Structure of matter: Homework to exercise 6

Waves and Particles

$J = C\,V$

Due on November 14th 2023

$e = 1.602\times10^{-19}\,A\cdot s$

Please indicate your name on the solution sheets and send it to your seminar leader! $= 1.602\times10^{-19}\,C$

$E = h\nu \Rightarrow \nu = \frac{0.5\,eV}{6.625\times10^{-34}\,W\cdot s^2} = \frac{0.5\times1.602\times10^{-19}\,C\cdot V}{6.625\times10^{-34}\,J\cdot s} = 1.2\times10^{14}\,s^{-1}$

1. Multiple-choice test: Please tick all box(es) with correct answer(s)!
   (correctly ticked box: +1/2 point; wrongly ticked box: -1/2 point)

A photon with a photon energy of 0.5eV corresponds to the	Visible spectral range	✓
	γ-ray spectral range	
	X-ray spectral range	
In the X-ray spectral region, the refractive index of a non-magnetic material is usually	Infinitely large	
	Larger than zero	✓
	Smaller than 1	✓
	Equal to one	
	Negative	

$\frac{3\times10^8\,m/s}{1.2\times10^{14}\,s^{-1}}$

$2.5\times10^{-6}\,m$

$2.5\,\mu m$

$\frac{\lambda}{T} = c$

$\lambda = cT = \frac{c}{\nu}$

2. True or wrong? Make your decision!

   (2 points): 1 point per correct decision, 0 points per wrong or no decision

$k^2 = \frac{\omega^2}{c^2}\varepsilon(\omega) \Rightarrow k = \frac{\omega}{c}\sqrt{\varepsilon(\omega)}$ $\quad \sqrt{\varepsilon(\omega)} = Re\sqrt{\varepsilon(\omega)} + i\,Im\sqrt{\varepsilon(\omega)}$

Assertion	true	wrong
The concentration of absorbing species in a medium may be estimated from the imaginary part of the dielectric function integrated over the (angular) frequency. $\varepsilon(\omega) = Re\,\varepsilon(\omega) + i\,Im\,\varepsilon(\omega)$		✓
The phase velocity may exceed c.	✓	

$k = \frac{\omega}{c}\left(Re\sqrt{\varepsilon(\omega)} + i\,Im\sqrt{\varepsilon(\omega)}\right)$

e^{-ikx} $\quad Im\sqrt{\varepsilon}$ $\quad e$

3. Estimate the so-called classical electron radius r_e by setting its classical electrostatic self-energy ($\approx \frac{e^2}{4\pi\varepsilon_0 r_e}$) equal to its relativistic total energy at rest ($m_e c^2$)! (2 Points)
4. A photon releases a photoelectron with a kinetic energy 2eV from a metal which has a work function of 2eV. What is the smallest possible energy of that photon? Indicate the wavelength of such a photon! (3 points)
5. If the kinetic energy of a relativistic electron is equal to its rest mass, what is its velocity? (4 points)
6. A resting atom with mass m absorbs a photon with angular frequency ω. As a result of momentum conservation, after absorption the atom will no more be at rest.
   a) When assuming a non-relativistic case ($v_{atom} << c$), find an expression for the velocity and the kinetic energy of the atom after absorption. Estimate the ratio of the kinetic energy and the photon energy for $\omega = 10^{15} s^{-1}$ and $m = 10^{-26}$kg. (3 Points)
   b) Find the expression for the atoms velocity assuming a relativistic case! (8 points)

③ $\frac{e^2}{4\pi\varepsilon_0 r_e} = m_e c^2 \Rightarrow r_e = \frac{e^2}{4\pi\varepsilon_0 m_e c^2} = \frac{(1.602\times10^{-19})^2 A^2 s^2}{4\times3.14\times8.86\times10^{-12}\frac{F}{m}\times9.108\times10^{-31}kg\times(3\times10^8 m/s)^2} =$

$10^{-9}m = nm \quad 10^{-12}m = pm \quad 10^{-15}m = fm$

$10^{-10}m = \mathring{A}$

$= \frac{2.566\times10^{-38}A^2s^2}{8.121\times10^{-24}\frac{C}{Vm}\cdot\frac{m^2}{s^2}\cdot kg} = 2.81\times10^{-15}\frac{As^3V}{m\,kg} = 2.81\times10^{-15}m$

$= 2.81\,fm$

$J = CV = N\cdot m = \frac{kg\,m^2}{s^2} \Rightarrow V = \frac{kg\,m^2}{s^2 C} = \frac{kg\,m^2}{As^3}$

④ $E_k = hf - \phi$ $\quad\phi$ is the work function of the material, E_k is the kinetic energy of the emitted electron

hf is the energy of the incident light

thus $\Rightarrow f = \frac{E_k+\phi}{h} \qquad f = \frac{1}{T} = \frac{c}{\lambda} \Rightarrow \lambda = \frac{ch}{E_k+\phi} = \frac{3\times10^8 m/s\cdot 6.625\times10^{-34}W\cdot s^2}{4\times1.602\times10^{-19}C\cdot V} \approx 3.1\times10^{-7}m = 310nm$

⑤ $p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow p_e = \frac{m_e v}{\sqrt{1-\frac{v^2}{c^2}}}$

$T_{kin} = \frac{p_e^2}{2m_e} = \frac{m_e v^2}{2(1-\frac{v^2}{c^2})} = m_e \Rightarrow v^2 = 2 - \frac{2v^2}{c^2} \Rightarrow (c^2+2)v^2 = 2c^2 \quad v^2 = \frac{2c^2}{c^2+2} \approx 2 \Rightarrow v \approx 1.41\,m/s$

⑥ (a) Phonon Energy $E_{ph} = h\nu = \hbar\omega = \frac{6.625\times10^{-34}W\cdot s^2\cdot10^{15}s^{-1}}{2\pi} \approx 1.05\times10^{-19}J$

According to momentum conservation $P_{ph} = P_a \qquad E = pc = \hbar\omega \Rightarrow p = \frac{\hbar\omega}{c}$

$T_{kin} = \frac{P_a^2}{2m} = \frac{P_{ph}^2}{2m} = \frac{\hbar^2\omega^2}{2mc^2} \qquad$ ratio $\alpha = \frac{T_{kin}}{E_{ph}} = \frac{\hbar\omega}{2mc^2} = 5.85\times10^{-11}$

$\frac{1}{2}mv^2 = \frac{\hbar^2\omega^2}{2mc^2} \Rightarrow v = \frac{\hbar\omega}{mc}$

(b) $P_{ph}c + mc^2 = \sqrt{P_a^2c^2 + m_i^2c^4}$ the mass of the atom increase after absorption

$P_a = \frac{m_i v}{\sqrt{1-\frac{v^2}{c^2}}} = P_{ph} = \frac{\hbar\omega}{c} \Rightarrow \frac{m_i^2v^2}{1-\frac{v^2}{c^2}} = \frac{\hbar^2\omega^2}{c^2} \Rightarrow m_i^2v^2 = \frac{\hbar^2\omega^2}{c^2} - \frac{\hbar^2\omega^2v^2}{c^4} \Rightarrow (m_i^2 + \frac{\hbar^2\omega^2}{c^4})v^2 = \frac{\hbar^2\omega^2}{c^2}$

$\Rightarrow (m_i^2c^4 + \hbar^2\omega^2)v^2 = \hbar^2\omega^2c^2 \Rightarrow v^2 = \frac{\hbar^2\omega^2c^2}{m_i^2c^4+\hbar^2\omega^2} \Rightarrow v = \frac{\hbar\omega c}{\sqrt{m_i^2c^4+\hbar^2\omega^2}} = \frac{\hbar\omega c}{\sqrt{P_a^2c^2+m_i^2c^4}} = \frac{\hbar\omega c}{\hbar\omega + mc^2}$