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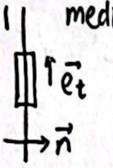
Task 1

1.5

a)

Solution:



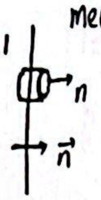
$\because$ We do not consider surface current and surface charges

$$\therefore \int \nabla \times \vec{H} \cdot d\vec{S} = \oint \vec{H} \cdot d\vec{l} = \hat{n} \times (\vec{H}_2 - \vec{H}_1) = 0$$

conclusion is right, but almost no derivation, so I have to ~~minu~~ some point.

-0.5

b)



$\because$ We do not consider surface ~~[illegible]~~ current and surface charges

$$\therefore \int \operatorname{div} \vec{D}\, dV = \oint_S \vec{D} \cdot d\vec{S} = \hat{n} \cdot (\vec{D}_2 - \vec{D}_1) = 0$$

E and B ? -2

Task 2

a) 3

The definition of potential $\varphi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$:

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_0} \nabla \times \vec{A}(\vec{r},t)$$

$$\vec{E}(\vec{r},t) = -\nabla\varphi(\vec{r},t) - \frac{\partial \vec{A}(\vec{r},t)}{\partial t}$$

Maxwell's equations:

① $\nabla \cdot \vec{D}(\vec{r},t) = \rho(\vec{r},t)$

② $\nabla \cdot \vec{B}(\vec{r},t) = 0$

③ $\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}$

④ $\nabla \times \vec{H}(\vec{r},t) = \vec{j}(\vec{r},t) + \frac{\partial \vec{D}(\vec{r},t)}{\partial t}$

put $\vec{H}(\vec{r},t) = \frac{1}{\mu_0} \nabla \times \vec{A}(\vec{r},t)$ into equation ②

$$\nabla \cdot (\nabla \times \vec{A}(\vec{r},t)) = 0$$

put $\vec{E}(\vec{r},t) = -\nabla\varphi(\vec{r},t) - \frac{\partial \vec{A}(\vec{r},t)}{\partial t}$ into equation ③

$$\nabla \times \left(-\nabla\varphi - \frac{\partial \vec{A}(\vec{r},t)}{\partial t}\right) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}$$

$$\because \vec{B}(\vec{r},t) = \nabla \times \vec{A}(\vec{r},t)$$

$$\therefore \nabla \times (-\nabla\varphi) = 0$$

Wave equations:

put $\vec{H}(\vec{r},t) = \frac{1}{\mu_0} \nabla \times \vec{A}(\vec{r},t)$ into equation ④

$$\nabla \times \nabla \times \vec{A}(\vec{r},t) = \mu_0 \vec{j}(\vec{r},t) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

put $\vec{E}(\vec{r},t) = -\nabla\varphi(\vec{r},t) - \frac{\partial \vec{A}(\vec{r},t)}{\partial t}$ into this equation

$$\nabla(\nabla \cdot \vec{A}(\vec{r},t)) - \nabla^2 \vec{A}(\vec{r},t) = \mu_0 \vec{j}(\vec{r},t) + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(-\nabla\varphi(\vec{r},t) - \frac{\partial \vec{A}(\vec{r},t)}{\partial t}\right)$$

when $\nabla \cdot \vec{A}(\vec{r},t) = -\frac{1}{c^2} \frac{\partial \varphi(\vec{r},t)}{\partial t}$

The wave equation is

$$\nabla^2 \vec{A}(\vec{r},t) - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2} = -\mu_0 \vec{j}(\vec{r},t)$$

Put $\vec{E}(\vec{r},t) = -\nabla\varphi(\vec{r},t) - \frac{\partial\vec{A}(\vec{r},t)}{\partial t}$ into equation ①

$$\nabla\cdot\left(-\nabla\varphi(\vec{r},t) - \frac{\partial\vec{A}}{\partial t}\right) = \frac{\rho(\vec{r},t)}{\varepsilon_0}$$ ✓

when $\nabla\cdot\vec{A}(\vec{r},t) = -\mu_0\varepsilon_0\frac{\partial\varphi(\vec{r},t)}{\partial t}$

The wave equation is

$$\nabla^2\varphi(\vec{r},t) - \mu_0\varepsilon_0\frac{\partial^2\varphi(\vec{r},t)}{\partial t^2} = -\frac{\rho(\vec{r},t)}{\varepsilon_0}$$

b)

Solution: 1

$$\vec{E}(\vec{r},t) = -\nabla\varphi'(\vec{r},t) - \partial_t\vec{A}'(\vec{r},t)$$

$$= -\nabla\varphi - \nabla\partial_t\lambda(\vec{r},t) - \partial_t\vec{A}(\vec{r},t) + \partial_t\nabla\lambda(\vec{r},t)$$

$$= -\nabla\varphi - \partial_t\vec{A}(\vec{r},t)$$ ✓

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\nabla\times\vec{A}'(\vec{r},t)$$

$$= \frac{1}{\mu_0}\nabla\times\left(\vec{A}(\vec{r},t) - \nabla\lambda(\vec{r},t)\right)$$

$$= \frac{1}{\mu_0}\nabla\times\vec{A}(\vec{r},t) - \frac{1}{\mu_0}\nabla\times\nabla\lambda(\vec{r},t)$$

$$= \frac{1}{\mu_0}\nabla\times\vec{A}(\vec{r},t) - 0$$ ($\because \lambda(\vec{r},t)$ is an arbitrary scalar, $\therefore \nabla\times\nabla\lambda(\vec{r},t) = 0$)

$$= \frac{1}{\mu_0}\nabla\times\vec{A}(\vec{r},t)$$ ✓

$\because$ $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$ are invariant under the transformation

$\therefore$ Maxwell's equations are invariant under the transformation too.

c) 0

d) Solution: 2

The wave equation for $\varphi(\vec{r},t)$:

$$\nabla^2\varphi(\vec{r},t) - \mu_0\varepsilon_0\frac{\partial^2\varphi(\vec{r},t)}{\partial t^2} = -\frac{\rho(\vec{r},t)}{\varepsilon_0}$$

$$\nabla^2\varphi(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2\varphi(\vec{r},t)}{\partial t^2} = -\frac{\rho(\vec{r},t)}{\varepsilon_0}$$ ✓

The wave equation for $\vec{A}(\vec{r},t)$

$$\nabla^2\vec{A}(\vec{r},t) - \mu_0\varepsilon_0\frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2} = -\mu_0\vec{j}(\vec{r},t)$$

$$\nabla^2\vec{A}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2} = -\mu_0\vec{j}(\vec{r},t)$$ ✓

e) 1

Solution:

$$\nabla\cdot\vec{A}'(\vec{r},t) = \nabla\cdot\vec{A}(\vec{r},t) - \nabla\cdot\nabla\lambda(\vec{r},t)$$

$$-\frac{1}{c^2}\partial_t\varphi'(\vec{r},t) = -\frac{1}{c^2}\frac{\partial\varphi(\vec{r},t)}{\partial t} - \frac{1}{c^2}\frac{\partial^2\lambda(\vec{r},t)}{\partial t^2}$$

$\therefore$ the condition to preserve the Lorenz gauge is

$$\nabla\cdot\nabla\lambda(\vec{r},t) = \frac{\partial^2\lambda(\vec{r},t)}{\partial t^2}$$ ✓

Task 3: 1+2+2+0

a) solution:

$$\chi(w) = \int_{-\infty}^{\infty} R(t) e^{iwt} dt$$

$$= \int_0^{\infty} \frac{f}{\Omega} e^{-\gamma t} \frac{e^{i\Omega t} - e^{-i\Omega t}}{2i} e^{iwt} dt$$

$$= \frac{f}{2i\Omega} \left(\int_0^{\infty} e^{[-\gamma + i(\Omega + w)]t} dt - \int_0^{\infty} e^{[-\gamma - i(\Omega - w)]t} dt \right)$$

$$= \frac{f}{2i\Omega} \left(- \frac{1}{-\gamma + i(\Omega + w)} + \frac{1}{-\gamma - i(\Omega - w)} \right)$$

$$= \frac{f}{2i\Omega} \frac{-[-\gamma - i(\Omega - w)] + [-\gamma + i(\Omega + w)]}{\gamma^2 - 2i\gamma w + \Omega^2 - w^2}$$

1 $$= \frac{f}{2i\Omega} \frac{2i\Omega}{\gamma^2 - 2i\gamma w + \Omega^2 - w^2}$$

$$= \frac{f}{\gamma^2 - 2i\gamma w + w_0^2 - \gamma^2 - w^2}$$

$$= \frac{f}{w_0^2 - w^2 - 2i\gamma w}$$

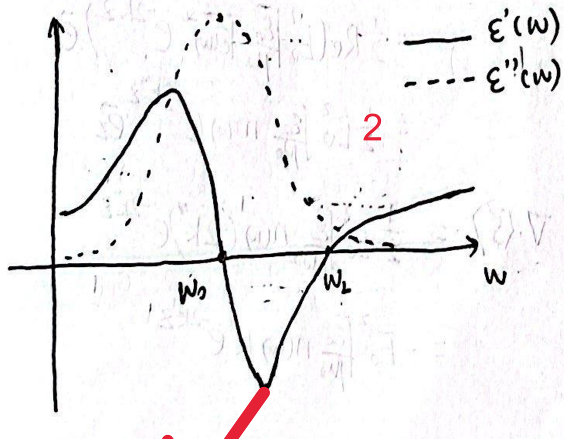
b) solution: $\varepsilon(w) = \varepsilon'(w) + i\varepsilon''(w)$

$$\varepsilon(w) = 1 + \chi(w) = 1 + \frac{f}{w_0^2 - w^2 - 2i\gamma w}$$

$$= 1 + \frac{(w_0^2 - w^2) f + 2i\gamma f w}{(w_0^2 - w^2)^2 + 4\gamma^2 w^2}$$

$$\therefore \varepsilon'(w) = 1 + \frac{(w_0^2 - w^2) f}{(w_0^2 - w^2)^2 + 4\gamma^2 w^2}$$

$$\varepsilon''(w) = \cancel{\frac{2\gamma f w}{(w_0^2 - w^2)^2 + 4\gamma^2 w^2}} \quad \frac{2\gamma f w}{(w_0^2 - w^2)^2 + 4\gamma^2 w^2}$$



2

$\boxed{\text{~~frequencies fre~~}}$

frequency range with normal dispersion: $\frac{d\varepsilon'(w)}{dw} > 0$

frequency range with anomalous dispersion: $\frac{d\varepsilon'(w)}{dw} < 0$

Strong absorption occur: $w = w_0$

c) solution:

$$\because \vec{P}(\vec{r}, w) = \varepsilon_0 \chi(w) \vec{E}(\vec{r}, w)$$

$$\therefore \vec{P}(\vec{r}, w) = \varepsilon_0 \chi(w) \vec{E}(\vec{r}) F\{\cos(w_{cw} t)\}$$

$$F\{\cos(w_{cw} t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(w_{cw} t) e^{iwt} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iw_{cw} t} + e^{-iw_{cw} t}}{2} e^{iwt} dt$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{i(w_{cw} + w)t} + e^{-i(w - w_{cw})t} dt$$

$$= \frac{1}{4\pi} [2\pi \delta(w + w_{cw}) + 2\pi \delta(w_{cw} - w)]$$

$$= \frac{1}{2} [\delta(w_{cw} + w) + \delta(w_{cw} - w)]$$

$$\therefore \vec{P}(\vec{r}, w) = \frac{\vec{E}(\vec{r}) \varepsilon_0}{2} [\delta(w_{cw} + w) + \delta(w_{cw} - w)] \frac{f}{w_0^2 - w^2 - 2i\gamma w}$$

$$\therefore \vec{P}(\vec{r}, t) = \int_{-\infty}^{\infty} \vec{P}(\vec{r}, w) e^{-iwt} dw$$

$$= \frac{\vec{E}(\vec{r}) \varepsilon_0 f}{2} \int_{-\infty}^{\infty} \delta(w_{cw} + w) \frac{e^{-iwt}}{w_0^2 - w^2 - 2i\gamma w} dw + \frac{\vec{E}(\vec{r}) \varepsilon_0 f}{2} \int_{-\infty}^{\infty} \delta(w_{cw} - w) \frac{e^{-iwt}}{w_0^2 - w^2 - 2i\gamma w} dw$$

$$= \frac{\vec{E}(\vec{r}) \varepsilon_0 f}{2} \left(\frac{e^{iw_{cw} t}}{w_0^2 - w_{cw}^2 + 2i\gamma w_{cw}} + \frac{e^{-iw_{cw} t}}{w_0^2 - w_{cw}^2 - 2i\gamma w_{cw}} \right)$$

$$= \frac{\vec{E}(\vec{r}) \varepsilon_0 f}{2} \left[\frac{(w_0^2 - w_{cw}^2)(e^{iw_{cw} t} + e^{-iw_{cw} t})}{(w_0^2 - w_{cw}^2)^2 + 4\gamma^2 w_{cw}^2} - \frac{2i\gamma w_{cw} (e^{iw_{cw} t} - e^{-iw_{cw} t})}{(w_0^2 - w_{cw}^2)^2 + 4\gamma w_{cw}^2} \right]$$

$$= \vec{E}(\vec{r}) \varepsilon_0 f \left[\frac{(w_0^2 - w_{cw}^2) \cos(w_{cw} t)}{(w_0^2 - w_{cw}^2)^2 + 4\gamma w_{cw}^2} + \frac{4\gamma w_{cw} \sin(w_{cw} t)}{(w_0^2 - w_{cw}^2)^2 + 4\gamma w_{cw}^2} \right]$$

2

Task 4

9

a)

Solution:

The wave equation is

$$\mathrm{rot}\,\mathrm{rot}\,\vec E(\vec r,w)=\frac{w^2}{c^2}\varepsilon(w)\vec E(\vec r,w)$$

∵ The wave is a monochromatic plane wave with the complex representation $\vec E(\vec r)=\vec E_0\exp[i\vec k\cdot\vec r]$

$$\mathrm{div}\,\vec E(\vec r,w)=0 \qquad \vec E(\vec r,w)=\vec E(w)\exp[i\vec k\cdot\vec r]$$

$$\therefore\ \Delta\vec E(\vec r,w)+\frac{w^2}{c^2}\varepsilon(w)\vec E(\vec r,w)=0$$

$$(ik)^2\vec E(w)+\frac{w^2}{c^2}\varepsilon(w)\vec E(w)=0$$

$$\left[-k^2+\frac{w^2}{c^2}\varepsilon(w)\right]\vec E(w)=0$$

$$\because \vec E(w)\neq 0\quad \therefore -k^2+\frac{w^2}{c^2}\varepsilon(w)=0 \text{ (dispersion relation)}$$

$$k(w)=\frac{w}{c}\sqrt{\varepsilon(w)}=\frac{w}{c}[n(w)+i\kappa(w)]$$

$$\sqrt{\varepsilon(w)}=n(w)+i\kappa(w)$$

$$\varepsilon(w)=[n(w)+i\kappa(w)]^2=n^2(w)+2in(w)\kappa(w)-\kappa^2(w)$$

$$\mathrm{Re}\,\varepsilon(w)=n^2(w)-\kappa^2(w)$$

$$\mathrm{Im}\,\varepsilon(w)=2n(w)\kappa(w)$$

$$\therefore\ n(w)=\frac{1}{\sqrt2}\sqrt{\sqrt{[\mathrm{Re}\,\varepsilon(w)]^2+[\mathrm{Im}\,\varepsilon(w)]^2}+\mathrm{Re}\,\varepsilon(w)}$$

2

b)

Solution:

$$\because \vec E(\vec r)=\vec E_0\exp[i\vec k\cdot\vec r]$$

The plane wave has a linear polarization along the y-direction and $\vec k$-vector is pointing in the z-direction

$$\therefore\ \vec E(\vec r,t)=\vec E_0 e^{i(kz-wt)}\vec e_y \qquad \vec E(\vec r)=E_0e^{ikz}\vec e_y$$

$$\because \vec H(\vec r,w)=\frac{1}{w\mu_0}[\vec k\times\vec E(w)]=\cancel{\frac{i}{w\mu_0}\vec E(\vec r,w)}-\frac{i}{w\mu_0}\mathrm{rot}\,\vec E(\vec r,w)$$

$$\vec E(\vec r,w)=F\{\vec E(\vec r,t)\}=E_0e^{ikz}\delta(w'-w)\vec e_y$$

$$\mathrm{rot}\,\vec E(\vec r,w)=\nabla\times\vec E(\vec r,w)$$

$$=i\vec k\times E_0e^{ikz}\delta(w'-w)\vec e_y$$

$$=-i\frac{w}{c}\sqrt{\varepsilon(w)}E_0e^{ikz}\delta(w'-w)\vec e_x$$

$\hat e_z\times\hat e_y=-\hat e_x$

$$\therefore\ \vec H(\vec r,w)=-\frac{i}{w\mu_0}\left[i\frac{w}{c}\sqrt{\varepsilon(w)}E_0e^{ikz}\delta(w'-w)\right]\vec e_x$$

$$=\sqrt{\frac{\varepsilon_0\varepsilon(w)}{\mu_0}}E_0e^{ikz}\delta(w'-w)\vec e_x$$

$$\vec H(\vec r,t)=\int_{-\infty}^{\infty}\vec e_x\sqrt{\frac{\varepsilon_0\varepsilon(w)}{\mu_0}}E_0e^{ikz}\delta(w'-w)e^{-iw't}dw'$$

$$=\sqrt{\frac{\varepsilon_0\varepsilon(w)}{\mu_0}}E_0e^{ikz}e^{-iwt}\vec e_x$$

$$\vec H(\vec r)=\sqrt{\frac{\varepsilon_0\varepsilon(w)}{\mu_0}}E_0e^{ikz}\vec e_x$$

2

c)

$$\langle\vec S(\vec r,t)\rangle=\frac12\mathrm{Re}[\vec E(\vec r)\times\vec H(\vec r)^*]$$

$$\vec E(\vec r)=E_0e^{ikz}\vec e_y$$

$$\vec H(\vec r)=\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\varepsilon(w)}E_0e^{ikz}\vec e_x$$

∵ $\vec k$ is a complex wavevector

$$\therefore\ \vec k=\vec k'+i\vec k''$$

$$\vec H(\vec r)^*=\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\varepsilon(w)}^*E_0e^{-k''z}e^{ik'z}\vec e_x$$

$$\cancel{\therefore\ \langle\vec S(\vec r,t)\rangle=\frac12\left(E_0e^{-k''z}e^{ik'z}\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\varepsilon(w)}^*E_0e^{-k''z}e^{-ik'z}\right)\vec e_z}$$

$$\cancel{E_0^2\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\varepsilon(w)}}$$

$$\therefore\ \langle\vec S(\vec r,t)\rangle=\frac12\mathrm{Re}\left(E_0e^{-k''z}e^{ik'z}\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\varepsilon(w)}^*E_0e^{-k''z}e^{-ik'z}\right)\vec e_z$$

$$=\frac12\mathrm{Re}\left(E_0^2\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\varepsilon(w)}^*e^{-2k''z}\right)\vec e_z$$

$$=\frac12E_0^2\sqrt{\frac{\varepsilon_0}{\mu_0}}n(w)e^{-2k''z}\vec e_z$$

$$\nabla\cdot\langle\vec S\rangle=\frac12E_0^2\sqrt{\frac{\varepsilon_0}{\mu_0}}n(w)(-2k'')e^{-2k''z}$$

$$=-E_0^2\sqrt{\frac{\varepsilon_0}{\mu_0}}n(w)k''e^{-2k''z}$$

3

d)

Solution:

i Propagating waves without loss

$\mathrm{div}\langle\vec{S}\rangle = 0 \quad \therefore \varepsilon'' = 0$

ii propagating waves with loss

$\mathrm{div}\langle\vec{S}\rangle > 0 \quad \therefore \varepsilon'' > 0$

iii nonpropagating waves ~~with loss~~ without loss

$\mathrm{div}\langle\vec{S}\rangle = 0 \quad \therefore \varepsilon'' = 0$

2

missing the ranges for ε'