Lesson 7: Solutions of the Schrödinger equation (II)

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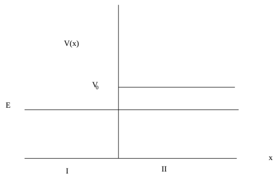
Potential step

- (a) $E < V_0$



All states are unbound

Classically: an ∞ force acts at $x = 0$ during an infinitesimal time, causing Δp finite

$$F = -\frac{dV(x)}{dx}$$

Quantum study

- we study particles falling into the step from the left
- we divide the x-axis into two regions I andII

  - (I) $-\frac{\hbar^2}{2m}\phi_I''(x) = E\phi_I(x)$

    $x \leq 0 \;\; ; \;\; k_1 = \frac{\sqrt{2mE}}{\hbar}$

$$\boxed{\phi_I(x) = A\, e^{ik_1 x} + B\, e^{-ik_1 x}}$$

- ...
  - (II) $-\frac{\hbar^2}{2m}\phi''_{II}(x) + V_0\phi_{II}(x) = E\phi_{II}(x)$

    $\phi''_{II}(x) + \bar{k}_2^2\,\phi_{II}(x) = 0 \quad \bar{k}_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$

    $\bar{k}_2 = ik_2 = i\frac{\sqrt{2m(V_0-E)}}{\hbar}$; k_2 real

$$\boxed{\phi_{II}(x) = C\,e^{k_2x} + De^{-k_2x}}$$

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We will determine the relation between $A,\ B,\ C$ and D requiring the continuity of the wave function and its derivative. The wave function is finite $\forall x$

$$\phi_I(0) = \phi_{II}(0) \quad ; \quad \phi'_I(0) = \phi'_{II}(0) \rightarrow$$

$$A = \frac{D}{2}\left[1 + i\frac{k_2}{k_1}\right] \quad ; \quad B = \frac{D}{2}\left[1 - i\frac{k_2}{k_1}\right]$$

$C = 0$ since $e^{k_2x} \rightarrow \infty$ when $x \rightarrow \infty$

$$\phi(x) = \begin{cases} \underbrace{\frac{D}{2}\left[1+i\frac{k_2}{k_1}\right]e^{ik_1x}}_{\text{incident wave}} + \underbrace{\frac{D}{2}\left[1-i\frac{k_2}{k_1}\right]e^{-ik_1x}}_{\text{reflected wave}} & x \leq 0 \\ \underbrace{De^{-k_2x}}_{\text{transmitted wave}} & x \geq 0 \end{cases}$$

$$\Psi(x,t) = \phi(x)e^{-\frac{iEt}{\hbar}}$$

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- We get no conditions limiting the possible values of E (unbound state)
- The wave function is a function of a unique constant D. It can not be determined by normalizing the wave function, since it is not normalizable (unbound state)
- The wave function does not represent a particle but a **beam**. $|D|^2$ can be determined from the incident flux

D can be chosen real

We write $\quad D\left[1+i\frac{k_2}{k_1}\right] \;=\; D'\, e^{i\delta} \quad (1)$

D' and δ real

$$\phi(x) \;=\; \begin{cases} D' \cos(k_1 x \;+\; \delta) & x \leq 0 \\ D e^{-k_2 x} & x \geq 0 \end{cases}$$

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We see that $\phi(x)$ is real if we choose D real. D can be solved from (1) to have the wave function as a function of D' and δ only

Probability current or flux (one dimension)

$$j \;=\; \frac{\hbar}{2im}\left(\phi^*\frac{d\phi}{dx} - \frac{d\phi^*}{dx}\phi\right) \;=\; \frac{\hbar}{m} Re\left(\frac{1}{i}\phi^*\frac{d\phi}{dx}\right)$$

If ϕ is real $\quad \rightarrow \quad j \;=\; 0$

We can calculate the incident flux (with the incident wave function)

$$j_i \;=\; \frac{\hbar k_1 |D|^2}{4m}\left[1 \;+\; \left(\frac{k_2}{k_1}\right)^2\right]$$

(incident particles per unit time)

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Reflected flux (calculated from the reflected wave function)

$$j_r \;=\; -\,\frac{\hbar k_1 |D|^2}{4m}\left[1\;+\;\left(\frac{k_2}{k_1}\right)^2\right] \leq 0$$

$$j \;=\; j_i \;+\; j_r \;=\; 0$$

We define **reflection coefficient**

$$R \;=\; -\,\frac{j_r}{j_i}$$

gives the fraction of the beam reflected in the step

Transmitted flux (calculated with the transmitted wave function)

$$j_t \;=\; 0 \quad \text{since} \quad \frac{1}{i}\phi_t^* \frac{d\phi_t}{dx} \quad \text{is purely imaginary}$$

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We define **transmission coefficient**

$$T \;=\; \frac{j_t}{j_i}$$

For $E < V_0$ $\quad R \;=\; 1 \;\; ; \;\; T \;=\; 0$ (as classically)

but quantically the probability of finding the particle with $x \geq 0$ is non-zero($\neq$ classical)

$$|\phi_{II}(x)|^2 \;=\; |D|^2 e^{-2k_2 x} \quad ; \quad x \geq 0$$

exponential decay, the probability drops by a factor e at a distance $\frac{1}{2k_2} \;=\; \frac{\hbar}{2\sqrt{2m(V_0-E)}}$

- Larger $V_0 - E$ implies less penetration of the wave function in the classically forbidden region

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$\frac{1}{2k_2}$ in this distance prob. decays to 37% of the value in $x = 0$

- Macroscopically $\frac{1}{2k_2}$ is very small

  Ex.: $m = 1$ kg; $E = 1\ J$; $V_0 = 2\ J \rightarrow \frac{1}{2k_2} \approx 4.10^{-26}$ fm

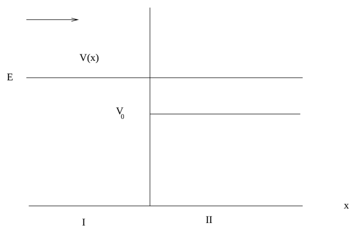
- To locate the particle in the classically forbidden region

$$\Delta x < \frac{\hbar}{2\sqrt{2m(V_0 - E)}} \rightarrow \Delta p \approx \frac{\hbar}{2\Delta x} \approx \sqrt{2m(V_0 - E)}$$

$$\rightarrow \Delta E \approx \frac{(\Delta p)^2}{2m} \approx V_0 - E$$ we can not say that

$V_0 > E$ (classically forbidden region)

- (b) $E > V_0$



  - **classically**: the particle slows down when going from I to II ($F_x < 0$). There is no reflection
  - **quantically**: if E is not much larger than V_0 there is probability that the particle is reflected

Ex. photoelectric cell: the e^- at the cathode receives the photon energy and tries to escape. If E is not much larger than V_0 (V_0 potential to which the e^- is subject on the surface of the metal) there is reflection and it does not escape (there is reduction of efficiency in photoelectric cells for light with $\nu \approx \nu_{\text{threshold}}$)

$$\phi_I(x) \;=\; A\, e^{ik_1 x} \;+\; B\, e^{-ik_1 x} \quad ; \quad k_1 \;=\; \frac{\sqrt{2mE}}{\hbar} \quad x \;\leq\; 0$$

$$\phi_{II}(x) \;=\; C\, e^{ik_2 x} \;+\; D\, e^{-ik_2 x} \;;\; k_2 \;=\; \frac{\sqrt{2m(E - V_0)}}{\hbar} \quad x \;\geq\; 0$$

$D \;=\; 0$ no reflected wave in region II (there are only **reflections** in the **discontinuities** of potential)

$$\phi_I(0) \;=\; \phi_{II}(0) \quad ; \quad A \,+\, B \;=\; C$$

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$$\phi_I'(0) \;=\; \phi_{II}'(0)$$

$$ik_1 A \;-\; ik_1 B \;=\; ik_2 C \;\rightarrow\; \frac{k_1}{k_2}(A - B) \;=\; C$$

$$C \;=\; \frac{2k_1}{k_1 \,+\, k_2} A$$

$$B \;=\; \frac{k_1 \,-\, k_2}{k_1 \,+\, k_2} A$$

$$\phi(x) \;=\; \begin{cases} A\left(\underset{\text{(i)}}{e^{ik_1 x}} \;+\; \underset{\text{(r)}}{\tfrac{k_1 - k_2}{k_1 + k_2} e^{-ik_1 x}}\right) & x \leq 0 \\[2ex] \underset{\text{(t)}}{A\, \tfrac{2k_1}{k_1 + k_2} e^{ik_2 x}} & x \geq 0 \end{cases}$$

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$$
\begin{aligned}
j_i &= \frac{\hbar}{m} Re \left[\frac{1}{i} A^* e^{-ik_1 x}(ik_1) A e^{ik_1 x}\right] \\
&= \frac{\hbar k_1}{m}|A|^2
\end{aligned}
$$

$$
\begin{aligned}
j_r &= \frac{\hbar}{m} Re \left[\frac{1}{i} A^* \frac{k_1 - k_2}{k_1 + k_2} e^{ik_1 x}(-ik_1) A \frac{k_1 - k_2}{k_1 + k_2} e^{-ik_1 x}\right] \\
&= \frac{-\hbar k_1}{m}|A|^2 \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2
\end{aligned}
$$

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$$
R = -\frac{j_r}{j_i} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \text{ does not change } k_1 \iff k_2
$$

$$
\begin{aligned}
j_t &= \frac{\hbar}{m} Re \left[\frac{1}{i} A^* \frac{2k_1}{k_1 + k_2} e^{-ik_2 x}(ik_2) A \frac{2k_1}{k_1 + k_2} e^{ik_2 x}\right] \\
&= \frac{\hbar k_2}{m}|A|^2 \left(\frac{2k_1}{k_1 + k_2}\right)^2
\end{aligned}
$$

$$
j_i + j_r = j_t \quad \rightarrow \quad R + T = 1
$$

$$
\begin{aligned}
T &= \frac{j_t}{j_i} = \frac{k_2}{k_1}\left(\frac{2k_1}{k_1 + k_2}\right)^2 \\
&= \frac{4k_1 k_2}{(k_1 + k_2)^2} \quad \text{it does not vary } k_1 \iff k_2
\end{aligned}
$$

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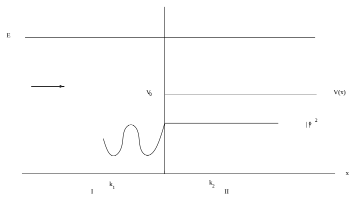
$$
\begin{aligned}
R + T &= \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 + \frac{4k_1k_2}{(k_1 + k_2)^2} \\
&= \left(\frac{k_1 + k_2}{k_1 + k_2}\right)^2 = 1
\end{aligned}
$$

- (i) particle coming from the region of negative x



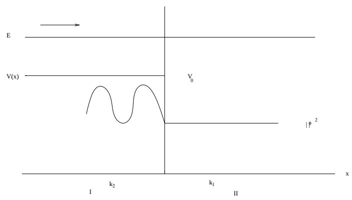
$$|\phi_I(x)|^2 = |A|^2 \left[1 + 2\tfrac{k_1 - k_2}{k_1 + k_2}\cos 2k_1x + \left(\tfrac{k_1 - k_2}{k_1 + k_2}\right)^2\right]$$

It takes values between $\frac{4|A|^2k_1^2}{(k_1 + k_2)^2}$ and $\frac{4|A|^2k_2^2}{(k_1 + k_2)^2}$

maximum minimum

$$|\phi_I(x)|^2_{x=0} = \frac{4|A|^2k_1^2}{(k_1 + k_2)^2} \quad ; \quad |\phi_{II}(x)|^2_{\forall x} = \frac{4|A|^2k_1^2}{(k_1 + k_2)^2}$$

- (ii) particle coming from the left



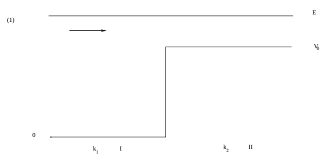
change $k_1 \iff k_2$; in (i)

$k_1 = \frac{\sqrt{2mE}}{\hbar}$; $k_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$; $k_1 > k_2$ c

$$|\phi_I(0)|^2 = |\phi_{II}(0)|^2 = \frac{4|A|^2 k_2^2}{(k_1 + k_2)^2}$$ minima in region I

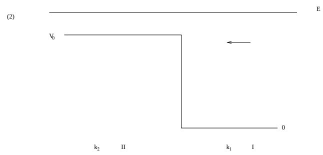
- if the particle enters from the right in (i), the solution is the same as in (ii) changing $x \iff -x$
- if the particle enters from the right in (ii), the solution is the same as in (i) changing $x \iff -x$

- (1)



we have solved it in (b) $k_1 = \frac{\sqrt{2mE}}{\hbar}$; $k_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$

- (2)



same as (1), changing $x \iff -x$

R and T as in (1)

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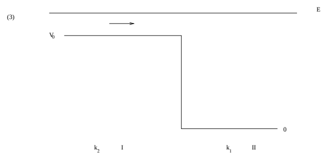
- (3)



same as (1), changing $k_1 \iff k_2$

R and T do not vary when changing $k_1 \iff k_2$

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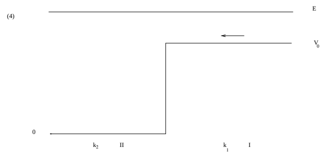
- (4)



W.F. as in (3) with change $x \iff -x$

R and T as in (1), (2) and (3)

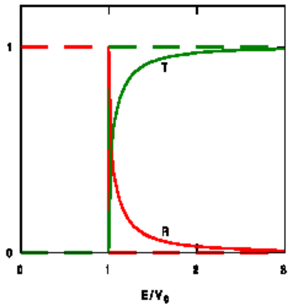
Once set $m,\ E\ and\ V_0 \ \rightarrow$ R and T are independent on whether the step goes up or down and where the beam comes from

As a function of E and V_0

$$R = \left(\frac{1 - \sqrt{1 - \frac{V_0}{E}}}{1 + \sqrt{1 - \frac{V_0}{E}}} \right)^2$$

where $\frac{E}{V_0} > 1$

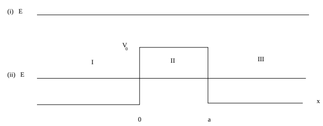
The m of the particles of the beam does not appear, R does not depend on it



case (a) $\rightarrow$ $\frac{E}{V_0} < 1$; case (b) $\rightarrow$ $\frac{E}{V_0} > 1$

$$\frac{E}{V_0} >> \quad \rightarrow \quad R \rightarrow 0$$

Potential barrier: barrier penetration



$$V(x) \;=\; \begin{cases} 0\,; & x < 0 \;\; and \;\; x > a \\ V_0\,; & 0 < x < a \end{cases}$$

- (i) $E > V_0$

$\phi_I \;=\; Ae^{ik_1x} \;+\; Be^{-ik_1x} \qquad ; \; k_1 \;=\; \frac{\sqrt{2mE}}{\hbar}$

$\phi_{II} \;=\; Pe^{ik_2x} \;+\; Qe^{-ik_2x} \qquad ; \; k_2 \;=\; \frac{\sqrt{2m(E-V_0)}}{\hbar}$

$\phi_{III} \;=\; Ce^{ik_1x} \;+\; De^{-ik_1x}$

If we assume that the beam comes from the left $\rightarrow\ D\ =\ 0$

$$\phi_I(0)\ =\ \phi_{II}(0)\ \ ;\ \ \phi'_I(0)\ =\ \phi'_{II}(0)$$

$A\ +\ B\ =\ P\ +\ Q\ \ ;\ \ k_2(P-Q)\ =\ k_1(A-B)$

$2A\ =\ P(1+\frac{k_2}{k_1})+Q(1-\frac{k_2}{k_1})\ \ ;\ \ 2B\ =\ P(1-\frac{k_2}{k_1})+Q(1+\frac{k_2}{k_1})$

$$\phi_{II}(a)\ =\ \phi_{III}(a)\ \ ;\ \ \phi'_{II}(a)\ =\ \phi'_{III}(a)$$

$Pe^{ik_2a}\ +\ Qe^{-ik_2a}\ =\ Ce^{ik_1a}\ k_2(Pe^{ik_2a}\ -\ Qe^{-ik_2a})\ =\ k_1\ Ce^{ik_1a}$

$$P\ =\ \frac{1}{2}Ce^{i(k_1-k_2)a}(1+\frac{k_1}{k_2})$$

$$Q\ =\ \frac{1}{2}Ce^{i(k_1+k_2)a}(1-\frac{k_1}{k_2})$$

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$$A\ =\ \left[\frac{1}{4}e^{i(k_1-k_2)a}\frac{(k_1+k_2)^2}{k_1k_2}-\frac{1}{4}e^{i(k_1+k_2)a}\frac{(k_1-k_2)^2}{k_1k_2}\right]C$$

$$B\ =\ \left[\frac{1}{4}e^{i(k_1-k_2)a}\frac{k_1^2-k_2^2}{k_1k_2}-\frac{1}{4}e^{i(k_1+k_2)a}\frac{k_1^2-k_2^2}{k_1k_2}\right]C$$

$$A\ =\ \frac{e^{ik_1a}}{4k_1k_2}\left[(k_1+k_2)^2e^{-ik_2a}-(k_1-k_2)^2e^{ik_2a}\right]C$$

$$B\ =\ \frac{e^{ik_1a}}{4k_1k_2}(k_1^2-k_2^2)\left(e^{-ik_2a}-e^{ik_2a}\right)C$$

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$$j = \frac{\hbar}{m} Re\left[\frac{1}{i}\phi^* \frac{d\phi}{dx}\right]$$

$$\begin{aligned}
\frac{C}{A} &= \frac{4k_1k_2e^{-ik_1a}}{(k_1+k_2)^2e^{-ik_2a} - (k_1-k_2)^2e^{ik_2a}} \\
&= \frac{4k_1k_2e^{-i(k_1-k_2)a}}{(k_1+k_2)^2 - (k_1-k_2)^2e^{i2k_2a}}
\end{aligned}$$

$$\begin{aligned}
\frac{B}{A} &= \frac{(k_1^2-k_2^2)(e^{-ik_2a} - e^{ik_2a})}{(k_1+k_2)^2e^{-ik_2a} - (k_1-k_2)^2e^{ik_2a}} \\
&= \frac{(k_1^2-k_2^2)(1 - e^{i2k_2a})}{(k_1+k_2)^2 - (k_1-k_2)^2e^{i2k_2a}}
\end{aligned}$$

$$\begin{aligned}
R &= \left|\frac{B}{A}\right|^2 \\
&= \frac{(k_1^2-k_2^2)^2(1 - e^{-i2k_2a} - e^{i2k_2a} + 1)}{\left[(k_1+k_2)^2 - (k_1-k_2)^2e^{i2k_2a}\right]\left[(k_1+k_2)^2 - (k_1-k_2)^2e^{-i2k_2a}\right]} \\
&= \frac{(k_1^2-k_2^2)^2 2(1-\cos 2k_2a)}{(k_1+k_2)^4 - (k_1+k_2)^2(k_1-k_2)^2\left(e^{i2k_2a} + e^{-i2k_2a}\right) + (k_1-k_2)^4} \\
&= \frac{4(k_1^2-k_2^2)^2\sin^2 k_2a}{2k_1^4 + 2k_2^4 + 12k_1^2k_2^2 - (k_1^2-k_2^2)^2 2\cos 2k_2a} \\
&= \frac{4(k_1^2-k_2^2)^2\sin^2 k_2a}{2(k_1^2-k_2^2)^2(1-2\cos 2k_2a) + 16k_1^2k_2^2} \\
&= \frac{1}{1 + \frac{4k_1^2k_2^2}{(k_1^2-k_2^2)^2\sin^2 k_2a}} \\
&= \left[1 + \frac{4k_1^2k_2^2}{(k_1^2-k_2^2)^2\sin^2 k_2a}\right]^{-1}
\end{aligned}$$

$$
\begin{aligned}
T &= \left|\frac{C}{A}\right|^2 \\
&= \frac{16k_1^2k_2^2}{(k_1+k_2)^4-(k_1+k_2)^2(k_1-k_2)^2\left(e^{i2k_2a}+e^{-i2k_2a}\right)+(k_1-k_2)^4} \\
&= \frac{16k_1^2k_2^2}{2(k_1^2-k_2^2)^2 2\sin^2 k_2a+16k_1^2k_2^2} \\
&= \left[1+\frac{(k_1^2-k_2^2)^2\sin^2 k_2a}{4k_1^2k_2^2}\right]^{-1}
\end{aligned}
$$

as a function of E and V_0 : $k_1^2 = \frac{2mE}{\hbar^2}$; $k_2^2 = \frac{2m(E-V_0)}{\hbar^2}$

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$$
R = \left[1+\frac{4E(E-V_0)}{V_0^2\sin^2 k_2a}\right]^{-1}
$$

$$
T = \left[1+\frac{V_0^2\sin^2 k_2a}{4E(E-V_0)}\right]^{-1}
$$

- $k_2a = n\pi$; $n=1,2,\cdots \rightarrow \lambda_2 = \frac{2a}{n}$

$$
a = n\,\frac{\lambda_2}{2}
$$

If the width of the barrier is an integral number of half-wavelengths $\rightarrow$ **perfect transmission** (the barrier is completely transparent to the incident particles, resonance condition)

Observed in the scattering of electrons by noble-gas atoms, **“Ramsauer effect”**

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- (ii) $E < V_0$

$\phi_{II} \; = \; Pe^{-k_2' x} \; + \; Qe^{k_2' x} \qquad ; \; k_2' \; = \; \frac{\sqrt{2m(V_0 - E)}}{\hbar}$

Change the previous k_2 by ik_2'

$$R \; = \; \left[1 + \frac{4k_1^2(-k_2'^2)}{(k_1^2 + k_2'^2)^2 \sin^2{(ik_2' a)}}\right]^{-1}$$

$$\sin\alpha \; = \; -\, i \sinh(i\alpha)$$

$$\sin^2\alpha \; = \; -\sinh^2(i\alpha)$$

$$\sin^2(i\alpha) \; = \; -\sinh^2(\alpha)$$

$$R \; = \; \left[1 + \frac{4k_1^2(k_2'^2)}{(k_1^2 + k_2'^2)^2 \sinh^2 k_2' a}\right]^{-1}$$

$$\begin{aligned} T \quad &= \quad \left[1 + \frac{(k_1^2 + k_2'^2)^2 \sin^2 ik_2' a}{4k_1^2(-k_2'^2)}\right]^{-1} \\ &= \quad \left[1 + \frac{(k_1^2 + k_2'^2)^2 \sinh^2 k_2' a}{4k_1^2(k_2'^2)}\right]^{-1} \\ &= \quad \left[1 + \frac{V_0^2 \sinh^2 k_2' a}{4E(V_0 - E)}\right]^{-1} \quad \neq \; 0 \end{aligned}$$

$\Downarrow$

TUNNELING

Some interesting limits

- 1) $E > V_0$

  If $E \to V_0 \Rightarrow k_2 \to 0$; $\sin^2(k_2 a) \approx (k_2 a)^2$

$$T^{E>V_0}_{E\to V_0} \to \left(1 + \frac{mV_0 a^2}{2\hbar^2}\right)^{-1}$$

- 2) $E < V_0$

  If $E << V_0$ and a is large / $k_2' a >> 1 \Rightarrow \sinh x \approx \cosh x = \frac{e^x}{2}$ if $x >> 1$

$$T^{E<V_0}_{k_2'a>>1} \to \left(1 + \frac{V_0^2 e^{2k_2' a}}{16E(V_0 - E)}\right)^{-1}$$

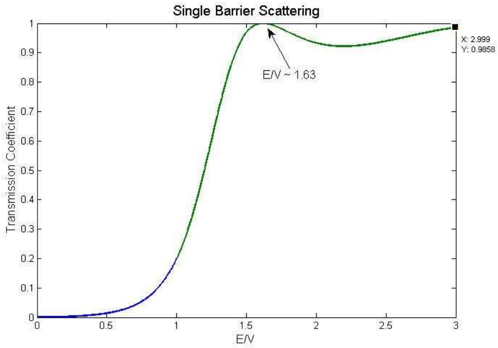
$$= \frac{1}{\frac{V_0^2 e^{2k_2' a}}{16E(V_0-E)}\left(1 + \frac{16E(V_0-E)}{V_0^2 e^{2k_2' a}}\right)} = \frac{x'}{1+x'}$$

with $x' = \frac{16E(V_0-E)}{V_0^2 e^{2k_2' a}} << 1$

Expanding around $x' = 0$

$$T^{E<V_0}_{k_2'a>>1} \approx x' = \frac{16E(V_0 - E)e^{-2k_2' a}}{V_0^2}$$

For $\frac{2mV_0a^2}{\hbar^2} = \frac{47}{3}$



$T = 1$ implies perfect transmission

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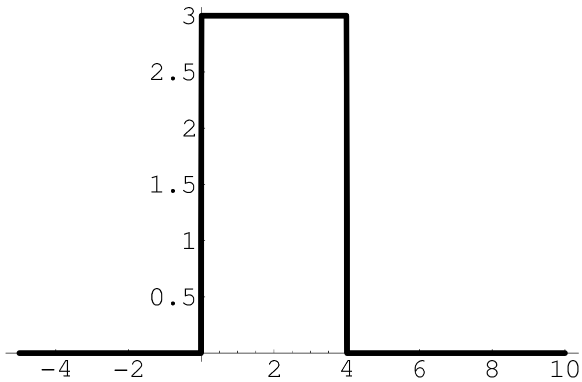
Ex. Barrier with $a = 4$ arbitrary units (a.u.) and $V_0 = 3$ a.u.



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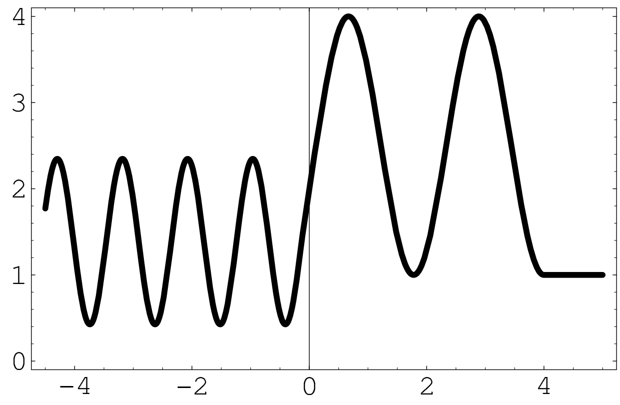
$$E/V_0 \;=\; 4/3 \quad ; \quad a \;=\; 4$$

$|\Psi|^2$ $(a.u.)$



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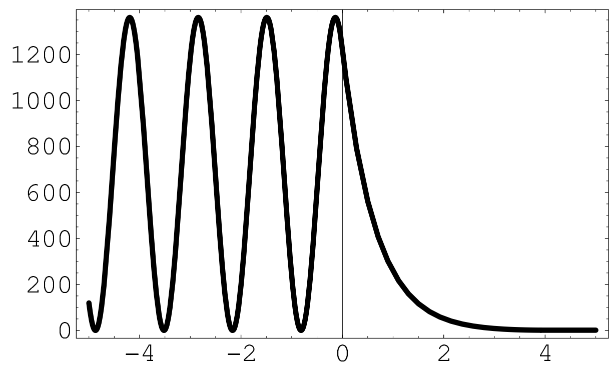
$$E/V_0 \;=\; 2.7/3 \quad ; \quad a \;=\; 4$$

$|\Psi|^2$ $(a.u.)$



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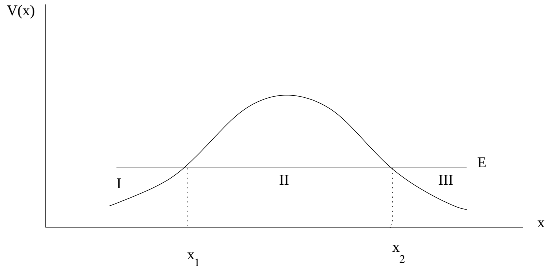
- For barriers of arbitrary shape
- We will calculate the **penetrability** $\mathcal{T}$



- For rectangular barriers we obtained $T_{k_2' a >> 1}$ $(E < V_0)$

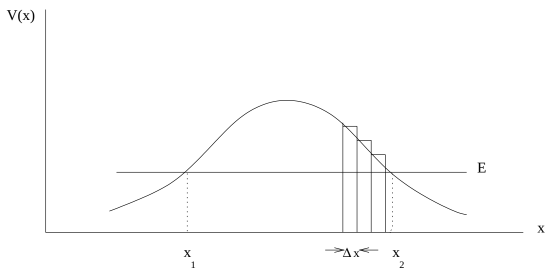
  $\mathcal{T}$ is the dominant factor $e^{-2k_2' a}$

$$T_{k_2' a >> 1} \sim \frac{16E(V_0 - E)}{V_0^2 e^{2k_2' a}} \quad ; \quad E < V_0$$

$$\frac{16E(V_0 - E)}{V_0^2} \begin{cases} \text{maximum 4} & E < V_0 \\ \sim 1 \quad \text{unless} & E \sim V_0 \text{ then } k_2' a \text{ is not } >> 1 \end{cases}$$

The dominant factor is the exponential

We approximate $T_{k_2' a >> 1} \approx e^{-2k_2' a}$



Approx.: succession of thin rectangular barriers

$$V(x_1) \;=\; V(x_2) \;=\; E$$

In x_1 and x_2, $k_2' a$ is not $>> 1$

$\mathcal{P}(x)$ probability of finding the particle at $x(x_1 \leq x \leq x_2)$

$\mathcal{P}(x+\Delta x)$ (probability of reaching x) $\times$ (probability of crossing the barrier of width Δx)

$$\mathcal{P}(x+\Delta x) \;=\; \mathcal{P}(x)\, e^{-2k(x)|\Delta x|} \;\; ; \;\; k(x) \;=\; \frac{\sqrt{2m(V(x)-E)}}{\hbar}$$

If $\Delta x \;\rightarrow\; 0$

$$\mathcal{P}(x+dx) \;=\; \mathcal{P}(x)\,(1-2k(x)\;|dx|\;)$$

$$\frac{\mathcal{P}(x+dx)-\mathcal{P}(x)}{|dx|} \;=\; -\,2k(x)\mathcal{P}(x)$$

$$\int_{x_1}^{x} \frac{d\mathcal{P}(x)}{\mathcal{P}(x)} \;=\; -\int_{x_1}^{x} 2k(x)|dx| \;\Rightarrow\; \ln\frac{\mathcal{P}(x)}{\mathcal{P}(x_1)} \;=\; -\left|\int_{x_1}^{x} 2k(x)dx\right|$$

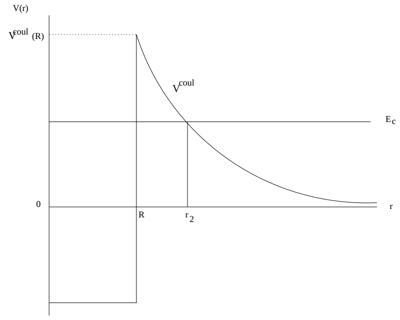
$$\mathcal{P}(x) \;=\; \mathcal{P}(x_1)e^{-2\left|\int_{x_1}^{x} k(x)dx\right|}$$

$$\boxed{\mathcal{T} \;=\; \frac{\mathcal{P}(x_2)}{\mathcal{P}(x_1)} \;=\; e^{-2\left|\int_{x_1}^{x_2} k(x)dx\right|}}$$

The value of $\mathcal{T}$ does not depend on the way the barrier is crossed

Applications: α decay and nuclear fusion

- we assume that the α particle had a continuing existence as distinct object inside the nucleus ($^4_2He_2$)
- it is subject to nuclear attractive potential plus Coulomb repulsion
- to exit the nucleus it has to cross the Coulomb barrier



- $^{238}_{92}$U $\rightarrow$ there is a barrier of ~ 37 MeV at $R \sim 7$ fm from the center of the nucleus
- α of $E_c = 4.3$ MeV are emitted

$$E^{coul} \simeq \frac{ZZ'e^2}{R} \simeq 37\,\text{M}eV$$

$$R \simeq 1.13A^{1/3}\,\text{f}m \simeq 7\,\text{f}m$$

Fine structure constant: $\alpha = \frac{e^2}{4\pi\epsilon_0\,\hbar\,c} = \frac{1}{137}$ (Gaussian units $4\pi\epsilon_0 = 1$)

- in the nucleus $E_c(\alpha) \simeq 10\,\text{M}eV \rightarrow v_d \simeq 10^7$ m$/s$
- classical image $\rightarrow$ α particle strikes the nucleus wall every

$$\Delta t = \frac{2R}{v_d} \simeq 10^{-22}\ \text{s/collision}$$

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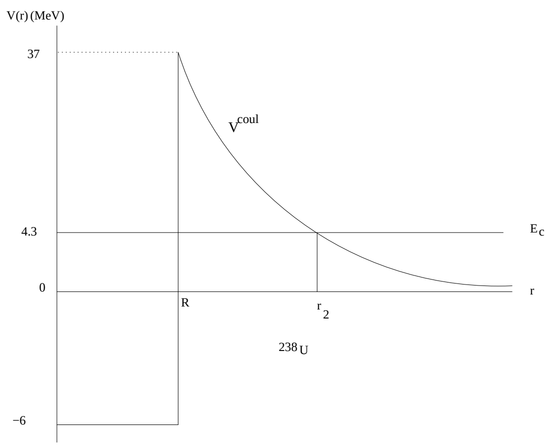

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- $\mathcal{T}$ chance, for each impact, that it will succeed in tunneling through the barrier to the outside
- probability of emission per unit time $\quad \rightarrow \quad \frac{\mathcal{T}}{\Delta t}$
- average time to cross the barrier $\tau \sim \frac{\Delta t}{\mathcal{T}}$

$$\mathcal{T} \;=\; e^{-2\int_{r_1}^{r_2}\; k(r)dr}$$

$$k(r) \;=\; \frac{\sqrt{2\mu(V^{coul}(r) - E_c)}}{\hbar} \quad ; \quad V^{coul}(r) \;=\; \frac{ZZ'e^2}{r}$$

$$Z' \;=\; 2 \quad ; \quad Z \;=\; Z_{\text{nucleus}} - 2 \simeq Z_{\text{nucleus}}$$

$$r_1 \;\simeq R \quad ; \quad r_2 \;=\; \frac{ZZ'e^2}{E_c}$$

$E_c \;=\; \frac{1}{2}m_\alpha v_f^2 \;\; ; \;\; v_f \;\rightarrow\;$ speed of emitted α

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$\mu \;\simeq\; m_\alpha$

$$\int_{r_1}^{r_2} k(r)dr \quad = \quad \frac{\sqrt{2\mu}}{\hbar}\int_R^{\frac{ZZ'e^2}{E_c}} dr\,\sqrt{\frac{ZZ'e^2}{r} - E_c}$$

$$= \quad \frac{2ZZ'e^2}{\hbar v_f}\left[-\frac{\sqrt{\gamma-1}}{\gamma} \;+\; \arccos\frac{1}{\sqrt{\gamma}}\right]$$

where $\gamma \;=\; \frac{ZZ'e^2}{E_c R}$

$$\mathcal{T} \;=\; e^{-\frac{4ZZ'e^2}{\hbar v_f}\left[-\frac{\sqrt{\gamma-1}}{\gamma} \;+\; \arccos\frac{1}{\sqrt{\gamma}}\right]}$$

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- if $\frac{ZZ'e^2}{R} >> E_c \;\; \rightarrow \;\; \gamma >> 1$

  $V^{coul}(R) \; >> E_c \; \rightarrow \; \arccos \frac{1}{\sqrt{\gamma}} \; = \; \frac{\pi}{2} \; ; \; \frac{\sqrt{\gamma - 1}}{\gamma} \; \rightarrow \; \frac{1}{\sqrt{\gamma}}$

$$\mathcal{T} \; \simeq \; e^{-(\pi \frac{ZZ'e^2}{\hbar} \sqrt{\frac{2\mu}{E_c}} - \frac{2}{\hbar} \sqrt{2\mu e^2 ZZ'R})}$$

  Applying it to ^{238}U

  $E_c \; \simeq \; 4$ MeV ; $\frac{ZZ'e^2}{R} \simeq 37$ MeV ; $\gamma >> 1$; $Z \; = \; 90$; $Z' \; = \; 2$; $\mu \; \simeq \; 4 \times 931 \frac{\text{MeV}}{c^2}$; $R \; = \; 7$ fm

$$\mathcal{T} \sim \; 10^{-39}$$

$$T_{1/2} \; \sim \; \frac{\Delta t}{\mathcal{T}} \; \simeq \; \frac{10^{-22}}{10^{-39}} \, s \; \simeq \; 10^{10} \text{ y}$$

  according to $T_{1/2}$ observed, 4.5×10^9y

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- tunneling in nuclear physics in reverse: penetration of charged particles $\rightarrow$ **FUSION**

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