Friedrich Schiller University Jena
Institute of Applied Physics
Prof. Dr. Thomas Pertsch
Anastasia Romashkina, Shreyas Ramakrishna, Mostafa Abasifard, Tina (Shiu Hei Lam), Xiao Chen, Pawan Kumar

WS 2022/2023

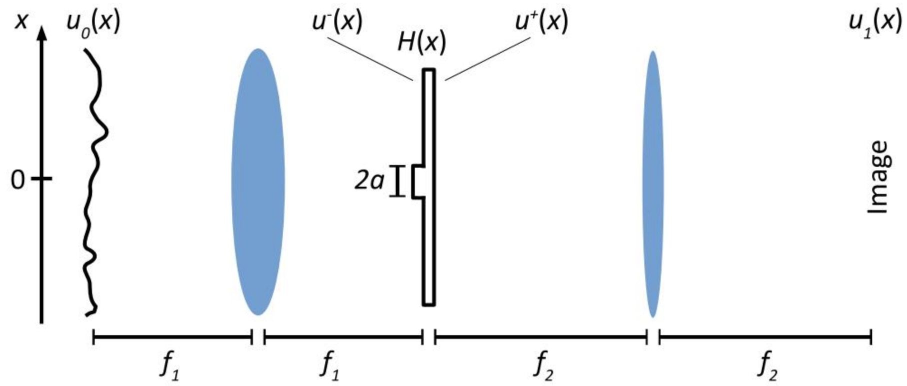
Series 10

Fundamentals of Modern Optics

to be returned on **19.01.2023**, at the beginning of the lecture

Task 1: Phase Contrast Microscopy (2+3 points)

Biological samples are often almost completely transparent. Consequently, they are very hard to see in a conventional microscope. However, these samples often have inhomogeneities of the refractive index and change the *phase* of the transmitted light. One elegant solution to make those phase profiles visible is the *phase contrast microscopy*. A simplified optical setup that implements it is shown below.



Essential is the phase plate in the centre whose effect can be described by the following transmission function

$$H(x) = \begin{cases} \exp(\mathrm{i}\varphi_0) & -a \le x \le a \\ 1 & \text{else} \end{cases}.$$

A part of the transmitted light is delayed compared to the rest. We will see in this task that this allows converting a phase profile into an intensity profile.

a) Derive an expression that describes how $u_1(x)$ depends on $u_0(x)$ and $H(x)$. Consider monochromatic light of wavelength λ. *Hint: Make use of the 2f-setup equation from the lecture.*

b) Calculate the image $u_1(x)$ of the initial phase-profile distribution:

$$u_0(x) = e^{\mathrm{i}A\cos(\alpha_0 x)} \approx 1 + \mathrm{i}A\cos(\alpha_0 x),$$

where $A \ll 1$ and $\alpha_0 = 2\pi/\Lambda$. This field can be thought to be caused approximately by a phase grating with a small amplitude and grating period Λ. Derive the conditions on a and φ_0 so that the setup converts the phase modulation in u_0 into an amplitude modulation in u_1.

Solution Task 1:

a) Using the expression for the 2f-setup from the lecture we have

$$u^-(x) = -\mathrm{i}\frac{2\pi}{\lambda f_1}\exp(2\mathrm{i}kf_1)U_0(kx/f_1),$$

where $U_0(\alpha)$ is the Fourier transform of $u_0(x)$. From that we obtain

$$u^+(x) = u^-(x)\,H(x) = -\mathrm{i}\frac{2\pi}{\lambda f_1}\exp(2\mathrm{i}kf_1)U_0(kx/f_1)\,H(x).$$

Finally, applying the 2f-setup formula again we obtain

$$u_1(x) = -\frac{(2\pi)^2}{\lambda^2 f_1 f_2}\exp[2\mathrm{i}k(f_1+f_2)]\underbrace{\mathcal{F}[U_0(kx/f_1)H(x)](kx/f_2)}_{\equiv I}$$

where

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} U_0\left(\frac{k}{f_1}x'\right) H(x') \exp(-\mathrm{i}\frac{k}{f_2}xx')\,\mathrm{d}x'.$$

b) First we calculate the Fourier transform of the initial field, which is given by

$$\begin{aligned}U_0(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}[1+\mathrm{i}A\cos(\alpha_0 x)]\exp(-\mathrm{i}\alpha x)\mathrm{d}x\\ &= \delta(\alpha)+\mathrm{i}\frac{A}{2}[\delta(\alpha-\alpha_0)+\delta(\alpha+\alpha_0)]\end{aligned}$$

This expression shows the $-1, 0, +1$ orders of the grating. Plugging that into the result from a) we have to solve the following integral

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\delta(kx'/f_1)+\mathrm{i}\frac{A}{2}[\delta(kx'/f_1-\alpha_0)+\delta(kx'/f_1+\alpha_0)]\right\}H(x')\exp(-\mathrm{i}\frac{k}{f_2}xx')\,\mathrm{d}x'.$$

Now what we want, is that $H(x)$ can somehow change the phase relation between the first delta function and the second and third delta functions, such that all delta functions have the same phase factor behind them, which then means the phase-variation has turned into an amplitude variation. With the given phase plate, we want the zeroth order of the grating to be delayed by the phase plate while the $+1, -1$ orders are not. Mathematically this means we require

$$\begin{aligned}\frac{k|a|}{f_1} &\le \alpha_0\\ \frac{2\pi|a|}{\lambda f_1} &\le \frac{2\pi}{\Lambda}\\ |a| &\le f_1\,\frac{\lambda}{\Lambda}.\end{aligned}$$

Under this assumption, we can expand the function $H(x)$ in the integral to obtain

$$\begin{aligned}I &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\exp(\mathrm{i}\varphi_0)\delta(kx'/f_1)+\mathrm{i}\frac{A}{2}[\delta(kx'/f_1-\alpha_0)+\delta(kx'/f_1+\alpha_0)]\right\}\exp(-\mathrm{i}\frac{k}{f_2}xx')\,\mathrm{d}x'\\ &= \frac{f_1}{2\pi k}\int_{-\infty}^{\infty}\left\{\exp(\mathrm{i}\varphi_0)\delta(x'')+\mathrm{i}\frac{A}{2}[\delta(x''-\alpha_0)+\delta(x''+\alpha_0)]\right\}\exp(-\mathrm{i}\frac{f_1}{f_2}xx'')\,\mathrm{d}x''\\ &= \frac{f_1}{2\pi k}\left\{\exp(\mathrm{i}\varphi_0)+\mathrm{i}\frac{A}{2}[\exp(-\mathrm{i}\frac{f_1}{f_2}\alpha_0 x)+\exp(\mathrm{i}\frac{f_1}{f_2}\alpha_0 x)]\right\}\\ &= \frac{f_1}{2\pi k}\left[\exp(\mathrm{i}\varphi_0)+\mathrm{i}A\cos\left(\alpha_0\frac{f_1}{f_2}x\right)\right]\end{aligned}$$

Now we plug this result in the expression for $u_1(x)$ and obtain

$$u_1(x) = -\frac{1}{\lambda f_2}\exp[2\mathrm{i}k(f_1+f_2)]\left[\exp(\mathrm{i}\varphi_0)+\mathrm{i}A\cos\left(\alpha_0\frac{f_1}{f_2}x\right)\right]$$

From this expression, it becomes clear that we need

$$\exp(\mathrm{i}\varphi_0)\stackrel{!}{=}\mathrm{i}\quad\rightarrow\quad\varphi_0=\pi/2$$

for the intensity to be

$$\begin{aligned}|u_1(x)|^2 &\propto 1+2A\cos\left(\alpha_0\frac{f_1}{f_2}x\right)+A^2\cos^2\left(\alpha_0\frac{f_1}{f_2}x\right)\\ &\approx 1+2A\cos\left(\alpha_0\frac{f_1}{f_2}x\right)\end{aligned}$$

Comparing this to

$$|u_0(x)|^2 \propto 1+A^2\cos^2(\alpha_0 x),$$

we see that the phase modulation has been magnified in space and amplitude.

Task 2: Prison Break (2+2+2 points)

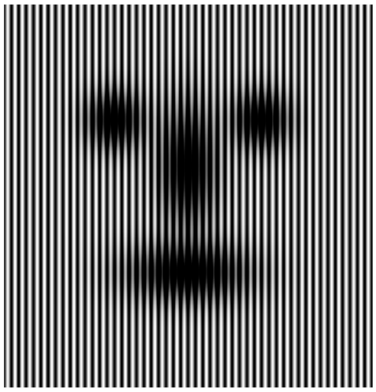


Figure 1: Hello! I am Günther! Please help me to get out of jail! I am innocent. I swear!

Your German friend Günther Gaußig was sent to prison for admitting that it is possible to overcome the resolution limit. He was locked by sheriff Peter Periodic using a grating in the x-direction which has the following form:

$$\text{PRISON} = g(x) = \frac{1}{2}\left[1 + \cos\left(\frac{2\pi}{d}x\right)\right]$$

Luckily, Günther is a very "Gaussian" person in real space, where he usually lives

$$\text{GÜNTHER} = f(x,y) = m(x,y) + n(x,y) + e_l(x,y) + e_r(x,y),$$

with

$$\text{Nose} = n(x,y) = \exp\left[-\frac{x^2}{w^2} - \frac{\left(y - \frac{R}{5}\right)^2}{l^2}\right]$$

$$\text{Mouth} = m(x,y) = \exp\left[-\frac{x^2}{l^2} - \frac{\left(y + \frac{R}{2}\right)^2}{w^2}\right]$$

$$\text{LeftEye} = e_l(x,y) = \exp\left[-\frac{\left(x - \frac{R}{2}\right)^2}{w^2} - \frac{\left(y - \frac{R}{2}\right)^2}{w^2}\right]$$

$$\text{RightEye} = e_r(x,y) = \exp\left[-\frac{\left(x + \frac{R}{2}\right)^2}{w^2} - \frac{\left(y - \frac{R}{2}\right)^2}{w^2}\right]$$

and

$$R > l > w \gg d$$

As an Abbe School of Photonics student you want to free your friend by optical means. You find Günther superimposed by the prison in the spatial domain as a starting point, i.e. the function $u_0(x,y) = f(x,y) + g(x)$ (see Figure).

a) Develop a plan to free Günther, i.e. to remove the prison, by Fourier optical filtering. Explain why it is needed the condition $R > l > w \gg d$ to be satisfied.

b) Construct and sketch an optical setup which is able to perform that task. Explain the mechanism from the initial optical field, the optics in between and the final field.

c) Why does Günther necessarily get battered (at least a little bit) by the escape from prison? How can you minimize that effect?

Hint: Notice that this task requires a physical understanding of the 4f-optical filtering. Try to be as precise as possible in your answers and avoid ambiguities. You can make sketches and write down general formulas in all questions to help the explanation.

Solution Task 2:

a) In the spatial domain, Günther is linearly superimposed by the grating. When transforming everything to Fourier space the result will be the sum of the Fourier transforms of the individual parts, since the FT is a linear operation.
So we can treat the different parts of Günther and the prison separately. The prison grating has the following Fourier transformation:

$$G(\alpha,\beta) = \frac{1}{2}\delta(\beta)\left[\delta(\alpha) + \delta\left(\alpha + \frac{2\pi}{d}\right) + \delta\left(\alpha - \frac{2\pi}{d}\right)\right].$$

This means it has three very sharp peaks in the Fourier domain. One is at $\alpha = \beta = 0$ but this is the constant contribution. It does not have to be removed, only the other two peaks contain information about the prison cell. Günther, however, consists of a sum of shifted and scaled Gaussian functions. We use the shifting theorem

$$\mathcal{F}[f(x-a)] = e^{-\mathrm{i}\alpha a}\mathcal{F}(\alpha)$$

and the scaling theorem

$$\mathcal{F}[f(ax)] = \frac{1}{|\alpha|}\mathcal{F}\left(\frac{\alpha}{a}\right)$$

to argue that the transformed Günther will eventually become a sum of (modulated) Gaussian functions centred in the Fourier domain. The maximum extent of this group will be given by the smallest feature in the spatial domain, i.e. $\approx 2/w$. Since w shall be much larger than d we can hope that the "prison peaks" will be sufficiently far away from Günther's defining Gaussian functions. The remaining task is to remove the "prison peaks" by an aperture in the Fourier domain and back-transform everything. Although the central peak of the grating will necessarily get transformed, it will just translate into a uniform background. The prison bars themselves will be gone. If sufficiently much information about Günther was preserved by the optical system, he shall come back to real space more or less as he was before.

b) The task shall be achieved by a standard $4f$ setup. The filtering could be achieved by a circular aperture in the Fourier plane. Its minimum radius to filter the grating shall be

$$\frac{k_0}{f}R_{\mathrm{min}} \leq \frac{2\pi}{d} \quad \rightsquigarrow \quad R_{\mathrm{min}} \leq \frac{f}{d}\lambda.$$

To be even more elegant, one could use the finite aperture a of the Fourier transforming lens to be too small to transmit the grating.

c) In Fourier space, Günther is infinitely extended. There are two reasons that this information cannot be transferred out of prison completely:
   a) The filtering of the grating happens inevitably at a finite spatial frequency. At least this particular information about Günther is lost.
   b) Any *real* $4f$-setup will just be able to operate on a finite range of frequencies due to its finite size. Any information in Fourier space greater than that cut-off will be lost. We can see, then, that although it was maybe too strict to send Günther to prison, the reason was not that wrong...

Task 3: Jones matrix formalism (2+2 points)

The Jones formalism is a powerful technique for the treatment of polarized light. To take a look at it, let's consider a monochromatic plane wave in the vacuum of the form $\mathbf{E}(r,t) = \mathbf{E_0}\,e^{i(kz-\omega t)}$, where the electric field is polarized in the (x,y)-plane, so $\mathbf{E_0} = (E_x, E_y, 0)$. We can write it in the form of the so-called "Jones vector":

$$\mathbf{J}_{\mathrm{in}} = \begin{pmatrix} E_x\,e^{i\varphi_x} \\ E_y\,e^{i\varphi_y} \end{pmatrix}.$$

Then light propagation through a polarizing optical element can be written as a linear transformation:

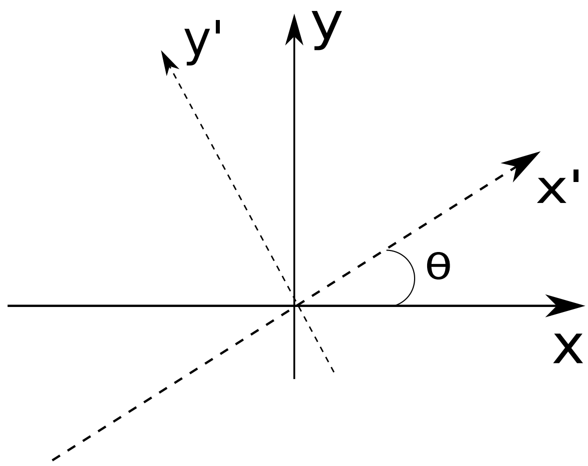
$$\mathbf{J}_{\text{out}} = \hat{\mathbf{T}} \cdot \mathbf{J}_{\text{in}}.$$

where $\hat{\mathbf{T}}$ denotes a "Jones matrix" of that element.
Recommendation: For those students who are further interested in the Jones matrix formalism we recommend to read chapter 6 of *"Fundamentals of Photonics"* by B.E.A. Saleh, M.C. Teich (*electronic copy of the book is available on Wiley online library for download*).

a) Elements of the Jones vectors and Jones matrices depend on the choice of the coordinate system. Consider, the Jones matrix of an x-polarizer is given by $\hat{\mathbf{T}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Now we rotate the polarizer around z axis by an angle θ. Using matrix methods, define the Jones matrix in the new coordinate system.
Hint: Make use of the rotation matrix.

b) Given is an optical element characterized by the Jones matrix $\hat{\mathbf{T}} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$. Which polarization state should the light have to pass the element without change?

Solution Task 3:



a) A system with such a Jones matrix $\hat{\mathbf{T}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ transforms a wave with components (A_x, A_y) into a wave with components $(A_x, 0)$, thus polarizing it along the x-direction. We use standard linear algebra to perform the coordinate transform. The rotation matrix in our case (rotation around z axis by the angle θ, at the picture) reads as:

$$\hat{\mathbf{R}} = \begin{pmatrix} \cos(\theta) & \mp\sin(\theta) \\ \pm\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

A coordinate transform is given by

$$\hat{\mathbf{T}}' = \hat{\mathbf{R}}\hat{\mathbf{T}}\hat{\mathbf{R}}^{-1}.$$

Since $\hat{\mathbf{R}}^{-1} = \hat{\mathbf{R}}^T$, we have

$$\hat{\mathbf{T}}' = \begin{pmatrix} \cos(\theta) & \mp\sin(\theta) \\ \pm\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) & \pm\sin(\theta) \\ \mp\sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$\hat{\mathbf{T}}' = \begin{pmatrix} \cos^2(\theta) & \pm\cos(\theta)\sin(\theta) \\ \pm\cos(\theta)\sin(\theta) & \sin^2(\theta) \end{pmatrix}$$

b) We need to find the Jones eigenvector to this Jones matrix. Again, basic linear algebra tells us how to do the job. First, we have to find the eigenvalues as

$$\det(\hat{\mathbf{T}} - \lambda\hat{\mathbf{I}}) = \left(\frac{1}{2} - \lambda\right)^2 - \frac{1}{4} \stackrel{!}{=} 0 \qquad \Longrightarrow \qquad \lambda_{1,2} = [0; 1].$$

Next, the eigenvectors are determined by the systems of equations

$$\hat{\mathbf{T}}\mathbf{J}_{1,2} = \lambda_{1,2}\mathbf{J}_{1,2}.$$

In the first case for $\lambda_1 = 0$ (the second equation is of course linear dependent on the first one and just stated for completeness)

$$\begin{aligned} J_x + iJ_y &= 0 \\ -iJ_x + J_y &= 0 \end{aligned}$$

which is solved by [1]:

$$\mathbf{J}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

The second eigenvector ($\lambda_2 = 1$) is found from

$$\begin{aligned} J_x + iJ_y &= 2J_x \\ -iJ_x + J_y &= 2J_y \end{aligned}$$

to be

$$\mathbf{J}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

They describe *left- and right-handed circular polarization*. In its diagonal form, $\hat{\mathbf{T}}$ would read as

$$\hat{\mathbf{T}}' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, it acts like a polarizer, but in its new eigenbasis $\mathbf{J}_1$ and $\mathbf{J}_2$. While the former is completely blocked (eigenvalue 0), the latter is completely transmitted (eigenvalue 1). The device that performs this is a 'circular polarizer'.

[1] We just need one solution out of the infinitely many linear dependent ones. I choose the vector to be normalized but any vector which is a product of this one by any complex number is correct.