Task 1: Response function of Fresnel approximation (5 pts.)

Consider the Fresnel approximation, under which the transfer function in the spatial frequency domain reads:

$$H_F(\alpha,\beta,z) = \exp(ikz)\exp\left[-i\frac{\alpha^2+\beta^2}{2k}z\right]$$

Derive the response function $h_F(x,y,z>z_0)$ in the spatial domain, as given in the lecture notes. Use the integral:

$$\int_{-\infty}^{+\infty} e^{-ix^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{i}}.$$

$$h_F = FT[H_F(\alpha,\beta,z)] = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty}\exp(ikz)\exp[-i\frac{\alpha^2+\beta^2}{2k}z]\,e^{i(\alpha x+\beta y)}\,d\alpha d\beta$$

$$= \frac{1}{4\pi^2}e^{ikz}\iint_{-\infty}^{\infty}\exp[-i\frac{z}{2k}(\alpha^2+\beta^2-\frac{2k}{z}\alpha x-\frac{2k}{z}\beta y)]\,d\alpha d\beta$$

$$= \frac{e^{ikz}}{4\pi^2}\iint_{-\infty}^{\infty}\exp[-i\frac{z}{2k}(\alpha^2-\frac{2kx}{z}\alpha+\frac{k^2x^2}{z^2}-\frac{k^2x^2}{z^2}+\beta^2-\frac{2ky}{z}\beta+\frac{k^2y^2}{z^2}-\frac{k^2y^2}{z^2})]\,d\alpha d\beta$$

$$= \frac{e^{ikz}}{4\pi^2}\iint_{-\infty}^{\infty}\exp\{-i\frac{z}{2k}[(\alpha-\frac{kx}{z})^2-\frac{k^2x^2}{z^2}+(\beta-\frac{ky}{z})^2-\frac{k^2y^2}{z^2}]\}\,d\alpha d\beta$$

$$= \frac{e^{ikz}}{4\pi^2}\int_{-\infty}^{\infty}\exp\{-i\frac{z}{2k}[(\alpha-\frac{kx}{z})^2-\frac{k^2x^2}{z^2}]\}\,d\alpha\int_{-\infty}^{\infty}\exp\{-i\frac{z}{2k}[(\beta-\frac{ky}{z})^2-\frac{ky^2}{z^2}]\}\,d\beta$$

$$= \frac{e^{ikz}}{4\pi^2}\exp(\frac{ikx^2}{2z})\exp(\frac{iky^2}{2z})\int_{-\infty}^{\infty}\exp\{i[\sqrt{\frac{z}{2k}}(\alpha-\frac{kx}{z})]^2\}\,d\alpha\int_{-\infty}^{\infty}\exp\{-i[\sqrt{\frac{z}{2k}}(\beta-\frac{ky}{z})]^2\}\,d\beta$$

$\eta = \sqrt{\frac{z}{2k}}(\alpha-\frac{kx}{z})$ $\quad d\alpha = \sqrt{\frac{2k}{z}}\,d\eta$ $\quad \xi = \sqrt{\frac{z}{2k}}(\beta-\frac{ky}{z}) \Rightarrow d\beta = \sqrt{\frac{2k}{z}}\,d\xi$

$$\Rightarrow = \frac{e^{ikz}}{4\pi^2}\exp[\frac{ik}{2z}(x^2+y^2)]\frac{2k}{z}\int_{-\infty}^{\infty}e^{-i\eta^2}d\eta\int_{-\infty}^{\infty}e^{-i\xi^2}d\xi = -\frac{ike^{ikz}}{2\pi z}\exp[\frac{ik}{2z}(x^2+y^2)]$$

Task 2: Fresnel spot (a=2, b=1, c=2, d*=1 pts.)

The Fresnel spot, also called Poisson/Arago spot, is a bright spot occurring at the center of an opaque disk's shadow. This spot illustrates the wave nature of light and arises due to the Fresnel diffraction when the observation of the diffraction occurs at a relatively short distance such that the curvature of the wavefront cannot be ignored. To calculate the diffraction pattern of the Fresnel spot on the optical axis $(x,y=0)$, we consider first the diffraction of a circular aperture with a radius a, that is the complementary structure of an opaque disk with the same radius. When such circular aperture is illuminated with a plane wave, the field distribution directly after the aperture can be written as

$$u_0(x,y) = \begin{cases} 1 & \text{if } x^2+y^2 \le a \\ 0 & \text{otherwise} \end{cases}.$$

a) Apply the Fresnel approximation and show that after a propagation distance of z, the field distribution at the centre is given as

$$u_F(0,0,z) = -\mathrm{i}\exp[ikz]\left(1-\mathrm{i}\exp\left[ika^2/2z\right]\right).$$

Hint: The integral needs to be calculated for $x=y=0$, and you may use the substitution $t=x^2$ to solve the integral of the form $\int_0^a \exp\left[ibx^2\right]x\,\mathrm{d}x$.

Babinet's principle states that the diffraction pattern of an opaque object is identical to that of its complementary object, but opposite in phase. Therefore, the sum of these two diffraction pattern amplitudes (of the opague object and its complementary object) at any given point must be equal to the case where the incident field is unobstructed and simply propagated for a distance z.

b) By using the beam propagation method and the Fresnel approximation, show that the diffraction pattern of an unobstructed plane wave at a propagation distance of z is simply $u_{PW}(x,y,z) = \exp[ikz]$.

c) Find the diffraction pattern amplitude and intensity (on axis at $x,y=0$) of the opaque disk with a radius a by using your results from a) and b).

d*) Explain your results intuitively. What is the intensity at this point? Why does this surprising bright spot occur at the center?

(a) $$u_F(x,y,z) = \iint_{-\infty}^{\infty}u_0(x',y')\,h_F(x-x',y-y')\,dx'dy' = -\frac{ike^{ikz}}{2\pi z}\iint_{-a}^{a}\exp\{\frac{ik}{2z}[(x-x')^2+(y-y')^2]\}\,dx'dy'$$

$$u_F(0,0,z) = -\frac{ike^{ikz}}{2\pi z}\iint_{-a}^{a}\exp[\frac{ik}{2z}(x'^2+y'^2)]\,dx'dy'$$ $\quad x'=r\cos\theta \quad y'=r\sin\theta$

$$\Rightarrow u_F(0,0,z) = -\frac{ike^{ikz}}{2\pi z}\int_0^{2\pi}d\theta\int_0^a\exp(\frac{ik}{2z}r^2)\,r\,dr = -\frac{ike^{ikz}}{z}\int_0^a\exp(\frac{ik}{2z}r^2)\,r\,dr$$ $\quad r^2=t \quad r=\sqrt{t} \quad dr=\frac{1}{2\sqrt{t}}dt$

$$\Rightarrow u_F(0,0,z) = -\frac{ike^{ikz}}{2z}\int_0^{a^2}\exp(\frac{ik}{2z}t)\,dt = -\frac{ike^{ikz}}{2z}\left[\frac{2z}{ik}\exp(\frac{ik}{2z}t)\Big|_0^{a^2}\right] = -\frac{ike^{ikz}}{2z}\left(\frac{2z}{ik}e^{\frac{ika^2}{2z}}-\frac{2z}{ik}\right)$$

$$= -ie^{ikz}\left(\frac{1}{i}e^{\frac{ika^2}{2z}}-\frac{1}{i}\right) = -ie^{ikz}[i-i\exp(\frac{ika^2}{2z})] = -e^{ikz}[\exp(\frac{ika^2}{2z})-1] = e^{ikz}[1-\exp(\frac{ika^2}{2z})]$$

(b) $u(x,y,z) = e^{ikz}$ $\quad u(x,y,0) = U_0(x,y) = 1$

$$U_{pw}(x,y,z) = \iint_{-\infty}^{\infty} u(x',y')\, h_F(x-x', y-y')\, dx'dy' = \iint_{-\infty}^{\infty} u(x-x', y-y')\, h_F(x',y')\, dx'dy'$$

$$= -\frac{ike^{ikz}}{2\lambda z}\iint_{-\infty}^{\infty} \exp\left[\frac{ik}{2z}(x'^2+y'^2)\right] dx'dy' \qquad x' = r\cos\theta \quad y = r\sin\theta$$

$$= -\frac{ike^{ikz}}{2\lambda z}\int_{-\infty}^{\infty}\exp\left(\frac{ik}{2z}x'^2\right)dx' \int_{-\infty}^{\infty}\exp\left(\frac{ik}{2z}y'^2\right)dy' = \frac{-ike^{ikz}}{2\lambda z}\int_{-\infty}^{\infty}\exp\left[-i\left(-\frac{k}{2z}x'^2\right)\right]dx'\int_{-\infty}^{\infty}\exp\left[-i\left(-\frac{k}{2z}y'^2\right)\right]dy'$$

$$\sqrt{-\frac{k}{2z}}\,x' = x \quad \sqrt{\frac{-k}{2z}}\,y' = y \quad dx' = \sqrt{-\frac{2z}{k}}\,dx \quad dy' = \sqrt{\frac{-2z}{k}}\,dy$$

$$\Rightarrow = -\frac{ike^{ikz}}{2\lambda z}\cdot\left(\sqrt{\frac{-2z}{k}}\right)^2\int_{-\infty}^{\infty}e^{-ix^2}dx\int_{-\infty}^{\infty}e^{-iy^2}dy = -\frac{ike^{ikz}}{2\lambda z}\left(-\frac{2z}{k}\right)\cdot\frac{\pi}{i} = e^{ikz}$$

(c) $U_{pw} = U_F(0,0,z) + U_{od}(0,0,z) \Rightarrow U_{od}(0,0,z) = e^{ikz} - e^{ikz}\left[1-\exp\left(\frac{ika^2}{2z}\right)\right] = \exp\left(\frac{ika^2}{2z}\right)e^{ikz}$

$$I_{pw} = \sqrt{|U_{pw}|^2} = 1 \qquad I_F = \sqrt{|U_F|^2} = \sqrt{\left\{e^{ikz} - \exp\left[ik\left(z+\frac{a^2}{2z}\right)\right]\right\}\left\{e^{-ikz}-\exp\left[-ik\left(z+\frac{a^2}{2z}\right)\right]\right\}}$$

$$= \sqrt{1+1-\exp\left(-ik\frac{a^2}{2z}\right) - \exp\left(ik\frac{a^2}{2z}\right)} = \sqrt{2-2\cos\frac{ka^2}{2z}} = \sqrt{2}\sqrt{1-\cos\frac{ka^2}{2z}} \approx 0$$

$$I_{od} = I_{pw} - I_F = 1$$

(d) The intensity at the spot is equal to 1.

The bright spot occurs because according to Huygens–Fresnel priciple, every spot at the wavefront can be seen as a point light source, when these light sources illuminate on a screen, superposition of light happens and the distances between the spot at the center and spots at the fringe are always identical, therefore lights superimpose at the spot and the bright spot appears.

Task 3: Talbot Effect, with and without the Fresnel approximation (a=5, b=1, c=5, d=5* pts.)

Assume an initial field $f(x,z=0)$ (with full translational symmetry in the y-direction), which is periodic along the x-direction with a period of a, such that $f(x+a,z=0) = f(x,z=0)$. We want to calculate the field $f(x,z)$ after propagation along the z-direction, in vacuum, where the vacuum wavelength of the field is λ. If we treat this diffraction problem in the Fresnel (paraxial) approximation, we will find that after a certain length L_T the initial field reappears except for an extra phase, such that

$$f(x,z=L_T) = f(x,z=0)\exp(ikL_T + i2\pi m_l) \quad \text{with } m_l \in \mathbb{Z}$$

This is known as the Talbot effect and L_T is known as the Talbot length.

a) Find the expression for L_T under the assumptions specified above. *Hint*: You do not need to know the specific expression for the function $f(x)$. Express $f(x)$ as a Fourier series, and then follow through with the standard approach for calculating beam diffraction. Keep in mind that we assume the paraxial approximation to be valid.

b) Calculate L_T for a wavelength of $\lambda = 1200\,\text{nm}$ and a period of $a = 3\,\text{mm}$.

The Talbot effect always holds true in the Fresnel approximation. In contrast, if the Fresnel approximation is not valid, for example when the period a is comparable to the wavelength λ, the Talbot effect does not necessarily take place. However, it can still occur for certain field patterns.

c) Show that for an initial field distribution of the form:

$$f(x,z=0) = A\cos(x\,2\pi/a_1)\cos(x\,2\pi/a_2)$$

the Talbot effect still takes place outside the paraxial regime and calculate the Talbot length. Find the value of L_T for the wavelength of $\lambda = 1200\,\text{nm}$ and periods $a_1 = 3\,\mu\text{m}$, $a_2 = 4\,\mu\text{m}$.

d*) Consider now an initial field, which is formed as the superposition of three periodic components

$$f(x,z=0) = A_1\cos(x\,2\pi/a_1) + A_2\cos(x\,2\pi/a_2) + A_3\cos(x\,2\pi/a_3).$$

Show that the Talbot effect in this case will only take place if a certain relation between λ, a_1, a_2, a_3 is satisfied and find this relation.

$$S(x) = A_0 + \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{2\pi nx}{P}\right) + B_n\sin\left(\frac{2\pi nx}{P}\right)\right]$$

$$= \sum_{n=-\infty}^{\infty} C_n \exp\left(\frac{i2\pi nx}{P}\right)$$

(a) $f(x) = \sum_{n=-\infty}^{\infty} a_n \exp\left[\frac{i2\pi nx}{a}\right]$

$$f(x,z) = h_F(x,y,z)\otimes f(x) = \iint_{-\infty}^{\infty} -\frac{ike^{ikz}}{2\lambda z}\exp\left[\frac{ik}{2z}(x'^2+y'^2)\right]\cdot\sum_{n=-\infty}^{\infty}a_n\exp\left[\frac{i2\pi n(x-x')}{a}\right]dx'dy'$$

$$= -\frac{ike^{ikz}}{2\lambda z}\sum_{n=-\infty}^{\infty}a_n\exp\left(\frac{i2\pi nx}{a}\right)\iint_{-\infty}^{\infty}\exp\left[\frac{ik}{2z}x'^2 + \frac{ik}{2z}y'^2 - \frac{i2\pi nx'}{a}\right)dx'dy'$$

$$\Rightarrow \iint_{-\infty}^{\infty} \exp\left[\frac{ik}{2z}\left(x'^2 - \frac{4\pi z n}{ka}x'\right) + \frac{ik}{2z}y'^2\right] dx' dy'$$

$$= \int_{-\infty}^{\infty} \exp\left[-i\left(-\frac{k}{2z}y'^2\right)\right]dy' \int_{-\infty}^{\infty} \exp\left[\frac{ik}{2z}\left(x'^2 - \frac{4\pi z n}{ka}x' + \frac{4\pi^2 z^2 n^2}{k^2 a^2} - \frac{4\pi^2 z^2 n^2}{k^2 a^2}\right)\right] dx'$$

$$= \sqrt{-\frac{2z}{k}}\cdot\sqrt{\frac{\pi}{i}} \int_{-\infty}^{\infty} \exp\left\{\frac{ik}{2z}\left[\left(x' - \frac{2\pi z n}{ka}\right)^2 - \frac{4\pi^2 z^2 n^2}{k^2 a^2}\right]\right\} dx'$$

$$= \sqrt{\frac{2iz\pi}{k}} \exp\left(-\frac{i2\pi^2 z n^2}{ka^2}\right) \int_{-\infty}^{\infty} \exp\left[\frac{ik}{2z}\left(x' - \frac{2\pi z n}{ka}\right)^2\right] dx'$$

$$= \sqrt{\frac{2iz\pi}{k}} \exp\left(\frac{-i2\pi^2 z n^2}{ka^2}\right) \int_{-\infty}^{\infty} \exp\left\{-i\left[\frac{-k}{2z}\left(x' - \frac{2\pi z n}{ka}\right)^2\right]\right\} dx' \quad \sqrt{\frac{-k}{2z}}\left(x' - \frac{2\pi z n}{ka}\right) = t$$

$$= \sqrt{\frac{4z^2 i\pi}{-k^2}} \exp\left(\frac{-i2\pi^2 z n^2}{ka^2}\right) \int_{-\infty}^{\infty} \exp(-it^2)\, dt = \sqrt{\frac{4z^2\pi^2}{-k^2}} \exp\left(\frac{-i2\pi^2 z n^2}{ka^2}\right) = -\frac{2z\pi}{ik} \exp\left(\frac{-i2\pi^2 z n^2}{ka^2}\right)$$

$$\Rightarrow f(x,z) = -\frac{ike^{ikz}}{2\pi z} \sum_{n=-\infty}^{\infty} a_n \exp\left(\frac{i2\pi n x}{a}\right)\cdot\left(-\frac{2z\pi}{ik}\right)\exp\left(\frac{-i2\pi^2 z n^2}{ka^2}\right)$$

$$= \sum_{n=-\infty}^{\infty} a_n \exp\left(\frac{i2\pi n x}{a}\right) e^{ikz} \exp\left(\frac{-i2\pi^2 z n^2}{ka^2}\right) = f(x, z=0) \exp\left(ikz - i\frac{2z n^2\pi^2}{ka^2}\right)$$

$$\Rightarrow f(x, z=L_T) = f(x, z=0) \exp\left[ikL_T + i2\pi\left(-\frac{L_T n^2\pi}{ka^2}\right)\right] \Rightarrow m_e = -\frac{L_T n^2\pi}{ka^2} = -\frac{L_T n^2\lambda}{2a^2} \in \mathbb{Z}$$

$$\Rightarrow L_T = \frac{2a^2}{\lambda}N \quad N = \text{integer}$$

(b) $L_T = \dfrac{2\times 9\,mm^2}{1200\,nm}N = \dfrac{18\,mm^2}{1200\times 10^{-6}\,mm}N = 15N\,(m)$

(c) $H_F = \exp(i\sqrt{k^2-\alpha^2-\beta^2}\,z)$ $\quad F(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx \quad f(x) = \int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x}d\alpha$

$f(x, z=0) = A\cos\left(\frac{2\pi}{a_1}x\right)\cos\left(\frac{2\pi}{a_2}x\right)$ $\quad F(\alpha, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A\cos\left(\frac{2\pi}{a_1}x\right)\cos\left(\frac{2\pi}{a_2}x\right)e^{i\alpha x}dx$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \Rightarrow \cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$

$$\Rightarrow F(\alpha, 0) = \frac{A}{4\pi}\int_{-\infty}^{\infty}\cos\left(\frac{2\pi}{a_1}+\frac{2\pi}{a_2}\right)x\, e^{i\alpha x}dx + \frac{A}{4\pi}\int_{-\infty}^{\infty}\cos\left(\frac{2\pi}{a_1}-\frac{2\pi}{a_2}\right)x\, e^{i\alpha x}dx$$

$$\int_{-\infty}^{\infty}\cos ax\, e^{i\alpha x}dx = \frac{1}{2}\int_{-\infty}^{\infty}\left(e^{iax}e^{i\alpha x} + e^{-iax}e^{i\alpha x}\right)dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{i(\alpha+a)x}dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{i(\alpha-a)x}dx$$

$$f(x) = \int_{-\infty}^{\infty}\delta(\alpha+a)e^{-i\alpha x}d\alpha = e^{iax} \Rightarrow F(\alpha) = \delta(\alpha+a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iax}e^{i\alpha x}dx \Rightarrow \underline{\int_{-\infty}^{\infty} e^{i(a+\alpha)x}dx = 2\pi\delta(\alpha+a)}$$

$$\Rightarrow \underline{\int_{-\infty}^{\infty}\cos ax\, e^{i\alpha x}dx = \frac{1}{2}[2\pi\delta(\alpha+a) + 2\pi\delta(\alpha-a)] = \pi[\delta(\alpha+a)+\delta(\alpha-a)]}$$

$$\Rightarrow \int_{-\infty}^{\infty}\cos\left(\frac{2\pi}{a_1}+\frac{2\pi}{a_2}\right)x\, e^{i\alpha x}dx = \pi\left[\delta\left(\alpha+\frac{2\pi}{a_1}+\frac{2\pi}{a_2}\right) + \delta\left[\alpha-\left(\frac{2\pi}{a_1}+\frac{2\pi}{a_2}\right)\right]\right]$$

$$\int_{-\infty}^{\infty}\cos\left(\frac{2\pi}{a_1}-\frac{2\pi}{a_2}\right)x\, e^{i\alpha x}dx = \pi\left[\delta\left(\alpha+\frac{2\pi}{a_1}-\frac{2\pi}{a_2}\right) + \delta\left(\alpha-\left(\frac{2\pi}{a_1}-\frac{2\pi}{a_2}\right)\right)\right]$$

$$\Rightarrow F(\alpha, 0) = \frac{A}{4}\left\{\delta\left(\alpha+\frac{2\pi}{a_1}+\frac{2\pi}{a_2}\right) + \delta\left[\alpha-\left(\frac{2\pi}{a_1}+\frac{2\pi}{a_2}\right)\right] + \delta\left(\alpha+\frac{2\pi}{a_1}-\frac{2\pi}{a_2}\right) + \delta\left[\alpha-\left(\frac{2\pi}{a_1}-\frac{2\pi}{a_2}\right)\right]\right\}$$

$$F(\alpha, z) = F(\alpha, 0)\cdot H_F = F(\alpha, 0)\exp\left(i\sqrt{k^2-\alpha^2}\,z\right)$$

$$f(x, z) = \int_{-\infty}^{\infty} F(\alpha, 0)\exp\left(i\sqrt{k^2-\alpha^2}\,z\right)e^{-i\alpha x}dx = \int_{-\infty}^{\infty} F(\alpha, 0)\exp\left[i\left(\sqrt{k^2-\alpha^2}\,z - \alpha x\right)\right]dx$$

$C_1=\frac{2\pi}{a_1}+\frac{2\pi}{a_2}$ $\quad C_2=\frac{2\pi}{a_1}-\frac{2\pi}{a_2}$

$\Rightarrow f(x,z)=\frac{A}{4}\exp[i\sqrt{k^2-C_1^2}z+C_1x]+\frac{A}{4}\exp[i\sqrt{k^2-C_1^2}z-C_1x]]+\frac{A}{4}\exp[i\sqrt{k^2-C_2^2}z+C_2x)+\frac{A}{4}\exp[i\sqrt{k^2-C_2^2}z-C_2x]$

$f(x,0)=A\cos\frac{2\pi}{a_1}x\cos\frac{2\pi}{a_2}x=\frac{A}{2}[\cos(\frac{2\pi}{a_1}+\frac{2\pi}{a_2})x+\cos(\frac{2\pi}{a_1}-\frac{2\pi}{a_2})x]=\frac{A}{2}(\cos C_1x+\cos C_2x)$

$=\frac{A}{4}(e^{iC_1x}+e^{-iC_1x}+e^{iC_2x}+e^{-iC_2x})$

$f(x,z)=\frac{A}{4}[e^{i\sqrt{k^2-C_1^2}z}(e^{iC_1x}+e^{-iC_1x})]+\frac{A}{4}[e^{i\sqrt{k^2-C_2^2}z}(e^{iC_2x}+e^{-iC_2x})]$

$=\frac{A}{2}\cos C_1x\,e^{i\sqrt{k^2-C_1^2}z}+\frac{A}{2}\cos C_2x\,e^{i\sqrt{k^2-C_2^2}z}=\frac{A}{2}e^{i\sqrt{k^2-C_1^2}z}[\cos C_1x+\cos C_2x\exp(i\sqrt{k^2-C_2^2}z-i\sqrt{k^2-C_1^2}z)]$

$\Rightarrow \exp[i(\sqrt{k^2-C_2^2}z-\sqrt{k^2-C_1^2}z)]=1\Rightarrow(\sqrt{k^2-C_2^2}-\sqrt{k^2-C_1^2})z=2\pi N$ N is integer

$k=\frac{2\pi}{\lambda}$ $\quad z=L_T\Rightarrow L_T=\frac{2\pi N}{\sqrt{\frac{4\pi^2}{\lambda^2}-C_2^2}-\sqrt{\frac{4\pi^2}{\lambda}-C_1^2}}$ $\quad \lambda=1200nm$ $=1.2\mu m$ $\quad a_1=3\mu m$ $\quad a_2=4\mu m$

$\Rightarrow C_1=2\pi(\frac{1}{a_1}+\frac{1}{a_2})$ $\quad C_2=2\pi(\frac{1}{a_1}-\frac{1}{a_2})\Rightarrow L_T=\frac{N}{\sqrt{\frac{1}{\lambda^2}-(\frac{1}{a_1}-\frac{1}{a_2})^2}-\sqrt{\frac{1}{\lambda^2}-(\frac{1}{a_1}+\frac{1}{a_2})^2}}$ ~ 37.19

(d) $f(x,z=0)=A_1\cos\frac{2\pi}{a_1}x+A_2\cos\frac{2\pi}{a_2}x+A_3\cos\frac{2\pi}{a_2}x$ $\quad H_F=-\frac{ike^{ikz}}{2\pi z}\exp[\frac{ik}{2z}(x^2+y^2)]$

$f_1(x,z=0)=A_1\cos\frac{2\pi}{a_1}x$ $\quad f_1(\alpha,z=0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}A_1\cos\frac{2\pi}{a}x\,e^{i\alpha x}dx=\frac{A_1}{2}[\delta(\alpha+\frac{2\pi}{a_1})+\delta(\alpha-\frac{2\pi}{a_1})]$

$f_1(\alpha,z)=f_1(\alpha,z=0)\cdot H_F=\frac{A_1}{2}[\delta(\alpha+\frac{2\pi}{a_1})+\delta(\alpha-\frac{2\pi}{a_1})]\cdot e^{ikz}\exp[\frac{-iz}{2k}\alpha^2]$

$f_1(x,z)=\frac{A_1}{2}e^{ikz}\int_{-\infty}^{\infty}[\delta(\alpha+\frac{2\pi}{a_1})+\delta(\alpha-\frac{2\pi}{a_1})]\exp(\frac{-iz}{2k}\alpha^2)e^{-i\alpha x}d\alpha$

$=\frac{A_1}{2}e^{ikz}\exp(\frac{-iz}{2k}\frac{4\pi^2}{a_1^2})(e^{i\frac{2\pi}{a_1}x}+e^{-i\frac{2\pi}{a_1}x})=A_1e^{ikz}\exp(-\frac{iz}{2k}\frac{4\pi^2}{a_1^2})\cos(\frac{2\pi}{a_1}x)$

$f_2(x,z)=A_2e^{ikz}\exp(-\frac{iz}{2k}\frac{4\pi^2}{a_2^2})\cos(\frac{2\pi}{a_2}x)$ $\quad f_3(x,z)=A_3e^{ikz}\exp(-\frac{iz}{2k}\frac{4\pi^2}{a_3^2})\cos\frac{2\pi}{a_3}x)$

$\Rightarrow f(x,z)=A_1e^{ikz}\exp(-\frac{iz}{2k}\cdot\frac{4\pi^2}{a_1^2})\cos\frac{2\pi}{a_1}x+A_2e^{ikz}\exp(-\frac{iz}{2k}\frac{4\pi^2}{a_2^2})\cos\frac{2\pi}{a_2}x+A_3e^{ikz}\exp(-\frac{iz}{2k}\frac{4\pi^2}{a_3^2})\cos\frac{2\pi}{a_3}x$

$=e^{ikz}\exp(\frac{-iz}{2k}\cdot\frac{4\pi^2}{a_1^2})\{A_1\cos\frac{2\pi}{a_1}x+A_2\cos\frac{2\pi}{a_2}x\exp[\frac{iz}{2k}4\pi^2(\frac{1}{a_1^2}-\frac{1}{a_2^2})]+A_3\cos\frac{2\pi}{a_3}x\exp[\frac{iz}{2k}4\pi^2(\frac{1}{a_1^2}-\frac{1}{a_3^2})]\}$

$z=L_T\Rightarrow$ $\quad \frac{L_T4\pi^2}{2k}(\frac{1}{a_1^2}+\frac{1}{a_2^2})=2\pi N$ $\quad \frac{L_T4\pi^2}{2k}(\frac{1}{a_1^2}-\frac{1}{a_3^2})=2\pi N$

$\Rightarrow L_T=\frac{2N_1}{\lambda(\frac{1}{a_1^2}+\frac{1}{a_2^2})}$ $\quad L_T=\frac{2N_2}{\lambda(\frac{1}{a_1^2}-\frac{1}{a_3^2})}$

Non-paraxial

$\Rightarrow f(x,z)=A_1\cos(\frac{2\pi}{a_1}x)\exp(i\sqrt{k^2-\frac{4\pi^2}{a_1^2}}z)+A_2\cos(\frac{2\pi}{a_2}x)\exp(i\sqrt{k^2-\frac{4\pi^2}{a_2^2}}z)+A_3\cos(\frac{2\pi}{a_3}x)\exp(i\sqrt{k^2-\frac{4\pi^2}{a_3^2}}z)$

$=\exp(i\sqrt{k^2-\frac{4\pi^2}{a_1^2}}z)[A_1\cos(\frac{2\pi}{a_1}x)+A_2\cos(\frac{2\pi}{a_2}x)\exp[i(\sqrt{k^2-\frac{4\pi^2}{a_2^2}}-\sqrt{k^2-\frac{4\pi^2}{a_1^2}})z]+A_3\cos(\frac{2\pi}{a_3}x)\exp[i(\sqrt{k^2-\frac{4\pi^2}{a_3^2}}-\sqrt{k^2-\frac{4\pi^2}{a_1^2}})z]]$

$\Rightarrow(\sqrt{k^2-\frac{4\pi^2}{a_2^2}}-\sqrt{k^2-\frac{4\pi^2}{a_1^2}})L_T=2\pi N$ and $(\sqrt{k^2-\frac{4\pi^2}{a_3^2}}-\sqrt{k^2-\frac{4\pi^2}{a_1^2}})L_T=2\pi N$

$$\Rightarrow \left(\sqrt{\frac{1}{\lambda^2}-\frac{1}{a_2^2}}-\sqrt{\frac{1}{\lambda^2}-\frac{1}{a_1^2}}\right)=\frac{N_1}{LT} \quad \text{and} \quad \sqrt{\frac{1}{\lambda^2}-\frac{1}{a_3^2}}-\sqrt{\frac{1}{\lambda^2}-\frac{1}{a_1^2}}=\frac{N_2}{LT}$$

$$\Rightarrow \frac{\sqrt{\frac{1}{\lambda^2}-\frac{1}{a_2^2}}-\sqrt{\frac{1}{\lambda^2}-\frac{1}{a_1^2}}}{\sqrt{\frac{1}{\lambda^2}-\frac{1}{a_3^2}}-\sqrt{\frac{1}{\lambda^2}-\frac{1}{a_1^2}}}=\frac{N_1}{N_2}$$