A gaussian pulse travels through a $l = 2\text{mm}$ long medium whose dispersive refractive index is defined as:

$$n(\omega) = B + C\omega^2$$

where $B = 1.5$ and $C = 8\times10^{-33}\text{s}^2$. Before entering the medium, the pulse has a bandwidth of $\omega_s = 100\times10^{12}\text{Hz}$ and is centered around the carrier frequency $\omega_0 = 2\times10^{15}\text{Hz}$.

a) What are the phase and group velocities of the pulse? You may leave your answers in terms of the velocity of light c_0.

b) Calculate the pulse width after propagating through $z = l$.

c) Another pulse was simultaneously launched in a different medium whose $n(\omega)$ is the same as before with a small difference that $C = 0$ now. Calculate the difference between the time it takes for the two pulses to reach $z = l$.

(a) $k^2 = \frac{w^2}{c_0^2}\varepsilon(w) \Rightarrow k = \frac{w}{c_0}n(w) = \frac{1}{c_0}(Bw + Cw^3)$

$$V_{ph} = \frac{w_0}{k} = \frac{c_0}{B + Cw_0^2} = \frac{c_0}{1.5 + 8\times10^{-33}s^2\cdot(2\times10^{15})^2 s^{-2}} = \frac{c_0}{1.532}$$

$$V_g = \frac{dw}{dk}\Big|_{w=w_0} \quad \frac{dk}{dw}\Big|_{w=w_0} = \frac{1}{c_0}(B + 3Cw_0^2) \Rightarrow V_g = \frac{c_0}{B + 3Cw_0^2} = \frac{c_0}{1.5 + 3\times8\times10^{-33}\times4\times10^{30}} = \frac{c_0}{1.596}$$

(b) $T(z) = T_0\sqrt{1+\frac{z^2}{z_0^2}}$ $\quad z_0 = -\frac{T_0^2}{2D}$ $\quad D = \frac{d^2k}{dw^2}\Big|_{w=w_0} = \frac{6Cw_0}{c_0} \Rightarrow |z_0| = \frac{c_0T_0^2}{12Cw_0}$ $\quad T_0 = \frac{2}{w_s}$

$$|z_0| = \frac{c_0T_0^2}{12Cw_0} = \frac{c_0}{3Cw_0w_s^2} = \frac{c_0}{3\times8\times10^{-33}s^2\cdot2\times10^{15}s^{-1}\cdot1\times10^{28}s^{-2}} = \frac{c_0}{48\times10^{10}s^{-1}}$$

$$\frac{l^2}{z_0^2} = \frac{4mm^2}{c_0^2}(48\times10^{10}s^{-1})^2 = \frac{4\times10^{-6}m^2}{(3\times10^8 m/s)^2}\times2.304\times10^{23}s^{-2} = \frac{4}{9}\times2.304\times10 = 10.24$$

$$T(l) = \frac{1}{w_s}\sqrt{1+\frac{l^2}{z_0^2}} = 1\times10^{-14}s\sqrt{1+10.24} \approx 3.35\times10^{-14}s \Rightarrow W(L) = 5.97\times10^{13}Hz$$

$$T(z)\cdot W(z) = 2 \Rightarrow W_z = \frac{2}{T_0\sqrt{1+\frac{z^2}{z_0^2}}} = \frac{w_s}{\sqrt{1+\frac{z^2}{z_0^2}}} = w_s\sqrt{\frac{z_0^2}{z_0^2+z^2}}$$

(c) $t_1 = \frac{l}{V_g} = \frac{1.596\,l}{c_0} = 1.064\times10^{-11}s$

$k'(w) = \frac{w}{c_0}n(w) = \frac{w}{c_0}B$ $\quad \frac{dk}{dw} = \frac{B}{c_0} \Rightarrow V_g = \frac{c_0}{B}$ $\quad t_2 = \frac{lB}{c_0} = \frac{3\times10^{-3}m}{3\times10^8 m/s} = 10^{-11}s$

$$\Delta t = t_1 - t_2 = 0.064\times10^{-11}s = 6.4\times10^{-13}s$$

A pulse is propagating in a homogeneous material with the dielectric function given by

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \quad , \qquad (1)$$

where the pulse's carrier frequency ω_0 is much larger than the plasma frequency ω_p.

a) Calculate the group velocity of the pulse in respect of ω_0.

b) The pulse is propagating towards a detector. Which frequencies arrive earlier, the ones higher than the carrier frequency or the lower ones? Prove with a (short) calculation.

c) Now consider a second pulse of the different carrier frequency $\omega_2 \gg \omega_p$, propagating in the same direction. Calculate the time delay between both pulses after a length L.

(a) $k^2 = \frac{w^2}{c^2}\varepsilon(w) \Rightarrow k(w) = \frac{w}{c}\sqrt{\varepsilon(w)} = \frac{w}{c}\sqrt{1-\frac{w_p^2}{w^2}}$

$$\frac{dk}{dw}\Big|_{w=w_0} = \frac{1}{c}\sqrt{1-\frac{w_p^2}{w^2}} + \frac{w}{c}\left[\frac{1}{2}\left(1-\frac{w_p^2}{w^2}\right)^{-\frac{1}{2}}\frac{2w_p^2}{w^3}\right] = \left[\frac{1}{c}\sqrt{1-\frac{w_p^2}{w^2}} + \frac{1}{c}\left(1-\frac{w_p^2}{w^2}\right)^{-\frac{1}{2}}\frac{w_p^2}{w^2}\right]_{w_0} \approx \frac{1}{c}$$

$$V_g = \frac{dw}{dk}\Big|_{w=w_0} \approx c$$

(b) $n(w) = \sqrt{\varepsilon(w)} = \sqrt{1-\frac{w_p^2}{w^2}}$ $\quad V = \frac{c}{n(w)} = \frac{c}{\sqrt{1-\frac{w_p^2}{w^2}}}$ $\quad w\uparrow \to \frac{w_p^2}{w^2}\downarrow \to \sqrt{1-\frac{w_p^2}{w^2}}\uparrow \to V\downarrow$

$$t = \frac{L}{V} \Rightarrow w\uparrow \to t\uparrow$$

Thus the waves with higher frequencies take more time to arrive and so waves with lower frequencies arrive earlier

(c) $V_1 = \frac{c}{\sqrt{1-\frac{w_p^2}{w^2}}}$ $V_2 = \frac{c}{\sqrt{1-\frac{w_p^2}{w_2^2}}}$ $\Delta t = \frac{L}{V_1} - \frac{L}{V_2} = L\left(\frac{1}{\sqrt{1-\frac{w_p^2}{w^2}}} - \frac{1}{\sqrt{1-\frac{w_p^2}{w_2^2}}}\right)$

Consider a laser source with an output power of 100 mW and a repetition rate of 100 MHz. The output of such a source is a sequence of transform-limited Gaussian pulses with central frequency ω_0. The envelope of each individual pulse in its co-moving frame is defined as $E(t') = E_0 \exp\left[-t'^2/\tau^2\right]$, where the pulse width is $\tau = 8$ ps. This pulse sequence is launched into a fiber characterized by

→ Pulse duration

→ the number of Pulses emitted per second N/s

$$k(\omega) = k_0 + \frac{1.5}{c}(\omega-\omega_0) + \frac{D}{2}(\omega-\omega_0)^2,$$

with $D = 0.08\ \mathrm{ps}^2/\mathrm{m}$.

$\frac{J/s}{N/s} = \frac{J}{N}$

a) Calculate the energy of each individual pulse.

b) Find the dispersion length L_D of each individual pulse. Is the red or the blue part of the spectrum appearing earlier at the end of the fiber?

c) After a fiber length of $L_1 = 4$ km, a second type of fiber is connected to the first. It has a dispersion of $-2000\ \mathrm{fs}^2/\mathrm{m}$ and a length of L_2. How long does L_2 have to be in order to fully restore the initial pulse sequence?

$1T = 10^{12}$

$1ps = 10^{-12}s$ $1fs = 10^{-15}s$

$1fs = 10^{-3}ps$

d) Now, a pulse sequence with a different frequency $\omega_1 = \omega_0 + \delta\omega$ is launched into the first fiber. Suppose that the detuning $\delta\omega = 1$ THz. Find the group index n_g in that case.

(a) $1\ MHz = 10^6\ Hz = 10^6\ s^{-1}$ $T = \frac{1}{100MHz} = 10^{-8}s$ $\tau = 8ps = 8\times10^{-12}s$

$I = E(t')^2 = E_0^2 \exp\left(\frac{-2t'^2}{\tau^2}\right)$ $E = \int_{-\infty}^{\infty} I\,dt = E_0^2 \int_{-\infty}^{\infty} \exp\left(-\frac{2t'^2}{\tau^2}\right)dt' = \frac{\tau E_0^2\sqrt{\pi}}{\sqrt{2}}$

For a pulsed laser, the output power is the averag power of the laser

Pulse Eenergy $= \frac{\text{Average Power}}{\text{Repetition Rate}}$ Peak Power $= \frac{\text{Pulse Energy}}{\text{Pulse Duration}}$

Pulse Energy $E = \frac{100mW}{100MHz} = \frac{0.1W}{10^8 s^{-1}} = 1\times10^{-9}J$.

(b) $L_D = 2|z_0| = \frac{T_0^2}{D} = \frac{64\ ps^2}{0.08\ ps^2/m} = 800m$ Red Part appear earlier

(c) $\tau(z) = \tau_0\sqrt{1+\left(\frac{z}{z_0}\right)^2} = \tau_0\sqrt{1+\left(\frac{2Dz}{\tau_0^2}\right)^2} \Rightarrow \tau(4) = \tau_0\sqrt{1+\left(\frac{2DL_1}{\tau_0^2}\right)^2} = 8\sqrt{101}\ (ps)$

$\Delta\tau = \tau(L_1) - \tau_0 = 8(\sqrt{101}-1)\ ps$

$\tau'(z) = \tau(L_1)\sqrt{1+\left(\frac{z}{z_0'}\right)^2} = \tau(L_1)\sqrt{1+\left(\frac{2D'z}{\tau(L_1)^2}\right)^2}$ $\tau'(L_2) = \tau(L_1)\sqrt{1+\left(\frac{2D'L_2}{\tau(L_1)^2}\right)^2}$

$\Delta\tau' = \tau'(L_2) - \tau(L_1) = \tau(L_1)\sqrt{1+\left(\frac{2D'L_2}{\tau(L_1)^2}\right)^2} - \tau(L_1) = \Delta\tau = 8(\sqrt{101}-1)ps$

$= \sqrt{1+\left(\frac{2D'L_2}{\tau(L_1)^2}\right)^2} - 1 = \frac{8(\sqrt{101}-1)ps}{8\sqrt{101}\ ps} = 1 - \frac{1}{\sqrt{101}} \Rightarrow 1 + \left(\frac{2D'L_2}{\tau(L_1)^2}\right)^2 = \left(2-\frac{1}{\sqrt{101}}\right)^2$

$\Rightarrow \frac{4D'^2L_2^2}{\tau(L_1)^4} = 3 + \frac{1}{101} - \frac{4}{\sqrt{101}} = \frac{304-4\sqrt{101}}{101}$ $-2000\ fs^2/m = -2\times10^{-3}\ ps^2/m$

$\Rightarrow \frac{4\times4\times10^{-6}\ ps^4/m^2\cdot L_2^2}{(64\times101)^2\ ps^4} = \frac{L_2^2}{4\times64\times101^2\times10^6\ m^2} = \frac{304-4\sqrt{101}}{101} \Rightarrow L \approx 8300020\ m$

(d) $n_g = \frac{c}{v_g}$ $k(w) = k_0 + \frac{1.5}{c}(w-w_0) + \frac{D}{2}(w-w_0)^2 = k_0 + \frac{1.5}{c}w - \frac{1.5}{c}w_0 + \frac{D}{2}w^2 - Dw_0w + \frac{D}{2}w_0^2$

$n_g = c\left.\frac{dk}{dw}\right|_{w=w_1}$ $\frac{dk}{dw} = \frac{1.5}{c} + Dw - Dw_0$ $\left.\frac{dk}{dw}\right|_{w_1=w_0+\delta w} = \frac{1.5}{c} + D(w_0+\delta w) - Dw_0$

$k(w) = \frac{w}{c}n(w)$ $n_g = c\left(\frac{1}{c}n(w) + \frac{w}{c}\frac{dn(w)}{dw}\right) = n(w) + w\frac{dn(w)}{dw}$ $= \frac{1.5}{c} + D\delta w$

$n_g = c\left.\frac{dk}{d\omega}\right|_{\omega=\omega_1} = 1.5 + cD\delta\omega = 1.5 + 3\times10^{8}\,m/s \cdot 0.08\,ps^2/m \cdot 10^{12}\,s^{-1} = 1.5 + 2.4\times10^{-5}$

a) Describe how the first three coefficients of the Taylor expansion of $k(\omega)$ with respect to ω at the center frequency ω_0 are connected to physical parameters of pulse propagation and explain their physical meaning.

b) A Gaussian pulse with center frequency ω_1 is launched into a material at $t = 0$. A second Gaussian pulse with center frequency ω_2 is launched into the same material at a later time $t = \Delta t$. The refractive index of the material is given by $n(\omega) = A + B\omega^2$ with $A > 0$ and $B > 0$. State under which conditions the second pulse can catch up to the first one. Derive the expression for the time t at which that happens in terms of ω_1 and ω_2.

c) By analogy of diffraction to pulse propagation in dispersive media argue how the temporal intensity profile $|v(t,z)|^2$ of a pulse envelope $v(t, z = 0)$ looks like for very large z when you neglect dispersion terms higher than second order.

(a) $k(\omega)\big|_{\omega\approx\omega_0} = k(\omega_0) + \frac{\partial k}{\partial\omega}(\omega-\omega_0) + \frac{1}{2}\frac{\partial^2 k}{\partial\omega^2}(\omega-\omega_0)^2$

phase velocity $V_{ph} = \frac{\omega}{k(\omega_0)} \Rightarrow k(\omega_0) = \frac{\omega}{V_{ph}}$ phase velocity is the speed at which phase of a wave propagates

Group Velocity $V_g = \frac{d\omega}{dk} \Rightarrow \frac{dk}{d\omega} = \frac{1}{V_g}$ Group velocity is the speed at which a envelop of waves with different frequencies propagate

Group velocity dispersion $D = \frac{\partial^2 k}{\partial\omega^2}$ D changes the Pulse shape upon Propagation.

(b) $k(\omega) = \frac{\omega}{c}n(\omega) = \frac{1}{c}(A\omega + B\omega^3)$ $\quad \frac{1}{V_{g_1}} = \left.\frac{dk}{d\omega}\right|_{\omega=\omega_1} = \frac{1}{c}(A+3B\omega_1^2) \Rightarrow V_{g_1} = \frac{c}{A+3B\omega_1^2}$

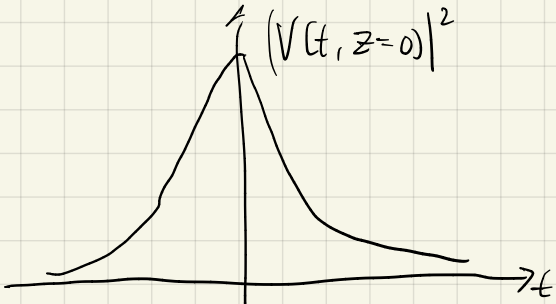
$\frac{1}{V_{g_2}} = \left.\frac{dk}{d\omega}\right|_{\omega=\omega_2} = \frac{1}{c}(A+3B\omega_2^2) \Rightarrow V_{g_2} = \frac{c}{A+3B\omega_2^2}$

The second Pulse catches up to the first one $\Rightarrow V_{g_2} > V_{g_1} \Rightarrow \underline{\omega_2 < \omega_1}$

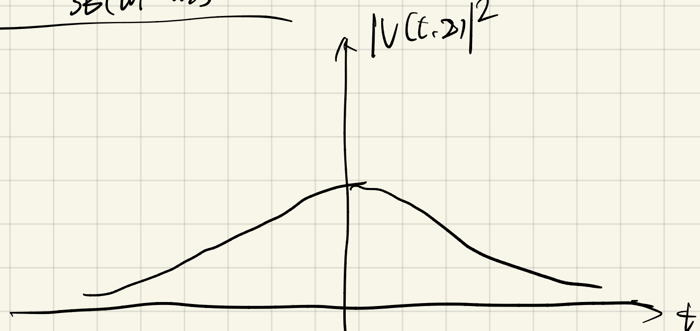
$V_{g_1}t = V_{g_2}(t-\Delta t) \quad \frac{ct}{A+3B\omega_1^2} = \frac{c(t-\Delta t)}{A+3B\omega_2^2} \Rightarrow \frac{A+3B\omega_2^2}{A+3B\omega_1^2} = \frac{t-\Delta t}{t} = 1-\frac{\Delta t}{t}$

$\Rightarrow \frac{\Delta t}{t} = 1 - \frac{A+3B\omega_2^2}{A+3B\omega_1^2} = \frac{3B(\omega_1^2-\omega_2^2)}{A+3B\omega_1^2} \Rightarrow \underline{t = \frac{(A+3B\omega_1^2)\Delta t}{3B(\omega_1^2-\omega_2^2)}}$

(c)



$\Rightarrow$



The evolution equation for the slowly varying envelope $\tilde{v}(z,\tau)$ of a pulse with central frequency ω_0 in the co-moving frame is given by

$$i\frac{\partial\tilde{v}(z,\tau)}{\partial z} - \frac{D}{2}\frac{\partial^2\tilde{v}(z,\tau)}{\partial\tau^2} = 0.$$

a) Which approximations are applied to derive the equation above from Maxwell's equations? Name them and state under which conditions they are applicable.

b) Define the co-moving frame τ and the dispersion D in the above equation and give their value in terms of the time t and the wavenumber $k(\omega)$.

c) What are the equivalents of $\partial/\partial\tau$ and D in the analogous beam diffraction equation?

d) By analogy to diffraction argue how $|\tilde{v}(z,\tau)|$ looks for very large z when the initial excitation is a temporally localized pulse $\tilde{v}_0(\tau)$.

(a) Paraxial Approximation: $k^2 \gg \alpha^2 + \beta^2$

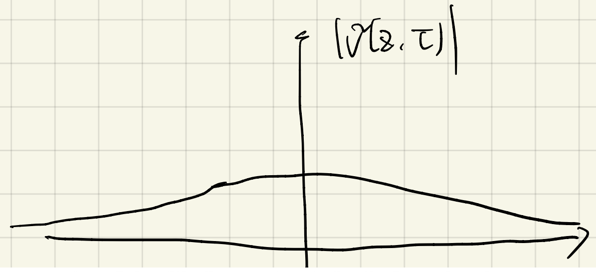
Parabolic Approximation: Pulse Beams are not too short.

(b) $\tau = (t - \frac{z}{V_g})$ comoving reference time.

$D = \frac{\partial^2 k(\omega)}{\partial\omega^2}$ (group velocity dispersion)

(c) $\frac{\partial}{\partial t} \rightarrow \nabla \qquad D \rightarrow -\frac{1}{k}$

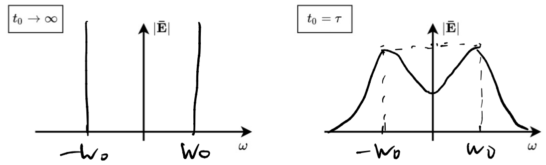
(d)



The given plane wave is modulated in the time domain by the Gauss-function:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \exp\left(-\frac{t^2}{t_0^2}\right)\cos(\mathbf{k}(\omega_0)\cdot\mathbf{r} + \omega_0 t)$$

a) Sketch the magnitude of the Fourier representation (with respect to time) $|\tilde{\mathbf{E}}(\mathbf{r},\omega)| = |FT(\mathbf{E}(\mathbf{r},t))|$ of the given wave schematically for the case $t_0 \to \infty$ **and** $t_0 = \tau$, where τ is a finite constant (calculation of Fourier representation is not necessary).



b) Would you expect changes of the given pulse-shapes when propagating through dispersive **and** non-dispersive media? Please, give a **short** explanation.

c) The polynomial dispersion term of the medium is assumed as: $n(\omega) = l + m\omega^2 + s\omega^3$, where $l, m, s \in \mathbb{R}$. Calculate the phase velocity, group velocity and group velocity dispersion in this medium. Which condition (relation between l, m, s and ω_0, **and** $T_0 = 2\pi/\omega_0$ and $t_0 = \tau$, see task a)) has to be fulfilled so that the first two quantities (phase and group velocity) remain applicable for the pulse propagation description?

(a) $t_0 \to \infty$: $\vec{E}(\vec{r},t) = \vec{E}_0 \cos(\vec{k}(w_0)\cdot\vec{r} + w_0 t) = \frac{E_0}{2}\left(e^{i\vec{k}(w_0)\cdot\vec{r}} e^{iw_0 t} + e^{-i\vec{k}(w_0)\cdot\vec{r}} e^{-iw_0 t}\right)$

$$\vec{E}(\vec{r},w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{E_0}{2}\left(e^{i\vec{k}(w_0)\cdot\vec{r}} e^{iw_0 t} + e^{-i\vec{k}(w_0)\cdot\vec{r}} e^{-iw_0 t}\right) e^{-iwt}\, dt$$

$$= \frac{E_0}{4\pi} e^{i\vec{k}(w_0)\cdot\vec{r}} \int_{-\infty}^{\infty} e^{i(w_0-w)t}\, dt + \frac{E_0}{4\pi} e^{-i\vec{k}(w_0)\cdot\vec{r}} \int_{-\infty}^{\infty} e^{-i(w_0+w)t}\, dt$$

$$= \frac{E_0}{2}\left[e^{i\vec{k}(w_0)\cdot\vec{r}}\, \delta(w_0-w) + e^{-i\vec{k}(w_0)\cdot\vec{r}}\, \delta(w_0+w)\right]$$

$$|\vec{E}(\vec{r},w)|^2 = \frac{E_0}{4}\left[e^{i\vec{k}(w_0)\cdot\vec{r}}\, \delta(w_0-w) + e^{-i\vec{k}(w_0)\cdot\vec{r}}\, \delta(w_0+w)\right]\left[e^{-i\vec{k}(w_0)\cdot\vec{r}}\, \delta(w_0-w) + e^{i\vec{k}(w_0)\cdot\vec{r}}\, \delta(w_0+w)\right]$$

$$= \frac{E_0}{4}\left[\delta^2(w_0-w) + e^{-2i\vec{k}(w_0)\cdot\vec{r}}\, \delta(w_0-w)\delta(w+w) + e^{2i\vec{k}(w_0)\cdot\vec{r}}\, \delta(w_0+w)\delta(w-w_0) + \delta^2(w_0+w)\right]$$

$$= \frac{E_0}{4}\left[\delta^2(w_0-w) + \delta^2(w_0+w) + 2\delta(w_0-w)\delta(w_0+w)\cos \vec{k}(w_0)\cdot\vec{r}\right]$$

(b) $t_0 = \tau$: $\vec{E}(\vec{r},t) = E_0 \exp\left(-\frac{t^2}{\tau^2}\right)\cos(\vec{k}(w_0)\cdot\vec{r} + w_0 t) = \frac{E_0}{2}\exp\left(-\frac{t^2}{\tau^2}\right)\left[e^{i\vec{k}(w_0)\cdot\vec{r}} e^{iw_0 t} + e^{-i\vec{k}(w_0)\cdot\vec{r}} e^{-iw_0 t}\right]$

$$\vec{E}(\vec{r},w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \vec{E}(\vec{r},t)\, dt = \frac{E_0}{4\pi} e^{i\vec{k}(w_0)\cdot\vec{r}} \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{\tau^2} + iw_0 t - iwt\right) dt + \frac{E_0}{4\pi} e^{-i\vec{k}(w_0)\cdot\vec{r}} \int_{-\infty}^{\infty} \exp\left[-\frac{t^2}{\tau^2} - iw_0 t - iwt\right] dt$$

$$= \frac{E_0}{4\pi} e^{i\vec{k}(w_0)\cdot\vec{r}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{\tau^2}\left[t^2 - i(w_0-w)\tau^2 t\right]\right\} dt + \frac{E_0}{4\pi} e^{-i\vec{k}(w_0)\cdot\vec{r}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{\tau^2}\left[t^2 + i(w_0+w)\tau^2 t\right]\right\} dt$$

$$= \frac{E_0}{4\pi} e^{i\vec{k}(w_0)\cdot\vec{r}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{\tau^2}\left[t - \frac{i(w_0-w)}{2}\tau^2\right]^2 - \frac{(w_0-w)^2\tau^2}{4}\right\} dt + \frac{E_0}{4\pi} e^{-i\vec{k}(w_0)\cdot\vec{r}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{\tau^2}\left[t + \frac{i(w_0+w)}{2}\tau^2\right]^2 - \frac{(w_0+w)^2\tau^2}{4}\right\} dt$$

$$= \frac{E_0 \tau\sqrt{\pi}}{4\pi} \exp\left[-\frac{(w_0-w)^2\tau^2}{4}\right] e^{i\vec{k}(w_0)\cdot\vec{r}} + \frac{E_0 \tau\sqrt{\pi}}{4\pi} \exp\left[-\frac{(w_0+w)^2\tau^2}{4}\right] e^{-i\vec{k}(w_0)\cdot\vec{r}}$$

(b) pulse-shapes change in dispersive media, because in dispersive media, waves with different frequencies travels at different speeds. So pulse-shapes broaden. But Pulse-shapes won't change in non-dispersive media, Because waves with different frequencies travel at the same speed.

(c) $k(w) = \frac{w}{c} n(w) = \frac{1}{c}(lw + mw^3 + sw^4)$

$V_{phase} = \frac{w}{k(w)}\Big|_{w=w_0} = \frac{c}{\ell + m w_0^2 + S w_0^3}$ $\quad \frac{dk}{dw}\Big|_{w=w_0} = \frac{1}{c}(\ell + 3m w_0^2 + 4 S w_0^3)$ $\quad v_g = \frac{dw}{dk} = \frac{c}{\ell+3mw_0^2+4Sw_0^3}$

$D = \frac{d^2k}{dw^2}\Big|_{w=w_0} = \frac{1}{c}(6m w_0 + 12 S w_0^2)$

No broadening $\rightarrow D = 0 \Rightarrow 6mw + 12 S w_0^2 = 0 \Rightarrow w_0 = -\frac{m}{2S}$

a) Describe the physical meanings of the phase velocity, the group velocity, and the group velocity dispersion in one sentence each. Write down the defining formulas as functions of the wave number in free space k_0, the angular frequency ω, and the frequency dependent refractive index of the medium $n(\omega)$.

b) There are striking similarities between the free-space propagated total field equations that describe the paraxial beam diffraction and the pulse dispersion after propagation of z–distance. Fill in the blank spaces (marked with underlines) in the following table by drawing analogies between the beam diffraction and the pulse dispersion.

Monochromaticy **Beam diffraction**	Plane wave **Pulse dispersion**
(x, y)	τ
(α, β) $\exp[-\frac{iz}{2k}(\alpha^2+\beta^2)]$	$\bar{w}$ $\exp(\frac{izD}{2}\bar{w}^2)$
$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$	$\partial/\partial\tau$
Width: W_0 at $z = 0$, $W(z)$	T_0 at $z=0$ $T(z)$
$-\frac{1}{k}$	D
Diffraction length: $L_B = 2z_0 = \frac{2\pi}{\lambda}W_0^2$	Dispersion length: $L_D = 2\lvert z_0\rvert = \frac{T_0^2}{D}$
broadening in spatial domain	broadening in **temporal** domain
Paraxial wave equation for a beam: $i\frac{\partial\tilde{v}(x,y,z)}{\partial z} + \frac{1}{2k_0}\Delta^2\tilde{v}(x,y,z) = 0$	Paraxial wave equation for a pulse: $i\frac{\partial\tilde{v}(z,\tau)}{\partial z} - \frac{D}{2}\frac{\partial^2}{\partial\tau^2}\tilde{v}(z,\tau)$

c) With the knowledge above in mind, write down the total field equation of an unchirped Gaussian pulse after a propagation of z–distance. Remember that when we propagated a Gaussian beam by a z–distance, we have obtained the total field as

$$v(x,y,z) = A_0\frac{1}{\sqrt{1+\left(\frac{z}{z_0}\right)^2}}\exp\left(-\frac{x^2+y^2}{W_0^2\left(1+\left(\frac{z}{z_0}\right)^2\right)}\right)\exp\left(i\frac{k}{2}\frac{x^2+y^2}{z\left(1+\left(\frac{z_0}{z}\right)^2\right)}\right)\exp(i\varphi(z)).$$

Hint: The phase curvature of a beam is analogous to the chirp parameter $C(z) = -\frac{z}{z_0\left(1+\frac{z^2}{z_0^2}\right)}$ that also describes the phase curvature in time.

(a) Phase velocity: The speed at which phase of wave propagate

Group velocity: The speed at which a envelope of waves propagate

Group velocity dispersion, changes of group velocity with respect to frequency

$V_{ph} = \frac{w}{k_0}$ $\quad V_{gr} = \frac{dw}{dk}\Big|_{k=k_0}$ $\quad D = \frac{d^2k}{dw^2}$

$\tilde{v}(x,y,z) \Rightarrow \tilde{v}(\tau,z) \Rightarrow \tilde{v}(z,\tau)$

$\tilde{v}(\tau,z) = \frac{A_0}{\sqrt{1+\frac{z^2}{z_0^2}}}\exp\left[-\frac{\tau^2}{T_0^2(1+\frac{z^2}{z_0^2})}\right]\exp\left[-\frac{i}{2D}\frac{\tau^2}{z(1+\frac{z_0^2}{z^2})}\right]$

$A(z) = A_0\sqrt{\frac{T_0}{T(z)}} = A_0\sqrt[4]{\frac{1}{1+\frac{z^2}{z_0^2}}}$ $\quad A^2(z)\cdot T(z) = A_0^2 T_0$

$\tilde{v}(x,y,z) = A(z)\exp(-\frac{x^2+y^2}{W(z)^2})\exp[\frac{ik}{2}\frac{x^2+y^2}{R(z)})e^{i\varphi(z)}$ $\quad A(z) = \frac{A}{\sqrt{1+\frac{z^2}{z_0^2}}}$

$\tilde{v}(z,\tau) = A(z)\exp(-\frac{\tau^2}{T(z)^2})\exp[-\frac{i}{2D}\frac{\tau^2}{R(z)}]$ $\quad A(z) = A_0\sqrt[4]{\frac{1}{1+\frac{z^2}{z_0^2}}}$

$\boxed{C(z) = -\frac{z_0}{R(z)} = \frac{T_0^2}{2DR(z)}} \Rightarrow \frac{1}{2D} = \frac{C(z)R(z)}{T_0^2}$

$\Rightarrow$ Chirp: $\tilde{v}(z,\tau) = A(z)\exp[-\frac{\tau^2}{T(z)^2}]\exp[-iC(z)\frac{\tau^2}{T_0^2}]$

Initial pulse $z=0 \Rightarrow \tilde{v}(z,\tau) = A_0\exp(-\frac{\tau^2}{T_0^2})\exp(-iC_0\frac{\tau^2}{T_0^2}) = A_0\exp[-(1+iC_0)\frac{\tau^2}{T_0^2}]$

$V_0(\bar{w}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{v}(z,\tau)e^{-i\bar{w}\tau}d\tau = \frac{A_0}{2\pi}\int_{-\infty}^{\infty}\exp[-(1+iC_0)\frac{\tau^2}{T_0^2} - i\bar{w}\tau]d\tau$

$= \frac{A_0}{2\pi}\int_{-\infty}^{\infty}\exp[-\frac{1+iC_0}{T_0^2}(\tau^2 + \frac{i\bar{w}T_0^2}{1+iC_0}\tau)] = \frac{A}{2\pi}\int_{-\infty}^{\infty}\exp\{-\frac{1+iC_0}{T_0^2}[\tau + \frac{i\bar{w}T_0^2}{2(1+iC_0)}\tau]^2 - \frac{\bar{w}^2T_0^2}{4(1+iC_0)}\}d\tau$

$= \frac{A}{2\pi}\exp[-\frac{\bar{w}^2T_0^2}{4(1+iC_0)}]\int_{-\infty}^{\infty}\exp\{-\frac{1+iC_0}{T_0^2}[\tau + \frac{i\bar{w}T_0^2}{2(1+iC_0)}]^2\}d\tau$

$= \frac{A\sqrt{\pi}T_0}{2\pi\sqrt{1+iC_0}}\exp[-\frac{\bar{w}^2T_0^2}{4(1+iC_0)}] = \frac{A\sqrt{\pi}T_0}{2\pi\sqrt{1+iC_0}}\exp[-\frac{\bar{w}^2T_0^2(1-iC_0)}{4(1+C_0^2)}]$

$\bar{w}_s^2 = \frac{4(1+C_0^2)}{T_0^2}$ $\quad$ Transform limited: $\bar{w}_s^2 = \frac{4}{T_0^2}$