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Series 2
Fundamentals of Modern Optics

to be returned on **03.11.2022**, at the beginning of the lecture

Task 1: Vector analysis (a=1, b=1, c=1, d=1, + 1* pts.)

Prove the following vector identities where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vector-valued functions of the coordinate vector $\mathbf{r}$ and α is a scalar function (In general, quantities set in bold font represent vectors, while non-bold quantities refer to scalars.)

a) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$
b) $\nabla \times (\alpha \mathbf{a}) = \alpha\, \nabla \times \mathbf{a} - \mathbf{a} \times \nabla \alpha$
c) $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$
d) $\nabla \cdot (\nabla \times \mathbf{a}) = 0$

Hint: You can prove each relation by brute-force, simply writing each side using all the Cartesian vector components and crossing out similar terms. While that is perfectly acceptable here, we encourage you to use the so called *Einstein summation notation*, that makes use of the *Levi-Civita symbol*, ε_{ijk}. Look it up in any reference on mathematical physics and try to make use of the identities of the Levi-Civita symbol (which you do not need to prove). The bonus point goes to anyone who proves at least one of the relations in (a-c) in this way.

Task 2: Stokes' theorem (4 pts.)

Inside a cylindrically symmetric conducting wire of radius R the current density $\mathbf{j}(\mathbf{r},\omega)$ depends just on the radial coordinate r and may be approximated as

$$\mathbf{j}(\mathbf{r},\omega) = j(r,\omega)\mathbf{e}_z = j_0 \cosh\left(\frac{r}{\delta}\right)\mathbf{e}_z,$$

with a frequency dependent parameter $\delta = \delta(\omega)$, called the *skin depth*. Assume Ohm's law $\mathbf{j}(\mathbf{r},\omega) = \sigma(\omega)\mathbf{E}(\mathbf{r},\omega)$ is valid inside the conductor and use Stokes' theorem on Maxwell's equation

$$\nabla \times \mathbf{H}(\mathbf{r},\omega) = \mathbf{j}(\mathbf{r},\omega) - i\omega\varepsilon_0\varepsilon(\omega)\mathbf{E}(\mathbf{r},\omega)$$

to find the magnetic field $\mathbf{H}(\mathbf{r},\omega)$ *inside* ($r < R$) the wire.
Something to think about: Try to convince yourself why this method cannot be used to find the magnetic field outside the wire. Would this change for a static field?

Task 3: The Continuity Equation (4 pts.)

Using Maxwell's equations, find an expression for the time derivative of the charge density and how it is related to the current density. Find this expression in both differential, as well as in integral notation. Try to explain the results in your own words.

Task 4: Maxwell's equations (a=2, b=2, c=2, d=1 pts.)

Consider Maxwell's equations for the electric field $\mathbf{E}(\mathbf{r},t)$ and the magnetic field $\mathbf{H}(\mathbf{r},t)$ in empty space with sources $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t) = \sigma\mathbf{E}(\mathbf{r},t)$.

a) Derive the wave equation for the magnetic field $\mathbf{H}(\mathbf{r},t)$ for that case.
b) Transform this equation in the frequency domain to obtain the wave equation for the spectrum $\overline{\mathbf{H}}(\mathbf{r},\omega)$.
c) In lecture script section 2.1.5, it is demonstrated that the vectorial wave equation for $\overline{\mathbf{E}}(\mathbf{r},\omega)$ can be separated into two wave equations for $\overline{\mathbf{E}}_\perp$ and $\overline{\mathbf{E}}_\parallel$, assuming translational invariance along one direction (e.g y-direction). Starting from the wave equation obtained in (b), derive the separated equations for $\overline{\mathbf{H}}_\perp$ and $\overline{\mathbf{H}}_\parallel$, respectively.
d) The derivation of the decoupled equations for the magnetic field is simpler than that for the electric field. Observe the difference in the derivation and explain it briefly.

Task 5: Heaviside Step Function (4* pts.)

The Heaviside step function,

$$\Theta(t) = \begin{cases} 0 & ,t < 0 \\ 1 & ,t > 0 \end{cases}$$

has many applications in physics and engineering. Similarly, knowing its Fourier transform can be useful, e.g., when you try to prove the properties of the Kramers-Kronig relation. However, the standard (Riemann) integral to obtain its Fourier transform

$$\overline{\Theta}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Theta(t)e^{i\omega t}dt,$$

does not converge as Θ does not vanish at $+\infty$. Still, in the context of distributions, i.e. generalized functions, the Fourier transform exists and is given by

$$\overline{\Theta}(\omega) = \frac{1}{2\pi}\left(\text{p.v.}\frac{i}{\omega} + \pi\delta(\omega)\right).$$

The above expression like the δ-distribution itself has to be interpreted inside an integral. p.v. stands for Cauchy's principal value and allows to assign values to integrals that otherwise would be undefined. If you have a function $f(x)$ that is unbounded at some point c, the integral over f is similarly unbounded. However, the integral over p.v.$f(x)$ that is defined by

$$\text{p.v.}\int_a^b f(x)dx \equiv \lim_{\varepsilon\to 0}\left(\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x)dx\right).$$

may be bounded. Use this definition to show that the inverse Fourier transform of $\overline{\Theta}(\omega)$ as given above indeed leads to the Heaviside function. Make use of the relation $\int_0^\infty \frac{\sin(ax)}{x}dx = \frac{\pi}{2}$ for $a > 0$.

Recommendation: Use this chance to take a look at distribution theory in physics. You may look at how to prove $\int_0^\infty \frac{\sin(ax)}{x}dx = \frac{\pi}{2}$ for $a > 0$ using Cauchy's integral theorem and Jordan's lemma. If you also want to study in more details, please go to mathematical physics literatures.