

c) Wave equation for $\bar{E}(\bar{r}, \omega), \bar{H}(\bar{r}, \omega)$

$$\bar{\nabla} \times \bar{\nabla} \times \bar{E}(\bar{r}, \omega) = i\omega\mu_0 \bar{\nabla} \times \bar{H}(\bar{r}, \omega)$$

$$\bar{\nabla}(\bar{\nabla} \cdot \bar{E}(\bar{r}, \omega)) - \bar{\nabla}^2 \bar{E}(\bar{r}, \omega) \stackrel{\text{IV}+b}{=} i\omega\mu_0 (-i\omega\epsilon_0 \epsilon(\bar{r}, \omega) \bar{E}(\bar{r}, \omega))$$

$$\otimes \quad \bar{\nabla} \bar{\nabla}(\bar{r}, \omega) = 0$$

$$\bar{\nabla} \epsilon_0 \epsilon(\bar{r}, \omega) \bar{E}(\bar{r}, \omega) = 0$$

$$\epsilon_0 (\bar{\nabla} \epsilon(\bar{r}, \omega)) \bar{E}(\bar{r}, \omega) + \epsilon_0 \epsilon(\bar{r}, \omega) \bar{\nabla} \bar{E}(\bar{r}, \omega) = 0$$

$$\bar{\nabla} \bar{E}(\bar{r}, \omega) = - \frac{\bar{\nabla} \epsilon(\bar{r}, \omega)}{\epsilon(\bar{r}, \omega)} \bar{E}(\bar{r}, \omega)$$

Wave equation E

$$-\bar{\nabla} \left(\frac{\bar{\nabla} \epsilon(\bar{r}, \omega)}{\epsilon(\bar{r}, \omega)} \bar{E}(\bar{r}, \omega) \right) = \bar{\nabla}^2 \bar{E}(\bar{r}, \omega) + \frac{\omega^2}{c^2} \epsilon(\bar{r}, \omega) \bar{E}(\bar{r}, \omega)$$

$$\boxed{\mu_0 \epsilon_0 = \frac{1}{c^2}}$$

Wave equation H

$$\bar{\nabla}^2 \bar{H}(\bar{r}, \omega) + \frac{\omega^2}{c^2} \epsilon(\bar{r}, \omega) \bar{H}(\bar{r}, \omega) = i\omega\epsilon_0 \bar{\nabla} \epsilon(\bar{r}, \omega) \times \bar{E}(\bar{r}, \omega)$$

Problem 2 Poynting vector $3+2+1+3$

$$\vec{E}(x, y, z, t) = E_0 \cos((y+z)k - \omega t) \vec{e}_x$$

a) $\vec{H}(\vec{r}, t)$ (MWE III)

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ 0 & \partial_y & \partial_z \\ E_x & 0 & 0 \end{vmatrix} = \partial_z E_x \vec{e}_y - \partial_y E_x \vec{e}_z$$

$$= E_0 k \sin((y+z)k - \omega t) \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\vec{H}(\vec{r}, t) = -\frac{1}{\mu_0} \int dt (\vec{\nabla} \times \vec{E}) = \frac{F_0 k}{\mu_0 \omega} \cos((y+z)k - \omega t) \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

b) $\langle S \rangle = \frac{1}{2} \text{Re} [\vec{E}(\vec{r}) \times \vec{H}(\vec{r})^*]$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \vec{E}_r(\vec{r}, t) \times \vec{H}_r(\vec{r}, t)$$

$$I(r) = |\langle S \rangle|$$

c) $\nabla \langle S \rangle = 0$ propagating, lossless

$\nabla \langle S \rangle > 0$ gain medium

$\nabla \langle S \rangle < 0$ lossy medium

d)

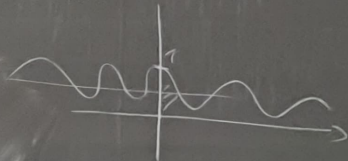
$$\langle S(\vec{r}, t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \vec{E}_r \times \vec{H}_r dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \frac{E_0^2 k}{\omega \mu_0} \cos^2((y+z)k - \omega t) \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} dt$$

$$= \frac{k E_0^2}{2 \mu_0 \omega} (\vec{e}_y + \vec{e}_z)$$

$$\nabla \langle S \rangle = 0$$

$$\cos^2(a) = \frac{1 + \cos(2a)}{2}$$



Problem 3

- a) homogeneous (position dependent)
isotropic (polarization independent)
dispersive (dependence on freq)

b) $\chi(\omega) = \int_{-\infty}^{\infty} R(t) e^{i\omega t} dt$

$$= \frac{f}{\omega_0^2 - \omega^2 - 2i\gamma\omega}$$

$$\chi' = \frac{f(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}$$

$$\chi'' = \frac{f 2\gamma\omega}{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}$$

$$\epsilon' = 1 + \chi'$$

$$\epsilon'' = \chi''$$

c)

c)

$$\vec{E}(\vec{r}) = E_0 \cos((g+ih)x) e^{(ig-h)z} \vec{e}_y$$

$$\Rightarrow \vec{k}_1 = \vec{k}_1' + i\vec{k}_1'' \quad \vec{k}_2 = \vec{k}_2' + i\vec{k}_2'' \quad e^{i\vec{k}\vec{r}}$$

eulers

$$\vec{E}(\vec{r}) = \frac{E_0}{2} \left[e^{i(g+ih)x + i(g+ih)z} + e^{i(-g-h)x + i(g+ih)z} \right]$$

$$i) \quad \vec{k}_1' = \begin{pmatrix} g \\ 0 \\ g \end{pmatrix} \quad \vec{k}_1'' = \begin{pmatrix} h \\ 0 \\ h \end{pmatrix} \quad \left| \quad \vec{k}_2' = \begin{pmatrix} -g \\ 0 \\ g \end{pmatrix} \quad \vec{k}_2'' = \begin{pmatrix} -h \\ 0 \\ h \end{pmatrix} \right.$$

ii) Helmholtz equation

$$\left[\Delta + \left(\frac{\omega}{c} \right)^2 \varepsilon(\omega) \right] E_0(\omega) e^{i\vec{k}\vec{r}} = 0$$

→

$$-(\vec{k}' + i\vec{k}'')(\vec{k}' + i\vec{k}'') + \left(\frac{\omega}{c} \right)^2 (\varepsilon(\omega) + i\varepsilon''(\omega)) = 0$$

$$-(\vec{k}^{\prime 2} - \vec{k}^{\prime\prime 2} + 2i\vec{k}'\vec{k}'') + \left(\frac{\omega}{c} \right)^2 (\varepsilon(\omega) + i\varepsilon''(\omega)) = 0$$

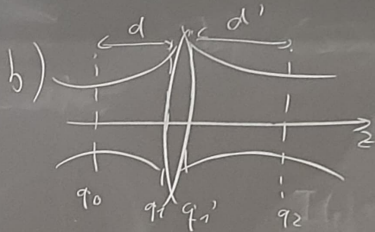
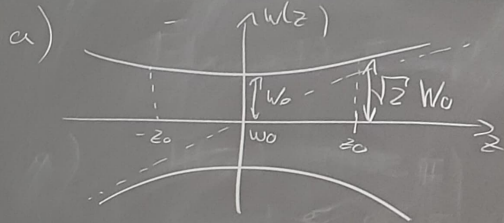
split into real & complex part

$$\text{I } \vec{k}^2 - \vec{k}^{\prime\prime 2} = + \left(\frac{\omega}{c} \right)^2 \varepsilon'(\omega) \quad \text{II } 2\vec{k}'\vec{k}'' = \left(\frac{\omega}{c} \right)^2 \varepsilon''(\omega)$$

$$\text{I } 2g^2 - 2h^2 = - \left(\frac{\omega}{c} \right)^2 \varepsilon'(\omega)$$

$$\text{II } 4gh = \left(\frac{\omega}{c} \right)^2 \varepsilon''(\omega)$$

Problem 2 Gaussian Beams



b)

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & d \\ -\frac{1}{f} & -\frac{d}{f} + 1 \end{pmatrix}$$

$R(z) \propto w(z)$ increases linearly

c)

$$z_c' = \frac{z_c}{\frac{z_c^2}{f^2} + \left(1 - \frac{d}{f}\right)^2} \quad d' = \left| \frac{d - \frac{d^2}{f} - \frac{z_c^2}{f}}{\left(\frac{z_c}{f}\right)^2 + \left(1 - \frac{d}{f}\right)^2} \right|$$

ges Rayleigh length $\ll d'$

$$q = z - iz_c \quad q_0 = -iz_c$$

$$q_1' = -\frac{iz_c + d}{\frac{iz_c}{f} - \frac{d}{f} + 1}$$

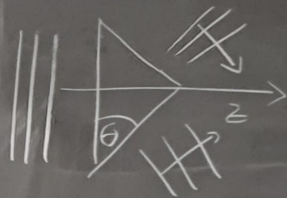
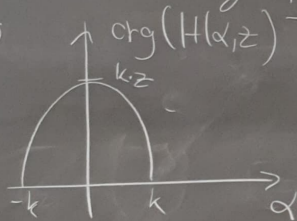
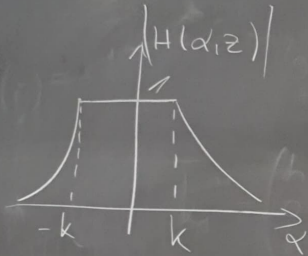
d) $d=f$ ges $M = w_0'/w_0 = \frac{f}{z_0}$

Problem 3 Beam Propagation

a) how to find $u(x, y, z > c)$ using

$$H(\alpha, \beta, z) = e^{i\sqrt{(\pi/\lambda)^2 - \alpha^2 - \beta^2} z}$$

b) $|H(\alpha, z)|$ $\beta=0$ $\arg(H(\alpha, z))$



c) $u(x) = 2h_0 \cos((n-1)\theta k_0 x)$ $h_0 = e^{ik_0 d_0}$

$$u_c(\alpha) = e^{ik_0 d_0} [S((n-1)\theta k_0 - \alpha) + S((n-1)\theta k_0 + \alpha)]$$

d) Bessel beam

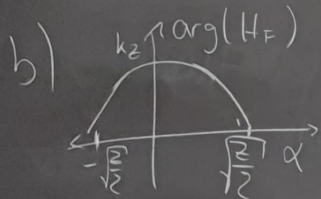


Problem 6 Diffraction

a) ges: condition for paraxial approximation $\boxed{\alpha, \beta^2 \ll k^2}$

$$H(\alpha, \beta, z) = e^{i\gamma z} \quad \gamma(\alpha, \beta) = \sqrt{k^2 - (\alpha^2 + \beta^2)} \quad \text{Taylor expansion}$$

$$H_F(\alpha, \beta, z) = e^{-ikz - i\left(\frac{\alpha^2 + \beta^2}{2k}\right)z} \approx k \left(1 - \frac{1}{2} \frac{\alpha^2 + \beta^2}{k^2}\right)$$



$$\beta = 0$$

$$\frac{\alpha^2}{2k} = kz$$

$$\alpha = \sqrt{\frac{2z}{k}}$$

c) $u(x, z) = e^{ikz}$

$$t(x) = \begin{cases} 1 & \text{if } \left(\frac{d}{2} - a\right) < |x| < \left(\frac{d}{2} + a\right) \\ 0 & \text{otherwise} \end{cases}$$

$$U(\alpha, z=c) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \, t(x) e^{-i\alpha x}$$

approximation $\boxed{\alpha^2, \beta^2 \ll k^2}$

$$\beta = \sqrt{k^2 - (\alpha^2 + \beta^2)}$$

$$\approx k(1 - \frac{1}{2}(\alpha^2 + \beta^2))$$

Taylor expansion

$$U(\alpha, z=0) = \frac{1}{2\pi} \left[\int_{-(\frac{d}{2}+a)}^{(\frac{d}{2}-a)} e^{-i\alpha x} dx + \int_{(\frac{d}{2}-a)}^{(\frac{d}{2}+a)} e^{-i\alpha x} dx \right]$$

$$0, y=0$$

$$= -\frac{1}{2\pi i \alpha} \left(e^{i\alpha(\frac{d}{2}-a)} - e^{-i\alpha(\frac{d}{2}+a)} - e^{-i\alpha(\frac{d}{2}-a)} + e^{i\alpha(\frac{d}{2}+a)} \right)$$

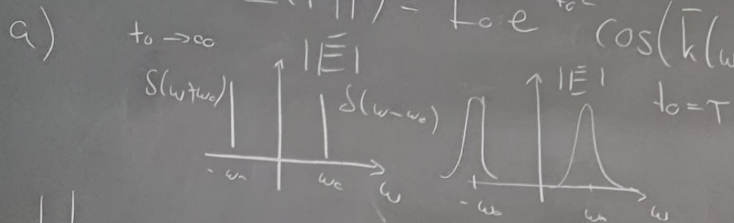
$$= \frac{2a}{\pi} \left(\frac{e^{i\alpha a} - e^{-i\alpha a}}{2i\alpha a} \left(\frac{e^{i\alpha \frac{d}{2}} + e^{-i\alpha \frac{d}{2}}}{2} \right) \right)$$

$$= \frac{2a}{\pi} \sin(\alpha a) \cos(\alpha \frac{d}{2})$$

$$= \frac{a}{\pi \alpha} \sin(\alpha a) \cos(\alpha \frac{d}{2})$$

Problem 7 Pulse Propagation

$$\vec{E}(\vec{r}, t) = E_0 e^{-\frac{t^2}{t_0^2}} \cos(\vec{k}(\omega) \vec{r} + \omega t)$$



b) dispersion \rightarrow Pulse changes shape

c)

$$n(\omega) = 1 + m\omega^2 + s\omega^3 \quad \lim_{\omega \rightarrow 0} s \in \mathbb{R}$$

$$\frac{1}{v_{\text{phase}}} = \frac{k(\omega)}{\omega} \Big|_{\omega=\omega_0} = \frac{n(\omega)}{c} \Big|_{\omega=\omega_0}$$

d) $d = f$ ges $M = \omega_0' / \omega_0 = \frac{f}{2\omega_0}$

$$\frac{1}{v_{\text{group}}} = \frac{\partial k(\omega)}{\partial \omega} \Big|_{\omega=\omega_0} = \frac{1}{c} (1 + 3m\omega_0^2 + 4s\omega_0^3)$$

$$D_v = \frac{\partial^2}{\partial \omega^2} k(\omega) = \frac{1}{c} 6\omega_0 (m + 2s\omega_0)$$

relation $\lim_{\omega \rightarrow 0} s$ & $T_0 = \frac{2\pi}{\omega_0}$ $t_0 = T$

$$D_v \rightarrow 0 \quad t_0 \gg T$$

$$0 = m + 2s\omega_0 \quad \omega_0 = -\frac{m}{2s}$$

