a) Calculate the intensity of the diffracted field pattern $I(x, z_B) = |u(x, z_B)|^2$ at $z = z_B$ in paraxial Fraunhofer approximation for two slits illuminated with a normally incident plane wave (just the diffraction pattern, no prefactors). The width of each slit is a ($a > \lambda$) and separated by a distance d ($d > a$):

$$u_0(x, z=0) = \begin{cases} 1, & \text{for } |x \pm d/2| \le a/2 \\ 0, & \text{elsewhere.} \end{cases}$$

b) Try to roughly sketch a figure of the intensity and identify the factor due to interference from the one due to slit diffraction.

Hint: The Fouriertransform of a single slit of width a is $\propto \mathrm{sinc}(\alpha a)$.

$FT[f(x,y,t)] = \frac{1}{(2\pi)^3}\iiint f(x,y,t)\, e^{i(k_x x + k_y y - \omega t)}$

$u(x,t) = e^{i(kx - \omega t)}$

(a) $h_F = -\frac{ike^{ikz}}{2\pi z}\exp\left[\frac{ik}{2z}(x^2+y^2)\right]$ $\quad \Delta z = z_B \Rightarrow h_F = -\frac{ie^{ikz_B}}{\lambda z}\exp\left[\frac{ik}{2z}(x^2+y^2)\right]$

$h_F(x-x', y-y') = -\frac{ie^{ikz_B}}{\lambda z_B}\exp\left[\frac{ik}{2z_B}[(x-x')^2+(y-y')^2]\right] \approx \frac{-ie^{ikz_B}}{\lambda z_B}\exp\left[\frac{ik}{2z_B}(x^2+y^2)\right]\exp\left[\frac{ikx}{z_B}x' + \frac{iky}{z_B}y'\right]$

$u_{FR} = -\frac{i2\pi}{\lambda z_B}e^{ikz_B}\exp\left(\frac{ik}{2z_B}x^2\right)U_+\left(\frac{kx}{z_B}\right)$ $\quad -\frac{a}{2} \le x \pm \frac{d}{2} \le \frac{a}{2}$ $\quad -\frac{a}{2}-\frac{d}{2} \le x \le \frac{a}{2}-\frac{d}{2}$ $\quad -\frac{a}{2}+\frac{d}{2} \le x \le \frac{a}{2}+\frac{d}{2}$

$U_+\left(\frac{kx}{z_B}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_+(x', z=0)\exp\left(-\frac{ikx}{z_B}x'\right) = \frac{1}{2\pi}\int_{-\frac{a}{2}-\frac{d}{2}}^{\frac{a}{2}-\frac{d}{2}}\exp\left(-\frac{ikx}{z_B}x'\right)dx' + \frac{1}{2\pi}\int_{-\frac{a}{2}+\frac{d}{2}}^{\frac{a}{2}+\frac{d}{2}}\exp\left(-\frac{ikx}{z_B}x'\right)dx'$

$= \frac{1}{2\pi}\left[-\frac{z_B}{ikx}\exp\left(-\frac{ikx}{z_B}x'\right)\Big|_{-\frac{a}{2}-\frac{d}{2}}^{\frac{a}{2}-\frac{d}{2}} - \frac{z_B}{ikx}\exp\left(-\frac{ikx}{z_B}x'\right)\Big|_{-\frac{a}{2}+\frac{d}{2}}^{\frac{a}{2}+\frac{d}{2}}\right]$

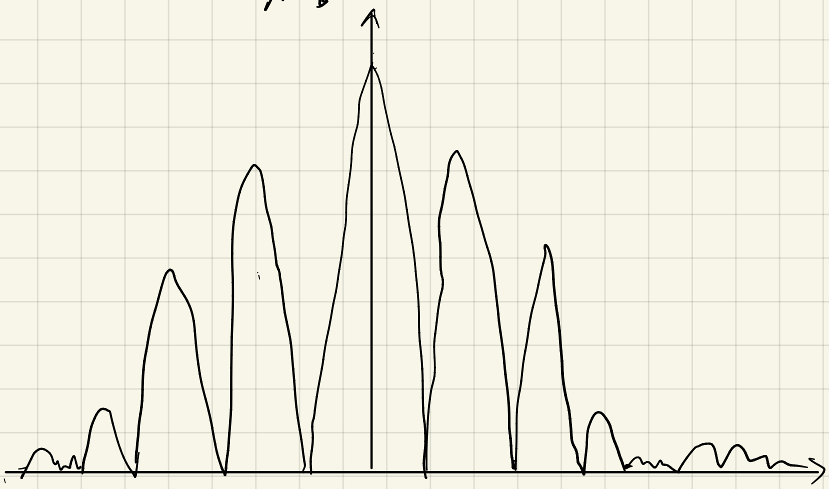
$= -\frac{z_B}{i2\pi kx}\left\{\exp\left(\frac{ikx}{z_B}\frac{d}{2}\right)\left[\exp\left(-\frac{ikx}{z_B}\frac{a}{2}\right) - \exp\left(\frac{ikx}{z_B}\frac{a}{2}\right)\right] + \exp\left(\frac{ikx}{z_B}\frac{d}{2}\right)\left[\exp\left(-\frac{ikx}{z_B}\frac{a}{2}\right) - \exp\left(\frac{ikx}{z_B}\frac{a}{2}\right)\right]\right\}$

$= \frac{z_B}{\pi kx}\left\{\left[\exp\left(\frac{ikx}{z_B}\frac{d}{2}\right) + \exp\left(\frac{ikx}{z_B}\frac{d}{2}\right)\right]\right\}\sin\left(\frac{ka}{2z_B}x\right) = \frac{2z_B}{\pi kx}\sin\left(\frac{ka}{2z_B}x\right)\cos\left(\frac{kd}{2z_B}x\right)$

$= \frac{2z_B}{\pi kx}\cdot\frac{kxa}{2z_B}\mathrm{sinc}\left(\frac{ka}{2z_B}x\right)\cos\left(\frac{kd}{2z_B}x\right) = \frac{a}{\pi}\mathrm{sinc}\left(\frac{ka}{2z_B}x\right)\cos\left(\frac{kd}{2z_B}x\right)$

$u_{FR} = -\frac{i2\pi}{\lambda z_B}\cdot\frac{a}{\pi}e^{ikz_B}\exp\left(\frac{ik}{2z_B}x^2\right)\mathrm{sinc}\left(\frac{ka}{2z_B}x\right)\cos\left(\frac{kd}{2z_B}x\right) = -\frac{i2a}{\lambda z_B}e^{ikz_B}\exp\left(\frac{ik}{2z_B}x^2\right)\mathrm{sinc}\left(\frac{ka}{2z_B}x\right)\cos\left(\frac{kd}{2z_B}x\right)$

$\Rightarrow I = |u_{FR}|^2 = \frac{4a^2}{\lambda^2 z_B^2}\mathrm{sinc}^2\left(\frac{ka}{2z_B}x\right)\cos^2\left(\frac{kd}{2z_B}x\right)$



a) Name two different approximations that can be made in diffraction theory and shortly explain their meaning in two sentences.

b) Calculate the intensity of the diffracted field pattern $I(x, z_B) = |u(x, z_B)|^2$ at $z = z_B$ in paraxial Fraunhofer approximation for one slit illuminated with a normally incident plane wave (just the diffraction pattern, no prefactors). The width of the slit is a ($a > \lambda$):

$$u_0(x, z=0) = \begin{cases} 1, & \text{for } |x| \le a/2 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Paraxial Approximation: Approximation when the angles between light and axis are very small.

Far field Approximation: The amplitude in the far field is given by the Fourier Transform of field behind aperture.

b. $u_{FR}(x, y, z_B) = -i\frac{(2\pi)^2}{\lambda z_B}e^{ikz_B}\exp\left[\frac{ik}{2z_B}(x^2+y^2)\right]U_+\left(\frac{kx}{z_B}, \frac{ky}{z_B}\right)$

$$U_+\left(\frac{k_x}{z_B}, \frac{k_y}{z_B}\right) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} u_0(x', z=0) \exp\left[-i\left(\frac{k_x}{z_B}x' + \frac{k_y}{z_B}y'\right)\right]$$

$$u_{FR}(x, z_B) = -i\frac{2\pi}{\lambda z_B} e^{ikz_B} \exp\left(\frac{ik}{2z_B}x^2\right) U_+\left(\frac{kx}{z_B}\right)$$

$$U_+\left(\frac{kx}{z_B}, z_B\right) = \frac{1}{2\pi}\int_{-\frac{a}{2}}^{\frac{a}{2}} u_0 \exp\left[-i\frac{kx}{z_B}x'\right]dx' = \frac{1}{2\pi}\int_{-\frac{a}{2}}^{\frac{a}{2}}\exp\left(-\frac{ikx}{z_B}x'\right)dx' = \frac{1}{2\pi}\cdot\frac{z_B}{-ikx}\exp\left(\frac{-ikx}{z_B}x'\right)\Big|_{-\frac{a}{2}}^{\frac{a}{2}}$$

$$= \frac{1}{2\pi}\cdot\frac{z_B}{-ikx}\left[\exp\left(\frac{-ikx}{z_B}\frac{a}{2}\right) - \exp\left(\frac{ikx}{z_B}\frac{a}{2}\right)\right] = \frac{z_B}{\pi kx}\sin\left(\frac{kxa}{2z_B}\right) = \frac{z_B}{\pi}\frac{\sin\left(\frac{kxa}{2z_B}\right)}{\frac{kxa}{2z_B}}\cdot\frac{a}{2z_B}$$

$$= \frac{a}{2\pi}\operatorname{sinc}\left(\frac{kxa}{2z_B}\right)$$

$$\Rightarrow u_{FR} = -\frac{ia}{\lambda z_B}e^{ikz_B}\exp\left(\frac{ik}{2z}x^2\right)\operatorname{sinc}\left(\frac{ka}{2z_B}x\right) \qquad I = |u_{FR}|^2 = \frac{a^2}{(\lambda z_B)^2}\operatorname{sinc}^2\left(\frac{ka}{2z_B}x\right)$$

A one-dimensional optical field directly behind a specific optical element is given as

$$u_0(x,0) = \begin{cases} A\exp\left[i\Phi_0\left(\frac{x}{a}+1\right)\right] & |x| \le a \\ 0 & \text{otherwise} \end{cases}$$

a) Find the optical intensity $I(x,z)$ in the case that the paraxial approximation holds and the distance $z \gg a$. You may omit possible prefactors.

b) How large does Φ_0 have to be so that the intensity on the optical axis $x = 0$ vanishes?

(a) $u_{FR} = -i\frac{2\pi}{\lambda z_B}e^{ikz_B}\exp\left(\frac{ik}{2z_B}x^2\right)U_+\left(\frac{kx}{z_B}\right)$

$$U_+\left(\frac{kx}{z_B}\right) = \frac{1}{2\pi}\int_{-a}^{a} u_0(x',0)\exp\left(-\frac{ikx}{z_B}x'\right)dx' = \frac{A}{2\pi}\int_{-a}^{a}\exp\left[i\phi_0\left(\frac{x'}{a}+1\right)\right]\exp\left(-\frac{ikx}{z_B}x'\right)dx'$$

$$= \frac{A}{2\pi}e^{i\phi_0}\int_{-a}^{a}\exp\left[i\left(\frac{\phi_0}{a}-\frac{kx}{z_B}\right)x'\right]dx' = \frac{A}{2\pi}e^{i\phi_0}\frac{\exp\left[i\left(\frac{\phi_0}{a}-\frac{kx}{z_B}\right)x'\right]}{i\left(\frac{\phi_0}{a}-\frac{kx}{z_B}\right)}\Bigg|_{-a}^{a}$$

$$= \frac{A}{2\pi}e^{i\phi_0}\frac{\exp\left[i\left(\frac{\phi_0}{a}-\frac{kx}{z_B}\right)a\right]-\exp\left[-i\left(\frac{\phi_0}{a}-\frac{kx}{z_B}\right)a\right]}{i\left(\frac{\phi_0}{a}-\frac{kx}{z_B}\right)} = \frac{A}{\pi}e^{i\phi_0}\frac{\sin\left[\left(\frac{\phi_0}{a}-\frac{kx}{z_B}\right)a\right]}{\left(\frac{\phi_0}{a}-\frac{kx}{z_B}\right)}$$

$$= \frac{Aa}{\pi}e^{i\phi_0}\operatorname{sinc}\left[\left(\frac{\phi_0}{a}-\frac{k}{z_B}x\right)a\right]$$

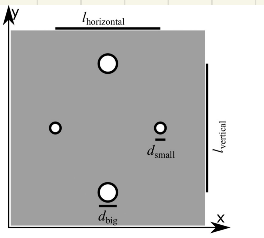
$$\Rightarrow u_{FR} = -i\frac{2Aa}{\lambda z_B}e^{i\phi_0}e^{ikz_B}\exp\left(\frac{ik}{2z_B}x^2\right)\operatorname{sinc}\left[\left(\frac{\phi_0}{a}-\frac{k}{z_B}x\right)a\right]$$

$$I = |u_{FR}|^2 = \frac{4A^2a^2}{\lambda^2 z_B^2}\operatorname{sinc}^2\left[\left(\frac{\phi_0}{a}-\frac{k}{z_B}x\right)a\right]$$

(b) $x=0 \Rightarrow I = \frac{4A^2a^2}{\lambda^2 z_B^2}\operatorname{sinc}^2(\phi_0) = \frac{4A^2a^2}{\lambda^2 z_B^2}\cdot\frac{\sin\phi_0}{\phi_0}$ $\quad \phi_0 = n\pi \quad n = \pm1, \pm2 \cdots \Rightarrow I = 0$

In Figure 1, a transmission mask is shown with the dimensions, indicated by d_{big} and d_{small} for the diameters of the holes and $l_{vertical}$ and $l_{horizontal}$ for the distances between the holes. They obey the following relation: $l_{vertical} > l_{horizontal} \gg d_{big} > d_{small}$.

b) Which of these dimensions defines the applicability of the Fresnel approximation and the Fraunhofer approximation, respectively? Explain the condition(s) for both approximations.



$N_F = \frac{a^2}{\lambda z_B} \qquad z_0 = \frac{\pi a^2}{\lambda}$

$0.1 < N_F < 10 \quad z_0 < z_B < 3z_0 \rightarrow$ Fresnel diffraction

$N_F < 0.1 \quad z_0 > 3z_0 \rightarrow$ Fraunhofer diffraction

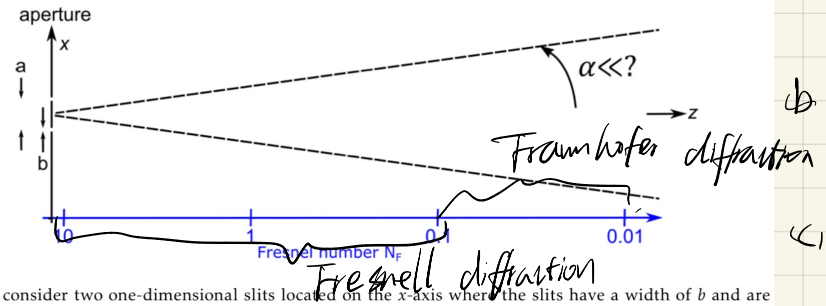
$l_{horizontal}$ and $l_{vertical}$ defines the applicabability of Fresnel Approximation.
When $z_B \ll l_{horizontal}$ and $z_B \ll l_{vertical}$, Fresnel Approximation can apply
d_{big} and d_{small} defines the applicability of Fraunhofer Approximation. When $d_{big} \ll z_B$

and $d_{small} \ll z_B$. Fraunhofer Approximation can apply.

And if $z_B \ll l_{horizontal}$ and $z_B \ll l_{vertical}$ are also met, it's Paraxial Fraunhofer diffraction.

if they are not met, it's non-paraxial Fraunhofer diffraction

a) Write down the conditions where 1) the Fresnel approximation, 2) the paraxial Fraunhofer approximation, and 3) the non-paraxial Fraunhofer approximation are valid. Mark the regions where the different diffraction approximations are valid in the following figure.



b) We consider two one-dimensional slits located on the x-axis where the slits have a width of b and are separated by a distance of a. Calculate the resulting far-field intensity when the field directly after the aperture is

$$u(x, z=0) = \begin{cases} 1 & \text{for } |x| < b/2 \\ 1 & \text{for } a - b/2 < x < a + b/2 \\ 0 & \text{otherwise} \end{cases}.$$

Hint: You may leave out the prefactors.

c) When $a \gg b$, which of these dimensions defines the applicability of the Fresnel approximation and the Fraunhofer approximation, respectively? For both approximations, state the condition(s) in relation with the wavelength λ and the observation distance z_B.

(a) Fresnel Approximation $0.1 < N_F < 10$ and $\alpha \ll \frac{a}{z}$ $N_F = \frac{b^2}{\lambda z_B}$

(b) Paraxial Fraunhofer Approximation: $N_F < 0.1$, and $\alpha \ll \frac{a}{z}$

(c) Non Paraxial Fraunhofer Approximation $N_F < 0.1$

(b) $u_{FR}(x, z) = \frac{-i2\pi}{\lambda z_B} e^{ikz_B} \exp\left[\frac{ik}{2z_B} x^2\right] U_0\left(\frac{kx}{z_B}\right)$

$$U_0\left(\frac{kx}{z_B}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_0(x', z=0)\, e\left(-\frac{ikx}{z_B}x'\right) dx' = \frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \exp\left(-\frac{ikx}{z_B}x'\right) dx' + \frac{1}{2\pi}\int_{a-\frac{b}{2}}^{a+\frac{b}{2}} \exp\left(-\frac{ikx}{z_B}x'\right) dx'$$

$$= \frac{z_B}{-i2\pi kx}\left[\exp\left(-\frac{ikx}{z_B}\frac{b}{2}\right) - \exp\left(\frac{ikx}{z_B}\frac{b}{2}\right)\right] - \frac{z_B}{i2\pi kx}\left\{\exp\left[-\frac{ikx}{z_B}\left(a+\frac{b}{2}\right)\right] - \exp\left[-\frac{ikx}{z_B}\left(a-\frac{b}{2}\right)\right]\right\}$$

$$= \frac{z_B}{\pi kx}\sin\left(\frac{kb}{2z_B}x\right) + \frac{z_B}{i2\pi kx}\exp\left(-\frac{ika}{z_B}x\right)\left[\exp\left(\frac{ikb}{2z_B}x\right) - \exp\left(-\frac{ikb}{2z_B}x\right)\right]$$

$$= \frac{z_B}{\pi kx}\sin\left(\frac{kb}{2z_B}x\right)\left[1 + \exp\left(-\frac{ika}{z_B}x\right)\right] = \frac{b}{2\pi}\,\mathrm{sinc}\left(\frac{kb}{2z_B}x\right)\left[1+\exp\left(-\frac{ika}{z_B}x\right)\right]$$

$$u_{FR} = \frac{-ib}{\lambda z_B} e^{ikz_B} \exp\left(\frac{ik}{2z_B}x^2\right) \mathrm{sinc}\left(\frac{kb}{2z_B}x\right)\left[1+\exp\left(-\frac{ika}{z_B}x\right)\right]$$

$$I = |u_{FR}|^2 = \left(\frac{b}{\lambda z_B}\right)^2 \mathrm{sinc}^2\left(\frac{kb}{2z_B}x\right)\left[1+\exp\left(-\frac{ika}{z_B}x\right)\right]\left[1+\exp\left(\frac{ika}{z_B}x\right)\right]$$

$$= \left(\frac{b}{\lambda z_B}\right)^2 \mathrm{sinc}^2\left(\frac{kb}{2z_B}x\right)\left[2+\exp\left(-\frac{ika}{z_B}x\right)+\exp\left(\frac{ika}{z_B}x\right)\right] = 2\left(\frac{b}{\lambda z_B}\right)^2 \mathrm{sinc}^2\left(\frac{kb}{2z_B}x\right)\left[1+\cos\left(\frac{ka}{z_B}x\right)\right]$$

$$= \left(\frac{2b}{\lambda z_B}\right)^2 \mathrm{sinc}^2\left(\frac{kb}{2z_B}x\right)\cos^2\left(\frac{ka}{2z_B}x\right)$$

(c) a defines the applicability of Fresnel Approximation.

$a \ll z_B$ and $0.1 < N_F < 10$ $N_F = \frac{b^2}{\lambda z_B}$ $\frac{b^2}{\lambda z_B} < 10$ $z_B > \frac{b^2}{10\lambda} \Rightarrow \frac{b^2}{10\lambda} < z_B < \frac{10b^2}{\lambda}$

b defines the applicability of Fraunhofer Approximation

$N_F < 0.1 \Rightarrow \frac{b^2}{\lambda z_B} < 0.1 \Rightarrow z_B > \frac{10b^2}{\lambda}$