Pr

Fundamentals of Modern Optics
Exercise 5
17.11.2014

to be returned: 24.11.2014, at the beginning of the lecture

Problem 1 - Normal modes in dieléctrics (2+2+2+1+1 points)

For transversal waves in an isotropic, homogeneous, and dielectric medium, the field amplitude $\vec{E}(\vec{r},\omega)$ is given by the Helmholtz equation in Fourier domain,

$$\left[\Delta + \frac{\omega^2}{c^2}\epsilon(\omega)\right]\vec{E}(\vec{r},\omega) = 0, \quad \text{with} \quad \epsilon(\omega) = \epsilon'(\omega) + i\epsilon''(\omega).$$

Assume an electric field $\vec{E}(\vec{r},\omega) = \vec{E}_0(\omega)e^{i\vec{k}\vec{r}}$, describing a plane wave with the complex wave vector $\vec{k} = \vec{k}' + i\vec{k}''$.

a) Derive the set of equations which connects $\vec{k}'$ and $\vec{k}''$ with ϵ' and ϵ''.

b) Discuss the conditions for propagating and evanescent waves with respect to $\epsilon(\omega)$ from that set of equations.

c) Now assume a dielectric function containing one sharp resonance at frequency ω_0,

$$\epsilon(\omega) = 1 + \frac{f_0}{(\omega_0^2 - \omega^2 - i\gamma\omega)}, \quad \text{where} \quad \gamma \ll \omega_0 \text{ (weak damping regime)}.$$

with

$$f_0 = 10 \text{ fs}^{-2}, \ \omega_0 = 5 \text{ fs}^{-1} \text{ and } \gamma = 10^{-3} \text{ fs}^{-1}.$$

Separate the real and imaginary parts of $\epsilon(\omega)$ to get $\epsilon'(\omega)$ and $\epsilon''(\omega)$ and calculate the ratio $\eta(\omega) \equiv \epsilon''(\omega)/\epsilon'(\omega)$. Sketch/plot this function and identify the frequency regions where $|\eta(\omega)| \ll 1$.

d) Just above the resonance frequency, we have $\epsilon'(\omega) < 0$. Determine the local minimum of $|\eta(\omega)|$ in this regime, assume $\vec{k}' \parallel \vec{k}'' \parallel \vec{e}_z$, and calculate the approximated refractive index $n(\omega_1) + i\kappa(\omega_1)$ at this point for $\epsilon''(\omega_1) \ll |\epsilon'(\omega_1)|$. Find the distance $z_0(\omega_1)$, in units of the vacuum wavelength λ, where the amplitude of the plane wave is reduced by $1/e$ (the so-called $1/e$ penetration depth).

穿透深度

e) Further away from ω_0, the regime with $0 < \epsilon'(\omega) < 1$ is reached. Calculate the approximated refractive index $n(\omega_2) + i\kappa(\omega_2)$ for $\vec{k}' \parallel \vec{k}'' \parallel \vec{e}_z$ at $\omega_2 = 6 \text{ fs}^{-1}$ as well as the $1/e$ penetration depth $z_0(\omega_2)$ in units of the vacuum wavelength λ. Compare your result with the one obtained in (d).

Problem 2 - Traveling and standing wave (2+2+1+1 points)

The electric-field complex amplitude vector for a monochromatic wave of wavelength λ_0 traveling in free space is $\vec{E}(\vec{r}) = E_0 \sin(\beta y)\, e^{i\beta z}\, \vec{e_x}$

a) Derive an expression for the magnetic-field complex amplitude vector $\vec{H}(\vec{r})$.

b) Determine the direction of the flow of optical power by calculating the time averaged Poyting Vector.

c) This wave may be regarded as the sum of two waves. Derive expressions for each wave and determine their directions of propagation. Draw a simple scetch of each plane wave.

d) Determine a relation between β and λ_0.

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Problem 3 - Evanescent waves (2+2 points)

Consider an amplitude distribution $u_0(x, y)$ of a Gaussian beam in cartesian coordinates,

$$u_0(x, y) = A \exp\left[-(x^2 + y^2)/W^2\right] \quad \text{with} \quad W > 0.$$

a) Compute its spatial frequency spectrum $U_0(k_x, k_y)$. Grafically explain your result in units of the wavelength and indicate homogenous and evanascent wave regions in the case of vacuum.

b) What is the minimum value for the waist W in order to transmit at least 90% of the beam energy in vacuum? Express your result in units of the wavelength.

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Problem 1 - Normal modes in dielectrics

a) $[\Delta + \frac{\omega^2}{c^2}\varepsilon(\omega)]\vec{E}(\vec{r},\omega) = 0$

$$[\Delta + \frac{\omega^2}{c^2}\varepsilon(\omega)]\vec{E}(\vec{r},\omega) = -k^2\vec{E}_0(\omega)\cdot e^{i\vec{k}\vec{r}} + \frac{\omega^2}{c^2}\varepsilon(\omega)\cdot\vec{E}_0(\omega)e^{i\vec{k}\cdot\vec{r}}$$

$$= [-k^2 + \frac{\omega^2}{c^2}\varepsilon(\omega)]\cdot\vec{E}_0\cdot e^{i\vec{k}\vec{r}} = 0$$

$\Rightarrow k^2 = \frac{\omega^2}{c^2}\varepsilon(\omega)$

since $\begin{cases}\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega) \\ \vec{k} = \vec{k}' + i\vec{k}''\end{cases}$

~~So k = w/c ε(w) =~~

So $\vec{k}'^2 - \vec{k}''^2 + 2i\vec{k}'\cdot\vec{k}'' = \frac{\omega^2}{c^2}\varepsilon'(\omega) + i\frac{\omega^2}{c^2}\varepsilon''(\omega)$

$\Rightarrow \begin{cases}\vec{k}'^2 - \vec{k}''^2 = \frac{\omega^2}{c^2}\varepsilon(\omega) \\ 2\vec{k}'\cdot\vec{k}'' = \frac{\omega^2}{c^2}\varepsilon''(\omega)\end{cases}$

b) (1) when $\varepsilon'(\omega) > 0$, $\varepsilon''(\omega) = 0$

$\begin{cases}\vec{k}'' = 0, |\vec{k}| = \frac{\omega}{c}\sqrt{\varepsilon'} & \text{propagating wave without loss} \\ \vec{k}' \perp \vec{k}'', \vec{k}^2 = \vec{k}'^2 - \vec{k}''^2 = \frac{\omega^2}{c^2}\varepsilon'(\omega) & \text{evanescent wave (only exist at interfaces)}\end{cases}$

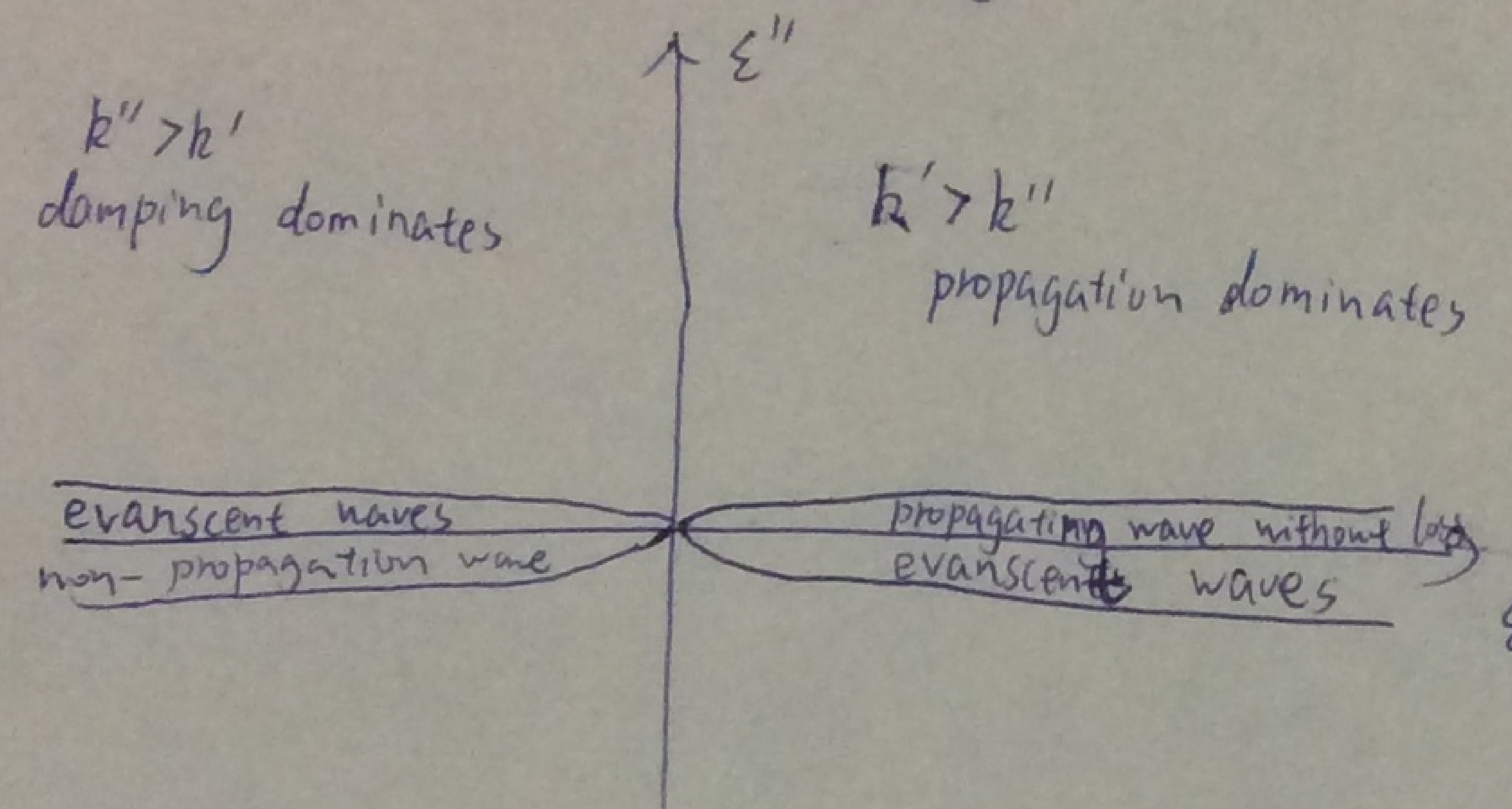
(2) when $\varepsilon'(\omega) < 0$, $\varepsilon''(\omega) = 0$

$\begin{cases}\vec{k}' = 0, |\vec{k}''| = \frac{\omega}{c}\sqrt{-\varepsilon'} & \text{non-propagating wave (strong damping)} \\ \vec{k}' \perp \vec{k}'', |\vec{k}''| > |\vec{k}'|, \vec{k}'^2 - \vec{k}''^2 = \frac{\omega^2}{c^2}\varepsilon'(\omega) & \text{evanescent wave (only exist at interfaces)}\end{cases}$

(3) ε - complex number , $\begin{cases} |\vec{k}'| = \frac{\omega}{c}\sqrt{\varepsilon'(\omega)} \\ |\vec{k}''| = \frac{\omega\,\varepsilon''(\omega)}{2c\sqrt{\varepsilon'(\omega)}} \end{cases}$

i) $\left.\begin{matrix}\varepsilon'>0 \\ \varepsilon''>0\end{matrix}\right\} \Rightarrow k'>k''$ propagating wave with loss (propagation dominates)

ii) $\left.\begin{matrix}\varepsilon'<0 \\ \varepsilon''>0\end{matrix}\right\} \Rightarrow k''>k'$ propagating wave with loss (damping dominates)



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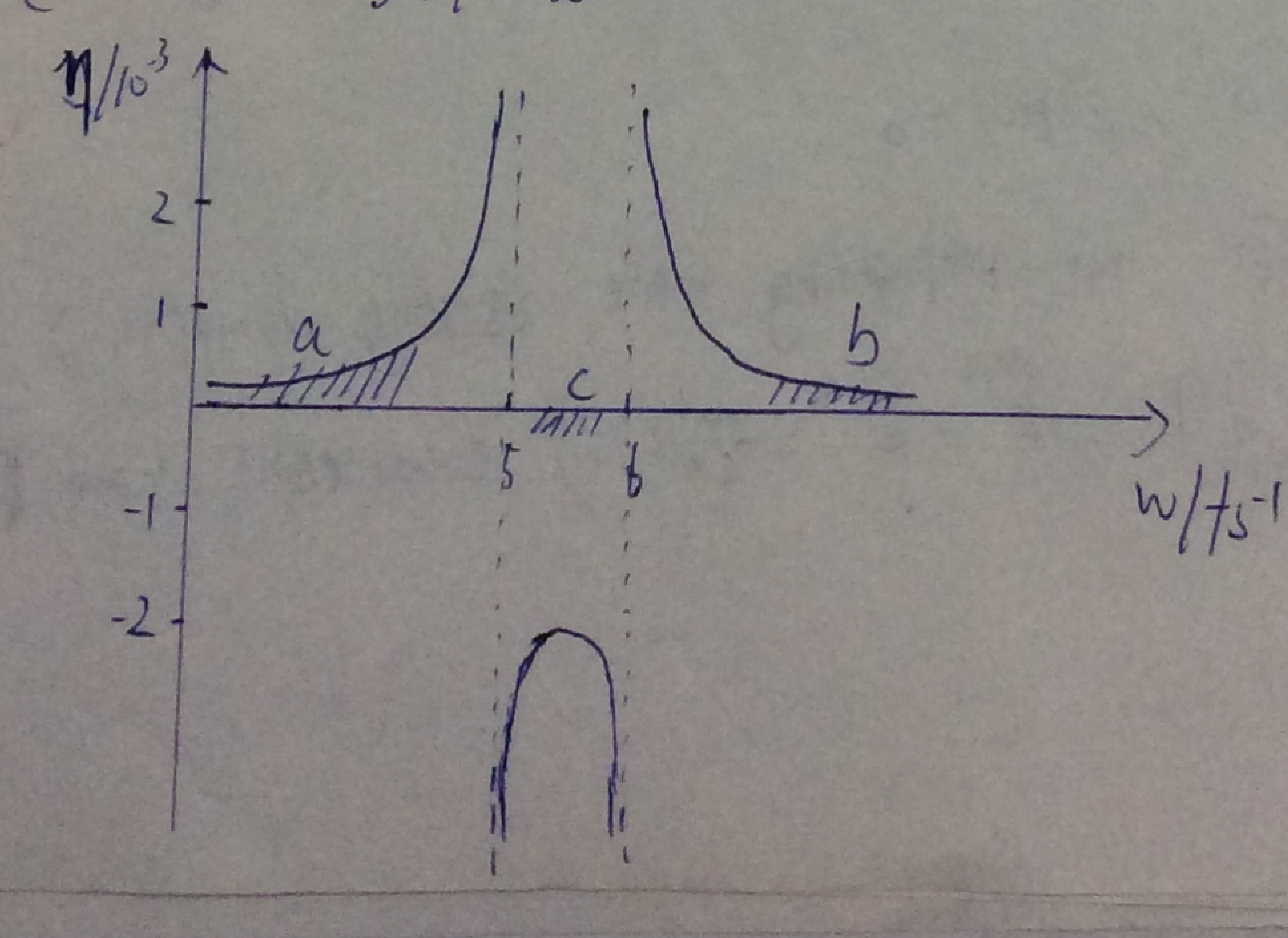
e) $\varepsilon(\omega) = 1 + \frac{f_0}{\omega_0^2-\omega^2-i\gamma\omega} = \underbrace{1 + \frac{f_0(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}_{\varepsilon'(\omega)} + i\underbrace{\frac{f_0\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}_{\varepsilon''(\omega)}$

$$\eta(\omega) \equiv \frac{\varepsilon''(\omega)}{\varepsilon'(\omega)} = \frac{f_0\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2+f_0(\omega_0^2-\omega^2)} = \frac{f_0\gamma\omega}{(\omega_0^2-\omega^2)(\omega_0^2-\omega^2+f_0)+\gamma^2\omega^2}$$

for $f_0 = 10\,fs^{-2}$, $\omega_0 = 5\,fs^{-1}$, $\gamma = 10^{-3}\,fs^{-1}$

$$\eta(\omega) = \frac{10^{-2}\omega}{(25-\omega^2)(35-\omega^2)+10^{-6}\omega^2}$$



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there are 2 asymptotic points, one is near resonance frequency w_0, another one is near $\sqrt{w_0^2+f_0}$

There are 3 regions, where $|\eta| \ll 1$, as the shadow region in sketch.

a) $w \ll w_0 = 5\ fs^{-1}$

b) $w \gg \sqrt{w_0^2+f_0} \approx 5.9\ fs^{-1}$

c) region c between w_0 and $\sqrt{w_0^2+f_0}$, where $|\eta|_{\substack{min \\ region\ c}} \approx 2\times10^{-3} \ll 1$ ✓

d) $|\eta|_{\substack{min \\ region\ c}} \approx 2\times10^{-3}$, $\varepsilon' < 0$

$$\eta' = \frac{f_0 \gamma [(w_0^2-w^2)^2 + (\gamma w)^2 + f_0(w_0^2-w^2)] - f_0 \gamma w [-2w(w_0^2-w^2+f_0) - 2w(w_0^2-w^2) + 2w\gamma^2]}{[(w_0^2-w^2)^2+(\gamma w)^2+f_0(w_0^2-w^2)]^2}$$

$= 0$

$\Rightarrow (w_0^2+w^2)(w_0^2-w^2+f_0) + 2w^2(w_0^2-w^2) - \gamma^2 w^2 = 0$

$\Rightarrow 3w^4 - w^2(f_0+2w_0^2-\gamma^2) - w_0^2(w_0^2+f_0) = 0$

$\Rightarrow w^2 = \dfrac{f_0+2w_0^2-\gamma^2+\sqrt{(f_0+2w_0^2-\gamma^2)^2+12w_0^2(w_0^2+f_0)}}{6} \approx 29.790$

$\Rightarrow w_1 \approx 5.458\ fs^{-1}$ ✓

$-|\eta|_{\substack{min \\ region\ c}} = \eta(w_1) = -0.0022$

for $\varepsilon''(w_1) \ll |\varepsilon'(w_1)|$, we can have

~~$\sqrt{\varepsilon} = \sqrt{\varepsilon'+i\varepsilon''} = \sqrt{|\varepsilon'|}\sqrt{1+\frac{\varepsilon''}{|\varepsilon'|}} \approx \sqrt{|\varepsilon'|}(1+\frac{i}{2}\frac{\varepsilon''}{|\varepsilon'|}) = \sqrt{|\varepsilon'|} + i\frac{1}{2}\frac{\varepsilon''}{\sqrt{|\varepsilon'|}} = \eta(w_1) + i\kappa(w_1))$~~

$\Rightarrow \sqrt{\varepsilon} = \sqrt{\varepsilon'+i\varepsilon''} = i\sqrt{|\varepsilon'|-i\varepsilon''} = i\sqrt{|\varepsilon'|}\sqrt{1-i\frac{\varepsilon''}{|\varepsilon'|}} = i\sqrt{|\varepsilon'|}\cdot(1-\frac{i}{2}\frac{\varepsilon''}{|\varepsilon'|}) = \frac{1}{2}\frac{\varepsilon''}{\sqrt{|\varepsilon'|}} + i\sqrt{|\varepsilon'|}$ ✓

$= n(w_1) + i\kappa(w_1)$

So $n(\omega_1) = \frac{1}{2}\frac{\varepsilon''(\omega)}{\sqrt{|\varepsilon'(\omega)|}}$ $\quad K(\omega_1) = \sqrt{|\varepsilon'(\omega)|}$

$\Rightarrow K(\omega_1) \approx 1.0428$

$n(\omega_1) \approx \frac{1}{2} \cdot |\eta(\omega_1)| \cdot K(\omega_1) = 0.0011$

refractive index is $\tilde{n}(\omega_1) = 0.0011 + i \cdot 1.0428$ ✓

$\vec{E} = \vec{E}_0(\omega) e^{ik' \cdot z} \cdot e^{-k'' z}$

$\vec{E}(z=0, \omega) = \vec{E}_0(\omega)$

$\vec{E}(z_0, \omega) = \vec{E}_0(\omega) \cdot e^{-1}$

$\Rightarrow k'' z_0 = 1$, $z_0 = \frac{1}{k''(\omega_1)}$

since $k'' = \frac{\omega_1}{c} \cdot K = \frac{2\pi}{\lambda} \cdot K$

so $z_0(\omega_1) = \frac{\lambda}{2\pi K(\omega_1)}$

In units of the vacuum wavelength λ,

$z_0(\omega_1) = \frac{1}{2\pi K(\omega_1)} \lambda \approx 0.152 \lambda$ ✓

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e) $\omega_2 = 6\ fs^{-1}$

in this case, $\varepsilon'(\omega) > 0$, $\eta > 0$ $\quad n(\omega_2) = \sqrt{\varepsilon'(\omega_2)} = 0.301$

and $K(\omega_2) = \frac{1}{2} \cdot \frac{\varepsilon''(\omega_2)}{\sqrt{\varepsilon'(\omega_2)}} \approx 8.223 \times 10^{-4}$ ✓ what about $n(\omega_2)$?

$z_0(\omega_2) = \frac{1}{2\pi K(\omega_2)} \cdot \lambda = 1.9354 \times 10^2 \lambda$ ✓

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Therefore, $z_0(\omega_2) >> z_0(\omega_1)$

roblem 2 Traveling and standing wave

a) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

in Fourier domain, $\nabla \times \vec{E}(\vec{r},\omega) = i\omega\mu_0 \vec{H}(\vec{r},\omega)$

$$\vec{H}(\vec{r}) = \frac{1}{i\omega\mu_0} \nabla \times \vec{E}(\vec{r}) = \frac{-E_0}{2\omega\mu_0} \nabla \times [(e^{i\beta y} - e^{i\beta y}) \cdot e^{i\beta z} \vec{e_x}]$$

$$= \frac{E_0 \cdot \beta}{\omega\mu_0} \cdot [\sin(\beta y) \cdot e^{i\beta z} \vec{e_y} + i\cos(\beta z) e^{i\beta z} \cdot \vec{e_z}]$$

$$\Rightarrow \vec{H}(\vec{r}) = \frac{E_0 \cdot \beta}{\omega\mu_0} \begin{pmatrix} 0 \\ \sin(\beta y) e^{i\beta z} \\ i\cos(\beta z) e^{i\beta z} \end{pmatrix}$$

✓

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b) $\langle \vec{S}(\vec{r},t) \rangle = \frac{1}{2} Re [\vec{E}(\vec{r}) \times \vec{H}^*(\vec{r})]$

$$= \frac{1}{2} Re \left[E_0 \cdot \begin{pmatrix} \sin(\beta y) e^{i\beta z} \\ 0 \\ 0 \end{pmatrix} \times \frac{E_0 \cdot \beta}{\omega\mu_0} \begin{pmatrix} 0 \\ \sin(\beta y) e^{-i\beta z} \\ -i\cos(\beta z) \cdot e^{-i\beta z} \end{pmatrix} \right]$$

$$= \frac{E_0^2 \beta}{2\omega\mu_0} Re \begin{pmatrix} 0 \\ i\cos(\beta y) \cdot \sin(\beta y) \\ \sin^2(\beta y) \end{pmatrix}$$

$$= \frac{E_0^2 \beta}{2\omega\mu_0} \begin{pmatrix} 0 \\ 0 \\ \sin^2(\beta y) \end{pmatrix}$$

✓

$\Rightarrow \langle \vec{S}(\vec{r},t) \rangle \parallel \vec{e_z}$

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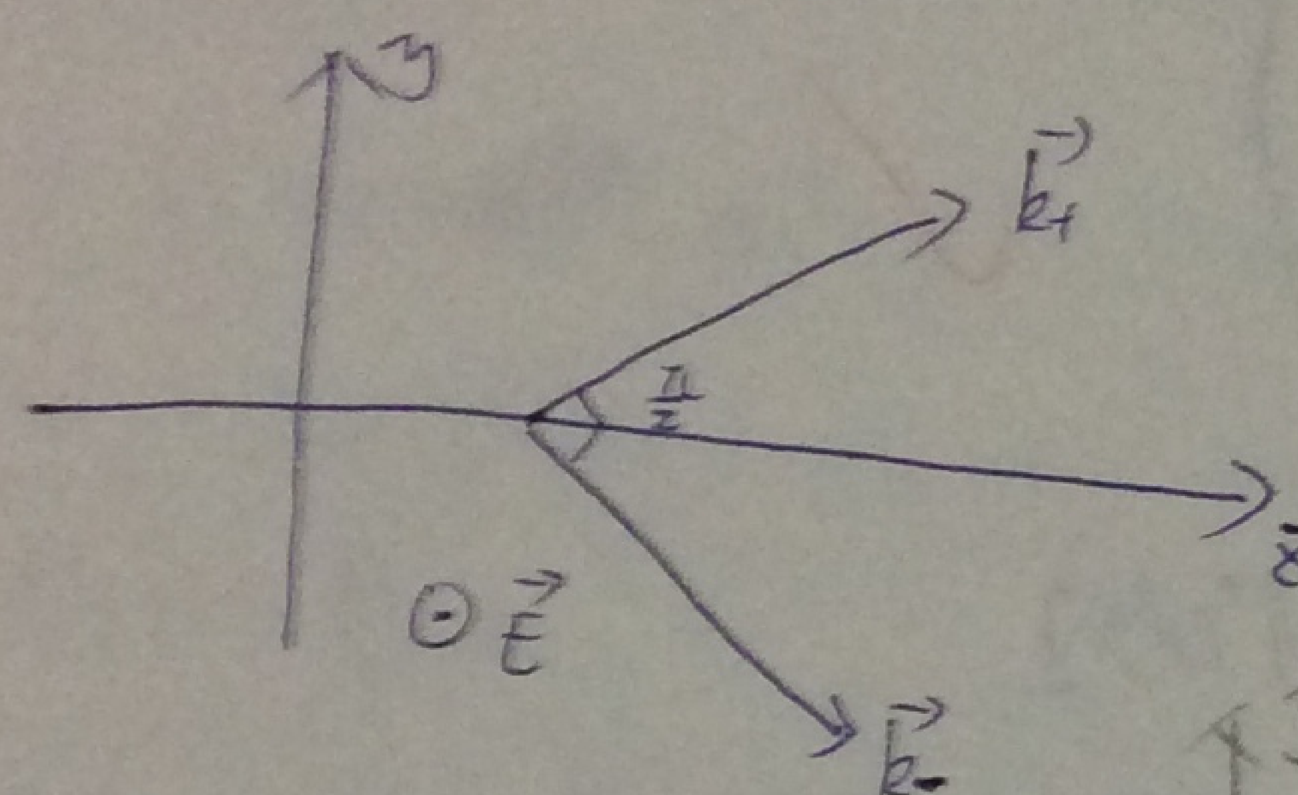
c) $\vec{E}(\vec{r}) = E_0 \cdot \sin(\beta y)\, e^{i\beta z}\, \vec{e_x}$

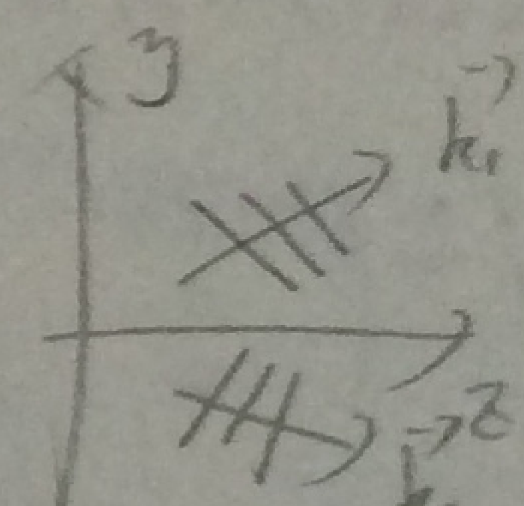
$= \frac{E_0}{2i}\left(e^{i(\beta y+\beta z)} - e^{i(\beta z-\beta y)}\right)\cdot \vec{e_x}$

$(e^{i\frac{\pi}{2}} = \pm i)$

$= \underbrace{\frac{E_0}{2} e^{i[\beta(z+y)-\frac{\pi}{2}]}\vec{e_x}}_{\vec{E_+}} + \underbrace{\frac{E_0}{2}\cdot e^{i[\beta(z-y)+\frac{\pi}{2}]}\vec{e_x}}_{\vec{E_-}}$ ~~$(e^{i\frac{\pi}{2}} = +i,\ e^{-i\frac{\pi}{2}} = -i)$~~

$\vec{E_+}$ and $\vec{E_-}$ are two plane waves, propagating along $\vec{k_+} = (0, \beta, \beta)$ and $\vec{k_-} = (0, -\beta, \beta)$, respectively.





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d) Helmholtz equation

$\Delta(\vec{E_+}+\vec{E_-}) + \frac{\omega_0^2}{c^2}(\vec{E_+}+\vec{E_-}) = 0$

$\Rightarrow -\vec{k_+}^2\vec{E_+} - \vec{k_-}^2 \vec{E_-} + \frac{\omega_0^2}{c^2}(\vec{E_+}+\vec{E_-}) = 0$

Since $\vec{k_+}^2 = \vec{k_-}^2 = 2\beta^2$, so $-\vec{k_+}^2\vec{E_+} - \vec{k_1}^2\vec{E_-} + \frac{\omega_0^2}{c^2}(\vec{E_+}+\vec{E_-}) = (-2\beta^2 + \frac{\omega_0^2}{c^2})(\vec{E_+}+\vec{E_-}) = 0$

$\Rightarrow -2\beta^2 + \frac{\omega_0^2}{c^2} \Rightarrow \beta = \frac{1}{\sqrt{2}}\frac{\omega_0}{c} = \frac{\sqrt{2}\pi}{\lambda_0}$ ✓

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Problem 3 - Evanescent waves

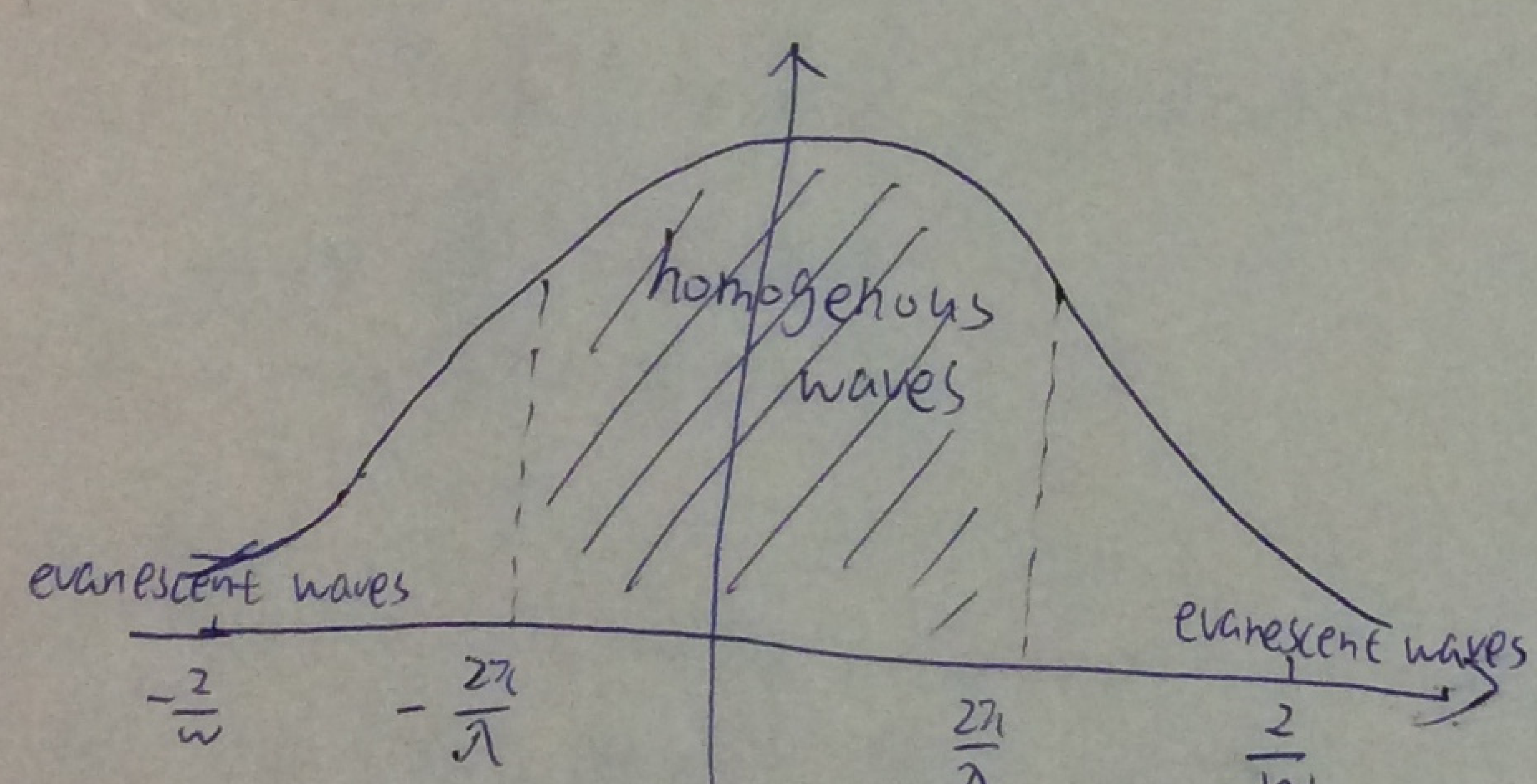
a) $U_0(k_x,k_y) = \frac{A}{(2\pi)^2}\int e^{-\frac{x^2+y^2}{w^2}} e^{-i(k_x x+k_y y)}\,dxdy$

$$= \frac{A}{(2\pi)^2}\underbrace{\int_{-\infty}^{\infty} e^{-\frac{x^2}{w^2}-ik_x x}\,dx}_{I_1(k_x)}\ \underbrace{\int_{-\infty}^{\infty} e^{-\frac{y^2}{w^2}-ik_y y}\,dy}_{I_2(k_y)}$$

$$I_1(k_x) = \int_{-\infty}^{\infty} e^{-(\frac{x}{w}+\frac{ik_x w}{2})^2-(\frac{k_x w}{2})^2}\,dx = e^{-(\frac{k_x w}{2})^2}\cdot w\sqrt{\pi}$$

$$I_2(k_y) = \int_{-\infty}^{\infty} e^{-(\frac{x}{w}+i\frac{k_y w}{2})^2-(\frac{k_y w}{2})^2}\,dy = e^{-(\frac{k_y w}{2})^2}\cdot w\sqrt{\pi}$$

$$\Rightarrow U_0(k_x,k_y) = \frac{A}{4\pi}w^2\cdot e^{-\frac{k_x^2+k_y^2}{4/w^2}}$$



$k_x^2+k_y^2 > (\frac{2\pi}{\lambda})^2$ - evanescent waves

$k_x^2+k_y^2 < (\frac{2\pi}{\lambda})^2$ - homogenous waves

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b) ~~only homogenous waves can propagated transmit in vaccum~~

b) only homogenous waves can propagate in vaccum

$\iint_{-k}^{k} U_0^2(k_x,k_y)\,dk_x dk_y$ ~ transmitted beam energy

So, $\iint_{-k}^{k} U_0^2(k_x,k_y)\,dk_x dk_y = 0.9\times\iint_{-\infty}^{\infty} U_0^2(k_x,k_y)\,dk_x dk_y = 0.9\times(\frac{A}{4\pi}w^2)^2\cdot(\frac{\sqrt{2}}{w})^2\times\pi$

$$\iint_{-k}^{k} U_0^2(k_x,k_y)\,dk_x\,dk_y = \left(\frac{A}{4\pi}w^2\right)^2 \iint_{-k}^{k} e^{-\frac{k_x^2+k_y^2}{2/w^2}}\,dk_x\,dk_y$$

$$\overset{\rho^2=k_x^2+k_y^2}{=} \left(\frac{A}{4\pi}w^2\right)^2 \cdot \int_0^{k} \int_0^{2\pi} e^{-\frac{\rho^2}{2/w^2}}\,\rho \cdot d\rho \cdot d\varphi = \left(\frac{A}{4\pi}w^2\right)^2 \cdot \pi \cdot \int_0^{k} e^{-\frac{\rho^2}{2/w^2}}\,d\rho^2$$

$$= \left(\frac{A}{4\pi}w^2\right)^2 \cdot \pi \cdot \frac{2}{w^2} \cdot \left(1-e^{-\frac{k^2}{2/w^2}}\right)$$

$$= 0.9 \cdot \left(\frac{A}{4\pi}w^2\right)^2 \cdot \pi \times \frac{2}{w^2}$$

$$\Rightarrow 1-e^{-\frac{k^2}{2/w^2}} = 0.9$$

$$-k^2 \cdot \frac{w^2}{2} = \ln(0.1) \approx -2.3$$

$$w_{min} = \sqrt{\frac{4.6}{k^2}} = \sqrt{\frac{4.6\times\lambda^2}{(2\pi)^2}} \approx \frac{2.146\lambda}{2\pi} \approx 0.342\lambda$$ ✓

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