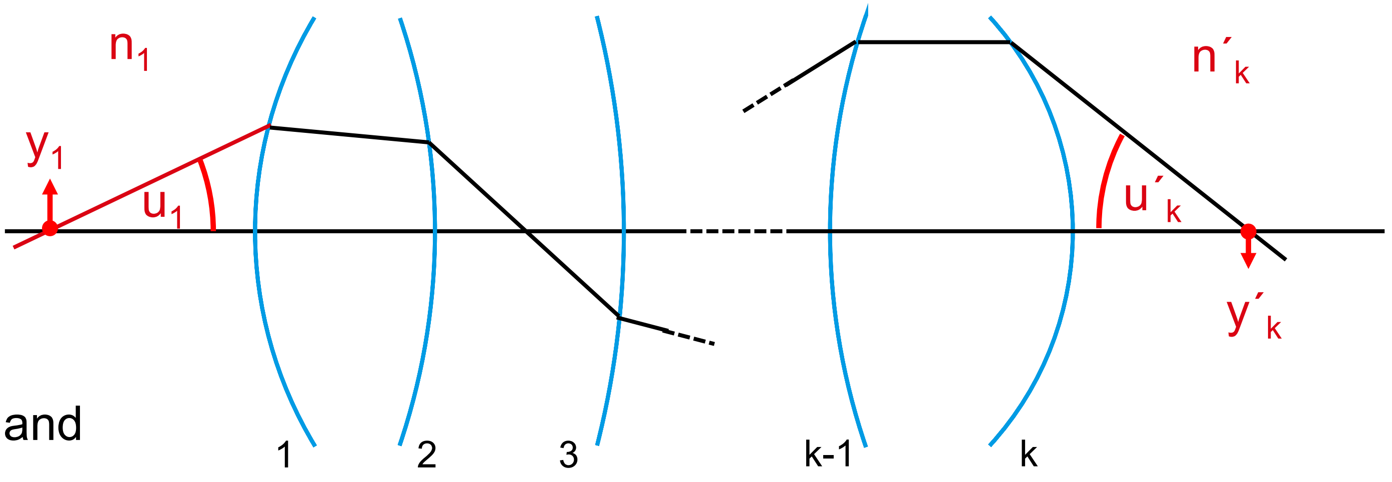
Paraxial numerical apertures in object and image space only!



Magnification / Helmholtz-Lagrange Invariant

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

$$m = \prod_{j=1}^{k} m_j = \prod_{j=1}^{k} \frac{n_j s_j'}{n_j' s_j} = \frac{n_1}{n_k'} \prod_{j=1}^{k} \frac{s_j'}{s_j}$$



$$m = \frac{n_1}{n_k'} \prod_{j=1}^{k} \frac{h_j' u_j'}{h_j u_j} = \frac{n_1 u_1}{n_k' u_k'}$$

Paraxial numerical apertures in object and image space only!

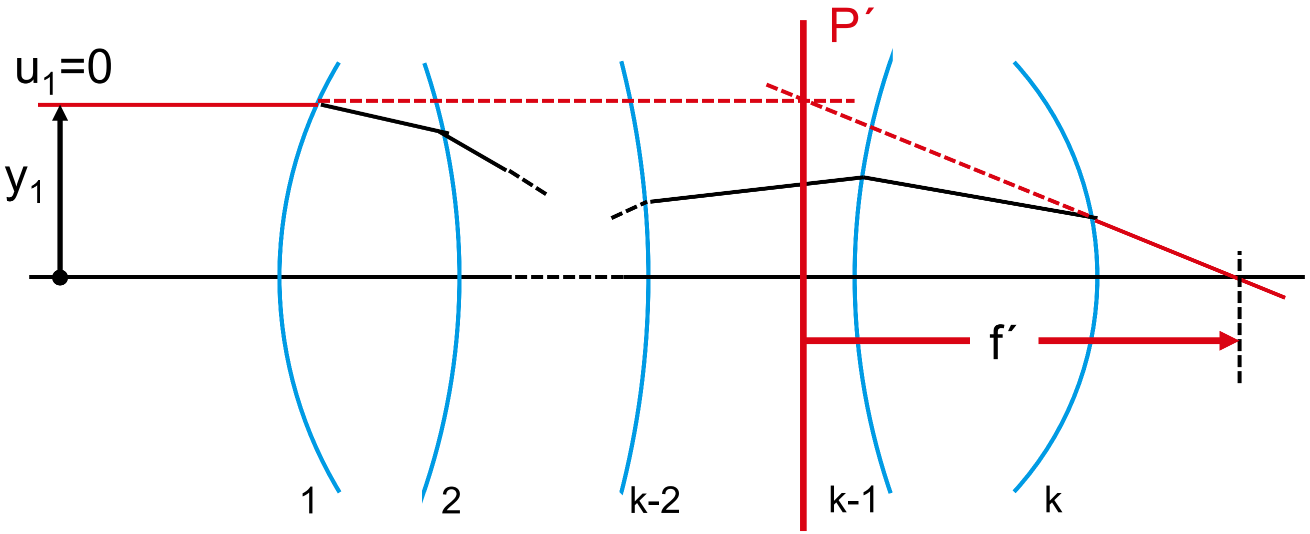
With $m = \frac{y_k'}{y_1}$ we get the **Helmholtz-Lagrange Invariant**, which holds between object and image space and also any other intermediate location inside the optical system:

$$n_1 u_1 y_1 = n_j u_j y_j = n_k' u_k' y_k'$$

Focal length, Calculation of Principal Plane positions

The focal length of the complete system f is calculated by paraxial tracing of a parallel object ray:

Multiplying the expression for magnification, $m = \frac{n_1}{n_k'} \prod_{j=1}^{k} \frac{s_j'}{s_j}$, with s$_1$, the result $m\, s_1$ remains finite even for an object at infinite distance $s_1 \rightarrow \infty$ (except for when the ray exits parallel to the optical axis representing an afocal system). The resulting system constant is the focal length f´ with respect to image space:



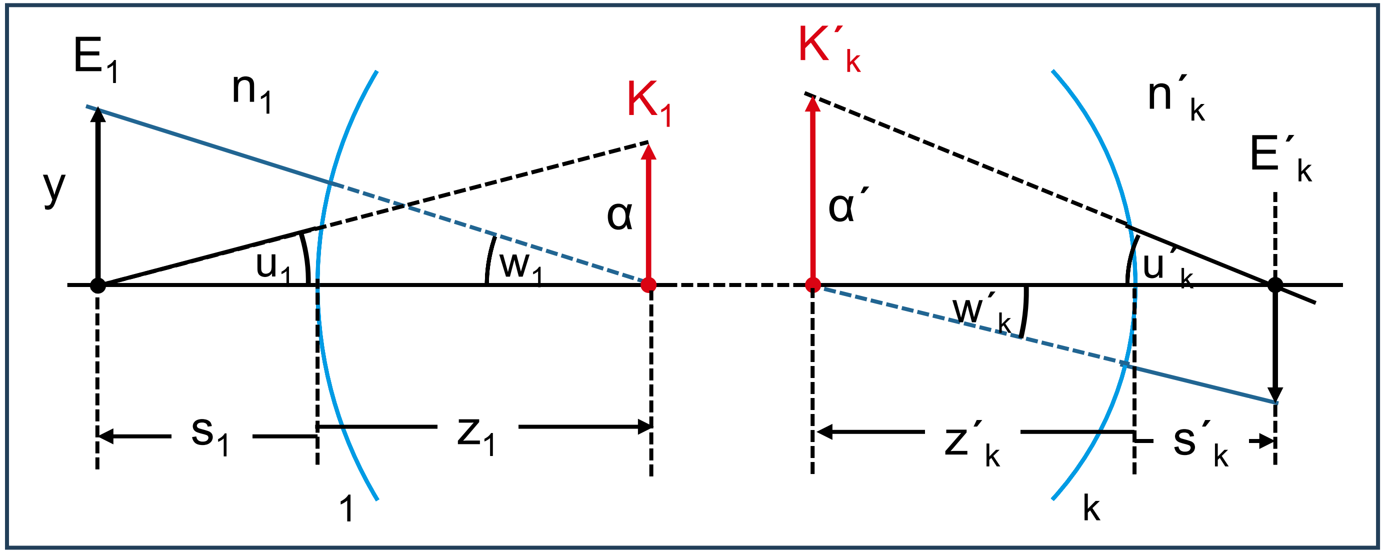
$$f' = (s_1 m)\Big|_{s_1 \rightarrow \infty} = \frac{n_1}{n_k'} \left(s_1' \prod_{j=2}^{k} \frac{s_j'}{s_j} \right)_{s_1 \rightarrow \infty}$$

Applying the surface imaging equation the image distance of the first surface can be explicitly written $s_1{'}|_{s_1 \rightarrow \infty} = \frac{n_1{'}}{n_1{'} - n_1} r_1$

With the focal length you automatically obtain the position of the **back principal plane** P´.
The position of the **front principal plane P** on the object side is obtained accordingly if one calculates from "right to left (i.e. parallel beam on the image side)".

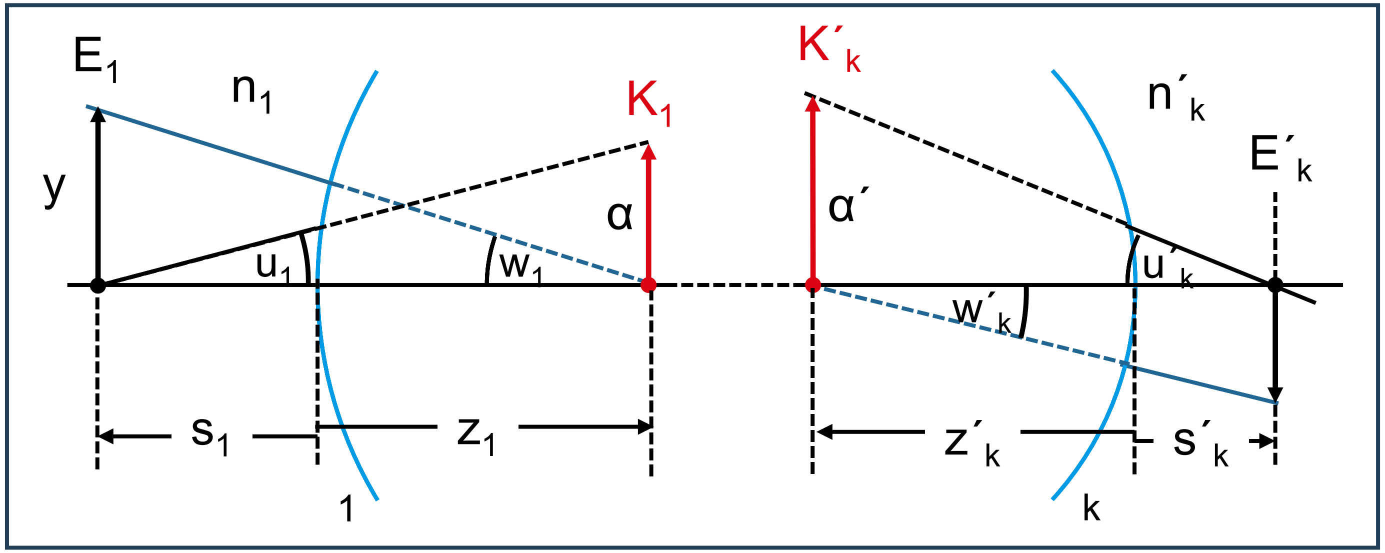
Depth magnification

We define <u>two pairs of conjugate planes</u>, E_1, E_k' and K_1, K_k' respectively, with respect to the complete optical system (spherical surfaces 1 to k).
They are different object-image planes; the fact that K_1, K_k' are drawn virtual is not relevant for generality.



Depth magnification

We define <u>two pairs of conjugate planes</u>, E_1, $E_k{}'$ and K_1, $K_k{}'$ respectively, with respect to the complete optical system (spherical surfaces 1 to k).
They are different object-image planes; the fact that K_1, $K_k{}'$ are drawn virtual is not relevant for generality.



From the figure we see that

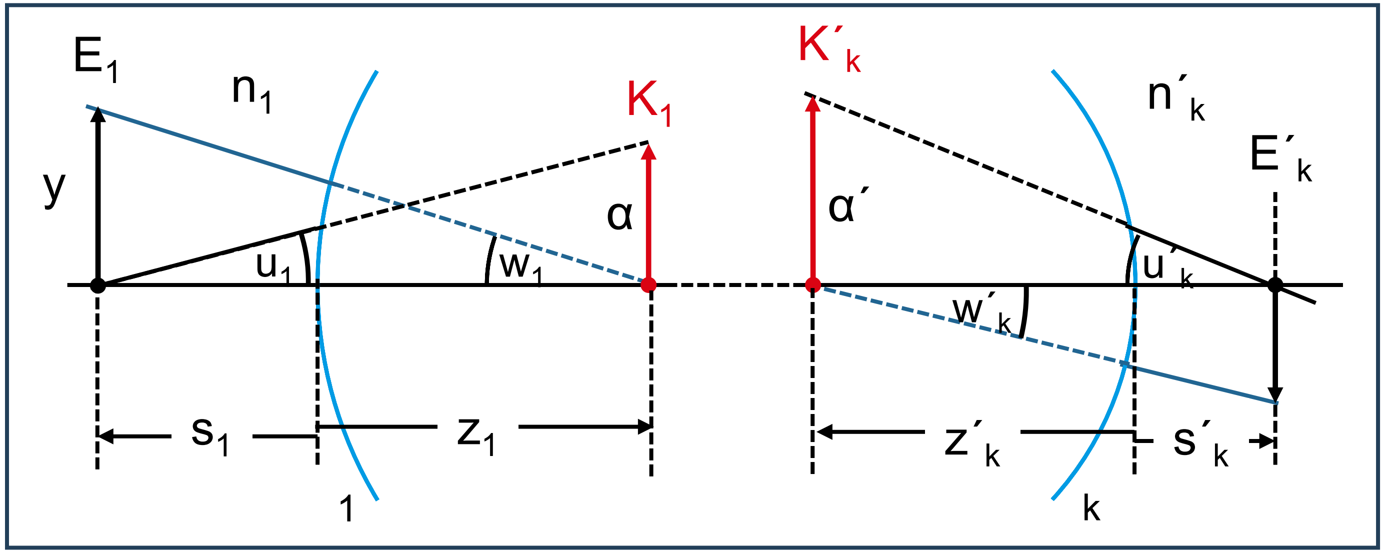
$$u_1 = \frac{\alpha}{z_1 - s_1}, \qquad u'_k = \frac{\alpha'}{z'_k - s'_k} \qquad \text{aperture angles}$$

$$w_1 = \frac{y}{z_1 - s_1}, \qquad w'_k = \frac{y'}{z'_k - s'_k} \qquad \text{field angles}$$

Therefore, the magnification m_z between the conjugate planes K_1, $K_k{}'$ is (with $m = \frac{n_1 u_1}{n'_k u'_k}$ and replacing u→w)

$$m_z = \frac{\alpha'}{\alpha} = \frac{n_1\, w_1}{n'_k\, w'_k} = \frac{n_1}{n'_k} \frac{y}{y'} \left(\frac{z'_k - s'_k}{z_1 - s_1} \right) = \frac{n_1}{n'_k} \frac{1}{m} \left(\frac{z'_k - s'_k}{z_1 - s_1} \right)$$

We define <u>two pairs of conjugate planes</u>, E_1, $E_k{'}$ and K_1, $K_k{'}$ respectively, with respect to the complete optical system (spherical surfaces 1 to k).
They are different object-image planes; the fact that K_1, $K_k{'}$ are drawn virtual is not relevant for generality.



From the figure we see that

$$u_1 = \frac{\alpha}{z_1 - s_1}\ , \qquad u'_k = \frac{\alpha'}{z'_k - s'_k} \qquad \text{aperture angles}$$

$$w_1 = \frac{y}{z_1 - s_1}\ , \qquad w'_k = \frac{y'}{z'_k - s'_k} \qquad \text{field angles}$$

Therefore, the magnification m_z between the conjugate planes K_1, $K_k{'}$ is (with $m = \frac{n_1 u_1}{n'_k u'_k}$ and replacing u→w)

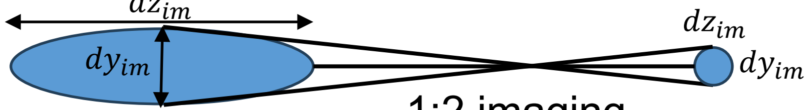
$$m_z = \frac{\alpha'}{\alpha} = \frac{n_1\, w_1}{n'_k\, w'_k} = \frac{n_1}{n'_k}\frac{y}{y'}\left(\frac{z'_k - s'_k}{z_1 - s_1}\right) = \frac{n_1}{n'_k}\frac{1}{m}\left(\frac{z'_k - s'_k}{z_1 - s_1}\right)$$

Rewriting this we have an expression for the <u>depth magnification</u>

$$\left(\frac{z'_k - s'_k}{z_1 - s_1}\right) = \frac{n'_k}{n_1}\, m\, m_z$$

If E_1, $E_k{'}$ and K_1, $K_k{'}$ are two closely adjacent object-image pairs m_z approaches m and for small longitudinal distances from object dz_{ob} and image dz_{im} respectively:

$$\frac{dz_{im}}{dz_{ob}} = \frac{n'_k}{n_1} m^2$$



1:2 imaging

Principal planes and their relation to focal length (1/2)

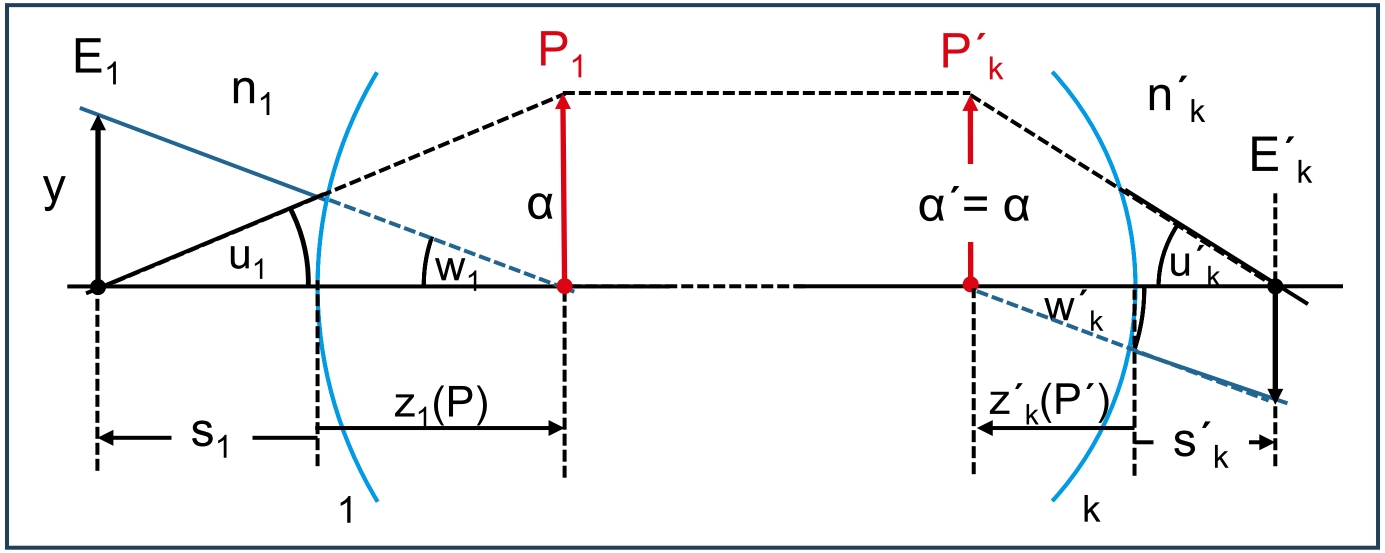
Institute of Applied Physics
Friedrich-Schiller-Universität Jena

The special pair of conjugate planes K_1, $K_k´$ with a magnification

$$m_z = 1$$

are called **principal planes**. We denote them P_1, $P_k´$ or for simplicity P, P´. With $m_z = \frac{\alpha´}{\alpha} = \frac{n_1\, w_1}{n´_k\, w´_k} = 1$ we have

$$n_1\; w_1 = n´_k\; w´_k$$



Principal planes and their relation to focal length (1/2)

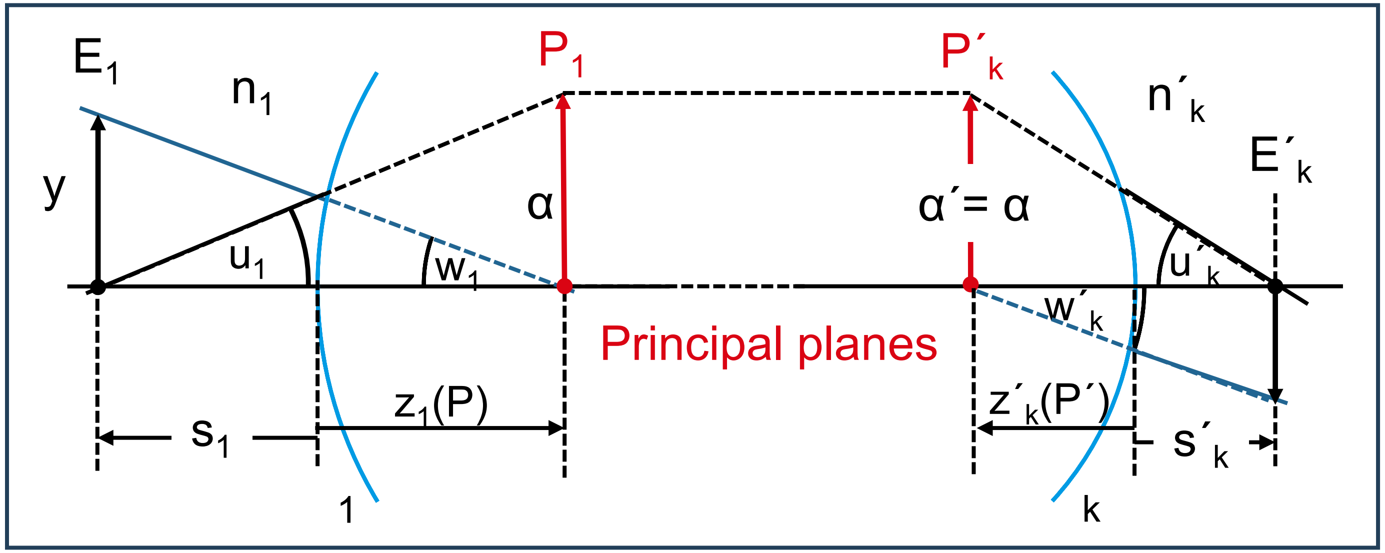
The special pair of conjugate planes K_1, $K_k´$ with a magnification

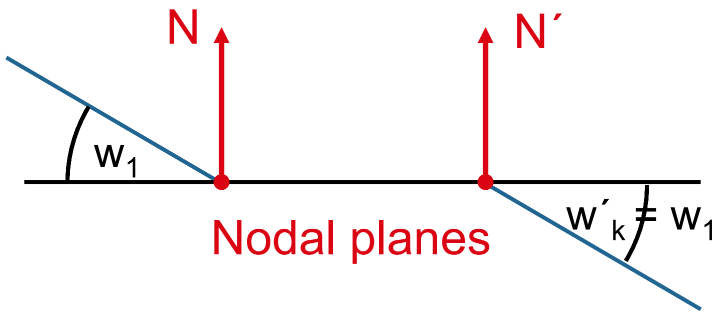
$$m_z = 1$$

are called **principal planes**. We denote them P_1, $P_k´$ or for simplicity P, P´. With $m_z = \frac{\alpha´}{\alpha} = \frac{n_1\, w_1}{n´_k\, w´_k} = 1$ we have

$$n_1\ w_1 = n´_k\ w´_k$$

If the object and image space refractive indices are equal $n_1 = n´_k$, the ray heading from object to center of P keeps its direction from P´ to image, $w_1 = w´_k$. In this case principal planes are equal to **nodal planes**, which are defined as pair of conjugates N_1, $N_k´$ with $w_1 = w´_k$ (in general also for different $n_1, n´_k$).





Principal planes and their relation to focal length (1/2)

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

The special pair of conjugate planes K_1, K_k' with a magnification

$$m_z = 1$$

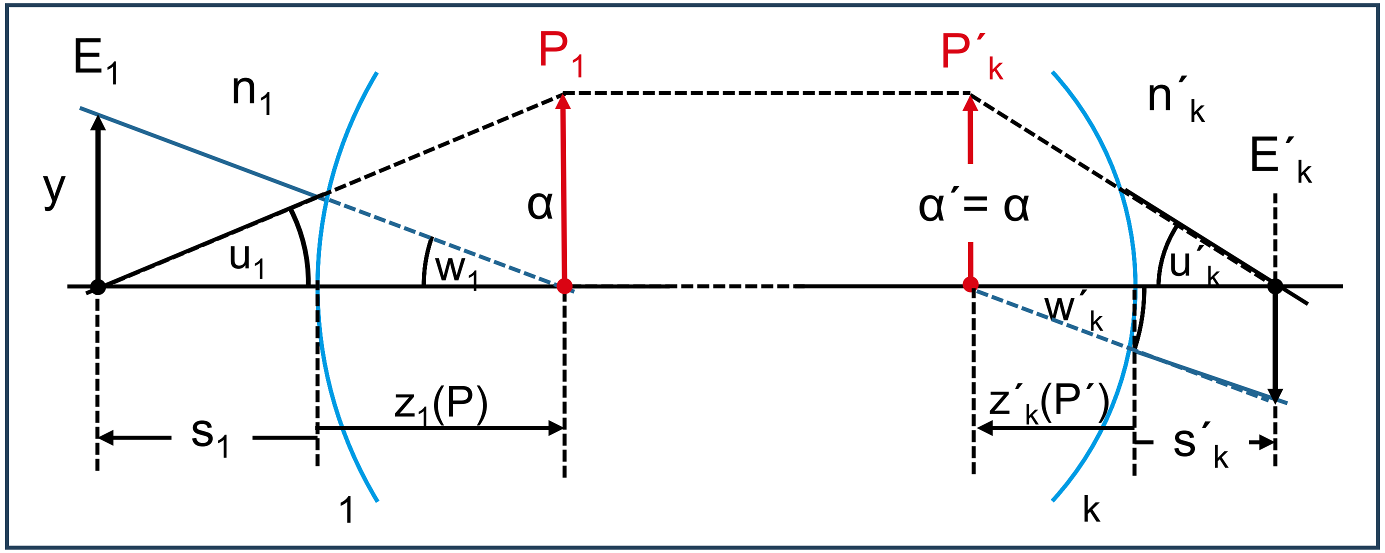
are called **principal planes**. We denote them P_1, P_k' or for simplicity P, P´. With $m_z = \frac{\alpha'}{\alpha} = \frac{n_1\, w_1}{n'_k\, w'_k} = 1$ we have

$$n_1\ w_1 = n'_k\ w'_k$$

If the object and image space refractive indices are equal $n_1 = n'_k$, the ray heading from object to center of P keeps its direction from P´ to image, $w_1 = w'_k$. In this case principal planes are equal to **nodal planes**, which are defined as pair of conjugates N_1, N_k' with $w_1 = w'_k$ (in general also for different n_1, n'_k).

For principal planes the depth magnification multiplied with object distance to first system surface s_1 is:

$$\frac{s_1(z'_k(P') - s'_k)}{z_1(P) - s_1} = \frac{n'_k}{n_1} s_1\, m$$



Principal planes and their relation to focal length (1/2)

The special pair of conjugate planes K_1, $K_k´$ with a magnification

$$m_z = 1$$

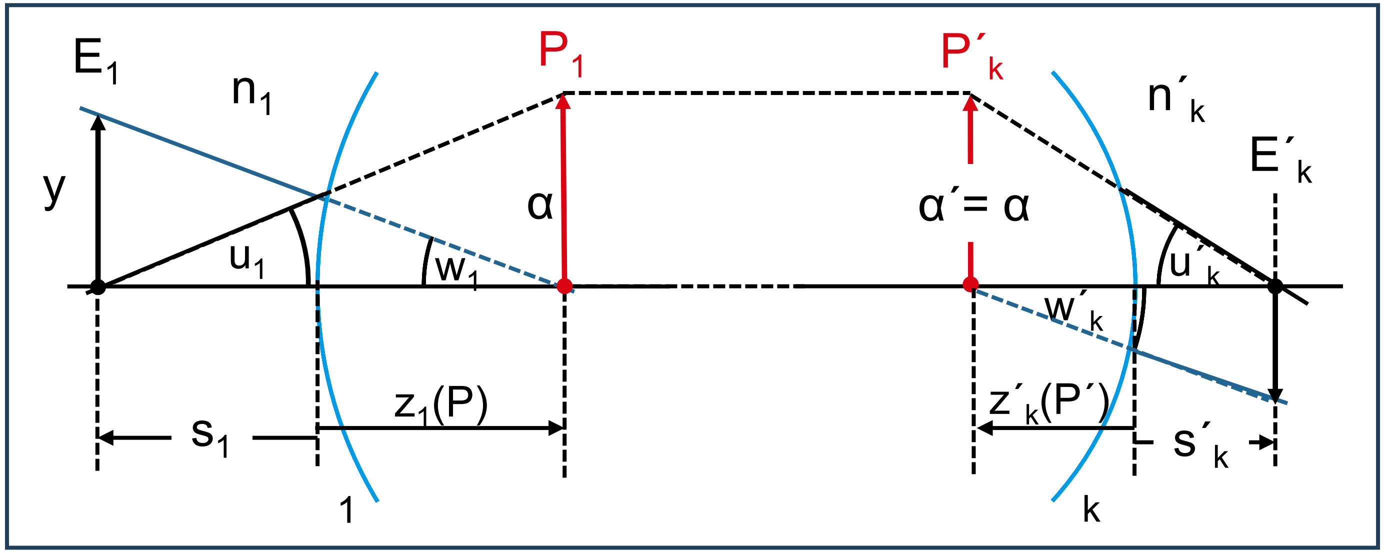
are called **principal planes**. We denote them P_1, $P_k´$ or for simplicity P, P´. With $m_z = \frac{\alpha´}{\alpha} = \frac{n_1\, w_1}{n´_k\, w´_k} = 1$ we have

$$n_1\; w_1 = n´_k\; w´_k$$

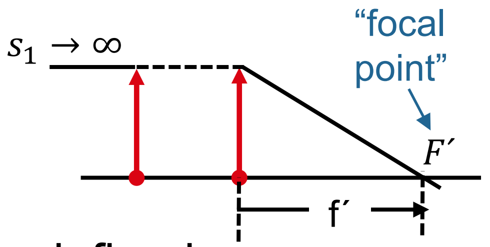
If the object and image space refractive indices are equal $n_1 = n´_k$, the ray heading from object to center of P keeps its direction from P´ to image, $w_1 = w´_k$. In this case principal planes are equal to **nodal planes**, which are defined as pair of conjugates N_1, $N_k´$ with $w_1 = w´_k$ (in general also for different $n_1, n´_k$).

For principal planes the <u>depth magnification</u> multiplied with object distance to first system surface s_1 is:

$$\frac{s_1(z´_k(P´) - s´_k)}{z_1(P) - s_1} = \frac{n´_k}{n_1} s_1\; m$$



Performing the limit $s_1 \rightarrow \infty$ gives for the left-hand side: $-(z´_k(P´) - s´_k)$



and for the right-hand side the previously defined system constant **focal length**: $\lim_{s_1 \to \infty}\left(\frac{n´_k}{n_1} s_1\; m\right) = f´$

so

$$f´ = (z´_k(P´) - s´_k)$$

that is <u>focal length corresponds the distance of P´ to image</u>.

Principal planes and their relation to focal length (2/2)

In order to get the corresponding relation for object space we divide the depth magnification by s'_k:

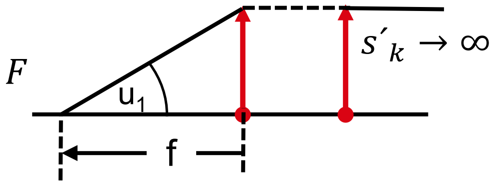
$$\frac{z'_k - s'_k}{s'_k\,(z_1 - s_1)} = \frac{n'_k}{n_1}\frac{m_z}{s'_k} = m_z\frac{u_1}{s'_k u'_k} \qquad [*]$$

Now with the relation $u'_k = \frac{\alpha'}{z'_k - s'_k}$ we get for $s'_k \to \infty$:

$$\lim_{s'_k\to\infty}(s'_k u'_k) = \lim_{s'_k\to\infty}\left(s'_k\frac{\alpha'}{z'_k - s'_k}\right) = -\alpha'$$

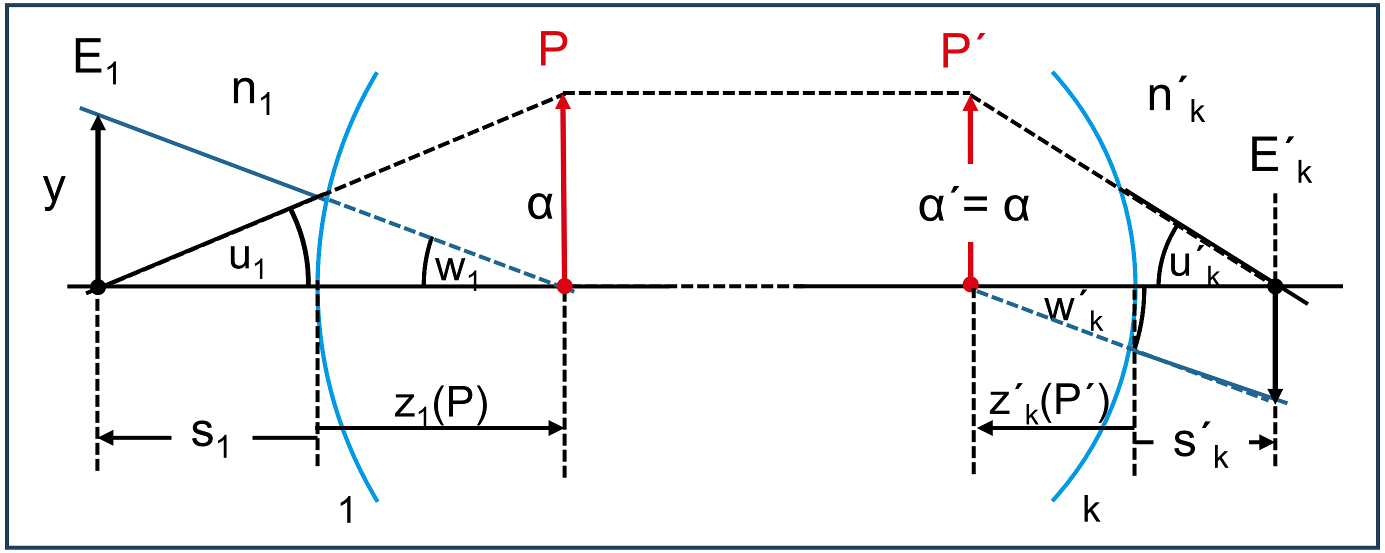
Thus [*] becomes in the limit $s'_k \to \infty$:

$$-\frac{1}{z_1 - s_1(F)} = -m_z\frac{u_1}{\alpha'} \qquad [**]$$



That is the rays entering the system from focal point F with the angle u_1 exit the system parallel to the axis.

The ratio u_1/α' remains constant for any distance z_1.



Multiplying [**] with z_1 and computing $\lim_{z_1\to\infty}(\ldots)$ we get, because $\lim_{s_1\to\infty}(z_1\, m) = f$, that because $-1 = f\frac{u_1}{\alpha'}$ and therefore $\frac{f}{z_1 - s_1(F)} = m_z$.

If z_1 is the distance from object-side principal plane P, $z_1 = z_1(P)$, then

$$z_1(P) - s_1(F) = f$$

And the relationship between object and image-side focal length is

$$f' = -\frac{n'_k}{n_1}f$$

Image construction of an optical system represented by principal planes: Newton Imaging Equation

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

The definition of the cardinal points of a system results in the following simple rules for image construction:

1. a) a ray passing through the focal point F on the object side leaves the system parallel to the axis
b) an object-side ray incident parallel to the axis passes through the focal point F´ on the image side

2. each ray intersects the object-side and image-side principal planes side at the same distance from the axis respectively

3. a ray aimed at the object-side nodal point exits the image-side nodal point without changing direction.

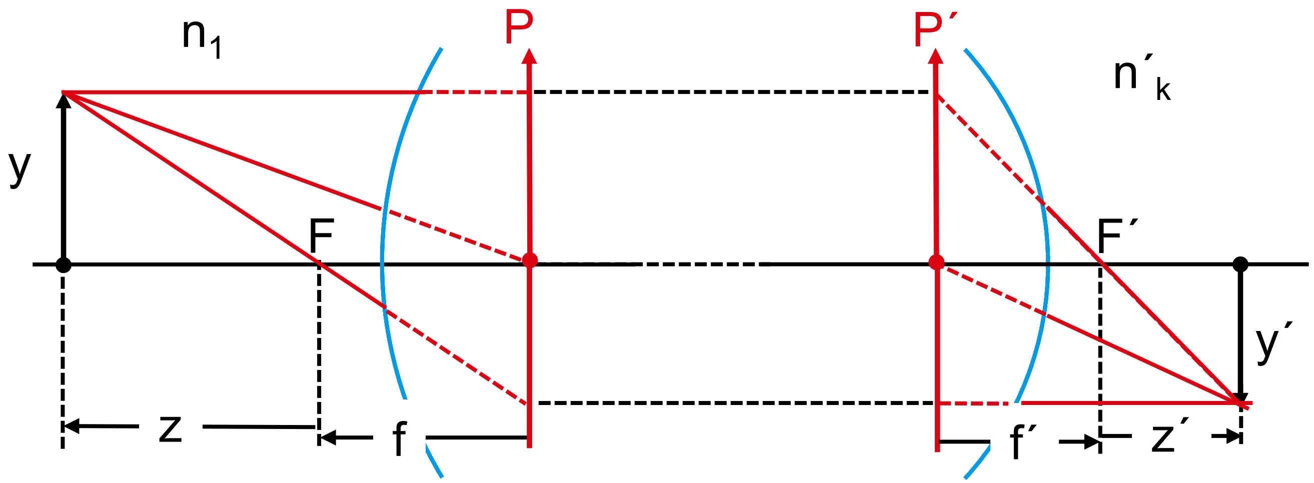


Image construction of an optical system represented by principal planes: Newton Imaging Equation

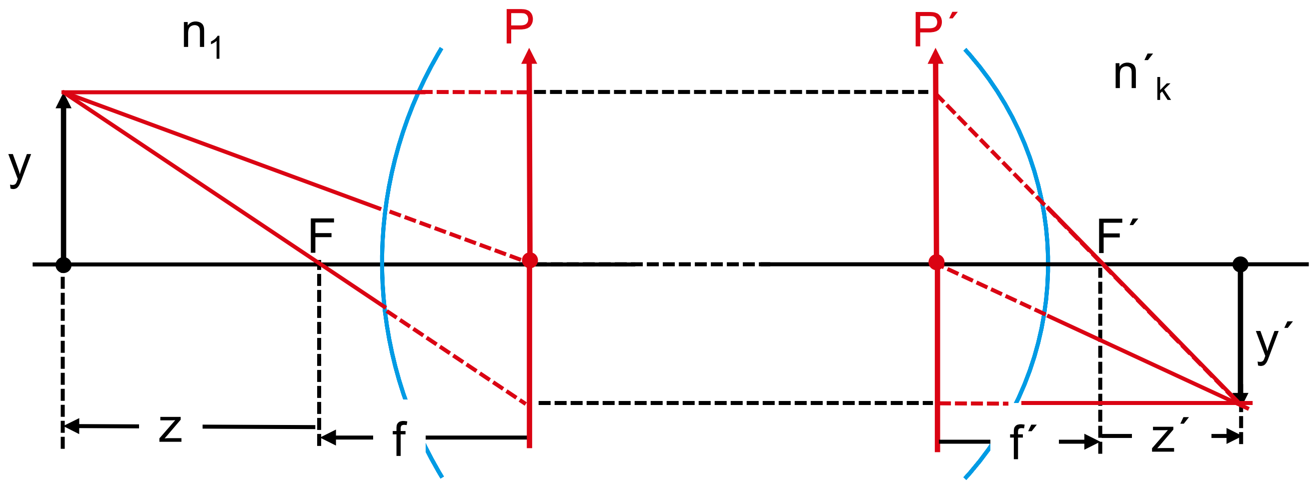
The definition of the cardinal points of a system results in the following simple rules for image construction:

1. a) a ray passing through the focal point F on the object side leaves the system parallel to the axis
b) an object-side ray incident parallel to the axis passes through the focal point F´ on the image side

2. each ray intersects the object-side and image-side principal planes side at the same distance from the axis respectively

3. a ray aimed at the object-side nodal point exits the image-side nodal point without changing direction.

The position of the back principal plane can be determined with $f' = \frac{n_1}{n'_k}\left(s'_1 \prod_{j=2}^{k} \frac{s'_j}{s_j} \right)_{s_1 \to \infty}$. The distances s_j, s'_j are a function of the system parameters (r_j, d_j, n_j) and s_1 (infinite).



The relation of object and image distances can be determined very simply by the proportionalities in the figure:

$$m = \frac{y'}{y} = \frac{z'}{f'} = \frac{f}{z}$$

or

$$\boxed{z\,z' = f f'} \quad \text{with} \quad f' = -\frac{n'_k}{n_1} f$$

In this form where object and image distances are related to the focal points the paraxial imaging equation is called **Newton´s imaging equation**.

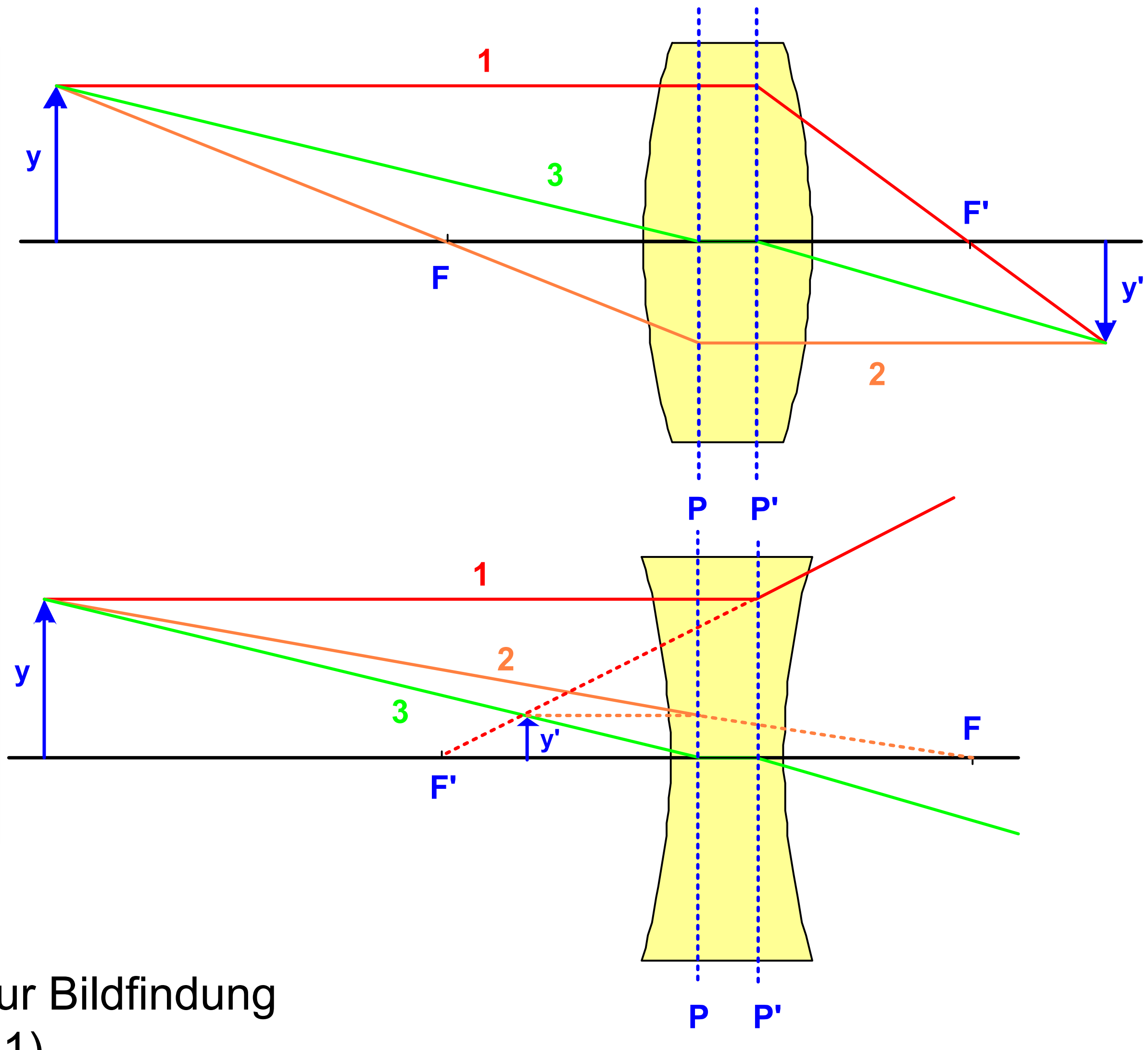
Graphical Image Construction (by J.B. Listing in 1851)

Graphical image construction according to Listing by 3 dedicated rays:

1. First parallel through axis, through focal point F´ in image space
2. First through focal point F, then parallel to optical axis
3. Through nodal points, leaves the lens with the same angle

Procedure works for positive and negative lenses.

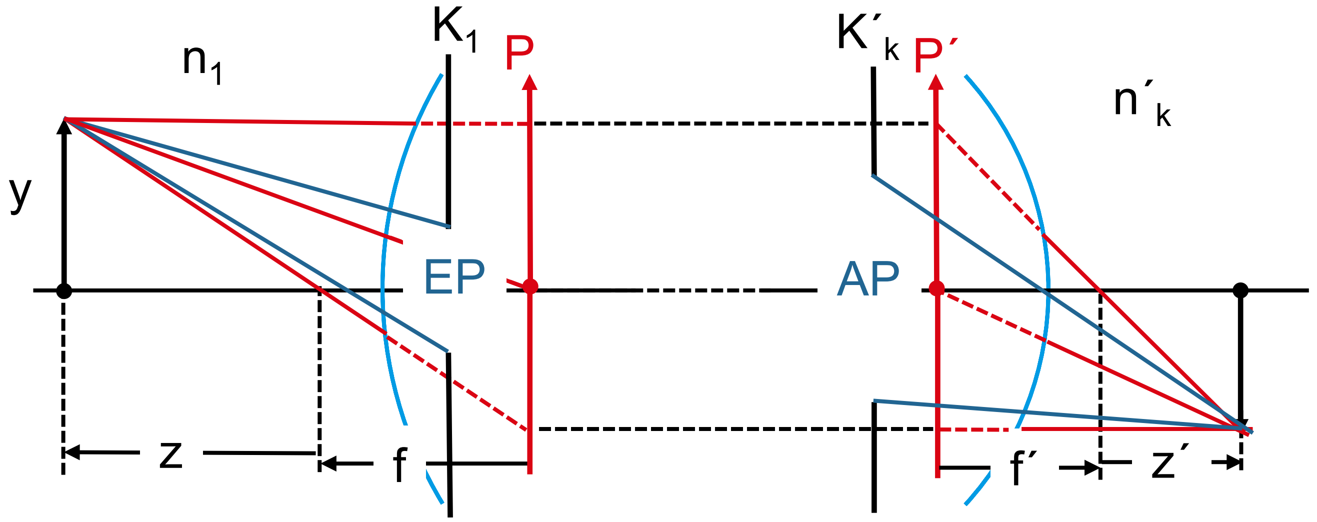
For negative lenses, the F / F‘ sequence is reversed.



Listing, J. B. (1808-1882). Zeichenverfahren zur Bildfindung [Graphical method to find image position] (1851)

General Imaging Equation via arbitrary conjugate planes (1/2)

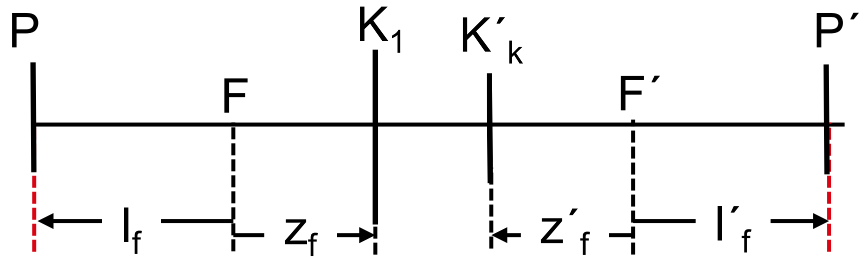
The construction of principal planes is a useful, simple way to (graphically) determine the image position, as we will show it is not well suited to describe the path of rays in a real optical system as those depend on the stop position which is not considered.



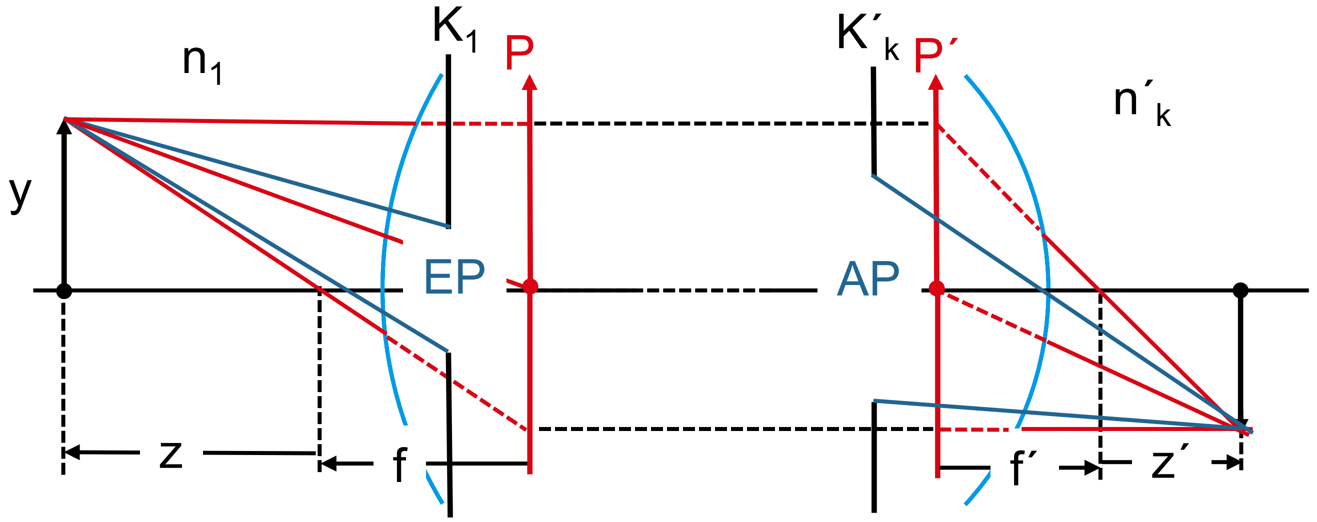
Institute of Applied Physics
Friedrich-Schiller-Universität Jena

The construction of principal planes is a useful, simple way to (graphically) determine the image position, as we will show it is not well suited to describe the path of rays in a real optical system as those depend on the stop position which is not considered.

Therefore, we return to the general case of arbitrary conjugate pair K_1, $K_k´$ which we will assign the pupil planes EP, AP later.



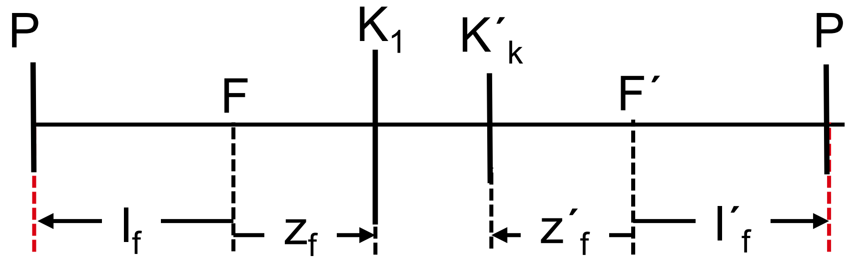
The distances l_f, $l´_f$ to P, P´ and the distances z_f, $z´_f$ of the conjugate pair K_1, $K_k´$ are referenced to the focal points (figure above). The object- and image distances of K_1, $K_k´$ are s, s´ respectively.



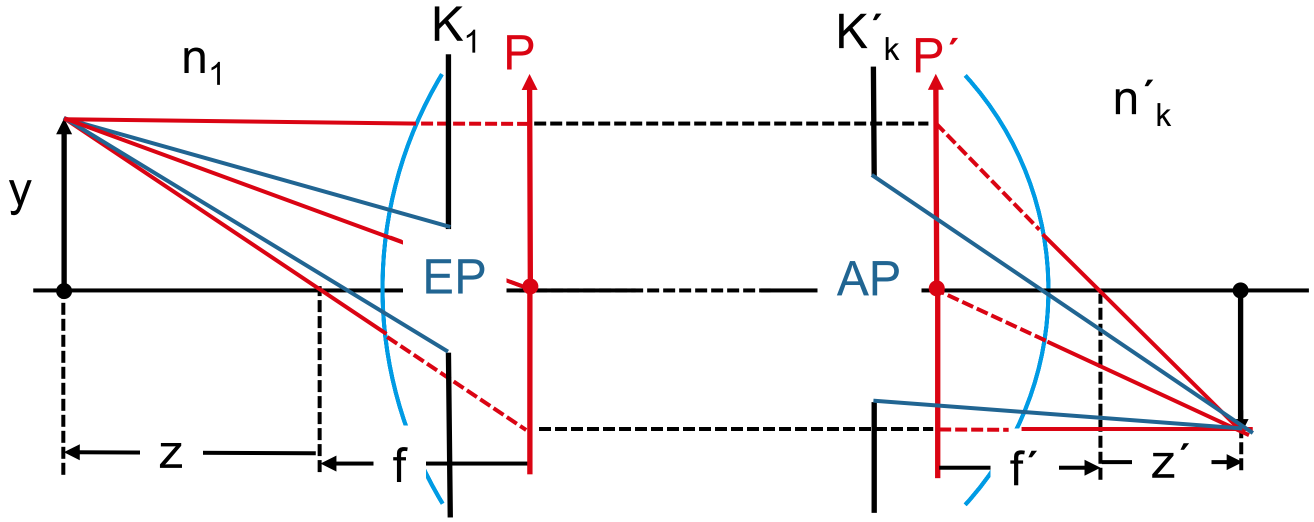
General Imaging Equation via arbitrary conjugate planes (1/2)

The construction of principal planes is a useful, simple way to (graphically) determine the image position, as we will show it is not well suited to describe the path of rays in a real optical system as those depend on the stop position which is not considered.

Therefore, we return to the general case of arbitrary conjugate pair K_1, $K_k´$ which we will assign the pupil planes EP, AP later.



The distances l_f, $l´_f$ to P, P´ and the distances z_f, $z´_f$ of the conjugate pair K_1, $K_k´$ are referenced to the focal points (figure above). The object- and image distances of K_1, $K_k´$ are s, s´ respectively.



Then $s = l_f - z_f$ and $s´ = l´_f - z´_f$. [1]

To be consistent with the sign convention the imaging of K, K´ via P, P´ is again referenced to the focal points.

Pupil magnification is

$$m_z = \frac{\alpha´}{\alpha} = \frac{f}{z_f} = \frac{z´_f}{f´} \qquad [2]$$

and with Newton´s imaging equation:

$$f\,f´ = l_f l´_f \overset{[1]}{=} \left(s + z_f\right)\left(s´ + z´_f\right) \overset{[2]}{=} \left(s + \frac{f}{m_z}\right)\left(s´ + m_z f´\right)$$

$$= s\,s´ + s\,m_z f´ + \frac{f}{m_z} s´ + ff´ \qquad [3]$$

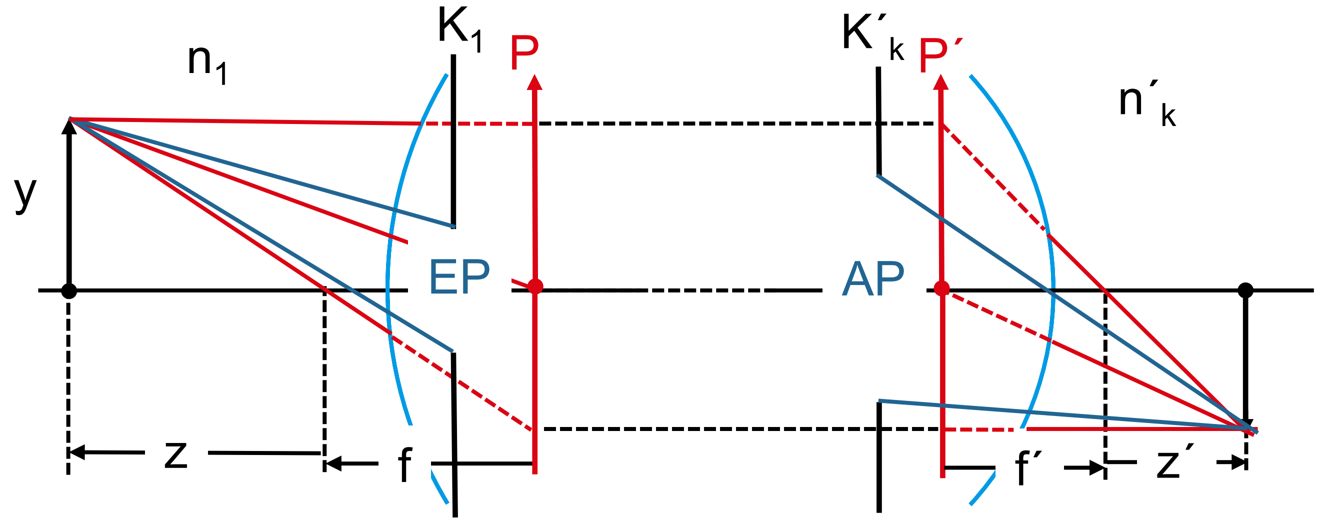
General Imaging Equation via arbitrary conjugate planes (2/2)

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

Division of [3] by fss´. $$-\frac{1}{m_z s}+\frac{n'_k}{n_1}\frac{m_z}{s'}=\frac{1}{f'}$$

Defining refractive power as $\Phi=-\frac{n_1}{f}=\frac{n'_k}{f'}$, the general imaging equation can also be written

$$-\frac{n_1}{s}+\frac{n'_k\, m_z^2}{s'}=m_z\Phi$$



If the conjugate pair K, K´ are principal planes P, P´, that is $m_z = 1$ the imaging equation is (object-/image distance to principal planes l, l´ respectively):

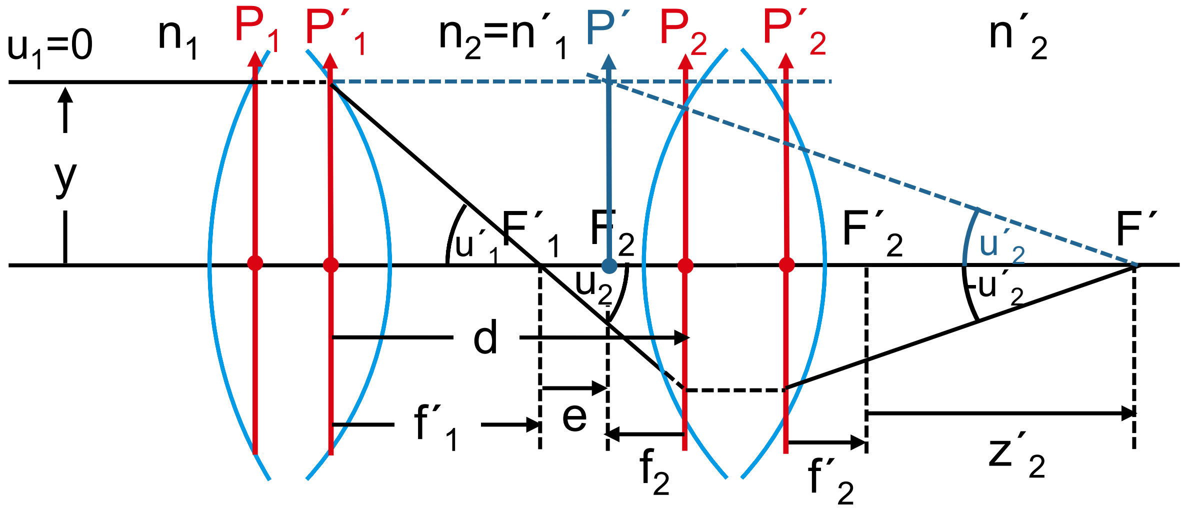
$$-\frac{n_1}{l}+\frac{n'_k}{l'}=\Phi$$

For refractive indices $n_1=n'_k=1$ it has the form of the “lens makers equation”:

$$-\frac{1}{l}+\frac{1}{l'}=\frac{1}{f'}$$

Two-Component System´s refractive power summation law

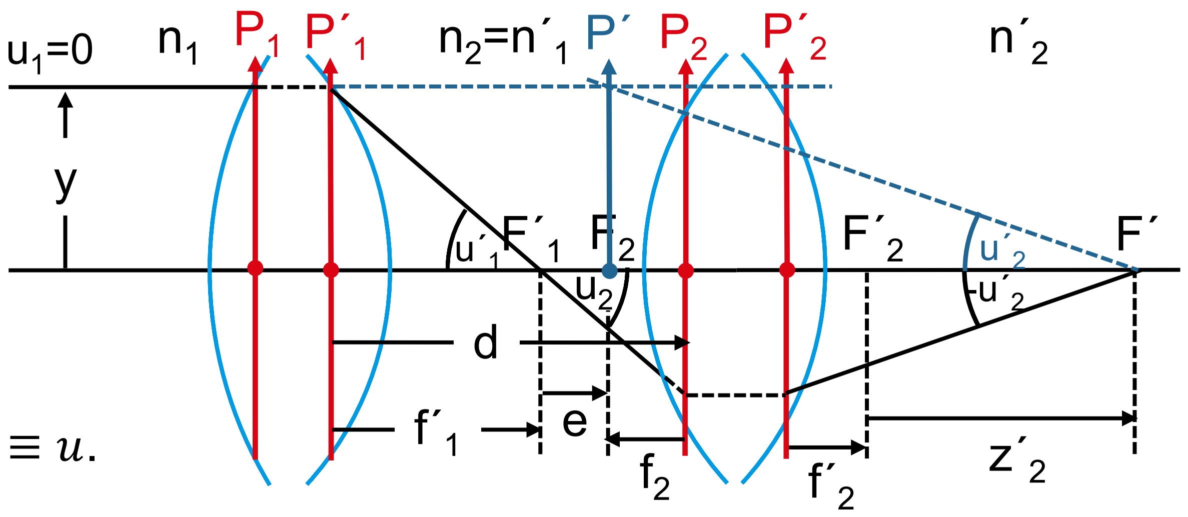
Optical system subdivided into two subsystems described by principal plane pairs P_1, P'_1 and P_2, P'_2. We want to find the equation for the total refractive power as a function of the subsystem powers and their distance d (or "optical tube length" e; see figure):



Two-Component System´s refractive power summation law

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

Optical system subdivided into two subsystems described by principal plane pairs P_1, P'_1 and P_2, P'_2. We want to find the equation for the total refractive power as a function of the subsystem powers and their distance d (or "optical tube length" e; see figure):



For a ray entering the system parallel: $u_1 = 0,\ u'_1 = u_2,\ u'_2 \equiv u.$

With the Helmholtz-Lagrange-eq. $\mathrm{m} = \frac{n_1 u_1}{n'_k u'_k}$ and $m = \frac{z'}{f'} = \frac{f}{z}$:

$$\frac{\tan u'_2}{\tan u'_1} = \frac{\tan u'_2}{\tan u_2} = \frac{n_2}{n'_2}\frac{z_2}{(-f_2)} = \frac{n_2}{n'_2}\frac{(-e)}{(-f_2)} = -\frac{e}{f'_2} \qquad [1]$$

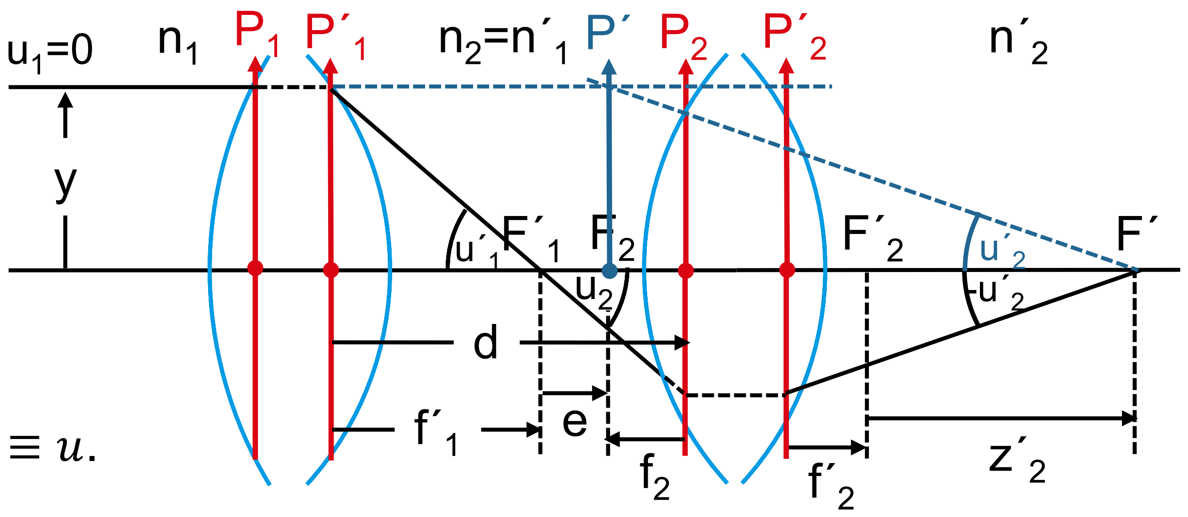
According to the figure we can construct the back principal plane P´ of the complete system (blue lines) and have following relationship:

$$f' = \frac{y}{\tan u'} = \frac{y}{\tan u'_2} = \frac{f'_1 \tan u'_1}{\tan u'_2}$$

and with [1]:

$$f' = -\frac{f'_1 f'_2}{e} \qquad [2]$$

Optical system subdivided into two subsystems described by principal plane pairs P_1, P'_1 and P_2, P'_2. We want to find the equation for the total refractive power as a function of the subsystem powers and their distance d (or "optical tube length" e; see figure):



For a ray entering the system parallel: $u_1 = 0,\ u'_1 = u_2,\ u'_2 \equiv u$.

With the Helmholtz-Lagrange-eq. $\mathrm{m} = \frac{n_1 u_1}{n'_k u'_k}$ and $m = \frac{z'}{f'} = \frac{f}{z}$:

$$\frac{\tan u'_2}{\tan u'_1} = \frac{\tan u'_2}{\tan u_2} = \frac{n_2}{n'_2}\frac{z_2}{(-f_2)} = \frac{n_2}{n'_2}\frac{(-e)}{(-f_2)} = -\frac{e}{f'_2} \qquad [1]$$

According to the figure we can construct the back principal plane P´ of the complete system (blue lines) and have following relationship:

$$f' = \frac{y}{\tan u'} = \frac{y}{\tan u'_2} = \frac{f'_1 \tan u'_1}{\tan u'_2}$$

and with [1]:

$$f' = -\frac{f'_1 f'_2}{e} \qquad [2]$$

The "optical tube length" e is related to the distance between P´1 and P2 as:

$$e = F'_1F_2 = P'_1P_2 - P'_1F'_1 - F_2P_2 = d - f'_1 - f'_2$$

and [2] becomes:

$$f' = \frac{f'_1 f'_2}{f'_1 + f'_2 - d}$$

Inversion and multiplication with n'_2 gives for the power:

$$\Phi = \frac{n'_2}{f'} = \frac{n'_2 n'_1}{f'_1 f'_2}\left(\frac{f'_1}{n'_1} + \frac{f'_2}{n'_2} - \frac{d}{n'_1}\right)$$
$$= \Phi_1 + \Phi_2 - \frac{d}{n'_1}\Phi_1\Phi_2$$

Focal length of a single lens in vacuum

Applying $\Phi = \Phi_1 + \Phi_2 - \frac{d}{n'_1}\Phi_1\Phi_2$

or $\frac{n'_2}{f'} = \frac{n'_1}{f'_1} + \frac{n'_2}{f'_2} - \frac{n'_2 d}{f'_1 f'_2}$

for a **lens in vacuum**:

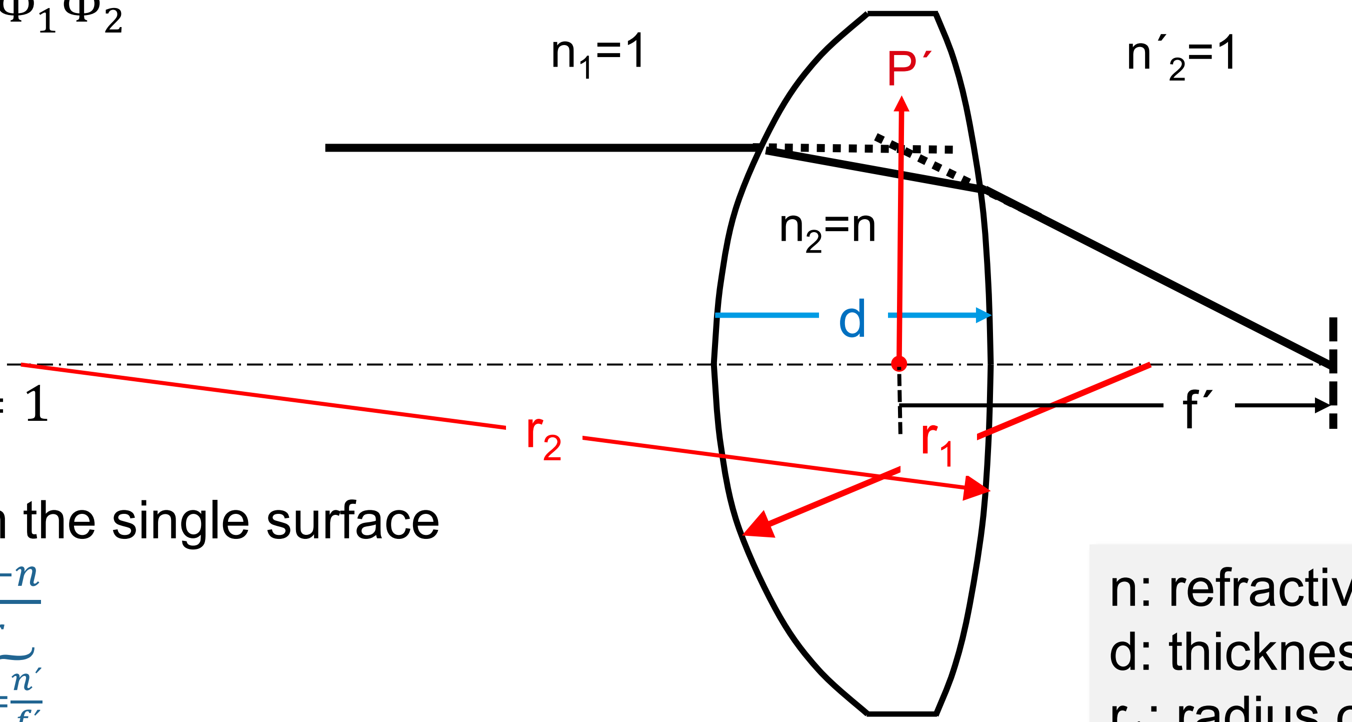
$n_1 = 1,\ n'_1 = n_2 = n$ and $n'_2 = 1$

And the refractive powers from the single surface imaging equation $-\frac{n}{s} + \frac{n'}{s'} = \underbrace{\frac{n'-n}{r}}_{\Phi = \frac{n'}{f'}}$

$$\Phi_1 = \frac{n-1}{r_1}$$

$$\Phi_2 = \frac{1-n}{r_2}$$

$$\Phi = \frac{1}{f'} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{d(n-1)^2}{n r_1 r_2}$$



n: refractive index
d: thickness
r_1: radius of surface 1
r_2: radius der surface 2

What is a diopter?

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

Definition refractive power: $$\Phi := \frac{1}{f}$$

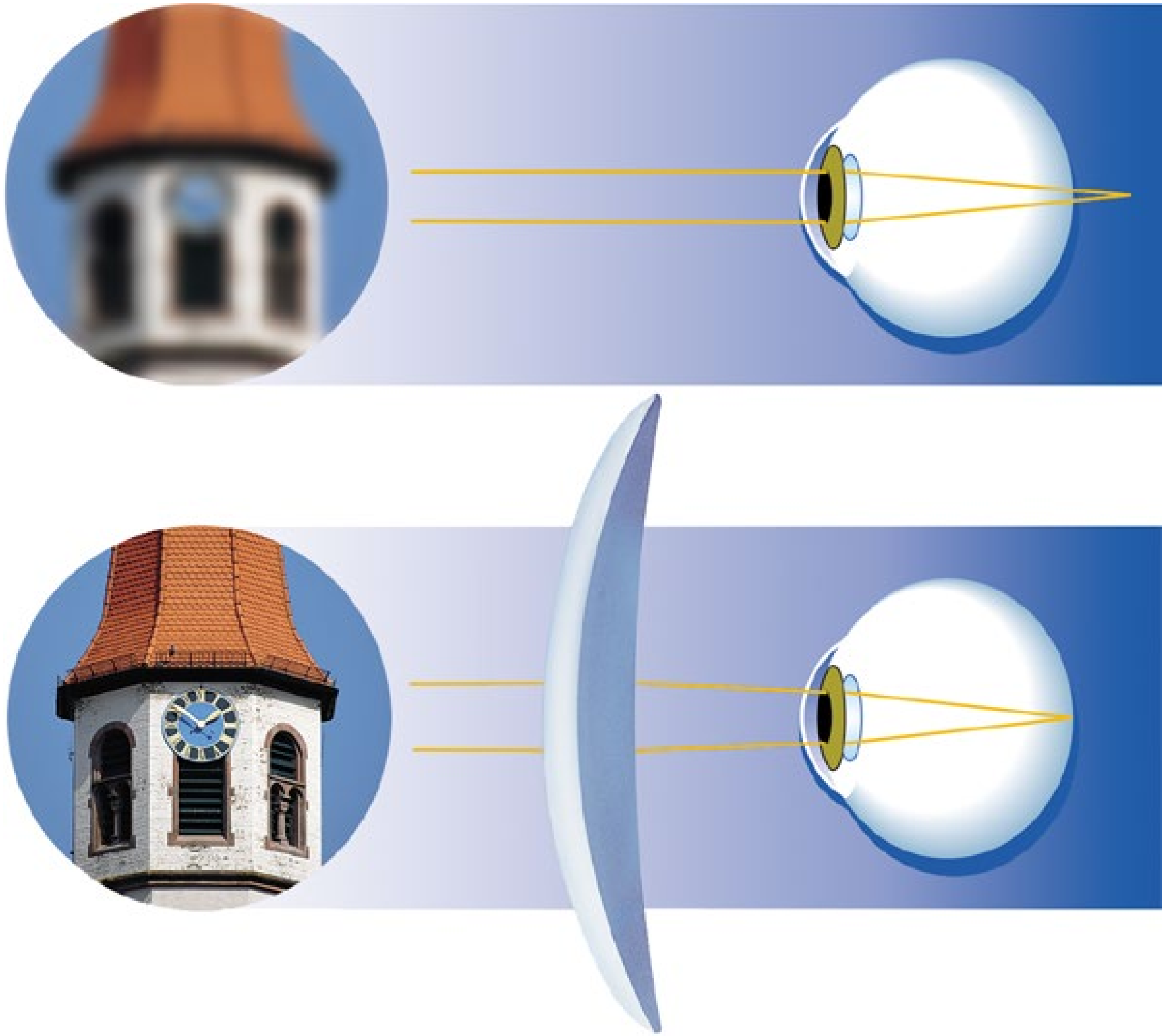
Focal length has dimension length [m].

Therefore, refractive power has dimension inverse length [1/m].

$$1\ \text{diopter} := \frac{1}{\text{m}}$$

Example:

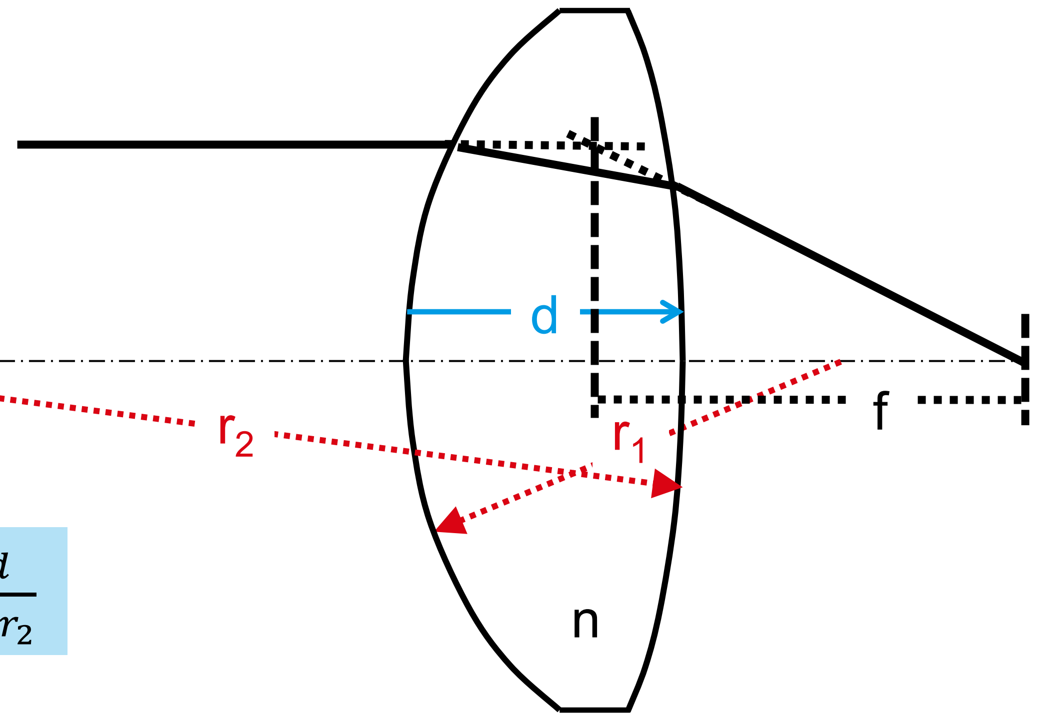
focal length of the human eye ca. 16.6mm, corresponding to a refractive power of 1/0.0166 m^{-1} = 60 diopter



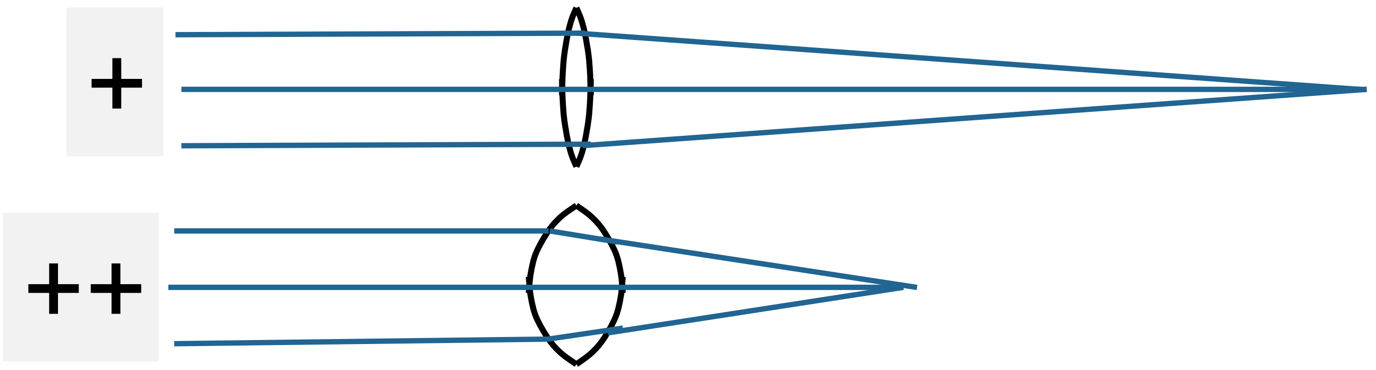
The focal length of a single lens

n: refractive index
d: thickness
r_1: radius of surface 1
r_2: radius der surface 2

Focal length f (or refractive power 1/f) depends on r_1, r_2, n, d:



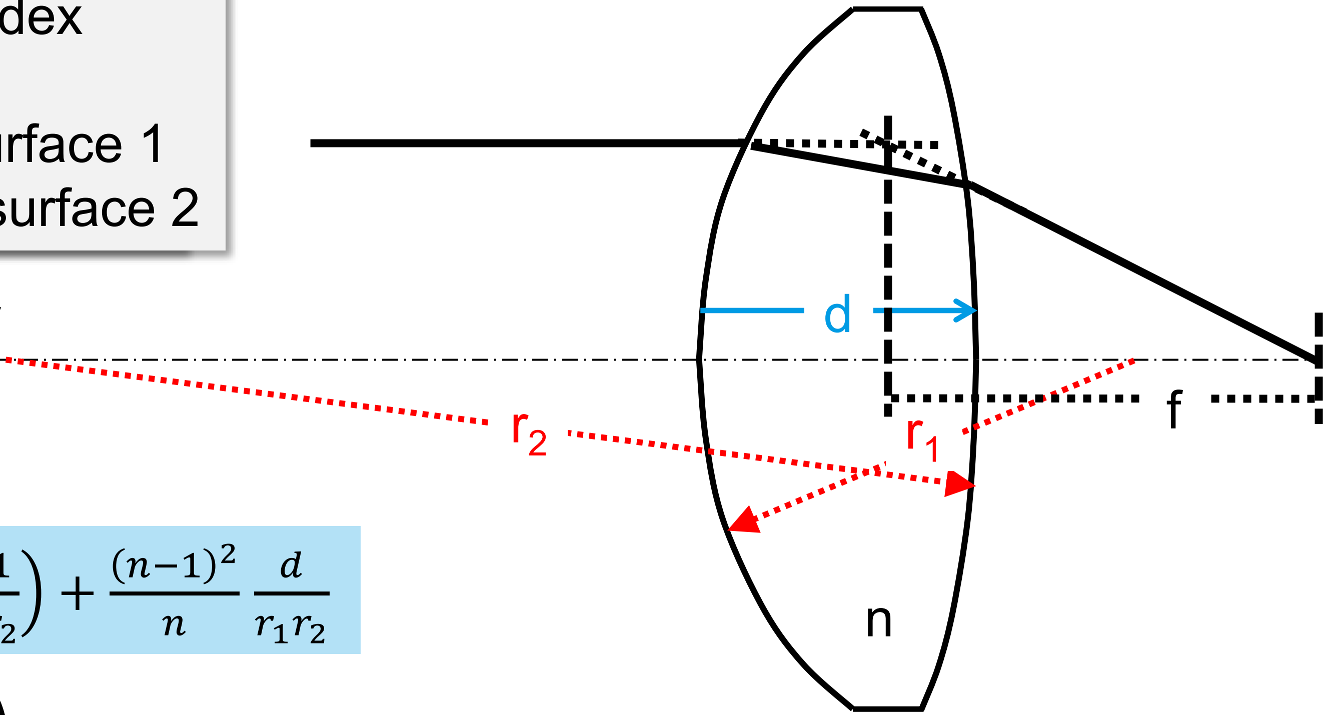
$$\frac{1}{f} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{(n-1)^2}{n}\frac{d}{r_1 r_2}$$



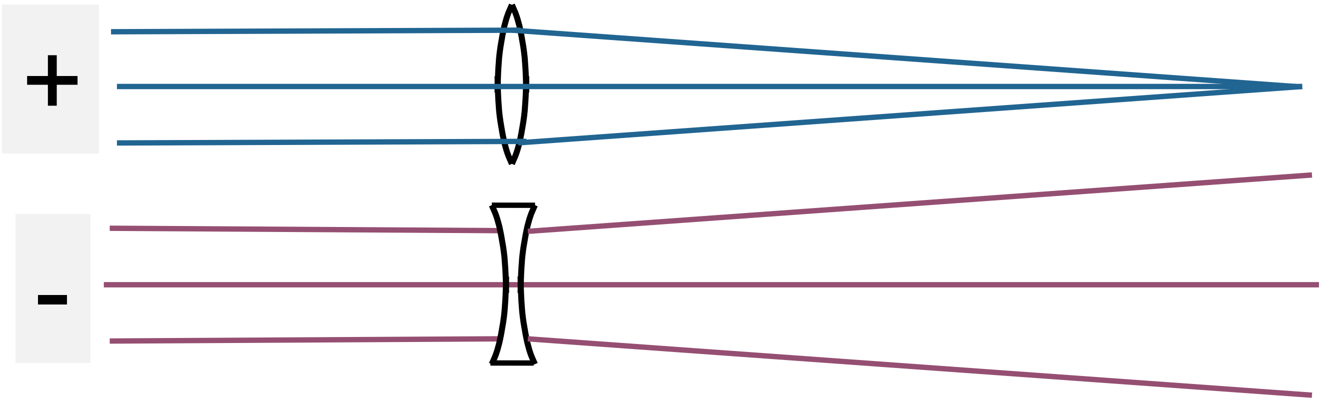
The focal length of a single lens

n: refractive index
d: thickness
r_1: radius of surface 1
r_2: radius der surface 2

Focal length f (or refractive power 1/f) depends on r_1, r_2, n, d:



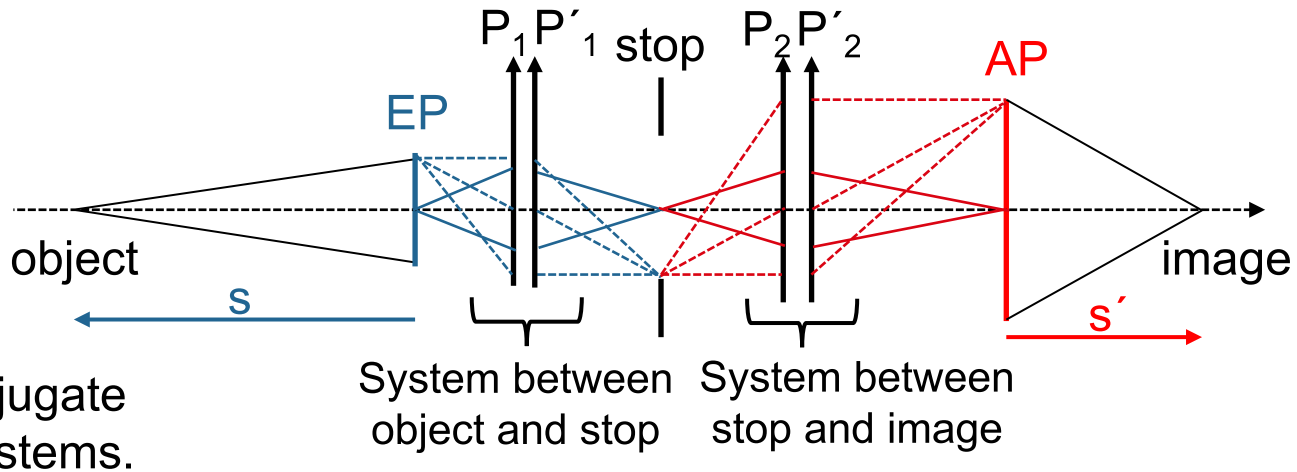
$$\frac{1}{f} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{(n-1)^2}{n}\frac{d}{r_1 r_2}$$



Imaging equation via entrance and exit pupil

Entrance pupil (EP, „Eintrittspupille") = image of stop towards object space

Exit pupil (AP, „Austrittspupille") = image of stop towards image space



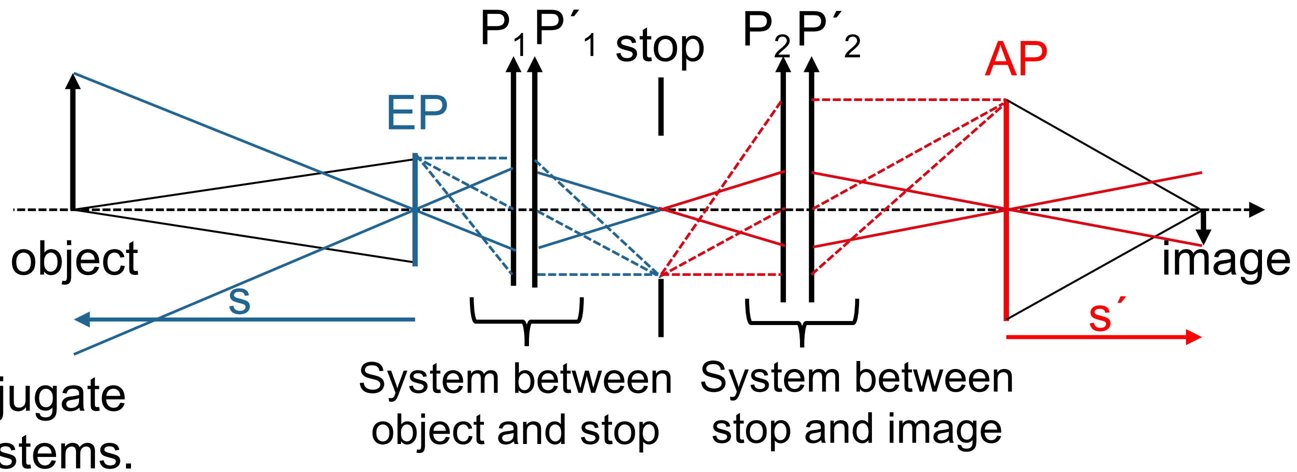
Consequently, Entrance pupil and Exit pupil are conjugate object and image planes via the complete optical systems.

Imaging equation via entrance and exit pupil

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

Entrance pupil (EP, „Eintrittspupille“) = image of stop towards object space

Exit pupil (AP, „Austrittspupille“) = image of stop towards image space



Consequently, Entrance pupil and Exit pupil are conjugate object and image planes via the complete optical systems.

In the general imaging equation via a pair of conjugate planes we denote the **pupil magnification** $m_z = \boldsymbol{m_p}$.
For refractive indices equal to one in object and image space:

$$\boxed{-\frac{1}{m_p s} + \frac{m_p}{s'} = \frac{1}{f'}}$$

Following relation between object / image distances and magnifications holds:

$$m_p m = \frac{s'}{s}$$

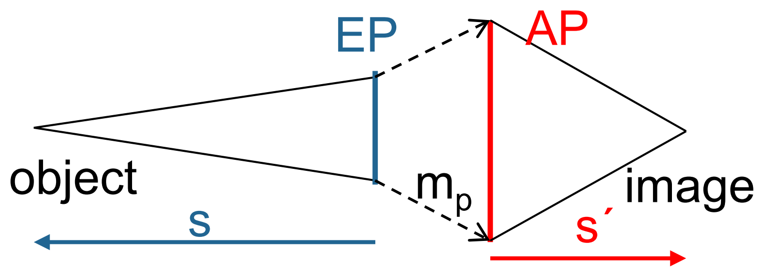
Note: In real optical systems (compact systems like camera lenses) between stop and entrance or exit pupil respectively there is not a real object-image relation (as drawn in the figure), but a virtual.

Imaging equation via entrance and exit pupil

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

Instead of using principal planes (1: 1 imaging), the imaging can also be described using any other conjugate pair of planes (with a different magnification) (Berek (1930), Fundamentals of Practical Optics, p. 24).

In particular, these conjugate reference planes can each be defined as an image of a plane in the optical system calculated on the object side or on the image side. The stop plane, i.e. the location of the (iris) diaphragm in the system, is of particular importance: The object-side image of the diaphragm is the entrance pupil. The image-side image of the diaphragm is the exit pupil. The magnification between the pupils is the pupil magnification m_p. The imaging equation is



$$\boxed{-\frac{1}{m_p s}+\frac{m_p}{s'}=\frac{1}{f'}}$$

Following relation between object / image distances and magnifications holds:

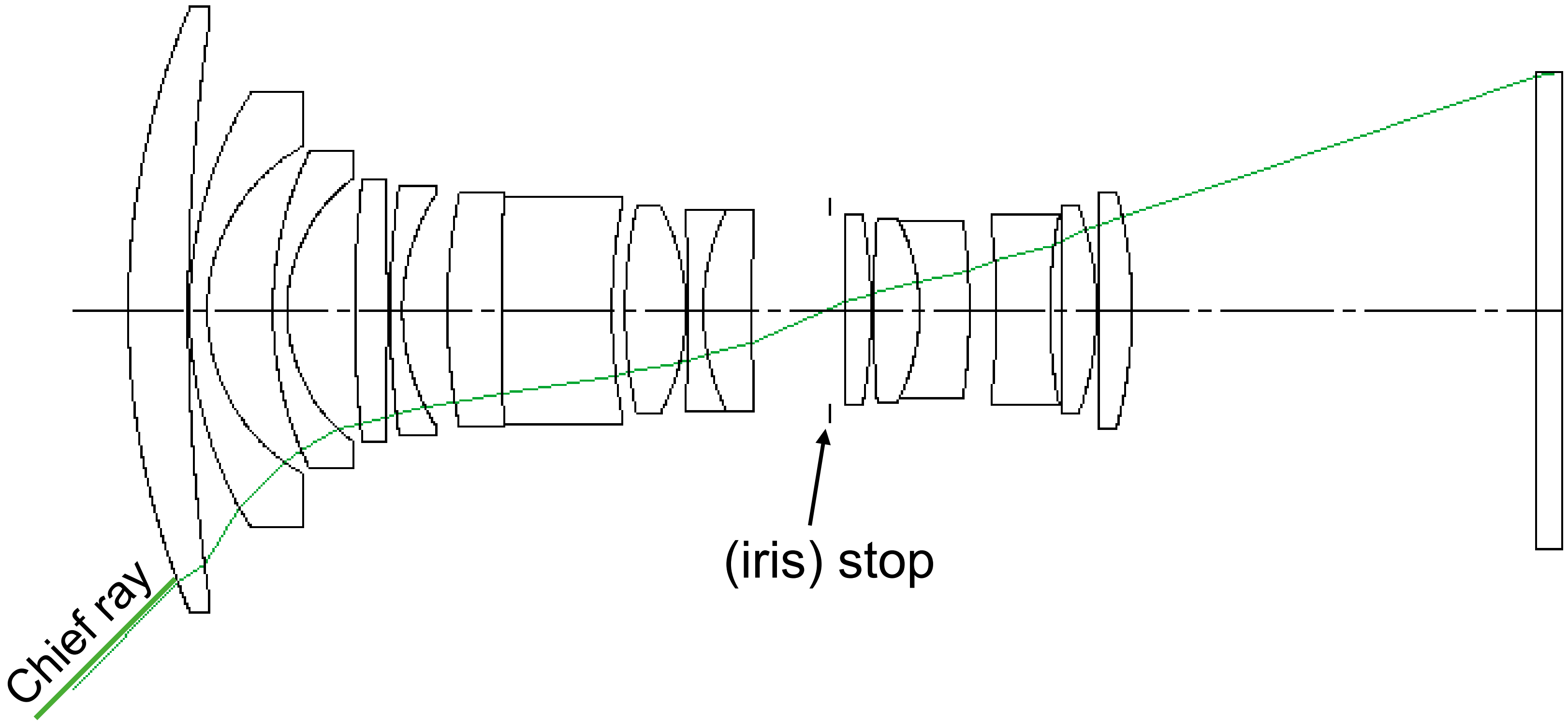
$$m_p m = \frac{s'}{s}$$

s is the distance between entrance pupil and object, s´ between exit pupil and image.

The path of the chief ray through a lens: Determination of entrance and exit position

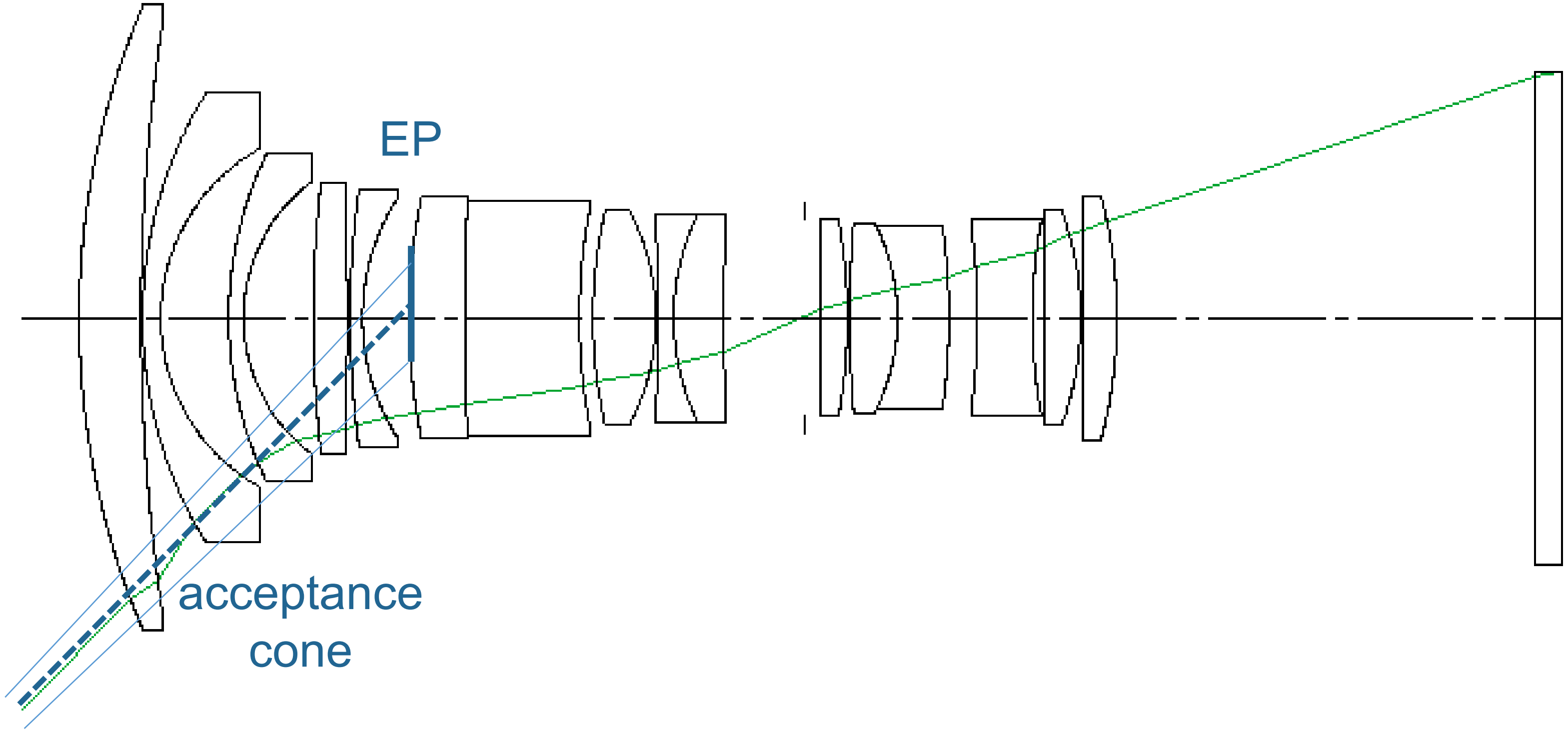
Institute of Applied Physics
Friedrich-Schiller-Universität Jena

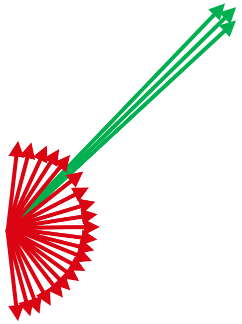
Chief ray (German: “Hauptstrahl”) = the ray through the center of the iris stop



Ray aiming in general optical system ray tracing procedure

For efficient ray tracing through optical systems **ray aiming** is the first step.



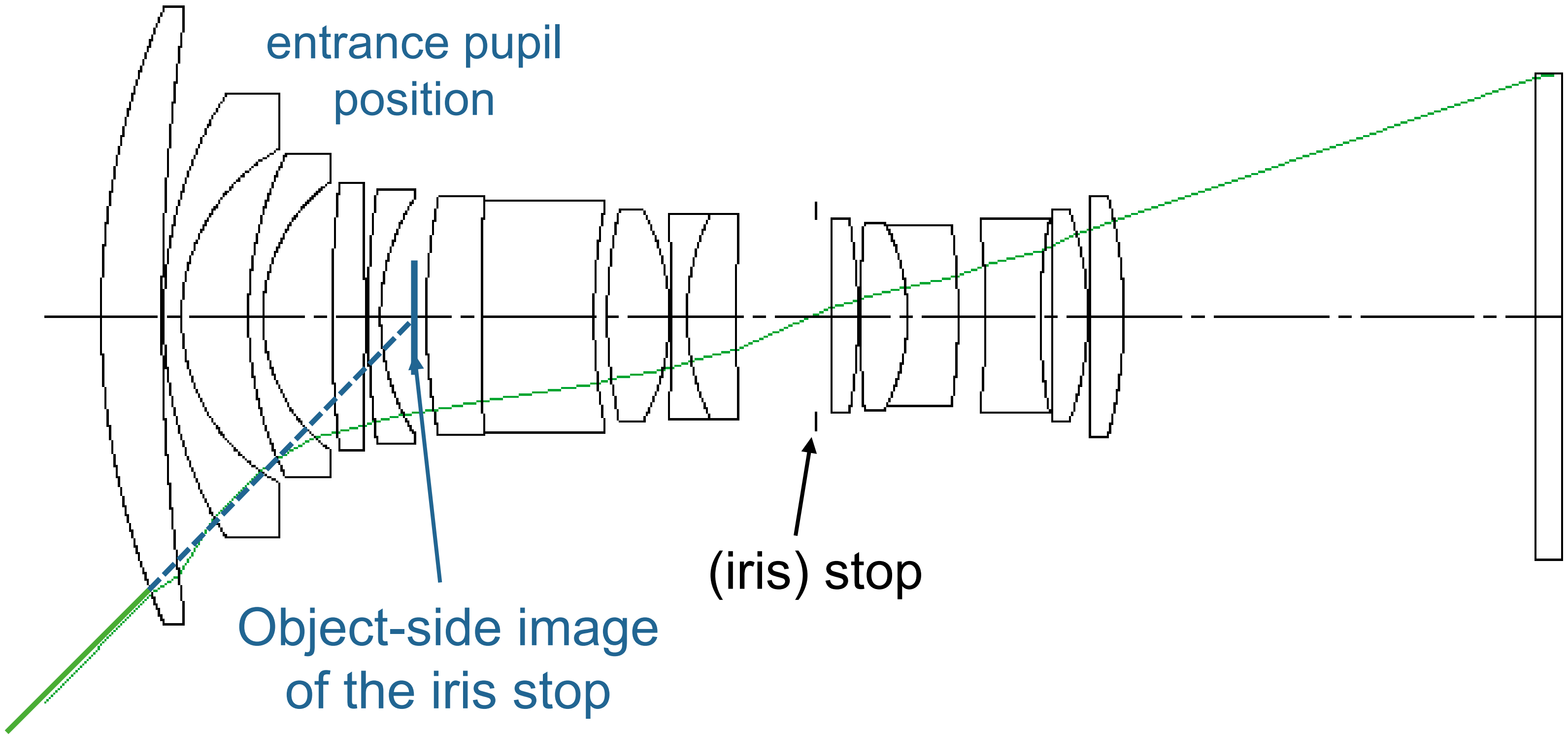


object point

In most systems the (paraxial) entrance pupil position is a good starting point to iteratively determine the complete ray bundle passing the lens.

The path of the chief ray through a lens: Determination of entrance and exit position

Chief ray (German: “Hauptstrahl”) = the ray through the center of the iris stop



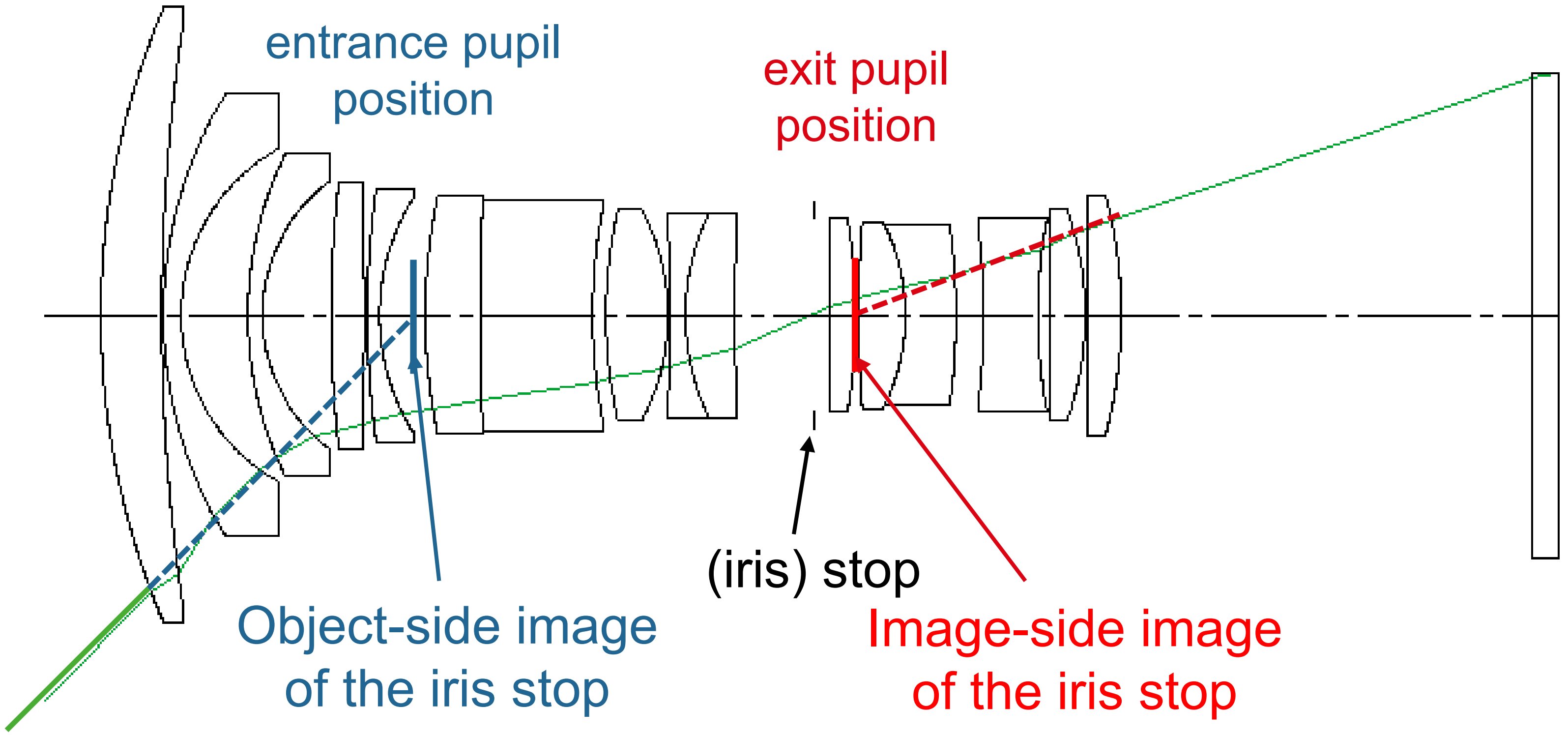
Consequently: As the entrance pupil is the image of the stop in object space, that is in front of the first optical system component its position on-axis is where all chief rays from object space are (apparently*) heading to*!

In other words: The **entrance pupil** corresponds to the **center of perspective of the optical system**.

The path of the chief ray through a lens: Determination of entrance and exit position

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

Chief ray (German: “Hauptstrahl”) = the ray through the center of the iris stop



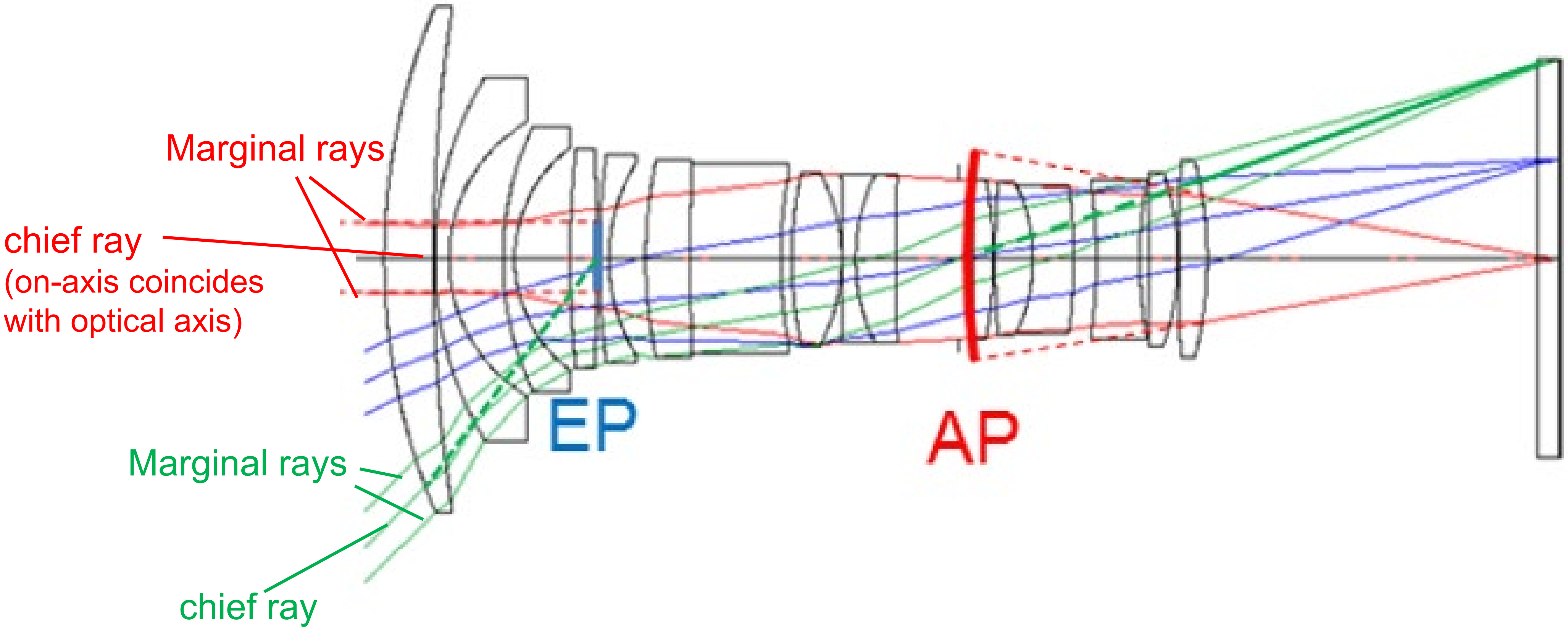
Consequently: As the entrance pupil is the image of the stop in object space, that is in front of the first optical system component its position on-axis is where all chief rays from object space are (apparently*) heading to*!

In other words: The **entrance pupil** corresponds to the **center of perspective of the optical system**.

* ”apparently” heading for the common case that the entrance pupil is inside the system…”

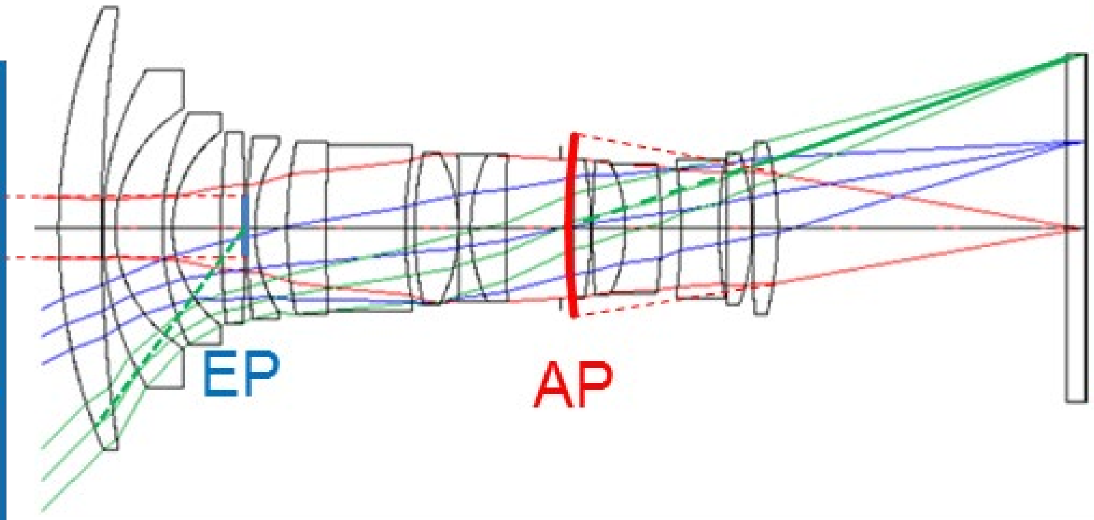
In photographic lenses the entrance and exit pupil are virtual images of the stop!

Chief ray and marginal rays



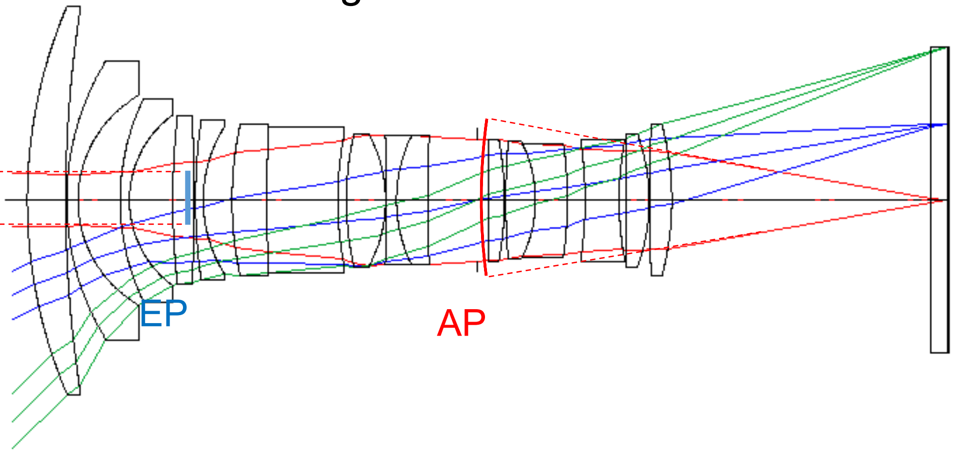
Entrance and exit pupil of a retrofocus lens

When you look into a lens from object-space or image-space in front of a bright surrounding you see the **entrance** or **exit pupil** respectively as a bright disk.
When you look obliquely into the lens the circular desk gets a “cat´s-eye shape” due to **vignetting** (truncation of light at different boundaries.)
You can also anticipate the EP or AP position when rotating the lens out of optical axis.

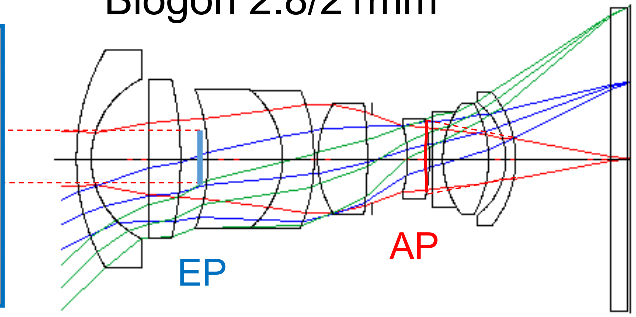


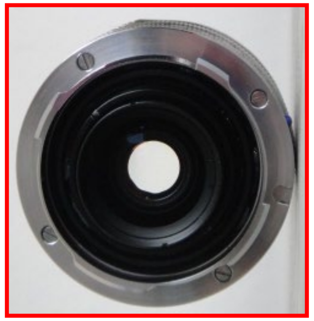
Pupils of symmetrical wide-angle and retro-focus lens types

Distagon 2.8/21mm



Biogon 2.8/21mm



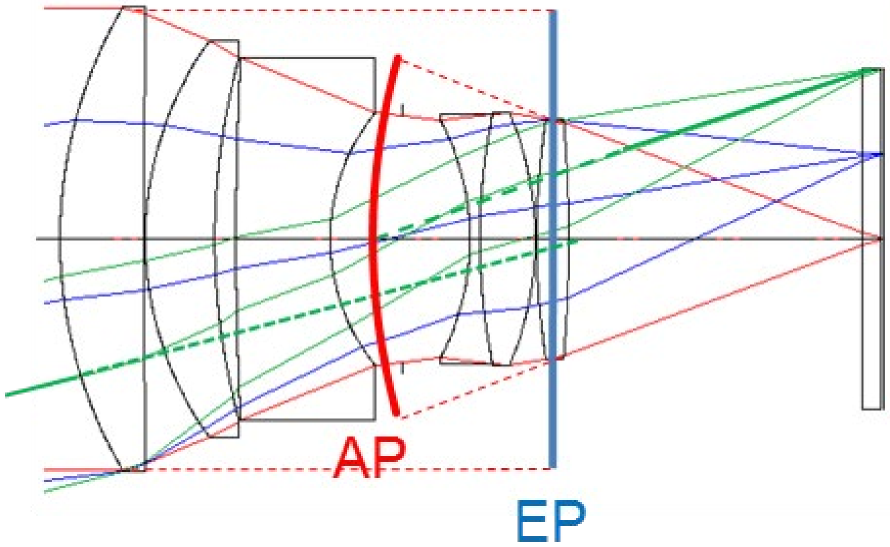


Same field-of-view and f-number.
However the relative pupil size gives insight on the lens structure…

	w [°]	$Ø_{EP}$ [mm]	$Ø_{AP}$ [mm]	β_p	s´ [mm]	t [°]	$\cos^4(t)$
Biogon 2.8/21mm ZM	45.2	7.5	10.1	1.3	28.3	37.4	0.4
Distagon 2.8/21mm ZE/ZF	45.2	7.5	22.6	3	64.3	18.6	0.81

Entrance and exit pupil

In real systems entrance and exit pupil position can have reversed position:

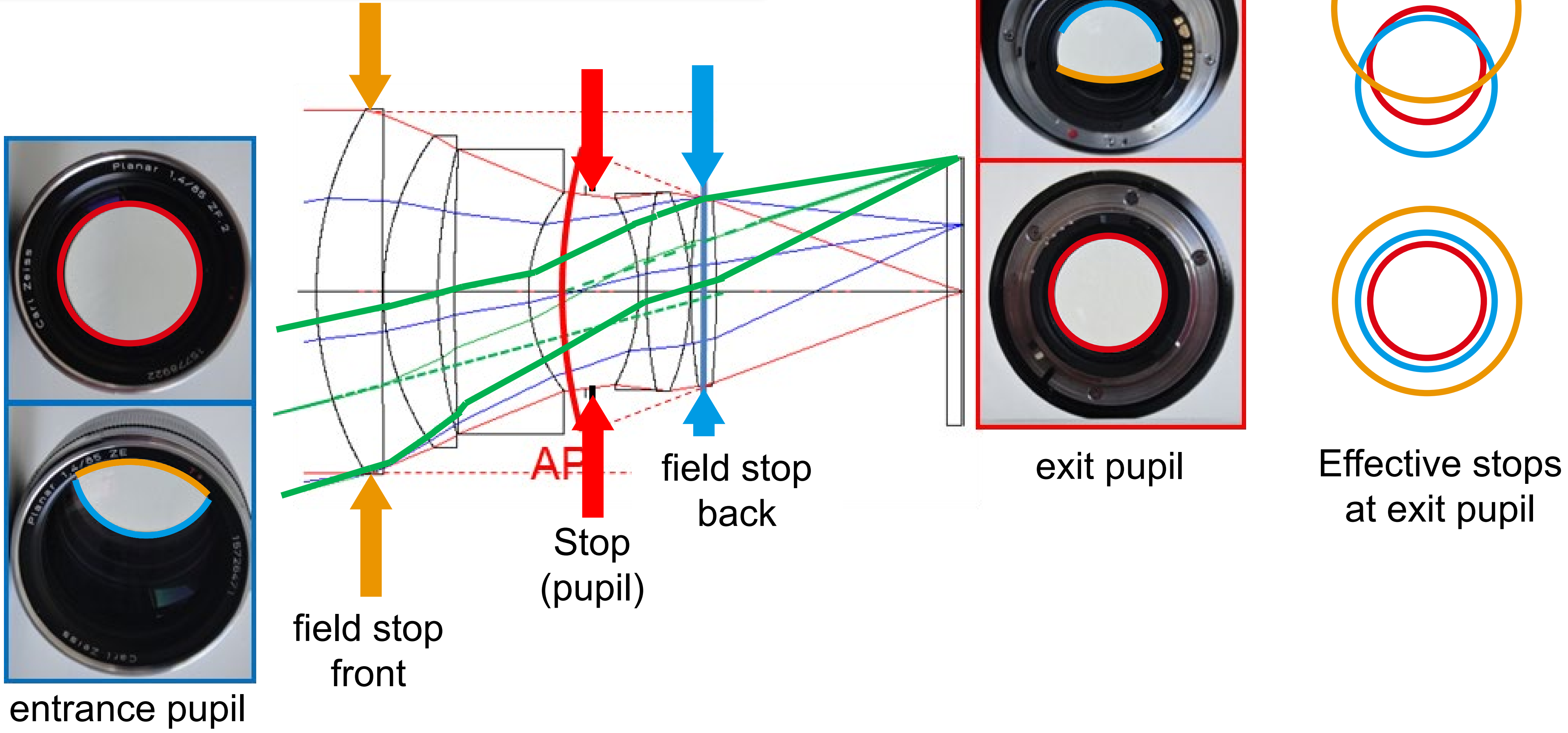


Compared to previous example the pupils are larger and, in this case, also more similar in size.
General structural properties of an optical system can be determined by pupil properties without detailed knowledge of lens data.

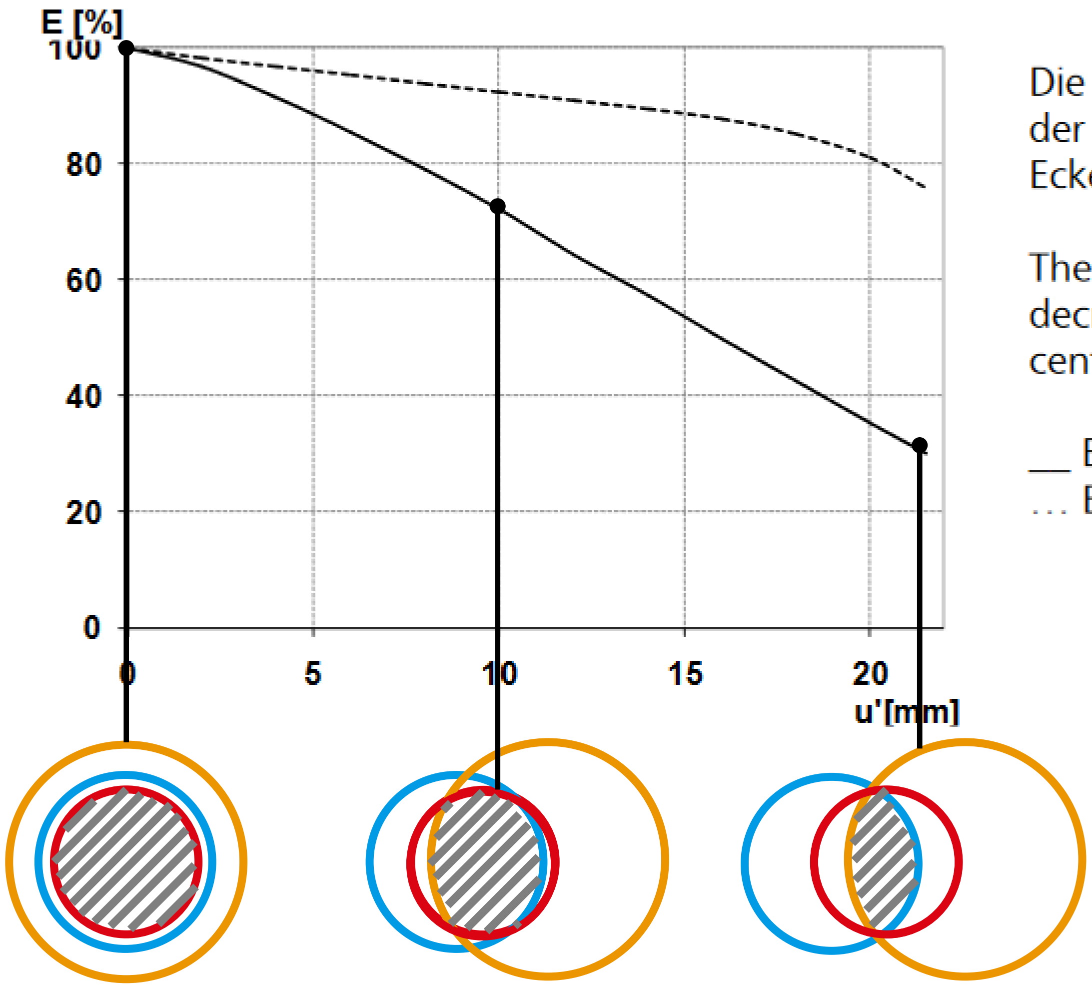
Vignetting and relative illumination

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

In camera lenses for off-axis light there are additional **field stops** which cut-off light of the pupil = **vignetting**.



Relative Beleuchtungsstärke/Relative Illuminance



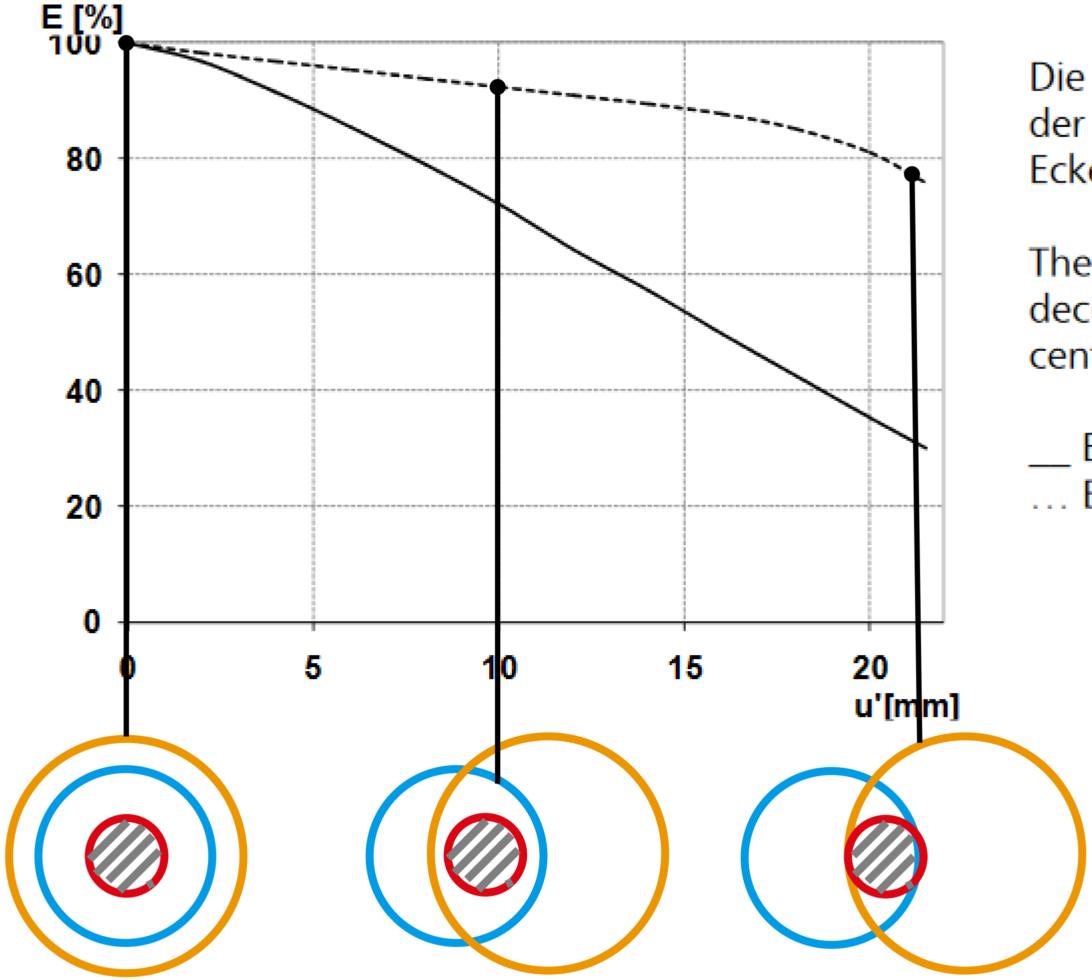
Die relative Beleuchtungsstärke zeigt die Abnahme der Bildhelligkeit von der Mitte des Bildes zu den Ecken. Angabe in Prozent.

The relative illumination shows in percent the decrease in image brightness from the image center to edge.

__ Blendenzahl: k = 1,4 / f-number = 1.4
… Blendenzahl: k = 2,8 / f-number = 2.8

https://www.zeiss.com/content/dam/consumer-products/downloads/photography/datasheets/en/classic-lenses/datasheet-zeiss-classic-planar-1485.pdf

Relative Beleuchtungsstärke/Relative Illuminance



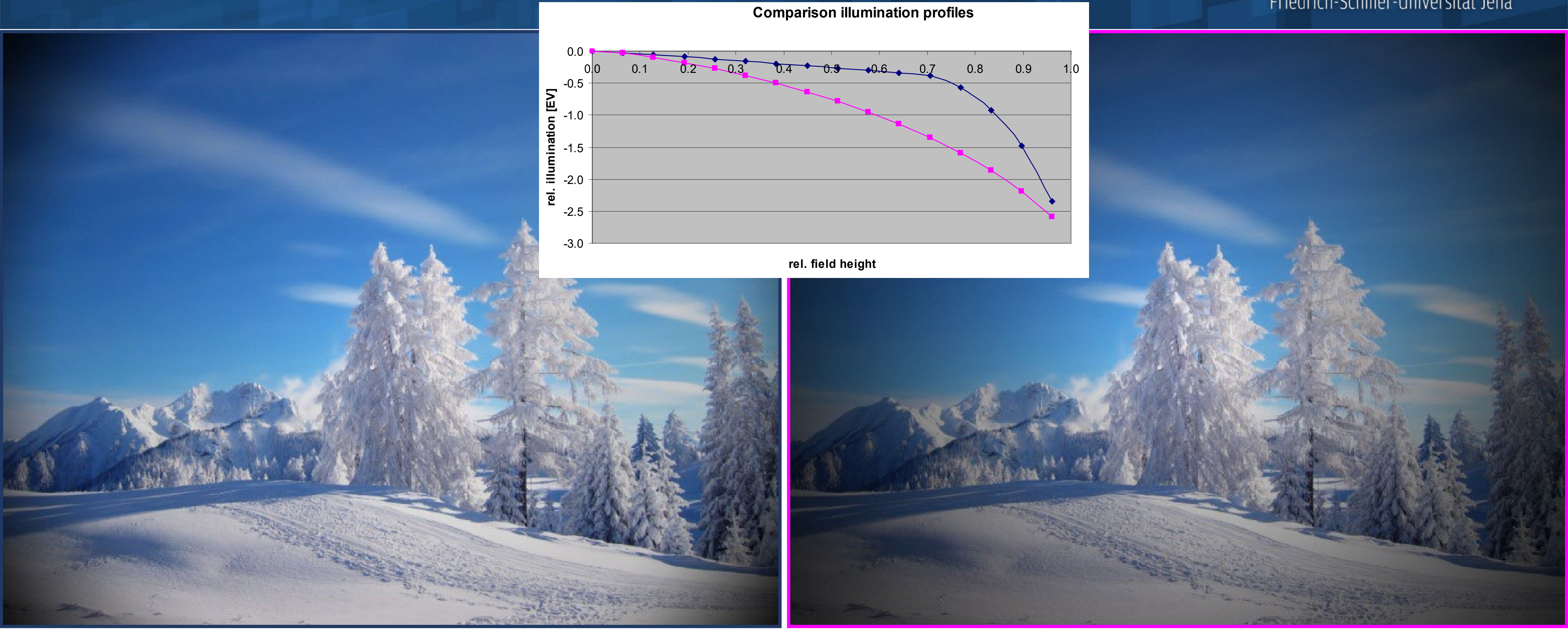
Die relative Beleuchtungsstärke zeigt die Abnahme der Bildhelligkeit von der Mitte des Bildes zu den Ecken. Angabe in Prozent.

The relative illumination shows in percent the decrease in image brightness from the image center to edge.

__ Blendenzahl: k = 1,4 / f-number = 1.4
… Blendenzahl: k = 2,8 / f-number = 2.8

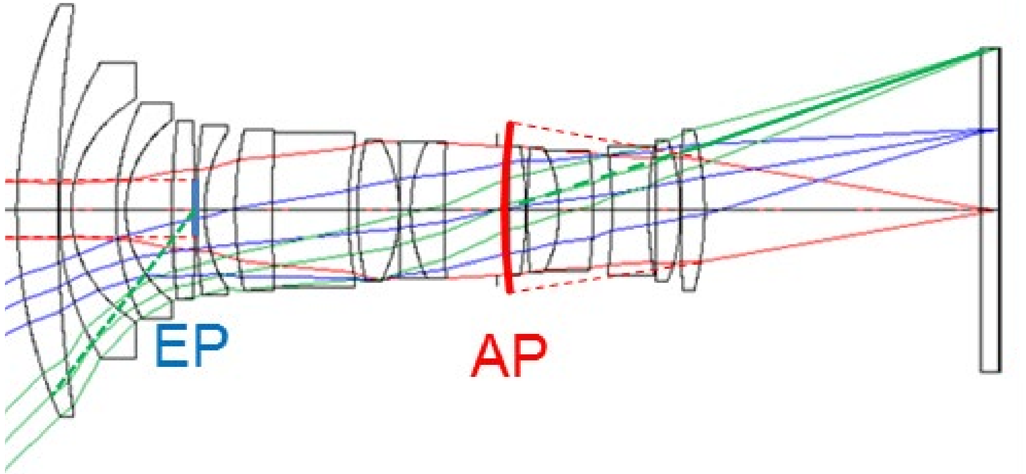
When the **iris stop is closed** the size of the projected pupil stop decreases, but not the field stops.
There is less vignetting and accordingly less variation of relative illuminance.
The radiometric $\cos^4$-contribution remains about equal.

Relative illumination



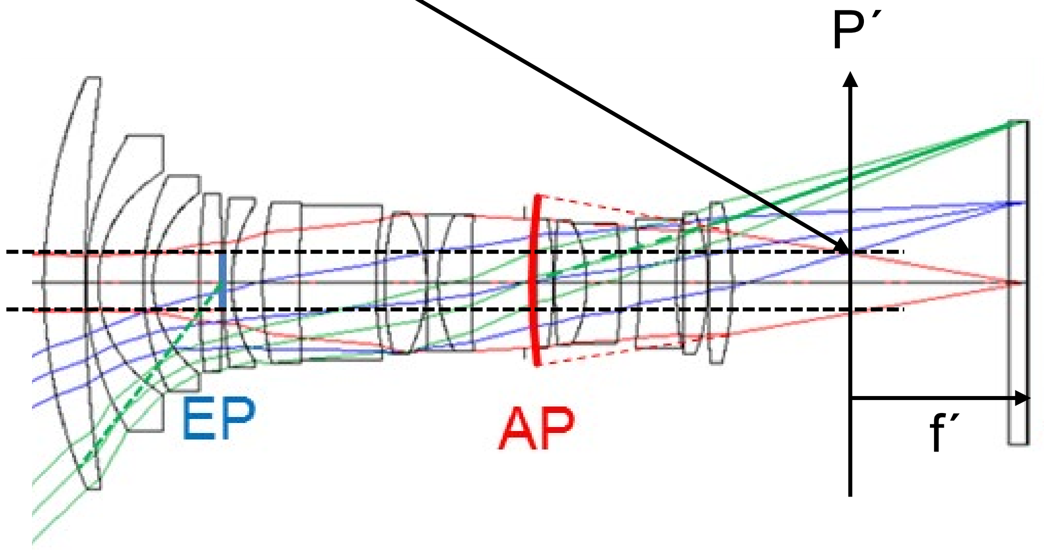
Relative illumination profile are optical design dependent. The maximum amount of relative illumination loss. The distribution depends significantly on the position of the field stops.

Entrance and exit pupil versus principal planes as system conjugates



Graphical determination of back principal plane P´ (or focal length f´)

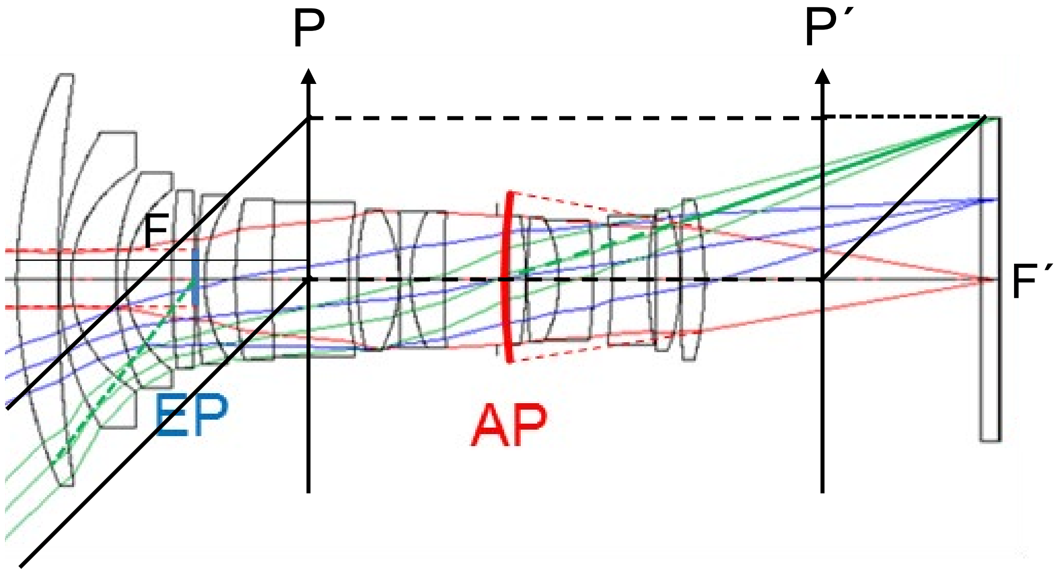
Intersection of a ray entering the optical system parallel in object (extended ray path) with the path of the same ray in image space (eventually backwards extended) = **back principal plane position**



Entrance and exit pupil versus principal planes as system conjugates

Principal planes are suited to (graphically) determine object and image position.

Principal planes are **not suited to make any prediction of <u>actual</u> ray paths**.
Necessarily for actual ray paths the **stop position must included**!

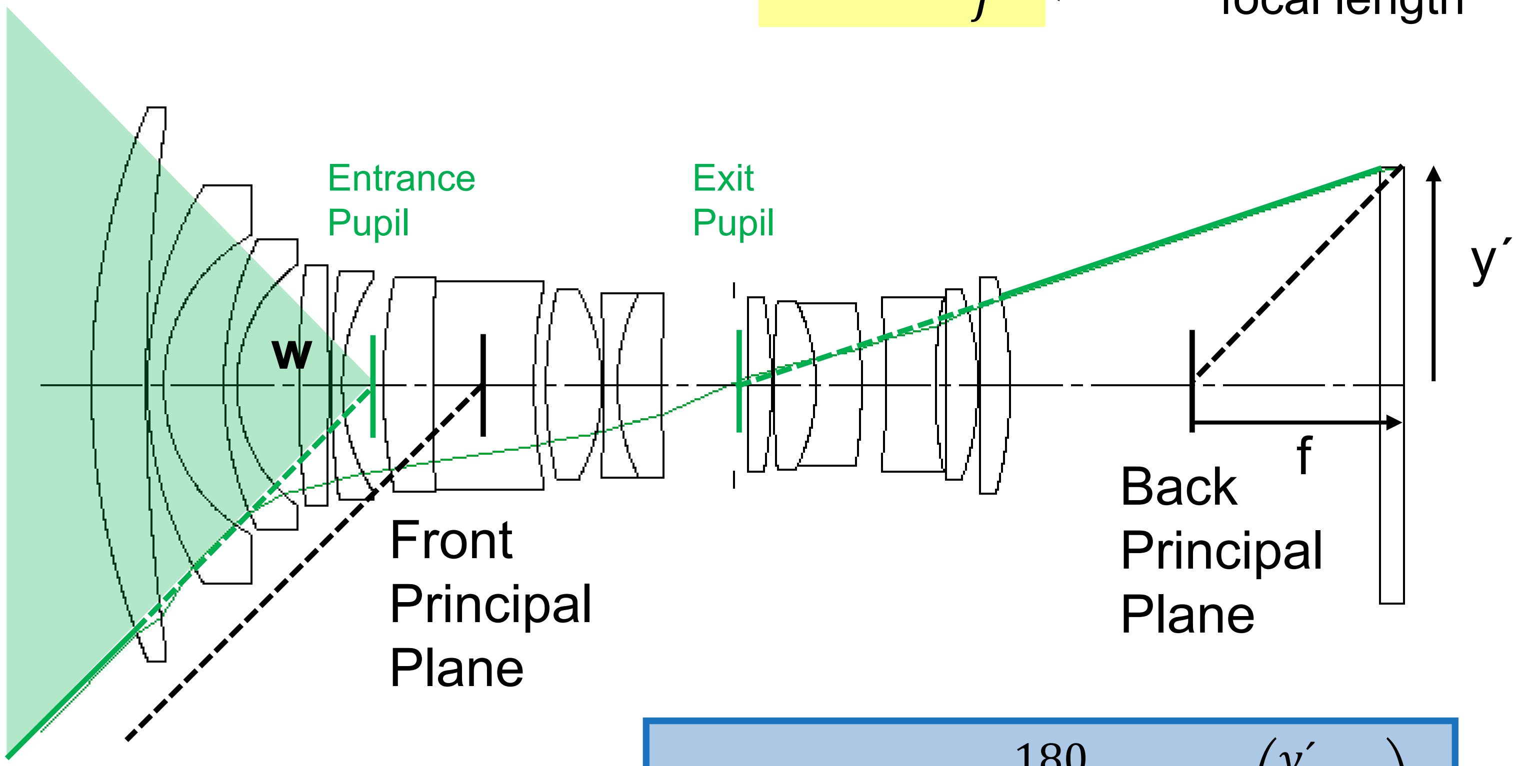


Here: object point at infinity!

Principal planes and corresponding light paths according to paraxial construction principles (although we are far out of paraxial regime here)

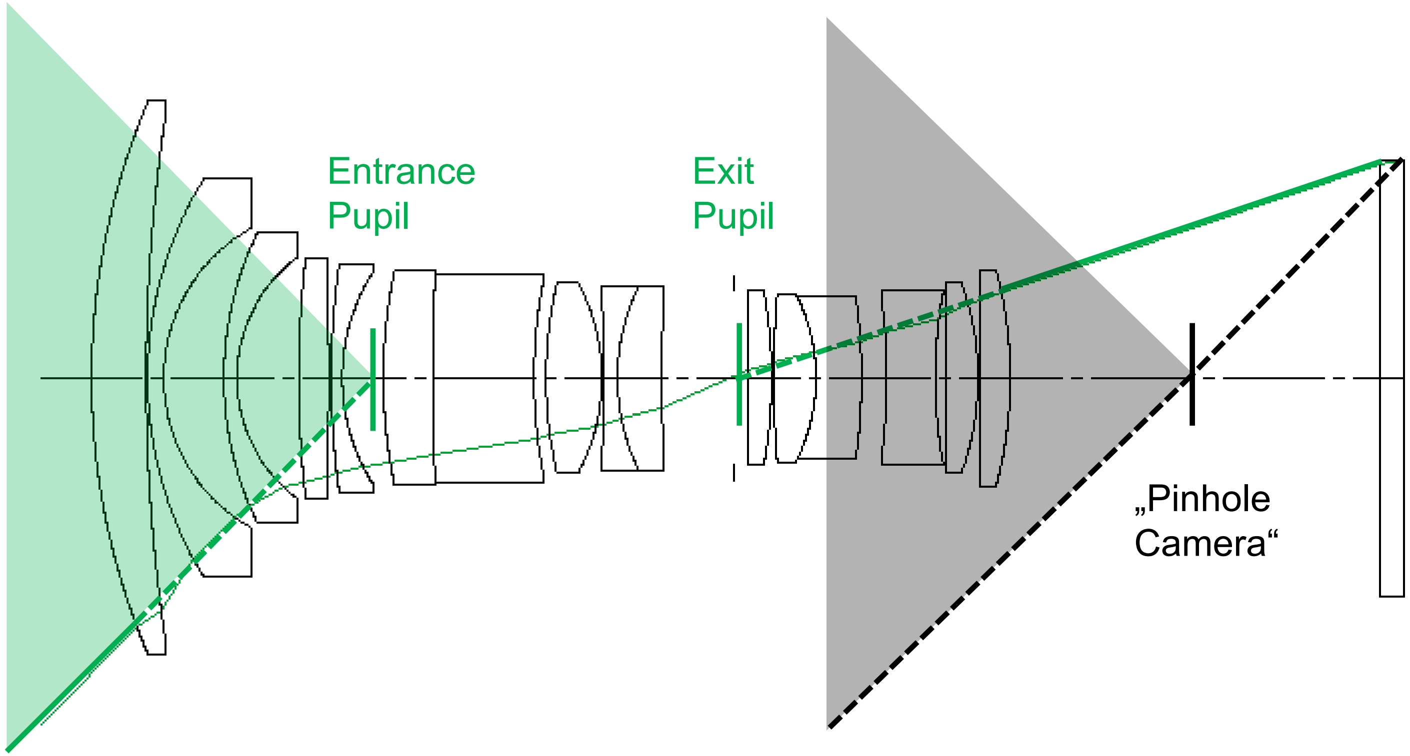
Field of view (FOV)

Semi-field of view angle $\tan w = \dfrac{y'}{f'}$ ← image height, ← focal length



$$diagFOV[°] = 2 \cdot \frac{180}{\pi} \cdot \arctan\left(\frac{y'_{max}}{f'}\right)$$

Real lens vs. Pinhole model (as used in computer graphics)



1. The (object-side) field of view angle is equal (only for an object at infinity distance!)
2. The center of projection (which for a real lens is the entrance pupil) is shifted when using the pinhole model
3. The telecentricity is given by the angle of the chief ray to the normal of the image plane, whereas in the pinhole model the ray which carries the information from object strikes the image plane under same angle as it has in object space

Sign conventions

There are different sign conventions in the literature.
This is most obvious regarding the „lens makers equation“ which is sometimes used with „-1/s“, sometimes „+1/s“.

$$-\frac{1}{s}+\frac{1}{s'}=\frac{1}{f'}$$

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f'}$$

If a convention is consistent and completely describes all possible cases everything is fine.

If formulas are taken somewhere from the literature and are inconsistently combined, errors will occur.

Sign errors are maybe the most occurring and time-consuming to repair.
Be careful to set up clear rules and follow those all the way!

Sign Convention “T” (commonly used Technical-optics texts*)

Imaging equation: $$-\frac{1}{s}+\frac{1}{s'}=\frac{1}{f'}$$

1. The direction of light in an optical system is from left to right.

2. Object distances to the right (left) of the refracting surface are measured positive (negative).

3. Image distances to the right (left) of the refracting surface are measured positive (negative).

4. The focal length to the right (left) hand side of the lens are measured positive (negative).

5. The radius of a convex (concave) refracting surface is counted as positive (negative).

6. Object and image sizes above (below) the optical axis are measured positive (negative).

The conventions 2, 3, 4, 5 can all be summarized in one: All distances, either to object, image, focal point or center of surface curvature, are measured relative to the surface as positive (negative) to the right (left) of the surface.

*e.g., Berek (1930), Picht (1955), Haferkorn (1994), Hopkins (1983)

Sign Convention “W” (often used in “wave-optics” texts*)

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

Imaging equation: $\frac{1}{s}+\frac{1}{s'}=\frac{1}{f'}$

1. The direction of light in an optical system is from left to right.

2. Object distances to the left (right) of the refracting surface are measured positive (negative).

3. Image distances to the right (left) of the refracting surface are measured positive (negative).

4. Both focal lengths are for a converging (diverging) lens are positive (negative).

5. The radius of a convex (concave) refracting surface is counted as positive (negative).

6. Object and image sizes above (below) the optical axis are measured positive (negative).

*e.g., Goodman Fourier-Optics (1968), Jenkins / White (1957)

Sign Convention "T" (commonly used Technical-optics texts*)

Imaging equation: $$-\frac{1}{s}+\frac{1}{s'}=\frac{1}{f'}$$

1. The direction of light in an optical system is from left to right.

2. Object distances to the right (left) of the refracting surface are measured positive (negative).

3. Image distances to the right (left) of the refracting surface are measured positive (negative).

4. The focal length to the right (left) hand side of the lens are measured positive (negative).

5. The radius of a convex (concave) refracting surface is counted as positive (negative).

6. Object and image sizes above (below) the optical axis are measured positive (negative).

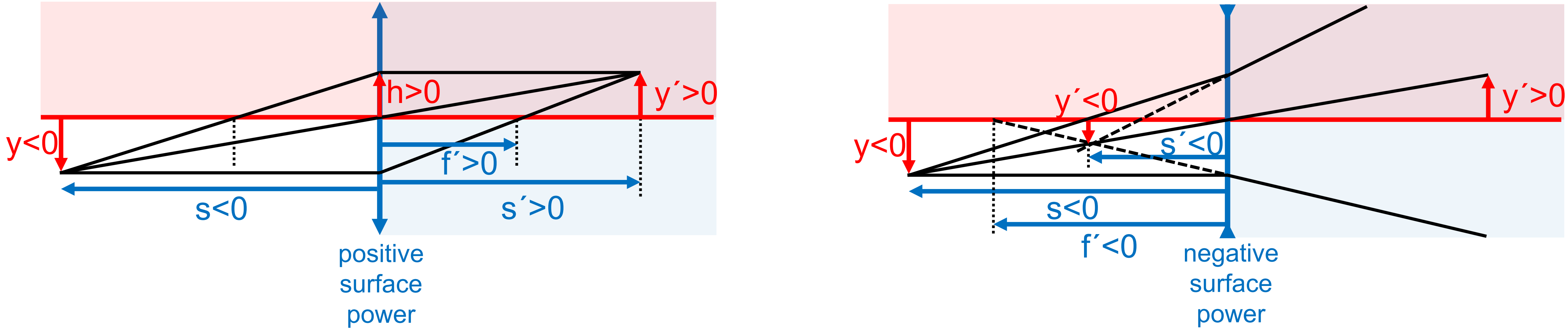
*e.g., Berek (1930), Picht (1955), Haferkorn (1994), Hopkins (1983)

Sign Convention

1. The direction of light in an optical system is from left to right.

2./ 3./ 4./ 5. All distances, either to object, image, focal point or center of surface curvature, are measured relative to the surface as positive (negative) to the right (left) of the surface.

6. Object and image sizes above (below) the optical axis are measured positive (negative).



Sign convention (1/2)

Distances s, s´ are measured relative to the lens position!
The light path is drawn from left to right.
For „normal imaging“ (real object) the object is at the left-hand side of the lens.
That is the **distance s is a negative number**.

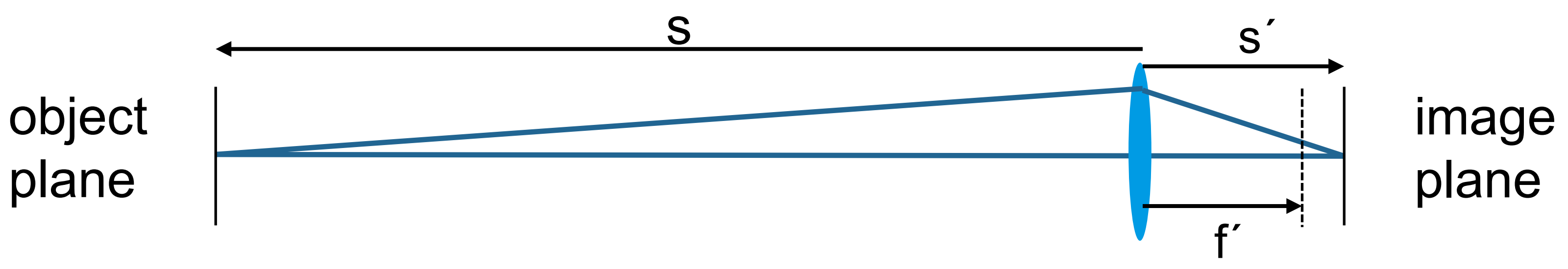
s = -1000mm

f = 100mm

$$\boxed{-\frac{1}{s}+\frac{1}{s'}=\frac{1}{f'}}$$

$$s'=\frac{1}{\frac{1}{f}+\frac{1}{s}}=\frac{f\ s}{f+s}=\frac{100\cdot\ (-1000)}{100-1000}=\frac{-100000}{-900}=\frac{1000}{9}=111.1$$

The (wrong) calculation with positive sign s=+1000mm yields $s'=\frac{1000}{11}=90.9<100=f$ (s positive means, that the object is virtual and is at the right-hand side of the lens)



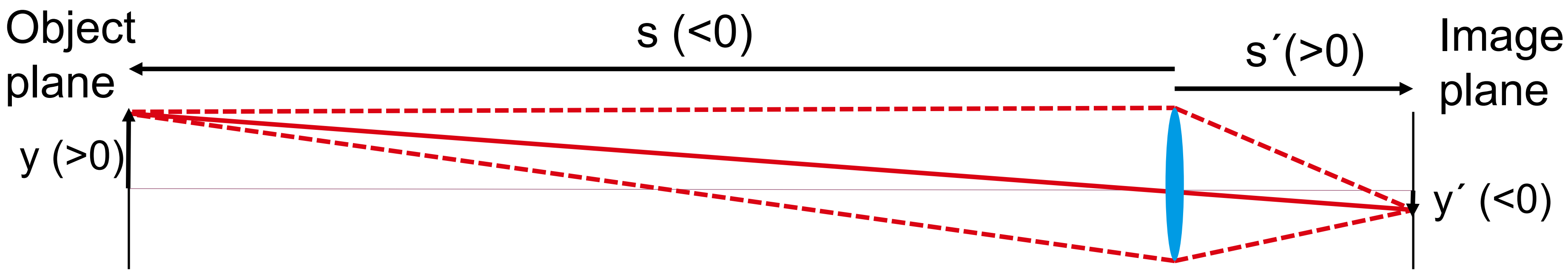
Sign convention (2/2)

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

The magnification is:

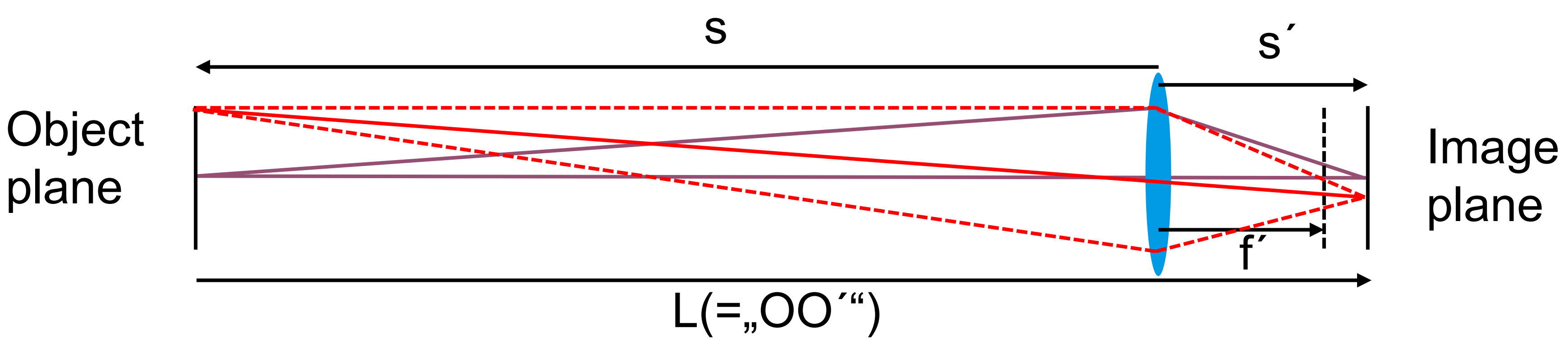
$$m = \frac{s'}{s} = \frac{111.1mm}{-1000mm} = -0.11$$

Magnification negative, that is, the image is upside down!
For camera lenses or visual imaging, the magnification is negative. (Usually left out for convenience or on engravings.)



Remark: Unfortunately several (mainly two) sign conventions are used in optics.
The exemplified sign convention is the most common, e.g. used in commercial optical design software or optics companies like ZEISS.

A useful table with paraxial relations



$$-\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$$

$$m=\frac{s'}{s}$$

	(s, s´)	(s, f´)	(s, m)	(s, L)	(s´, f´)	(s´, m)	(s´, L)	(f´, m)	(f´, L)	(m, L)
s					$\frac{s'f'}{f'-s'}$	$\frac{s'}{\beta}$	$s'-L$	$f'\frac{1-m}{m}$	$-\frac{L}{2}\pm\sqrt{\frac{L^2}{4}-f'L}$	$\frac{L}{m-1}$
s´		$\frac{sf'}{f'+s}$	ms	$s+L$				$f'(1-m)$	$\frac{L}{2}\pm\sqrt{\frac{L^2}{4}-f'L}$	$L\frac{m}{m-1}$
f´	$\frac{ss'}{s-s'}$		$s\frac{m}{1-m}$	$-\frac{s(s+L)}{L}$		$\frac{s'}{1-m}$	$\frac{s'(L-s')}{L}$			$-L\frac{m}{(m-1)^2}$
m	$\frac{s'}{s}$	$\frac{f'}{f'+s}$		$\frac{L+s}{s}$	$\frac{f'-s'}{f'}$		$\frac{s'}{s'-L}$		$1-\frac{L}{2f'}\pm\sqrt{\frac{L}{f'}\left(\frac{L}{4f'}-1\right)}$	
L	$-s+s'$	$-\frac{s^2}{f'+s}$	$(m-1)s$		$\frac{s'^2}{s'-f'}$	$\left(1-\frac{1}{m}\right)s'$		$-f'\frac{(m-1)^2}{m}$		

A useful table with paraxial relations

	(s, s´)	(s, f´)	(s, m)	(s, L)	(s´, f´)	(s´, m)	(s´, L)	(f´, m)	(f´, L)	(m, L)
s					$\frac{s'f'}{f'-s'}$	$\frac{s'}{\beta}$	$s'-L$	$f'\frac{1-m}{m}$	$-\frac{L}{2}\pm\sqrt{\frac{L^2}{4}-f'L}$	$\frac{L}{m-1}$
s´		$\frac{sf'}{f'+s}$	ms	$s+L$				$f'(1-m)$	$\frac{L}{2}\pm\sqrt{\frac{L^2}{4}-f'L}$	$L\frac{m}{m-1}$
f´	$\frac{ss'}{s-s'}$		$s\frac{m}{1-m}$	$-\frac{s(s+L)}{L}$		$\frac{s'}{1-m}$	$\frac{s'(L-s')}{L}$			$-L\frac{m}{(m-1)^2}$
m	$\frac{s'}{s}$	$\frac{f'}{f'+s}$		$\frac{L+s}{s}$	$\frac{f'-s'}{f'}$		$\frac{s'}{s'-L}$		$1-\frac{L}{2f'}\pm\sqrt{\frac{L}{f'}\left(\frac{L}{4f'}-1\right)}$	
L	$-s+s'$	$-\frac{s^2}{f'+s}$	$(m-1)s$		$\frac{s'^2}{s'-f'}$	$\left(1-\frac{1}{m}\right)s'$		$-f'\frac{(m-1)^2}{m}$		

Object-image distance OO´ by magnification $$L=-f\,\frac{(m-1)^2}{m}$$

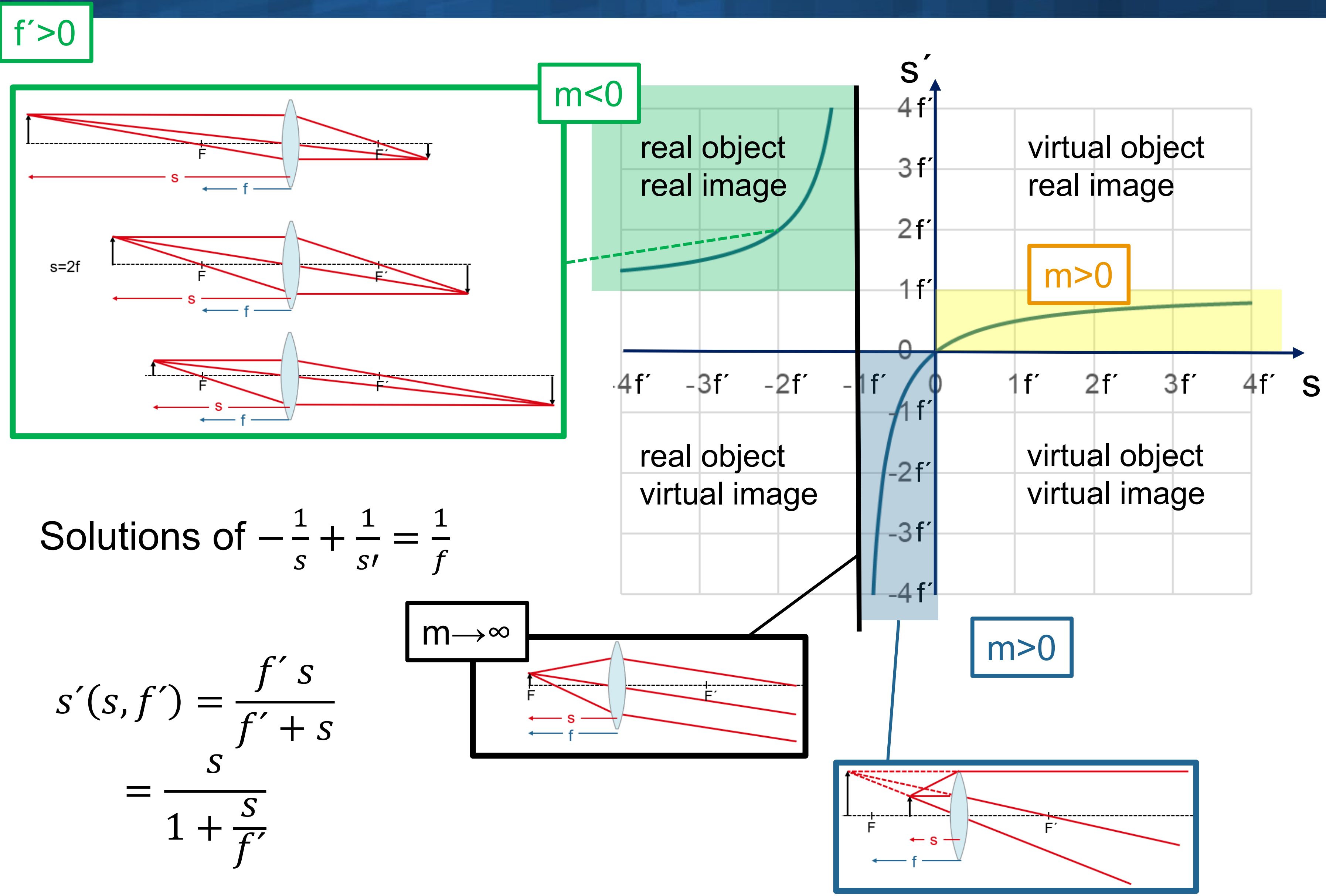
$$L_{non-macro}\approx-\frac{f}{m}$$

e.g.: m = -0.1
L ≈ 10·f

Non-Macro-Distance
(in photography typical MOD at closest distance m ≈ -0.05 – (-0.1))

Caution: The use of these formulas only applies approximately to real lenses, as these are not thin and often have internal focusing. In the non-macro distance range, however, the “thin lens approximation” is often sufficient for an estimate.

Solutions of the imaging equation



Solutions of $-\frac{1}{s}+\frac{1}{s\prime}=\frac{1}{f}$

$$s\acute{}(s,f\acute{}) = \frac{f\acute{}\, s}{f\acute{}+s} = \frac{s}{1+\frac{s}{f\acute{}}}$$

Imaging by a lens in air:
lens makers formula

$$\frac{1}{s'}-\frac{1}{s}=\frac{1}{f}$$

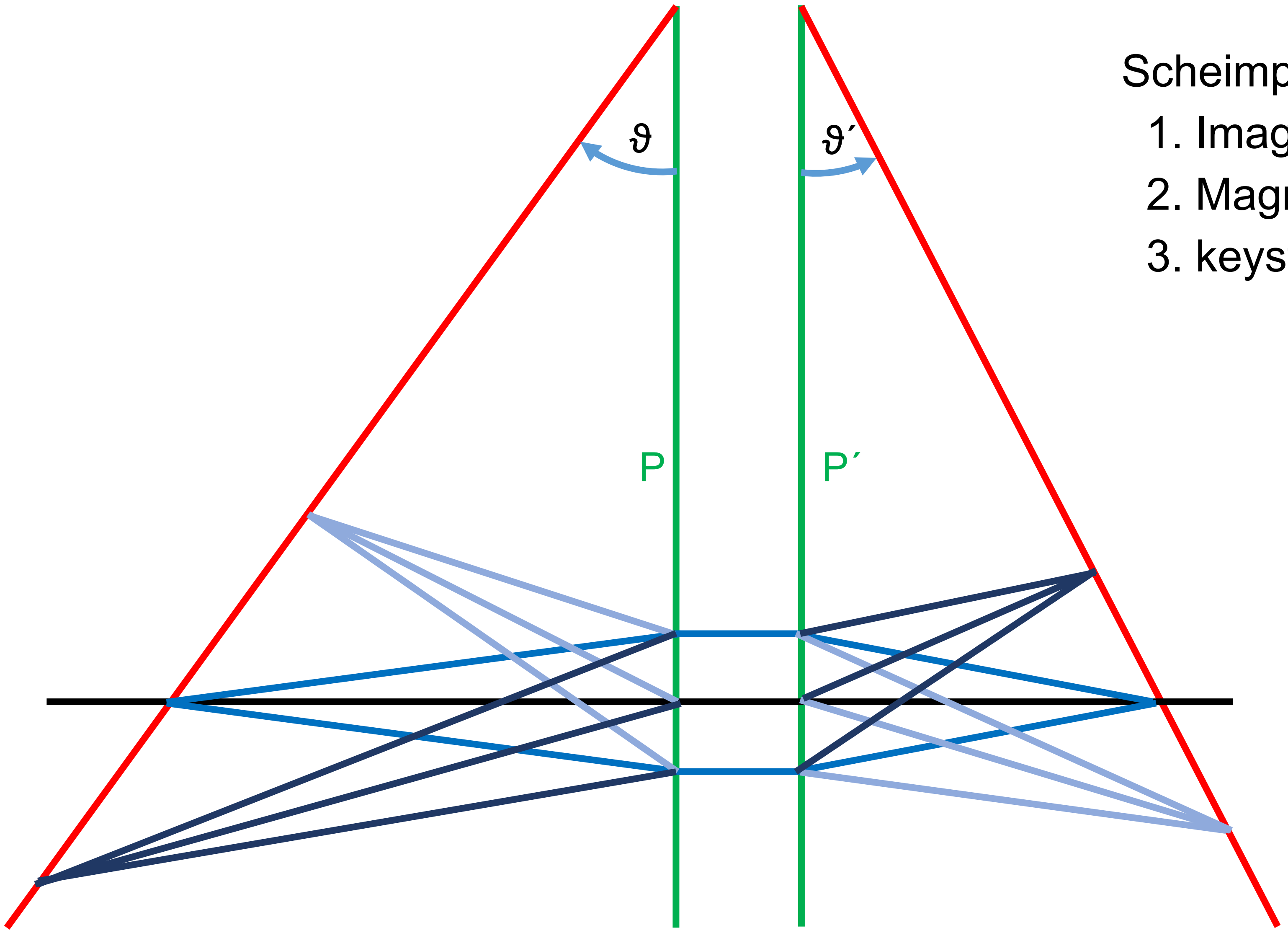
Magnification

$$m=\frac{s'}{s}$$

Real imaging:
s < -f´ , s' > 0 (or m<0)

Intersection lengths s, s'
measured with respect to
principal planes P, P'

Scheimpflug principle



Scheimpflug-Imaging, tilted object plane:

1. Image plane is tilted
2. Magnification is anamorphic
3. keystone distortion in tilted image plane

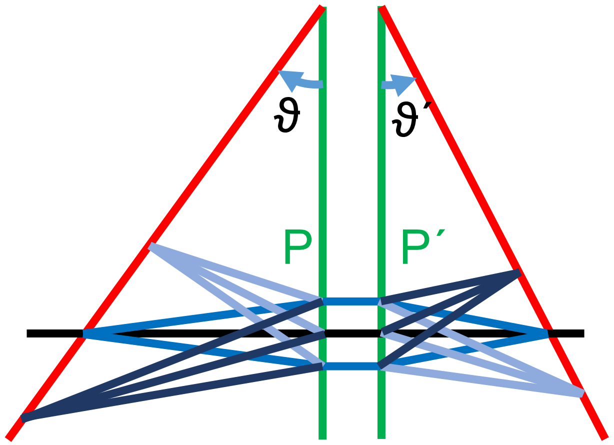
$$m_x = m_o = \frac{s'}{s} = \frac{\tan\theta\,'}{\tan\theta}$$

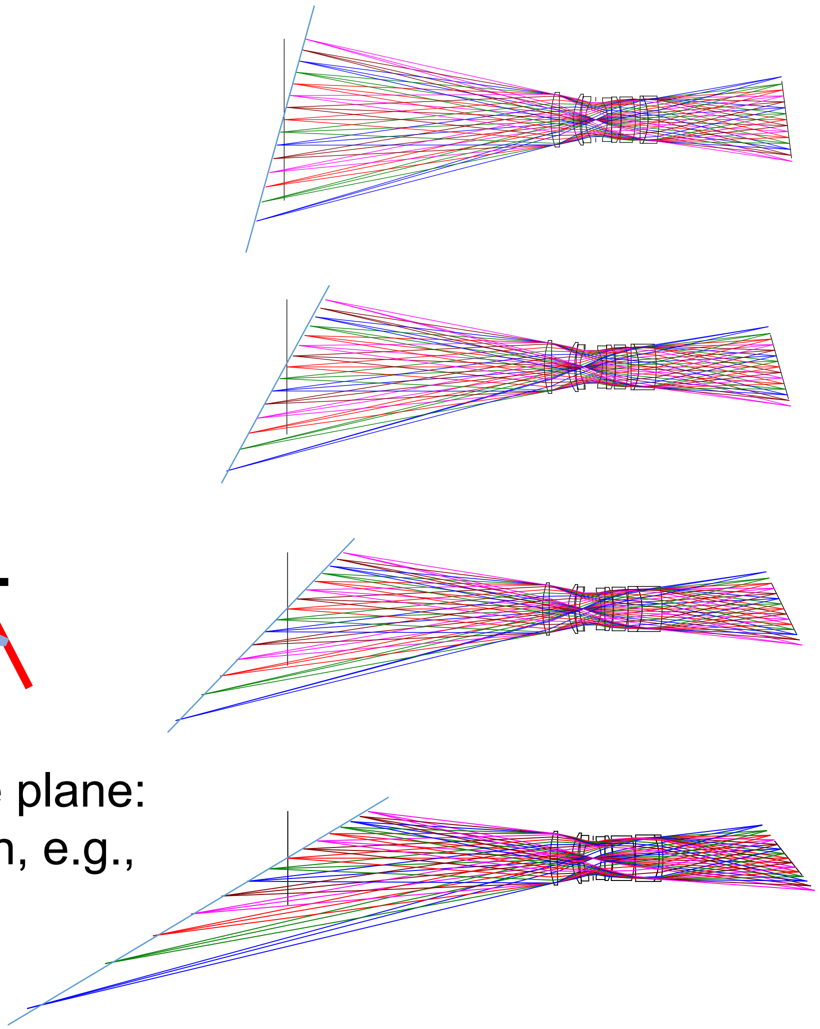
$$m_y = m_o^2 \cdot \frac{\sin\theta}{\sin\theta\,'}$$

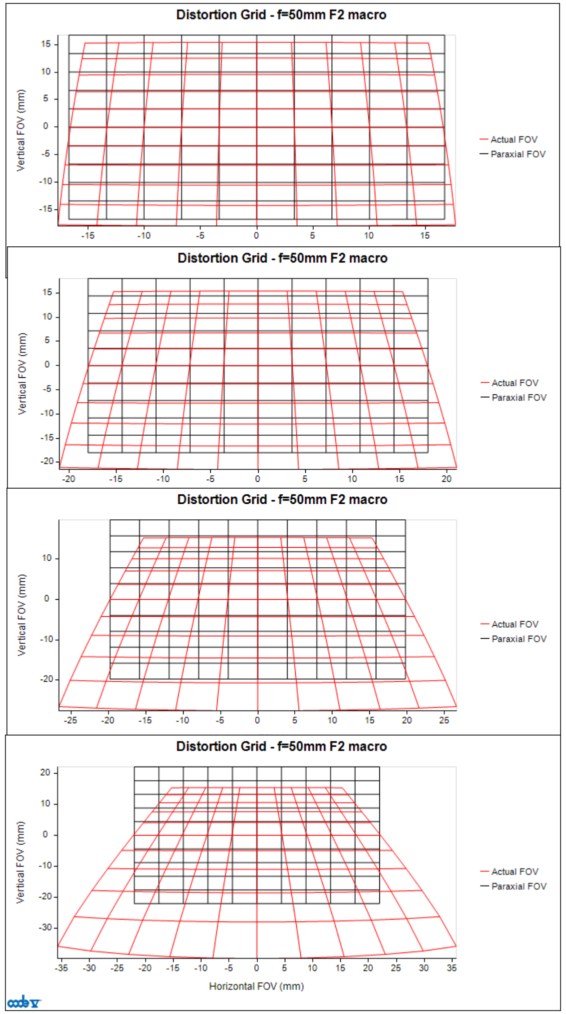
Scheimpflug principle demonstrated with macro camera lens design

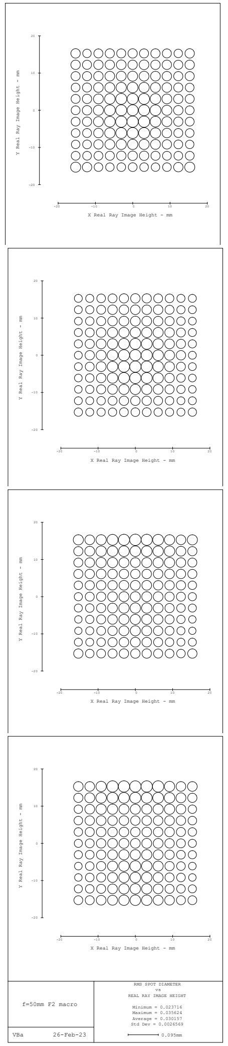
Scheimpflug relation valid over large tilt range

$$m_x = m_o = \frac{s'}{s} = \frac{\tan\theta\,'}{\tan\theta}$$









„Keystone distortion“ in image plane:
Desired perspective correction, e.g., looking up skyscrapers.
→ Tilt-shift –lenses.

*Residual aberrations present, which require non-rotational symmetric correction

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

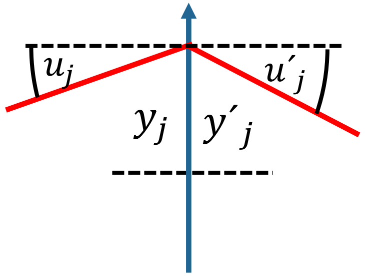
Alternatively paraxial ray tracing can be formulated in **matrix notation**: Transfer equation are linear regarding height **y** (and h, denoting y at refractive surface) and angle **u**=y/s : $\begin{pmatrix} y \\ u=y/s \end{pmatrix} \rightarrow \begin{pmatrix} y' \\ u'=y'/s' \end{pmatrix}$

Refraction / reflection:

$$\boxed{y'_j = y_j}$$

$$\boxed{u'_j = \frac{n'-n}{n'\,r_j} y_j + \frac{n}{n'} u_j}$$

y invariant,
u changes



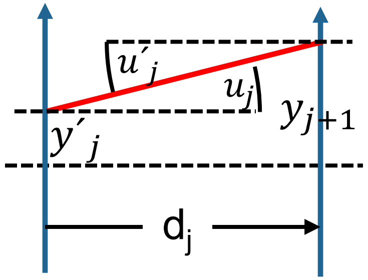
$$\begin{pmatrix} y' \\ u' \end{pmatrix} = R_{n,n',r} \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{n'-n}{n'\,r} & \frac{n}{n'} \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} y \\ \frac{n'-n}{n'\,r} y + \frac{n}{n'} u \end{pmatrix}$$

Transition to next surface at distance d_j:

$$\boxed{y_{j+1} = y'_j - d_j\, u_j}$$

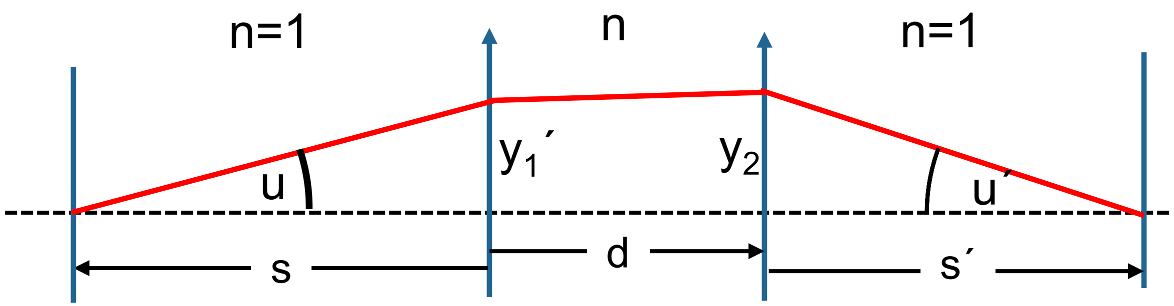
$$\boxed{u'_j = u_j}$$

y changes,
u invariant



$$\begin{pmatrix} y' \\ u' \end{pmatrix} = T_d \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} 1 & -d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} y - d\,u \\ u \end{pmatrix}$$

Example: Transfer through thick lens in air



$$R_{n,n',r} = \begin{pmatrix} 1 & 0 \\ \frac{n'-n}{n'\,r} & \frac{n}{n'} \end{pmatrix}$$

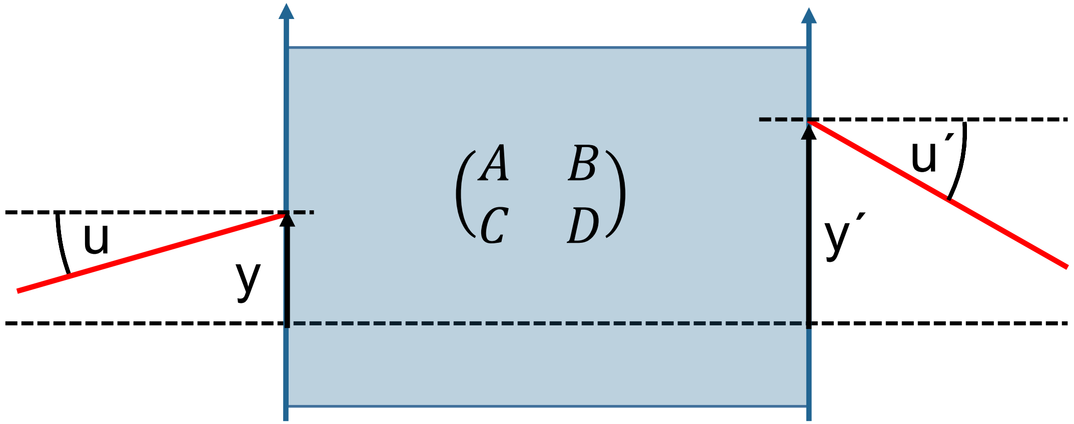
$$T_d = \begin{pmatrix} 1 & -d \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} y' \\ u' \end{pmatrix} = T_{s'} R_{n,1,r_2} T_d R_{1,n,r_1} T_{-s} \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} 1 & s' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1-n}{r_2} & n \end{pmatrix} \begin{pmatrix} 1 & -d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{n-1}{n\,r_1} & \frac{1}{n} \end{pmatrix} \begin{pmatrix} 1 & -(-s) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix}$$

General case:

paraxial segment with matrix „ABCD-matrix”:

$$\begin{pmatrix} y' \\ u' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix}$$



Matrix Formulation of Paraxial Optics

Linear transfer of spatial coordinate y and aperture angle u

$$y' = Ay + Bu$$
$$u' = Cy + Du$$

Matrix formulation:

$$\begin{pmatrix} y' \\ u' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} y \\ u \end{pmatrix} = \underline{M} \cdot \begin{pmatrix} y \\ u \end{pmatrix}$$

Lateral magnification for u=0

$$A = \frac{y'}{y} = m$$

Angular magnification of conjugated planes

$$D = \frac{u'}{u} = \Gamma$$

Refractive power for u=0

$$C = \frac{u'}{y} = \Phi$$

Composition of systems

$$\underline{M} = \underline{M}_k \cdot \underline{M}_{k-1} \cdot \ldots \cdot \underline{M}_2 \cdot \underline{M}_1$$

Determinante

$$\det \underline{M} = AD - BC = \frac{n}{n'}$$

3 free variables only!

Matrix Notation: System parameter calculation by ABCD matrix

Intersection length $$s' = \frac{A \cdot s + B}{C \cdot s + D}$$

Magnifications:

- Lateral $$m = \frac{AD - BC}{C \cdot s + D}$$
- Angular $$\Gamma = C \cdot s + D = \frac{AD - BC}{A - C \cdot s'}$$
- Depth (longitudinal) $$\alpha = \frac{ds'}{ds} = \frac{AD - BC}{(C \cdot s + D)^2}$$

Principal planes $$a_P = \frac{AD - BC - D}{C} \qquad a_{P'} = \frac{A - 1}{C}$$

Focal points $$a_{F'} = \frac{A}{C} \qquad a_F = -\frac{D}{C}$$

Matrix Formulation of Paraxial Optics: Examples

System inversion
$$S^{-1} = \frac{1}{AD - BC}\begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$$

Transition over distance L
$$T = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Thin lens with focal length f
$$R = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

Dielectric plane interface
$$R = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n}{n'} \end{pmatrix}$$

Afocal telescope
$$R = \begin{pmatrix} \frac{1}{\Gamma} & L \\ 0 & \Gamma \end{pmatrix}$$

Summary / Take-Away´s

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

- **Imaging equation** / invariant law for **single refracting surface** $-\frac{n}{s}+\frac{n´}{s´}=\frac{n´-n}{r}$
- Image location can be found by paraxial ray tracing (analytical expression of system parameters n_j, r_j, d_j)
- System characteristics **focal length**, **lateral** & **depth magnification** (interpretation and expression of n_j, r_j, d_j)
- Imaging equations of complete optical system as „**black box**“ via pair of conjugate planes: namely **principal planes** or another arbitrary one, via **entrance** and **exit pupil** $-\frac{1}{m_p s}+\frac{m_p}{s'}=\frac{1}{f´}$
- **Imaging equation** $-\frac{1}{m_p s}+\frac{m_p}{s'}=\frac{1}{f´}$ describes the actual system structure **generalizing the pinhole scheme**: entrance and exit pupil are the **crossing points with the optical axis of object and image chief ray**
- **Listing´s image construction** scheme **follows from principal plane construction properties**
- It can be proven that paraxial imaging defines a **stigmatic and distortion-free projection between planes**

- Paraxial imaging can be formulated in **matrix notation** as well, the system transfer being described by 4 parameters “**ABCD**” which are coupled (3 parameters are independent)

- an optical system imaging between planes perpendicular to the axis also images between tilted object plane according to the relation $m_x=\frac{\tan\theta\prime}{\tan\theta}$ (**Scheimpflug principle**)

Literature (Lecture 2)

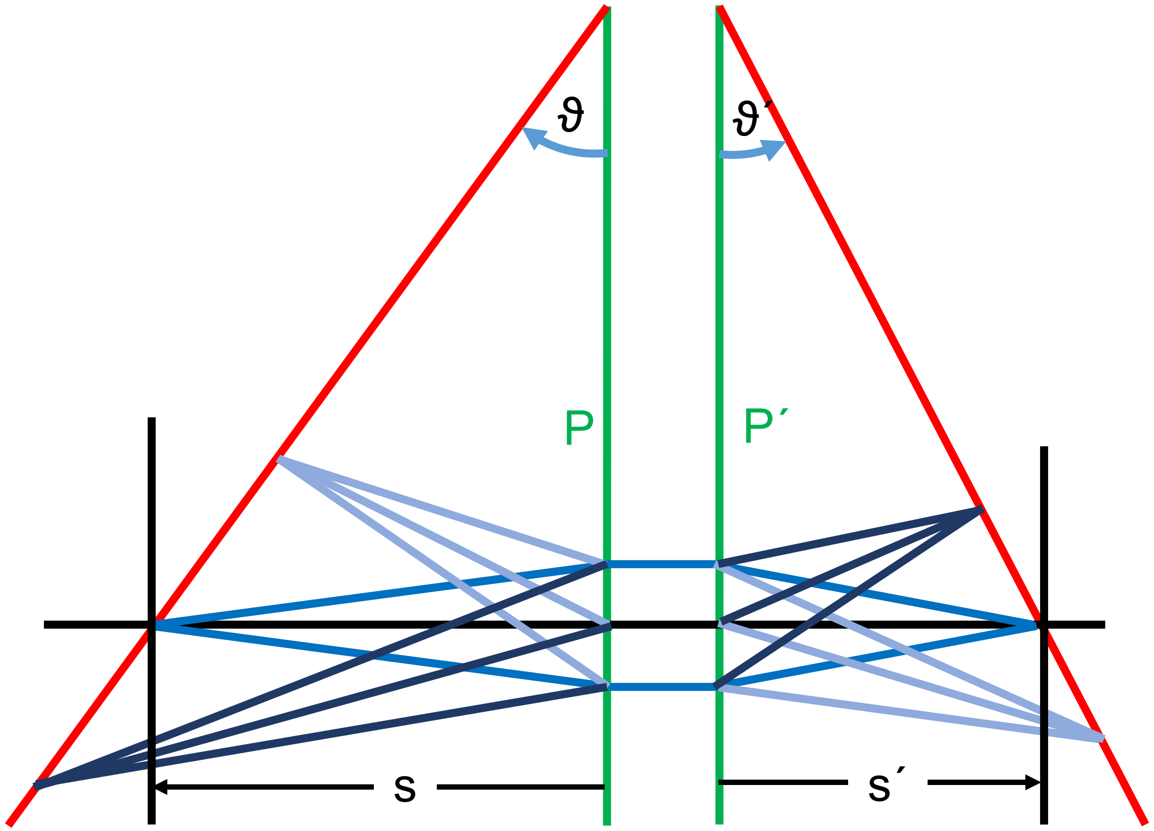
Berek, M. (1930). *Grundlagen der praktischen Optik. Analyse und Synthese optischer Systeme*, de Gruyter, Berlin. [german only]

Blahnik, V. (2014). About the irradiance and apertures of camera lenses (zeiss.com)
https://lenspire.zeiss.com/photo/app/uploads/2022/02/technical-article-about-the-irradiance-and-apertures-of-camera-lenses.pdf

Braat, J., Török, P. (2019). *Imaging Optics*, Cambridge University Press.

Gross, H. (2005). *Handbook of Optical Systems*, vol. 1. Wiley.

Institute of Applied Physics
Friedrich-Schiller-Universität Jena



Scheimpflug-Imaging, tilted object plane:

1. Image plane is tilted
2. Magnification is anamorphic
3. keystone distortion in tilted image plane

$$m_x = m_o = \frac{s'}{s} = \frac{\tan\theta\,'}{\tan\theta}$$

$$m_y = m_o^2 \cdot \frac{\sin\theta}{\sin\theta\,'}$$