4. Description of Laser Processes by Rate Equations

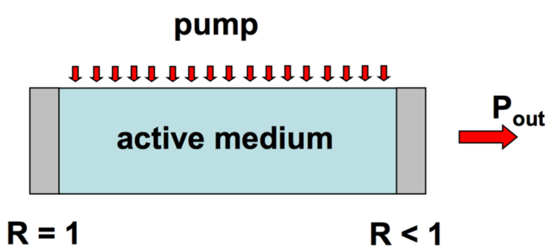
4.1 Setting up of Rate Equations

The goal of the following chapter is a quantitative description of the laser process. That description should not only include the effect of stimulated emission on the electromagnetic field but also on the inversion (i.e. saturation effects).

Each theory needs a few assumptions and approximation to be manageable:

- The description is based on the photon content of a mode. Any spatial distribution of resonator modes is neglected.
- All the signal photons possess the same frequency. This means that the fields are considered monochromatic, but the signal and pump photons still have different frequencies.
- We disregard the spatial dependence of the energy density, i.e. an average energy density is considered. This means, for instance, that the power evolution along the cavity or a standing wave pattern is neglected.
- The laser active medium is homogeneously distributed within the resonator.
- The pump process is position independent.
- Hence, we describe the state of the active medium via position-independent population densities of the levels that participate in the laser process.

The following picture illustrates that simplified laser model.

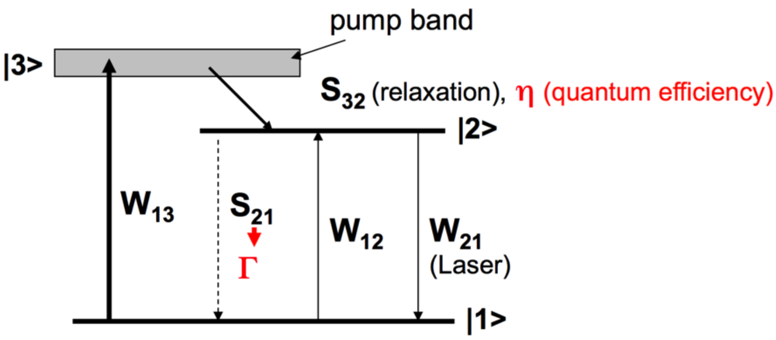


Additionally, we will use the average photon density p instead of the average energy density ρ. These parameters are related to each other by:

$$\rho = p \cdot h\nu \tag{4.1}$$

<u>Rate equations of the three-level system</u>

Let's start with the derivation of the rate equations for a 3-level system. The following picture summarizes all the relevant transitions that we have to consider.



Herein, W_{13} describes the transition probability per unit time for one particle to make the transition from level 1 to level 3 through absorption (of pump photons). Hence, the number of pump processes per unit time and unit volume is given by $W_{13} \cdot N_1$. η defines the quantum efficiency, which is the probability of ending up in level 2 after an excitation into level 3 (through a pump process). As discussed in chapter 3, the lifetime of level 3 has to be short, i.e. a relaxation out of this level (and ideally into level 2) should immediately follow the excitation. This implies that level 3 will, ideally, be unpopulated.

In order to describe the change of population density in level 2, we need to consider all the possible processes which populate and depopulate this level. These are the pump, the spontaneous emission and the induced (a.k.a. stimulated) emission. The pump process (transition from 1 to 2 via 3) occurs at a rate given by:

$$\left.\frac{dN_2}{dt}\right|_{pump\ process} = \eta \cdot W_{13} \cdot N_1 = W_P \cdot N_1$$

with

$$W_P = \eta \cdot W_{13},$$

which is *the pump rate, i.e. the probability per unit time for the excitation of particles from level 1 to level 2 via level 3*.

Additionally, spontaneous emission depopulates level 2 with the rate:

$$\left.\frac{dN_2}{dt}\right|_{spontan} = -\Gamma \cdot N_2 \qquad \text{with} \qquad G = \frac{1}{t_2}.$$

In the equations above τ_2 is the average lifetime of level 2.

Besides, using equations (4.1), (2.11a) and (3.3), it is possible to write the change of population density in level 2 through induced emission and absorption processes (between level 1 and 2) as:

$$\left.\frac{dN_2}{dt}\right|_{induced} = \sigma \cdot c \cdot p \cdot (N_1 - N_2) \tag{4.2a}$$

Finally, adding up all the contributions discussed above, it is possible to *express the overall temporal behavior of the population in level 2* as:

$$\frac{dN_2}{dt} = \sigma \cdot c \cdot p \cdot (N_1 - N_2) - \Gamma \cdot N_2 + W_P \cdot N_1$$

In analogy we also need to understand the temporal behavior of population in level 1. However, *since we consider a closed system of particles (where $N_3 = 0$ is assumed) the relevant processes in level 1 have the opposite rates than in level 2*. Thus, the rate of the pump process is:

$$\left.\frac{dN_1}{dt}\right|_{pump\ process} = -\left.\frac{dN_2}{dt}\right|_{pump\ process}.$$

The rate of the spontaneous emission from level 2 to level 1 is:

$$\left.\frac{dN_1}{dt}\right|_{spontan} = -\left.\frac{dN_2}{dt}\right|_{spontan}.$$

And the rate of induced emission is:

$$\left.\frac{dN_1}{dt}\right|_{induced} = -\left.\frac{dN_2}{dt}\right|_{induced}.$$

Hence, *the overall temporal behavior of the population in level 1* is given by:

$$\frac{dN_1}{dt} = -\sigma \cdot c \cdot p \cdot (N_1 - N_2) + \Gamma \cdot N_2 - W_P \cdot N_1 . \tag{4.2 b}$$

From lecture 1 we know that *each absorption process annihilates one photon and that each emission process creates one photon*. Hence, the rate of change in the photon density due to the induced processes (emission and absorption) can be written as:

$$\left.\frac{dp}{dt}\right|_{induced} = -\left.\frac{dN_2}{dt}\right|_{induced} = -\sigma \cdot c \cdot p \cdot (N_1 - N_2)$$

On the other hand, the leakage through the out-coupling mirror together with any other losses will cause a decay of the photon density with time according to:

$$\left.\frac{dp}{dt}\right|_{losses} = -\frac{p}{\tau_{ph}}$$

where τ_{ph} is a characteristic time known as the *photon lifetime* of the resonator.

In addition, spontaneous emission adds photons to the radiation field. However, the *spontaneous emission is a small noise term* (denoted with S) on top of the coherent signal in the cavity. Nevertheless, *spontaneous emission is important in a laser since it is the initial photon density that triggers the laser oscillation*.

The overall change of photon density (adding up all three contributions discussed above) is given by:

$$\frac{dp}{dt} = \left\{\sigma \cdot c \cdot (N_2 - N_1) - \frac{1}{\tau_{ph}}\right\} \cdot p + S \tag{4.3}$$

Equations (4.2a), (4.2b) and (4.3) constitute a system of coupled differential equations for the unknown quantities p, N_2 and N_1. The structure of these equations suggests introducing new variables:

- The *total particle density* (i.e. the doping concentration), defined as:

  $n_{tot} = N_2 + N_1$ (4.4a)

- and *the inversion density*:

  $n = N_2 - N_1$ (4.4b)

Additionally, n_{tot} has to fulfill the principle of particle conservation in a closed system, i.e.:

$$\frac{dn_{tot}}{dt} = 0 \quad . \tag{4.5}$$

With these definitions we can derive the following relations for N_1 and N_2:

$$N_1 = (n_{tot} - n)/2 \qquad\qquad N_2 = (n_{tot} + n)/2$$

Subtracting equation (4.2b) from (4.2a) and using the relations above, the rate equations of a three-level-system are given by:

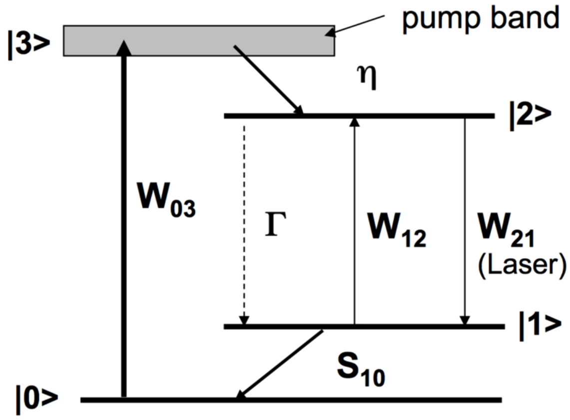
- for the inversion density:

$$\frac{dn}{dt} = -2 \cdot \sigma \cdot c \cdot p \cdot n - \Gamma \cdot (n + n_{tot}) + W_P \cdot (n_{tot} - n) \tag{4.5b}$$

- for the photon density:

$$\frac{dp}{dt} = \left(\sigma \cdot c \cdot n - \frac{1}{\tau_{ph}} \right) \cdot p + S \tag{4.5c}$$

However, before we move on to solve these coupled differential equations, we should derive the rate equations of the four-level system. This is illustrated in the following figure:



The temporal behavior of the population density in level 2 (now pumping from level 0) is, in analogy to (4.2 a), given by:

$$\frac{dN_2}{dt} = \sigma \cdot c \cdot p \cdot (N_1 - N_2) - \Gamma \cdot N_2 + W_P \cdot N_0 \tag{4.6 a}$$

Now we simplify this treatment by assuming that $N_3 = 0$ (due to the fast relaxation from level 3 to level 2). In addition, a fast relaxation from the lower laser level 1 back to the ground state 0 is also assumed. This, in turn, leads to a negligible population in level 1, i.e. $N_1 = 0$.

The rate of change of the photon density due to transitions between level 2 and level 1 is still given by equation (4.5 c), i.e. it is identical to that in a 3-level laser:

$$\frac{dp}{dt} = \left(\sigma \cdot c \cdot n - \frac{1}{\tau_{ph}} \right) \cdot p + S \tag{4.6 b}$$

In the 4-level system the inversion density is given by:

$n = N_2 - N_1 \approx N_2$,

i.e. the inversion density is approximately equal to the population density of the upper laser level.

On the other hand, the total population density is:

$n_{tot} = N_0 + N_1 + N_2 + N_3 \approx N_0 + N_2$.

This means that the overall particle density is roughly given by the ground state population plus the population density of the upper laser level. Hence, the population in the ground state is:

$N_0 = n_{tot} - N_1 - N_2 - N_3 \approx n_{tot} - n$

i.e. it is given by the overall particle density minus the inversion density.

With all this knowledge, equation (4.6 a) can be rewritten to describe the temporal evolution of the inversion density:

$$\frac{dn}{dt} = -\sigma \cdot c \cdot p \cdot n - \Gamma \cdot n + W_P \cdot (n_{tot} - n) \qquad (4.6\ c)$$

As a matter of fact, equations (4.6b) and (4.6c) are very similar to the rate equations of the 3-level system; so much so that we can summarize them all into one generalized expression:

Generalized rate equations of three- and four-level-system

- for the inversion density:

$$\frac{dn}{dt} = -\gamma \cdot \sigma \cdot c \cdot p \cdot n - \Gamma \cdot \{n + n_{tot}(\gamma - 1)\} + W_P \cdot (n_{tot} - n) \qquad (4.7a)$$

with

$$\gamma = \begin{cases} 2 & for \quad 3-level laser \\ 1 & for \quad 4-level laser \end{cases}$$

- for the photon density:

$$\frac{dp}{dt} = \left(\sigma \cdot c \cdot n - \frac{1}{\tau_{ph}} \right) \cdot p + S \qquad (4.7b)$$

4.2 Solution of the Rate Equations

Equations (4.7a) and (4.7b) are a pair of coupled differential equations which determine the inversion density n and the photon density p in the system. These equations are non-linear as both contain the product n·p. Analytical solutions are not known and, hence, numerical approaches are needed to solve the rate equations. Nevertheless, even without numerically solving the equations, it is still possible to discuss several special cases. At first, we will try to draw some general conclusions, e.g. calculate the threshold condition for laser oscillation.

For a build-up of signal intensity from noise or in order to maintain a certain power level constant, the change of the photon density with time should be permanently above or equal to zero, i.e.

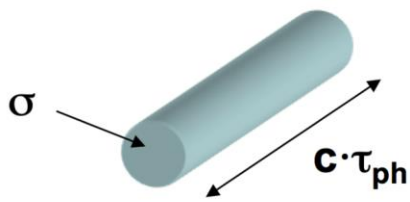
$$\frac{dp}{dt} \geq 0 \quad .$$

We can obtain the threshold condition for the inversion density required to achieve laser oscillation from (4.7 b), neglecting spontaneous emission, as:

$$n \geq n_{th} = \left(\sigma \cdot c \cdot \tau_{ph}\right)^{-1} \tag{4.8 a}$$

As can be seen in (4.8 a), in order to get laser oscillation it is not sufficient to just create inversion, but the inversion level has to be larger than a threshold value! Such threshold value depends on the optical transition (σ) and on the properties of the resonator (τ_{ph}).

There is an interesting interpretation of equation (4.8 a) which provides some insight in the physical meaning of this threshold condition. In order to understand this we need to remember that the cross sections represent a probability of a particular light-matter interaction projected to an area (i.e. the cross-sections represent an area that is proportional to the probability of the interaction). In (4.8 a) the term $c \cdot \tau_{ph}$ represents the distance that a photon travels during its average lifetime in the cavity. Hereby it is assumed that, during this propagation, each photon interacts with all atoms located in a cylinder with length $c \cdot \tau_{ph}$ and diameter σ, i.e. this is a kind of interaction or collision cylinder.



Thus, equation (4.8 a) can be interpreted as the overall inversion in such a cylinder (i.e. inversion density times the cylinder volume), which has to be at least 1, as seen when rewriting (4.8 a) in the following form:

$$n \cdot \sigma \cdot c \cdot \tau_{ph} \geq 1 \quad .$$

This means that each photon has to reproduce itself during the propagation through the collision cylinder (i.e. during its average lifetime). In other words, if we want to build-up of intensity from noise or maintain the average power level constant, at least one new photon has to be effectively created during the lifetime of each photon in the system.

Typically in most practical cases, the cross section σ and the photon lifetime τ_{ph} are known quantities and the threshold inversion density n_{th} has to be calculated from them. Some other times, though, it is the achievable inversion density being given and the conditions related to the cavity design (τ_{ph}) have to be obtained. In this context, solving (4.8 a) for the photon lifetime:

$$\tau_{ph} \geq (\sigma \cdot c \cdot n)^{-1} , \tag{4.8 c}$$

we can learn that laser oscillation is possible even with arbitrarily small inversion levels if the resonator has sufficiently low losses (i.e. long lifetime of the photons in the resonator).

4.2.1 Stationary Laser Operation

Stationary laser operation is the simplest special case for the solution of (4.7 a) and (4.7 b). Here we assume that the creation of inversion occurs with a constant pump rate and until an equilibrium state is reached. Such an equilibrium state implies that there is no change of the inversion density and of the photon density with time:

$$\frac{dn}{dt} = 0 \tag{4.9a}$$

$$\frac{dp}{dt} = 0 . \tag{4.9b}$$

Using the expression (4.9 a) in equation (4.7 a), it is possible to derive the inversion density in stationary regime as:

$$n = n_{tot} \cdot \{W_P - (\gamma - 1) \cdot \Gamma\} \cdot \{\gamma \cdot \sigma \cdot c \cdot p + W_P + \Gamma\}^{-1} . \tag{4.10}$$

Since below the threshold condition there is no build-up of photon density (i.e. p = 0), equation (4.10) can be simplified for this case in which it will describe the inversion below threshold n_0 by:

$$n_0 = n(p = 0) = n_{tot} \cdot \{W_P - (\gamma - 1) \cdot \Gamma\} \cdot \{W_P + \Gamma\}^{-1} . \tag{4.11}$$

Which conclusions can be drawn from equation (4.11)?
Assuming a 3-level laser (γ = 2), it can be seen that a minimum pump rate $W_{P,min}$ is required to obtain population inversion (i.e. $n_0 > 0$), where:

$$W_{P\min} = \Gamma = \frac{1}{\tau_2} . \tag{4.12 a}$$

As we already know, a metastable level is helpful for this (Γ small, τ_2 large). In contrast to that, in a 4-level system ($\gamma = 1$) inversion is always given as soon as a pump mechanism is at work ($W_P > 0$). This allows using in practice small pump rates ($W_P << \Gamma$). Within this approximation (i.e. small pump rates) the inversion below threshold n_0 in a 4-level system is given by:

$$n_0 \cong n_{tot} \cdot \frac{W_P}{\Gamma} \quad . \qquad (4.12\ b)$$

At this point we should remember that the definition of the gain coefficient was:

$$g = n \cdot \sigma \quad .$$

Hereby, if the photon density is small ($p \sim 0$), then we will consider a small-signal gain coefficient g_0 which can be written (using 4.11) as:

$$g_0 = n_0 \cdot \sigma = n_{tot} \cdot \sigma \cdot \{W_P - (\gamma - 1) \cdot \Gamma\} \cdot \{W_P + \Gamma\}^{-1} \quad . \qquad (4.13)$$

However, as soon as the photon density increases we have to use equation (4.10) to describe the inversion density above the threshold. In this context it appears helpful to express n as a function of n_0, i.e.

$$n = n_0 \cdot \left\{1 + \gamma \cdot \sigma \cdot c \cdot p \cdot \left(W_p + \Gamma\right)^{-1}\right\}^{-1} \qquad (4.14\ a)$$

Now we can rewrite equation (4.14 a) as:

$$n = n_0 \cdot \left(1 + \frac{p}{p_{Sat}}\right)^{-1} \qquad (4.14\ b)$$

by defining the saturation photon density p_{Sat} as:

$$p_{Sat} = \frac{W_P + \Gamma}{\gamma \cdot \sigma \cdot c} \quad . \qquad (4.15)$$

With that, the gain coefficient (for arbitrarily large photon densities) is given by:

$$g = g_0 \cdot \left(1 + \frac{p}{p_{Sat}}\right)^{-1} \quad . \qquad (4.16\ a)$$

At this point we can use the known relation between the photon density and the intensity

$$I = c \cdot h \cdot \nu \cdot p$$

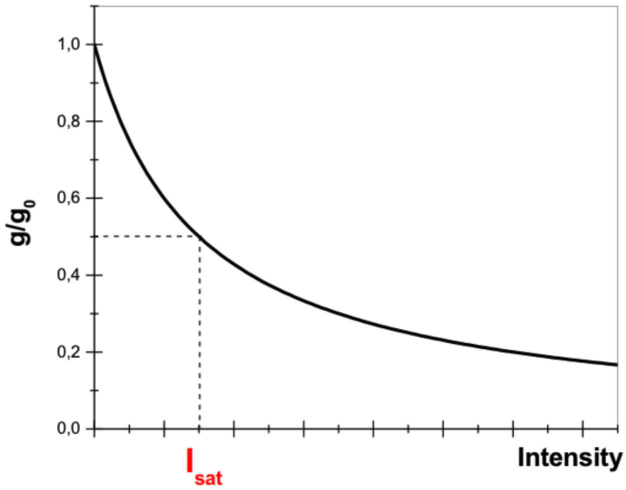
to write (4.14 a) as:

$$g = g_0 \cdot \left(1 + \frac{I}{I_{Sat}}\right)^{-1} \quad , \qquad (4.16\ b)$$

and, consequently, the saturation intensity I_{Sat} as:

$$I_{Sat} = \frac{(W_P + \Gamma) \cdot h \cdot \nu}{\gamma \cdot \sigma} \quad . \qquad (4.17)$$

The gain coefficient as a function of intensity is plotted in the figure below. It can be seen that the gain coefficient is reduced to 50% of the small signal gain coefficient at an intensity equal to the saturation intensity I_{Sat}.



Thus, by starting from the rate equations (4.7) and by assuming a stationary case, we have just learned that the active medium shows a saturation behavior in its gain.

Assuming again a 3-level-system, a pump rate W_P larger than the spontaneous emission rate Γ is required to reach inversion, according to equation (4.12 a). However, according to (4.17), an increase of W_p leads to an increase of I_{Sat}. Thus, the more pump power is brought into the system, the higher the intensity needed to reduce the gain.

In a 4-level-system, on the other hand, the pump rates W_P are typically significantly smaller than Γ. Within this approximation I_{Sat} becomes independent of the pump rate W_P. Thus, the saturation intensity and the saturation photon density can be simplified to:

$$p_{Sat} = \frac{\Gamma}{\sigma \cdot c} \qquad (4.18)$$

and

$$I_{Sat} = \frac{\Gamma \cdot h \cdot \nu}{\sigma} \quad .$$

By reminding that $\sigma \cdot c \cdot p$ describes the transition probability from level 2 to level 1 due to induced emission processes and that Γ is the transition probability from level 2 to level 1 due to spontaneous emission processes, the interpretation of (4.18) is the following:

If $p = p_{Sat}$, then both processes (i.e. induced and spontaneous emission) possess an equal probability. However, if $p << p_{Sat}$, then the depopulation of the upper laser level is dominated by spontaneous emission. On the other hand, if $p >> p_{Sat}$, then the depopulation of the upper laser level is dominated by induced emission processes and, consequently, the particles stay in this level for a time significantly shorter than the average lifetime of the upper laser level (τ_2).

Which inversion level is present in the active medium in the stationary laser regime?

In order to find this out, we will use equation (4.7 b) neglecting the noise term S and assuming a non-zero photon density $p \neq 0$.

$$\frac{dp}{dt} = 0 = \left(\sigma \cdot c \cdot n - \frac{1}{\tau_{ph}} \right) \cdot p$$

Now we can find the inversion density in the stationary regime by solving the expression above:

$$n = \left(\sigma \cdot c \cdot \tau_{ph} \right)^{-1} = n_{th}$$

Hence, in the stationary regime (equilibrium) no inversion larger than the threshold inversion is present in the active medium. This means that, if the pump rate is increased, then the photon density increases and, thereby, keeps the inversion at the threshold value (i.e. the inversion is clamped at n_{th}).

Which is the photon density in the stationary regime?

To find this out we use equation (4.10), i.e. the inversion density in the stationary regime, and replace n by n_{th}. Then by solving the resulting expression for p, the photon density in stationary regime becomes:

$$p = \left\{ W_P \cdot \left(n_{tot} - n_{th} \right) - \Gamma \cdot \left\{ \left(\gamma - 1 \right) \cdot n_{tot} + n_{th} \right\} \right\} \cdot \frac{\tau_{ph}}{\gamma} \tag{4.19 a}$$

Which can be simplified to

$$p = \left(n_{tot} - n_{th} \right) \left(W_P - W_{th} \right) \cdot \frac{\tau_{ph}}{\gamma}$$

by introducing the definition of pump rate at the threshold as:

$$W_{th} = \Gamma \cdot \{(\gamma - 1) \cdot n_{tot} + n_{th}\} \cdot (n_{tot} - n_{th})^{-1}$$

Further discussions in this chapter will be restricted to 4-level lasers (γ = 1). It is worth mentioning, though, that the general statements that will be extracted are also valid for 3-level lasers, but the equations are easier to handle for a 4-level system.

Therefore, for a 4-level system, we can assume:

$n_{th} << n_{tot}$ and $W_P << \Gamma$

With these approximations the equations (4.19 a) and (4.19 b) can be reduced to:

photon density:

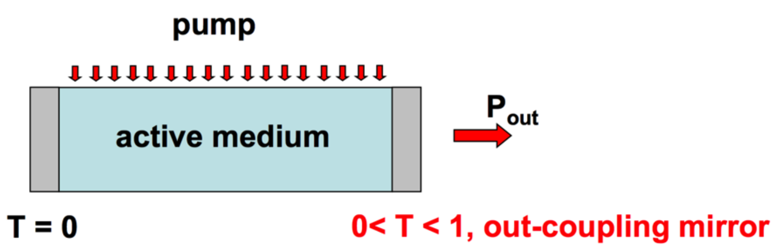
$$p = n_{tot} \cdot (W_P - W_{th}) \cdot \tau_{ph} \tag{4.20 a}$$

threshold pump rate:

$$W_{th} = \Gamma \cdot \frac{n_{th}}{n_{tot}} \tag{4.20 b}$$

Unfortunately, the photon density and the pump rate are not quantities that are directly accessible or measurable. However, the photon density can be linked to the laser output power and the pump rate can be related to the pump power. The advantage of this is that both the pump and laser output power can be directly measured. Thus, in the following these relations are derived.

We consider a linear cavity, such as the one shown in the schematic below:



Hereby we assume that the out-coupling of photons out of the resonator is the only loss in the cavity, which is described by:

$$\left.\frac{dp}{dt}\right|_{out-coupling} = -\frac{p}{\tau_{ph}}$$

Restricting the consideration to small out-coupling ratios (i.e. T is small) the decrease of photon density per round-trip time τ_R is then approximately given by:

$$\left.\frac{dp}{dt}\right|_{out-coupling} = -\frac{p \cdot T}{\tau_R}$$

At this point it is worth stressing that: *In the stationary regime the out-coupling loss is exactly compensated by the laser process (gain).*

The number of photons in the cavity is given by the photon density p times the cavity volume V. Additionally, each photon possesses an energy $h \cdot \nu_{21}$ and, hence, the intra-cavity stored energy (in the form of a radiation field) is $E_{Resonator} = p \cdot V \cdot h \cdot \nu_{21}$.

The output power is the energy loss in the cavity per unit time due to a transmission through the out-coupling mirror, i.e.

$$P_{out} = -\left.\frac{dE}{dt}\right|_{out-coupling} = -V \cdot h \cdot \nu_{21} \cdot \left.\frac{dp}{dt}\right|_{out-coupling}$$
$$= V \cdot h \cdot \nu_{21} \cdot p \cdot \frac{T}{\tau_R} \qquad (4.21\text{ a})$$

with equation (4.20 a) the output power in stationary regime (4-level) can be written as:

$$P_{out} = V \cdot h \cdot \nu_{21} \cdot T \cdot n_{tot} \cdot (W_P - W_{th}) \cdot \frac{\tau_{ph}}{\tau_R} \qquad (4.21\text{ b})$$

In addition, we use the relation between the photon lifetime, the cavity round-trip time and the losses (including transmission losses T as well as any other losses L). Note that the losses L describe the probability that a photon is lost due to scattering, absorption, etc. per round trip. Thus, the relation between the photon lifetime and the cavity round-trip time is:

$$\frac{\tau_{ph}}{\tau_R} = (T + L)^{-1} \qquad (4.21\text{ c})$$

Now we can finally establish a link between the pump power P_P and the pump rate W_P. We know that the relation between the pump rate W_P, the probability per unit time for one particle to make the transition 0 -> 3 by absorption W_{03} and the quantum efficiency η (probability to end up in level 2 after excitation in 3) is

$$W_P = \eta \cdot W_{03}$$

The corresponding photon energy is $h \cdot \nu_{03}$ and the number of active particles in the resonator volume is given by $n_{tot} \cdot V$. On the other hand, the pump power is defined as the energy load per unit time. Using the known approximation $N_0 >> N_2$ and, hence $n_{tot} \approx N_0$, the pump power can be written as:

$$P_P = n_{tot} \cdot V \cdot h \cdot \nu_{03} \cdot \frac{W_P}{\eta} \tag{4.22a}$$

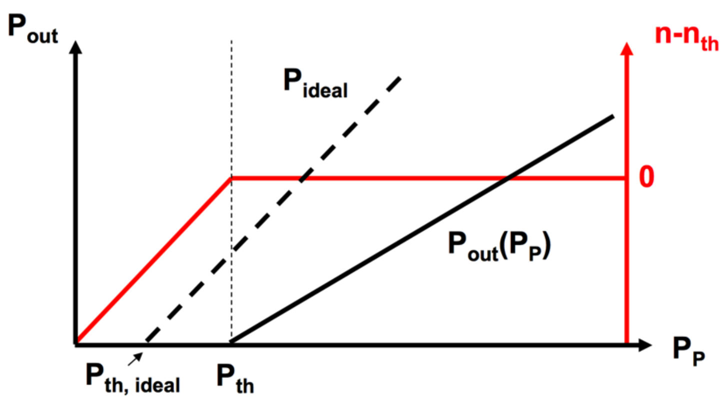
In analogy, the threshold pump power P_{th} is related to the threshold pump rate W_{th} by:

$$P_{th} = n_{tot} \cdot V \cdot h \cdot \nu_{03} \cdot \frac{W_{th}}{\eta} \tag{4.22b}$$

Using equations (4.21 b), (4.21 c), (4.22 a) and (4.22 b), the output power (at the signal wavelength) is given by:

$$P_{out} = \eta \cdot \frac{\nu_{21}}{\nu_{03}} \cdot T \cdot (T + L)^{-1} \cdot (P_P - P_{th}) \tag{4.22c}$$

Hence, the output power increases linearly with the pump power above the threshold pump power. This behavior is illustrated in the following figure.



The relative increase of the output power with respect to the pump power is referred to as slope efficiency:

$$\sigma_S = \eta \cdot \frac{\nu_{21}}{\nu_{03}} \cdot T \cdot (T + L)^{-1}$$

which cannot be larger than the slope efficiency of an ideal loss-less lasers (L = 0):

$$\sigma_S^{ideal} = \eta \cdot \frac{\nu_{21}}{\nu_{03}}$$

The ratio of signal frequency to pump frequency ν_{21}/ν_{03} is referred to as Stokes-shift and the quantity $(1- \nu_{21}/\nu_{03})$ as quantum defect, which quantifies the relative loss of photon energy due to the laser process (since an emitted signal photon has a lower energy, i.e. longer wavelength, than a pump photon).As an example: Ytterbium doped glass lasers are pumped at 980nm and emit at 1030nm, consequently, the quantum defect is about 5%.

4.2.2 Time-dependent behavior

We have considered the solution of rate equations in the stationary regime (i.e. time independent) so far, i.e. we have considered that the inversion and photon density are in equilibrium. Now we investigate how the system approaches this stationary case when deviations from the steady state (i.e. equilibrium situation) exist. In reality, these deviations from the equilibrium are unavoidable. Some examples of such deviations are the switch-on process of the laser, fluctuations in the pump process or mechanical perturbations of the resonator.

For the time being, let's assume that there are only small deviations from the steady-state situation. The rate equations for the change of inversion and photon density with time (assuming a 4-level system: $\gamma = 1$ and $n \ll n_{tot}$) are:

Equation (4.7a)

$$\frac{dn}{dt} = -\gamma \cdot \sigma \cdot c \cdot p \cdot n - \Gamma \cdot \{n + n_{tot}(\gamma - 1)\} + W_P \cdot (n_{tot} - n),$$

which can be simplified to

$$\frac{dn}{dt} = -\sigma \cdot c \cdot p \cdot n - \Gamma \cdot n + W_P \cdot n_{tot} . \qquad (4.24\ a)$$

And equation (4.7b)

$$\frac{dp}{dt} = \left(\sigma \cdot c \cdot n - \frac{1}{\tau_{ph}} \right) \cdot p + S$$

whic, using the equation for the threshold condition:

$$\sigma \cdot c = (n_{th} \cdot \tau_{ph})^{-1}$$

and neglecting the noise term S, can be written as:

$$\frac{dp}{dt} = \frac{p \cdot \left(\frac{n}{n_{th}} - 1 \right)}{\tau_{ph}} . \qquad (4.24\ b)$$

Herein we assume that p and n differ only slightly from their steady state values.Hence:

$$n = \bar{n} + \Delta n \qquad |\Delta n| << \bar{n} \tag{4.25 a}$$

$$p = \bar{p} + \Delta p \qquad |\Delta p| << \bar{p} \tag{4.25 b}$$

where $\bar{p}$ and $\bar{n}$ are the steady state values of p and n. Additionally Δn and Δp are the small deviations from the steady state values. By inserting equations (4.25 a/b) in (4.24 a/b), and by neglecting the product $\Delta n \cdot \Delta p$ while taking into account that the steady state values are not time-dependent, i.e. that we are dealing with a stationary regime with small deviations:

$$\frac{d\bar{n}}{dt} = 0 \qquad \frac{d\bar{p}}{dt} = 0 \qquad \bar{n} = n_{th}$$

it is possible to derive the following expressions for the temporal evolution of the deviations from the steady state:

$$\frac{d(\Delta n)}{dt} = -(\sigma \cdot c \cdot \bar{p} + \Gamma) \cdot \Delta n - \frac{\Delta p}{\tau_{ph}} \tag{4.26 a}$$

$$\frac{d(\Delta p)}{dt} = \bar{p} \cdot c \cdot \sigma \cdot \Delta n = \frac{1}{\tau_{ph}} \left(\frac{\Delta n}{n_{th}} \right) \cdot \bar{p} \tag{4.26 b}$$

These two equations constitute a system of two first-order linear differential equations, which can be converted into one second-order differential equation:

$$\frac{d^2(\Delta n)}{dt^2} + (\bar{p} \cdot c \cdot \sigma + \Gamma) \cdot \frac{d(\Delta n)}{dt} + \frac{\bar{p}}{\tau_{ph}^2 \cdot n_{th}} \cdot \Delta n = 0 \tag{4.27}$$

which is a linear homogeneous differential equation with constant coefficients. The functional shape of this equation is identical to the well-known differential equation of a damped harmonic oscillator:

$$\frac{d^2(\Delta n)}{dt^2} + 2 \cdot \delta \cdot \frac{d(\Delta n)}{dt} + \omega^2 \cdot \Delta n = 0$$

with δ being the damping constant and ω being the oscillation frequency. At this point we just need to compare this equations with (4.27) and use a few known expressions, such as:

$$\bar{p} = n_{tot} \cdot (W_P - W_{th}) \cdot \tau_{ph} \tag{4.20 a}$$

$$W_{th} = \Gamma \cdot \frac{n_{th}}{n_{tot}} \tag{4.20 b}$$

$$\bar{n} = n_{th} = (\sigma \cdot c \cdot \tau_{ph})^{-1}$$

in order to extract the damping constant and oscillation frequency relevant for the temporal behavior of the inversion and photon density as:

$$\delta = \frac{\Gamma}{2} \cdot \frac{W_P}{W_{th}} = \frac{1}{2 \cdot \tau_2} \cdot \frac{W_P}{W_{th}}$$

$$\omega = \sqrt{\frac{1}{\tau_2 \cdot \tau_{ph}} \left(\frac{W_P}{W_{th}} - 1 \right)}$$

The character of the solution depends on the relative values of δ and ω. Similar to the damped harmonic oscillator, one can observe an overdamped, a critically damped and a damped harmonic oscillation.

As an example, we consider the so-called <u>relaxation oscillations</u> of a gas and a solid-state laser. For this we assume a pump rate two times higher than the threshold and a cavity length of 5 cm, i.e.

$$\frac{W_P}{W_{th}} = 2 \qquad\qquad \delta = \frac{1}{\tau_2}$$

and apply the expression for the photon lifetime in the cavity (with c: speed of light, l: cavity length and T: transmission of out-coupling mirror), given by

$$\tau_{ph} = \frac{2 \cdot l}{c \cdot T}.$$

The following table summarizes are relevant quantities for a HeNe laser and a Nd:YAG laser.

laser	τ_2	$\delta[Hz]$	**T**	$\tau_{ph}^{-1}[Hz]$	$\omega[Hz]$	**oscillation**
HeNe	**10 ns**	**10^8**	**0.01**	**$3\cdot10^7$**	**$5\cdot10^7$**	**no (δ > ω)**
Nd:YAG	**250 µs**	**$4\cdot10^3$**	**0.2**	**$6\cdot10^8$**	**$1.5\cdot10^6$**	**yes (δ < ω)**

We can recognize that, due to the long lifetime of their excited state, solid-state lasers tend to relaxation oscillations. The damping via spontaneous emission (to get rid of excessive inversion) is too weak in these lasers. In that case (i.e. when $\delta < \omega$) the temporal evolution of the deviation from the steady state of the inversion density can be written as (this expression is known from damped harmonic oscillators):

$$\Delta n = \Delta n_{\max} \cdot \exp(-\delta \cdot t)\cos\left(\sqrt{\omega^2 - \delta^2}\, t\right)$$

And, according to (4.26 b), a similar oscillation of the deviation of photon density Δp can be observed.

Let's consider the physical origin of these relaxation oscillations in more detail. For this we assume at a time zero (t = 0) a small perturbation in the pump process, meaning that the pump rate W_P is slightly increased for a moment. Hence, the inversion is slightly increased above its steady state value. However, at t = 0 the photon density is still unchanged.

$$n(t \geq 0) > n(t < 0) = \bar{n} \qquad \bar{n} = n_{th} \qquad \Delta n(t = 0) > 0$$

With equation (4.26 b):

$$\frac{d(\Delta p)}{dt} = \bar{p} \cdot c \cdot \sigma \cdot \Delta n > 0$$

We can learn that the change of the deviation of the photon density from the steady state is positive, i.e. the photon density increases:

$$\Delta p(t > 0) > 0$$

So, after some time, the photon density and the inversion density will be above their steady state value (i.e. the one given by the steady state pump rate). Hence, in this situation with (4.26a):

$$\frac{d(\Delta n)}{dt} = -(\sigma \cdot c \cdot \bar{p} + \Gamma) \cdot \Delta n - \frac{\Delta p}{\tau_{ph}}$$

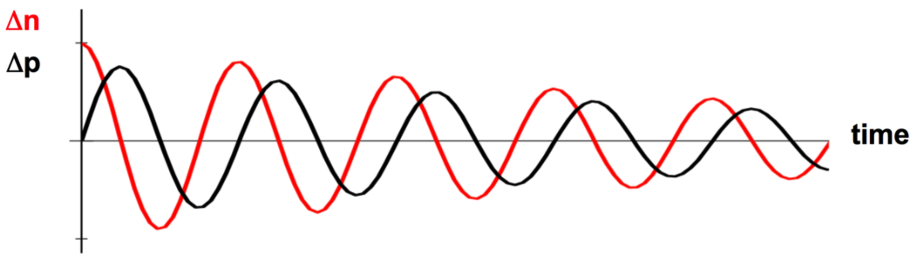
we recognize that the change of the deviation of inversion with respect to its steady state value becomes negative:

$$\frac{d(\Delta n)}{dt} < 0$$

This means that the deviation of the inversion will become smaller with time and, therefore, the inversion value will approach the steady state value $\bar{n}$. However, when the inversion value reaches its steady state value $\bar{n}$, the photon density is still above its steady state value $\bar{p}$. This implies that the inversion density will decrease below its steady state value $\bar{n}$ and, hence, below the threshold value. However, below the threshold the gain is not sufficient to sustain laser oscillation. Thus, from equation (4.26 b):

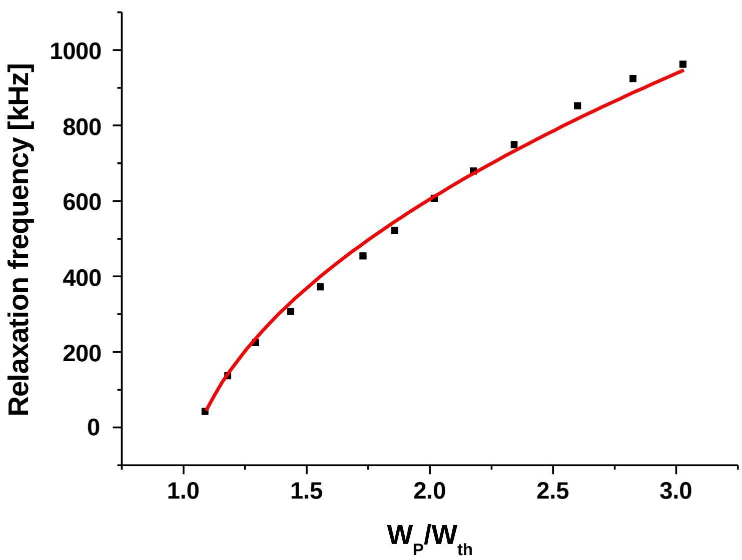
$$\frac{d(\Delta p)}{dt} = \bar{p} \cdot c \cdot \sigma \cdot \Delta n < 0$$

we see that the photon density will start to decrease with time. When the photon density reaches its equilibrium value $\bar{p}$ the deviation of the inversion density Δn is still negative (i.e. the inversion is below threshold) and, consequently, the photon density will be further reduced below the steady state value $\bar{p}$. Now, both the inversion and the photon density are below their steady state values $\bar{n}$ and $\bar{p}$, respectively. Using equation (4.26 a), it can be seen that, in this situation, the inversion will build up, reaching levels even beyond its steady state value $\bar{n}$. The behavior discussed up to now corresponds to just one cycle of a relaxation oscillation. The oscillatory behavior of the inversion and photon densities is schematically depicted in the figure below.



At this point it is important to take into account that, as seen in the figure above, such an oscillation is a damped oscillation which is triggered by any small perturbation of the laser process.

Example: By measuring the output signal of a solid-state laser e.g. a Nd:YAG laser with a photo-diode over time and by performing a Fourier analysis of that signal, it is possible to reveal the following trend of the relaxation oscillation frequency as a function of the pump rate / power, where a square root dependence is clearly visible.



Spiking of Lasers – nonlinear oscillations

We now assume that the deviations from steady state are large, potentially even in the order of the steady state values, i.e.

$$|\Delta n| \sim \bar{n} \qquad\qquad |\Delta p| \sim \bar{p}$$

In this case, an analytical solution of the rate equations is not possible anymore with that assumption, because we have to consider the product $\Delta n \cdot \Delta p$ in all the equations. In principle, the behavior of a laser in such a case is similar to the one discussed above in the context of relaxation oscillations, but instead of damped oscillations one can observe nonlinear oscillations. In any case, numerical approaches are required to describe the temporal evolution of the inversion density n(t) and the photon density p(t).

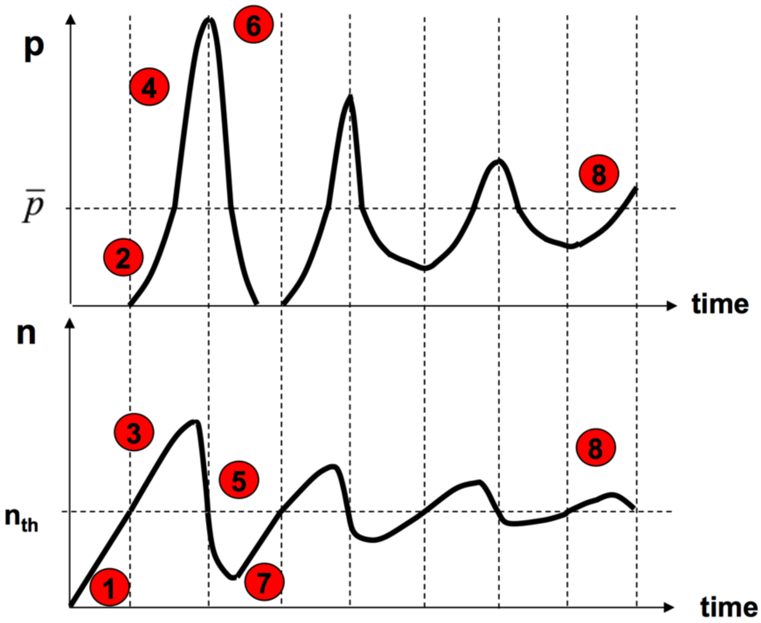
However, one simple special case that can be analyzed is the switch-on process of the laser itself. Obviously, just after the pump power is switched on, a large deviation from steady state is present (since there is pump and inversion, but no photon density). To simplify the consideration, we restrict the problem to a 4-level laser ($\gamma = 1$ and $n << n_{tot}$). The evolution of the inversion density is then described by:

$$\frac{dn}{dt} = -\sigma \cdot c \cdot p \cdot n - \Gamma \cdot n + W_P \cdot n_{tot} \qquad \text{(was 4.24 a)}$$

If the laser is below the threshold ($p \sim 0$) and for periods of time smaller than the lifetime of the upper laser level (τ_2), spontaneous emission processes can be neglected. Thus the equation above can be simplified to

$$\frac{dn}{dt} = W_P \cdot n_{tot}$$

Consequently, below the threshold the inversion increases linearly with time (1). Hereby, as soon as the inversion reaches the threshold value ($n > n_{th}$) the photon density starts to build up (2). However, it takes a certain time until the steady state photon density value $\bar{p}$ is reached and, hence, in that time the inversion increases beyond the threshold value n_{th} (3). This provides enough gain so that the photon density can overshoot its steady state value $\bar{p}$ (4). As a consequence, the inversion rapidly decreases towards the threshold value (5). Exactly at the threshold value the laser cannot support a further increase of the photon density (i.e. the peak photon density is reached (6)) because the net gain in the cavity becomes 1. However, since at this point the photon density is maximum, the inversion decreases below the threshold value (7) and this results in a quick drop of the photon density (because the net gain in the cavity becomes <1). Actually, at this point the emission can even die down. Several cycles of this process (8) are needed to approach the steady state values of both the inversion and photon density which belong to the given pump rate W_P. The behavior described above is called spiking and it is illustrated below.



Q-switching of Lasers – Giant Pulse Regime

Q-switching of lasers is widely used method to generate short and energetic optical pulses. It is based on changing the quality factor of a laser resonator that can be defined as:

$$Q = \frac{Energy\ stored\ in - cavity}{Energy\ loss\ per\ round - trip}$$

Hence, a high-quality cavity implies low losses and a long photon lifetime τ_{ph}.

The conceptual idea of Q-switching is to change the quality factor of the laser cavity (from low to high) when enough energy has been stored in the active medium. Thus, the process starts with a low-Q resonator. Under these circumstances a high inversion can be built up in the active medium without the occurrence of laser oscillation (due to the high lasing threshold of the low-Q cavity). Laser oscillations should be avoided while accumulating inversion in the active medium because they would reduce this parameter. Once that the inversion in the active medium is high enough, a sudden switch to a high-Q resonator leads to a situation in which the inversion density is significantly larger than the threshold inversion density (of the high-Q cavity). Thus, a very fast build-up of photon density is possible, which extracts most of the energy stored in the active medium (in the form of inversion) as a giant optical pulse.

In order to describe the Q-switch process, we simulate the ideal switching behavior by means of a time-dependent photon lifetime:

$$\tau_{ph}(t) = \begin{cases} 0 & for \quad t < 0 \\ \tau_{ph} & for \quad t \geq 0 \end{cases}$$

Furthermore, we assume that at the time zero (t = 0) an initial inversion density has been created ($n(0) = n_i$). Moreover, we assume that this initial inversion density ($n(0) = n_i$) is significantly larger than the threshold value in the high-Q cavity ($n_i >> n_{th}$). During the fast build-up of the photon density p, the pump process and the spontaneous emission can be neglected. In addition, the following discussion will be restricted to a 4-level system.

Consequently, after the switching event (i.e. for t > 0) the rate equations (4.24 a/b) can be reduced to:

$$\frac{dn}{dt} = -\frac{n}{n_{th}} \cdot \frac{p}{\tau_{ph}} \qquad (4.29\ a)$$

and

$$\frac{dp}{dt} = \left(\frac{n}{n_{th}} - 1 \right) \cdot \frac{p}{\tau_{ph}} \qquad (4.29\ b = 4.24\ b)$$

As long as the photon density p is small, it can be assumed that the inversion density is constant and stays at its initial value ($n = n_i$). Hence, we can re-write equation (4.29 b) as

$$\frac{dp}{p} = \left(\frac{n_i}{n_{th}} - 1 \right) \cdot \frac{dt}{\tau_{ph}}.$$

The solution of the equation above can be directly obtained with the ansatz:

$$p(t) = \exp\left(\frac{t}{\tau'} \right)$$

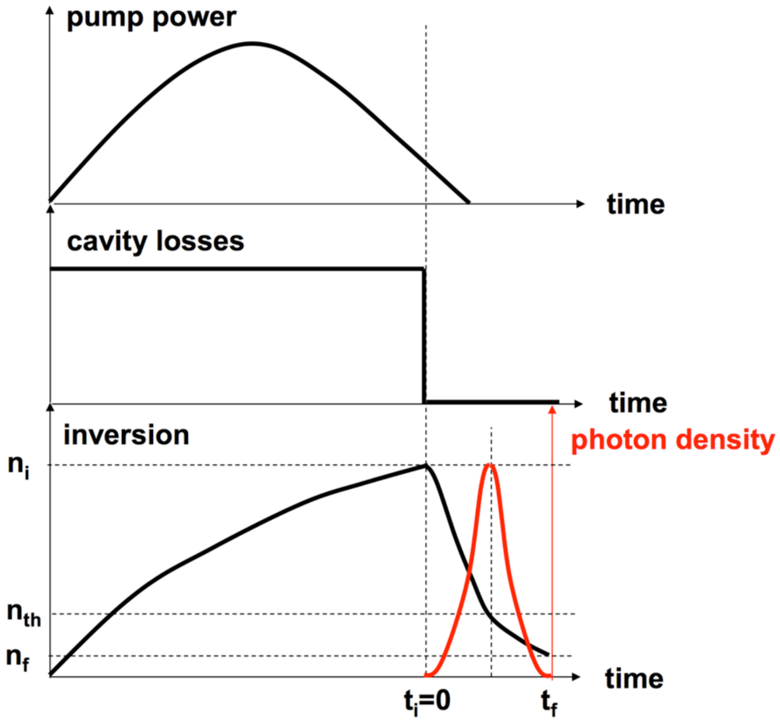
where

$$\tau' = \tau_{ph} \Big/ \left(\frac{n_i}{n_{th}} - 1 \right)$$

Thus, an exponential increase of the photon density can be observed in the cavity right after the switch of the Q-factor. Because the initial inversion density is significantly larger than the threshold value ($n_i >> n_{th}$), the time constant of that exponential growth of photon density is much smaller than the photon lifetime in the cavity ($\tau' << \tau_{ph}$).

However, as the time elapses and the photon density p increases, the inversion is reduced (due to saturation). Hereby, when the inversion level reaches the threshold value (i.e. $n = n_{th}$), the derivative of the photon density with respect to time becomes zero. This implies that the photon density reaches its maximum ($p = p_{max}$ at $n = n_{th}$) at that point in time. At this point the high photon density will keep on depleting the inversion in the active medium. Thus, as soon as the inversion drops below the threshold value ($n < n_{th}$), there is a decay of the photon density, which occurs with a time constant τ_{ph}. This rapid increase and decay of the photon density in the cavity gives rise to the emission of a giant optical pulse from the Q-switched laser.

The following figure summarizes the temporal behavior of the pump power (even though in this example a pulsed pump is shown, it is important to remark that continuous pumping works as well), the cavity losses, the inversion and the photon density during the emission of the giant pulse.



For many applications the generated pulse energy plays a crucial role. So it is important to determine the energy of the giant pulse, which will be done in the following.

In order to calculate this, we assume that the inversion density n(t) is a monotonically decaying function. This implies that the reversal function t(n) exists and, therefore, we can write p{t(n)} instead of p(t). With that, the derivative of the photon density with respect to the inversion is given by:

$$\frac{dp}{dn} = \left(\frac{dp}{dt}\right) \cdot \left(\frac{dn}{dt}\right)^{-1}$$

Using the known equations (4.29 a) and (4.29b), i.e.

$$\frac{dn}{dt} = -\frac{n}{n_{th}} \cdot \frac{p}{\tau_{ph}} \qquad \text{and} \qquad \frac{dp}{dt} = \left(\frac{n}{n_{th}} - 1\right) \cdot \frac{p}{\tau_{ph}}$$

the following differential equation describing the evolution of the photon density as a function of the inversion density can be obtained:

$$\frac{dp}{dn} = \frac{n_{th}}{n} - 1 \quad . \tag{4.31 a}$$

A solution of (4.31 a) is possible with the ansatz:

$$p = n_{th} \cdot \ln n - n + C \tag{4.31 b}$$

whereby the initial conditions n(0) = n_i and p(0) = p_i determine the parameter C through the expression:

$$C = p_i + n_i - n_{th} \cdot \ln n_i \, .$$

Hence the photon density as a function of the inversion density is given by:

$$p = n_{th} \cdot \ln \frac{n}{n_i} - \left(n - n_i\right) + p_i \quad . \tag{4.31 c}$$

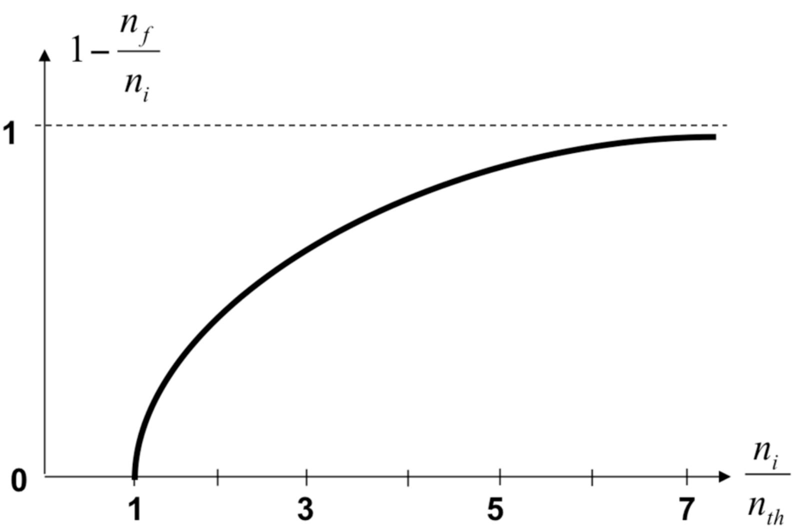
At this point, in order to determine the energy content of the emitted pulse, it is necessary to calculate how strongly the inversion is reduced by the giant pulse. For this we assume that the initial photon density is zero and that after the pulse emission no photon density is present, i.e. p_i = 0 and p_f = 0. With those assumptions equation (4.31c) can be used to obtain the inversion n_f after the emission of the pulse:

$$p_f = 0 = n_{th} \cdot \ln \frac{n_f}{n_i} - n_f + n_i \tag{4.32 a}$$

Thus, from here we can obtain:

$$\frac{n_f}{n_i} = \exp\left\{\left(\frac{n_i}{n_{th}}\right) \cdot \left(\frac{n_f}{n_i} - 1\right)\right\} \tag{4.32 b}$$

which is a transcendental equation, i.e. a type of equation that is typically solved with a graphic representation:



We can learn from this plot is that if the initial inversion is sufficiently high, then the inversion is efficiently extracted from the active medium by the emitted pulse, i.e.

$$\frac{n_f}{n_i} \to 0 \quad if \quad \frac{n_i}{n_{th}} \to \infty$$

Finally, the energy of the emitted pulse is given by the number of released photons (obtained from the depleted inversion) times their photon energy, i.e.

$$E_{out} = V \cdot (n_i - n_f) \cdot h\nu \, .$$

Now, using with some known expressions such as

$$n_i - n_f = n_{th} \cdot \ln\frac{n_i}{n_f}$$ from (4.32 a)

as well as:

$$(\sigma \cdot c \cdot \tau_{ph})^{-1} = n_{th}$$ and $$\frac{t_{ph}}{t_R} = T^{-1} ,$$

it is possible to derive the pulse energy of a Q-switched emission as:

$$E_{out} = \frac{V \cdot T}{\sigma \cdot 2 \cdot l} \cdot \ln\left(\frac{n_i}{n_f}\right) \cdot h\nu \, .$$

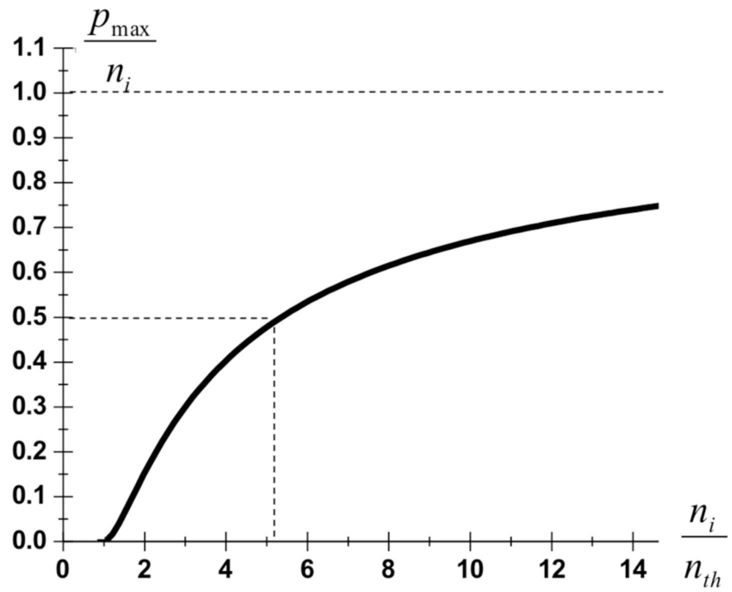
Another important quantity that should be calculated is the peak photon density p_{max}, from which the peak intensity I_{max} can be derived. We know already that the peak photon density is reached when the inversion density is reduced to its threshold value ($p = p_{max}$

at n = n_{th}). With equation (4.31 c) we can, therefore, directly obtain the peak photon density as:

$$p_{\max} = p(n_{th}) = n_{th} \cdot \ln \frac{n_{th}}{n_i} - (n_{th} - n_i) \tag{4.32 c}$$

Plotting this equation, i.e the peak value of the photon density as a function of the initial inversion, we can reveal that the peak photon density approaches the initial inversion density only if the initial inversion is significantly above the threshold inversion:

$$p_{\max} \to n_i \quad if \quad n_i >> n_{th}$$



It is worth mentioning at this point that this behavior implies that the higher the initial inversion is, the shorter the emitted pulse duration will be. This, in principle, is due to the fact that with higher initial inversion levels the photon density will grow very quickly out of the spontaneous emission. Thus, the front edge of the pulse will be very steep and short. On the other hand, the duration of the back edge of the pulse (i.e. the portion of the pulse after its maximum) is mainly governed by the photon lifetime in the cavity, so it is not significantly affected by the initial inversion level. Therefore, for high initial inversion levels the pulses will become shorter (because of the shorter initial front edge) and temporally asymmetric.

Example: Q-switching of a Nd:Glass Laser

Recall the assumption that the laser active medium is homogeneously distributed within the resonator. The cross-section of stimulated emission for the 1060nm line in a Nd:YAG laser is:

$$\sigma \approx 10^{-20} \; cm^2$$

The photon lifetime is in the cavity is assumed to be 10ns. This, in turn, determines the value of the threshold inversion density, which is given by

$$\left(\sigma \cdot c \cdot \tau_{ph}\right)^{-1} = n_{th} \cong 3.3 \cdot 10^{17}\ cm^{-3}$$

Besides, the initial inversion is set to fulfill:

$$\frac{n_i}{n_{th}} = 5$$

Thus, the peak photon density can be extracted from the figure above as:

$$p_{\max} = 0.5 \cdot n_i = 8.3 \cdot 10^{17}\ cm^{-3}$$

Furthermore, we assume that the transmission of the out-coupling mirror is 50% (T = 0.5). Thus, the peak intensity is:

$$I_{\max} = \frac{1}{2} T \cdot h\nu \cdot c \cdot p_{\max} = 1.25 \cdot 10^{9}\ W/cm^{2}$$

The additional factor 0.5 has its origin in the two counter-directional propagating parts of the radiation field in the cavity.

Another important parameter to characterize a radiation field is the radiance, i.e. the brightness, of the light source, defined as the power per surface area and solid angle. We assume that the emission of the Nd:glass laser occurs within an aperture angle of a few mrad, corresponding to a solid angle of about 10^{-5} sr (steradian). Hence, the brightness of the Q-switched Nd:glass laser is:

$$1.25 \cdot 10^{14}\ W/\left(cm^{2} \cdot sr\right)$$

Comparing that value to the brightness of the Sun at its surface:

$$2 \times 10^{3}\ W/\left(cm^{2} \times sr\right)$$

we can recognize a difference of many order of magnitudes. That difference becomes even more pronounced when comparing the brightness per unit frequency of a Nd:glass laser (spectral bandwidth of 0.1 THz):

$$1.25 \cdot 10^{3}\ W/\left(cm^{2} \cdot sr \cdot Hz\right)$$

with the brightness per unit frequency of the Sun (emission bandwidth of sun ~20 THz):

$$\sim 1.0 \times 10^{-10}\ W/\left(cm^{2} \times sr \times Hz\right)$$

This comparison illustrates the exceptional properties of a laser as a light source. In Q-switched lasers two facts contribute to this outstanding performance:

A) The laser active medium stores energy as inversion for a long time (limited by τ_2) and releases that energy in a short time in the form of an optical pulse. As an example: Nd:glass has an upper-state lifetime of $\tau_2 \sim 300$ µs and the typical Q-switched pulse duration is in the order of ~ 30 ns. Thus, the laser peak power can be orders of magnitude higher than the pump power.
B) The laser emission possesses a high spatial coherence, i.e. a large transversal coherence length. Therefore, the laser emission has a low divergence and, consequently, a high brightness.

Experimental realization of Q-switching

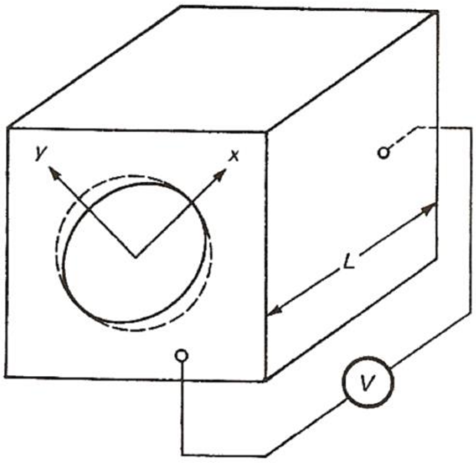
There are various possibilities to Q-switch a laser. They can be classified as active methods (such as mechanical shutters, AOMs, Pockels-cells) and passive approaches (e.g. by means of saturable absorbers). In the following, the most important experimental realizations of Q-switching are discussed in more details.

Pockels-cells (EOMs)

Pockels cells can perform a fast switch of the polarization state exploiting the electro-optical effect. In these devices, a static electric field induces birefringence in an electro-optical crystal, such as KDP (which is a negative uniaxial crystal).

In an idealized description one of the refractive indices of the crystal changes its value when applying an electric field E_0 (i.e. an static electric field in z-direction) and, therewith, the crystal becomes birefringent.

$$n_y = \begin{cases} n_x & for \quad E_0 = 0 \\ n_x + \Delta n & for \quad E_0 \neq 0 \end{cases}$$



To understand the polarization rotation, we define the absolute value of the wave-vector (i = x,y) as:

$$k_i = \frac{\omega}{c} \cdot n_i$$

Furthermore, we assume that the considered light-wave at position z = 0 has an angular frequency ω and is linearly polarized with an angle that is 45° inclined with respect to the x-axis. Hence, the electric field can be written as:

$$\vec{E}(0,t) \sim (\vec{e}_x + \vec{e}_y) \cdot \cos(\omega \cdot t - \varphi)$$

with $\vec{e}_x, \vec{e}_y$ being the unit vectors in the x- and y-direction.

After propagation in the crystal through a layer of thickness l, the electric field evolves to:

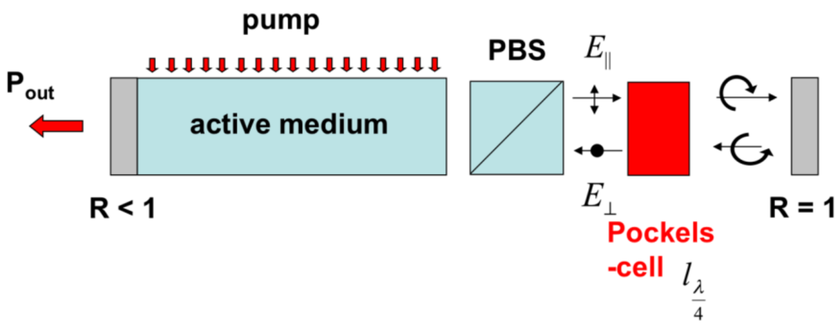
$$\vec{E}(l,t) \sim \vec{e}_x \cdot \cos(\omega \cdot t - k_x \cdot l - \varphi) + \vec{e}_y \cdot \cos(\omega \cdot t - k_y \cdot l - \varphi)$$

In general (and in particular in a birefringent material) the x- and y-component accumulate different phases as they propagate through the medium, which leads to the polarization state evolving in the general case towards an elliptical polarization. There are "special" layer thicknesses know as half-wave ($l_{\frac{\lambda}{2}}$) and quarter-wave thickness ($l_{\frac{\lambda}{4}}$), where a half-wave layer thickness is twice the quarter-wave layer thickness. In particular, a quarter-wave thickness ($l_{\frac{\lambda}{4}}$) causes a phase shift of 90° between the x- and y-components of the electric field:

$$|k_x - k_y| \cdot l_{\frac{\lambda}{4}} = \frac{\pi}{2}$$

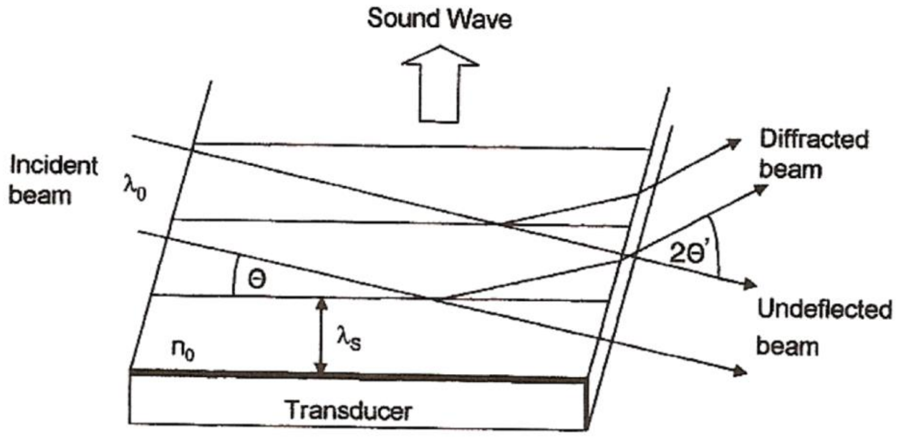
This turns a linearly polarized wave into a circularly polarized one. A phase shift of 180° between the x- and y- components of the electric field (achieved when using a half-wave thickness or two consecutive quarter-wave thicknesses of the material) leads to a 90° rotation of the polarization orientation of the 45° linearly polarized light.

The implementation of a Pockel's cell as a fast Q-switch for lasers is illustrated in the figure below. Assuming that a voltage is applied at the Pockel's cell, such that the cell acts as a quarter-wave-layer, the cavity will have a low-Q value (which allows building up high inversion levels in the active medium). After a sudden switch-off of the voltage applied to the Pockel's cell, the cavity is switched to its high-Q state and a giant pulse is released.



Acousto-optical Modulators (AOMs)

By applying a high-frequency voltage (RF), via a piezo-element attached to a transparent material (e.g. Quartz), an acoustic wave can be excited in such materials. This acoustic wave propagates in the material causing a modulation of the density of the medium, which is transferred to a modulation of the refractive index via the photo-elastic effect. The result is a traveling index-grating that leads to Bragg-diffraction of the incident light.



Interestingly, there is the same relation between frequency, wavelength and propagation speed for electro-magnetic and acoustic waves:

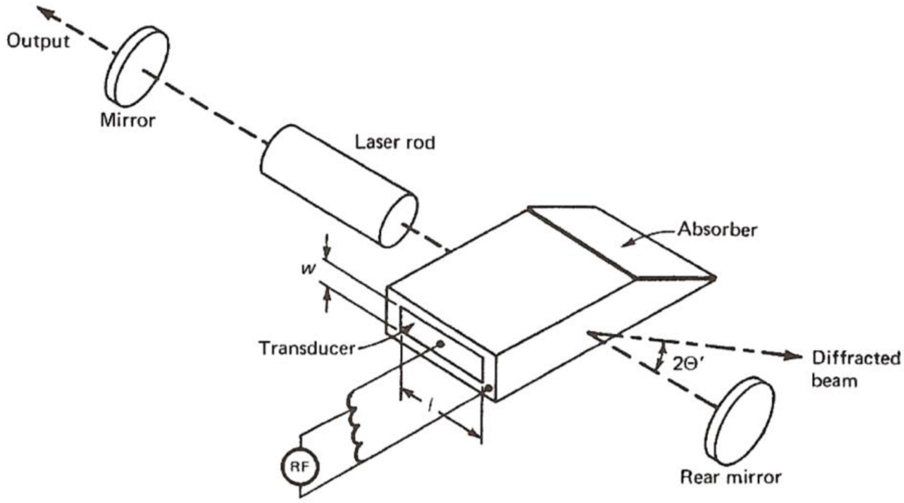
$$c_S = \nu_S \cdot \lambda_S$$

where c_S is the speed of sound, ν_S is the frequency of the applied RF voltage and λ_S is the wavelength of the index grating (i.e. the acoustic wave), which finally causes the Bragg diffraction at an angle Θ. This diffraction angle is determined by:

$$\sin\Theta = \frac{1}{2} \cdot \frac{\lambda}{\lambda_S}$$

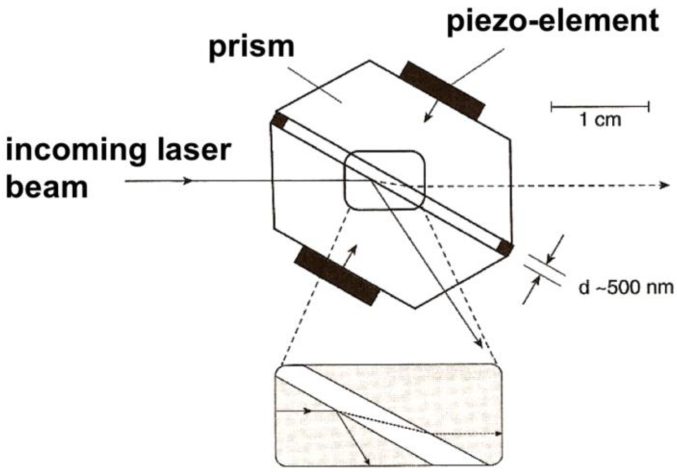
Where λ is the wavelength in the medium, i.e. the vacuum wavelength divided by the refractive index.

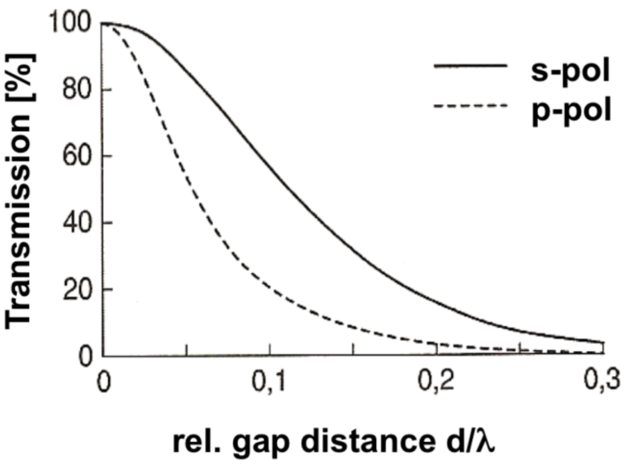
The implementation of an AOM Q-switch in a laser is possible in a very similar way as with the Pockels-cell technique (see figure below): If a RF voltage is applied to the AOM, the cavity experiences high losses; Switching the RF voltage off, the resonator is switched to its high-Q state and the giant pulse is created.



FTIR Modulators

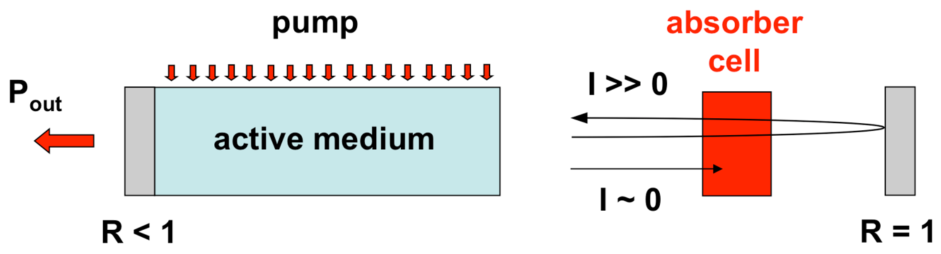
FTIR stands for frustrated total internal reflection. The basic idea of that kind of switch is to overcome total internal reflection by optical tunneling. A FTIR modulator consists of two prims separated by a very small gap. Pressure waves induced by piezo elements control the gap distance. If a voltage is applied, the transmission increases due to optical tunneling.





Saturable Absorbers – Passive Q-switching

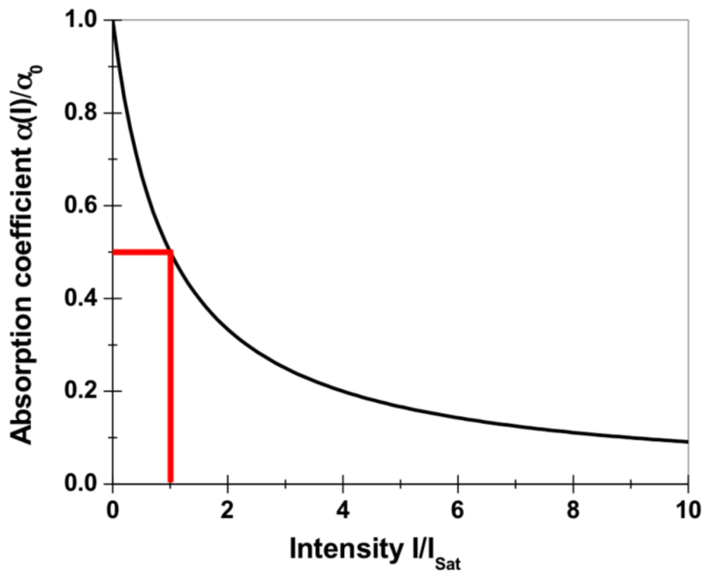
Passive Q-switching is experimentally easier to implement than all the active approaches discussed so far. One simply places a saturable absorber in the resonator in addition to the active medium.



In this case, the switching time is not externally defined, but the radiation intensity itself determines the switching behavior instead. The saturable absorber can be, for instance, a cell with an appropriate absorber (e.g. a dye solution, which absorbs at the laser wavelength) possessing an intensity dependent absorption coefficient (often described by a 2-level system):

$$\alpha(I) = \alpha_0 \cdot \left(1 + \frac{I}{I_{Sat}}\right)^{-1}$$

with $a_0 = a(I = 0)$ being the small-signal absorption coefficient and I_{Sat} being the saturation intensity.



The value of the initial absorption is chosen (e.g. by the doping concentration) so that, at the moment of maximum inversion in the active medium, the switching threshold is reached. From this moment on the photon density p increases and the absorber is bleached, which implies that its transmission increases rapidly (typically in a time frame of ns), i.e. the passive switch is opened. After a relaxation time (typ. µs), the absorber is refreshed and a new cycle of inversion build-up can start. The obvious advantage of a passive Q-switch is its simplicity and fast optical switching characteristics. The drawbacks are related to the rather poor chemical stability of dye solutions. Therefore, doped crystals or saturable semiconductor mirrors are used today as passive switches. Furthermore, the moment of pulse emission occurs not at a predefined time, but it is triggered by a statistical event (spontaneous emission). Hence, no triggering is possible and the emission of the pulse train is characterized by a rather large temporal jitter (i.e. a fluctuation of the pulse-to-pulse distance).