Task 1

a)

Solution:

$$I = |\langle \vec{S}(\vec{r},t)\rangle|$$

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}Re[\vec{E}(\vec{r}) \times \vec{H}^*(\vec{r})]$$

$$= \frac{1}{2}Re[\frac{\alpha A_0^2}{w_0\mu_0}\vec{e}_z + \frac{\beta A_1^2}{w_0\mu_0}\vec{e}_z]$$

$$= \frac{\alpha A_0^2 + \beta A_1^2}{2w_0\mu_0}\vec{e}_z$$

b)

$$\vec{E}(z,t) = A_0 \exp[i(\alpha z - w_0 t)]\,\vec{e}_x + A_1 \exp[i(\beta z - w_0 t)]\,\vec{e}_y$$

$$\therefore \vec{E}(\vec{r},w) = A_0 e^{i\alpha z}\delta(w-w_0)\vec{e}_x + A_1 e^{i\beta z}\delta(w-w_0)\vec{e}_y$$

From Maxwell's equations $\vec{H}(\vec{r},w) = -\frac{i}{w\mu_0}\,\mathrm{rot}\,\vec{E}(\vec{r},w)$

$$\therefore \vec{H}(\vec{r},w) = -\frac{i}{w\mu_0}[i\alpha A_0 e^{i\alpha z}\delta(w-w_0)\vec{e}_y - i\beta A_1 e^{i\beta z}\delta(w-w_0)\vec{e}_x]$$

$$= \frac{\alpha}{w\mu_0}A_0 e^{i\alpha z}\delta(w-w_0)\vec{e}_y - \frac{\beta}{w\mu_0}A_1 e^{i\beta z}\delta(w-w_0)\vec{e}_x$$

$$\vec{H}(\vec{r},t) = \frac{\alpha A_0}{w_0\mu_0}e^{[i(\alpha z - w_0 t)]}\vec{e}_y - \frac{\beta A_1}{w_0\mu_0}e^{[i(\beta z - w_0 t)]}\vec{e}_x$$

$$I = E(\vec{r})\cdot E^*(\vec{r})$$

$$= A_0^2 + A_1^2$$

Task 2

a)

Solution:

$$U_0(\alpha,\beta) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} u_0(x,y)\exp[-i(\alpha x + \beta y)]\,dxdy$$

$$= \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} A e^{-\frac{x^2+y^2}{w^2}} e^{-i(\alpha x+\beta y)}\,dxdy$$

$$= \frac{A}{4\pi^2}\int_{-\infty}^{\infty} e^{-\frac{x^2}{w^2} - i\alpha x}dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{w^2} - i\beta y}dy$$

$$= \frac{A}{4\pi^2}\int_{-\infty}^{\infty} e^{-\frac{1}{w^2}(x+\frac{i\alpha w^2}{2})^2}\cdot e^{-\frac{\alpha^2 w^2}{4}}dx \int_{-\infty}^{\infty} e^{-\frac{1}{w^2}(y+\frac{i\beta w^2}{2})^2}\cdot e^{-\frac{\beta^2 w^2}{4}}dy$$

$$= \frac{A}{4\pi^2}[e^{-\frac{\alpha^2 w^2}{4}}\int_{-\infty}^{\infty} e^{-[\frac{1}{w}(x+\frac{i\alpha w^2}{2})]^2}dx\; e^{-\frac{\beta^2 w^2}{4}}\int_{-\infty}^{\infty} e^{-[\frac{1}{w}(y+\frac{i\beta w^2}{2})]^2}dy]$$

$$= \frac{A}{4\pi^2}[e^{-\frac{\alpha^2 w^2}{4}}\int_{-\infty}^{\infty} w e^{-[\frac{1}{w}(x+\frac{i\alpha w^2}{2})]^2}d[\frac{1}{w}(x+\frac{i\alpha w^2}{2})]\; e^{-\frac{\beta^2 w^2}{4}}\int_{-\infty}^{\infty} w e^{-[\frac{1}{w}(y+\frac{i\beta w^2}{2})]^2}d[\frac{1}{w}(y+\frac{i\beta w^2}{2})]]$$

From Gaussian integral

$$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$$

$$\therefore U_0(\alpha,\beta) = \frac{A}{4\pi^2}(w\sqrt{\pi}e^{-\frac{\alpha^2 w^2}{4}}\cdot w\sqrt{\pi}e^{-\frac{\beta^2 w^2}{4}}) = \frac{Aw^2}{4\pi}e^{-\frac{(\alpha^2+\beta^2)w^2}{4}}$$

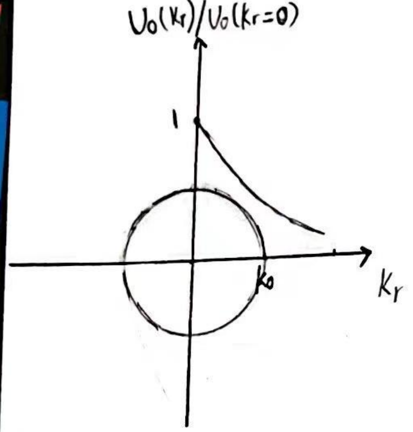
b)

Solution: $\because k_r = \sqrt{k_x^2+k_y^2}$

$$U_0(k_r=k_0)/U_0(k_r=0) = \frac{\frac{AW^2}{4\pi} e^{-\frac{k_0^2 W^2}{4}}}{\frac{AW^2}{4\pi}} = e^{-\frac{k_0^2 W^2}{4}} = e^{-\frac{4\pi^2 W^2}{4\lambda^2}} = e^{-\frac{\pi^2 W^2}{\lambda^2}}$$

$W = \lambda/\pi$

$$U_0(k_r)/U_0(k_r=0) = e^{-\frac{k_r^2 \lambda^2}{4\pi^2}} \qquad U_0(k_r=k_0)/U_0(k_r=0) = \frac{1}{e}$$



(c)

Solution:

Draw a circle with radius $= k_0$

The area in the circle ($k_x^2+k_y^2 \leq k_0^2$) contributes to propagating wave (real part of k_z)

The area beyond the ~~circle~~ circle ($k_x^2+k_y^2 > k_0^2$) contributes to evanescent wave (imaginary part of k_z)

Task 3

a)

Solution:

In Talbot Effect, we can get

$$f(x, z=L_T) = f(x, z=0)\exp(ikL_T + i2\pi m_\ell)$$

$$f(x+a) = f(x)$$

Express $f(x)$ as a Fourier Series

$$f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{in\omega_0 x} \quad (n \in Z)$$

$\because T = a \quad \therefore \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{a}$

$$f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{in\frac{2\pi}{a}x}$$

$$F\{f(x,z=0)\} = F(\alpha, z=0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} C_n e^{in\frac{2\pi}{a}x} e^{-i\alpha x} dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} C_n e^{-i(\alpha - n\frac{2\pi}{a})x} dx$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} C_n \delta(\alpha - n\frac{2\pi}{a})$$

$$F(\alpha, z) = F(\alpha, z=0)\cdot H(\alpha, z)$$

$$H(\alpha, z) = \exp[i\sqrt{k^2-\alpha^2}\, z]$$

$$\therefore F(\alpha, z) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} C_n \delta(\alpha - n\frac{2\pi}{a}) e^{i\sqrt{k^2-\alpha^2}\, z}$$

$$f(x,z) = \int_{-\infty}^{+\infty} F(\alpha, z) e^{i\alpha x} d\alpha$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} C_n \delta(\alpha - n\frac{2\pi}{a}) e^{i\sqrt{k^2-\alpha^2}\, z} e^{i\alpha x} d\alpha$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} C_n e^{i\sqrt{k^2-(\frac{2n\pi}{a})^2}\, z} e^{i\frac{2\pi n}{a}x}$$

$$\because f(x, z=L_T) = f(x, z=0) e^{i(kL_T + 2\pi m_\ell)}$$

$$\therefore \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} e^{i\sqrt{k^2-(\frac{2n\pi}{a})^2}\, L_T} e^{i\frac{2n\pi}{a}x} = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} C_n e^{i\frac{2n\pi}{a}x} e^{i(kL_T + 2\pi m_\ell)}$$

$$\therefore \quad e^{i\sqrt{k^2-(\frac{2n\pi}{a})^2}\, L_T} = e^{i(kL_T + 2\pi m_\ell)}$$

$$\sqrt{k^2-(\frac{2n\pi}{a})^2}\, L_T = kL_T + 2\pi m_\ell$$

$$L_T = \frac{2\pi m_\ell}{\sqrt{k^2-(\frac{2n\pi}{a})^2} - k}$$

$\because$ The paraxial is valid

$$\therefore \quad \alpha^2 \ll k^2$$

$$\sqrt{k^2-(\frac{2n\pi}{a})^2} \approx k - \frac{(\frac{2n\pi}{a})^2}{2k}$$

$$\therefore L_T \approx \frac{-ka^2 m_\ell}{n^2\pi}$$

b)

solution:

From (a) we can get

$$L_T = -\frac{ka^2 m_\ell}{n^2\pi}$$

$\because m_\ell, n \in Z$

$\therefore$ let $m_\ell = -n^2$

$$L_T = \frac{ka^2}{\pi} = \frac{2a^2}{\lambda}$$

$\because L_T = 25.7\,m \quad a = 3\times10^{-3}\,m$

$$\therefore \lambda = 0.7\times10^{-6}\,m$$

c)

Solution:

$$f(x,z=0) = A\cos(x\,2\pi/a_1)\cos(x\,2\pi/a_2)$$

$$= \frac{A}{2}\left[\cos\left(\frac{2\pi}{a_1}x+\frac{2\pi}{a_2}x\right)+\cos\left(\frac{2\pi}{a_1}x-\frac{2\pi}{a_2}x\right)\right]$$

$$F(\alpha,z=0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{A}{2}\left[\cos\left(\frac{2\pi}{a_1}x+\frac{2\pi}{a_2}x\right)+\cos\left(\frac{2\pi}{a_1}x-\frac{2\pi}{a_2}x\right)\right]e^{-i\alpha x}dx$$

$$= \frac{A}{4\pi}\int_{-\infty}^{+\infty}\left[\frac{e^{i\left(\frac{2\pi}{a_1}+\frac{2\pi}{a_2}\right)x}+e^{-i\left(\frac{2\pi}{a_1}+\frac{2\pi}{a_2}\right)x}}{2}+\frac{e^{i\left(\frac{2\pi}{a_1}-\frac{2\pi}{a_2}\right)x}+e^{-i\left(\frac{2\pi}{a_1}-\frac{2\pi}{a_2}\right)x}}{2}\right]e^{-i\alpha x}dx$$

suppose $\frac{2\pi}{a_1}+\frac{2\pi}{a_2}=M \quad \frac{2\pi}{a_1}-\frac{2\pi}{a_2}=N$

$$F(\alpha,z=0) = \frac{A}{4\pi}\int_{-\infty}^{+\infty}\frac{1}{2}\left[e^{i(M-\alpha)x}+e^{-i(M+\alpha)x}+e^{i(N-\alpha)x}+e^{-i(N+\alpha)x}\right]dx$$

$$= \frac{A}{8\pi}\left[\delta(M-\alpha)+\delta(M+\alpha)+\delta(N-\alpha)+\delta(N+\alpha)\right]$$

$$F(\alpha,z) = F(\alpha,z=0)e^{i\sqrt{k^2-\alpha^2}\,z}$$

$$f(x,z) = \int_{-\infty}^{+\infty}F(\alpha,z=0)e^{i\sqrt{k^2-\alpha^2}\,z}e^{i\alpha x}d\alpha$$

$$= \int_{-\infty}^{+\infty}\frac{A}{8\pi}\left[\delta(M-\alpha)+\delta(M+\alpha)+\delta(N-\alpha)+\delta(N+\alpha)\right]e^{i\sqrt{k^2-\alpha^2}\,z}e^{i\alpha x}d\alpha$$

$$= \frac{A}{8\pi}\left[e^{i\sqrt{k^2-M^2}\,z}e^{iMx}+e^{i\sqrt{k^2-M^2}\,z}e^{-iMx}+e^{i\sqrt{k^2-N^2}\,z}e^{iNx}+e^{i\sqrt{k^2-N^2}\,z}e^{-iNx}\right]$$

$$= \frac{A}{4\pi}\left[e^{i\sqrt{k^2-M^2}\,z}\cos(Mx)+e^{i\sqrt{k^2-N^2}\,z}\cos(Nx)\right]$$

$$\boxed{\cancel{f(x,z=L_T) = f(x,z=0)\exp(ikL_T}}$$

$\because f(x,z=L_T) = f(x,z=0)\exp(ikL_T+i2\pi m_\ell)$ the Talbot effect still takes place outside the paraxial regime

$$\therefore \frac{A}{4\pi}\left[e^{i\sqrt{k^2-M^2}L_T}\cos(Mx)+e^{i\sqrt{k^2-N^2}\,z}\cos(Nx)\right] = \frac{A}{4\pi}\left[\cos(Mx)+\cos(Nx)\right]e^{i(kL_T+2\pi m_\ell)}$$

$$\therefore \boxed{\cancel{e^{i\sqrt{k^2-M^2}L_T} = e^{i(kL_T+2\pi m_\ell)}} \quad \cancel{e^{i\sqrt{k^2-N^2}L_T}}} \qquad L_T =$$