# Final 15/16

a) $\mathrm{rot}\vec{E}(\vec{r},\omega) = i\omega\vec{B}(\vec{r},\omega)$ $\quad$ $\mathrm{div}\vec{D}(\vec{r},\omega) = \rho(\vec{r},\omega)$

$\mathrm{rot}\vec{H}(\vec{r},\omega) = \vec{j}(\vec{r},\omega) - i\omega\vec{D}(\vec{r},\omega)$ $\quad$ $\mathrm{div}\vec{B}(\vec{r},\omega) = 0$

$\vec{D}(\vec{r},\omega) = \varepsilon_0\varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega)$

$\frac{1}{\mu_0}\vec{B}(\vec{r},\omega) = [\vec{H}(\vec{r},\omega) - M(\vec{r},\omega)]$ ?

b) $\mathrm{rot}\vec{E}(\vec{r},\omega) = i\omega\mu_0\vec{H}(\vec{r},\omega)$

$\mathrm{rot}\,\mathrm{rot}\vec{E}(\vec{r},\omega) = i\omega\mu_0[\vec{j}(\vec{r},\omega) - i\omega(\varepsilon_0\varepsilon(\vec{r},\omega)E(\vec{r},\omega))]$

$= i\omega\mu_0\sigma(\omega)\vec{E}(\vec{r},\omega) + \frac{\omega^2}{c^2}\varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega)$

$\varepsilon_0\,\mathrm{div}(\varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega)) = \rho(\vec{r},\omega)$

$\mathrm{grad}\,\varepsilon(\vec{r},\omega)\cdot\vec{E}(\vec{r},\omega) + \mathrm{div}\vec{E}(\vec{r},\omega)\cdot\varepsilon(\vec{r},\omega) = \frac{\rho(\vec{r},\omega)}{\varepsilon_0}$

$\mathrm{div}\vec{E}(\vec{r},\omega) = \dfrac{\frac{\rho(\vec{r},\omega)}{\varepsilon_0} - \mathrm{grad}\,\varepsilon(\vec{r},\omega)\cdot E(\vec{r},\omega)}{\varepsilon(\vec{r},\omega)}$

$\mathrm{grad}\left\{\dfrac{\frac{\rho(\vec{r},\omega)}{\varepsilon_0} - \mathrm{grad}\,\varepsilon(\vec{r},\omega)\cdot\vec{E}(\vec{r},\omega)}{\varepsilon(\vec{r},\omega)}\right\} - \Delta\vec{E}(\vec{r},\omega) - \frac{\omega^2}{c^2}\varepsilon(\vec{r},\omega) = i\omega\mu_0\sigma(\omega)\vec{E}(\vec{r},\omega)$

c) $\iiint \nabla\times S\,ds = \oint_c S\,dc$,

$\oint E\cdot dl = -\iint_S \frac{\partial B}{\partial t}ds \rightarrow \oint\vec{E}(\vec{r},\omega)dl = i\omega\iint_S \vec{B}(\vec{r},\omega)ds$

$\oint_c H\cdot dl = \iint_S J\cdot ds + \iint_S \frac{\partial D}{\partial t}ds \rightarrow \oint\vec{H}(\vec{r},\omega)dl = \iint_S \vec{j}(\vec{r},\omega)ds - i\omega\iint_S\vec{D}(\vec{r},\omega)ds$

$\oint_S \vec{B}(\vec{r},\omega)dS = 0$

$\oint_S \vec{D}(\vec{r},\omega)ds = \iiint \rho(\vec{r},\omega)dV$

d) $\nabla\cdot\nabla\times\vec{H}(\vec{r},\omega) = \nabla\cdot\vec{j}(\vec{r},\omega) - i\omega\,\mathrm{div}\vec{D}(\vec{r},\omega) = 0$

$\therefore\ \mathrm{div}\vec{j}(\vec{r},\omega) = i\omega\rho(\vec{r},\omega)$

$\oint_S \vec{j}(\vec{r},\omega)ds = i\omega\iiint_V \rho(\vec{r},\omega)dV$

Problem 2

a) $E_1 = E_1 \cdot \exp(-i\omega t)\cdot \exp(ikz) = E_1 \cdot \exp(-i\frac{2\pi c t}{\lambda}) \cdot \exp(i\frac{2\pi}{\lambda} z)$

$E_2 = E_2 \cdot \exp(-i\omega t)\cdot \exp[i(k\cos\theta z + k\sin\theta y)] = E_2 \exp(-i\frac{2\pi c t}{\lambda}) \cdot \exp[i\frac{2\pi}{\lambda}(\cos\theta z + \sin\theta y)]$

b) if $x = z = 0$.

$E_1 = 2E_2 \exp(-i\omega t)\cdot 1$

$E_2 = E_2 \exp(-i\omega t) \cdot \exp(i\frac{2\pi}{\lambda}\sin\theta y)$

$I = |E_1 + E_2|^2 = 4E_2^2 + E_2^2 + 4E_2^2 \cos(\frac{2\pi}{\lambda}\sin\theta \cdot y)$

$= E_2^2 [5 + 4\cos(\frac{2\pi}{\lambda}\sin\theta \cdot y)]$

c) $\frac{2\pi \sin\theta}{\lambda} \cdot y = 2\pi$ the distance of two maxima will ~~be smaller~~ decrease.

$\Delta y = \frac{\lambda}{\sin\theta}$

d) $\langle S(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}[E \times H^*]$, $H = -\frac{i}{\omega\mu_0}\mathrm{rot}E$

$H_1 = -\frac{i}{\omega\mu_0}\mathrm{rot}E_1 = -\frac{i}{\omega\mu_0}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial x & \partial y & \partial z \\ E_x & 0 & 0 \end{vmatrix} = -\frac{i}{\omega\mu_0}\begin{pmatrix} 0 \\ \partial z E_x \\ -\partial y E_x \end{pmatrix} = -\frac{i}{\omega\mu_0}\begin{pmatrix} 0 \\ ik \\ 0 \end{pmatrix} E_1$

$\vec{S}_1 = \frac{1}{2}\mathrm{Re}\left[\vec{E}_1 \times \frac{i}{\omega\mu_0}\begin{pmatrix} 0 \\ -ik \\ 0 \end{pmatrix} \vec{E}_1^*\right] = \frac{|E_1|^2}{2\omega\mu_0} k = \frac{|E_1|^2}{2\omega\mu_0}\cdot\frac{2\pi}{\lambda} = \frac{|E_1|^2}{2\mu_0 c} \cdot \hat{z}$

$H_2 = -\frac{i}{\omega\mu_0}\mathrm{rot}E_2 = -\frac{i}{\omega\mu_0}\begin{pmatrix} 0 \\ \partial z E_x \\ -\partial y E_x \end{pmatrix} = -\frac{i}{\omega\mu_0}\begin{pmatrix} 0 \\ ik\cos\theta \\ -ik\sin\theta \end{pmatrix} \vec{E}_2$

$\vec{S}_2 = \frac{1}{2}\mathrm{Re}\left[\vec{E}_2 \times \frac{i}{\omega\mu_0}\begin{pmatrix} 0 \\ -ik\cos\theta \\ ik\sin\theta \end{pmatrix} \vec{E}_2\right] = \frac{|E_2|^2}{2\omega\mu_0}\cdot\begin{pmatrix} 0 \\ k\cos\theta \\ -k\sin\theta \end{pmatrix} = \frac{|E_2|^2}{2\mu_0 c}\begin{pmatrix} 0 \\ \cos\theta \\ -\sin\theta \end{pmatrix}$

$\vec{S} = \vec{S}_1 + \vec{S}_2 = \frac{1}{2\mu_0 c}\begin{pmatrix} 0 \\ |E_2|^2\cos\theta \\ |E_1|^2 - |E_2|^2\sin\theta \end{pmatrix}$

blem 4

a) $\frac{1}{v_{ph}} = \frac{k(w_0)}{w_0} = \alpha_1 \frac{w_0}{w_1} + \beta_1\left(1-\frac{w_0}{w_1}\right) + \gamma_1\left(\frac{w_1}{w_0} - 2 + \frac{w_0}{w_1}\right)$

$w_0/w_1 = \frac{2}{3}$, $w_1/w_0 = \frac{3}{2}$

$\frac{1}{v_{ph}} = \frac{3}{2c} \times \frac{2}{3} + \frac{3}{4c}\left(1-\frac{2}{3}\right) + \frac{3}{8c}\left(\frac{3}{2} - 2 + \frac{2}{3}\right) = \cancel{\frac{16}{21}c}\ \frac{21}{16c}$

$v_{ph} = \frac{16}{21}c = \frac{16}{7} \times 10^8 \,(m/s)$

b) $\frac{1}{v_{gr}} = \frac{\partial k}{\partial w}\Big|_{w_0} = \beta_1 + \frac{2\gamma_1}{w_0}(w - w_0)$ when $w = w_1$

$\frac{1}{v_{gr}} = \frac{9}{8c}$, $v_{gr} = \frac{8c}{9} = \frac{8}{3} \times 10^8 \, m/s$

$t = \frac{L}{v_{gr}} = \frac{1}{\frac{8}{3} \times 10^8 \, m/s} = \frac{3}{8} \times 10^{-8} \,(s)$

c) $v_{gr_1} = \beta_1 + \frac{2\gamma_1}{w_0}(w - w_0)$, $v'_{gr_1} = \frac{8c}{9}$, $v''_{gr_1} = \frac{2}{3}c$

$\frac{1}{v_{gr_2}} = \frac{\partial k_2}{\partial w}\Big|_{w_1} = \beta_2 + \frac{2\gamma_2}{w_0}(w - w_0)$

$\frac{1}{v'_{gr_2}} = \frac{2}{c} - \frac{8}{9c}\left(\frac{w_1}{w_0} - 1\right) = \frac{14}{9c}$, $v'_{gr_2} = \cancel{\frac{14}{9c}}\ \frac{9}{14}c$

$\frac{1}{v''_{gr_2}} = \frac{2}{c} - \frac{8}{9c}\left(\frac{w_2}{w_0} - 1\right) = \frac{10}{9c}$, $v''_{gr_2} = \frac{9}{10}c$

$t_1 = t_2$

$\frac{L_1}{v'_{gr_1}} + \frac{L_2}{v'_{gr_2}} = \frac{L_1}{v''_{gr_1}} + \frac{L_2}{v''_{gr_2}}$

$\frac{9}{8c} + \frac{14}{9c}L_2 = \frac{3}{2c} + \frac{10}{9c}L_2$

$L_2 = \frac{27}{32} \,(m)$

Problem 3

a) i) paraxial case. $\alpha^2+\beta^2 \ll k^2$. $N_F \lesssim 10$

ii) long distance. $N_F = \frac{a^2}{\lambda z_B} \lesssim 1$

b) No. Because Fraunhofer approximation was derived from Response function of Fresnel Transferfunction.

c) i) $u_0(x,y) = \iint_{-\infty}^{\infty} \exp(i\sqrt{k^2-\alpha^2-\beta^2}\,z) \cdot U_0(\alpha,\beta)\exp[i(\alpha x+\beta y)]\,d\alpha\, d\beta.$

$U_0(\alpha,\beta) = FT[u_0(x,y)]$

ii) $u_0(x,y) = \iint_{-\infty}^{\infty} \exp(ikz)\exp\left(-i\frac{\alpha^2+\beta^2}{2k}z\right)U_0(\alpha,\beta)\exp[i(\alpha x+\beta y)]\,d\alpha\, d\beta$

iii) $u_0(x,y) = -i\frac{(2\pi)^2}{\lambda z_B}\exp(ikz_B)\,U_0\left(k\frac{x}{z_B}, k\frac{y}{z_B}\right)\exp\left[i\frac{k}{2z_B}(x^2+y^2)\right]$

Problem 5

a) $U_{00}(x,y) = rect(\frac{a}{2}) rect(\frac{b}{2})$

$U_{10}(x,y) = rect(\frac{a}{2}-\Delta) rect(\frac{b}{2})$

$U_{01}(x,y) = rect(\frac{a}{2}) rect(\frac{b}{2}+\Delta)$

$\cdots$

$U_{mn}(x,y) = rect(\frac{a}{2}-m\Delta) rect(\frac{b}{2}+n\Delta)$

$U(x,y) = \sum_{m=0}^{3}\sum_{n=0}^{3} rect(\frac{a}{2}-m\Delta) rect(\frac{b}{2}+n\Delta)$

b) According to the shift theorem: $f(x-a, y+b) \rightarrow F(\alpha,\beta)\cdot \exp[-i\alpha a + i\beta b]$

$U(x,y,2f) = -i\frac{(2\pi)^2}{\lambda f}\exp(2ikf)\, U_0(\frac{k}{f}x, \frac{k}{f}y)$

$\therefore \cancel{U_{00}(x,y,2f) = -i\frac{(2\pi)^2}{\lambda f}\cdot ex} \sim \quad sinc(\frac{kxa}{f2}) sinc(\frac{kxb}{2f})$

$U_{mn}(x,y,2f) \sim sinc(\frac{kxa}{2f}) sinc(\frac{kyb}{2f})\cdot \exp\{-i[\frac{kx}{f}m\Delta - \frac{ky}{f}n\Delta]\}$

$U(x,y,2f) = \sum_{m=0}^{3}\sum_{n=0}^{3} sinc(\frac{kxa}{2f}) sinc(\frac{kyb}{2f}) \exp\{-i[\frac{kx}{f}m\Delta - \frac{ky}{f}n\Delta]\}$

c) $\cancel{W(x,y) = \begin{cases} 1 & |y|<W \\ 0 & elsewhere \end{cases} \rightarrow p(x,y) = \begin{cases} 1 & |\frac{f}{k}\beta|<W \\ 0 & elsewhere \end{cases}}$

~~$W < 2\pi$~~

~~The highest frequency in y- direction is: $\frac{ky\,3\Delta}{f}$~~

$W(x,y) = \begin{cases} 1 & |y|<W \\ 0 & elsewhere. \end{cases}$ The width of sinc function is:

$|\frac{kyb}{2f}| < \pi$

$|y| < |\frac{f\lambda}{b}|$

$\therefore$ The width of aperture: $|W| < |\frac{f\lambda}{b}|$ ?

Problem 6.

a) $E = E_0 e^{ikz}\begin{pmatrix}1\\i\\0\end{pmatrix}$ $\quad \cancel{E_{\parallel} = \frac{\sqrt{2}}{2}(\vec{E_x}+\vec{E_y})\cdot\exp(ik_e z)}$

$\cancel{=\frac{\sqrt{2}}{2}}$

$E_{\parallel} = \frac{\sqrt{2}}{2}(\vec{E_x}+\vec{E_y})\cdot E_0\exp(ik_e z)$ . $\quad k_e = k n_e = \frac{2\pi}{\lambda} n_e$

$= \frac{\sqrt{2}}{2}(1+i)E_0\exp(i\frac{2\pi n_e}{\lambda}z)$.

$E_{\perp} = \frac{\sqrt{2}}{2}(\vec{E_x}-\vec{E_y})\cdot\exp(ik_o z)$. $\quad k_o = k n_o = \frac{2\pi}{\lambda} n_o$

$= \frac{\sqrt{2}}{2}(1-i)\exp(i\frac{2\pi n_o}{\lambda}z)$

b) ○ ○ ○ ○ ○ ○ ○

c) $k_e d = \frac{2\pi}{\lambda}\cdot n_e d = 21.5\lambda \quad \exp(ik_e d) = i$

$k_o d = \frac{2\pi}{\lambda} n_o d = 22\lambda$. $\exp(ik_o d) = 1$

$\cancel{X: \begin{cases} E_{\parallel} = \frac{\sqrt{2}}{2} i\cdot E_0 e^{i\frac{2\pi}{\lambda}z} \\ E_{\perp} = \frac{\sqrt{2}}{2}\cdot E_0 e^{i\frac{2\pi}{\lambda}z}\end{cases}} = \frac{\sqrt{2}}{2}(i+1)E_0 e^{i\frac{2\pi}{\lambda}z}$

$\cancel{Y: E_{\parallel} =}$ $\quad E_{\parallel} = \frac{\sqrt{2}}{2}(1+i)E_0\cdot i = \frac{\sqrt{2}}{2}E_0(i-1)\exp(i\frac{2\pi}{\lambda}z)$.

$E_{\perp} = \frac{\sqrt{2}}{2}(1-i)E_0\cdot 1 = \frac{\sqrt{2}}{2}E_0(1-i)\exp(i\frac{2\pi}{\lambda}z)$

$E_x = E_{\parallel}\cdot\cos 45° + E_{\perp}\sin 45° = -E_0\exp(i\frac{2\pi}{\lambda}z)$. $\cancel{(1+1)}$

$E_y = E_{\parallel}\cdot\sin 45° + E_{\perp}\cos 45° = E_0\exp(i\frac{2\pi}{\lambda}z)$.

$E = E_0\cdot\exp(i\frac{2\pi}{\lambda}z)\begin{pmatrix}-1\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}E_0\exp(i\frac{2\pi}{\lambda}z)\begin{pmatrix}-\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{pmatrix}$. linear polarization.

# Problem 8

a) $k_{zs} = k_{z_1} = k_{z_2} = \cdots = k_{zc}$.

$$\frac{w}{c}\sqrt{\varepsilon_s}\cdot\sin\phi_s = \frac{w}{c}\sqrt{\varepsilon_c}\sin\phi_c$$

$$\sin\phi_c = \sqrt{\frac{\varepsilon_s}{\varepsilon_c}}\,\sin\phi_s$$

$$\phi_c = \sin^{-1}\left(\sqrt{\frac{\varepsilon_s}{\varepsilon_c}}\sin\phi_s\right)$$

b)
$$\begin{pmatrix} F_n(x) \\ G_n(x) \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} F_0(x) \\ G_0(x) \end{pmatrix}$$

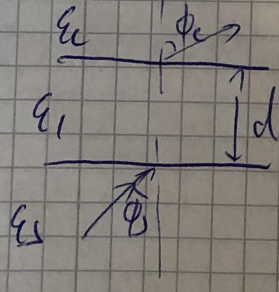
$F(x) = E, H$

$G(x) = \alpha_f \frac{\partial}{\partial x} F(x)$

$\hat{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ is the matrix of the layer system.

$$\hat{m} = \begin{pmatrix} \cos(k_{fx}x) & \frac{1}{k_{fx}\alpha_f}\sin(k_{fx}x) \\ -k_{fx}\alpha_f\sin(k_{fx}x) & \cos(k_{fx}x) \end{pmatrix}. \qquad k_{fx} = \sqrt{\frac{w^2}{c^2}\varepsilon_f(w) - k_z^2}$$

c)




$$k_{fx} = \sqrt{\frac{w^2}{c^2}\varepsilon_1 - \frac{w^2}{c^2}\varepsilon_s\sin^2\phi_s} = \frac{w}{c}\sqrt{\varepsilon_1 - \varepsilon_s\sin^2\phi_s}$$

$$\alpha_{fTE} = 1.$$

$$M = \begin{pmatrix} \cos\left(\frac{w}{c}\sqrt{\varepsilon_1 - \varepsilon_s\sin^2\phi_s}\cdot d\right) & \frac{1}{\frac{w}{c}\sqrt{\varepsilon_1 - \varepsilon_s\sin^2\phi_s}}\sin\left(\frac{w}{c}\sqrt{\varepsilon_1 - \varepsilon_s\sin^2\phi_s}\cdot d\right) \\ -\frac{w}{c}\sqrt{\varepsilon_1 - \varepsilon_s\sin^2\phi_s}\,\sin\left(\frac{w}{c}\sqrt{\varepsilon_1 - \varepsilon_s\sin^2\phi_s}\,d\right) & \cos\left(\frac{w}{c}\sqrt{\varepsilon_1 - \varepsilon_s\sin^2\phi_s}\cdot d\right) \end{pmatrix}.$$