Beam Propogation

$$U(\alpha,\beta,t)=\frac{1}{8\pi^3}\iiint u(x,y,t)\exp[-i(\alpha x+\beta y-\omega t)]$$

Given is the field directly behind a two dimensional phase mask

$$u_0(x,y,z=0)=A\exp\left(-i\pi\frac{x^2+y^2}{\lambda f}\right),$$

where $f>0$. The field is propagating through vacuum.

a) Calculate the spatial frequency spectrum $U_0(\alpha,\beta;z=0)$.

b) By introducing the paraxial approximation, derive the free space transfer function $(H_F(\alpha,\beta;z))$. Indicate propagating and evanescent wave regions.

c) Calculate the field $u(x,y,z=f)$.

$$U_0(\alpha,\beta,z=0)=\frac{1}{4\pi^2}\iint_{-\infty}^{\infty}u_0(x,y,z=0)e^{-i(\alpha x+\beta y)}dxdy$$

$$=\frac{A}{4\pi^2}\iint_{-\infty}^{\infty}\exp\left[-\frac{i\pi}{\lambda f}(x^2+y^2)-i\alpha x-i\beta y\right]dxdy$$

$$=\frac{A}{4\pi^2}\iint_{-\infty}^{\infty}\exp\left\{\frac{-i\pi}{\lambda f}\left[\left(x^2+\frac{\lambda f\alpha}{\pi}x\right)+\left(y^2+\frac{\lambda f\beta}{\pi}y\right)\right]\right\}dxdy$$

$$=\frac{A}{4\pi^2}\iint_{-\infty}^{\infty}\exp\left\{\frac{-i\pi}{\lambda f}\left[\left(x+\frac{\lambda f}{2\pi}\alpha\right)^2-\frac{\lambda^2f^2\alpha^2}{4\pi^2}+\left(y+\frac{\lambda f\beta}{2\pi}\right)^2-\frac{\lambda^2f^2\beta^2}{4\pi^2}\right]\right\}dxdy$$

$$=\frac{A}{4\pi^2}\exp\left[\frac{i\lambda f}{4\pi}(\alpha^2+\beta^2)\right]\iint_{-\infty}^{\infty}\exp\left\{-\left[\sqrt{\frac{i\pi}{\lambda f}}\left(x+\frac{\lambda f}{2\pi}\alpha\right)\right]^2-\left[\sqrt{\frac{i\pi}{\lambda f}}\left(y+\frac{\lambda f\beta}{2\pi}\right)\right]^2\right\}dxdy$$

$$=\frac{A\pi}{4\pi^2}\cdot\frac{\lambda f}{i\pi}\exp\left[\frac{i\lambda f}{4\pi}(\alpha^2+\beta^2)\right]=-\frac{iA\lambda f}{4\pi^2}\exp\left[\frac{i\lambda f}{4\pi}(\alpha^2+\beta^2)\right]$$

(b) $e^{i\gamma z}=\exp[i\sqrt{k^2-\alpha^2-\beta^2}\,z]=\exp\left[ik\sqrt{1-\frac{\alpha^2+\beta^2}{k^2}}\,z\right]=\exp\left[ik\left(1-\frac{\alpha^2+\beta^2}{2k^2}\right)z\right]=e^{ikz}\exp\left[\frac{-iz}{2k}(\alpha^2+\beta^2)\right]$

(c) $U_0(\alpha,\beta,z=f)=-\frac{iA\lambda f}{4\pi^2}\exp\left[\frac{i\lambda f}{4\pi}(\alpha^2+\beta^2)\right]e^{ikf}\exp\left[-\frac{if\lambda}{4\pi}(\alpha^2+\beta^2)\right]=\frac{-iA\lambda f}{4\pi^2}e^{ikf}$

$$u_0(x,y,z=f)=\iint_{-\infty}^{\infty}U_0(\alpha,\beta,z=f)\,e^{i(\alpha x+\beta y)}d\alpha d\beta$$

$$=\frac{-iA\lambda f}{4\pi^2}e^{ikf}\int_{-\infty}^{\infty}e^{i\alpha x}d\alpha\int_{-\infty}^{\infty}e^{i\beta y}d\beta=-iA\lambda f e^{ikf}\delta(x)\delta(y)$$

A certain circularly symmetric object, infinite in extent, has amplitude transmittance

$$u_0=2\pi aJ_0(2\pi ar)+4\pi aJ_0(4\pi ar)$$

where J_0 is a bessel function of the first kind, zero order, r is the radius in the two dimensional plane and a is a constant. This object is illuminated by a normally incident, unit amplitude plane wave of wavelength λ.

a) Calculate the spatial frequency spectrum $U_0(\rho;z=0)$, where ρ is the spatial frequency in the radial direction given by $\rho=\sqrt{\alpha^2+\beta^2}$

b) Write down the free space transfer function $H(\rho;z)$. Indicate propagating and evanescent wave regions.

c) Calculate the field $u(r,z)$.

d) At what distances behind this object will we find a field distribution that is of the same form as that of the object, upto a possible complex constant?

e) Please provide a condition on a with respect to wavelength λ for the periodic modulation to appear. Explain the physics behind it.

$$FT[J_0(2\pi ar)]=\frac{1}{2\pi a}\delta(\rho-a)$$

$$J_0(x)=\frac{1}{2\pi}\int_0^{2\pi}e^{-ix\cos(\theta-\varphi)}d\theta$$

$$G(\rho)=2\pi\int_0^{\infty}g(r)J_0(2\pi\rho r)rdr$$

$$g(r)=2\pi\int_0^{\infty}G(\rho)J_0(2\pi\rho r)\rho d\rho$$

(a) $FT(u_0)=2\pi a\,FT[J_0(2\pi ar)]+4\pi a\,FT[J_0(4\pi ar)]=\delta(\rho-a)+\delta(\rho-2a)=U_0(\rho,z)$

(b) $H(\rho,z)=e^{ikz}\exp\left[-\frac{iz}{2k}(\alpha^2+\beta^2)\right]=e^{ikz}\exp\left(\frac{-iz}{2k}\rho^2\right)$

(c) $U(\rho,z)=U_0(\rho,z=0)\cdot H(\rho,z)=[\delta(\rho-a)+\delta(\rho-2a)]\,e^{ikz}\exp\left[\frac{-iz}{2k}\rho^2\right]$

$$u(r,z)=2\pi\int_{-\infty}^{\infty}[\delta(\rho-a)+\delta(\rho-2a)]\,e^{ikz}\exp\left(\frac{-iz}{2k}\rho^2\right)J_0(2\pi\rho r)\rho d\rho$$

$$=2\pi e^{ikz}\exp\left(\frac{-iz}{2k}a^2\right)J_0(2\pi ar)a+2\pi e^{ikz}\exp\left[\frac{-iz}{2k}4a^2\right]J_0(4\pi ar)2a$$

$$=e^{ikz}\exp\left(-\frac{iz}{2k}a^2\right)\cdot 2\pi aJ_0(2\pi ar)+e^{ikz}\exp\left[\frac{-iz}{2k}4a^2\right]4\pi aJ_0(4\pi ar)$$

d) $u(r,z) = e^{ikz}\exp\left(\frac{-iz}{2k}a^2\right)\left[2\pi a J_0(2\pi a r) + 4\pi a J_0(4\pi a r)\exp\left(\frac{-iz}{2k}2a^2\right)\right]$ $\quad \frac{z a^2}{k} = 2\pi N \Rightarrow z = \frac{2\pi N k}{a^2}$

e) $z = \frac{2\pi N 2\pi}{a^2\lambda} \Rightarrow a = \sqrt{\frac{\lambda z}{4\pi^2 N}} = \frac{1}{2\pi}\sqrt{\frac{\lambda z}{N}}$

a) Describe an algorithm which makes use of the transfer function $H(\alpha, \beta, z)$ and is capable of calculating an optical field $u(x, y, d)$ at position $z = d$ from a field $u_0(x, y, 0)$ given at a position $z = 0$.

b) What is the explicit mathematical form of $H(\alpha, \beta, z)$ for free space? How can it be approximated for the paraxial case?

c) An initial field $u_0(x, y, 0)$ is given as the superposition of two fields

$$u_0(x, y, 0) = u_0^{(1)}(x, y, 0) + u_0^{(2)}(x, y, 0)$$

How will $u(x, y, z)$ depend on the two input fields and why?

d) Prove that the algorithm in a) corresponds to a convolution operation in real space!

d) $U(x,y,d) = \iint_{-\infty}^{\infty} U_0(x-x', y-y', 0)\, h_F(x', y', d)\, dx'dy'$

$U(\alpha,\beta,d) = \iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty} U_0(x-x', y-y', 0)\, h_F(x', y', d)\, dx'dy'\, e^{-i(\alpha x + \beta y)}\, dx\,dy$

$x - x' = \xi \quad y - y' = \eta \Rightarrow x = x' + \xi \quad y = y' + \eta$

$\Rightarrow U(\alpha,\beta,d) = \iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty} U_0(\xi,\eta,0)\, h_F(x', y', d)\, dx'dy'\, e^{-i(\alpha x' + \beta y')}\, e^{-i(\alpha\xi + \beta\eta)}\, d\xi\, d\eta$

$= \iint_{-\infty}^{\infty} U_0(\xi,\eta,0)\, e^{-i(\alpha\xi+\beta\eta)}\, d\xi\, d\eta \iint_{-\infty}^{\infty} h_F(x', y', d)\, e^{-i(\alpha x'+\beta y')}\, dx'dy'$

$= U(\alpha,\beta,0)\, H_F(\alpha,\beta,d)$

a) Write down the expression for the general transfer function $H(\alpha, \beta, z)$, which describes the scalar field propagation in free space along z-direction.

b) Specify the conditions of paraxial (Fresnel) approximation in the spatial frequency domain and derive the transfer function in Fresnel approximation $H_F(\alpha, \beta, z)$ from the general transfer function $H(\alpha, \beta, z)$ in part (a).

c) Consider an initial field distribution in the plane $z = 0$ of the form:

$$u_0(x, z = 0) = A + B\cos\left(\frac{2\pi x}{L}\right), \quad L > \lambda_0,$$

where λ_0 is the vacuum wavelength. Calculate the field distribution after propagation for an arbitrary distance $z > 0$ *without* Fresnel approximation.

d) Show that the field periodically reappears upon propagation up to a constant phase factor and calculate the distance along the z-axis until the first reappearance of the initial field.

e) Specify the conditions of applicability of Fresnel approximation for the field distribution from part (c) and calculate the value of the distance of field reappearance from part (d) under these conditions.

f) Show that for certain distances z the transverse *intensity* distribution will be periodic with twice the spatial frequency of the original field and find the distance along z-axis where this happens for the first time.

(a) $H = \exp[i\gamma(\alpha,\beta)z]$

b) $H = \exp[i\sqrt{k^2 - (\alpha^2+\beta^2)}\, z] = e^{ikz}\exp\left[\frac{-iz}{2k}(\alpha^2+\beta^2)\right]$

c) $U_0(\alpha, z=0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[A + B\cos\left(\frac{2\pi x}{L}\right)\right]e^{-i\alpha x}\, dx = \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-i\alpha x}\, dx + \frac{B}{4\pi}\int_{-\infty}^{\infty}\left(e^{\frac{i2\pi x}{L}} + e^{\frac{-i2\pi x}{L}}\right)e^{-i\alpha x}\, dx$

$= A\delta(\alpha) + \frac{B}{4\pi}\int_{-\infty}^{\infty}\exp\left[i\frac{2\pi}{L}\left(x - \frac{L\alpha}{2\pi}x\right)\right]dx + \frac{B}{4\pi}\int_{-\infty}^{\infty}\exp\left[-i\left(\frac{2\pi}{L}+\alpha\right)x\right]dx$

$= A\delta(\alpha) + \frac{B}{2}\delta\left(\alpha - \frac{2\pi}{L}\right) + \frac{B}{2}\delta\left(\alpha + \frac{2\pi}{L}\right)$

$U_0(\alpha, z) = \left[A\delta(\alpha) + \frac{B}{2}\delta\left(\alpha - \frac{2\pi}{L}\right) + \frac{B}{2}\delta\left(\alpha + \frac{2\pi}{L}\right)\right]\exp[i\sqrt{k^2-\alpha^2}\, z]$

$u(x,z) = \int_{-\infty}^{\infty}\left[A\delta(\alpha) + \frac{B}{2}\delta\left(\alpha - \frac{2\pi}{L}\right) + \frac{B}{2}\delta\left(\alpha - \frac{2\pi}{L}\right)\right]\exp[i\sqrt{k^2-\alpha^2}\, z]\, e^{i\alpha x}\, d\alpha$

$= Ae^{ikz} + \frac{B}{2}\exp\left[i\sqrt{k^2 - \frac{4\pi^2}{L^2}}\, z\right]e^{i\frac{2\pi}{L}x} + \frac{B}{2}\exp\left[i\sqrt{k^2 - \frac{4\pi^2}{L^2}}\, z\right]e^{-i\frac{2\pi}{L}x}$

$= Ae^{ikz} + B\exp\left[i\sqrt{k^2 - \frac{4\pi^2}{L^2}}\, z\right]\cos\frac{2\pi}{L}x$

d) $u(x,z) = \left\{A + B\exp\left[i\sqrt{k^2 - \frac{4\pi^2}{L^2}}\, z - ikz\right]\cos\frac{2\pi}{L}x\right\}e^{ikz}$

Reappearance $\Rightarrow \left(\sqrt{k^2 - \frac{4\pi^2}{L^2}} - k\right)z = 2\pi N \Rightarrow z = \frac{2\pi N}{\sqrt{k^2 - \frac{4\pi^2}{L^2}} - k} = \frac{2\pi N}{\gamma - k}$

e) $k^2 \gg \frac{4\pi^2}{L^2} \Rightarrow |\Delta x|\cdot|\Delta y| > \frac{10\lambda}{n} \gg \frac{\lambda}{n}$

$$u(x,z) = Ae^{ikz} + B\exp\left[ik\left(1-\frac{4\pi^2}{2k^2L^2}\right)z\right] = Ae^{ikz} + Be^{ikz}\exp\left[-\frac{iz}{2k}\frac{4\pi^2}{L^2}\right]\cos\frac{2\pi}{L}x$$

$$\Rightarrow u(x,z) = \left[A + B\cos\frac{2\pi}{L}x\exp\left(\frac{-iz}{2k}\frac{4\pi^2}{L^2}\right)\right]e^{ikz} \Rightarrow \frac{z}{2k}\frac{4\pi^2}{L^2} = 2\pi N \Rightarrow z = \frac{kN}{\pi L^2} = \frac{2N}{\lambda L^2}$$

f) $I = |u(x,z)|^2 = \left[Ae^{ikz} + B\cos\frac{2\pi}{L}x\, e^{ikz}\exp\left(\frac{-iz}{2k}\frac{4\pi^2}{L^2}\right)\right]\left[Ae^{-ikz} + B\cos\frac{2\pi}{L}x\, e^{-ikz}\exp\left(\frac{iz}{2k}\frac{4\pi^2}{L^2}\right)\right]$

$$= A^2 + B^2\cos^2\frac{2\pi}{L}x + AB\cos\frac{2\pi}{L}x\exp\left(\frac{iz}{2k}\frac{4\pi^2}{L^2}\right) + AB\cos\frac{2\pi}{L}\exp\left(\frac{-iz}{2k}\frac{4\pi^2}{L^2}\right)$$

$$= A^2 + B^2\cos^2\frac{2\pi}{L}x + 2AB\cos\frac{2\pi}{L}x\cos\left(\frac{z}{2k}\cdot\frac{4\pi^2}{L^2}\right) = A^2 + B^2\cos^2\frac{2\pi}{L}x + 2AB\cos\frac{2\pi}{L}x\cos\left(\frac{\pi z\lambda}{L^2}\right)$$

$$I = |u(x,z=0)|^2 = A^2 + B^2\cos^2\frac{2\pi}{L}x + 2AB\cos\frac{2\pi}{L}x$$

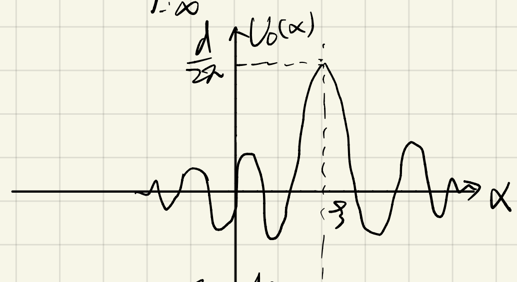
$\frac{\pi z\lambda}{L^2} = 2\pi \Rightarrow z = \frac{2L^2}{\lambda}$ when $z = \frac{2L^2}{\lambda}$, this happens for the first time

In the following we consider a slit with a width d and a linear phase profile $\varphi = \xi x$ in x-direction. Assume that the field after the slit can be written as

$$u(x, z=0) = u_0(x) = \begin{cases} \exp(\mathrm{i}\xi x) & \text{for } |x| < d/2, \\ 0 & \text{else.} \end{cases}$$

c) Calculate the Fourier transform $U_0(\alpha)$ of the initial field. Sketch the resulting spectrum and explain what effect the linear phase mask has on the spectrum.
d) Find the range of ξ such that the main spectral lobe (region from spectral maximum to the first zeros) is fully propagating assuming a wavelength of $\lambda = 2d/3$.

$$U_0(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U_0(x, z=0)\, e^{-i\alpha x}\, dx = \frac{1}{2\pi}\int_{-\frac{d}{2}}^{\frac{d}{2}} e^{i\xi x} e^{-i\alpha x}\, dx = \frac{1}{2\pi}\frac{e^{i(\xi-\alpha)x}}{i(\xi-\alpha)}\Big|_{-\frac{d}{2}}^{\frac{d}{2}} = \frac{\sin[(\xi-\alpha)\frac{d}{2}]}{\pi(\xi-\alpha)}$$



The linear phase mask cause a shift in frequency domain

d) $\frac{\sin[(\xi-\frac{2\pi}{\lambda})\frac{d}{2}]}{\pi(\xi-\frac{2\pi}{\lambda})} \Rightarrow (\xi - \frac{2\pi}{\lambda})\frac{d}{2} = \pm\pi \Rightarrow \xi = \pm\frac{2\pi}{d} + \frac{2\pi}{\lambda} = \pm\frac{2\pi}{d} + \frac{3\pi}{d} \Rightarrow \frac{\pi}{d} < \xi < \frac{5\pi}{d}$

Given is a monochromatic field, of vacuum wavelength λ, directly behind a one dimensional amplitude mask

$$u_0(x, z_0) = A\left[1 + \cos\left(\frac{2\pi}{G}x\right)\right].$$

The field is propagating through vacuum.
a) Calculate the spatial frequency spectrum $U_0(\alpha, z_0)$.
b) Calculate the field $u(x,z)$ for all $z > z_0$ without approximation.
c) The field will reproduce itself periodically except for a constant phase factor $e^{i\Phi}$ after a certain propagation length z_T (Talbot effect). Calculate the shortest repetition length z_T as a function of G and the wavelength λ.
d) Specify the condition for the applicability of the Fresnel approximation for the given field distribution. What will z_T approximately be under the Fresnel approximation?

a) $U_0(\alpha, z_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U_0(x, z_0)\, e^{-i\alpha x}\, dx = \frac{A}{2\pi}\int_{-\infty}^{\infty}\left[1 + \frac{1}{2}\left(e^{i\frac{2\pi}{G}x} + e^{-i\frac{2\pi}{G}x}\right)\right]e^{-i\alpha x}\, dx$

$$= \frac{A}{2\pi}\left[\int_{-\infty}^{\infty} e^{-i\alpha x}\, dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{i(\frac{2\pi}{G}-\alpha)x}\, dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{-i[(\frac{2\pi}{G}+\alpha)x]}\, dx\right]$$

$$= A\left[\delta(\alpha) + \frac{1}{2}\delta\left(\alpha - \frac{2\pi}{G}\right) + \frac{1}{2}\delta\left(\alpha + \frac{2\pi}{G}\right)\right]$$

b) $H_F = \exp[i\gamma(\alpha,\beta)(z-z_0)] = \exp[i\sqrt{k^2-\alpha^2}(z-z_0)]$

$U(\alpha, z) = U_0(\alpha, z_0)H_F = A\left[\delta(\alpha) + \frac{1}{2}\delta(\alpha - \frac{2\pi}{G}) + \frac{1}{2}\delta(\alpha + \frac{2\pi}{G})\right]\exp[i\sqrt{k^2-\alpha^2}(z-z_0)]$

$u(x, z) = A\int_{-\infty}^{\infty}\left[\delta(\alpha) + \frac{1}{2}\delta(\alpha - \frac{2\pi}{G}) + \frac{1}{2}\delta(\alpha + \frac{2\pi}{G})\right]\exp[i\sqrt{k^2-\alpha^2}(z-z_0)]\, e^{i\alpha x}\, d\alpha$

$$= Ae^{ik(z-z_0)} + \frac{A}{2}e^{-i\frac{2\pi}{G}x}\exp\left[i\sqrt{k^2-\frac{4\pi^2}{G^2}}(z-z_0)\right] + \frac{A}{2}e^{i\frac{2\pi}{G}x}\exp\left[i\sqrt{k^2-\frac{4\pi^2}{G^2}}(z-z_0)\right]$$

$$= Ae^{ik(z-z_0)} + A\exp\left[i\sqrt{k^2-\frac{4\pi^2}{G^2}}(z-z_0)\right]\cos\frac{2\pi}{G}x = Ae^{ik(z-z_0)}\left\{1 + \cos\left(\frac{2\pi}{G}x\right)\exp\left[i\sqrt{k^2-\frac{4\pi^2}{G^2}}(z-z_0) - ik(z-z_0)\right]\right\}$$

(c) $\sqrt{k^2-\frac{4\pi^2}{G^2}}\,z_T - kz_T = 2\pi \Rightarrow z_T = \dfrac{2\pi}{\sqrt{k^2-\frac{4\pi^2}{G^2}}-k} = \dfrac{1}{\sqrt{\frac{1}{\lambda^2}-\frac{1}{G^2}}-\frac{1}{\lambda}}$

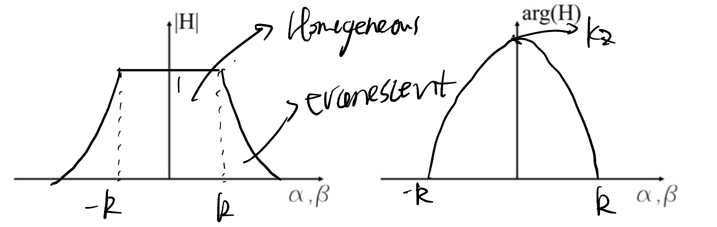
(d) $k^2 \gg \frac{4\pi^2}{G^2}$ $\quad |\Delta x| > |D\lambda / n|$

$u(x,z) = Ae^{ik(z-z_0)}\left\{1+\cos\frac{2\pi}{G}x\,\exp\left[-\frac{i(z-z_0)}{2k}\frac{4\pi^2}{G^2}\right]\right\}$ $\quad \frac{z_T}{2k}\frac{4\pi^2}{G^2} = 2\pi \Rightarrow z_T = \frac{kG^2}{\pi} = \frac{2G^2}{\lambda}$

$$\frac{1}{\sqrt{\frac{1}{\lambda^2}-\frac{1}{G^2}}-\frac{1}{\lambda}} = \frac{1}{\frac{1}{\lambda}\left(1-\frac{\lambda^2}{2G^2}\right)-\frac{1}{\lambda}} = \frac{1}{-\frac{\lambda}{2G^2}} = -\frac{2G^2}{\lambda}$$

a) What are the properties of homogeneous and evanescent waves in terms of amplitude and energy transfer? Write down the complex transfer function $H(\alpha,\beta;z)$ in a homogeneous space and define the evanescent and homogeneous wave regions depending on the spatial frequencies α and β.

b) Plot the amplitude and phase of the transfer function on the below graphs. Indicate characteristic points or dimensions of the drawn transfer function (such as the amplitudes, radius *etc.*) on the graphs.



(a) $H(\alpha,\beta;z) = \exp[i\gamma(\alpha,\beta)z] = \exp[i\sqrt{k^2-\alpha^2-\beta^2}\,z]$

Honegeneous wave region: $k^2 > \alpha^2+\beta^2$

$\Rightarrow H(\alpha,\beta;z) = \exp[i\sqrt{k^2-\alpha^2-\beta^2}\,z]$

Thus Honogeneous wave keeps its amplitude and transport Energy / information

Evanescent wave region: $k^2 \le \alpha^2+\beta^2$

$\Rightarrow H(\alpha,\beta;z) = \exp[-\sqrt{\alpha^2+\beta^2-k^2}\,z] \le 1$

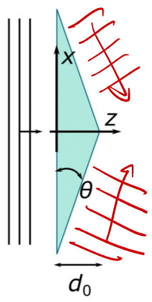
Thus evanescent wave keeps its phase, and doesn't transport Energy / information. The amplitude decrease

c) The beam propagation in free-space can be formulated as a superposition of plane waves with different phase evolution due to the propagation. For a given initial field $u_0(x,y)$, we want to calculate the field at a distance z. By considering $U_0(\alpha,\beta) = \mathrm{FT}\{u_0(x,y)\}$, fill the blanks in the following formula.

$$u(x,y,z) = \iint_{-\infty}^{\infty} \mathrm{d}\alpha\mathrm{d}\beta \times \underbrace{U_0(\alpha,\beta)}_{\text{Amplitude of the excited eigenstates}} \times \underbrace{\exp[i\gamma(\alpha,\beta)z]}_{\text{phase factor which is accumulated by the eigenstates during propagation}} \times \underbrace{e^{i(\alpha x+\beta y)}}_{\text{Shape of eigenstates (plane waves)}}$$

interference of eigenstates to form the field pattern after propagation

Consider the propagation of a monochromatic, scalar field $u(x,y,z)$ at a wavelength λ along the z-direction starting from a given initial field distribution $u(x,y,z=0) = u_0(x,y)$.



a) Describe the steps necessary to calculate the field distribution $u(x,y,z)$ for any $z>0$ by using the free space transfer function. Name and define all functions and quantities that you use and give the necessary formulas for this calculation.

b) Sketch the amplitude and phase of the free space transfer function $H(\alpha,\beta=0,z)$ for a finite distance z. Interpret your results when $\alpha > k_0$ and $\alpha \le k_0$.

Now, we consider a 2 dimensional space for the sake of simplicity. As illustrated in the figure, we have a thin phase mask with an edge angle of $\theta \ll 1°$ (so that it is sufficiently large along x-axis), and a height of d_0, and made of a material with a refractive index of n. The phase mask can be described in some approximation as:

$$t(x) = 2h_0\cos[(n-1)\theta k_0 x],$$

where $h_0 = e^{ik_0 d_0}$.

c) Assuming that a monochromatic plane wave is incident on the phase mask, calculate the Fourier transform $U_0(\alpha)$ of the initial field right after the phase mask. Describe the fields after the phase mask: What do you get after the phase mask? What will happen upon propagation?

d) Now, imagine the phase mask in the figure is rotated around z-axis to create a 3D phase mask which has a conical shape. Based on your result in c), what conclusion can you draw for the case of this 3D phase mask? What shape of electromagnetic field will you get after the phase mask, if a plane wave is incident on it?

(a) First, transform $u_0(x,y)$ from spatial domain to frequency domain by Fourier transform:

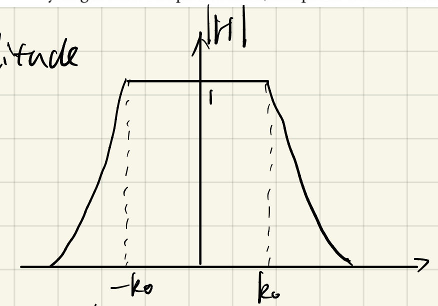
$U_0(\alpha,\beta) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} u_0(x,y)\,e^{-i(\alpha x+\beta y)}\,dxdy$

And then, multiply $U_0(\alpha,\beta)$ with the transfer function to get $U(\alpha,\beta,z) = U_0(\alpha,\beta)H_F(\alpha,\beta,z)$.

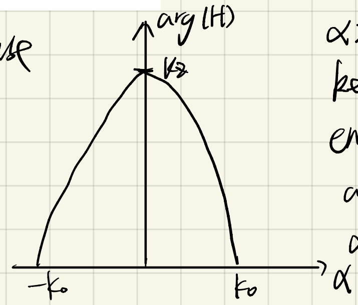
Finally, transform $U(\alpha,\beta,z)$ back to space domain by inverse Fourier transform

$u(x,y,z) = \iint_{-\infty}^{\infty} U_0(\alpha,\beta)H_F(\alpha,\beta,z)\,e^{i(\alpha x+\beta y)}\,d\alpha d\beta$

(b) Amplitude



phase



$\alpha > k_0$: Evanescent wave. the wave keeps its phase, and doesn't transport energy and time, And amplitude decrease along z-axis

$\alpha < k_0$. Homogeneous wave, the wave keeps its amplitude and transport Energy and information

(c) $t(x) = 2e^{ik_0d_0}\cos[(n-1)\alpha x]$ $\quad U(x, z=0) = 1$

$u(x, z=d_0) = \int_{-\infty}^{\infty} u(x-x', z=0)\,t(x')\,dx' = 2e^{ik_0d_0}\int_{-\infty}^{\infty}\cos[(n-1)\alpha x']\,dx'$

$= e^{ik_0d_0}\int_{-\infty}^{\infty} e^{i(n-1)\alpha x'} + e^{-i(n-1)\alpha x}\,dx' = 2\pi e^{ik_0d_0}\{\delta[(n-1)\alpha] + \delta[(1-n)\alpha]\}$

$U_0(x) = U_0'(x)\, t(x) \qquad U_0'(x) = 1 \Rightarrow U_0(x) = t(x)$

$$U_0(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} t(x)\, e^{-i\alpha x}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2h_0 \cos[(n-1)k_0\theta x]\, e^{-i\alpha x}\, dx = \frac{h_0}{2\pi}\int_{-\infty}^{\infty} e^{i[(n-1)k_0\theta-\alpha]x} + e^{-i[(n-1)k_0\theta+\alpha]x}\, d\alpha$$

$$= e^{ik_0 d_0}\{\delta[(n-1)k_0\theta-\alpha] + \delta[(n-1)k_0\theta+\alpha]\}$$

$$U_0(\alpha, z) = U_0(\alpha)\, H_F(\alpha; z) = e^{ik_0 d}\, e^{ik_0 z}\{\delta[(n-1)k_0\theta-\alpha] + \delta[(n-1)k_0\theta+\alpha]\} \exp\left(\frac{-iz}{2k}\alpha^2\right)$$

$$u(x,z) = \int_{-\infty}^{\infty} U_0(\alpha, z)\, e^{i\alpha x}\, d\alpha = e^{ik_0(d+z)}\int_{-\infty}^{\infty}\{\delta[(n-1)k_0\theta-\alpha] + \delta[(n-1)k_0\theta+\alpha]\} \exp\left(\frac{-iz}{2k}\alpha^2\right) e^{i\alpha x}\, d\alpha$$

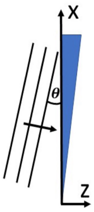
$$= e^{ik_0(d+z)} \exp\left\{\frac{-iz}{2k_0}[(n-1)k_0\theta]^2\right\}\left[e^{i(n-1)k_0\theta x} + e^{-i(n-1)k_0\theta x}\right]$$

$$= 2e^{ik_0(d+z)} \exp\left\{\frac{-iz}{2k_0}[(n-1)k_0\theta]^2\right\} \cos[(n-1)k_0\theta x]$$

Now, we consider a 2 dimensional space for the sake of simplicity. As shown in the figure, an incoming plane wave described by its field distribution $u(x,z)$

$$u(x, z \le 0) = Ae^{i(-xk_0 \sin\theta + zk_0 \cos\theta)},$$

is incident on a thin wedge of a transparent optical material. The effect of the wedge can be mathematically described by a phase mask function $t(x) = e^{i\sigma x}$, such that the field just after the wedge is given by $u(x,z)t(x)$.



b) Calculate the Fourier transform $U_0(\alpha)$ of the field right after the optical wedge.
c) Using $U_0(\alpha)$ calculate the field $u(x,z)$ for arbitrary $z > 0$.
d) What should be the value of σ so that the outgoing wavefronts become parallel to the x-axis?

(b) $$U_0(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,z)\, t(x)\, e^{-i\alpha x}\, dx = \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{i(-xk_0\sin\theta + zk_0\cos\theta)}\, e^{-i\alpha x}\, e^{i\sigma x}\, dx$$

$$= \frac{A}{2\pi}\, e^{izk_0\cos\theta}\int_{-\infty}^{\infty} e^{-i(k_0\sin\theta + \alpha - \sigma)x}\, dx = A\, e^{izk_0\cos\theta}\, \delta[\alpha - (\sigma - k_0\sin\theta)]$$

$$U(\alpha, z) = U_0(\alpha)\, H_F(\alpha; z) = Ae^{izk_0\cos\theta}\, \delta[\alpha - (\sigma - k_0\sin\theta)]\, e^{ikz} \exp\left[\frac{-iz}{2k}\alpha^2\right)$$

$$u(x,z) = \int_{-\infty}^{\infty} U(\alpha, z)\, e^{i\alpha x}\, d\alpha = Ae^{izk_0\cos\theta}\, e^{ik_0 z}\int_{-\infty}^{\infty} \delta[\alpha - (\sigma - k_0\sin\theta)] \exp\left(\frac{-iz}{2k}\alpha^2\right) e^{i\alpha x}\, dx$$

$$= Ae^{izk_0(1+\cos\theta)} \exp\left[\frac{-iz}{2k}(\sigma - k_0\sin\theta)^2\right] \exp[i(\sigma - k_0\sin\theta)x]$$

(d) $\sigma - k_0\sin\theta = 0 \Rightarrow \sigma = k_0\sin\theta$