Friedrich Schiller University Jena WS 2022/2023
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Series 6
Fundamentals of Modern Optics

to be returned on **01.12.2022**, at the beginning of the lecture

Example solution

Task 1: Response function of Fresnel approximation (5 points)

Consider the Fresnel approximation, under which the transfer function in the spatial frequency domain reads:

$$H_F(\alpha,\beta,z) = \exp(ikz)\exp\left[-i\frac{\alpha^2+\beta^2}{2k}z\right]$$

Derive the response function $h_F(x,y,z>z_0)$ in the spatial domain, as given in the lecture notes. Use the integral:

$$\int_{-\infty}^{+\infty} e^{-ix^2}\mathrm{d}x = \sqrt{\frac{\pi}{i}}.$$

Solution Task 1:

For the transfer function, we have a factor of $1/4\pi^2$ (which comes not from the Fourier transform!) in the definition of the lecture to maintain the form of linear system theory.

$$\begin{aligned} h_F(x,y,z) &= \frac{1}{4\pi^2}\iint_{\mathbb{R}^2} H_F(\alpha,\beta,z)e^{i\alpha x+i\beta y}\,\mathrm{d}\alpha\mathrm{d}\beta \\ &= \frac{1}{4\pi^2}\exp(ikz)\iint \exp\left[-i\frac{\alpha^2+\beta^2}{2k}z\right]\exp(i\alpha x+i\beta y)\,\mathrm{d}\alpha\mathrm{d}\beta \end{aligned}$$

Completing the square, e.g. for α.

$$\begin{aligned} \frac{\alpha^2}{2k}z-\alpha x &= \frac{z}{2k}\left(\alpha^2-2\alpha\frac{kx}{z}\right) = \frac{z}{2k}\left[\left(\alpha-\frac{kx}{z}\right)^2-\left(\frac{kx}{z}\right)^2\right] \\ &= \frac{z}{2k}\left(\alpha-\frac{kx}{z}\right)^2-\frac{kx^2}{2z}, \end{aligned}$$

and similar for β. The second term does not depend on α and can be pulled out of the integral. We get

$$h_F(x,y,z) = \frac{1}{4\pi^2}\exp(ikz)\exp\left(ik\frac{x^2+y^2}{2z}\right)\iint_{\mathbb{R}^2}\exp\left[-i\frac{z}{2k}(\alpha'^2+\beta'^2)\right]\mathrm{d}\alpha'\mathrm{d}\beta',$$

where α and β have been shifted by kx/z and ky/z, respectively to obtain α',β' (has no influence on infinite integral).

For the remaining integral, one substitutes

$$\alpha'' := \sqrt{\frac{z}{2k}}\alpha',\ d\alpha'' := \sqrt{\frac{z}{2k}}d\alpha' \qquad \beta'' := \sqrt{\frac{z}{2k}}\beta',\ d\beta'' := \sqrt{\frac{z}{2k}}d\beta'$$

$$\iint_{\mathbb{R}^2}\exp\left[-i\frac{z}{2k}(\alpha'^2+\beta'^2)\right]\mathrm{d}\alpha'\mathrm{d}\beta' = \int\exp\left[-i\alpha''^2\right]\mathrm{d}\alpha''\sqrt{\frac{2k}{z}}\int\exp\left[-i\beta''^2\right]\mathrm{d}\beta''\sqrt{\frac{2k}{z}}$$

We know:

$$\int_{-\infty}^{+\infty}\exp\left[-i\alpha''^2\right]\mathrm{d}\alpha'' = \int_{-\infty}^{+\infty}\exp\left[-i\beta''^2\right]\mathrm{d}\beta'' = \sqrt{\frac{\pi}{i}},$$ *The students do not have to prove this integral's result*

Hence, the whole integral yields

$$\iint_{\mathbb{R}^2} \exp\left[-i\frac{z}{2k}(\alpha'^2+\beta'^2)\right] \mathrm{d}\alpha'\mathrm{d}\beta' = \frac{2\pi k}{iz}$$

and the resulting transfer function reads as

$$\boxed{h_F(x,y,z) = -\frac{ik}{2\pi z}\exp\left[ikz\left(1+\frac{x^2+y^2}{2z^2}\right)\right] = \frac{-i}{\lambda z}\exp\left[ikz\left(1+\frac{x^2+y^2}{2z^2}\right)\right].}$$

Task 2: Gaussian beam (2+2 points)

In the lecture we defined the Gaussian beam as

$$v(x,y,z) = A(z)\exp\left[-\frac{x^2+y^2}{w(z)^2}\right]\exp\left[ikz + i\frac{k}{2}\frac{x^2+y^2}{R(z)} + i\varphi(z)\right].$$

a) Derive a spherical wave in paraxial approximation and show that for which condition the wavefront of a Gaussian beam is the same as a wavefront of the spherical wave . *Hint: Neglect Guoy phase shift of the Gaussian beam.*

b) How far can a Gaussian beam with $\lambda = 630$ nm and $W_0 = 8$ mm stay collimated(we consider maximum 10% broadening after propagating z_1 from waist)?

Solution Task 2:

a) In the lecture, we defined the Gaussian beam wavefront as:

$$\Phi(x,y,z) = k\left(z+\frac{x^2+y^2}{2R(z)}\right) + \underbrace{\varphi(z)}_{\text{Guoy phase} \implies \text{Neglect!}} = const$$

$$\implies \boxed{z+\frac{x^2+y^2}{2R(z)} = const}; \quad \text{where } R(z) = z\left(1+\left(\frac{z_0}{z}\right)^2\right) \text{ from lecture.}$$

The equation in the box is always true for the Gaussian beam.

Now lets consider an arbitrary sphere with radius R in cartesian coordinate:

$$x^2+y^2+z^2 = R^2$$

$$\implies z = \sqrt{R^2-x^2-y^2} = R\sqrt{1-\frac{x^2}{R^2}-\frac{y^2}{R^2}} = R\sqrt{1-\frac{x^2+y^2}{R^2}} = R\sqrt{1-\frac{\rho^2}{R^2}},$$

where ρ is radius in cylindrical coordinate system. If we use paraxial approximation $\frac{x^2}{R^2}, \frac{y^2}{R^2} \ll 1 \implies \rho^2/R^2 \ll 1$, we can Taylor expand the root with ρ/R variable

$$\implies \quad z \approx R\left[1-\frac{1}{2}\left(\frac{\rho^2}{R^2}\right)\right] = R\left[1-\frac{1}{2}\left(\frac{x^2+y^2}{R^2}\right)\right] = R - \frac{x^2+y^2}{2R}$$

For sphere in paraxial approximation

$$\implies z+\frac{x^2+y^2}{2R} = R$$

This resembles the formula in the box; thus, it indicates that the Gaussian beam phase evolves in the spherical fashion and the $R(z)$ in boxed formula is radius of curvature of a sphere.

b) We need to start with the calculation of Rayleigh length z_0

$$z_0 = \frac{\pi W_0^2}{\lambda} = 319.15 \text{ m}$$

Therefore the Rayleigh length is $z_0 = 319.15$ m Since we consider 10% broadening, a beam width $W(z) = 1.1W_0$. So that we write down the wavefront once again using z_1

$$W(z) = W_0\sqrt{1+\frac{z_1^2}{z_0^2}}$$

Hence at $z_1 = 146.25$ m still beam is collimated only by 10% broadening

Task 3: Focusing a Gaussian Beam (4+2 points)

A collimated Gaussian beam of wavelength λ with a waist W_0 (the waist is just behind the lens) is focused by a lens with a focal distance f, as shown in Figure 1. The Rayleigh length of the beam before the lens, $z_0 = \frac{\pi W_0^2}{\lambda}$, is much larger than f. The focused Gaussian beam after the lens would have the waist W_1 at the distance d after the lens.

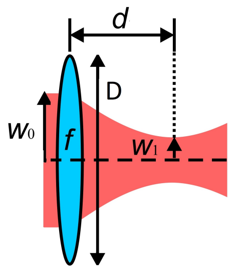

Figure 1: Focusing with a lens.

a) Use the ABCD matrix of the system to calculate W_1 and d exactly. Then use the fact that $z_0 >> f$ to simplify your results.

b) How small can $2W_1$ be? In other words, how small can the focal spot after the lens be? Use the approximate result of (a) in the $z_0 >> f$ regime.
Hint: To make a statement about this, you have to make some assumptions. Firstly, you have to notice that for the calculation in (a) to be correct, you are assuming that the lens aperture D is large enough to let a substantial part of the Gaussian beam to pass through it. Let us say that $2W_0$ should be smaller than D for a substantial part of the beam to pass through the lens. Moreover, for a thin lens, the size of the aperture is also limited based on its focal length. So let us assume that $D/2$ is smaller than f, such that the ratio $D/2f$ is always smaller than 1. Put all these statements together, to be able to find a limit on how small the size of the focused beam can be.

Solution Task 3:

a) The q-parameter for the Gaussian beam right before the lens is $q_0 = -iz_0$. The ABCD matrix of the lens is $\begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}$, so q_1 right after the lens is

$$q_1 = \frac{Aq_0 + B}{Cq_0 + D} = \frac{q_0}{-q_0/f + 1} = \frac{-iz_0 f}{f + iz_0} = \frac{-iz_0 f^2}{z_0^2 + f^2} - \frac{z_0^2 f}{z_0^2 + f^2}$$

and after the propagation length d we have $q_2 = q_1 + d$. For the beam to be at its waist there, we need to have $q_2 = -iz_1$ be a purely imaginary number. To get this, we need to have $\boxed{d = \frac{z_0^2 f}{z_0^2 + f^2}}$. We then have $-iz_1 = -i\frac{\pi W_1^2}{\lambda} = -i\frac{z_0 f^2}{z_0^2 + f^2}$. Hence we have $\boxed{W_1 = \sqrt{\frac{\lambda}{\pi}\frac{z_0 f^2}{z_0^2 + f^2}}}$.

In the limit of $z_0 >> f$, we get $\boxed{d \approx f}$ and $W_1 \approx \sqrt{\frac{\lambda}{\pi}\frac{f^2}{z_0}} = \sqrt{\frac{\lambda}{\pi}\frac{f^2}{\pi W_0^2/\lambda}}$ which finally gives $\boxed{W_1 \approx \frac{\lambda f}{\pi W_0}}$.

b) We start from $2W_1 = 2\frac{\lambda f}{\pi W_0}$. Now it is clear that the bigger W_0 is the smaller W_1 gets. But we said W_0 could only be as big as $D/2$. So the smallest spot size is $2W_1 = 2\frac{\lambda f}{\pi (D/2)} = 2\frac{\lambda}{\pi (D/2f)}$. Now it is clear that the larger the factor $\frac{D}{2f}$ is the smaller W_1 gets. But we have also said that the maximum value of $\frac{D}{2f}$ is 1. So the smallest possible value for the spot size is $\boxed{2W_1 = \frac{\lambda}{\pi/2}}$.