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WS 2010/2011

Exam
FUNDAMENTALS OF MODERN OPTICS

Final exam on February 25

Exercise 1 (Maxwell, linear media, plane waves, TE/TM) **5+2+10+10 Points**

Before you start with this exam, please have a look at the second sheet which provides you with some useful formulas. Good luck!

a) Write down the macroscopic Maxwell equations in temporal Fourier domain. ✓

b) Specify (in Fourier domain) the general relation between $\overline{\mathbf{D}}(\mathbf{r},\omega)$ and $\overline{\mathbf{E}}(\mathbf{r},\omega)$ for a linear, dispersive, inhomogeneous, and anisotropic medium. ✓

c) Let us now consider a linear, dispersive, homogeneous and isotropic dielectric medium in transparent regime: $\overline{\mathbf{D}}(\mathbf{r},\omega) = \epsilon_0\epsilon(\omega)\overline{\mathbf{E}}(\mathbf{r},\omega)$ with $\epsilon(\omega) > 0$, $\overline{\mathbf{B}}(\mathbf{r},\omega) = \mu_0\overline{\mathbf{H}}(\mathbf{r},\omega)$, $\overline{\mathbf{j}}(\mathbf{r},\omega) = 0$, and $\rho = 0$. Show that plane waves of the form $\overline{\mathbf{E}}(\mathbf{r},\omega) = \mathbf{E}(\omega)\exp(i\mathbf{k}(\omega)\cdot\mathbf{r})$ exist which are solutions to Maxwell's equations. Derive the neccessary conditions for $\mathbf{k}(\omega)$ and $\mathbf{E}(\omega)$ [length and direction of $\mathbf{k}(\omega)$ with respect to $\mathbf{E}(\omega)$] and compute the corresponding $\overline{\mathbf{H}}(\mathbf{r},\omega)$. ✓

d) For linear, *inhomogeneous* and isotropic dielectric media things become more complicated, however, in the case of translational invariance in one direction (e.g., y) we can decouple Maxwell's equations into TE and TM polarization. Derive the resulting wave equations for $\overline{\mathbf{E}}_{\mathrm{TE}}(x,z,\omega)$ and $\overline{\mathbf{H}}_{\mathrm{TM}}(x,z,\omega)$, i.e., one equation containing $\overline{E}_y(x,z,\omega)$ and $\epsilon(x,z,\omega)$ only, and one equation containing $\overline{H}_y(x,z,\omega)$ and $\epsilon(x,z,\omega)$ only. ?

Exercise 2 (uniaxial crystal, quarter-wave plate) **7 Points**

Suppose you want to use a thin slice of an uniaxial crystal (quartz, $n_0 \simeq 1.54$, $n_e \simeq 1.55$) to transform circularly polarized light at a wavelength of $\lambda_{vac} = 500$ nm to linear polarization. How do you have to cut the crystal with respect to the optical axis, to make the slice as thin as possible? How thick the resulting "quarter-wave plate" will be? ✓

Exercise 3 (4f-setup, optical integration) **7+10 Points**

A Master of Science in Photonics student wants to build an optical integrator by using a 4f-setup. He/she knows that by placing an appropriate pupil $p(x)$ at distance $z = 2f$, one can perform useful transformations on the entrance field $u_0(x)$.

a) What would be the ideal pupil $p(x)$ to obtain $u(-x, z = 4f) \propto \int_{-\infty}^{x} u_0(x')dx'$? ✓

b) Unfortunately, this ideal pupil function for optical integration has a singularity at $x = 0$. Show that

$$p(x) = \operatorname{sinc}\left(\frac{kb}{f}x\right),$$

✓

which is free of singularities, leads to $u(-x, z = 4f) \propto \int_{x-b}^{x+b} u_0(x')dx'$.

✓

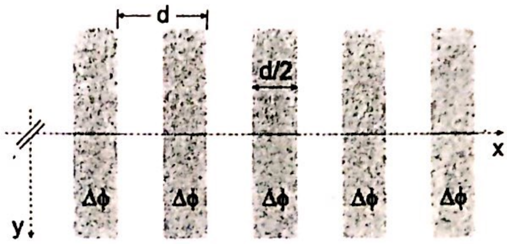
Exercise 4 (dielectric multi-layer systems, reflectivity) **7 Points**

Consider a very complicated (absorption-free in the relevant frequency range) multi-layer system between a substrate ($\epsilon_s = 2$) and cladding ($\epsilon_c = 1$) material. Give the value of the reflectivity for an incident plane wave in the substrate, when the angle of incidence (between k-vector and normal of the layer-system) is $\phi = \pi/3$.

(p.t.o)

Exercise 5 (dielectric function, response function, Kramers-Kronig relations) **7 Points**
A friend of yours claims to have a homogeneous medium characterized by the dielectric function $\epsilon(\omega) = 1 + i\exp(-\omega^2\tau_0^2)$, with $\tau_0 = 42$ fs. Do you believe him? Give reasons for your answer.

Exercise 6 (Fraunhofer approximation, phase grating) **15 Points**
A monochromatic plane wave propagates along $\vec{e}_z$ in vacuum and is diffracted by a thin non-absorbing phase grating of period d, whose N slits (white) and bars (gray) have a width $d/2$. The phase shift of the incident light within the bars amounts to $\Delta\phi = \pi$ relative to the light which passes directly through the slits. Assume an infinite extension along $\vec{e}_y$, i.e. a one-dimensional structure.



Show that the transmitted intensity in paraxial Fraunhofer approximation may be written as

$$I(x,z) \propto \operatorname{sinc}^2\left(\frac{Nkxd}{2z}\right)\tan^2\left(\frac{kxd}{4z}\right).$$

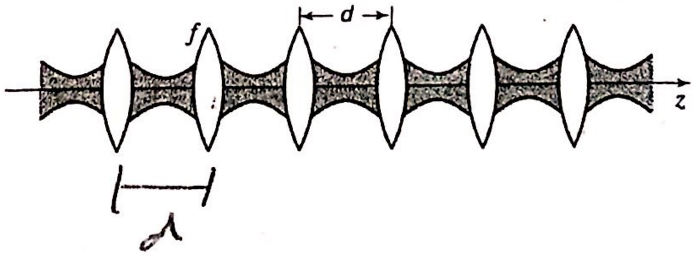
Which diffraction orders m can be observed, which ones vanish? ?

Exercise 7 (pulse propagation in dispersive media) **10 Points**
The group velocity in a homogeneous dielectric used for manufacturing glass fibers can be well approximated in a relatively wide range in the infrared wavelength regime by

$$v_g(\omega) = \frac{c}{n_0}\frac{1}{1+\left(\frac{\omega-\omega_0}{\delta}\right)^2}$$

with the center frequency ω_0, the center group index n_0, a parameter δ, and the speed of light c. Let us consider a plane wave with a Gaussian pulse envelope $u(t, z=0) = u_0\exp\{-(t/T_0)^2\}\exp(-i\omega t)$ launched into this material at the plane $z = 0$ and propagating in z-direction. Derive a formula for the propagation length in dependence on the frequency ω, after which the pulse duration has doubled. Consider dispersion up to second order (GVD) only. At which ω the pulse does not broaden upon propagation?

Exercise 8 (propagation of Gaussian beams, beam relaying) **10 Points**
A Gaussian beam is focused by a periodic sequence of identical lenses, each of focal length f and separated by a distance d (see Figure below). The beam is consecutively focused to the same waist and propagates periodically in this sequence of lenses. Show that the condition of periodic transmission ("relaying") can arise only if the inequality $d < 4f$ is satisfied. Where is the beam waist located between the individual lenses?



Formulas which may be useful:

- $\mathbf{rot}\,\mathbf{E}(\mathbf{r}) = \nabla \times \mathbf{E}(\mathbf{r})$, $\mathbf{div}\,\mathbf{E}(\mathbf{r}) = \nabla \cdot \mathbf{E}(\mathbf{r})$, $\mathbf{grad}\,\epsilon(\mathbf{r}) = \nabla\epsilon(\mathbf{r})$, $\nabla^2 = \Delta$
- $\nabla \times [\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r})] = \epsilon(\mathbf{r})\nabla \times \mathbf{E}(\mathbf{r}) - \mathbf{E} \times \nabla\epsilon(\mathbf{r})$
- $\nabla \times [\nabla \times \mathbf{E}(\mathbf{r})] = \nabla[\nabla \cdot \mathbf{E}(\mathbf{r})] - \Delta\mathbf{E}(\mathbf{r})$
- $\nabla \times \left[\frac{1}{\epsilon(\mathbf{r})}\nabla \times \mathbf{H}(\mathbf{r})\right] = \frac{1}{\epsilon(\mathbf{r})}\nabla \times [\nabla \times \mathbf{H}(\mathbf{r})] - [\nabla \times \mathbf{H}(\mathbf{r})] \times \nabla\left[\frac{1}{\epsilon(\mathbf{r})}\right]$
- For a $4f$-system we have

$$u(-x, z = 4f) \propto \int_{-\infty}^{\infty} p\left(\frac{f}{k}\alpha\right) U(\alpha, z = 0)e^{i\alpha x}d\alpha \propto \int_{-\infty}^{\infty} P\left[\frac{k}{f}(x' - x)\right] u(x', z = 0)dx',$$

  where $U(\alpha, z = 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x, z = 0)\exp(-i\alpha x)dx$ and $P(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} p(x)\exp(-i\alpha x)dx$ is the Fourier transform of the pupil $p(x)$ situated at distance $z = 2f$.

- $\int_{-\infty}^{\infty} \mathrm{sinc}(x)\exp(-i\alpha x)dx = \pi\theta(\alpha + 1)\theta(\alpha - 1)$ with $\theta(x) = \begin{cases} 0 & x < 0 \\ 1/2 & x = 0 \\ 1 & x > 0 \end{cases}$
- $\sin(\pi/3) = \sqrt{3}/2$
- The linear Kramers-Kronig relations read

$$\Re\chi(\omega) = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} \frac{\Im\chi(\overline{\omega})}{\overline{\omega} - \omega}d\overline{\omega}, \qquad \Im\chi(\omega) = -\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} \frac{\Re\chi(\overline{\omega})}{\overline{\omega} - \omega}d\overline{\omega}.$$

- $\int_{\infty}^{\infty}\exp(-x^2)\exp(-i\alpha x)dx = \sqrt{\pi}\exp(-\alpha^2/4)$
- $\int_{\infty}^{\infty}\exp(-i\alpha x)dx = 2\pi\delta(\alpha)$
- The intensity distribution created by a one-dimensional periodic mask in paraxial Fraunhofer approximation is given by

$$I(x, z) \propto \left|T_S\left(\frac{kx}{z}\right)\right|^2 \frac{\sin^2\left(\frac{Nkxd}{2z}\right)}{\sin^2\left(\frac{kxd}{2z}\right)},$$

  where d is the period length, N the number of periods, and $T_S(\alpha) = \frac{1}{2\pi}\int_0^d t(x)\exp(-i\alpha x)dx$ the Fourier transform of one period of the mask.

- $\sin(x) = 2\sin(x/2)\cos(x/2)$
- The propagation of pulse envelopes $v(z, \tau)$ in linear dispersive media is governed by

$$i\partial_z v(z, \tau) = \frac{D}{2}\partial_\tau^2 v(z, \tau),$$

  where $\tau = t - z/v_g(\omega_0)$ is the co-moving time, and the wavenumber

$$k(\omega) = \frac{\omega_0 n_0}{c} + \frac{1}{v_g(\omega_0)}(\omega - \omega_0) + \frac{D}{2}(\omega - \omega_0)^2$$

  is expanded in a Taylor series around the central frequency ω_0.

(p.t.o)