Institute of Applied Physics
Friedrich-Schiller-Universität Jena

Laser Physics problem sheet 8

Summer semester 2023

Problem 1 (5 points)

Problem 1

We consider an active medium with gain $G = \exp(\sigma n d)$, which is placed inside a linear cavity with roundtrip loss L and outcoupling mirror transmission T. The steady state condition is $(1-T)(1-L)G^2 = 1$. Additionaly, we assume a spatially constant inversion distribution n, so the 4-level system rate equation is (indexes: p...pump, s...signal):

$$\frac{\partial n}{\partial t} = -\frac{n}{\tau_2} + W_p(n_{tot} - n) - \frac{I_s \sigma}{h\nu_s} n$$

τ_2 is the upper state lifetime, n_{tot} the total inversion density, and σ the emission cross section. The pump rate W_p is:

$$W_p = \frac{\eta P_p}{h\nu_p A d(n_{tot} - n)}$$

which depends on the thickness d of the medium, the pump power P_p, the quantum efficiency η, and the pumped area A. Use the equations above and

a) Knowing that the saturation intensity is $I_{sat} = \frac{h\nu_s}{\sigma\tau_2}$, derive the equation for the linear cavity cw-laser output power $P_l = \sigma_s(P_p - P_{th})$. Assume that the pump- and laser-fields cover the same area A. Demonstrate that the slope efficiency $\sigma_s = \frac{\eta \cdot \eta_{Stokes} T}{-Ln[(1-L)(1-T)]}$ and that the threshold pump power $P_{th} = \frac{-Ln[(1-L)(1-T)] \cdot A \cdot I_{sat}}{2 \cdot \eta \cdot \eta_{Stokes}}$, where $\eta_{Stokes} = \nu_s/\nu_p$. (3 points)

$$(1-L)(1-T)G^2 = 1$$

$$G = [(1-L)(1-T)]^{-\frac{1}{2}} \mid \ln(\dots)$$

$$with\ G = e^{\sigma n d}$$

$$n = \frac{-1}{2\sigma d} \ln[(1-L)(1-T)] \quad (*)$$

$$\eta' = \eta \eta_{stokes} \quad (**)$$

$$\frac{\partial n}{\partial t} = -\frac{n}{\tau_2} + \underbrace{W_P(n_{tot} - n)} - \frac{I_s\sigma}{h\nu_s} n \overset{!}{=} 0$$

$\uparrow$ steady state

$$0 = -\frac{n}{\tau_2} + \overbrace{\frac{\eta P_p}{h\nu_p A d}} - \frac{I_s \sigma}{h\nu_s} n$$

$$I_s = \frac{\frac{\eta P_p}{h\nu_p A d} - \frac{n}{\tau_2}}{\frac{n\sigma}{h\nu_s}} = \frac{\eta P_p \cancel{h}\nu_s}{\cancel{h}\nu_p A d n \sigma} - \frac{\cancel{n} h\nu_s}{\tau_2 \cancel{n} \sigma} = \frac{P_p \eta \eta_{stokes}}{A d n \sigma} - \frac{h\nu_s}{\tau_2 \sigma} \overset{(*)(**)}{=} \frac{-2\cancel{\sigma}\cancel{d}\eta' P_p}{A \cancel{d} n \cancel{\sigma} \ln[(1-L)(1-T)]} - \overbrace{\frac{h\nu_s}{\tau_2\sigma}}^{I_{sat}} = \frac{-2\eta' I_p}{\ln[(1-L)(1-T)]} - I_{sat}$$

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$$(1-L)(1-T)G^2 = 1$$

$$G = [(1-L)(1-T)]^{-\frac{1}{2}} \mid \ln(\ldots)$$

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$$n = \frac{-1}{2\sigma d} \ln[(1-L)(1-T)] \quad (*)$$

$$\eta' = \eta \eta_{stokes} \quad (**)$$

$$I_s = \frac{-2\eta' I_p}{\ln[(1-L)(1-T)]} - I_{sat}$$ … equals intensity inside the resonator

$$I_{out} = \frac{1}{2} T I_s = \frac{-\eta' T}{\ln[(1-L)(1-T)]} [I_p - \underbrace{\frac{-\ln[(1-L)(1-T)]}{2\eta'} I_{sat}}]$$

$$I_{out} = \frac{-\eta' T}{\ln[(1-L)(1-T)]} [I_p - \overbrace{I_{th}}] \mid * A$$

$$P_l = \frac{-\eta' T}{\ln[(1-L)(1-T)]} [P_p - P_{th}] = \underline{\underline{\sigma_s [P_p - P_{th}]}}$$

a) ✓

b)

$$P_{th} = \frac{-A \ln[(1-L)(1-T)]}{2\eta'} I_{sat} \ \ldots$$

$$= -\frac{A I_{sat}}{2\eta'} \ln[(1-L)] - \frac{A I_{sat}}{2\eta'} \ln[(1-T)]$$

$$= -C \underbrace{\ln[(1-L)]}_{\text{y-axis}} - C \underbrace{\ln[(1-T)]}_{\text{x-axis}} \text{ with } C = \frac{A I_{sat}}{2\eta'}$$

Problem 1

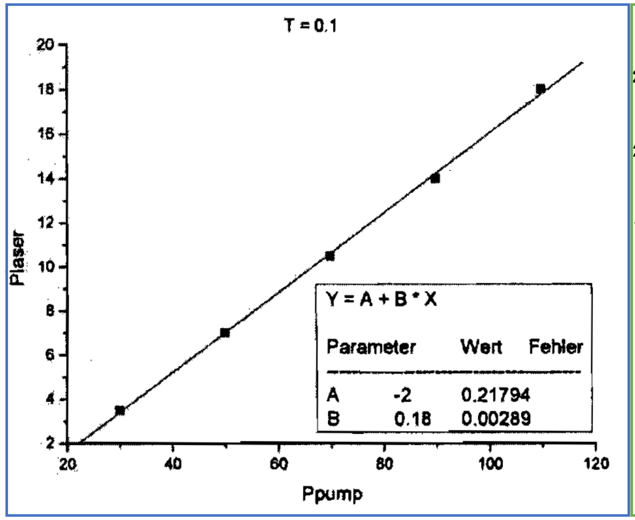
b) We want to experimentally determine the cavity losses L by using the Findlay Clay method. For this laser slopes for different output transmissions were measured:

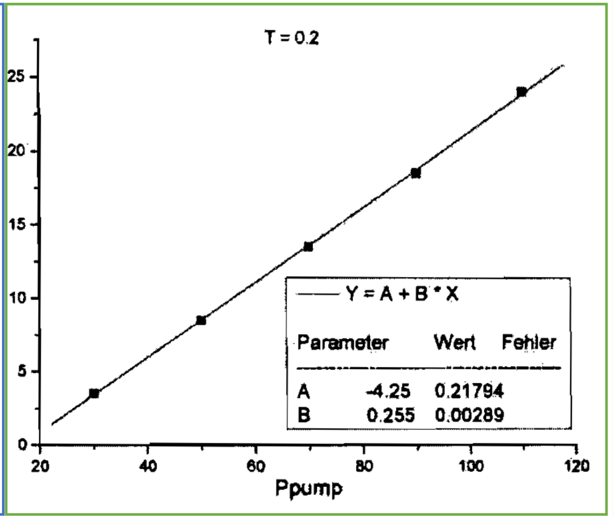
P_p / W	P_l / W $T = 0.1$	P_l / W $T = 0.2$	P_l / W $T = 0.4$
30	3.5	3.5	1.0
50	7.0	8.5	6.5
70	10.5	13.5	12.5
90	14.0	18.5	18.5
110	18.0	24.0	24.0

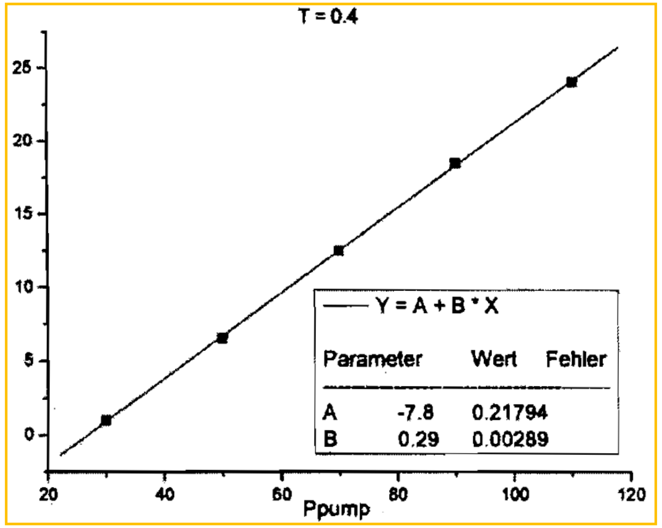
Calculate P_{th} by linear regression and use your results from a) to plot P_{th} in a way that L can also be derived by linear regression (Hint: in this plot the x-axis should be $-Ln(1-T)$). (2 points)

$$P_{th} = -C\ln[(1-L)] - C\ln[(1-T)]$$

$$with\ C = \frac{AI_{sat}}{2\eta'}$$







$P_{th1} = 11{,}4W$ $\quad$ $P_{th2} = 16{,}7W$ $\quad$ $P_{th3} = 26{,}9W$

Problem 1

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$$P_{th} = -C\ln[(1-L)] - C\ln[(1-T)]$$

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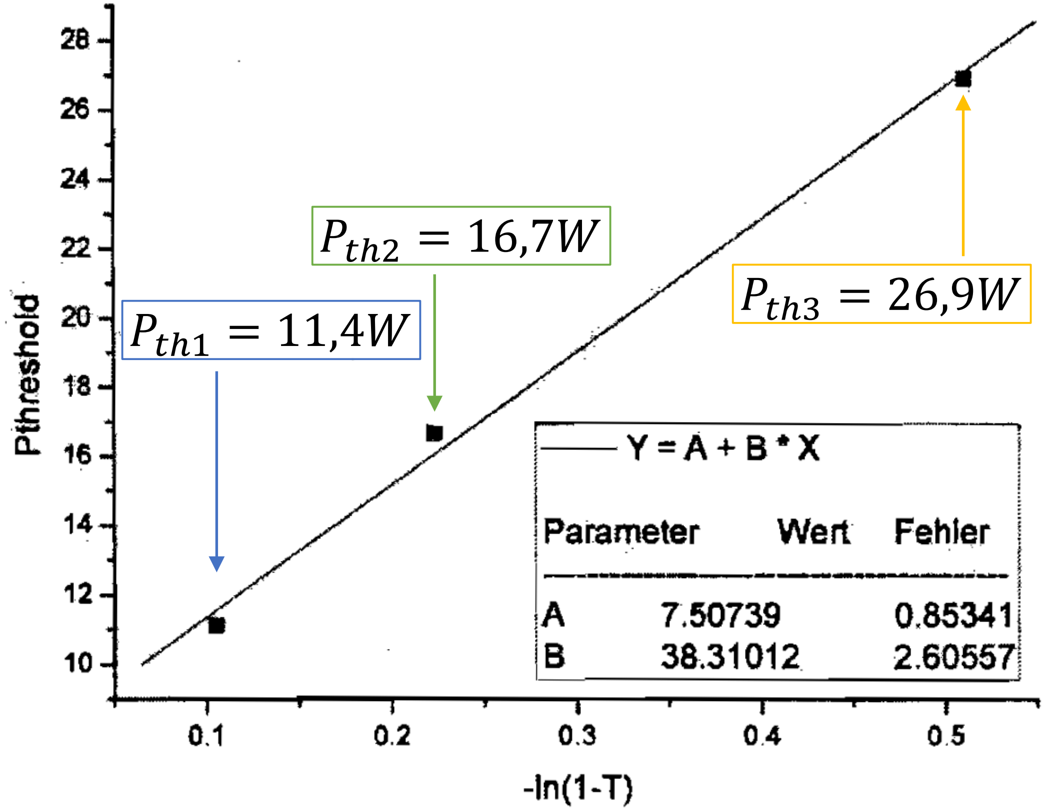

$$\underline{\underline{L = 17{,}8\%}}$$

b) ✓

Problem 2 (4 points)

Problem 2

The fundamental limit to the monochromaticity of a continuous-wave single-mode laser is given by the Schawlow-Townes formula. This limit originates from the quantum nature of the electromagnetic field. It is written as

$$\Delta\nu_{21} = \frac{2\pi h\nu_{21}\left(\delta\nu_R\right)^2}{P}$$

with $\Delta\nu_{21}$ being the minimum laser linewidth, $\delta\nu_R$ being the half width of a cavity resonance in frequency domain, ν_{21} the laser emission frequency and P the output power, respectively.

a) Obtain the half width $\delta\nu_R$ of a Lorentzian shaped cavity resonance with respect to the photon lifetime τ_{ph}. (1 point)

$$F_l(\nu) = \frac{C_1}{(\nu-\nu_{21})^2 + \boxed{\left(\frac{\Delta\nu}{2}\right)^2}}\text{; lorentzian funcion with } \Delta\nu \text{ as the lorentzians FWHM}$$

$$E = E_0 e^{-\frac{t}{2\tau_{ph}}}\text{; decaying electric field}$$

$$P(\nu) = E(\nu)E^*(\nu) \propto \frac{1}{4}\frac{1}{[2\pi(\nu-\nu_{21})]^2 + \left(\frac{1}{2\tau_{ph}}\right)^2} = \frac{1}{4(2\pi)^2}\frac{1}{[(\nu-\nu_{21})]^2 + \boxed{\left(\frac{1}{4\pi\tau_{ph}}\right)^2}}\text{; spectral power density}$$

→ compare $F_l(\nu)$ to $P(\nu)$: $\tau_{ph} = \frac{1}{2\pi\Delta\nu}$

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$$\boxed{\text{a) Obtain the half width } \delta\nu_R \text{ of a Lorentzian shaped cavity resonance with respect to the photon lifetime } \tau_{ph}\text{. (1 point)}}$$

→ compare $F_l(\nu)$ to $P(\nu)$: $\tau_{ph} = \frac{1}{2\pi\Delta\nu}$

→ from lecture: $\tau_{ph} = \frac{1}{2\pi\delta\nu_R}$ (+)

$$\Delta\nu_{21_{min}} = \frac{BV_aN_2}{2\pi\Phi} = \frac{1}{2\pi\underbrace{\Phi}_{\Phi = \frac{\tau_{ph}P}{h\nu_{21}}}\tau_{ph}} = \frac{h\nu_{21}}{2\pi P\tau_{ph}^2} \overset{(+)}{=} \frac{2\pi h\nu_{21}(\delta\nu_R)^2}{P}$$

$$BV_a = \frac{1}{N_2\tau_{ph}}$$

a) ✓

$$\boxed{\text{b) Why is the Schawlow-Townes formula inversely proportional to the laser power? (1 point)}}$$

Because it takes gain narrowing into account!

Problem 2 (4 points)

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The fundamental limit to the monochromaticity of a continuous-wave single-mode laser is given by the Schawlow-Townes formula. This limit originates from the quantum nature of the electromagnetic field. It is written as

$$\Delta\nu_{21} = \frac{2\pi h\nu_{21}\left(\delta\nu_R\right)^2}{P}$$

with $\Delta\nu_{21}$ being the minimum laser linewidth, $\delta\nu_R$ being the half width of a cavity resonance in frequency domain, ν_{21} the laser emission frequency and P the output power, respectively.

c) Compare the Schawlow-Townes linewidth of a He-Ne laser emitting at 633 nm and of a GaAs semiconductor laser emitting at 850 nm. Both lasers have an ouput power of 1mW. In the case of the He-Ne laser the cavity is 20 cm long, has an index of refraction of $n = 1$ and resonator mirrors with $R_1 = 100\%$ and $R_2 = 99\%$. For the GaAs laser the cavity length is 300 μm and the resonator mirrors are formed by the interface between semiconductor material with $n_{GaAs} = 3.5$ and air. (2 points)

→ from task:

$$R_1 = 1; R_2 = 0.99$$
$$\lambda_{HeNe} = 633nm \;\rightarrow \nu_{21} = 4.7393 * 10^{14}H$$
$$\lambda_{GaAs} = 850nm \;\rightarrow \nu_{21} = 3.5294 * 10^{14}Hz$$
$$L_{HeNe} = 300\mu m; L_{GaAs} = 300\mu m$$
$$n = 1; n_{GaAs} = 3.5$$

He-Ne:

$$\tau_{ph} = \frac{-\tau_R}{\ln(R_1R_2)} = \frac{2nL}{c\ln(R_1R_2)} \approx 132{,}7ns$$

$$\Delta\nu_{21_{min}} = \frac{h\nu_{21}}{2\pi P\tau_{ph}^2} \approx 2.8mHz$$

Not achievable in a real system.

GaAs:

$$R_1 = R_2 = \left(\frac{n_{GaAs}-1}{n_{GaAs}+1}\right)^2 = 0.3$$

$$\tau_{ph} \approx 2.74ps$$

$$\Delta\nu_{21_{min}} \approx 4.95MHz$$

Achievable in a real system.

Problem 3

For achieving high power emission in the $2\mu m$ wavelength range Thulium (Tm)-doped fibers are a often used platform. An energy level diagram reduced to the most relevant energy levels and the corresponding cross-sections are shown below. In this simplified case, the Tm-system can be seen as a three level laser system.

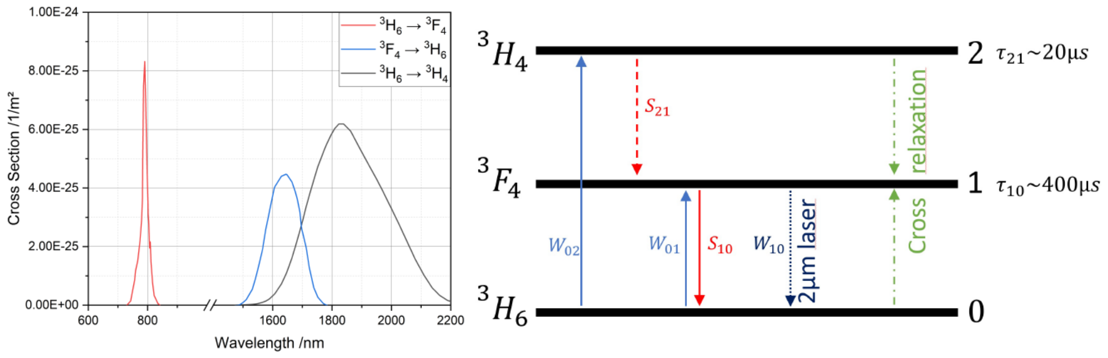


Fig. 1: Relevant cross-sections, transitions and energy levels for a simplified 3 level Tm-laser.

a) Write down the rate equation for the inversion in the system given above. Disregard the transitions due to cross-relaxation for this task. (2 points)

*Also it is again possible to introduce the already known quantities: $\Gamma = S_{10}$, quantum efficiency η, and the pump rate W_p

** $N_0 = \frac{n_{tot}-n}{2}$ and $N_1 = \frac{n_{tot}+n}{2}$

Goal: $$\frac{dn}{dt} = \frac{dN_1}{dt} - \frac{dN_0}{dt}$$

All transitions taken into account:
$$\frac{dN_0}{dt} = -W_{02}N_0 - W_{01}N_0 + S_{10}N_1 + W_{10}N_1$$
$$\frac{dN_1}{dt} = S_{21}N_2 - S_{10}N_1 + W_{01}N_0 - W_{10}N_1$$

We still assume that $n_{tot} \approx N_0 + N_1$ because $\tau_{21} < \tau_{10}$ and all ions pumped into level 2 end up in 1 anyway!

$$\xrightarrow{*} \quad \frac{dn}{dt} = 2W_P N_0 - 2\Gamma N_1 - 2\sigma c p n \qquad \xrightarrow{**} \quad \frac{dn}{dt} = W_P(n_{tot}-n) - \Gamma(n_{tot}+n) - 2\sigma c p n$$

a) ✓

Problem 3

For achieving high power emission in the $2\mu m$ wavelength range Thulium (Tm)-doped fibers are a often used platform. An energy level diagram reduced to the most relevant energy levels and the corresponding cross-sections are shown below. In this simplified case, the Tm-system can be seen as a three level laser system.

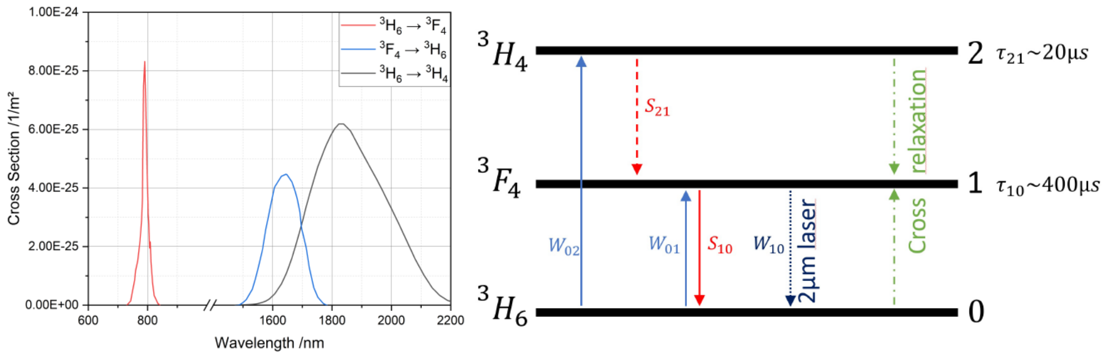


Fig. 1: Relevant cross-sections, transitions and energy levels for a simplified 3 level Tm-laser.

b) Calculate the quantum defect for the cases of a system emitting at 2000nm that is pumped at 790nm or at 1600nm. Also calculate the slope efficiency under the assumption that there only losses in the system are causes by the out-coupling mirros, which has 50% reflectivity. What would the output power be in these two cases, if you were using a 1W pump power at each of the previously mentioned pump wavelengths? (2 points)

$$QD = 1 - \frac{\nu_{signal}}{\nu_{pump}} = 1 - \frac{\lambda_{pump}}{\lambda_{signal}}$$

$$QD_{1600\,\mathrm{nm}} = 1 - \frac{1600\,\mathrm{nm}}{2000\,\mathrm{nm}} = 0.2 = 20\,\%$$

$$QD_{790\,\mathrm{nm}} = 0.605 = 60.5\,\%$$

$$\sigma = \eta \frac{\nu_{signal}}{\nu_{pump}} \frac{T}{T+L}$$

$T = 0.5$ and $L = 0$

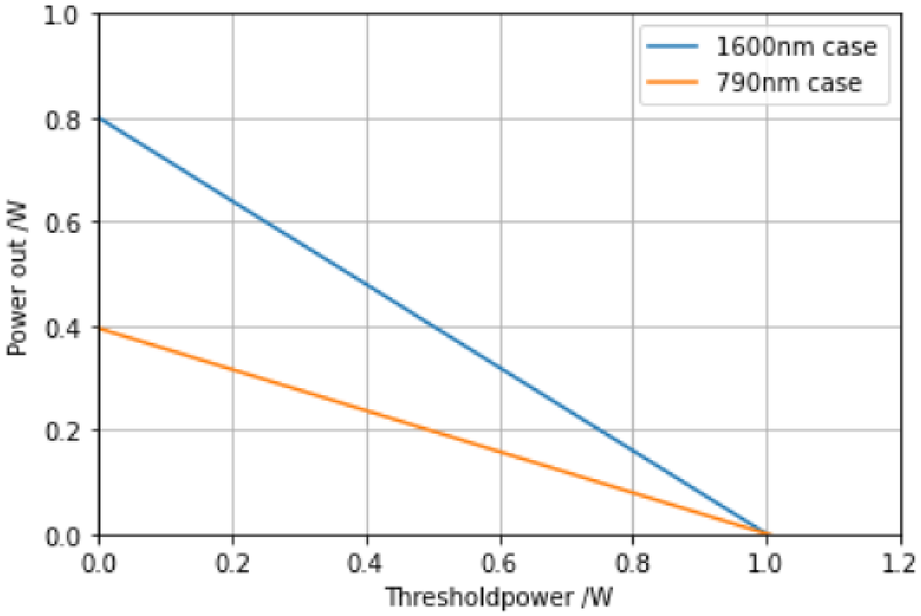
$$\sigma = \eta \frac{\nu_{signal}}{\nu_{pump}} = \eta \frac{\lambda_{pump}}{\lambda_{signal}}$$

$$\sigma_{1600\,\mathrm{nm}} = \frac{1600\,\mathrm{nm}}{2000\,\mathrm{nm}} = 0.8 = 80\,\%$$

$$\sigma_{790\,\mathrm{nm}} = 0.395 = 39.5\,\%$$

$$P_{out;790\,\mathrm{nm}} = 0.395 * (1\,\mathrm{W} - P_{th})$$

$$P_{out;1600\,\mathrm{nm}} = 0.8 * (1\,\mathrm{W} - P_{th})$$



Note: The threshold powers for the two pump cases is not equal!

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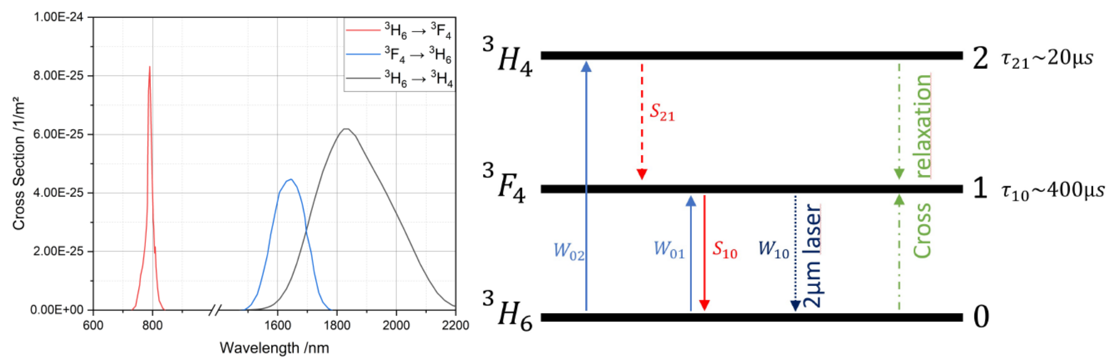

Fig. 1: Relevant cross-sections, transitions and energy levels for a simplified 3 level Tm-laser.

c) Now take the effect of the cross-relaxations into account. This effect describes the possibility for a Tm-ion, which is excited to the pump level (|2>), to excite a neighboring Tm-ion in the ground state (|0>), because the energy level difference between levels |0> and |1> is similar to the difference between levels |1> and |2>. This effect can, therefore, only happen when using pump lasers in the $790nm$ absorption band. The practical consequence of this effect is that it is possible toget two Tm-ions excited to the upper laser level with one absorbed $790nm$ photon, effectively increasing the maximum quantum efficiency to 200%. Assume two cases for the quantum efficiency under the presence of cross relaxations ($\eta_{CR} = C_{CR}\eta$), namely $C_{CR} = 1.5$ and $C_{CR} = 2$ (perfect case). Under these circumstances calculate the quantum defect and slope efficiencies again of the system described in b) again. Compare these new results to the 1600nm pumped case from b). (2 points)

$$\sigma_{790\,\mathrm{nm}}^{CR} = \eta_{CR}\frac{\lambda_{pump}}{\lambda_{signal}}$$

$$\sigma_{790\,\mathrm{nm}}^{CR1.5} = 1.5\frac{790\,\mathrm{nm}}{2000\,\mathrm{nm}} = 0.5925 = 59.25\,\%$$

$$\sigma_{790\,\mathrm{nm}}^{CR2} = 0.79 = 79\,\%$$

$$QD_{790\,\mathrm{nm}}^{CR1.5} = QD_{790\,\mathrm{nm}}^{CR2} = QD_{790\,\mathrm{nm}} = 0.605$$

- Quantum defect only wavelenght dependent, therefore unchanged!
- Slope efficiencies closer to the 1600nm pumped case possible if CR works!

d) What can prevent the cross relaxation effect from working perfectly in the real world? (1 point)

- Doping concentration: Tm ions are not close enough to one another for efficent interaction
- Relaxation due to not infinitely long life time from level 2 to 1
- …