a) Write down the Maxwell's equations in material in time domain in general form.

b) Write down the material equations relating the vector fields $\boldsymbol{D}, \boldsymbol{E}$, both in time domain and in frequency domain.

c) Derive the wave equation for the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ in a source-free, non-magnetic, isotropic, homogeneous medium with constant real-valued susceptibility χ and constant real-valued conductivity σ (so that the induced electric current density is $\boldsymbol{j}(\boldsymbol{r}, t) = \sigma \boldsymbol{E}(\boldsymbol{r}, t)$).

d) Derive the dispersion relation $k = k(\omega)$ for a plane wave solving the wave equation from part (c) and find how the complex dielectric function $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$ is defined from χ and σ.

e) Derive the wave equation in frequency domain for the electric field $\boldsymbol{E}(\boldsymbol{r}, \omega)$ in source-free, non-magnetic, non-conducting, inhomogeneous medium with spatially-varying dielectric permittivity

$$\varepsilon(\boldsymbol{r}) = \tilde{\varepsilon} + \boldsymbol{\kappa r},$$

where $\tilde{\varepsilon}$ is some constant value and $\boldsymbol{\kappa}$ is a constant vector.

(a) $\nabla\times\vec{E}(\vec{r},t) = -\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t}$ $\quad \nabla\cdot\vec{D}(\vec{r},t) = \vec{\rho}(\vec{r},t)$

$\nabla\times\vec{H}(\vec{r},t) = \vec{j}(\vec{r},t) + \dfrac{\partial\vec{D}(\vec{r},t)}{\partial t}$ $\quad \nabla\cdot\vec{H}(\vec{r},t) = 0$

(b) $\vec{D}(\vec{r},t) = \varepsilon_0\vec{E}(\vec{r},t) + P(\vec{r},t) = \varepsilon_0\vec{E}(\vec{r},t) + \varepsilon_0\int_{-\infty}^{t} R(\vec{r},t-t')\vec{E}(\vec{r},t')dt'$

$\vec{D}(\vec{r},\omega) = \varepsilon_0\vec{E}(\vec{r},\omega) + \vec{P}(\vec{r},\omega) = \varepsilon_0\vec{E}(\vec{r},\omega) + \varepsilon_0\chi(\vec{r},\omega)\vec{E}(\vec{r},\omega) = \varepsilon_0[1+\chi(\omega)]\vec{E}(\vec{r},\omega) = \varepsilon_0\varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega)$

(c) $\nabla\times\nabla\times\vec{E}(\vec{r},t) = -\mu_0\frac{\partial}{\partial t}\nabla\times\vec{H}(\vec{r},t) = -\mu_0\frac{\partial}{\partial t}[\vec{j}(\vec{r},t) + \frac{\partial\vec{D}(\vec{r},t)}{\partial t}]$

$\vec{R}(\vec{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\chi(\vec{r},\omega)e^{-i\omega t}d\omega$ $\quad \chi\to\text{constant} \Rightarrow \vec{R}(\vec{r},t) = \frac{\chi}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega t}dt = \frac{\chi}{2\pi}\delta(t)$

$\vec{P}(\vec{r},t) = \varepsilon_0\int_{-\infty}^{t}\frac{\chi}{2\pi}\delta(t-t')\vec{E}(\vec{r},t')dt' = \varepsilon_0\frac{\chi}{2\pi}\vec{E}(\vec{r},t) \Rightarrow \vec{D}(\vec{r},t) = \varepsilon_0\vec{E}(\vec{r},t) + \varepsilon_0\frac{\chi}{2\pi}\vec{E}(\vec{r},t)$

$\Rightarrow \nabla\times\nabla\times\vec{E}(\vec{r},t) = \nabla[\nabla\cdot\vec{E}(\vec{r},t)] - \Delta\vec{E}(\vec{r},t) = -\mu_0\frac{\partial}{\partial t}[\sigma\vec{E}(\vec{r},t) + \varepsilon_0(1+\frac{\chi}{2\pi})\frac{\partial}{\partial t}\vec{E}(\vec{r},t)]$

$\nabla\cdot\vec{D}(\vec{r},t) = 0 \Rightarrow (1+\frac{\chi}{2\pi})\nabla\cdot\vec{E}(\vec{r},t) = 0 \Rightarrow \nabla\cdot\vec{E}(\vec{r},t) = 0$

$\Rightarrow -\Delta\vec{E}(\vec{r},t) = -\mu_0\sigma\frac{\partial}{\partial t}\vec{E}(\vec{r},t) - \mu_0\varepsilon_0(1+\frac{\chi}{2\pi})\frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t)]$

$$\Rightarrow \underline{\Delta\vec{E}(\vec{r},t) - \frac{1}{c^2}(1+\frac{\chi}{2\pi})\frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t) = \mu_0\sigma\frac{\partial}{\partial t}\vec{E}(\vec{r},t)}$$

(d) Plane wave $\Rightarrow e^{i(\vec{k}\vec{r}-\omega t)}$

$\Rightarrow \Delta\vec{E}(\vec{r},t) = -k^2\vec{E}(\vec{r},t)$ $\quad \frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t) = -\omega^2\vec{E}(\vec{r},t)$ $\quad \frac{\partial}{\partial t}\vec{E}(\vec{r},t) = -i\omega\vec{E}(\vec{r},t)$

$\Rightarrow -k^2 + \frac{\omega^2}{c^2}(1+\frac{\chi}{2\pi}) = -i\omega\mu_0\sigma \Rightarrow k^2(\omega) = \frac{\omega^2}{c^2}(1+\frac{\chi}{2\pi}) + i\omega\mu_0\sigma = \frac{\omega^2}{c^2}(1+\frac{\chi}{2\pi} + i\frac{c^2\mu_0\sigma}{\omega})$

$\Rightarrow k^2(\omega) = \frac{\omega^2}{c^2}(1+\frac{\chi}{2\pi} + i\frac{\sigma}{\omega\varepsilon_0}) \Rightarrow \varepsilon' = 1+\frac{\chi}{2\pi}$ $\quad \varepsilon'' = \frac{\sigma}{\omega\varepsilon_0}$

(e) $\nabla\times\vec{E}(\vec{r},\omega) = i\omega\mu_0\vec{H}(\vec{r},\omega)$ $\quad \nabla\times\vec{H} = -i\omega\vec{D}(\vec{r},\omega)$ $\quad \vec{D}(\vec{r},\omega) = \varepsilon_0\varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega)$

$\nabla\cdot\vec{D}(\vec{r},\omega) = \varepsilon_0[\nabla\cdot\varepsilon(\vec{r},\omega)]\vec{E}(\vec{r},\omega) + \varepsilon_0\varepsilon(\vec{r},\omega)\nabla\cdot\vec{E}(\vec{r},\omega) = 0$

$\Rightarrow [\nabla\cdot(\tilde{\varepsilon}+\vec{\kappa}\vec{r})]\vec{E}(\vec{r},\omega) = -(\tilde{\varepsilon}+\vec{\kappa}\vec{r})\nabla\cdot\vec{E}(\vec{r},\omega)$

$\Rightarrow \nabla\cdot\vec{E}(\vec{r},\omega) = -\dfrac{(\nabla\cdot\vec{\kappa}\vec{r})\vec{E}(\vec{r},\omega)}{\tilde{\varepsilon}+\vec{\kappa}\vec{r}} \Rightarrow \nabla[\nabla\cdot\vec{E}(\vec{r},\omega)] = -\nabla\left[\dfrac{(\nabla\cdot\vec{\kappa}\vec{r})\vec{E}(\vec{r},\omega)}{\tilde{\varepsilon}+\vec{\kappa}\vec{r}}\right]$

$\nabla\times\nabla\times\vec{E}(\vec{r},\omega) = i\omega\mu_0\nabla\times\vec{H}(\vec{r},\omega) = \omega^2\mu_0\vec{D}(\vec{r},\omega) = \frac{\omega^2}{c^2}\varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega)$

$\nabla\times\nabla\times\vec{E}(\vec{r},\omega) = \nabla[\nabla\cdot\vec{E}(\vec{r},\omega)] - \Delta\vec{E}(\vec{r},\omega) = -\nabla\left[\dfrac{(\nabla\cdot\vec{\kappa}\vec{r})\vec{E}(\vec{r},\omega)}{\tilde{\varepsilon}+\vec{\kappa}\cdot\vec{r}}\right] - \Delta\vec{E}(\vec{r},\omega)$

$\Rightarrow \Delta\vec{E}(\vec{r},\omega) + \frac{\omega^2}{c^2}(\tilde{\varepsilon}+\vec{\kappa}\cdot\vec{r})\vec{E}(\vec{r},\omega) = -\nabla\left[\dfrac{(\nabla\cdot\vec{\kappa}\cdot\vec{r})\vec{E}(\vec{r},\omega)}{\tilde{\varepsilon}+\vec{\kappa}\cdot\vec{r}}\right] = -\nabla\left[\dfrac{\vec{\kappa}\cdot\vec{E}(\vec{r},\omega)}{\tilde{\varepsilon}+\vec{\kappa}\cdot\vec{r}}\right]$