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WS 2009/2010

Exam
Fundamentals of Modern Optics

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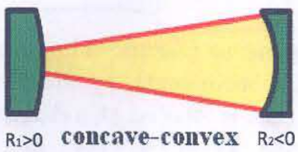
Exercise 1 — 8 Points

A resonator consists of two spherical mirrors: one of them is a concave mirror with radius $R_1 > 0$, the other one is a convex mirror with radius $R_2 < 0$, and the distance between them is d. Define the conditions for this resonator to be stable, and sketch a stable configuration. Do not forget to mark the positions of the mirrors and their respective centers of curvature in your sketch.



Exercise 2 — 10 Points

In an experiment you have an input Gaussian beam profile with width W_{in} and flat phase (phase curvature $R_{in} = \infty$). Your task is to obtain a beam profile with the same parameters (W_{in} and R_{in}), but at a distance L from the input position. You are allowed to use just a single lens. Calculate the focal length of the lens you need and its position.

Exercise 3 — 15 Points

Illumination of a cross grating produces a light distribution

$$u_0(x,y) = \frac{A}{4}\left(1+\cos\frac{2\pi}{a}x\right)\left(1+\cos\frac{2\pi}{a}y\right),$$

with period length $a = 1\,\mathrm{mm}$. This light field is now imaged by a $4f$-setup, where in the plane $z = 2f$ a slit with the filter function

$$p(x,y) = \begin{cases} 1 & ,|x| < D/2 \\ 0 & ,\text{elsewhere} \end{cases}$$

is applied. The focal length is $f = 1\,\mathrm{m}$ and the wavelength used is $\lambda = 1\,\mu\mathrm{m}$.
Calculate the field $u(x,y,4f)$ at the end of the $4f$-setup for a slit width $D = 1\,\mathrm{mm}$.

Exercise 4 — 10+5 Points

a) Consider a dielectric medium with response function

$$R(t) = \frac{f}{\sqrt{a^2-b^2}}\, e^{-bt}\sin\left(\sqrt{a^2-b^2}\;t\right),$$

where $a > b > 0$. Calculate the electric susceptibility $\chi(\omega)$.

b) A company producing optical instruments is looking for a new homogeneous, isotropic material which should have the optical property

$$\chi(\omega) = Ae^{-\frac{(\omega-\omega_0)^2}{B^2}} + i\, C\delta(\omega - \omega_0),$$

where $A = 0.542$, $B = 1.02 \cdot 10^{15}\, s^{-1}$, $C = 3.29$ and $\omega_0 = 4.71 \cdot 10^{15}\, s^{-1}$. Do you think their research can be successful? Explain with the help of the Kramers Kronig relations.

Exercise 5

7 Points

Consider an uniaxial crystal with refractive indices for the ordinary wave n_o and the extraordinary wave n_e. The crystal's optical axis is parallel to its surface. A monochromatic, circularly polarized wave is normally incident on the crystal. Compute the propagation lengths after which the light is linearly polarized.

Exercise 6

10 Points

Two pulses are propagating in a homogeneous plasma. They have a carrier frequency much larger than the plasma frequency. The corresponding wavelengths of the carrier waves are λ_1 and λ_2.
The signals are recorded by a detector that is located at a distance L from the source. Use the dielectric function of the plasma

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

to compute the time delay between the two pulses.

Exercise 7

5+10 Points

Consider an interface between two media, $n(x < 0) = n_1 = 2$ and $n(x > 0) = n_2 = 1$.

a) Compute the angle of incidence ϕ_{Itot} above which total internal reflection occurs.

b) If the second medium (n_2) is not infinite, but forms a layer of thickness d $[n(0 < x < d) = n_2 = 1]$, and $n(x > d) = n_3 = 2$, do you still expect reflectivity $\rho = 1$ at the interface at $x = 0$? Compute the reflectivity for this layer (TE-polarization) to prove your answer.

Exercise 8

10+10 Points

Investigate the propagation of a 1-dimensional initial field distribution $u_0(x) = A\cos^2(x/\Lambda)$ in paraxial (Fresnel) approximation.

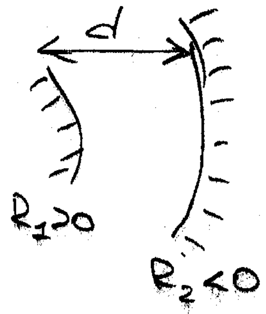
a) At which propagation distances z_T do we observe $|u(x, z_T)|^2 = |u_0(x)|^2$?

b) Now consider the above $u_0(x)$ with a finite aperture $a = N\pi\Lambda/2$, so that

$$\tilde{u}_0(x) = \begin{cases} u_0(x) & , |x| < a \\ 0 & , \text{elsewhere.} \end{cases}$$

Compute the far field intensity distribution of $\tilde{u}_0(x)$, and discuss the necessary propagation distances to apply Fraunhofer approximation with respect to N.

Resonator



$$0 < \left(1 + \frac{d}{R_1}\right)\left(1 + \frac{d}{R_2}\right) \le 1$$

$$0 < \left(1 + \frac{d}{|R_1|}\right)\left(1 - \frac{d}{|R_2|}\right) \le 1$$

$$0 < (|R_1| + d)(|R_2| - d) \le |R_2| \cdot |R_1|$$

1) $0 < (|R_1| + d)(|R_2| - d)$

$|R_1| + d > 0$ always

$|R_2| - d > 0 \Rightarrow \boxed{|R_2| > d} \Rightarrow \underline{|R_2| = d + \varepsilon}, \ \varepsilon > 0$

2) $(|R_1| + d)(|R_2| - d) \le |R_2||R_1|$

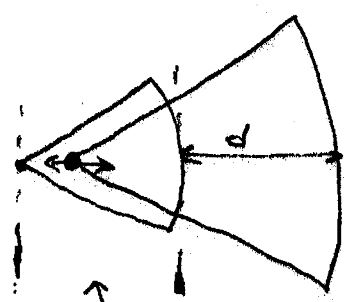
$$\cancel{|R_1|\cdot|R_2|} - d^2 - |R_1|\cancel{d} + |R_2|\cdot\cancel{d} \le \cancel{|R_2|\cdot|R_1|}$$

$$\boxed{|R_2| - |R_1| \le d}$$

$$d + \varepsilon - |R_1| \le \cancel{d}$$

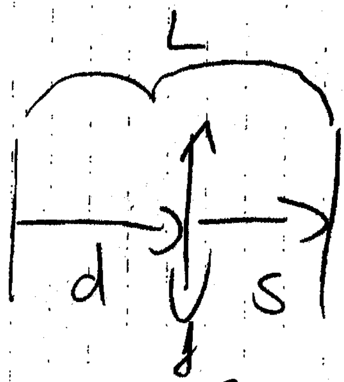
$$\underline{|R_1| \ge \varepsilon}$$

$\Rightarrow$



center of the convex sphere should be between these dot lines

Gaussian beams:



$$q_2 = \frac{A q_1 + B}{C q_1 + D}$$

$$\frac{1}{q_1} = \underbrace{\frac{1}{R_1}}_{=0} + i \frac{\lambda}{\pi w^2} \Rightarrow q_1 = -i \frac{\pi w^2}{\lambda}$$

$$q_1 = q_2 \Rightarrow -i \frac{\pi w^2}{\lambda} = \frac{-i A \frac{\pi w^2}{\lambda} + B}{-i C \frac{\pi w^2}{\lambda} + D}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = M_s M_f M_d = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{s}{f} & d + s\left(1 - \frac{d}{f}\right) \\ -\frac{1}{f} & 1 - \frac{d}{f} \end{pmatrix}$$

$$\Rightarrow \frac{1}{f} \frac{\pi^2 w^4}{\lambda^2} - i \left(1 - \frac{d}{f}\right) \frac{\pi w^2}{\lambda} = -i \left(1 - \frac{s}{f}\right) \frac{\pi w^2}{\lambda} + d + s\left(1 - \frac{d}{f}\right)$$

$$=$$

$$\Rightarrow d = s = \frac{L}{2}$$

$$\Rightarrow \frac{\pi^2 w^4}{\lambda^2} = f \left(\frac{L}{2} + \frac{L}{2}\left(1 - \frac{L}{2f}\right)\right) = fL - \frac{L^2}{4}$$

$$\Rightarrow f = \frac{1}{L} \left(\frac{\pi^2 w^2}{\lambda^2} + \frac{L^2}{4} \right) \quad \square$$

Exercise 1 *** **Points**

Illumination of a cross grating produces a light distribution

$$u_0(x,y) = \frac{A}{4}\left(1+\cos\frac{2\pi}{a}x\right)\left(1+\cos\frac{2\pi}{a}y\right),$$

with period length $a = 1\,\mathrm{mm}$. This light field is now imaged by a $4f$-setup, where in the plane $z = 2f$ a slit with the pupil function

$$p(x,y) = \begin{cases} 1 & , |x| < D/2 \\ 0 & , \text{else} \end{cases}$$

is applied. The focal length is $f = 1\,\mathrm{m}$ and the wavelength used is $\lambda = 1\,\mu\mathrm{m}$.
Calculate or accurately sketch the field $u(x,y,4f)$ at the end of the $4f$-setup for a slit width $D = 1\,\mathrm{mm}$. What would be the maximal width of the slit in order to obtain the same image?

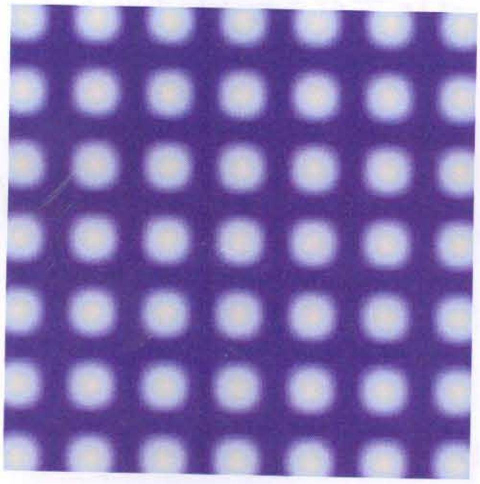


Figure 1: Light distribution $u_0(x,y)$.

Exercise 2 *** **Points**

a) Consider a dielectric medium with response function

$$R(t) = \frac{f}{\sqrt{a^2-b^2}}\, e^{-bt} \sin\left(\sqrt{a^2-b^2}\; t\right),$$

where $a > b > 0$. Calculate the electric susceptibility $\chi(\omega)$.

b) A company producing optical instruments is looking for a new isotropic material which should have the optical property

$$\chi(\omega) = Ae^{-\frac{(\omega-\omega_0)^2}{B^2}} + i\; C\delta(\omega-\omega_0),$$

where $A = 0.542$, $B = 1.02\cdot 10^{15}\,s^{-1}$, $C = 3.29$ and $\omega_0 = 4.71\cdot 10^{15}\,s^{-1}$. Do you think their search can be successful? Explain with the help of Kramers Kronig relations.

Exercise 1 - Solution

$$u(-x,-y,4f) = \int\int H_A(\alpha,\beta;4f)\, U_0(\alpha,\beta)\, e^{i(\alpha x+\beta y)}\, d\alpha d\beta$$
$$H_A(\alpha,\beta;4f) = \tilde{A}p\left(\frac{f}{k}\alpha, \frac{f}{k}\beta\right)$$

Need FT of $u_0(x,y)$:

$$U_0(\alpha,\beta) = \frac{1}{(2\pi)^2}\int\int u_0(x,y)\, e^{-i(\alpha x+\beta y)}\, dxdy$$

Therefore:

$$\begin{aligned}\frac{1}{2\pi}\int\left(1+\cos\frac{2\pi}{a}x\right)e^{-i\alpha x}\,dx &= \frac{1}{2\pi}\int\left(1+\frac{1}{2}e^{i\frac{2\pi}{a}x}+\frac{1}{2}e^{-i\frac{2\pi}{a}x}\right)e^{-i\alpha x}\,dx\\ &= \delta(\alpha)+\frac{1}{2}\delta\left(\alpha-\frac{2\pi}{a}\right)+\frac{1}{2}\delta\left(\alpha+\frac{2\pi}{a}\right)\end{aligned}$$

$\Rightarrow$

$$U_0(\alpha,\beta) = \frac{A}{4}\left(\delta(\alpha)+\frac{1}{2}\delta\left(\alpha-\frac{2\pi}{a}\right)+\frac{1}{2}\delta\left(\alpha+\frac{2\pi}{a}\right)\right)\left(\delta(\beta)+\frac{1}{2}\delta\left(\beta-\frac{2\pi}{a}\right)+\frac{1}{2}\delta\left(\beta+\frac{2\pi}{a}\right)\right)$$

The pupil with slit width $D = 1\,\mathrm{mm}$ is placed at $z = 2f$. The scaling of the spatial frequency there is given by

$$x = \frac{f}{k}\alpha.$$

We ask ourselves where the delta-peaks of the cross grating are in the $2f$ plane:

$$x_{\text{peak}} = \frac{f}{k}\frac{2\pi}{a} = \frac{\lambda f}{2\pi}\frac{2\pi}{a} = \frac{f\lambda}{a} = 1\,\mathrm{m}\cdot 10^{-6}\,\mathrm{m}\cdot 10^{3}\,\mathrm{m}^{-1} = 10^{-3}\,\mathrm{m}$$

The slit in fact shadows these peaks since $D/2 = 0.5\cdot 10^{-3}\,\mathrm{m}$. Hence, $u(-x,-y,4f)$ is given by

$$\begin{aligned}u(-x,-y,4f) &= \int\int \frac{A}{4}\delta(\alpha)\left(\delta(\beta)+\frac{1}{2}\delta\left(\beta-\frac{2\pi}{a}\right)+\frac{1}{2}\delta\left(\beta+\frac{2\pi}{a}\right)\right)e^{i(\alpha x+\beta y)}\,d\alpha d\beta\\ &= \frac{A}{4}\left(1+\cos\frac{2\pi}{a}y\right)\\ &= u(x,y,4f).\end{aligned}$$

This means the structural information of the grating in x-direction gets lost.

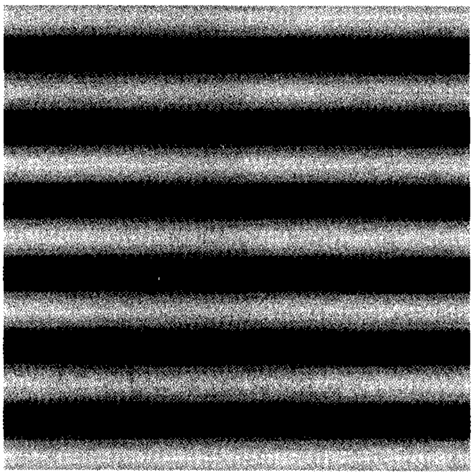


Figure 2: Light distribution $u(x, y, 4f)$

Exercise 2 - Solution

a)

Susceptibility is FT of Response function (note sign convention). Replace a and b with physical quantities ω_0 and γ and use abbreviation $\Omega = \sqrt{\omega_0^2 - \gamma^2}$.

$$\begin{aligned}
\chi(\omega) &= \int_{-\infty}^{\infty} R(t) e^{i\omega t} dt \\
&= \frac{f}{\Omega} \int_0^{\infty} e^{-\gamma t} \frac{1}{2i} \left(e^{i\Omega t} - e^{-i\Omega t}\right) e^{i\omega t} dt \qquad \text{because } R(t) = 0 \text{ for } t < 0 \\
&= \frac{f}{\Omega} \frac{1}{2i} \int_0^{\infty} \left(e^{t(-\gamma + i(\omega+\Omega))]} - e^{t[-\gamma + i(\omega - \Omega)]}\right) dt \\
&= \frac{f}{\Omega} \frac{1}{2i} \left(\frac{e^{t[-\gamma+i(\omega+\Omega)]}}{-\gamma + i(\omega+\Omega)} - \frac{e^{t[-\gamma+i(\omega-\Omega)]}}{-\gamma + i(\omega - \Omega)} \right) \Bigg|_0^{\infty} \\
&= \frac{f}{\Omega} \frac{1}{2i} \left(\frac{-1}{-\gamma + i(\omega+\Omega)} - \frac{-1}{-\gamma + i(\omega-\Omega)} \right) \\
&= \frac{f}{\Omega} \frac{1}{2i} \frac{(\gamma - i(\omega - \Omega)) - (\gamma - i(\omega + \Omega))}{(\gamma - i(\omega+\Omega))(\gamma - i(\omega - \Omega))} \\
&= \frac{f}{\Omega} \frac{1}{2i} \frac{2i\Omega}{\gamma^2 - i\gamma(\omega - \Omega) - i\gamma(\omega + \Omega) - (\omega - \Omega)(\omega + \Omega)} \\
&= \frac{f}{\gamma^2 + \Omega^2 - \omega^2 - i2\gamma\omega} \\
&= \frac{f}{\omega_0^2 - \omega^2 - i2\gamma\omega}
\end{aligned}$$

b)

Besides troubles with an experimental realization of the delta distribution the students should recognize that the Kramers Kronig relations are violated:

$$\mathrm{Re}(\varepsilon'(\omega - 1)) = \mathrm{Re}(\chi(\omega)) = \frac{2}{\pi} \int_0^{\infty} \frac{\bar{\omega}\chi''(\bar{\omega})}{\bar{\omega}^2 - \omega^2} d\bar{\omega}.$$

In our case the integration yields

$$\mathrm{Re}(\chi(\omega)) = \frac{2}{\pi} C \frac{\bar{\omega}_0}{\bar{\omega}_0^2 - \omega^2},$$

which is obviously not the Gaussian-like real part demanded.

Note: The constants $A = 0.542$, $B = 1.02 \cdot 10^{15}\, s^{-1}$, $C = 3.29$ and $\omega_0 = 4.71 \cdot 10^{15}\, s^{-1}$ in this task are chosen such that a physically realistic refractive index results, the medium is lossy (and not the opposite) and the resonance position is at $\lambda = 400\,\mathrm{nm}$.

ordinary : $\vec{e}_o e^{i \frac{\omega}{c} n_o z}$

extraordinary: $\vec{e}_e e^{i \frac{\omega}{c} n_e z} e^{i \frac{\pi}{2}}$

$$\frac{\omega}{c} n_o z = \frac{\omega}{c} n_e z + \frac{\pi}{2} + m\pi$$

$$\Rightarrow z = \frac{c\pi(m+\frac{1}{2})}{\omega(n_o - n_e)} \qquad m \in \mathbb{Z}$$

Two short pulses are propagating in a homogenous plasma. They have a carrier frequency much larger than the plasma frequency. The corresponding wavelengths are $\lambda_1$und λ_2.
The signals are recorded by a detector that is located at a distance L from the source. Use the dielectric function of the plasma

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

to compute the time delay between the two pulses.

Lösung zu 2.)

1)

$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$

$\frac{dk}{d\omega} = \frac{d}{d\omega}\left(\frac{\omega}{c}\sqrt{\varepsilon(\omega)}\right) = \frac{1}{c}\left(\sqrt{\varepsilon(\omega)} + \frac{1}{2}\frac{\omega\left(\frac{2\omega_p^2\omega}{\omega^4}\right)}{\sqrt{\varepsilon(\omega)}}\right) =$

$= \frac{1}{c}\left(\sqrt{\varepsilon(\omega)} + \frac{\omega_p^2}{\omega^2\sqrt{\varepsilon(\omega)}}\right) = \frac{1}{c}\left(\frac{\varepsilon(\omega)\omega^2 + \omega_p^2}{\omega^2\sqrt{\varepsilon(\omega)}}\right) =$

$= \frac{1}{c}\left(\frac{1}{\sqrt{\varepsilon(\omega)}}\right) = \frac{1}{c\sqrt{\varepsilon(\omega)}} = \frac{1}{v_{gr}}$

2)$t = \frac{L}{v_{gr}1} - \frac{L}{v_{gr}2} = \frac{L}{c}\left(\frac{1}{\sqrt{\varepsilon(\omega_1)}} - \frac{1}{\sqrt{\varepsilon(\omega_2)}}\right)$

a) $\sin \varphi_{I\,tot} = \frac{n_2}{n_1} = \frac{1}{2} \Rightarrow \varphi_{I\,tot} = 30° \triangleq \frac{\pi}{6}$

b) $$\hat{M} = \begin{pmatrix} \cos k_{dx} d & \frac{1}{k_{dx}} \sin k_{dx} d \\ -k_{dx} \sin k_{dx} d & \cos k_{dx} d \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$R_{TE} = \frac{k_{sx} M_{22} - k_{cx} M_{11} - i(M_{21} + k_{sx} k_{cx} M_{12})}{k_{sx} M_{22} + k_{cx} M_{11} + i(M_{21} - k_{sx} k_{cx} M_{12})}$$

$$k_{sx} = k_{cx} = 2\frac{\omega}{c}\sqrt{1 - \sin^2 \varphi_I}$$

$$k_{dx} = \frac{\omega}{c}\sqrt{1 - 4\sin^2 \varphi_I}$$

$$\Rightarrow R_{TE} = \frac{-i\left(-k_{dx} \sin k_{dx} d + \frac{k_{sx}^2}{k_{dx}} \sin k_{dx} d\right)}{2k_{sx} \cos k_{dx} d + i\left(-k_{dx} \sin k_{dx} d - \frac{k_{sx}^2}{k_{dx}} \sin k_{dx} d\right)}$$

$$= \frac{i(k_{dx}^2 - k_{sx}^2)}{2k_{dx} k_{sx} \cot k_{dx} d - i(k_{dx}^2 + k_{sx}^2)}$$

for $\varphi_I > \varphi_{I\,tot}$: $k_{dx} = i\kappa_d$

$\cos ix = \cosh x$
$\sin ix = i \sinh x$
$\cot ix = -i \coth x$

$$= \frac{\kappa_d^2 + k_{sx}^2}{k_{sx}^2 - \kappa_d^2 + i\,2\kappa_d k_{sx} \coth \kappa_d d}$$

$$S_{TE} = |R_{TE}|^2 = \frac{(\kappa_d^2 + k_{sx}^2)^2}{(k_{sx}^2 - \kappa_d^2) + 4\kappa_d^2 k_{sx}^2 \coth^2 \kappa_d d} < 1$$

for $\kappa_d d <$

[redacted]

$$\frac{1}{2\pi}\int A\left(\frac{e^{i\frac{x}{\Lambda}}+e^{-i\frac{x}{\Lambda}}}{2}\right)^2 e^{-i\alpha x}dx = \frac{A}{2\pi}\int\left(\frac{1}{2}+\frac{e^{i\frac{2x}{\Lambda}}+e^{-i\frac{2x}{\Lambda}}}{4}\right)e^{-i\alpha x}dx$$ 8

$$= A\left\{\frac{\delta(\alpha)}{2}+\frac{1}{4}\delta\left(\alpha-\frac{2}{\Lambda}\right)+\frac{1}{4}\delta\left(\alpha+\frac{2}{\Lambda}\right)\right\} = U_0(\alpha)$$

$$U(\alpha,z) = U_0(\alpha)e^{ik_0 z}e^{-i\frac{\alpha^2}{2k_0}z}$$

$$u(x,z) = e^{ik_0 z}\int U_0(\alpha)e^{-i\frac{\alpha^2 z}{2k_0}}e^{i\alpha x}d\alpha = e^{ik_0 z}A\left\{\frac{1}{2}+\frac{1}{4}e^{-i\frac{2z}{\Lambda^2 k_0}}e^{i\frac{2x}{\Lambda}} + \frac{1}{4}e^{-i\frac{2z}{\Lambda^2 k_0}}e^{-i\frac{2x}{\Lambda}}\right.$$

$$|u(x,z_T)|^2 = |u_0(x)|^2 \Rightarrow e^{-i\frac{2z_T}{\Lambda^2 k_0}} = 1 \Rightarrow \frac{2z_T}{\Lambda^2 k_0} = m2\pi$$

$$\Rightarrow z_T = m\pi k_0\Lambda^2 = m\pi^2\frac{2\Lambda^2}{\lambda}$$

$$U_0(\alpha) = \frac{1}{2\pi}\int_{-\frac{N\pi\Lambda}{2}}^{\frac{N\pi\Lambda}{2}} A\left(\frac{1}{2}+\frac{e^{i\frac{2x}{\Lambda}}+e^{-i\frac{2x}{\Lambda}}}{4}\right)e^{-i\alpha x}dx$$

$$= \frac{A}{2\pi}\left[\frac{-e^{-i\alpha x}}{2i\alpha}+\frac{-e^{-i(\alpha-\frac{2}{\Lambda})x}}{4i(\alpha-\frac{2}{\Lambda})}+\frac{-e^{-i(\alpha+\frac{2}{\Lambda})x}}{4i(\alpha+\frac{2}{\Lambda})}\right]_{-\frac{N\pi\Lambda}{2}}^{\frac{N\pi\Lambda}{2}}$$

$$= -\frac{iA}{2\pi}\left[\frac{\sin\frac{N\pi\Lambda}{2}\alpha}{\alpha}+\frac{\sin\frac{N\pi\Lambda}{2}(\alpha-\frac{2}{\Lambda})}{2(\alpha-\frac{2}{\Lambda})}+\frac{\sin\frac{N\pi\Lambda}{2}(\alpha+\frac{2}{\Lambda})}{2(\alpha+\frac{2}{\Lambda})}\right]$$

Fraunhofer: $|u(x,z)|^2 \sim |U_0(k\frac{x}{z})|^2$

$$N_F = \frac{a^2}{\lambda z}, \quad a = \frac{N\pi\Lambda}{2} \Rightarrow N_F \sim \frac{N^2}{z}$$

$$\Rightarrow z \sim N^2$$