Introduction to Optical Modeling

Friedrich-Schiller-University Jena
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Lecture 2

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Course Overview

Part 1: Geometrical optics based modeling and design (U.D. Zeitner)

1. Introduction
2. **Paraxial approximation / Gaussian optics**
3. **ABCD-matrix formalism**
4. Real lenses
5. Optical materials
   - glass types, dispersion
   - chromatic aberrations
6. Imaging systems
   - apertures/stops, entrance-/exit-pupil
   - wavefront aberrations

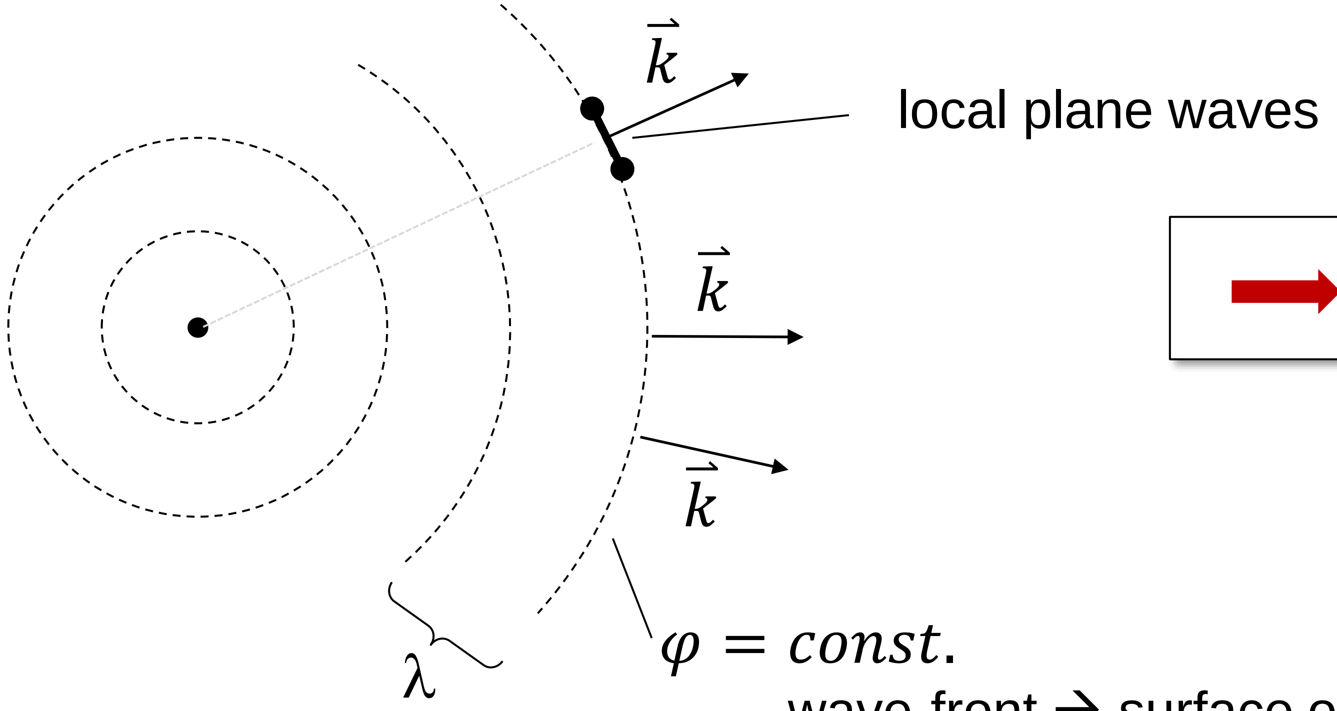
Part 2: Wave-optics based modeling (F. Wyrowski)

Recap: Propagation of Spherical Waves

Basic solution of Helmholtz Equation:

$$E(\vec{r}) = \frac{E_0}{r} \cdot e^{ik \cdot r} \tag{1.5}$$

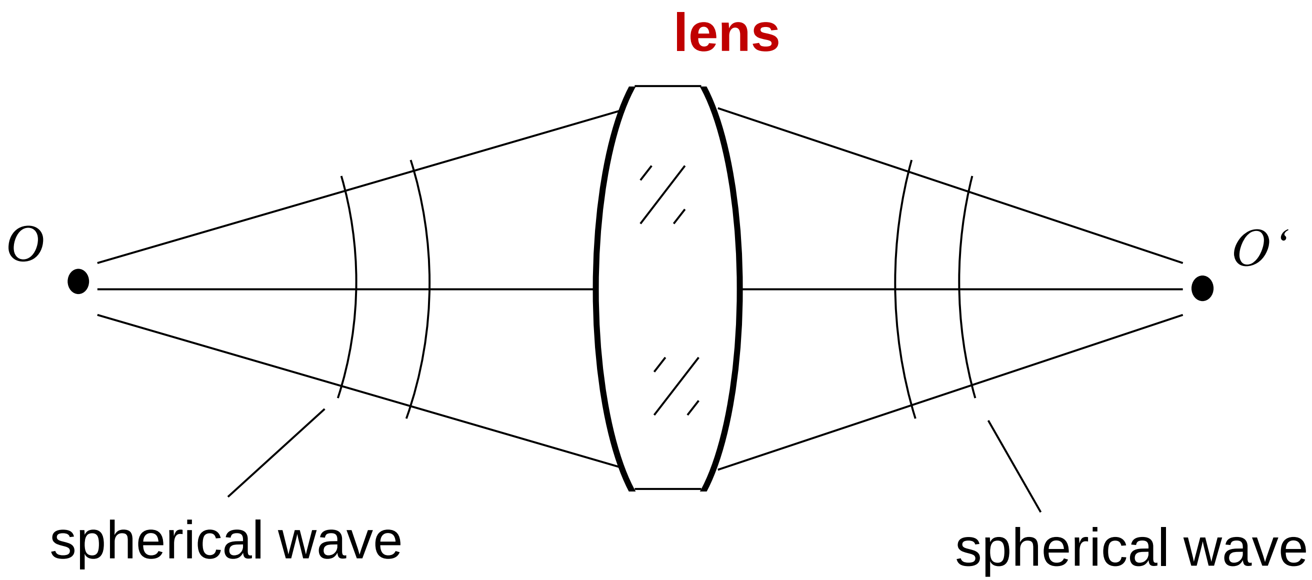
→ spherical wave (from a point source)



➡ representation of rays !

wave-front → surface of equal phase

2 Paraxial Imaging / Gaussian Optics

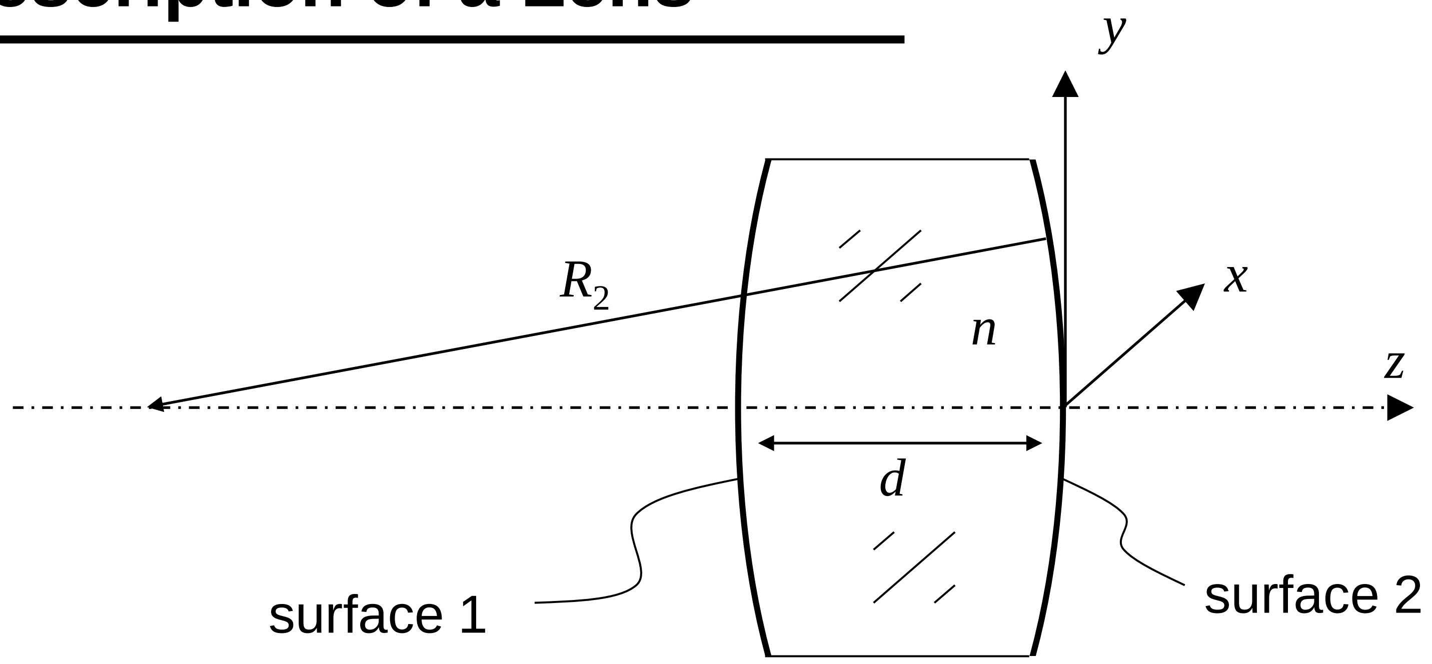


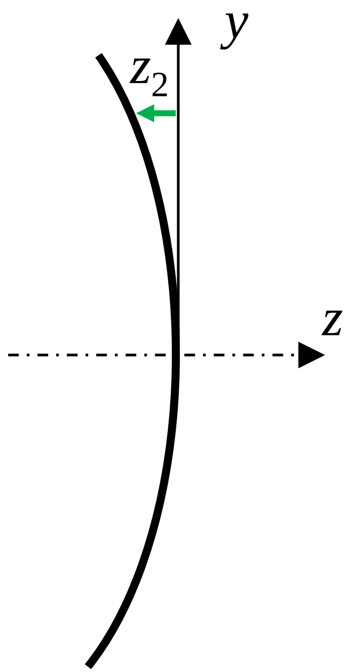
→ emerging from the object point

→ converging to form the image point

Imaging: transformation of a spherical wave into another spherical wave

Description of a Lens





Lens sag (description in the local coordinate system):

$$z_{1/2} = \frac{c_{1/2} \cdot r^2}{1 + \sqrt{1 - (1 + k_{1/2}) c_{1/2}^2 r^2}} + a_2 r^2 + a_4 r^4 + \cdots \qquad (2.1)$$

$r^2 = x^2 + y^2$ … radius

$c_{1/2} = \frac{1}{R_{1/2}}$ … curvature

R … radius of curvature

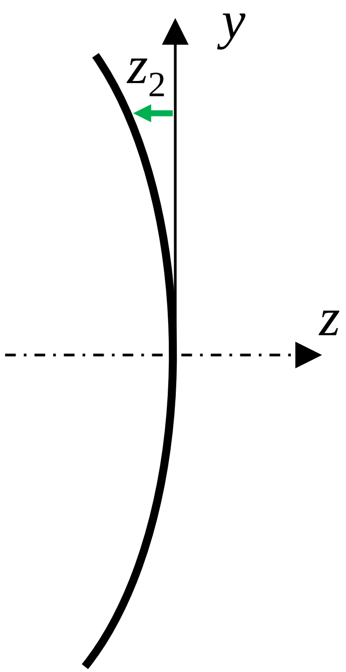
k … conic constant

d … lens thickness (distance of vertex points)

Description of a lens, conic constant

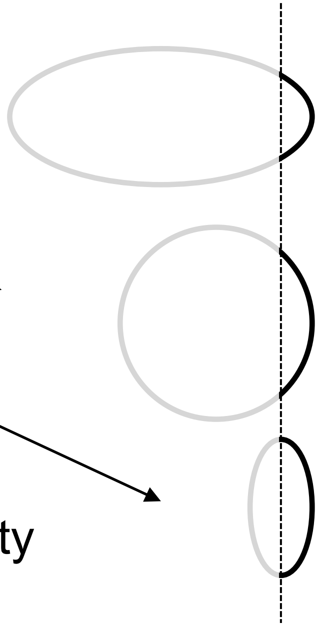
Sag formula:

$$z_{1/2} = \frac{c_{1/2} \cdot r^2}{1 + \sqrt{1 - \left(1 + k_{1/2}\right) c_{1/2}^2 r^2}} + a_2 r^2 + a_4 r^4 + \cdots$$



k … conic constant → describes surface as conic section

$k < -1$ → hyperbola
$k = -1$ → parabola
$-1 < k < 0$ → ellipse
$k = 0$ → sphere
$k > 0$ → oblate ellipse

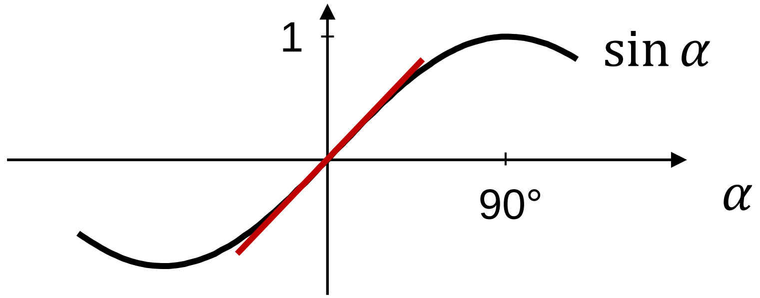


$k = -\varepsilon^2 = -\left(1 - {}^{b^2}/_{a^2}\right)$ ε… eccentricity

${}^{1}/_{c} = R = \pm\, {}^{b^2}/_{a}$

2.1 Paraxial Approximation

Law of refraction: $n \cdot \sin\alpha = n' \cdot \sin\alpha'$ (1.6)



Paraxial approximation: $\alpha,\ \alpha'$ small → linear approx. of sin-Fct.

$$n \cdot \alpha = n' \cdot \alpha'$$ paraxial law of refraction (2.2)

$$\cos\alpha \approx 1$$

Simplification used in:

- law of refraction
- corresponding ray angles
- equations describing optical surfaces

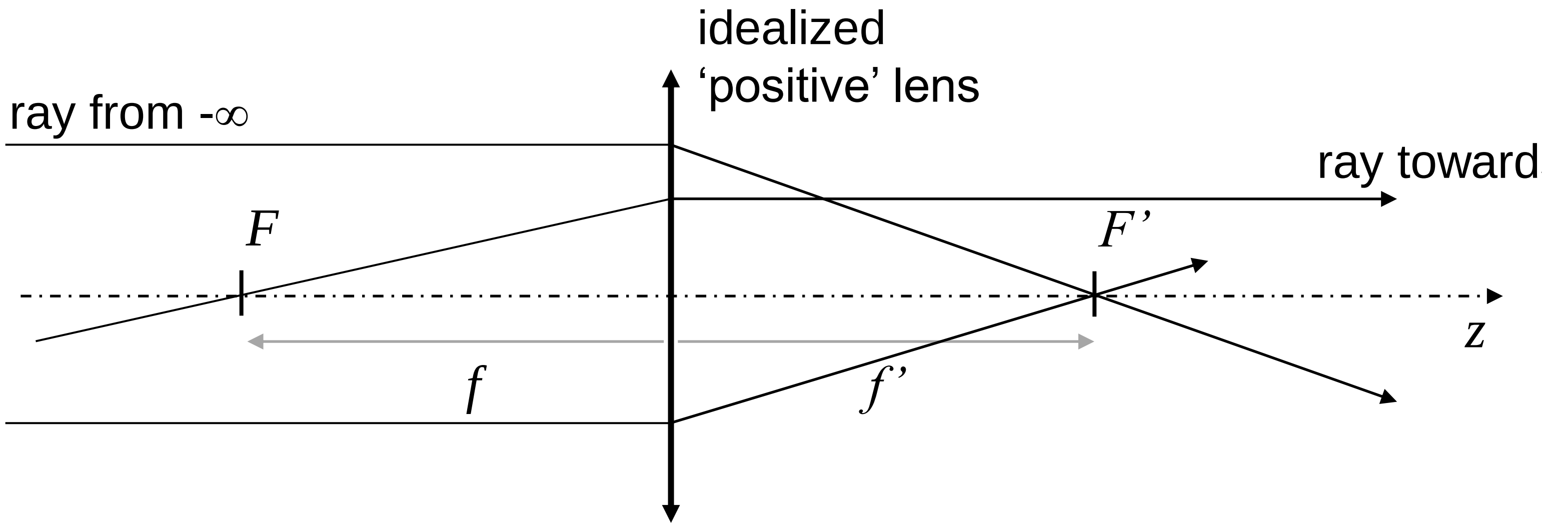
- most **equations** become **linear**
- **no aberrations** during imaging

2.2 Ideal Lens

Optical function: transforming a spherical wave into another spherical wave

Ideal Lens: assumption that optical effect takes place in the "plane of the lens"

Considering two special cases: object / image point at infinite distance from the lens



$f < 0$
$f' > 0$

Sign convention in optics:

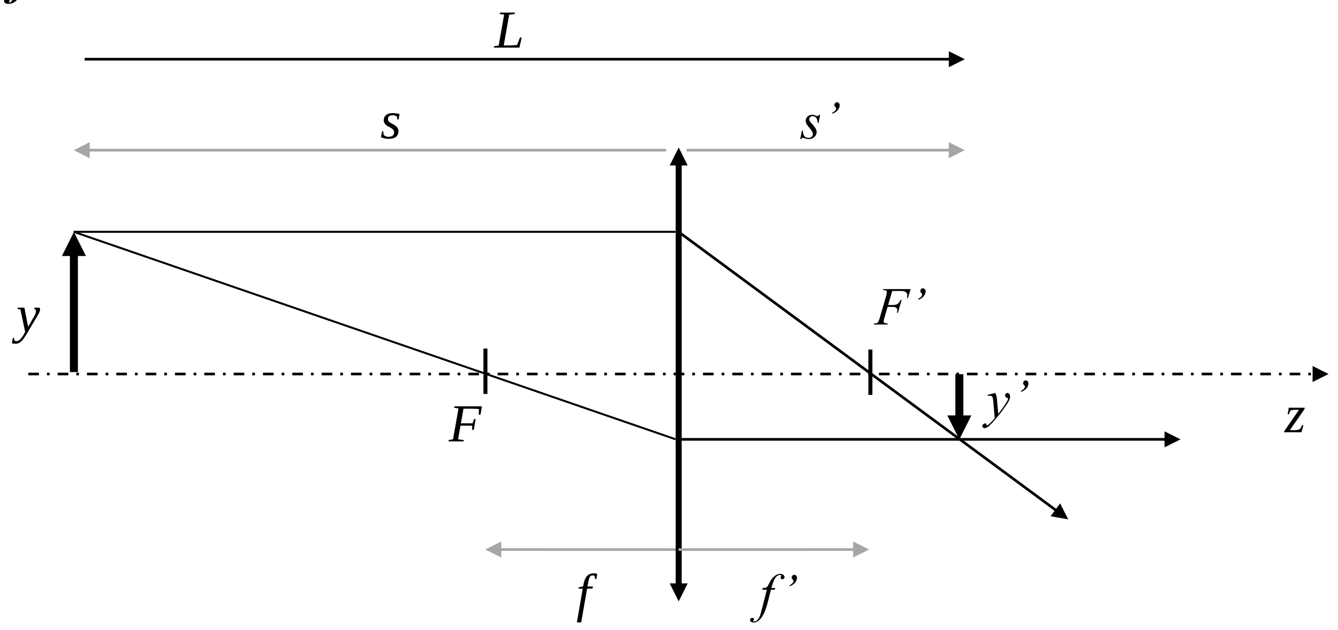
distances are counted positive if oriented in sense of a vector in positive axial direction

Radii: direction is oriented from the surface towards the center of curvature

Simple Image Construction

for a finite object distance

Positive lens: $f' > 0$



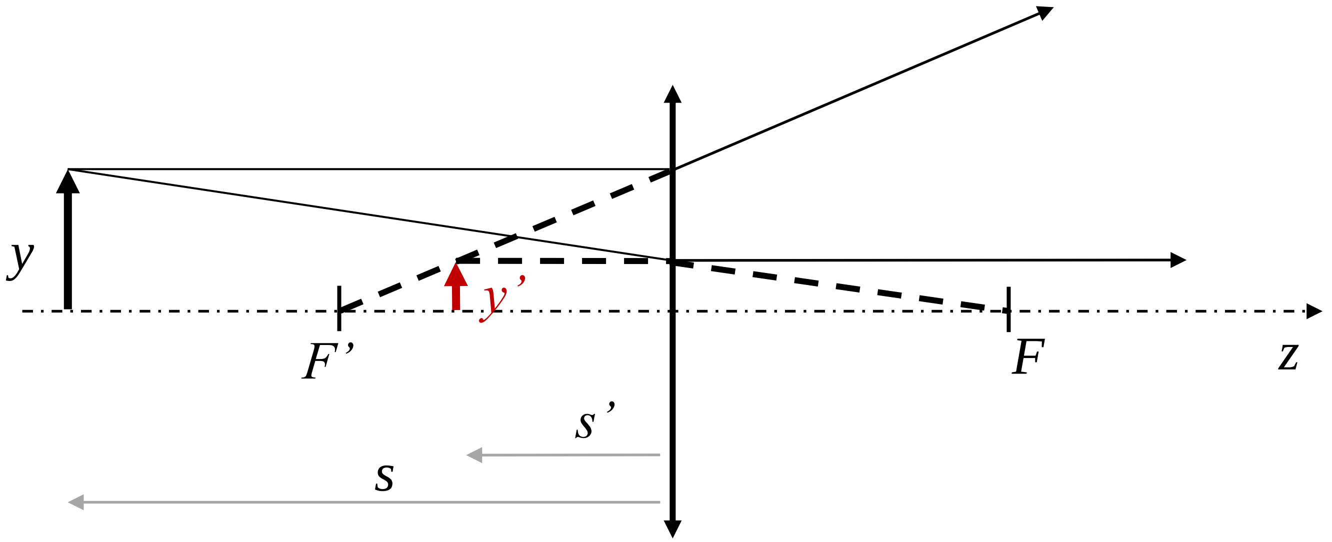
y … object height (> 0)
y' … image height (< 0)
s … object distance
s' … image distance
$L = s' - s$ … object – image distance

here: equal n before and behind the lens

Simple Image Construction

for a finite object distance

Negative lens: $f' < 0$ → reversed position of F and F'



virtual image !!

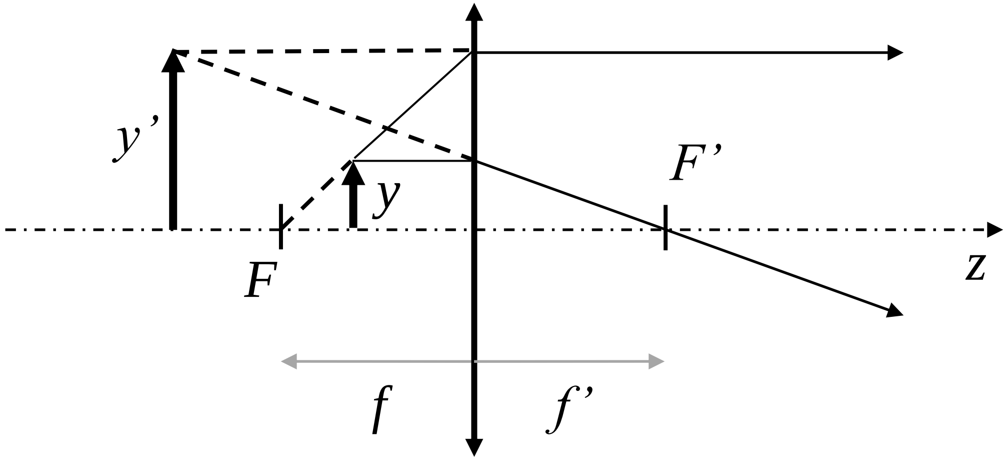
y … object height (> 0)
y'… image height (> 0)
s … object distance
s'… image distance

here: equal n before and behind the lens

Two important optical "systems"

a) Magnifying Lens

$$|s| < |f|$$

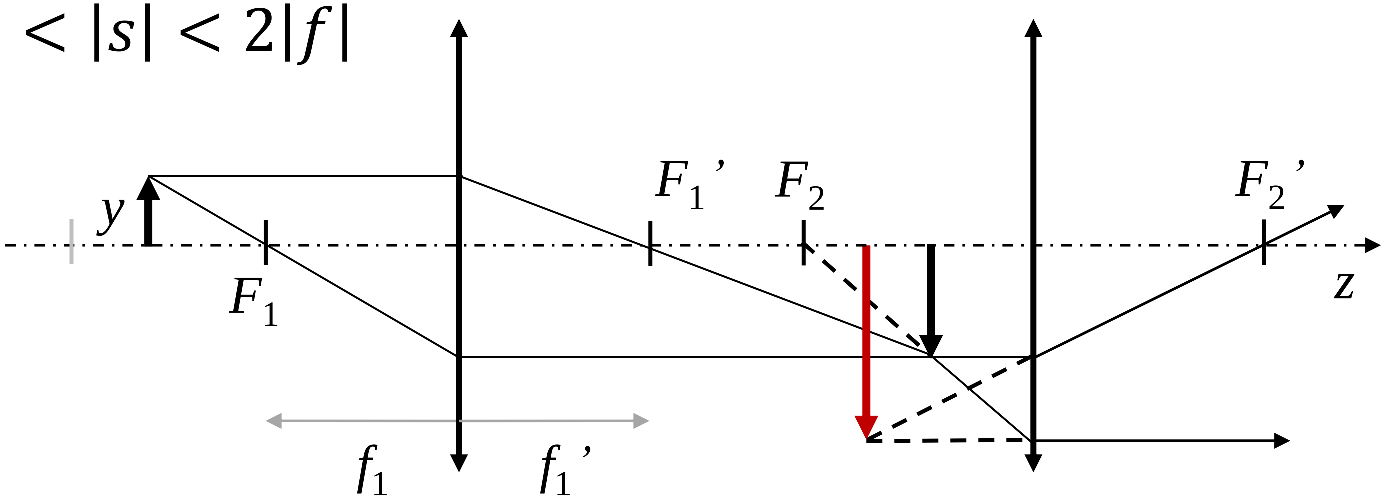


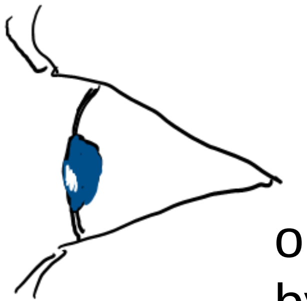
→ virtual magnified image

→ observable by eye

b) Microscope → combination of a magnified real image and a magnifying lens

$$|f| < |s| < 2|f|$$

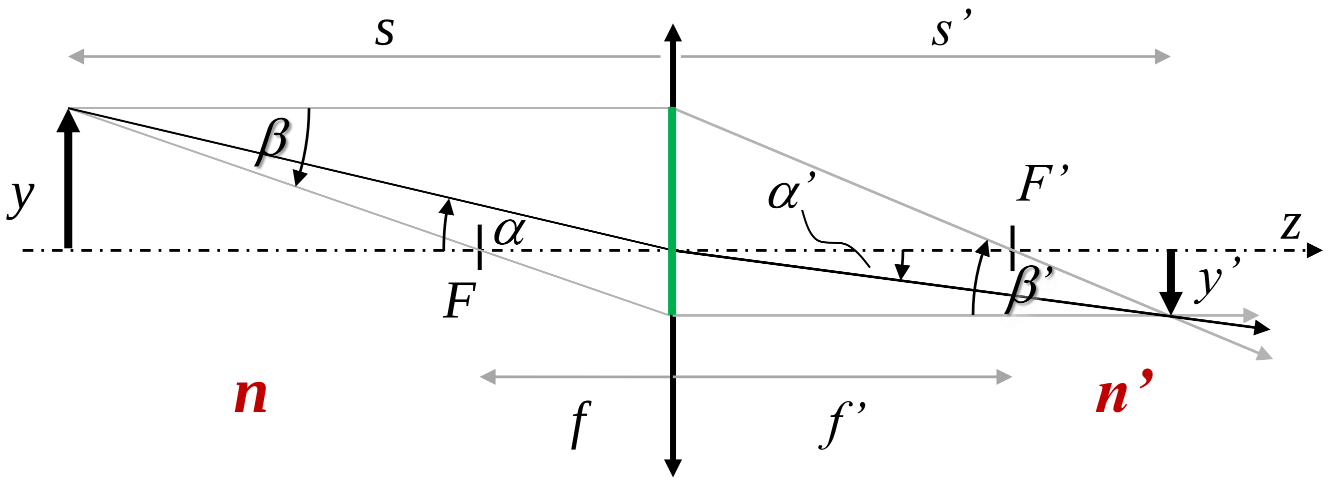




observation by eye

More general relation between *f* and *f'*

more general situation: different n in object and image space



$\Rightarrow \quad f \neq f'$

$$n \cdot \alpha = n' \cdot \alpha' \qquad (2.2)$$

$\beta = \dfrac{y - y'}{s}$ and $\beta' = \dfrac{y - y'}{s'}$

$\Rightarrow \quad s \cdot \beta = s' \cdot \beta'$

with $\beta = \dfrac{y'}{f}$; $\beta' = -\dfrac{y}{f'}$

and $\alpha = \dfrac{y}{s}$; $\alpha' = \dfrac{y'}{s'}$

$\xRightarrow{(2.2)} \boxed{\dfrac{f'}{n'} = -\dfrac{f}{n}} \qquad (2.3)$

similar consideration:

$$\frac{f'}{s'} + \frac{f}{s} = 1 \qquad (2.4)$$

imaging equation

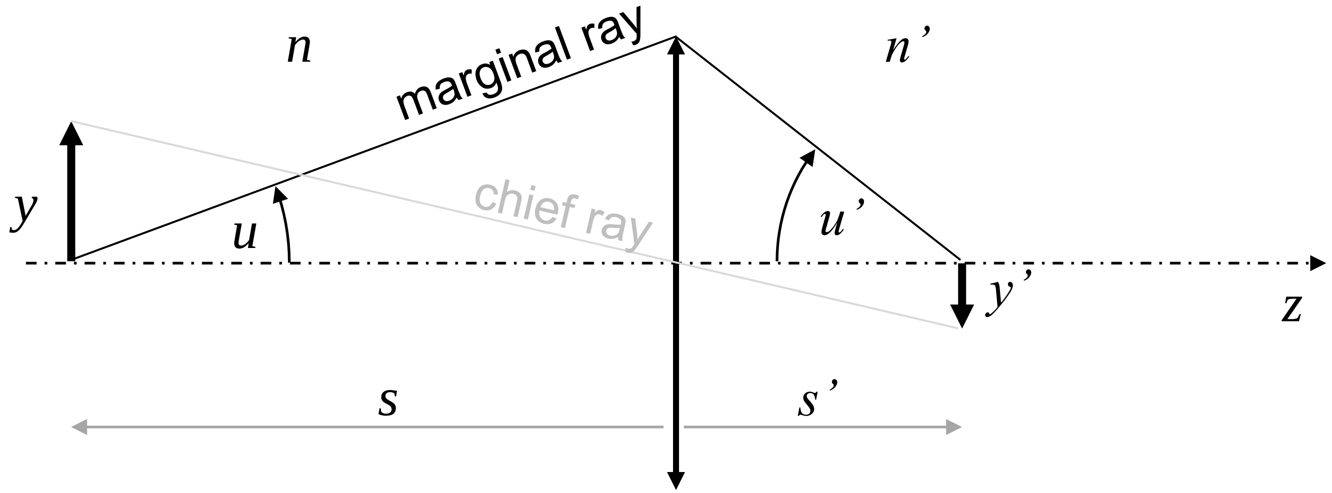
or with $n = n'$

$$\frac{1}{s'} - \frac{1}{s} = \frac{1}{f'} \qquad (2.5)$$

"lens makers equation"

Magnification

Definition: $$\boxed{m = \frac{y'}{y}} \qquad (2.6)$$



from sketch: $$m = \frac{n \cdot s'}{n' \cdot s}$$

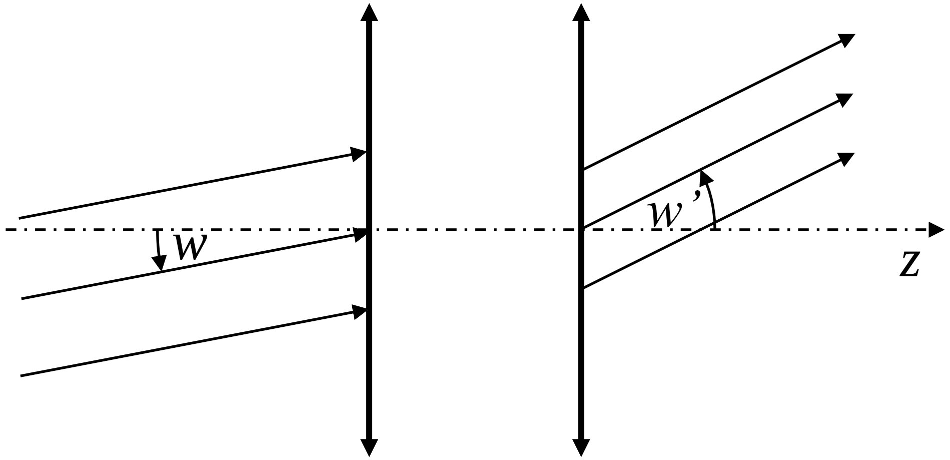
with aperture angles *u, u'*: $$m = \frac{n \cdot \tan u}{n' \cdot \tan u'} \approx \frac{n \cdot u}{n' \cdot u'}$$

Special case: object and image at <u>infinity</u> (afocal system) ➡ Telescope

→ *y, y'* are not defined

→ definition of the <u>angular magnification</u>
by the chief ray angles *w, w'*

$$\Gamma = \frac{w'}{w}$$



Paraxial Imaging Equations

set of 30 equations for the calculation of imaging parameters

Quantity to be calculated	Calculation equations		
s	$s = \frac{s' f'}{f' - s'}$	$s = s' - L$	$s = \frac{s'}{m}$
	$s = -\frac{L}{s} \pm \sqrt{\frac{L^2}{4} - f' \cdot L}$	$s = \frac{(1-m) \cdot f'}{m}$	$s = \frac{L}{m-1}$
s'	$s' = \frac{s \cdot f'}{f' + s}$	$s' = s + L$	$s' = \frac{L}{m-1}$
	$s' = \frac{L}{2} \pm \sqrt{\frac{L^2}{4} - f' \cdot L}$	$s' = f' \cdot (1 - m)$	$s' = \frac{L \cdot m}{m-1}$
f'	$f' = \frac{s \cdot s'}{s - s'}$	$f' = -\frac{s \cdot (L+s)}{L}$	$f' = \frac{s \cdot m}{1-m}$
	$f' = \frac{s' \cdot (L - s')}{L}$	$f' = \frac{s'}{1-m}$	$f' = -\frac{L \cdot m}{(1-m)^2}$
L	$L = s \cdot (m - 1)$	$L = s - s'$	$L = -\frac{s^2}{s + f'}$
	$L = \frac{s'^2}{s' - f'}$	$L = \frac{s' \cdot (m-1)}{m}$	$L = f' \cdot \left(2 - m - \frac{1}{m}\right)$
m	$m = \frac{s'}{s}$	$m = \frac{f'}{s + f'}$	$m = \frac{f' - s'}{f'}$
	$m = \frac{L + s}{s}$	$m = \frac{s'}{s' - L}$	$m = 1 - \frac{L}{2f'} \pm \sqrt{\frac{L}{f'} \cdot \left(\frac{L}{4f'} - 1\right)}$

2.3 ABCD-Matrix Formalism

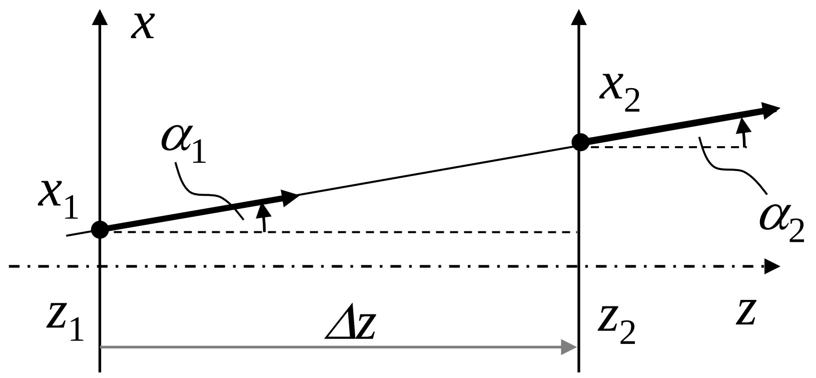
- based on a geometrical optics consideration of field propagation
- simple but powerful method for the (paraxial) treatment of complex optical systems
- ABCD-matrices can also be used for studying paraxial diffraction phenomena!

2.3.1 Derivation of the formalism

→ see blackboard

ABCD-matrices for common optical elements I

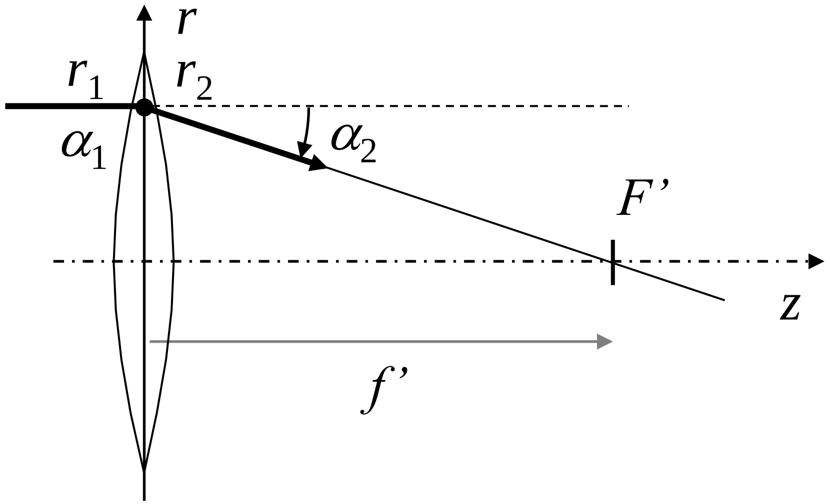
a) Free-space propagation:



$$x_2 = x_1 + \Delta z \cdot \tan\alpha_1$$
$$\alpha_2 = \alpha_1$$

$$\Rightarrow \quad M_{\Delta z} = \begin{pmatrix} 1 & \Delta z \\ 0 & 1 \end{pmatrix} \tag{2.8}$$

b) Thin lens:

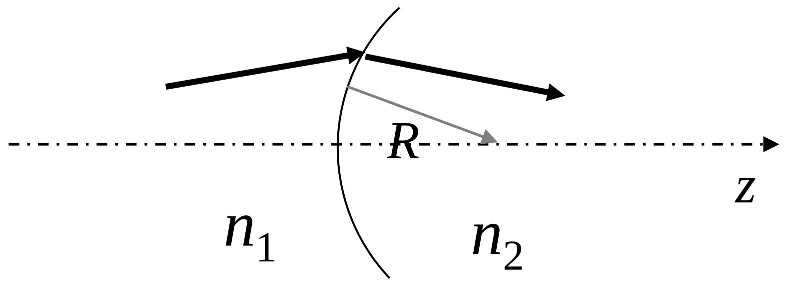


$$r_2 = r_1$$
$$\alpha_2 = -\frac{r_1}{f'} + \alpha_1$$

$$\Rightarrow \quad M_f = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f'} & 1 \end{pmatrix} \tag{2.9}$$

ABCD-matrices for common optical elements II

c) Transition at a spherical interface $n_1 \rightarrow n_2$; ROC R:

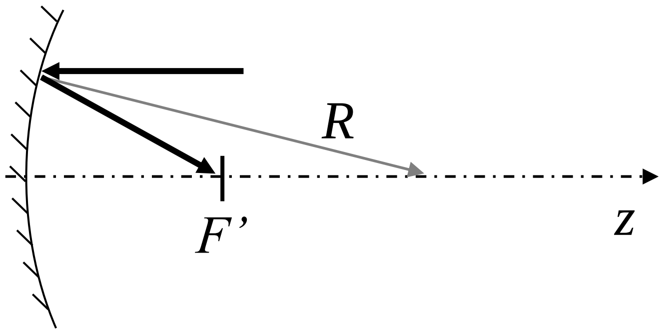


$$M_{SI} = \begin{pmatrix} 1 & 0 \\ \dfrac{n_1 - n_2}{n_2 \cdot R} & \dfrac{n_1}{n_2} \end{pmatrix} \qquad (2.10)$$

special case: $R \rightarrow \infty$ (plane interface)

$$M_{ref} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{n_1}{n_2} \end{pmatrix} \qquad (2.11)$$

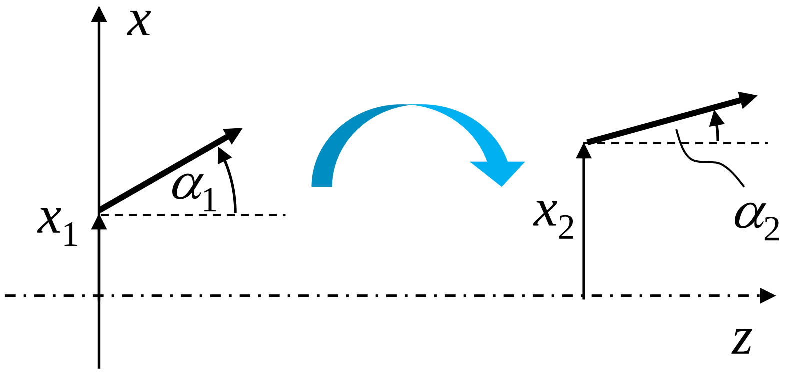
d) Curved mirror, radius $R = 2f$:



$$M_M = \begin{pmatrix} 1 & 0 \\ \dfrac{2}{R} & 1 \end{pmatrix} \qquad (2.12)$$

ABCD-matrices for common optical elements III

e) Magnification m:



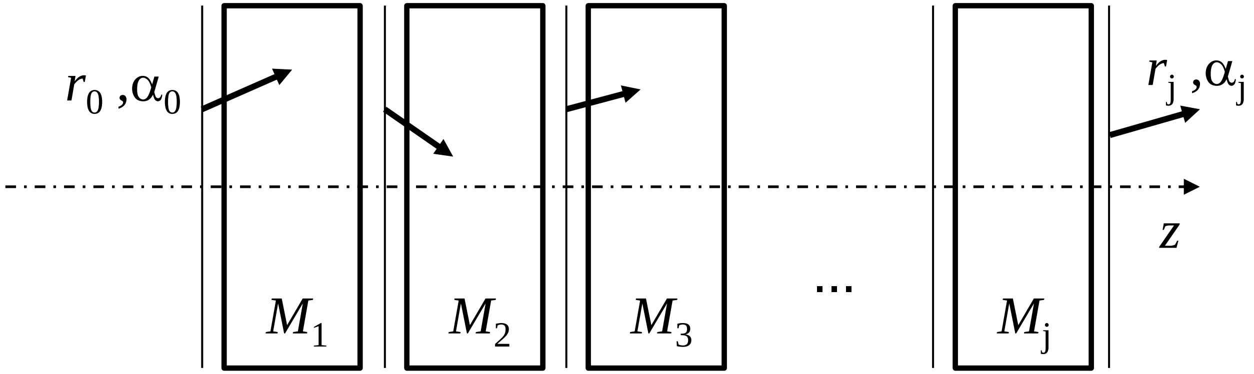
$$m = \frac{x_2}{x_1}$$

$$M_m = \begin{pmatrix} m & 0 \\ 0 & \frac{1}{m} \end{pmatrix} \qquad (2.13)$$

→ a number of other matrices for other elements can be found in literature

ABCD-matrices for systems

optical system → sequence of single optical elements



$$M_{sys} = M_j \cdot \ \dots \cdot M_3 \cdot M_2 \cdot M_1 \qquad (2.14)$$

The system matrix can be easily obtained by multiplication of the matrices of the single elements the system is composed of.

2.3.2 General properties of ABCD-matrices I

1) Determinant of M

n_1 ... refractive index in front of the system
n_2 ... refractive index behind the system

$$|M| = AD - BC = \frac{n_1}{n_2} \tag{2.15}$$

equal indices: $AD - BC = 1$

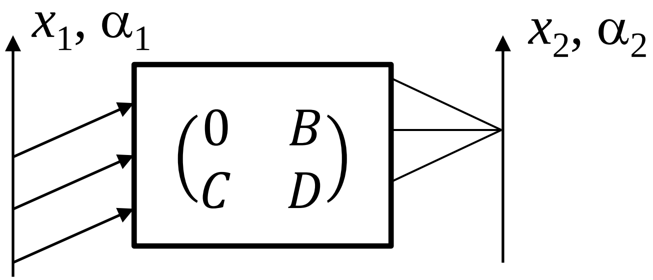
→ only 3 independent variables possible!

2) Inversion of the light direction / backwards propagation through the same system

$$M^{-1} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \tag{2.16}$$

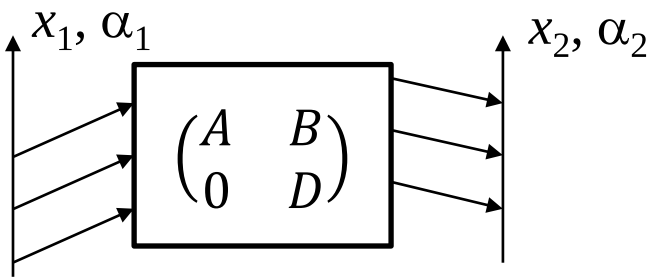
General properties if one matrix element is zero

a) $A = 0 \quad \Rightarrow \; x_2 = B\alpha_1$



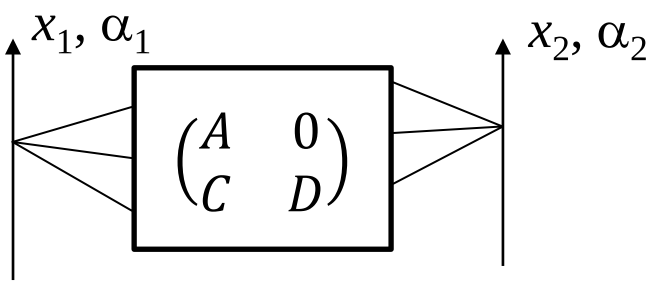
→ focusing

c) $C = 0 \quad \Rightarrow \; \alpha_2 = D\alpha_1$



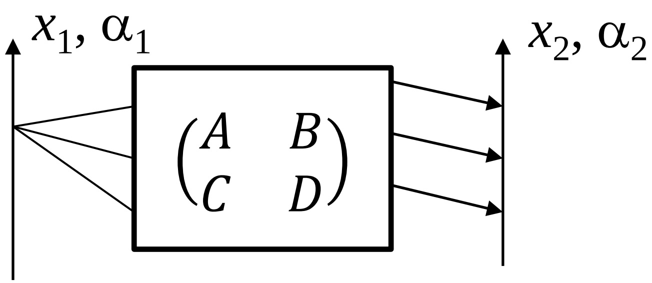
→ deflection

b) $B = 0 \quad \Rightarrow \; x_2 = Ax_1$



→ imaging

d) $D = 0 \quad \Rightarrow \; \alpha_2 = Cx_1$



→ defocusing

General properties of ABCD-matrices II

3) **Equivalent optical system**: → systems having the same ABCD-matrix

→ showing the same optical behavior

Decomposition of a given M into a “fixed” series of four basic operations:

- magnification change m
- change of refractive index $n_1 \rightarrow n_2$
- thin lens of optical power $\Phi = 1/f'$
- propagation along distance Δz

equivalent matrix:

$$M_{eq} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & \Delta z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f' \cdot n_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & \frac{1}{m} \end{pmatrix}$$

➡ 4 free variables in terms of the given ABCD-matrix:

$$\frac{n_1}{n_2} = AD - BC \qquad \frac{1}{f' \cdot n_2} = \frac{CD}{AD - BC} = -\frac{C}{m}$$

$$m = \frac{AD - BC}{D} \qquad \Delta z = \frac{B}{D} \tag{2.17}$$

Attention: if $D = 0$ → re-arrange the sequence of the 4 operations

General properties of ABCD-matrices III

4) ABCD-matrices and fields, **Collins-Integral:**

→ the 4 operations of the equivalent matrix (magnification, material transition, thin lens phase, propagation) can easily be applied on arbitrary fields $\vec{E}(x, y, z)$
→ not only a ray-optical consideration
→ propagation of wave-optical fields through complex systems possible

Other option: integrate ABCD-matrix directly into a diffraction integral

$$E_2(x_2, y_2) = \frac{i}{\lambda B} e^{-ikL} \iint E_1\,(x_1, y_1) \cdot e^{-i\frac{\pi}{\lambda B}\left(A\left(x_1^2+y_1^2\right)+D\left(x_2^2+y_2^2\right)-2\left(x_1x_2+y_1y_2\right)\right)} dx_1 dy_1 \qquad (2.18)$$

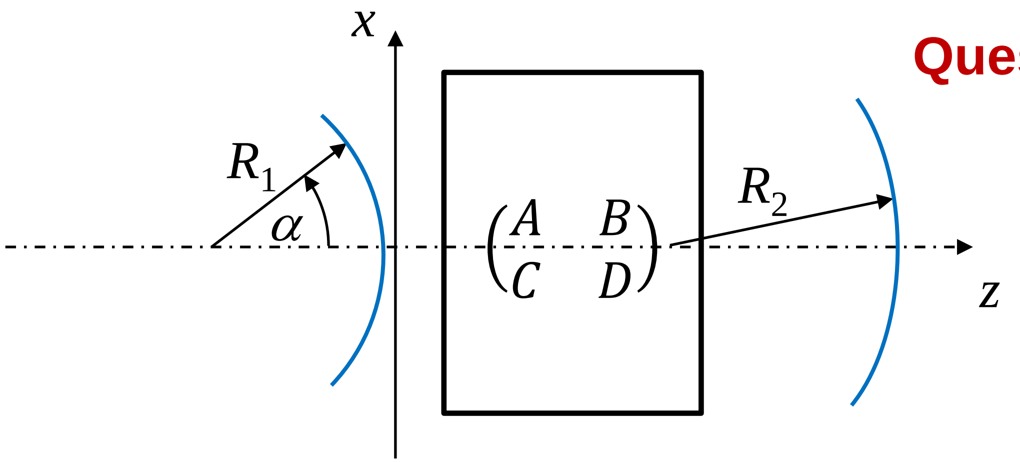
→ **Collins Integral**

E_1 ... field at the input plane of the system
E_2 ... field at the output plane of the system
L ... optical path along the optical axis

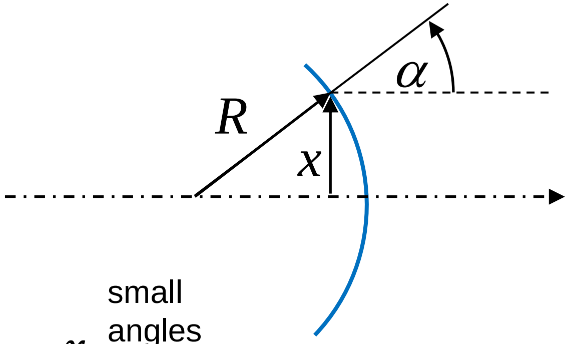
General properties of ABCD-matrices IV

5) Transformation of a spherical wave:

illumination of the system with spherical wave, ROC = R_1



Question: $R_2 = ?$



relation: $\tan\alpha = \frac{x}{R} \overset{\text{small angles}}{\approx} \alpha$

relation between x and α

$$\left.\begin{aligned} x_1 &= R_1 \cdot \alpha_1 \\ x_2 &= R_2 \cdot \alpha_2 \end{aligned}\right\} \qquad R_2 = \frac{AR_1 + B}{CR_1 + D} \qquad (2.19)$$

Example: lens with focal length f'

$$\rightarrow \frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f'}$$