Problem 1 Jinsong Liu 206216

(a)

solution:

$$\sigma_\nu = \sigma_0 \cdot F(\nu) \qquad F(\nu) - \text{line shape function}$$

$$F(\nu) = A \cdot \frac{\left(\frac{\Delta\nu}{2}\right)^2}{(\nu-\nu_{21})^2 + \left(\frac{\Delta\nu}{2}\right)^2} \qquad A - \text{constant}$$

Before frequency shift:

$$F(\nu) = A$$

After frequency shift:

$$F'(\nu) = A \cdot \frac{\left(\frac{\Delta\nu}{2}\right)^2}{(\Delta\nu)^2 + \left(\frac{\Delta\nu}{2}\right)^2} = A \cdot \frac{1}{5} = \frac{A}{5}$$

$$\therefore \eta = \frac{F'(\nu)}{F(\nu)} = \frac{\frac{A}{5}}{A} = 20\%$$

(b)

solution:

$$\bar{F}(\nu) = \frac{1}{T}\int_T F(\nu)\,dt = \frac{1}{T}\int_T A\frac{\left(\frac{\Delta\nu}{2}\right)^2}{(\nu-\nu_{21})^2 + \left(\frac{\Delta\nu}{2}\right)^2}dt$$

$$= \frac{1}{10}\int_0^{10} A\frac{25}{100t^2+25}dt$$

$$= \frac{A}{10}\int_0^{10}\frac{1}{4t^2+1}dt$$

$$= \frac{A}{20}\left[\tan^{-1}(20) - \tan^{-1}(0)\right] = 0.076A$$

$$\bar{\eta} = \frac{\bar{F}(\nu)}{A} = 7.6\%$$

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solution:

$\because$ The laser is an inhomogeneously broadened laser

$\therefore$ $$P(\nu) = B \cdot \exp\left[-4\ln 2\left(\frac{\nu - \nu_{21}}{\Delta\nu_L}\right)^2\right] \qquad \Delta\nu_L = 1\,\text{GHz}$$

When the laser is a monochromatic laser

$$P'(\nu) = \delta(\nu - \nu_{21})$$

Suppose the average power of excitation laser is 1.

$$\int_{-\infty}^{\infty} P'(\nu)\,d\nu = \int_{-\infty}^{\infty} \delta(\nu - \nu_{21})\,d\nu = 1$$

$$\int_{-\infty}^{\infty} P(\nu)\,d\nu = \int_{-\infty}^{\infty} B \cdot \exp\left[-4\ln 2\left(\frac{\nu - \nu_{21}}{\Delta\nu_L}\right)^2\right]d\nu = 1$$

from Gaussian integral $\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$

$\therefore$ $$B = \sqrt{\frac{4\ln 2}{\pi\Delta\nu_L^2}}$$

from (a), we know that $\sigma_\nu = \sigma_0 \cdot F(\nu)$

$$F(\nu) = A \cdot \frac{\left(\frac{\Delta\nu}{2}\right)^2}{(\nu - \nu_{21})^2 + \left(\frac{\Delta\nu}{2}\right)^2}$$

$\therefore$ $$\int_{-\infty}^{\infty} P'(\nu)\,F(\nu)\,d\nu = A$$

$$\int_{-\infty}^{\infty} P(\nu)\,F(\nu)\,d\nu = A \cdot B \cdot \frac{\pi\Delta\nu}{2}$$

$$\eta = \frac{A \cdot B \cdot \frac{\pi \Delta\nu}{2}}{A} = B \cdot \frac{\pi \cdot \Delta\nu}{2} = \sqrt{\frac{4\ln 2}{\pi \Delta\nu_L^2}} \frac{\pi \Delta\nu}{2} = \sqrt{\pi \ln 2} \frac{\Delta\nu}{\Delta\nu_L}$$

$\Delta\nu = 10\,MHz \quad \Delta\nu_L = 1\,GHz$

$\therefore \eta \approx 0.0148 = 1.48\%$

(d) solution:

$$\Delta\nu_D = \frac{2\nu_{21}}{c} \cdot \sqrt{\frac{2kT\ln 2}{M}} = 1.07\,GHz$$

(e) solution:

now $F(\nu) = A\exp\left[-4\ln 2\left(\frac{\nu-\nu_{21}}{\Delta\nu_D}\right)^2\right]$

$$\int_{-\infty}^{\infty} P'(\nu) F(\nu)\, d\nu = 1$$

$$\int_{-\infty}^{\infty} P(\nu) F(\nu)\, d\nu = A \cdot B \int_{-\infty}^{\infty} \exp\left[-4\ln 2\left(\frac{1}{\Delta\nu_L^2} + \frac{1}{\Delta\nu_D^2}\right)(\nu-\nu_{21})^2\right] d\nu$$

$$= \sqrt{\frac{4\ln 2}{\pi \Delta\nu_L^2}} \sqrt{\frac{\pi}{4\ln 2}} \sqrt{\frac{1}{\frac{1}{\Delta\nu_L^2} + \frac{1}{\Delta\nu_D^2}}} = \sqrt{\frac{\Delta\nu_D^2}{\Delta\nu_L^2 + \Delta\nu_D^2}}$$

$$= \sqrt{\frac{1.07^2}{1+1.07^2}} \approx 0.73$$

$\therefore \eta = 0.73$

Problem 2 Jinsong Liu 206216

(a) solution:

$$\begin{cases} \alpha = N \cdot \sigma_\nu \\ N = 1\times10^{19}\,cm^{-3} = 1\times10^{25}\,m^{-3} \\ \sigma_\nu = 2.3\times10^{-24} \cdot \left[1-\left(\frac{\nu-\nu_0}{\frac{1}{2}\Delta\nu}\right)^2\right] m^2 \end{cases}$$

$$\lambda = \frac{c}{\nu}$$

$$\Rightarrow \ln\lambda = \ln c - \ln\nu$$

$$\Rightarrow \frac{1}{\lambda}d\lambda = -\frac{1}{\nu}d\nu$$

$$\Rightarrow \left|\frac{\Delta\nu}{\nu}\right| = \left|\frac{\Delta\lambda}{\lambda}\right|$$

$\because$ the peak wavelength is 976nm

$c = \lambda\nu \qquad \Delta\lambda = 10nm$

$\therefore \quad \nu_0 = 3.07\times10^{14}\,Hz$

$\Delta\nu = 3.15\times10^{12}\,Hz$

$$\therefore \alpha = 23 \cdot \left[1-\left(\frac{\nu - 3.07\times10^{14}\,Hz}{1.57\times10^{12}\,Hz}\right)\right]$$

(b) solution:

$$\frac{I_\nu(x)}{I_\nu(0)} = \exp(-\alpha_\nu \cdot x)$$

For laser pump diode

$$\frac{I_\nu(x)}{I_\nu(0)} = \frac{\int \delta(\nu-\nu_0)e^{(-\alpha_\nu\cdot 1)}\,d\nu}{\int \delta(\nu-\nu_0)\,d\nu} = e^{-2.3} \approx 0.1$$

$\eta = 0.9$

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For LED (suppose the value of rectangular spectrum is 1)

$$\frac{I_\nu(x)}{I_\nu(0)} = \frac{I_\nu(x)}{1 \cdot \Delta\nu_{LED}}$$

$$= \frac{1}{\Delta\nu_{LED}}\left[\int_{\nu_0-\frac{\Delta\nu_{LED}}{2}}^{\nu_0-\frac{\Delta\nu}{2}} \exp(-0\cdot l)\,d\nu + \int_{\nu_0-\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu}{2}} \exp(-\alpha_\nu l)\,d\nu \right.$$

$$\left. + \int_{\nu_0+\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu_{LED}}{2}} \exp(-0\cdot l)\,d\nu\right]$$

$$= \frac{1}{\Delta\nu_{LED}}\left[\int_{\nu_0-\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu}{2}} \exp(-\alpha_\nu\cdot l)\,d\nu + \Delta\nu_{LED} - \Delta\nu\right]$$

$$= \frac{1}{\Delta\nu_{LED}}\left[\int_{\nu_0-\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu}{2}} \exp\left[-2.3\cdot\left(1-\left(\frac{\nu-\nu_0}{\Delta\nu/2}\right)^2\right)l\right]d\nu + \Delta\nu_{LED} - \Delta\nu\right]$$

$$= \frac{1}{\Delta\nu_{LED}}\left[\int_{-\frac{\Delta\nu}{2}}^{\frac{\Delta\nu}{2}} \exp\left[-2.3\cdot\left(1-\left(\frac{\nu}{\Delta\nu/2}\right)^2\right)l\right]d\nu + \Delta\nu_{LED} - \Delta\nu\right]$$

$$= \frac{1}{\Delta\nu_{LED}}\left[\exp(-2.3)\frac{\sqrt{\pi}\,\Delta\nu\,\mathrm{erfi}(\sqrt{2.3})}{2\sqrt{2.3}} + \Delta\nu_{LED} - \Delta\nu\right]$$

$$\simeq 85.6\%$$

$$\therefore \eta = 14.4\%$$

(d) Solution: Jinsong Liu 206216

Advantage:

High pump efficiency

Disadvantage:

Expensive

(e) Solution:

Laser:

$$P_{output} = \frac{P_{input} \cdot \eta_1}{\bar{E}_{pump}} \cdot \bar{E}_{signal} = P_{in} \cdot \eta_1 \cdot \frac{\lambda_{pump}}{\lambda_{output}} = 850 \mu W$$

LED:

$$P_{output} = \frac{P_{input} \cdot \eta_2}{\bar{E}_{pump}} \cdot \bar{E}_{signal} = P_{in} \cdot \eta_2 \cdot \frac{\lambda_{pump}}{\lambda_{output}} = 137 \mu W$$

Problem 3 Jinsong Liu 206216

(a) Solution:

$\because$ R is the reflectivity (power) of the beam splitter,

M_1 and M_2 are the (power) reflectivities of the mirrors.

$\therefore$

$$E_{out} = E_{in}\sqrt{(1-R)\cdot M_1 \cdot R} \cdot \exp(-ik\cdot 2L_1) + E_{in}\sqrt{(1-R)\cdot M_2 \cdot R}\exp(-ik\cdot 2L_2)$$

$$= E_{in}\sqrt{(1-R)\cdot R}\left[\sqrt{M_1}\exp(-2ikL_1) + \sqrt{M_2}\exp(-2ikL_2)\right]$$

$$I_{out} = \langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}\left[\vec{E}(\vec{r}) \times \vec{H}^*(\vec{r})\right]$$

From Maxwell's equations, (in frequency domain)

$$\mathrm{rot}\,\vec{E}(\vec{r},\omega) = i\omega\mu_0\vec{H}(\vec{r},\omega) \Rightarrow \vec{H}(\vec{r},\omega) = -\frac{i}{\omega\mu_0}\mathrm{rot}\,\vec{E}(\vec{r},\omega)$$

$$\therefore \langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}\left[\vec{E}(\vec{r}) \times \frac{i}{\omega\mu_0}\mathrm{rot}\,\vec{E}^*(\vec{r})\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[\frac{i}{\omega\mu_0}\cdot(-i\vec{k})\,\vec{E}(\vec{r})\,\vec{E}^*(\vec{r})\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[\frac{1}{c\mu_0}\vec{E}(\vec{r})\cdot\vec{E}^*(\vec{r})\right]$$

$$= \frac{1}{2c\mu_0}\left|\vec{E}(\vec{r})\right|^2 = \frac{\sqrt{\varepsilon_0}}{2\sqrt{\mu_0}}\left|\vec{E}(\vec{r})\right|^2$$

$$\therefore I_{out} = \frac{\sqrt{\varepsilon_0}}{2\sqrt{\mu_0}}\vec{E}_{in}\cdot\vec{E}_{in}^*$$

$$I_{out} = \frac{\sqrt{\varepsilon_0}}{2\sqrt{\mu_0}} E_{in}^2 (1-R)R\left[\left(\sqrt{M_1}\exp(2ikL_1)+\sqrt{M_2}\exp(2ikL_2)\right)\cdot\left(\sqrt{M_1}\exp(-i2kL_1)+\sqrt{M_2}\exp(-i2kL_2)\right)\right]$$

$$= \frac{\sqrt{\varepsilon_0}}{2\sqrt{\mu_0}} E_{in}^2 (1-R)\cdot R\left[M_1+M_2+\sqrt{M_1M_2}\left\{\exp[2ik(L_1-L_2)]+\exp[-2ik(L_1-L_2)]\right\}\right]$$

$$= \frac{\sqrt{\varepsilon_0}}{2\sqrt{\mu_0}} E_{in}^2 (1-R)\cdot R\left[M_1+M_2+2\sqrt{M_1M_2}\cos(2k(L_1-L_2))\right]$$

(b) Solution:

$$V = \frac{I_{max}-I_{min}}{I_{max}+I_{min}}$$

$$\because \begin{cases} I_{max} = \dfrac{\sqrt{\varepsilon_0}}{2\sqrt{\mu_0}} E_{in}^2 (1-R)R\left[M_1+M_2+2\sqrt{M_1M_2}\right] \\ I_{min} = \dfrac{\sqrt{\varepsilon_0}}{2\sqrt{\mu_0}} E_{in}^2 (1-R)R\left[M_1+M_2-2\sqrt{M_1M_2}\right] \end{cases}$$

$$\therefore \quad V = \frac{2\sqrt{M_1M_2}}{M_1+M_2}$$

(c) Solution:

$$\bar{E}_{in} = \frac{E_0}{2} \exp\left[-2\ln 2 \frac{t^2}{\tau^2} + i\omega_0 t\right] + c.c.$$

$$\bar{E}_{out} =$$