Task 1

a) prove:

$\vec{a} \times (\vec{b} \times \vec{c})$

$= \epsilon_{ijk} a_j (\vec{b} \times \vec{c})_k$

$= \epsilon_{ijk} a_j \epsilon_{klm} b_l c_m$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_j b_l c_m$

$= \delta_{il}\delta_{jm} a_j b_l c_m - \delta_{im}\delta_{jl} a_j b_l c_m$

$= b_i a_j c_j - a_j b_j c_i$

$= \vec{b}(\vec{a}\cdot\vec{c}) - (\vec{a}\cdot\vec{b})\vec{c}$

b) prove:

$\vec{\nabla} \times (\alpha \vec{a})$

$= \epsilon_{ijk} \nabla_j (\alpha a_k)$

$= \epsilon_{ijk} (\alpha \nabla_j a_k + a_k \nabla_j \alpha)$

$= \alpha \epsilon_{ijk} \nabla_j a_k + \epsilon_{ijk} a_k \nabla_j \alpha$

$= \alpha \epsilon_{ijk} \nabla_j a_k - \epsilon_{ikj} a_k \nabla_j \alpha$

$= \alpha (\vec{\nabla} \times \vec{a}) - (\vec{a} \times \vec{\nabla}) \alpha$

$= \alpha \vec{\nabla} \times \vec{a} - \vec{a} \times \vec{\nabla} \alpha$

c)

prove:

$\vec{\nabla} \cdot (\vec{a} \times \vec{b})$

$= \nabla_i (\vec{a} \times \vec{b})_i$

$= \nabla_i \epsilon_{ijk} a_j b_k$

$= \epsilon_{ijk} (b_k \nabla_i a_j + a_j \nabla_i b_k)$

$= \epsilon_{ijk} b_k \nabla_i a_j + \epsilon_{ijk} a_j \nabla_i b_k$

$= b_k \epsilon_{kij} \nabla_i a_j - a_j \epsilon_{jik} \nabla_i b_k$

$= \vec{b} (\vec{\nabla} \times \vec{a}) - \vec{a} (\vec{\nabla} \times \vec{b})$

d) prove:

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{a})$

$= \nabla_i (\vec{\nabla} \times \vec{a})_i$

$= \nabla_i \epsilon_{ijk} \nabla_j a_k$

$= \epsilon_{ijk} (\nabla_i \nabla_j a_k + \nabla_j \nabla_i a_k)$

$= \epsilon_{ijk} \nabla_i \nabla_j a_k + \epsilon_{ijk} \nabla_j \nabla_i a_k$

$= \nabla_i \epsilon_{ijk} \nabla_j a_k - \nabla_j \epsilon_{jik} \nabla_i a_k$

$= \vec{\nabla} \cdot (\vec{\nabla} \times \vec{a}) - \vec{\nabla} \cdot (\vec{\nabla} \times \vec{a}) = 0$

Task 2:

$\vec{j}(\vec{r},\omega) = j_0 \cosh\left(\frac{r}{\delta}\right)\vec{e}_z$

$\delta = \delta(\omega)$

$\vec{j}(\vec{r},\omega) = \sigma(\omega)\vec{E}(\vec{r},\omega)$

$\vec{\nabla}\times\vec{H}(\vec{r},\omega) = \vec{j}(\vec{r},\omega) - i\omega\varepsilon_0\varepsilon(\omega)\vec{E}(\vec{r},\omega) = j_0\cosh\left(\frac{r}{\delta}\right)\vec{e}_z\left(1 - i\omega\varepsilon_0\frac{\varepsilon(\omega)}{\sigma(\omega)}\right)$

According to Stoke's theorem

$$\int \vec{\nabla}\times\vec{H}(\vec{r},\omega)\, d\vec{s} = \int j_0\cosh\left(\frac{r}{\delta}\right)\left(1 - i\omega\varepsilon_0\frac{\varepsilon(\omega)}{\sigma(\omega)}\right)\vec{e}_z\cdot d\vec{s} = \oint\vec{H}\cdot d\vec{l} = 2\pi r\,\vec{H}(\vec{r},\omega)$$

$$\therefore\ j_0\left(1 - i\omega\varepsilon_0\frac{\varepsilon(\omega)}{\sigma(\omega)}\right)\int_0^r r\,\frac{e^{\frac{r}{\delta}}+e^{-\frac{r}{\delta}}}{2}\,dr\int_0^{2\pi}d\varphi = \pi j_0\left(1 - i\omega\varepsilon_0\frac{\varepsilon(\omega)}{\sigma(\omega)}\right)\int_0^r\left(re^{\frac{r}{\delta}} + re^{-\frac{r}{\delta}}\right)dr$$

$$= \pi j_0\left(1 - i\omega\varepsilon_0\frac{\varepsilon(\omega)}{\sigma(\omega)}\right)\left[\left(\delta re^{\frac{r}{\delta}} - \delta^2 e^{\frac{r}{\delta}}\right) - \left(\delta re^{-\frac{r}{\delta}} - \delta^2 e^{-\frac{r}{\delta}}\right)\right]_0^r$$

$$= 2\pi j_0\left(1 - i\omega\varepsilon_0\frac{\varepsilon(\omega)}{\sigma(\omega)}\right)\left[\delta r\sinh\left(\frac{r}{\delta}\right) - \delta^2\cosh\left(\frac{r}{\delta}\right) + \delta^2\right]$$

$$\therefore\ \vec{H}(\vec{r},\omega) = \frac{j_0}{r}\left(1 - i\omega\varepsilon_0\frac{\varepsilon(\omega)}{\sigma(\omega)}\right)\left[\delta^2 - \delta^2\cosh\left(\frac{r}{\delta}\right) + \delta r\sinh\left(\frac{r}{\delta}\right)\right]$$

Task 3:

Solution:

Maxwell's equation:

① $\text{rot}\ \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}$ ③ $\text{div}\ \vec{D}(\vec{r},t) = \rho$

② $\text{rot}\ \vec{H}(\vec{r},t) = \vec{j}(\vec{r},t) + \frac{\partial \vec{D}(\vec{r},t)}{\partial t}$ ④ $\text{div}\ \vec{B}(\vec{r},t) = 0$

According to equation ②, ③

we can get:

$$\text{div}[\text{rot}\ \vec{H}(\vec{r},t)] = \text{div}\ \vec{j}(\vec{r},t) + \frac{\partial \rho}{\partial t}$$

$\because \text{div}[\text{rot}\ \vec{H}(\vec{r},t)] = 0$

$\therefore$ ⑤ $\text{div}\ \vec{j}(\vec{r},t) + \frac{\partial \rho}{\partial t} = 0$ (Differential notation)

Explaination of the equation ⑤:

The varying charges is the source of divergence of the current.

According to the definition of divergence:

$$\text{div}\ \vec{A} = \lim_{V \to 0} \frac{\oint_S \vec{A} \cdot d\vec{s}}{V}$$

$$\therefore \int_V \text{div}\ \vec{A}\ dv = \oint_S \vec{A} \cdot d\vec{s}$$

So the equation ⑤ can be written as:

⑥ $\oint_S \vec{j}(\vec{r},t)\, d\vec{s} + \int_V \frac{\partial \rho}{\partial t}\, dv = 0$ (Integral notation)

Explaination of the equation ⑥:

In the space of volume V enclosed by a closed surface S, the charge flowing into the closed surface S per unit time is the same as the charge lost in space V.

Both ⑤ and ⑥ mean the law of charge conservation.

Task 4 :

a)

Solution:

Maxwell's equations : (in empty space)

① $\text{rot}\, \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}$ ③ $\text{div}\, \vec{D}(\vec{r},t) = \rho$

② $\text{rot}\, \vec{H}(\vec{r},t) = \vec{j}(\vec{r},t) + \frac{\partial \vec{D}(\vec{r},t)}{\partial t}$ ④ $\text{div}\, \vec{B}(\vec{r},t) = 0$

In empty space ,

applying the curl operator a second time on equation ②

$$\text{rot rot}\, \vec{H}(\vec{r},t) = \text{rot}\, \vec{j}(\vec{r},t) + \varepsilon_0\, \text{rot} \frac{\partial \vec{E}(\vec{r},t)}{\partial t}$$

$$⑤ \quad \text{rot rot}\, \vec{H}(\vec{r},t) = -\varepsilon_0 \mu_0 \frac{\partial^2 \vec{H}(\vec{r},t)}{\partial t^2} + \text{rot}\, \vec{j}(\vec{r},t)$$

The equation ⑤ is the wave equation for the magnetic field.

b)

Solution:

Fourier domain: $\text{rot rot}\, \vec{H}(\vec{r},w) = \frac{w^2}{c^2} \vec{H}(\vec{r},w) + \text{rot}\, \vec{j}(\vec{r},w)$

c)

Solution:

$$\text{rot rot}\, \vec{H} = \text{grad div}\, \vec{H} - \Delta \vec{H} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \Delta^{(2)} \vec{H}_x \\ \Delta^{(2)} \vec{H}_y \\ \Delta^{(2)} \vec{H}_z \end{bmatrix}$$

$$\vec{H} = \vec{H}_\perp + \vec{H}_\parallel$$

$$\vec{H}_\perp = \begin{bmatrix} 0 \\ H_y \\ 0 \end{bmatrix} \quad \vec{H}_\parallel = \cancel{\begin{bmatrix} H_x \\ 0 \\ H_z \end{bmatrix}} \quad \begin{bmatrix} \vec{H}_x \\ 0 \\ \vec{H}_z \end{bmatrix}$$

Two decoupled equations :

$$\Delta^{(2)} \vec{H}_\perp(\vec{r},w) + \frac{w^2}{c^2} \vec{H}_\perp(\vec{r},w) + \text{rot}\, \vec{j}_\perp(\vec{r},w) = 0$$

$$\Delta^{(2)} \vec{H}_\parallel(\vec{r},w) + \frac{w^2}{c^2} \vec{H}_\parallel(\vec{r},w) + \text{rot}\, \vec{j}_\parallel(\vec{r},w) = 0$$

d)

Solution:

Because $\text{div}\, \vec{H}(\vec{r},w) = 0$, the derivation of the decoupled equations for the magnetic field is simpler.

Task 5:

$$F\{\tilde{\theta}(w)\} = \int_{-\infty}^{+\infty} \tilde{\theta}(w) e^{-iwt} dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} P.V. \frac{i}{w} e^{-iwt} dw + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \pi \delta(w) e^{-iwt} dw$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^{0-\varepsilon} \frac{i}{w} e^{-iwt} dw + \int_{0+\varepsilon}^{+\infty} \frac{i}{w} e^{-iwt} dw \right] + \frac{1}{2}$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^{-\varepsilon} -\frac{i}{w} e^{i(-w)t} d(-w) + \int_{\varepsilon}^{+\infty} \frac{i}{w} e^{-iwt} dw \right] + \frac{1}{2}$$

$$= \frac{1}{2\pi} \int_{\varepsilon}^{+\infty} \frac{i}{w} e^{-iwt} - \frac{i}{w} e^{iwt} dw + \frac{1}{2}$$

$$= \frac{1}{2\pi} \int_{\varepsilon}^{+\infty} - \frac{i \cdot 2i}{w} \left(\frac{e^{iwt} - e^{-iwt}}{2i} \right) dw + \frac{1}{2}$$

$$= \frac{1}{\pi} \int_{\varepsilon}^{+\infty} \frac{\sin(wt)}{w} dw + \frac{1}{2}$$

when $t > 0$, $F\{\hat{\theta}(w)\} = \frac{\pi}{2} \cdot \frac{1}{\pi} + \frac{1}{2} = 1$

when $t = 0$, $F\{\hat{\theta}(w)\} = 0 + \frac{1}{2} = \frac{1}{2}$

when $t < 0$, $F\{\hat{\theta}(w)\} = -\frac{1}{2} + \frac{1}{2} = 0$