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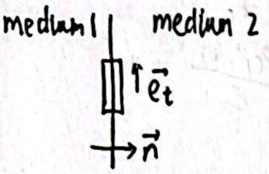
Task 1

a)

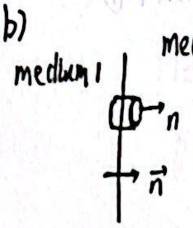
Solution:



$\because$ We do not consider surface current and surface charges

$$\therefore \int \nabla\times\vec{H}\cdot d\vec{s} = \oint \vec{H}\cdot d\vec{l} = \hat{n}\times(\vec{H}_2-\vec{H}_1) = 0$$

b)



$\because$ We do not consider surface ~~[illegible]~~ current and surface charges

$$\therefore \int \operatorname{div}\vec{D}\,dV = \oint_s \vec{D}\cdot d\vec{s} = \hat{n}\cdot(\vec{D}_2-\vec{D}_1) = 0$$

Task 2

a)

The definition of potential $\varphi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$:

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\nabla\times\vec{A}(\vec{r},t)$$

$$\vec{E}(\vec{r},t) = -\nabla\varphi(\vec{r},t) - \frac{\partial\vec{A}(\vec{r},t)}{\partial t}$$

Maxwell's equations:

$$\cancel{\nabla\times\vec{H}(\vec{r},t) = \vec{j}(\vec{r},t) + \frac{\partial\vec{D}(\vec{r},t)}{\partial t}}$$

① $\nabla\cdot\vec{D}(\vec{r},t) = \rho(\vec{r},t)$

② $\nabla\cdot\vec{B}(\vec{r},t) = 0$

③ $\nabla\times\vec{E}(\vec{r},t) = -\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t}$

④ $\nabla\times\vec{H}(\vec{r},t) = \vec{j}(\vec{r},t) + \dfrac{\partial\vec{D}(\vec{r},t)}{\partial t}$

put $\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\nabla\times\vec{A}(\vec{r},t)$ into equation ②

$$\nabla\cdot(\nabla\times\vec{A}(\vec{r},t)) = 0$$

put $\vec{E}(\vec{r},t) = -\nabla\varphi(\vec{r},t) - \frac{\partial\vec{A}(\vec{r},t)}{\partial t}$ into equation ③

$$\nabla\times\left(-\nabla\varphi - \frac{\partial\vec{A}(\vec{r},t)}{\partial t}\right) = -\frac{\partial\vec{B}(\vec{r},t)}{\partial t}$$

$$\because \vec{B}(\vec{r},t) = \nabla\times\vec{A}(\vec{r},t)$$

$$\therefore \nabla\times(-\nabla\varphi) = 0$$

Wave equations:

put $\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\nabla\times\vec{A}(\vec{r},t)$ into equation ④

$$\nabla\times\nabla\times\vec{A}(\vec{r},t) = \mu_0\vec{j}(\vec{r},t) + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

put $\vec{E}(\vec{r},t) = -\nabla\varphi(\vec{r},t) - \frac{\partial\vec{A}(\vec{r},t)}{\partial t}$ into this equation

$$\cancel{\nabla(\nabla\cdot\vec{A}(\vec{r},t)) - \nabla^2\vec{A}(\vec{r},t) = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\nabla\varphi - \frac{\partial\vec{A}}{\partial t}\right)}$$

~~when~~ $\cancel{\nabla\cdot\vec{A}(\vec{r},t) = -}$

$$\nabla(\nabla\cdot\vec{A}(\vec{r},t)) - \nabla^2\vec{A}(\vec{r},t) = \mu_0\vec{j}(\vec{r},t) + \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\nabla\varphi(\vec{r},t) - \frac{\partial\vec{A}(\vec{r},t)}{\partial t}\right)$$

when $\nabla\cdot\vec{A}(\vec{r},t) = -\dfrac{1}{c^2}\dfrac{\partial\varphi(\vec{r},t)}{\partial t}$

The wave equation is

$$\nabla^2\vec{A}(\vec{r},t) - \mu_0\varepsilon_0\frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2} = -\mu_0\vec{j}(\vec{r},t)$$

Put $\vec{E}(\vec{r},t) = -\nabla\varphi(\vec{r},t) - \frac{\partial \vec{A}(\vec{r},t)}{\partial t}$ into equation ①

$$\nabla\cdot\left(-\nabla\varphi(\vec{r},t) - \frac{\partial \vec{A}}{\partial t}\right) = \frac{\rho(\vec{r},t)}{\varepsilon_0}$$

when $\nabla\cdot\vec{A}(\vec{r},t) = -\mu_0\varepsilon_0 \frac{\partial \varphi(\vec{r},t)}{\partial t}$

The wave equation is

$$\nabla^2\varphi(\vec{r},t) - \mu_0\varepsilon_0 \frac{\partial^2\varphi(\vec{r},t)}{\partial t^2} = -\frac{\rho(\vec{r},t)}{\varepsilon_0}$$

b)

Solution:

$$\vec{E}(\vec{r},t) = -\nabla\varphi'(\vec{r},t) - \partial_t \vec{A}'(\vec{r},t)$$

$$= -\nabla\varphi - \nabla\partial_t \lambda(\vec{r},t) - \partial_t \vec{A}(\vec{r},t) + \partial_t \nabla\lambda(\vec{r},t)$$

$$= -\nabla\varphi - \partial_t \vec{A}(\vec{r},t)$$

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\nabla\times\vec{A}'(\vec{r},t)$$

$$= \frac{1}{\mu_0}\nabla\times\left(\vec{A}(\vec{r},t) - \nabla\lambda(\vec{r},t)\right)$$

$$= \frac{1}{\mu_0}\nabla\times\vec{A}(\vec{r},t) - \frac{1}{\mu_0}\nabla\times\nabla\lambda(\vec{r},t)$$

$$= \frac{1}{\mu_0}\nabla\times\vec{A}(\vec{r},t) - 0$$

$(\because \lambda(\vec{r},t)$ is an arbitrary scalar, $\therefore \nabla\times\nabla\lambda(\vec{r},t) = 0)$

$$= \frac{1}{\mu_0}\nabla\times\vec{A}(\vec{r},t)$$

$\because$ $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$ are invariant under the transformation

$\therefore$ Maxwell's equations are invariant under the transformation too.

c)

d) Solution:

The wave equation for $\varphi(\vec{r},t)$:

$$\nabla^2\varphi(\vec{r},t) - \mu_0\varepsilon_0 \frac{\partial^2\varphi(\vec{r},t)}{\partial t^2} = -\frac{\rho(\vec{r},t)}{\varepsilon_0}$$

$$\nabla^2\varphi(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2\varphi(\vec{r},t)}{\partial t^2} = -\frac{\rho(\vec{r},t)}{\varepsilon_0}$$

The wave equation for $\vec{A}(\vec{r},t)$

$$\nabla^2\vec{A}(\vec{r},t) - \mu_0\varepsilon_0 \frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2} = -\mu_0\vec{j}(\vec{r},t)$$

$$\nabla^2\vec{A}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2} = -\mu_0\vec{j}(\vec{r},t)$$

e)

Solution:

$$\nabla\cdot\vec{A}(\vec{r},t) = \nabla\cdot\vec{A}(\vec{r},t) - \nabla\cdot\nabla\lambda(\vec{r},t)$$

$$-\frac{1}{c^2}\partial_t\varphi'(\vec{r},t) = -\frac{1}{c^2}\frac{\partial\varphi(\vec{r},t)}{\partial t} - \frac{1}{c^2}\frac{\partial^2\lambda(\vec{r},t)}{\partial t^2}$$

$\therefore$ the condition to preserve the Lorenz gauge is

$$\nabla\cdot\nabla\lambda(\vec{r},t) = \frac{\partial^2\lambda(\vec{r},t)}{\partial t^2}$$

Task 3:

a) solution:

$$\chi(\omega) = \int_{-\infty}^{\infty} R(t) e^{i\omega t} dt$$

$$= \int_0^{\infty} \frac{f}{\Omega} e^{-\gamma t} \frac{e^{i\Omega t} - e^{-i\Omega t}}{2i} e^{i\omega t} dt$$

$$= \frac{f}{2i\Omega}\left(\int_0^{\infty} e^{[-\gamma + i(\Omega+\omega)]t} dt - \int_0^{\infty} e^{[-\gamma - i(\Omega-\omega)]t} dt \right)$$

$$= \frac{f}{2i\Omega}\left(-\frac{1}{-\gamma + i(\Omega+\omega)} + \frac{1}{-\gamma - i(\Omega-\omega)} \right)$$

$$= \frac{f}{2i\Omega} \frac{-[-\gamma - i(\Omega-\omega)] + [-\gamma + i(\Omega+\omega)]}{\gamma^2 - 2i\gamma\omega + \Omega^2 - \omega^2}$$

$$= \frac{f}{2i\Omega} \frac{2i\Omega}{\gamma^2 - 2i\gamma\omega + \Omega^2 - \omega^2}$$

$$= \frac{f}{\gamma^2 - 2i\gamma\omega + \omega_0^2 - \gamma^2 - \omega^2}$$

$$= \frac{f}{\omega_0^2 - \omega^2 - 2i\gamma\omega}$$

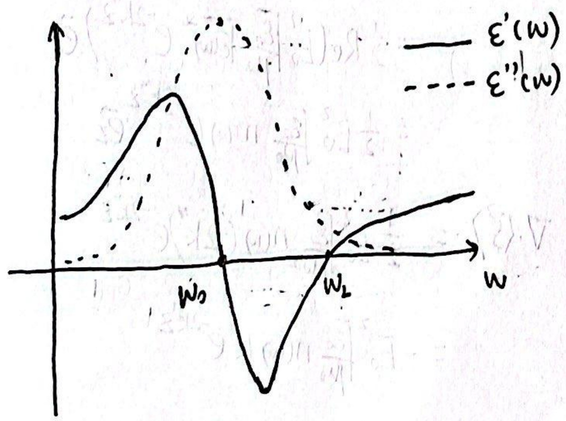
b) solution: $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$

$$\varepsilon(\omega) = 1 + \chi(\omega) = 1 + \frac{f}{\omega_0^2 - \omega^2 - 2i\gamma\omega}$$

$$= 1 + \frac{(\omega_0^2-\omega^2) f + 2i\gamma f\omega}{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}$$

$$\therefore \varepsilon'(\omega) = 1 + \frac{(\omega_0^2-\omega^2) f}{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}$$

$$\varepsilon''(\omega) = \cancel{\frac{2\gamma f\omega}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}} \quad \frac{2\gamma f\omega}{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}$$



~~frequencies ra~~

frequency range with normal dispersion: $\frac{d\varepsilon'(\omega)}{d\omega} > 0$

frequency range with anomalous dispersion: $\frac{d\varepsilon'(\omega)}{d\omega} < 0$

Strong absorption occur: $\omega = \omega_0$

c) solution:

$$\because \vec{P}(\vec{r},\omega) = \varepsilon_0 \chi(\omega) \vec{E}(\vec{r},\omega)$$

$$\therefore \vec{P}(\vec{r},\omega) = \varepsilon_0 \chi(\omega) \vec{E}(\vec{r}) F\{\cos(\omega_{cw} t)\}$$

$$F\{\cos(\omega_{cw} t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(\omega_{cw} t) e^{i\omega t} dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega_{cw} t} + e^{-i\omega_{cw} t}}{2} e^{i\omega t} dt$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{i(\omega_{cw}+\omega)t} + e^{-i(\omega-\omega_{cw})t} dt$$

$$= \frac{1}{4\pi}\left[2\pi\delta(\omega+\omega_{cw}) + 2\pi\delta(\omega_{cw}-\omega)\right]$$

$$= \frac{1}{2}\left[\delta(\omega_{cw}+\omega) + \delta(\omega_{cw}-\omega)\right]$$

$$\therefore \vec{P}(\vec{r},\omega) = \frac{\vec{E}(\vec{r})\varepsilon_0}{2}\left[\delta(\omega_{cw}+\omega) + \delta(\omega_{cw}-\omega)\right]\frac{f}{\omega_0^2-\omega^2-2i\gamma\omega}$$

$$\therefore \vec{P}(\vec{r},t) = \int_{-\infty}^{\infty} \vec{P}(\vec{r},\omega) e^{-i\omega t} d\omega$$

$$= \frac{\vec{E}(\vec{r})\varepsilon_0 f}{2}\int_{-\infty}^{\infty} \delta(\omega_{cw}+\omega)\frac{e^{-i\omega t}}{\omega_0^2-\omega^2-2i\gamma\omega} d\omega + \frac{\vec{E}(\vec{r})\varepsilon_0 f}{2}\int_{-\infty}^{\infty} \delta(\omega_{cw}-\omega)\frac{e^{-i\omega t}}{\omega_0^2-\omega^2-2i\gamma\omega} d\omega$$

$$= \frac{\vec{E}(\vec{r})\varepsilon_0 f}{2}\left(\frac{e^{i\omega_{cw} t}}{\omega_0^2-\omega_{cw}^2+2i\gamma\omega_{cw}} + \frac{e^{-i\omega_{cw} t}}{\omega_0^2-\omega_{cw}^2-2i\gamma\omega_{cw}} \right)$$

$$= \frac{\vec{E}(\vec{r})\varepsilon_0 f}{2}\left[\frac{(\omega_0^2-\omega_{cw}^2)(e^{i\omega_{cw} t}+e^{-i\omega_{cw} t})}{(\omega_0^2-\omega_{cw}^2)^2+4\gamma^2\omega_{cw}^2} - \frac{2i\gamma\omega_{cw}(e^{i\omega_{cw} t}-e^{-i\omega_{cw} t})}{(\omega_0^2-\omega_{cw}^2)^2+4\gamma\omega_{cw}^2} \right]$$

$$= \vec{E}(\vec{r})\varepsilon_0 f\left[\frac{(\omega_0^2-\omega_{cw}^2)\cos(\omega_{cw} t)}{(\omega_0^2-\omega_{cw}^2)^2+4\gamma\omega_{cw}^2} + \frac{4\gamma\omega_{cw}\sin(\omega_{cw} t)}{(\omega_0^2-\omega_{cw}^2)^2+4\gamma\omega_{cw}^2} \right]$$

Task 4

a)

Solution:

The wave equation is

$$\text{rot rot } \bar{E}(\vec{r},\omega)=\frac{\omega^2}{c^2}\varepsilon(\omega)\bar{E}(\vec{r},\omega)$$

∵ The wave is a monochromatic plane wave with the complex representation $\vec{E}(\vec{r})=\vec{E}_0\exp[i\vec{k}\cdot\vec{r}]$

∵ $\mathrm{div}\,\bar{E}(\vec{r},\omega)=0 \quad \bar{E}(\vec{r},\omega)=\bar{E}(\omega)\exp[i\vec{k}\cdot\vec{r}]$

∴ $\Delta\bar{E}(\vec{r},\omega)+\frac{\omega^2}{c^2}\varepsilon(\omega)\bar{E}(\vec{r},\omega)=0$

$(ik)^2\bar{E}(\omega)+\frac{\omega^2}{c^2}\varepsilon(\omega)\bar{E}(\omega)=0$

$\left[-k^2+\frac{\omega^2}{c^2}\varepsilon(\omega)\right]\bar{E}(\omega)=0$

∵ $\bar{E}(\omega)\neq 0$ ∴ $-k^2+\frac{\omega^2}{c^2}\varepsilon(\omega)=0$ (dispersion relation)

$k(\omega)=\frac{\omega}{c}\sqrt{\varepsilon(\omega)}=\frac{\omega}{c}[n(\omega)+i\kappa(\omega)]$

$\sqrt{\varepsilon(\omega)}=n(\omega)+i\kappa(\omega)$

$\varepsilon(\omega)=[n(\omega)+i\kappa(\omega)]^2=$ [redacted] $n^2(\omega)+2in(\omega)\kappa(\omega)-\kappa^2(\omega)$

$\mathrm{Re}\,\varepsilon(\omega)=n^2(\omega)-\kappa^2(\omega)$

$\mathrm{Im}\,\varepsilon(\omega)=2n(\omega)\kappa(\omega)$

∴ $n(\omega)=\frac{1}{\sqrt{2}}\sqrt{\sqrt{[\mathrm{Re}\,\varepsilon(\omega)]^2+[\mathrm{Im}\,\varepsilon(\omega)]^2}+\mathrm{Re}\,\varepsilon(\omega)}$

b)

Solution:

∵ $\vec{E}(\vec{r})=\vec{E}_0\exp[i\vec{k}\cdot\vec{r}]$

The plane wave has a linear polarization along the y-direction and $\vec{k}$-vector is pointing in the z-direction

∴ $\vec{E}(\vec{r},t)=\vec{E}_0e^{i(kz-\omega t)}\vec{e}_y \qquad \vec{E}(\vec{r})=E_0e^{ikz}\vec{e}_y$

∵ $\vec{H}(\vec{r},\omega)=\frac{1}{\omega\mu_0}[\vec{k}\times\vec{E}(\omega)]=\cancel{\frac{i}{\omega\mu_0}\vec{E}(\vec{r},\omega)}-\frac{i}{\omega\mu_0}\text{rot}\,\vec{E}(\vec{r},\omega)$

$\vec{E}(\vec{r},\omega)=F\{\vec{E}(\vec{r},t)\}=E_0e^{ikz}\delta(\omega'-\omega)\vec{e}_y$

$\text{rot}\,\vec{E}(\vec{r},\omega)=\nabla\times\vec{E}(\vec{r},\omega)$

$=i\vec{k}\times E_0e^{ikz}\delta(\omega'-\omega)\vec{e}_y$

$=i\frac{\omega}{c}\sqrt{\varepsilon(\omega)}E_0e^{ikz}\delta(\omega'-\omega)\vec{e}_x$

∴ $\vec{H}(\vec{r},\omega)=-\frac{i}{\omega\mu_0}\left[i\frac{\omega}{c}\sqrt{\varepsilon(\omega)}E_0e^{ikz}\delta(\omega'-\omega)\right]\vec{e}_x$

$=\sqrt{\frac{\varepsilon_0\varepsilon(\omega)}{\mu_0}}E_0e^{ikz}\delta(\omega'-\omega)\vec{e}_x$

$\vec{H}(\vec{r},t)=\int_{-\infty}^{\infty}\vec{e}_x\sqrt{\frac{\varepsilon_0\varepsilon(\omega)}{\mu_0}}E_0e^{ikz}\delta(\omega'-\omega)e^{-i\omega't}d\omega'$

$=\sqrt{\frac{\varepsilon_0\varepsilon(\omega)}{\mu_0}}E_0e^{ikz}e^{-i\omega t}\vec{e}_x$

$\vec{H}(\vec{r})=\sqrt{\frac{\varepsilon_0\varepsilon(\omega)}{\mu_0}}E_0e^{ikz}\vec{e}_x$

c) $\langle\vec{S}(\vec{r},t)\rangle=\frac{1}{2}\mathrm{Re}[\vec{E}(\vec{r})\times\vec{H}(\vec{r})^*]$

$\vec{E}(\vec{r})=E_0e^{ikz}\vec{e}_y$

$\vec{H}(\vec{r})=\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\varepsilon(\omega)}E_0e^{ikz}\vec{e}_x$

∵ $\vec{k}$ is a complex wavevector

∴ $\vec{k}=k'+ik''$

$\vec{H}(\vec{r})^*=\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\varepsilon(\omega)}^*E_0e^{-k''z}e^{ik'z}\vec{e}_x$

$\cancel{∴ \langle\vec{S}(\vec{r},t)\rangle=\frac{1}{2}\left(E_0e^{-k''z}e^{ik'z}\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\varepsilon(\omega)}^*E_0e^{-k''z}e^{-ik'z}\right)\vec{e}_z}$

$\cancel{E_0^2\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\varepsilon(\omega)}}$

∴ $\langle\vec{S}(\vec{r},t)\rangle=\frac{1}{2}\mathrm{Re}\left(E_0e^{-k''z}e^{ik'z}\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\varepsilon(\omega)}^*E_0e^{-k''z}e^{-ik'z}\right)\vec{e}_z$

$=\frac{1}{2}\mathrm{Re}\left(E_0^2\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\varepsilon(\omega)}^*e^{-2k''z}\right)\vec{e}_z$

$=\frac{1}{2}E_0^2\sqrt{\frac{\varepsilon_0}{\mu_0}}n(\omega)e^{-2k''z}\vec{e}_z$

$\nabla\cdot\langle\vec{S}\rangle=\frac{1}{2}E_0^2\sqrt{\frac{\varepsilon_0}{\mu_0}}n(\omega)(-2k'')e^{-2k''z}$

$=-E_0^2\sqrt{\frac{\varepsilon_0}{\mu_0}}n(\omega)k''e^{-2k''z}$

d)

Solution:

i Propagating waves without loss

$\mathrm{div}\langle \vec{S} \rangle = 0 \quad \therefore \varepsilon'' = 0$

ii propagating waves with loss

$\mathrm{div}\langle \vec{S} \rangle > 0 \quad \therefore \varepsilon'' > 0$

ii) nonpropagating waves ~~with loss~~ without loss

$\mathrm{div}\langle \vec{S} \rangle = 0 \quad \therefore \varepsilon'' = 0$