Task 1 Jinsong Liu

Solution:

a)

$$I = I_0 \exp\left[-\frac{2(x^2+y^2)}{w_0^2}\right]\exp(-2t^2/\tau_0^2)$$

From the definition for the optical intensity $I = |\langle \vec{S}(\vec{r},t)\rangle|$

$\because$ The pulsed beam has a total energy of $E = 10\,mJ$

$$\therefore \iiint I \, dx dy dt = 10^{-3} J$$

$$\iiint I_0 \exp\left[-\frac{2(x^2+y^2)}{w_0^2}\right]\exp(-2t^2/\tau_0^2)\, dx dy dt = 10^{-3} J$$

$$I_0 \cdot w_0\sqrt{\frac{\pi}{2}} \cdot w_0\sqrt{\frac{\pi}{2}} \cdot \tau_0\sqrt{\frac{\pi}{2}} = 10^{-3} J$$

$$I_0 = \frac{10^{-3}}{w_0^2 \frac{\pi\sqrt{2\pi}}{4} \cdot 50\times 10^{-12}} J/m^2\cdot s$$

$$I_0 = \frac{10^{-3}}{25\times 10^{-6} \times \frac{\pi\sqrt{2\pi}}{4} \times 50\times 10^{-12}} J/m^2\cdot s$$

$$I_0 \approx 2.88\times 10^{13} W/m^2$$

b)

solution:

$$I(\alpha,\beta,\bar{\omega}) = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} I_0 \exp\left[-\frac{2(x^2+y^2)}{w_0^2}\right]\exp\left(-\frac{2t^2}{\tau_0^2}\right)\exp[-i(\alpha x+\beta y)]\exp(i\bar{\omega}t)\, dx dy dt$$

$$= \frac{I_0}{8\pi^3}\iint_{-\infty}^{\infty} \exp\left[-\frac{2(x^2+y^2)}{w_0^2}\right]\exp[-i(\alpha x+\beta y)]\, dx dy \int_{-\infty}^{\infty}\exp\left(-\frac{2t^2}{\tau_0^2}+i\bar{\omega}t\right)dt$$

$$= \frac{I_0}{8\pi^3}\cdot w_0^2 \frac{\pi\sqrt{\pi}}{2\sqrt{2}}\tau_0 e^{-\frac{w_0^2(\alpha^2+\beta^2)}{8}} \cdot e^{-\frac{\tau_0^2\bar{\omega}^2}{8}} \quad \text{(Gaussian integral)}$$

$$= \frac{I_0\sqrt{2\pi}}{32\pi^2} w_0^2 \tau_0 e^{-\frac{w_0^2(\alpha^2+\beta^2)}{8}} \cdot e^{-\frac{\tau_0^2\bar{\omega}^2}{8}}$$

$$I(\alpha,\beta,\bar{\omega};z) = I(\alpha,\beta,\bar{\omega})\cdot \tilde{H}_F \cdot \tilde{H}_p$$

$$= \frac{I_0\sqrt{2\pi}}{32\pi^2} w_0^2 \tau_0 e^{-\frac{w_0^2(\alpha^2+\beta^2)}{8}} \cdot e^{-\frac{\tau_0^2\bar{\omega}^2}{8}} \cdot e^{-i\frac{\alpha^2+\beta^2}{2k}z} \cdot e^{i\frac{D\bar{\omega}^2}{2}z}$$

$$I(x,y,t,z) = \frac{I_0\sqrt{2\pi}}{32\pi^2} w_0^2 \tau_0 \iiint_{-\infty}^{\infty} \exp\left[-(\alpha^2+\beta^2)\left(\frac{w_0^2}{8}+i\frac{z}{2k}\right)\right]\cdot \exp[i(\alpha x+\beta y)]\cdot \exp\left[-\bar{\omega}^2\left(\frac{\tau_0^2}{8}-i\frac{D}{2}z\right)\right]\cdot \exp(-i\bar{\omega}t)\, d\alpha d\beta d\bar{\omega}$$

$$I(x,y,\tau,z) = \frac{I_0\sqrt{2\pi}}{32\pi^2} W_0^2 \tau_0 \sqrt{\frac{\pi}{\frac{W_0^2}{8}+i\frac{z}{2k}}} \cdot \sqrt{\frac{\pi}{\frac{W_0^2}{8}+i\frac{z}{2k}}}\, e^{-\frac{x^2+y^2}{\frac{W_0^2}{2}+i\frac{2z}{k}}} \cdot \sqrt{\frac{\pi}{\frac{\tau_0^2}{8}-i\frac{D}{2}z}}\, e^{-\frac{\tau^2}{\frac{\tau_0^2}{2}-iDz}}$$

$$I(x,y,z,\tau) = I_0 \frac{1}{\sqrt{1+(\frac{2z}{z_0})^2}} \exp\left\{-\frac{2(x^2+y^2)}{W_0^2[1+(\frac{2z}{z_0})^2]}\right\} \exp\left\{i\frac{2k(x^2+y^2)}{z(4+\frac{z_0^2}{z^2})}\right\} \exp[i\varphi(z)] \frac{1}{\sqrt[4]{1+(\frac{4z}{L_D})^2}} \exp\left\{-\frac{2\tau^2}{\tau_0^2[1+(\frac{4z}{L_D})^2]}\right\}$$

$$\exp\left\{-\frac{2}{D}\frac{i\tau^2}{z(4+(\frac{L_D}{2z})^2)}\right\}$$

$$\therefore I(z) = \frac{I_0}{\sqrt{1+(\frac{2z}{z_0})^2}} \cdot \frac{1}{\sqrt[4]{1+(\frac{4z}{L_D})^2}} \exp\left\{-\frac{2(x^2+y^2)}{W_0^2[1+(\frac{2z}{z_0})^2]}\right\} \exp\left\{-\frac{2\tau^2}{\tau_0^2[1+(\frac{4z}{L_D})^2]}\right\}$$

c)

solution:

$\because W(z) = W_0\sqrt{1+(\frac{2z}{z_0})^2}$

$z_0 = \frac{kW_0^2}{2} = \frac{\pi}{\lambda}W_0^2 = \frac{\pi}{520\times10^{-9}} 25\times10^{-6}\ m \approx 151 m$

$\therefore$ when $\frac{W(z)}{W_0} = \sqrt{2}$ $\quad z = 75.5 m$

$\because \tau(z) = \tau_0\sqrt{1+(\frac{4z}{L_D})^2}$

$z_0' = -\frac{\tau_0^2}{2D} = -\frac{50^2\times10^{-24}}{2\times0.05\times10^{-24}} m = -25000 m$

$\therefore$ when $\frac{\tau(z)}{\tau_0} = \sqrt{2}$ $\quad z' = 12500 m$

$\because z < z'$

$\therefore$ spatial broadening is dominant.

d)

if $\frac{W(z)}{W_0} = \frac{\tau(z)}{\tau_0}$

$\sqrt{1+(\frac{2z}{z_0})^2} = \sqrt{1+(\frac{4z}{L_D})^2}$

$\Leftrightarrow \frac{kW_0^2}{2} = \left|-\frac{1}{2}\frac{\tau_0^2}{D}\right|$

$\therefore \tau_0 = \sqrt{151\times2\times0.05\times10^{-24}}\ s$

$\tau_0 \approx 3.89\ ps$

Task 2 Jinsong Liu

a)

Solution:

$$U_1(t) = U(t)\cdot \exp(i\zeta t^2) = B_0 \exp(-\frac{t^2}{\tau_0^2} + i\zeta t^2)$$

$$U_1(t) = B_{10}\exp(-(1-iC_1)\frac{t^2}{\tau_1^2}) = B_0\exp(-\frac{t^2}{\tau_1^2} + iC_1\frac{t^2}{\tau_1^2})$$

$\therefore\ B_{10} = B_0$

$\tau_0 = \tau_1$

$C_1\frac{t^2}{\tau_1^2} = \zeta t^2$

$\therefore\ C_1 = \zeta\tau_0^2$

$$FT[U_1(t)] = \frac{B_{10}\tau_1\sqrt{1+iC_1}}{2\sqrt{\pi(1+C_1^2)}}\exp\left[-\frac{\omega^2\tau_1^2(1+iC_1)}{4(1+C_1^2)}\right]$$

$$\therefore\ \omega_1^2 = \frac{4(1+C_1^2)}{\tau_1^2} = \frac{4(1+\zeta^2\tau_0^4)}{\tau_0^2}$$

$$\omega_1 = \frac{2\sqrt{1+\zeta^2\tau_0^4}}{\tau_0}$$

b)

Solution:

from problem a), we can get

$$U_1(\omega) = \frac{B_{10}\tau_1\sqrt{1+iC_1}}{2\sqrt{\pi(1+C_1^2)}}\exp\left[-\frac{\omega^2\tau_1^2(1+iC_1)}{4(1+C_1^2)}\right]$$

$$U_2(\omega) = H_e(\omega)U_1(\omega) = \frac{B_{10}\tau_1\sqrt{1+iC_1}}{2\sqrt{\pi(1+C_1^2)}}\exp(-\frac{ib\omega^2}{4})\exp\left[-\frac{\omega^2\tau_1^2(1+iC_1)}{4(1+C_1^2)}\right]$$

$$U_2(t) = \int_{-\infty}^{\infty}\frac{B_{10}\tau_1\sqrt{1+iC_1}}{2\sqrt{\pi(1+C_1^2)}}\exp\left[-\frac{ib(1+C_1^2)\omega^2+\omega^2\tau_1^2(1+iC_1)}{4(1+C_1^2)}\right]\exp(-i\omega t)\,d\omega$$

$$= \frac{B_{10}\tau_1\sqrt{1+iC_1}}{2\sqrt{\pi(1+C_1^2)}}\sqrt{\frac{\pi\cdot 4(1+C_1^2)}{\tau_1^2+i[C_1\tau_1^2+b(1+C_1^2)]}}\exp\left[-\frac{t^2(1+C_1^2)}{\tau_1^2+i[C_1\tau_1^2+b(1+C_1^2)]}\right]$$

$$= \frac{B_{10}\tau_1\sqrt{1+iC_1}}{2\sqrt{\pi(1+C_1^2)}}\sqrt{\frac{4\pi(1+C_1^2)[\tau_1^2-i[C_1\tau_1^2+b(1+C_1^2)]]}{\tau_1^4+[C_1\tau_1^2+b(1+C_1^2)]^2}}\exp\left\{-\frac{t^2(1+C_1^2)\{\tau_1^2-i[C_1\tau_1^2+b(1+C_1^2)]\}}{\tau_1^4+[C_1\tau_1^2+b(1+C_1^2)]^2}\right\}$$

$\because\ U_2(t) = B_{20}\exp(-(1-iC_2)\frac{t^2}{\tau_2^2})$ is transform-limited

$\therefore\ C_2 = 0$

$\therefore\ C_1\tau_1^2 + b(1+C_1^2) = 0 \qquad b = -\frac{C_1\tau_1^2}{1+C_1^2} = -\frac{\zeta\tau_0^4}{1+\zeta^2\tau_0^4}$

c)

solution:

From problem b)

$$U_2(t) = \frac{B_0 \tau_1 \sqrt{1+iC_1}}{2\sqrt{\pi(1+C_1^2)}} \sqrt{\frac{4\pi(1+C_1^2)\{\tau_1^2 - i[C_1\tau_1^2 + b(1+C_1^2)]\}}{\tau_1^4 + [C_1\tau_1^2 + b(1+C_1^2)]^2}} \exp\left\{ -\frac{t^2(1+C_1^2)\{\tau_1^2 - i[C_1\tau_1^2 + b(1+C_1^2)]\}}{\tau_1^4 + [C_1\tau_1^2 + b(1+C_1^2)]^2} \right\}$$

$$U_2(\omega) = \frac{B_{10}\tau_1\sqrt{1+iC_1}}{2\sqrt{\pi(1+C_1^2)}} \exp\left(-i\frac{b\omega^2}{4}\right) \exp\left[-\frac{\omega^2\tau_1^2(1+iC_1)}{4(1+C_1^2)}\right]$$

$$\because C_1\tau_1^2 + b(1+C_1^2) = 0$$

$$\therefore \tau_2^2 = \frac{\tau_1^4}{(1+C_1^2)\tau_1^2} = \frac{\tau_1^2}{1+C_1^2} = \frac{\tau_0^2}{1+\zeta^2\tau_0^4}$$

$$\tau_2 = \sqrt{\frac{\tau_0^2}{1+\zeta^2\tau_0^4}} = \frac{\tau_0}{\sqrt{1+\zeta^2\tau_0^4}}$$

$$B_{20} = \frac{B_{10}\tau_1\sqrt{1+iC_1}}{2\sqrt{\pi(1+C_1^2)}} \cdot \frac{2\tau_1\sqrt{\pi(1+C_1^2)}}{\tau_1^2} = B_{10}\sqrt{1+iC_1} = B_0\sqrt{1+i\zeta\tau_0^2}$$

$$\omega_2^2 = \frac{4(1+C_1^2)}{\tau_1^2} = \frac{4(1+\zeta^2\tau_0^4)}{\tau_0^2}$$

$$\omega_2 = \frac{2\sqrt{1+\zeta^2\tau_0^4}}{\tau_0}$$

d)

solution:

QPM: It works in time domain,

$\because \tau_0 = \tau_1 \neq \tau_2$

$\therefore$ it doesn't change the pulse duration

Chirp filter: It works in frequency domain,

$\because \omega_1 = \omega_2 \neq \omega_0$

$\therefore$ it doesn't change the spectral width

Task 3 Jinsong Liu

a)

Solution:

$\because$ $n(\omega)$ the refractive index changes with ω, it causes different group velocity of different frequency components

$\therefore$ The initial pulse will in general undergo temporal broadening.

b)

solution:

$$\frac{1}{V_{ph}} = \frac{n(\omega_0)}{c} = (a_1 + a_2\omega_0^2 + a_3\omega_0^3)\cdot\frac{1}{c}$$

$$\therefore V_{ph} = \frac{c}{a_1 + a_2\omega_0^2 + a_3\omega_0^3}$$

$$\frac{1}{V_g} = \frac{\partial k}{\partial \omega}\Big|_{\omega_0} = \frac{1}{c}\left[n(\omega_0) + \omega_0 \frac{\partial n}{\partial \omega}\Big|_{\omega_0}\right] = \frac{1}{c}\left[a_1 + a_2\omega_0^2 + a_3\omega_0^3 + \omega_0(2a_2\omega_0 + 3a_3\omega_0^2)\right]$$

$$\frac{1}{V_g} = \frac{1}{c}\left[a_1 + a_2\omega_0^2 + a_3\omega_0^3 + 2a_2\omega_0^2 + 3a_3\omega_0^3\right]$$

$$\frac{1}{V_g} = \frac{1}{c}(a_1 + 3a_2\omega_0^2 + 4a_3\omega_0^3)$$

$$V_g = \frac{c}{a_1 + 3a_2\omega_0^2 + 4a_3\omega_0^3}$$

c)

solution:

$\because$ The temporal broadening is minimized at $\omega = \omega_0$

$$\therefore \quad D = \frac{\partial}{\partial \omega}\left(\frac{1}{V_g}\right) = -\frac{1}{V_g^2}\frac{\partial V_g}{\partial \omega} = 0$$

$$\therefore \frac{\partial V_g}{\partial \omega} = 0 \quad \text{at } \omega = \omega_0$$

$$-\frac{c(6a_2\omega_0 + 12a_3\omega_0^2)}{(a_1 + 3a_2\omega_0^2 + 4a_3\omega_0^3)^2} = 0$$

$$\therefore \quad a_2\omega_0 + 2a_3\omega_0^2 = 0$$

$$\omega_0 = \frac{-a_2 \pm \sqrt{a_2^2}}{4a_3}$$

$$\because \omega_0 \neq 0 \qquad \omega_0 = \frac{-2a_2}{4a_3} = -\frac{a_2}{2a_3}$$

d)

solution: $k(w) = w\dfrac{n(w)}{c} = w\dfrac{(a_1 + a_2 w^2 + a_3 w^3)}{c}$

Taylor expansion of $k(w)$: (at $w = w_0$)

$$k(w) \approx k(w_0) + \left.\frac{\partial k}{\partial w}\right|_{w_0}(w-w_0) + \frac{1}{2}\left.\frac{\partial^2 k}{\partial w^2}\right|_{w_0}(w-w_0)^2$$

$$= \frac{1}{c}(a_1 w_0 + a_2 w_0^3 + a_3 w_0^4) + \frac{1}{c}(a_1 + 3a_2 w_0^2 + 4a_3 w_0^3)(w-w_0) + \frac{1}{c}(6a_2 w_0 + 12a_3 w_0^2)(w-w_0)^2$$

$$= \frac{1}{c}\cdot a_2 w_0^4\left(\frac{a_1}{a_2}\frac{1}{w_0^3} + \frac{1}{w_0} + \frac{a_3}{a_2}\right) + \frac{a_2 w_0^3}{c}\left(\frac{a_1}{a_2}\frac{1}{w_0^3} + 3\frac{1}{w_0} + 4\frac{a_3}{a_2}\right)(w-w_0) + \frac{a_2 w_0^2}{c}\left(6\frac{1}{w_0} + \frac{12a_3}{a_2}\right)(w-w_0)^2$$

$\because \dfrac{1}{w_0} << \dfrac{a_3}{a_2}$

$$\therefore k(w) \approx \frac{1}{c} a_2 w_0^4\left(\frac{a_1}{a_2}\frac{1}{w_0^3} + \frac{a_3}{a_2}\right) + \frac{a_2 w_0^3}{c}\left(\frac{a_1}{a_2}\frac{1}{w_0^3} + 4\frac{a_3}{a_2}\right)(w-w_0) + \frac{a_2 w_0^2}{c}\cdot\frac{12a_3}{a_2}(w-w_0)^2$$

$$= \frac{1}{c}(a_1 w_0 + a_3 w_0^4) + \frac{1}{c}(a_1 + 4a_3 w_0^3)(w-w_0) + \frac{1}{c}\cdot 12a_3 w_0^2 (w-w_0)^2$$