Institute of
Applied Physics
Friedrich-Schiller-Universität Jena

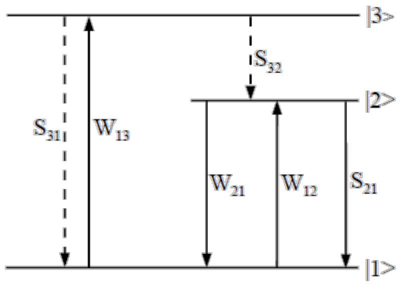
Laser Physics problem sheet 7

Summer semester 2023

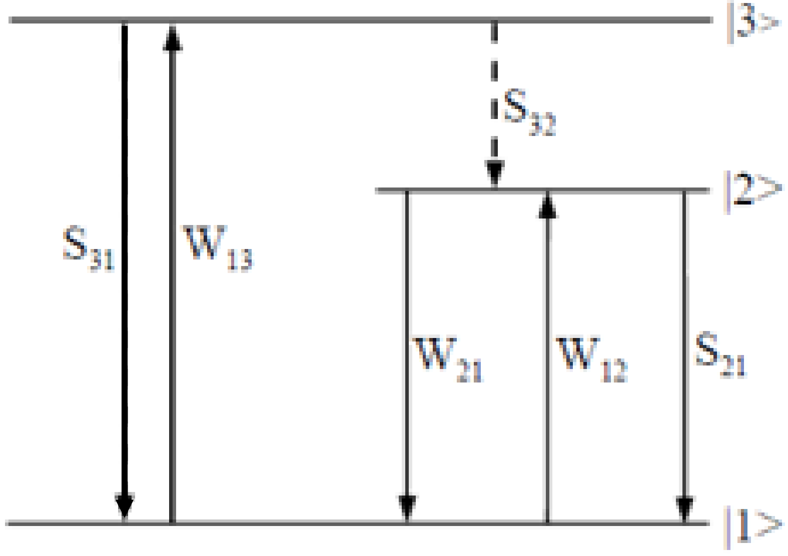
Problem 1

Problem 3 (5 points)

In this exercise we will calculate the equivalent 2-level rate equations of a 3-level system. The energy diagram of a generic 3-level system with all the relevant transitions is shown in the picture. Non-radiative transitions are represented by dashed lines:



a) Write down the 3-level rate equations for the energy diagram shown above (i.e. the rate equations for all three levels). (1 point)



Recap of 3-level system:

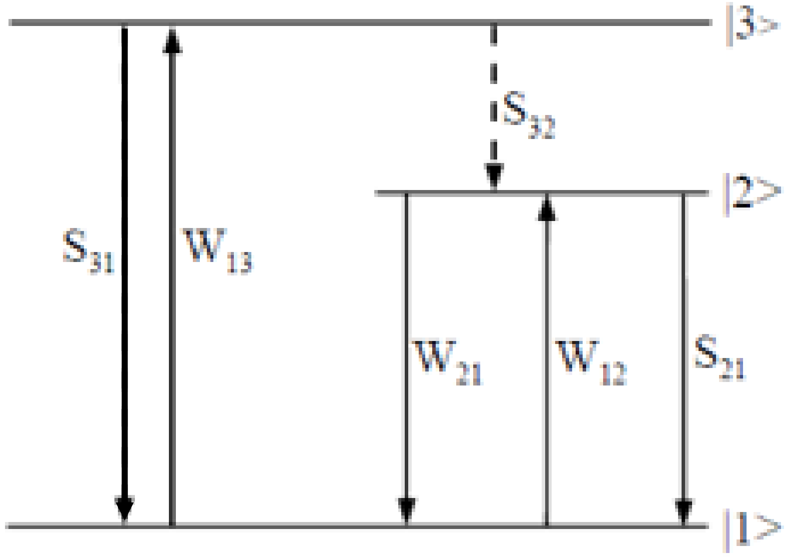
$\boldsymbol{W_{13}/W_{12}}$: one-particle transition probability per unit time for absorption $|\mathbf{1}> \rightarrow |\mathbf{3}>/\ |\mathbf{2}>$

$\boldsymbol{W_{21}}$: one-particle transition probability per unit time for stimulated emission $|\mathbf{2}> \rightarrow |\mathbf{1}>$

$\boldsymbol{S_{31}}/\ \boldsymbol{S_{32}}/\ \boldsymbol{S_{21}}$: decay rate due to spontaneous emission (relaxation)

$\boldsymbol{N}$: population density at each energy level

The rate equations for all three levels:



$|3>:$

$$\frac{dN_3}{dt} = W_{13}N_1 - S_{31}N_3 - S_{32}N_3$$

number of spont. emission process $|3> \rightarrow |2>$ per unit time and volume

number of spont. emission process $|3> \rightarrow |1>$ per unit time and volume

number of pump process $|1> \rightarrow |3>$ per unit time and volume

The rate equations for all three levels:

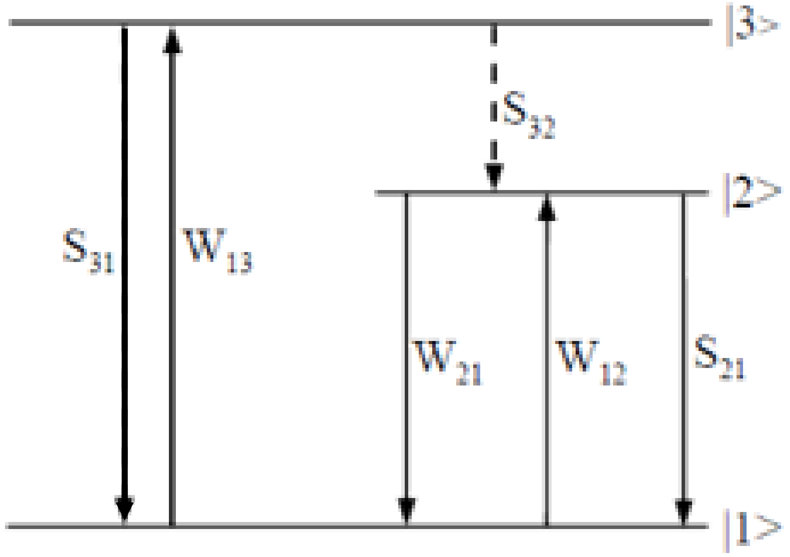
$|2>$:

$$\frac{dN_2}{dt} = S_{32}N_3 - \underbrace{W_{21}N_2 + W_{12}N_1} - S_{21}N_2$$

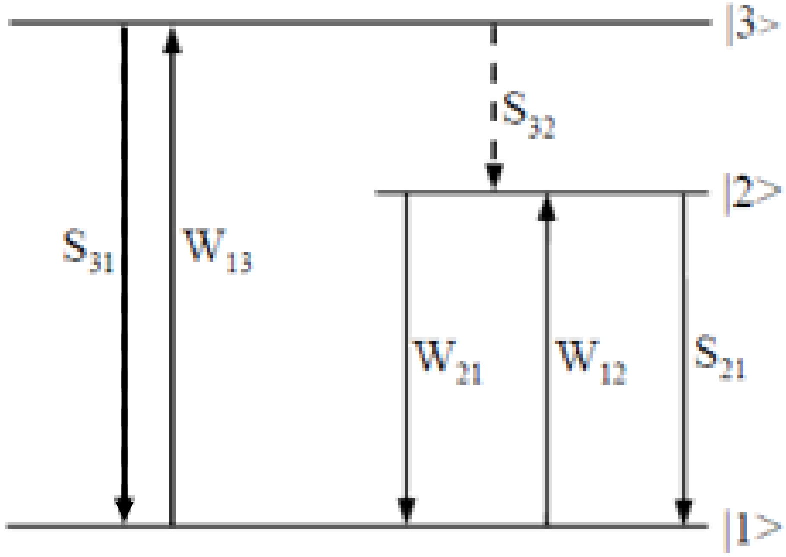
number of spont. emission process $|2> \rightarrow |1>$ per unit time and volume

number of stimulated process $|3> \leftrightarrow |1>$ per unit time and volume

number of spont. emission process $|3> \rightarrow |2>$ per unit time and volume



The rate equations for all three levels:



$|1>:$

$$\frac{dN_1}{dt} = S_{31}N_3 - W_{13}N_1 - \underbrace{W_{12}N_1 + W_{21}N_2}_{} + S_{21}N_2$$

number of spont. emission process $|2> \rightarrow |1>$ per unit time and volume

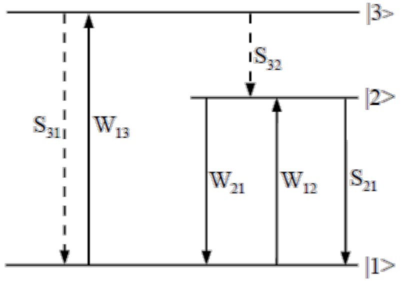
number of stimulated process $|2> \leftrightarrow |1>$ per unit time and volume

number of absorption process $|1> \rightarrow |3>$ per unit time and volume

number of spont. emission process $|3> \rightarrow |1>$ per unit time and volume

Problem 3 (5 points)

In this exercise we will calculate the equivalent 2-level rate equations of a 3-level system. The energy diagram of a generic 3-level system with all the relevant transitions is shown in the picture. Non-radiative transitions are represented by dashed lines:



b) Using the equations above, obtain the condition required to keep $N_3 = 0$ (assuming that this level is unpopulated before starting the pump process). Explain why this condition makes physical sense. (2 points)

Rate equations we wrote in a):

$$\frac{dN_3}{dt} = W_{13}N_1 - S_{31}N_3 - S_{32}N_3$$

$$\frac{dN_2}{dt} = S_{32}N_3 - W_{21}N_2 + W_{12}N_1 - S_{21}N_2$$

$$\frac{dN_1}{dt} = S_{31}N_3 - W_{13}N_1 - W_{12}N_1 + W_{21}N_2 + S_{21}N_2$$

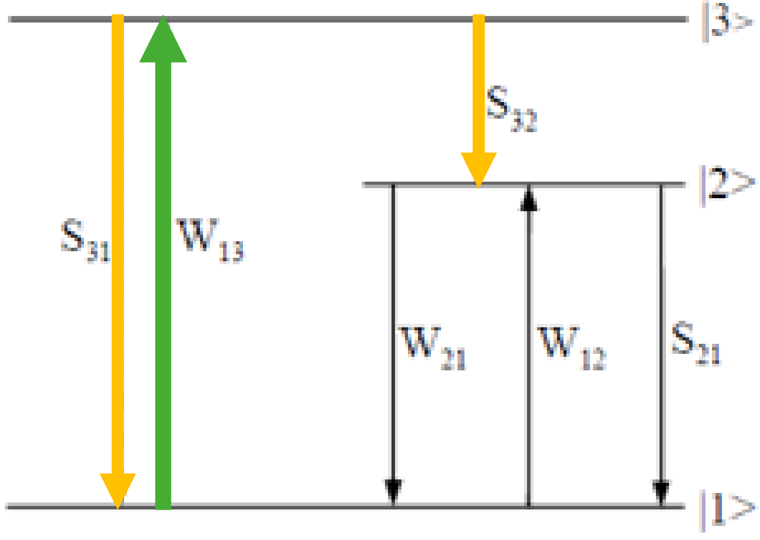
Unpopulated pump band $N_3 = 0$ before pump process $(W_{13} = 0)$

$\Rightarrow$ $$\frac{dN_3}{dt} = W_{13}N_1 - S_{31}N_3 - S_{32}N_3 = 0$$

$\rightarrow$ $|3>$ will remain unpopulated during pump process

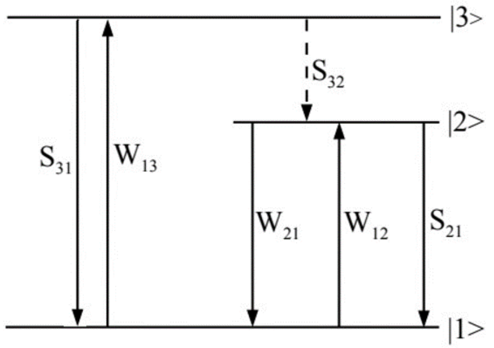
$\Rightarrow$ Condition to keep $N_3 = 0$:

$$\boxed{W_{13}N_1} = \boxed{(S_{31}+S_{32})N_3}$$



- Left term describes the one possible absorption path to |3>
- Right terms describe the two possible decay paths from |3>
- To keep N_3 = 0, the absorption rate must equal the rate of both spontaneous emission paths

c) Now, considering that N_3 remains $N_3 \approx 0$, obtain the 2-level rate equations equivalent to the 3-level system depicted above. These equations should contain the quantum efficiency η as a parameter.
Hint: $\eta = \frac{S_{32}}{S_{31}+S_{32}}$



Equivalent 2-level rate equations: eliminate N_3 in $\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$ from a):

$$\frac{dN_1}{dt} = -W_{12}N_1 - W_{13}N_1 + S_{21}N_2 + W_{21}N_2 + \boxed{S_{31}N_3}$$

$$\big\downarrow (2)$$

$$\frac{dN_1}{dt} = -W_{12}N_1 - \cancel{W_{13}N_1} + S_{21}N_2 + W_{21}N_2 + \boxed{(\cancel{1} - \eta)\, W_{13}N_1}$$

$$= -W_{12}N_1 + S_{21}N_2 + W_{21}N_2 - \eta\, W_{13}N_1$$

$$\frac{dN_2}{dt} = W_{12}\, N_1 - S_{21}N_2 - W_{21}N_2 + \boxed{S_{32}N_3}$$

$$\big\downarrow (1)$$

$$\frac{dN_2}{dt} = W_{12}\, N_1 - S_{21}N_2 - W_{21}N_2 + \boxed{\eta\, W_{13}N_1}$$

From b)

$$N_3(S_{31} + S_{32}) = W_{13}N_1$$

Re-arange the quantum efficiency term:

$$\eta = \frac{S_{32}}{S_{31} + S_{32}} \rightarrow S_{31} + S_{32} = \frac{S_{32}}{\eta}$$

Combine:

$$N_3(S_{31} + S_{32}) = \underbrace{N_3 \frac{S_{32}}{\eta} = W_{13}N_1}$$

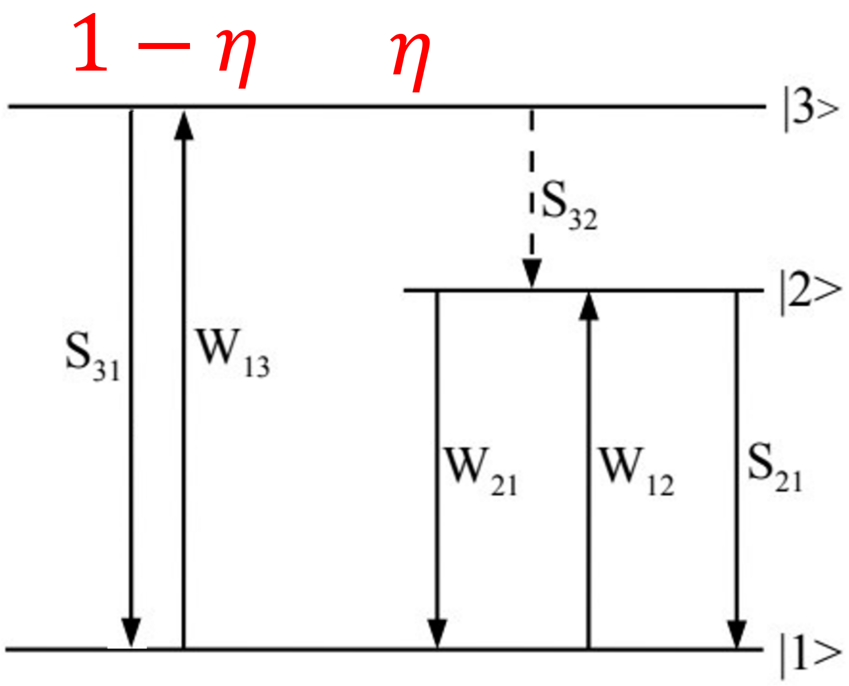
$$N_3S_{32} = \eta W_{13}N_1 \quad (1)$$

$$N_3S_{31} = W_{13}N_1 - N_3S_{32} \overset{(1)}{=} (1 - \eta)W_{13}N_1 \quad (2)$$

c) Now, considering that N_3 remains $N_3 \approx 0$, obtain the 2-level rate equations equivalent to the 3-level system depicted above. These equations should contain the quantum efficiency η as a parameter.
Hint: $\eta = \frac{S_{32}}{S_{31}+S_{32}}$

If N_3=0, why can't we just cross out the terms with N_3?

- We are assuming that τ_3 is very short, so no population N_3 can accumulate
- However, as ions are pumped, they must still pass through |3>
- This means they still have a probability to move to either |2> or back to |1>
- This process is instantaneous and independent of N_3 so we have to find a way to represent this in the rates $\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$
- This is represented using the quantum efficiency η



Every ion pumped from |1> to |3> has a probability η of getting to |2> and a probability (1- η) of going back to |1>. This must be included in the 2-level rates

d) When considering the rate equations from point c), it can be seen that the terms involved in the pump process fulfill

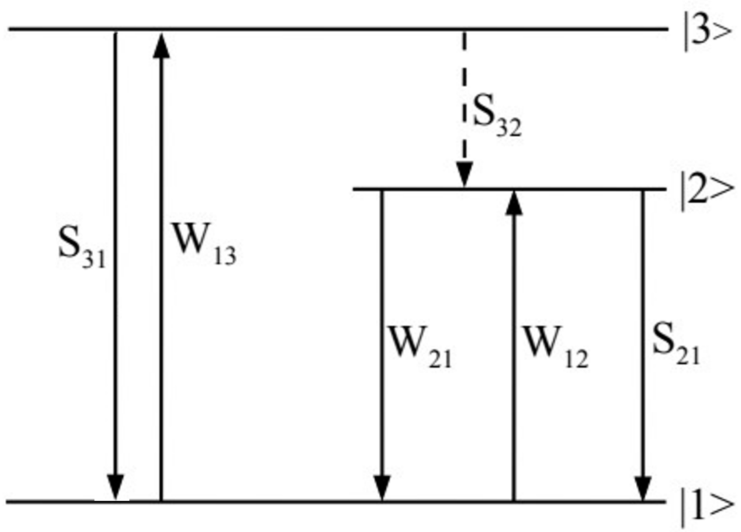
$$\frac{dN_1}{dt}\Big|_{\text{Pump process}} = -\frac{dN_2}{dt}\Big|_{\text{Pump process}}$$

Why is this like that?

From c)

$$\frac{dN_1}{dt} = -W_{12}N_1 + S_{21}N_2 + W_{21}N_2 - \eta\, W_{13}N_1$$

$$\frac{dN_2}{dt} = W_{12}\, N_1 - S_{21}N_2 - W_{21}N_2 + \eta\, W_{13}N_1$$

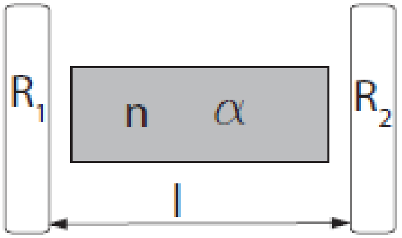


$$\frac{dN_1}{dt}\Big|_{\text{Pump process}} = -\eta\, W_{13}N_1 = -\frac{dN_2}{dt}\Big|_{\text{Pump process}}$$

- Since we assumed $N_3 = 0$ at all times, we were able to reduce the equations for $\frac{dN_2}{dt}$ and $\frac{dN_1}{dt}$ to a two-level dependency.
- These new equations imply that the only population sources are |1> and |2>
  - Any ion successfully pumped to |2> must come from |1>
- Therefore the rate of change of N_1 due to pumping must be the inverse of the rate of change of N_2 due to pumping

Problem 2

Problem 2 (4 Points)



In the linear cavity depicted above n is the index of refraction of the material filling the cavity, and α represents the one-pass losses in this material. R_1 and R_2 are the reflectivities of the mirrors and l is the cavity length.

a) Calculate the general expression of the photon lifetime τ_{ph} in the cavity as a function of the cavity round-trip time τ_R. (2 points)

b) Assuming that $R_1 = 1$, obtain the expression given in the lecture:

$$\frac{\tau_{ph}}{\tau_R} \approx (T + L)^{-1}$$

where T is the transmissivity of the outcoupling mirror and L are the round-trip losses in the cavity. Under which circumstances is the expression above valid? (2 points)

The round-trip time is

$$\tau_R = \frac{2nl}{c}$$

From the lecture we know that the photon density in a passive cavity changes according to

$$\frac{dp}{dt} = -\frac{p}{\tau_{ph}} \quad \rightarrow \quad p(t) = p_0 exp(-t/\tau_{ph}) \quad \rightarrow \quad p(N\tau_R) = p_0 exp(-N\tau_R/\tau_{ph})$$

On the other hand, we have that every round-trip (τ_R) the photon density is reduced by a factor $R_1 R_2 (1-\alpha)^2$, thus:

$$p(\tau_R) = p_0 R_2(1-\alpha) R_1(1-\alpha) = p_0 R_1 R_2 (1-\alpha)^2$$

$$p(2\tau_R) = p(\tau_R) R_2(1-\alpha) R_1(1-\alpha) = p_0 [R_1 R_2 (1-\alpha)^2]^2$$

$$p(N\tau_R) = p_0 [R_1 R_2 (1-\alpha)^2]^N$$

Therefore,

$$p(N\tau_R) = p_0 exp(-N\tau_R/\tau_{ph}) = p_0 [R_1 R_2 (1-\alpha)^2]^N$$

So,

$$\tau_{ph} = \frac{-\tau_R}{ln[R_1 R_2 (1-\alpha)^2]}$$

b) Assuming that $R_1 = 1$, obtain the expression given in the lecture:

$$\frac{\tau_{ph}}{\tau_R} \approx (T + L)^{-1}$$

where T is the transmissivity of the outcoupling mirror and L are the round-trip losses in the cavity. Under which circumstances is the expression above valid? (2 points)

b) if $R_1 = 1$, then $T = 1 - R_2 \Rightarrow R_2 = 1 - T$

Additionally, $\underset{\substack{\uparrow \\ \text{round-trip losses}}}{L} = 1 - (1-\alpha)^2 \Rightarrow (1-\alpha)^2 = 1 - L$

Thus from a):

$$\boxed{\tau_{ph}} = \frac{-\tau_R}{\ln[(1-T)\cdot(1-L)]} = \frac{-\tau_R}{\ln(1-T) + \ln(1-L)} \approx \boxed{\frac{\tau_R}{T+L}}$$

Small transmisivity, small losses $\Leftarrow$ Validity conditions $\rightarrow$ $\boxed{\begin{array}{l} \ln(1-T) \simeq -T \text{ if } T \ll 1 \\ \ln(1-L) \simeq -L \text{ if } L \ll 1 \end{array}}$

Problem 3

Problem 3 (6 Points)

A system described by the rate equations deviates from the equilibrium state by a small margin. The change of the inversion (Δn) and photon density (Δp) with respect to their steady state values ($\bar{n}$ and $\bar{p}$) can be mathematically modelled by:

$$n = \bar{n} + \Delta n \qquad p = \bar{p} + \Delta p$$

In the following calculations please neglet the product $\Delta n \Delta p$ as well as the spontaneous emission term S.

a) Consider a 4-level system with $n << n_{tot}$ and show that the temporal change of the deviations in inversion and photon density can be written as:

$$\frac{d(\Delta n)}{dt} = -(\sigma c \bar{p} + \Gamma)\Delta n - \frac{\Delta p}{\tau_{ph}}$$

$$\frac{d(\Delta p)}{dt} = \sigma c \bar{p} \Delta n$$

Hint: $\bar{n} = n_{th}$, $\bar{p} = n_{tot}(W_p - W_{th})\tau_{ph}$, $W_{th} = \Gamma \frac{n_{th}}{n_{tot}}$. (2 points)

b) Derive $\frac{d(\Delta n)}{dt}$ analogous to a) but for a 3-level system, where $n << n_{tot}$ is not valid anymore. (2 points)

a) – rate equation for a 4-level system ($\gamma = 1$) where $n << n_{tot}$

$$\frac{dn}{dt} = -\sigma c p n - \Gamma n + W_P n_{tot}$$

$$\frac{dp}{dt} = \sigma c p (n - n_{th})$$

– insert $n = \bar{n} + \Delta n$ and $p = \bar{p} + \Delta p$ into these equations and simplify.

– start with p

– calculation

$$\frac{d(\bar{p} + \Delta p)}{dt} = \sigma c(\bar{p} + \Delta p)((n + \Delta n) - n_{th}) \qquad \bar{n} = n_{th}$$

$$\frac{d\bar{p}}{dt} + \frac{d(\Delta p)}{dt} = \sigma c \bar{p} \Delta n + \sigma c \Delta p \Delta n \qquad \frac{d\bar{p}}{dt} = 0, \quad \Delta n \cdot \Delta p = 0$$

$$\frac{d(\Delta p)}{dt} = \bar{p} c \sigma \Delta n$$

– next: n

$$\frac{d\bar{n}}{dt} + \frac{d(\Delta n)}{dt} = -\sigma c\bar{p}\bar{n} - \sigma c\bar{p}\Delta n - \sigma c\bar{n}\Delta p - \sigma c\Delta p\Delta n - \Gamma\bar{n} - \Gamma\Delta n + W_P n_{tot}$$

$$\frac{d(\Delta n)}{dt} = -(\sigma cp + \Gamma)\Delta n - \sigma cp\bar{n} - \sigma c\Delta p\bar{n} - \Gamma\bar{n} + W_P n_{tot}$$

– here the term $\sigma c\Delta p\bar{n}$ can be phrased as $\frac{\Delta p}{\tau_{ph}}$

– the last 3 remaining terms cancel each other

$$W_P n_{tot} = \Gamma\bar{n} + \sigma c\bar{p}\bar{n}$$

– to show this this last step you need the hint equations

$$\frac{d(\Delta n)}{dt} = -(\sigma c\bar{p} + \Gamma)\Delta n - \frac{\Delta p}{\tau_{ph}}$$

1) Use gamma = 2 to find rate equation
2) Insert terms for n and p analogous to a) and simplify.

b)

$$\frac{d(\bar{n}+\Delta n)}{dt} = -2\sigma c(\bar{p}+\Delta p)(\bar{n}+\Delta n) - \Gamma n_{tot} - \Gamma(\bar{n}+\Delta n) + W_P n_{tot} - W_P(\bar{n}+\Delta n)$$

$$\frac{d(\Delta n)}{dt} = -(2\sigma c\bar{p} + \Gamma + W_P)\Delta n - 2\frac{\Delta p}{\tau_{ph}} - [2\sigma c\bar{p}\bar{n} + \Gamma n_{tot} + \Gamma\bar{n} - W_P n_{tot} + W_P\bar{n}]$$

$$\frac{d(\Delta n)}{dt} = -(2\sigma c\bar{p} + \Gamma + W_P)\Delta n - \frac{\bar{p}+2\Delta p}{\tau_{ph}} + \Gamma n_{tot} + W_P n_{th}$$

Solution is similar to result from a), but contains more terms, as certain terms do no longer cancel each other with n << n_tot no longer valid

Problem 3: c) and d)

c) The two linear differential equations obtained in question a) can be converted into one differential equation of second order:

$$\frac{d^2(\Delta n)}{dt^2} + 2\delta\frac{d(\Delta n)}{dt} + \omega^2\Delta n = 0$$

where

$$\delta = \frac{W_p}{2\tau_2 W_{th}} \qquad \omega = \sqrt{\frac{1}{\tau_2\tau_{ph}}(\frac{W_p}{W_{th}} - 1)}.$$

What is the physical meaning of the parameters δ and ω? What will happen to Δn and Δp in the cases that δ is much higher or lower than ω? (1 point)

d) Assume two laser systems. The first one has $\tau_2 = 100ns$ and $\tau_{ph} = 10^{-4}s$. The second one has $\tau_2 = 100\mu s$ and $\tau_{ph} = 10^{-8}s$. Additionally, assume that $\frac{W_p}{W_{th}} = 4$. In which system would you expect to see relaxation oscillations? Why? (1 point)

c) δ... damping constant, suppresses oscillations due to spontaneous emission and pump
ω ... oscillation frequency of Δn if damping is weak
$\delta >> \omega$ ... strong damping means Δn will go back to 0 without oscillations
$\delta << \omega$ ... damped oscillations of Δn with frequency ω
Change of Δn will impact Δp. Strong damping of Δn leads to a strongly damped Δp. Same in the case of oscillations.

d) Calculate delta and omega in the 2 cases using the equation from c). Compare their values.

System 1: delta = 20*10^6 Hz, omega = 547*10^3 Hz; delta is bigger → no oscillations

System 2: delta = 20*10^3 Hz, omega = 1.7*10^6 Hz; delta is smaller → oscillations