Fundamentals of Modern Optics WS 2013/14
Midterm Exam

to be written December 9, 8:15 – 9:45 a.m.

Problem 1 – Maxwell's equations **$4+5=9$ points**

a) Write down Maxwell's equations in the frequency domain for a linear, isotropic, non-magnetizable, inhomogeneous dielectric medium in absence of free charges and current density ($\rho = 0$ and $\mathbf{j} = 0$).

b) Show that for such a medium the wave equation for the electric field can be written as:

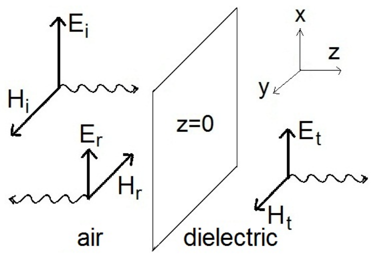
$$\Delta\mathbf{E}(\mathbf{r},\omega) + \frac{\omega^2}{c^2}\,\epsilon(\mathbf{r},\omega)\,\mathbf{E}(\mathbf{r},\omega) = -\mathbf{grad}\left\{\frac{\mathbf{grad}\,\epsilon(\mathbf{r},\omega)}{\epsilon(\mathbf{r},\omega)}\cdot\mathbf{E}(\mathbf{r},\omega)\right\}.$$

Problem 2 – Normal modes **$3+4+4=11$ points**

A semi-infinite block of some dielectric (with relative permittivity of $\epsilon' + i\epsilon''$) is illuminated perpendicularly on its surface with a plane monochromatic wave of frequency ω from air. The electric field of the incoming and the reflected and the transmitted waves have the form:

$$\mathbf{E}_\mathrm{i} = E_\mathrm{i}e^{i(k_0 z)}\hat{\mathbf{x}} \quad , \quad \mathbf{E}_\mathrm{r} = E_\mathrm{r}e^{i(-k_0 z)}\hat{\mathbf{x}} \quad , \quad \mathbf{E}_\mathrm{t} = E_\mathrm{t}e^{i(k_1 z)}\hat{\mathbf{x}}$$

respectively, where $k_0 = \omega/c$ and $k_1 = \frac{\omega}{c}\sqrt{\epsilon' + i\epsilon''} = \frac{\omega}{c}(n + i\kappa)$.



a) Find out the three corresponding magnetic fields using the Maxwell's equations.

b) Use the continuity of the tangential components of the electric and magnetic field at the interface between the two media to find E_t as a function of E_i.

c) Calculate the time averaged Poynting vector in the dielectric medium (transmitted power).
Hint: This energy flux in the dielectric medium should be a function of z.
Hint: If you failed to find E_t as a function of E_i from part b, you can write the transmitted Poynting vector as function of E_t.

Problem 3 – Beam propagation (Imaging) **$3+3+3=9$ points**

Given is the field directly behind a two dimensional phase mask

$$u_0(x,y,z=0) = A\exp\left(-i\pi\frac{x^2+y^2}{\lambda\, f}\right),$$

where $f > 0$. The field is propagating through vacuum.

a) Calculate the spatial frequency spectrum $U_0(\alpha,\beta;z=0)$.

b) By introducing the paraxial approximation, derive the free space transfer function ($H_F(\alpha,\beta;z)$). Indicate propagating and evanescent wave regions.

c) Calculate the field $u(x,y,z=f)$.

Problem 4 – Gaussian beams **4 + 2 = 6 points**

A laser at $\lambda_0 = 1000$ nm emits a beam with Gaussian profile and waist radius $W_0 = 2$ mm.

a) Use the matrix approach to calculate the waist position z' and waist radius W_0' of the beam after the lens. Approximate the result assuming that the waist of the incoming beam is on the lens and that its Rayleigh range $z_0 = \frac{\pi W_0^2}{\lambda_0}$ is much longer than the focal length f.

b) Choose a lens such that the waist radius of the focused beam is equal to $W_0' = 50\mu$m.

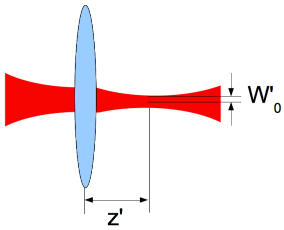


Figure 1: Sketch of the focusing arrangement.

Problem 5 – Pulse propagation **3 + 6 + 2 = 11 points**

A gaussian pulse travels through a $l = 2$mm long medium whose dispersive refractive index is defined as:

$$n(\omega) = B + C\omega^2$$

where $B = 1.5$ and $C = 8\times10^{-33}\text{s}^2$. Before entering the medium, the pulse has a bandwidth of $\omega_\text{s} = 100\times10^{12}$Hz and is centered around the carrier frequency $\omega_0 = 2 \times 10^{15}$Hz.

a) What are the phase and group velocities of the pulse? You may leave your answers in terms of the velocity of light c_0.

b) Calculate the pulse width after propagating through $z = l$.

c) Another pulse was simultaneously launched in a different medium whose $n(\omega)$ is the same as before with a small difference that $C = 0$ now. Calculate the difference between the time it takes for the two pulses to reach $z = l$.

Problem 6 – Fraunhofer diffraction **4 + 2 = 6 points**

a) Calculate the intensity of the diffracted field pattern $I(x, z_\text{B}) = |u(x, z_\text{B})|^2$ at $z = z_B$ in paraxial Fraunhofer approximation for two slits illuminated with a normally incident plane wave (just the diffraction pattern, no prefactors). The width of each slit is a ($a > \lambda$) and separated by a distance d ($d > a$):

$$u_0(x, z = 0) = \begin{cases} 1, & \text{for } |x \pm d/2| \leq a/2 \\ 0, & \text{elsewhere.} \end{cases}$$

b) Try to roughly sketch a figure of the intensity and identify the factor due to interference from the one due to slit diffraction.

Hint: The Fouriertransform of a single slit of width a is $\propto \text{sinc}(\alpha a)$.

— **Useful formulas** —

$\nabla \times \nabla \times \mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) - \Delta\mathbf{a}$

$\nabla \cdot (\nabla \times \mathbf{a}) = 0$

$\nabla \cdot (c\mathbf{a}) = \mathbf{a} \cdot \nabla c + c\, \nabla \cdot \mathbf{a}$

Gaussian q-parameter transform law:

$$q' = \frac{Aq + B}{Cq + D}$$

ABCD matrix for a thin lens:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$