Problem 1 MWE

a) linear, isotropic, dispersive inhomogeneous

$\bar{M}=0,\ \rho=0,\ j=0$

$$\bar{\nabla}\bar{D}(\bar{r},t)=0$$

$$\bar{\nabla}\bar{H}(\bar{r},t)=0$$

$$\bar{\nabla}\times\bar{E}(\bar{r},t)=-\mu_0\frac{\partial\bar{H}(\bar{r},t)}{\partial t}$$

$$\bar{\nabla}\times\bar{H}(\bar{r},t)=\frac{\partial\bar{D}(\bar{r},t)}{\partial t}$$

frequency Domain

b)

$$\bar{D}=\varepsilon_0\bar{E}(\bar{r},t)+\bar{P}(\bar{r},t)$$

Response of Material

$$=\varepsilon_0\bar{E}(\bar{r},t)+\varepsilon_0\int_{-\infty}^{\infty}R(\bar{r},t)\,\bar{E}(\bar{r},t-t')\,dt'$$

$$\bar{D}(\bar{r},\omega)=\varepsilon_0 E(\bar{r},\omega)+\varepsilon_0\chi(\bar{r},\omega)\bar{E}(\bar{r},\omega)$$

$$=\varepsilon_0\varepsilon(\bar{r},\omega)\bar{E}(\bar{r},\omega)$$

c) Waveequation for $\bar{E}(\bar{r},\omega), \bar{H}(\bar{r},\omega)$

$$\bar{\nabla}\times\bar{\nabla}\times \bar{E}(\bar{r},\omega) = i\omega\mu_0 \bar{\nabla}\times \bar{H}(\bar{r},\omega)$$

$$\bar{\nabla}(\bar{\nabla}\bar{E}(\bar{r},\omega)) - \bar{\nabla}^2\bar{E}(\bar{r},\omega) \overset{\text{IV}+b}{=} i\omega\mu_0\left(-i\omega\varepsilon_0\varepsilon(\bar{r},\omega)\bar{E}(\bar{r},\omega)\right)$$

$$\circledast\ \bar{\nabla}\bar{D}(\bar{r},\omega) = 0$$

$$\bar{\nabla}\varepsilon_0\varepsilon(\bar{r},\omega)\bar{E}(\bar{r},\omega) = 0$$

$$\varepsilon_0\left(\nabla\varepsilon(\bar{r},\omega)\right)\bar{E}(\bar{r},\omega) + \varepsilon_0\varepsilon(\bar{r},\omega)\bar{\nabla}\bar{E}(\bar{r},\omega) = 0$$

$$\nabla\bar{E}(\bar{r},\omega) = -\frac{\nabla\varepsilon(\bar{r},\omega)}{\varepsilon(\bar{r},\omega)}\bar{E}(\bar{r},\omega)$$

Waveequation $\bar{E}$

$$-\bar{\nabla}\left(\frac{\bar{\nabla}\varepsilon(\bar{r},\omega)}{\varepsilon(\bar{r},\omega)}\bar{E}(\bar{r},\omega)\right) = \bar{\nabla}^2\bar{E}(\bar{r},\omega) + \frac{\omega^2}{c^2}\varepsilon(\bar{r},\omega)\bar{E}(\bar{r},\omega)$$

$$\boxed{\mu_0\varepsilon_0 = \frac{1}{c^2}}$$

Waveequation $\bar{H}$

$$\bar{\nabla}^2\bar{H}(\bar{r},\omega) + \frac{\omega^2}{c^2}\varepsilon(\bar{r},\omega)\bar{H}(\bar{r},\omega) = i\omega\varepsilon_0\bar{\nabla}\varepsilon(\bar{r},\omega)\times\bar{E}(\bar{r},\omega)$$

Problem 2 Poynting vector   3+2+1+3

$$\bar{E}(x,y,z,t) = E_0 \cos((y+z)k - \omega t)\, \bar{e}_x$$

a) $\bar{H}(\bar{r},t)$ (MWE III)

$$\bar{\nabla} \times \bar{E} = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ 0 & \partial_y & \partial_z \\ E_x & 0 & 0 \end{vmatrix} = \partial_z E_x \bar{e}_y - \partial_y E_x \bar{e}_z$$

$$= E_0 k \sin((y+z)k - \omega t) \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\bar{H}(\bar{r},t) = -\frac{1}{\mu_0} \int dt (\bar{\nabla} \times \bar{E}) = \frac{E_0 k}{\mu_0 \omega} \cos((y+z)k - \omega t) \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

b) $$\langle S \rangle = \frac{1}{2} \mathrm{Re}\left[\bar{E}(\bar{r}) \times \bar{H}(\bar{r})^* \right]$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \bar{E}_r(\bar{r},t) \times \bar{H}_r(\bar{r},t)$$

$$I(r) = |\langle S \rangle|$$

c) $\nabla \langle S \rangle = 0$ propagating, lossless

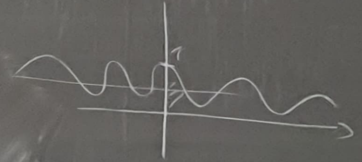
$\nabla \langle S \rangle > 0$ gain medium

$\nabla \langle S \rangle < 0$ lossy medium d)

$$\langle S(\vec{r},t)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \vec{E}_r \times \vec{H}_r \, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{E_0^2 k}{\omega\mu_0} \cos^2((y+z)k-\omega t)\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

$$\cos^2(a) = \frac{1+\cos(2a)}{2}$$



$$= \frac{kE_0^2}{2\mu_0\omega}\left(\vec{e_y}+\vec{e_z}\right)$$

$$\nabla\langle S\rangle = 0$$

Problem 3

a) homogeneous (position dependent)
isotropic (polarization independent)
dispersive (dependence on freq)

b) $\chi(\omega) = \int_{-\infty}^{\infty} R(t)\, e^{i\omega t}\, dt$

$$= \frac{f}{\omega_0^2 - \omega^2 - 2i\gamma\omega}$$

$$\chi' = \frac{f(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}$$

$$\chi'' = \frac{f\, 2\gamma\omega}{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}$$

$$\varepsilon' = 1 + \chi'$$
$$\varepsilon'' = \chi''$$

c)

c)

$$\bar{E}(\bar{r}) = E_0 \cos((g+ih)x) e^{(ig-h)z} \bar{e_y}$$

ges $\bar{k}_1 = k_1' + ik_1''$ $k_2' = k_2' + ik_2''$ $e^{(ik\bar{r})}$

eulers

$$\bar{E}(\bar{r}) = \frac{E_0}{2}\left[e^{i(g+ih)x + i(g+ih)z} + e^{i(-g-ih)x + i(g+ih)z} \right)$$

i) $\bar{k}_1' = \begin{pmatrix} g \\ 0 \\ g \end{pmatrix}$ $\bar{k}_1'' = \begin{pmatrix} h \\ 0 \\ h \end{pmatrix}$ | $\bar{k}_2' = \begin{pmatrix} -g \\ 0 \\ g \end{pmatrix}$ $\bar{k}_2 = \begin{pmatrix} -h \\ 0 \\ h \end{pmatrix}$

ii) Helmtholz equation

$$\left[\Delta + \left(\frac{\omega}{c}\right)^2 \varepsilon(\omega)\right] E_0(\omega) e^{ikr} = 0 \rightarrow$$

$$-(\bar{k}' + ik'')(\bar{k}' + ik'') + \left(\frac{\omega}{c}\right)^2 (\varepsilon(\omega) + i\varepsilon''(\omega)) = 0$$

$$-(\bar{k}'^2 - \bar{k}''^2 + 2i\bar{k}'\bar{k}'') + \left(\frac{\omega}{c}\right)^2 (\varepsilon(\omega) + i\varepsilon''(\omega)) = 0$$

split into real & complex part

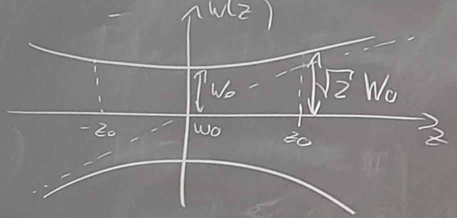
I $\bar{k}^2 - \bar{k}''^2 = +\left(\frac{\omega}{c}\right)^2 \varepsilon'(\omega)$ II $2\bar{k}'k'' = \left(\frac{\omega}{c}\right)^2 \varepsilon''(\omega)$

I $2g^2 - 2h^2 = -\left(\frac{\omega}{c}\right)^2 \varepsilon'(\omega)$

II $4gh = \left(\frac{\omega}{c}\right)^2 \varepsilon''(\omega)$

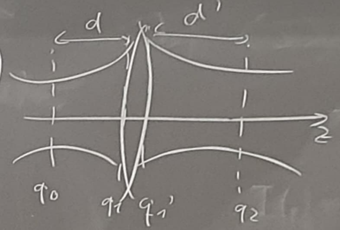
Problem 2 Gaussian Beams

a)



$R(z)$ & $w(z)$ increases linearily

b)



b)

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & d \\ -\frac{1}{f} & -\frac{d}{f} + 1 \end{pmatrix}$$

$$q = z - iz_0 \qquad q_0 = -iz_0$$

$$q_1' = -\frac{iz_0 + d}{\frac{iz_0}{f} - \frac{d}{f} + 1}$$

c) $$z_0' = \frac{z_0}{\frac{z_0^2}{f^2} + \left(1 - \frac{d}{f}\right)^2} \qquad d' = \left| \frac{d - \frac{d^2}{f} - \frac{z_0^2}{f}}{\left(\frac{z_0}{f}\right)^2 + \left(1 - \frac{d}{f}\right)^2} \right|$$

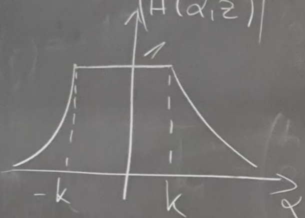
<u>ges</u> Rayleigh length & d'

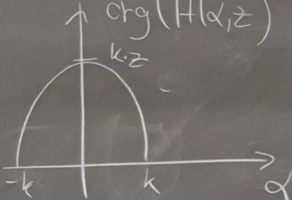
d) $d = f$ <u>ges</u> $M = w_0'/w_0 = \frac{f}{z_0}$

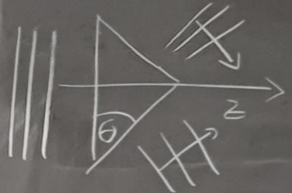
Problem 3 Beam Propagation

a) how to find $u(x,y,z>0)$ using
$$\boxed{H(\alpha,\beta,z) = e^{i\sqrt{(2\pi/\lambda)^2-\alpha^2-\beta^2}\, z}}$$

b) $\beta = 0$



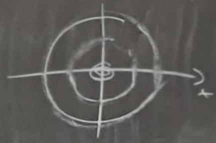




c) $t(x) = 2h_0 \cos((n-1)\theta k_0 x)$ $\qquad h_0 = e^{ik_0 d_0}$

$$U_0(\alpha) = e^{ik_0 d_0}\left[\delta((n-1)\theta k_0 - \alpha) + \delta((n-1)\theta k_0 + \alpha)\right]$$

d) Besselbeam

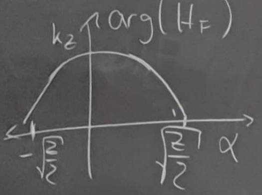


Problem 6 Diffraction

a) ges: condition for paraxial approximation $\boxed{\alpha^2, \beta^2 \ll k^2}$

$H(\alpha,\beta,z) = e^{i\gamma z}$ $\qquad \gamma(\alpha,\beta) = \sqrt{k^2-(\alpha^2+\beta^2)}$ $\qquad$ Taylor expansion

$H_F(\alpha,\beta,z) = e^{ikz - i\left(\frac{\alpha^2+\beta^2}{2k}\right)z}$ $\qquad \approx k\left(1-\frac{1}{2}\frac{\alpha^2+\beta}{k^2}\right)$

b) 

$\beta = 0$

$\frac{\alpha^2}{2k} = kz$

$\alpha = \frac{\sqrt{z}}{\sqrt{2}}$

c) $u(x,z) = e^{ikz}$

$$t(x) = \begin{cases} 1 & \text{if } \left(\frac{d}{2}-a\right) < |x| < \left(\frac{d}{2}+a\right) \\ 0 & \text{otherwise} \end{cases}$$

$$U(\alpha, z=0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, t(x)\, e^{-i\alpha x}$$

approximation $\boxed{\alpha^2, \beta^2 << k^2}$

$$\ldots) = \sqrt{k^2 - (\alpha^2 + \beta^2)}$$

Taylor expansion

$$\approx k\left(1 - \frac{1}{2}\,\alpha^2 + \beta \right)$$

$$U(\alpha, z=0) = \frac{1}{2\pi}\left[\int_{-(\frac{d}{2}+a)}^{-(\frac{d}{2}-a)} e^{-i\alpha x}\,dx + \int_{(\frac{d}{2}-a)}^{(\frac{d}{2}+a)} e^{-i\alpha x}\,dx\right.$$

$0,\ y = 0$

$$= -\frac{1}{2\pi i\alpha}\left(e^{i\alpha(\frac{d}{2}-a)} - e^{i\alpha(\frac{d}{2}+a)} + e^{-i\alpha(\frac{d}{2}+a)} - e^{-i\alpha(\frac{d}{2}-a)}\right)$$

$$= \frac{2a}{\pi}\left(\left(\frac{e^{i\alpha a} - e^{-i\alpha a}}{2i\alpha a}\right)\left(\frac{e^{i\alpha\frac{d}{2}} + e^{-i\alpha\frac{d}{2}}}{2}\right)\right)$$

$$= \frac{2a}{\pi}\,\mathrm{sinc}(\alpha a)\cos\left(\alpha\frac{d}{2}\right)$$

$$= \frac{a\,2}{a\,\pi\alpha}\sin(\alpha a)\cos\left(\alpha\frac{d}{2}\right)$$

$$\left(\tfrac{\ }{f}\right) + \left(1-\tfrac{d}{f}\right)^{-1} \qquad q_1' = \frac{\ldots + d}{\frac{iz_0}{f} - \frac{d}{f} + 1}$$

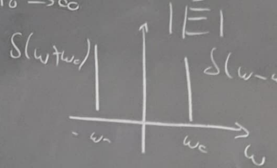
Rayleigh length $\ll d'$

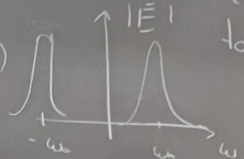
d) $d = f$ ges $\quad M = w_0'/w_0 = \frac{f}{z_0}$

Problem 7 Pulse Propagation

$$\bar{E}(\bar{r},t) = E_0\, e^{-\frac{t^2}{t_0^2}} \cos\left(\bar{k}(\omega_0)\bar{r} + \omega_0 t\right)$$

a) $t_0 \to \infty$





$t_0 = T$

b) dispersion $\to$ Pulse changes shape

c) $n(\omega) = l + m\omega^2 + s\omega^3 \quad l,m,s \in \mathbb{R}$

$$\frac{1}{v_{phase}} = \frac{k(\omega)}{\omega}\bigg|_{\omega=\omega_0} = \frac{n(\omega)}{c}\bigg|_{\omega=\omega_0}$$

$$\frac{1}{v_{group}} = \frac{\partial k(\omega)}{\partial \omega}\bigg|_{\omega=\omega_0} = \frac{1}{c}\left(l + 3m\omega_0^2 + 4s\omega_0^3\right)$$

$$D_\omega = \frac{\partial^2}{\partial \omega^2} k(\omega) = \frac{1}{c}\, 6\omega_0 (m + 2s\omega_0)$$

relation l, m, s & $T_0 = \frac{2\pi}{\omega_0} \quad t_0 = T$

$D_\omega \to 0 \quad t_0 > T$

$$0 = m + 2s\omega_0 \qquad \omega_0 = -\frac{m}{2s}$$