Calculate the diffraction pattern in Fraunhofer approximation for:

a) A pinhole with radius a.
(*Hint:* Use polar coordinates for **k** and **r** to solve the Fourier transform. Useful formulas are:

$$\frac{i^{-n}}{2\pi}\int_0^{2\pi} e^{ix\cos\alpha}e^{in\alpha}\,d\alpha = J_n(x)$$
$$\frac{d}{dx}\left[x^{n+1}J_{n+1}(x)\right] = x^{n+1}J_n(x)$$

where J_i are the Bessel functions of first kind).

b) A ring-shaped aperture which is bounded by two circles of radius a_1 and a_2 with $a_2 > a_1$.

c) A sequence of N pinholes with radius a placed along the x-axis with distances of $b > 2a$.

d) A set of multiple non-overlapping pinholes (number of pinholes $N \gg 1$) with radius a that are randomly placed in the screen with no particular order.

e) A set of multiple $N \gg 1$ non-overlapping pinholes of different radiuses that are randomly placed in the screen with no particular order and have the radii between a_1 and a_2 with distribution function $w(a)$ (so that $\int_{a_1}^{a_2} w(a)da = 1$).

$$F(\alpha,\beta,\omega) = \left(\frac{1}{2\pi}\right)^3 \iiint_{-\infty}^{\infty} f(x,y,t)\exp[-i(\alpha x+\beta y-\omega t)]\,dx\,dy\,dt$$

(a) $$I_{FR} = \frac{1}{(\lambda z_B)^2}\left|U_t\left(\frac{kx}{z_B}, \frac{ky}{z_B}\right)\right|^2$$

$$U_t\left(\frac{kx}{z_B}, \frac{ky}{z_B}\right) = \left(\frac{1}{2\pi}\right)^2\iint_{-\infty}^{\infty} t(x,y)\exp\left[-i\left(\frac{kx}{z_B}x'+\frac{ky}{z_B}y'\right)\right]dx'dy'$$

$$\Rightarrow U_t(\alpha,\beta) = \iint_{-\infty}^{\infty} t(x,y)\, e^{-i(\alpha x+\beta y)}\,dx\,dy$$

radius $a \Rightarrow x^2+y^2 \le a^2$ $\quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta\end{cases}$ $\quad \begin{cases} \alpha = \rho\cos\varphi \\ \beta = \rho\sin\varphi\end{cases}$

$$\Rightarrow U_t(\rho,\varphi) = \iint_{-\infty}^{\infty} t(r,\theta)\, e^{-i(\rho r\cos\theta\cos\varphi+\rho r\sin\theta\sin\varphi)}\, r\,dr\,d\theta$$

$$= \iint_{-\infty}^{\infty} t(r,\theta)\, e^{-i\rho r\cos(\theta-\varphi)}\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^a e^{-i\rho r\cos(\theta-\varphi)}\, r\,dr\,d\theta$$

$$J_n(x) = \frac{i^{-n}}{2\pi}\int_0^{2\pi} e^{ix\cos\alpha}e^{in\alpha}\,d\alpha \qquad J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\alpha}\,d\alpha$$

$x = -\rho r \quad \alpha = \theta-\varphi \Rightarrow J_0(-\rho r) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\rho r\cos(\theta-\varphi)}\,d\theta$

$$U_t(\rho,\varphi) = \int_0^a r\,dr\int_0^{2\pi} e^{-i\rho r\cos(\theta-\varphi)}\,d\theta = 2\pi\int_0^a r J_0(-\rho r)\,dr$$

$$\frac{d}{dx}\left[x^{n+1}J_{n+1}(x)\right] = x^{n+1}J_n(x) \Rightarrow xJ_0(x) = \frac{d}{dx}[xJ_1(x)] \Rightarrow xJ_1(x) = \int xJ_0(x)\,dx$$

assume $x = -\rho r \Rightarrow -\rho r J_1(-\rho r) = \int -\rho r J_0(-\rho r)\, d(-\rho r) = \rho^2\int r J_0(-\rho r)\,dr$

$$\Rightarrow \int r J_0(-\rho r)\,dr = -\frac{r}{\rho}J_1(-\rho r) \qquad J_{2n+1}(-x) = -J_{2n+1}(x) \qquad J_{2n}(-x) = J_{2n}(x)$$

$$\Rightarrow U_t(\rho,\varphi) = 2\pi\int_0^a r J_0(-\rho r)\,dr = -\frac{2\pi a}{\rho}J_1(-\rho a) = \frac{2\pi a}{\rho}J_1(\rho a)$$

$$\Rightarrow I_{FR} = \frac{4\pi^2}{(\lambda z_B)^2}\frac{a^2}{\rho^2}J_1^2(\rho a) \qquad \rho = \sqrt{\alpha^2+\beta^2} \qquad \alpha = \frac{kx}{z_B} \qquad \beta = \frac{ky}{z_B} \Rightarrow \rho = \frac{k}{z_B}\sqrt{x^2+y^2} = \frac{k}{z_B}r$$

$$\Rightarrow I_{FR} = \frac{4\pi^2}{(\lambda z_B)^2}\cdot\frac{a^2 z_B^2}{k^2r^2}J_1^2\left(\frac{ka}{z_B}r\right) = \frac{a^2}{r^2}J_1^2\left(\frac{ka}{z_B}r\right)$$

(b) $$U_t(\rho,\varphi) = 2\pi\int_{a_1}^{a_2} r J_0(\rho r)\,dr = \frac{2\pi}{\rho}[a_2J_1(\rho a_2) - a_1J_1(\rho a_1)] = \frac{2\pi z_B}{kr}\left[a_2J_1\left(\frac{ka_2}{z_B}r\right) - a_1J_1\left(\frac{ka_1}{z_B}r\right)\right]$$

$$I_{FR} = \frac{1}{(\lambda z_B)^2}\cdot\frac{4\pi^2 z_B^2}{k^2r^2}\left[a_2J_1\left(\frac{ka_2}{z_B}r\right) - a_1J_1\left(\frac{ka_1}{z_B}r\right)\right]^2 = \frac{1}{r^2}\left[a_2J_1\left(\frac{ka_2}{z_B}r\right) - a_1J_1\left(\frac{ka_1}{z_B}r\right)\right]^2$$

(c) $$t_1(x,y) = \begin{cases} 1 & x^2+y^2 \le a^2 \\ 0 & \text{others.}\end{cases} \qquad t(x) = \sum_{n=0}^{N-1} t_1(x-nb,y)$$

$$U_t(\alpha,\beta) = \iint_{-\infty}^{\infty} t(x)\, e^{-i(\alpha x+\beta y)}\,dx\,dy = \iint_{-\infty}^{\infty}\sum_{n=0}^{N-1} t_1(x-nb,y)\, e^{-(\alpha x+\beta y)}\,dx\,dy \qquad x-nb = \xi$$

$$\Rightarrow U_+(\alpha,\beta)=\iint_{-\infty}^{\infty}\sum_{n=0}^{N-1} t(\xi,y)\, e^{-i(\alpha\xi+\beta y)} e^{-i\alpha nb}\, d\xi dy = \iint_{-\infty}^{\infty} t(\xi,y) e^{-i(\alpha\xi+\beta y)} d\xi dy \sum_{n=0}^{N-1} e^{-i\alpha nb}$$

$$\iint_{-\infty}^{\infty} t(\xi,y) e^{-i(\alpha\xi+\beta y)} d\xi dy = \int_0^{2\pi}\int_0^{a} e^{-i\rho r\cos(\theta-\varphi)}\, r\, dr d\theta = \frac{2\pi a}{\rho} J_1(\rho a)$$

$$\sum_{n=0}^{N-1} e^{-i\alpha nb} = \frac{e^{-i\alpha Nb}-1}{e^{-i\alpha b}-1} = \frac{\sin N\alpha\frac{b}{2}}{\sin\alpha\frac{b}{2}} e^{i(1-N)\alpha b} \xlongequal{\alpha=\rho\cos\varphi} \frac{\sin N\rho\cos\varphi\frac{b}{2}}{\sin\rho\cos\varphi\frac{b}{2}} e^{i(1-N)\rho\cos\varphi b}$$

$$U_+(\rho,\varphi) = \frac{2\pi a}{\rho} J_1(a\rho) \frac{\sin N\rho\cos\varphi\frac{b}{2}}{\sin\rho\cos\varphi\frac{b}{2}} e^{i(1-N)\rho\cos\varphi b}$$

$$I_{FR} = \frac{1}{(\lambda z_B)^2}\cdot\frac{4\pi^2 a^2}{\rho^2} J_1^2(a\rho)\frac{\sin^2 N\rho\cos\varphi\frac{b}{2}}{\sin\rho\cos\varphi\frac{b}{2}} \qquad \rho=\sqrt{\alpha^2+\beta^2} = \frac{k}{z_B}\sqrt{x^2+y^2} = \frac{kr}{z_B} \qquad \alpha = \frac{kx}{z_B}$$

$$= \frac{a^2}{r^2} J_1^2\left(\frac{kr}{z_B}a\right)\frac{\sin^2\frac{Nxkb}{2z_B}}{\sin^2\frac{xkb}{2z_B}}$$

(d) a pinhole at (x_i, y_i)

$$t(x,y) = \begin{cases} 1 & x^2+y^2\le a^2 \\ 0 & \text{others}\end{cases} \qquad t_i(x_i,y_i) = t(x-x_i, y-y_i) = \begin{cases} 1 & (x-x_i)^2+(y-y_i)^2\le a^2 \\ 0 & \text{others}\end{cases}$$

$$U_+(x,y) = \iint t_i(x',y')\exp\left[-i\left(\frac{kx}{z_B}x'+\frac{ky}{z_B}y'\right)\right] dx'dy'$$

$$= \iint t(x'-x_i, y'-y_i)\exp\left[-i\left(\frac{kx}{z_B}x'+\frac{ky}{z_B}y'\right)\right] dx'dy'$$

$$x'-x_i=\xi \quad y'-y_i=\eta \quad \Rightarrow x'=\xi+x_i \quad y'=\eta+y_i$$

$$\Rightarrow U_+(x,y) = \iint t(\xi,\eta)\exp\left[-i\left(\frac{kx}{z_B}(\xi+x_i)+\frac{ky}{z_B}(\eta+y_i)\right)\right] d\xi d\eta$$

$$= \iint t(\xi,\eta)\exp\left[-i\left(\frac{kx}{z_B}\xi+\frac{ky}{z_B}\eta\right)\right]\exp\left[-i\left(\frac{kx}{z_B}x_i+\frac{ky}{z_B}y_i\right)\right] d\xi d\eta$$

According to the result in a.

$$\Rightarrow U_+(\rho,\varphi) = \frac{2\pi a}{\rho} J_1(\rho a)\exp\left[-i\left(\frac{kx}{z_B}x_i+\frac{ky}{z_B}y_i\right)\right]$$

$$U_{+N}(\rho,\varphi) = \frac{2\pi a}{\rho} J_1(\rho a)\sum_{i=1}^{N}\exp\left[-i\left(\frac{kx}{z_B}x_i+\frac{ky}{z_B}y_i\right)\right]$$

$$I_{FR} = \frac{4\pi^2 a^2}{\lambda^2 z_B^2\rho^2} J_1^2(\rho a)\left\{\sum_{i=1}^{N}\exp\left[-i\left(\frac{kx}{z_B}x_i+\frac{ky}{z_B}y_i\right)\right]\right\}^2 = \frac{a^2}{r^2} J_1^2\left(\frac{kra}{z_B}\right)\left\{\sum_{i=1}^{N}\exp\left[-i\left(\frac{kx}{z_B}x_i+\frac{ky}{z_B}y_i\right)\right]\right\}^2$$

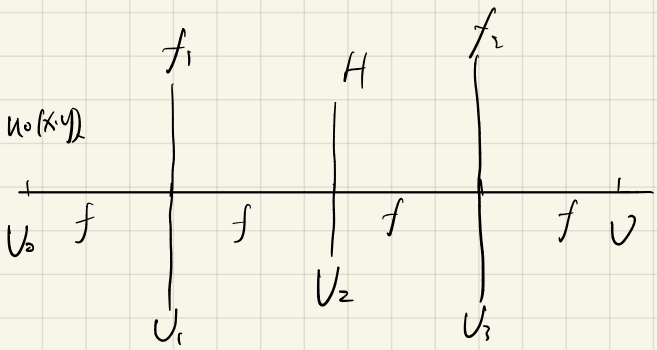
(e) $t_i'(x_i,y_i) = t'(x-x_i, y-y_i) = \begin{cases} 1 & (x-x_i)^2+(y-y_i)^2\le a_i^2 \\ 0 & \text{others}\end{cases}$

$$\Rightarrow U_{+i} = \frac{2\pi a_i}{\rho} J_1(\rho a_i)\exp\left[-i\left(\frac{kx}{z_B}x_i+\frac{ky}{z_B}y_i\right)\right]$$

$$U_+ = \sum_{i=1}^{N}\frac{2\pi a_i}{\rho} J_1(\rho a_i)\exp\left[-i\left(\frac{kx}{z_B}x_i+\frac{ky}{z_B}y_i\right)\right]$$

$$I_{FR} = \frac{1}{r^2}\left\{\sum_{i=1}^{N} a_i J_1\left(\frac{kr}{z_B}a_i\right)\exp\left[-i\left(\frac{kx}{z_B}x_i+\frac{ky}{z_B}y_i\right)\right]\right\}^2$$

Consider a 4f-setup consisting of two identical lenses with focal length f placed at a distance of $2f$ illuminated with light of wavelength λ. Halfway between the two lenses, centered on the optical axis, we place a filter $H(x,y)$. We shall now consider this "device" as a black box, which takes a certain input field $A_0(x,y)$ incident at the focal plane before the first lens and transforms it into an output field $A_1(x,y)$ at a distance of f after the second lens.



a) The action of the device can be described by a convolution with a response function $h_R(x,y)$ such that

$$A_1(-x,-y) = \iint h_R(x-x', y-y') A_0(x', y') \mathrm{d}x' \mathrm{d}y'$$

Why is that so? How are $h_R(x,y)$ and $H(x,y)$ related?

b) Assuming that the filter could also include optical gain (transmission > 1) one could implement the following two filters: $H(x,y) = \mathrm{i}qx$ and $H(x,y) = -q(x^2+y^2)$. What effect do they have on the optical field?

c) The above mentioned $4f$ setup can be used to build an optical correlator 相关器, which measures the similarity between two wave fields $A_0(x,y)$ and a reference wave field $B(x,y)$. The correlation $A^{\mathrm{corr}}(x,y)$ between these two fields is calculated by

$$A^{\mathrm{corr}}(x,y) = A_1(-x,-y) = A_0(x,y) \otimes B^*(-x,-y),$$

where $\otimes$ denotes a convolution operation. How does the aperture function have to look like in order to implement this special filtering operation? How could it be implemented practically?

$$t_L(x,y) = \exp\left[\frac{-ik}{2f}(x^2+y^2)\right] \qquad T_L(\alpha,\beta) = -i\frac{\lambda f}{(2\lambda)^2}\exp\left[\frac{if}{2k}(\alpha^2+\beta^2)\right] \qquad H_F = e^{ikf}\exp\left[\frac{-if}{2k}(\alpha^2+\beta^2)\right]$$

$$U_0(\alpha,\beta) = FT[u_0(x,y)]$$

$$U_{1-}(\alpha,\beta,f) = U_0(\alpha,\beta)\cdot H_F = e^{ikf}U_0(\alpha,\beta)\exp\left[\frac{-if}{2k}(\alpha^2+\beta^2)\right]$$

$$U_{1+}(\alpha,\beta,f) = T_L(\alpha,\beta)\otimes U_{1-}(\alpha,\beta,f) = \frac{-i\lambda f}{(2\lambda)^2}e^{ikf}\iint_{-\infty}^{\infty}\exp\left\{\frac{if}{2k}\left[(\alpha-\alpha')^2+(\beta-\beta')^2\right]\right\}\exp\left[\frac{-if}{2k}(\alpha'^2+\beta'^2)\right]U_0(\alpha',\beta')\,d\alpha' d\beta'$$

$$= \frac{-i\lambda f}{(2\lambda)^2}e^{ikf}\iint_{-\infty}^{\infty}\exp\left\{\frac{if}{2k}\left[(\alpha^2-2\alpha\alpha')+\beta^2-2\beta\beta'\right]\right\}U_0(\alpha,\beta)\,d\alpha' d\beta'$$

$$U_{2-}(\alpha,\beta,2f) = U_{1+}(\alpha,\beta,f)\cdot H_F$$

$$= \frac{-i\lambda f}{(2\lambda)^2}e^{2ikf}\iint_{-\infty}^{\infty}\exp\left\{\frac{if}{2k}\left[(\alpha^2-2\alpha\alpha')+(\beta^2-2\beta\beta')\right]\right\}\exp\left[\frac{-if}{2k}(\alpha^2+\beta^2)\right]U_0(\alpha,\beta)\,d\alpha' d\beta'$$

$$= \frac{-i\lambda f}{(2\lambda)^2}e^{2ikf}\iint_{-\infty}^{\infty}U_0(\alpha,\beta)\exp\left[-i\left(\frac{f\alpha}{k}\alpha'+\frac{f\beta}{k}\beta'\right)\right]d\alpha' d\beta' = \frac{-i\lambda f}{(2\lambda)^2}e^{2ikf}u_0\left(-\frac{f\alpha}{k}, -\frac{f\beta}{k}\right)$$

$$u_{2-}(x,y,2f) = -\frac{i\lambda f}{(2\lambda)^2}e^{2ikf}\iint_{-\infty}^{\infty}u_0\left(\frac{-f\alpha}{k}, -\frac{f\beta}{k}\right)e^{i(\alpha x+\beta y)}\,d\alpha d\beta$$

$$x' = -\frac{f\alpha}{k}, \quad y' = -\frac{f\beta}{k} \Rightarrow \alpha = \frac{kx'}{-f} \quad \beta = \frac{ky'}{-f}$$

$$\Rightarrow u_{2-}(x,y,2f) = -\frac{i\lambda f}{(2\lambda)^2}\frac{k^2}{f^2}e^{2ikf}\iint_{-\infty}^{\infty}u_0(x',y')\exp\left[-i\left(\frac{k}{f}x'x+\frac{ky}{f}y'\right)\right]dx' dy'$$

$$= \frac{-i(2\lambda)^2}{\lambda f}e^{2ikf}U_0\left(\frac{k}{f}x, \frac{k}{f}y\right)$$

$$u_{2+}(x,y,2f) = u_{2-}(x,y,2f)H(x,y) = \frac{-i(2\lambda)^2}{\lambda f}e^{2ikf}U_0\left(\frac{kx}{f}, \frac{ky}{f}\right)H(x,y)$$

Similarly: $u(x,y,4f) = \dfrac{-i(2\lambda)^2}{\lambda f}e^{2ikf}U_{2+}\left(\dfrac{kx}{f}, \dfrac{ky}{f}, 2f\right)$

$$U_{2+}\left(\frac{kx}{f}, \frac{ky}{f}, 2f\right) = \frac{1}{(2\lambda)^2}\iint u_{2+}(x',y',2f)\exp\left[-i\left(\frac{kx}{f}x'+\frac{ky}{f}y'\right)\right]dx' dy'$$

$$= \frac{1}{(2\lambda)^2}\iint\frac{-i(2\lambda)^2}{\lambda f}e^{2ikf}U_0\left(\frac{kx'}{f}, \frac{ky'}{f}\right)H(x',y')\exp\left[-i\left(\frac{kx}{f}x'+\frac{ky}{f}y'\right)\right]dx' dy'$$

$$= \frac{-i}{\lambda f}e^{2ikf}\iint U_0\left(\frac{kx'}{f}, \frac{ky'}{f}\right)H(x',y')\exp\left[-i\left(\frac{kx}{f}x'+\frac{ky}{f}y'\right)\right]dx' dy'$$

$$\Rightarrow u(x,y,4f) = -\frac{(2\lambda)^4}{(\lambda f)^2}e^{4ikf}\iint U_0\left(\frac{kx'}{f}, \frac{ky'}{f}\right)H(x',y')\exp\left[-i\left(\frac{kx}{f}x'+\frac{ky}{f}y'\right)\right]dx' dy'$$

$$\alpha = \frac{kx'}{f}, \quad \beta = \frac{ky'}{f} \quad x' = \frac{f\alpha}{k} \quad y' = \frac{f\beta}{k}$$

$$\Rightarrow u(x,y,4f) = -\frac{(2\lambda)^4}{\lambda^2 f^2}\frac{f^2}{k^2}e^{4ikf}\iint U_0(\alpha,\beta)H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right)\exp\left[-i(\alpha x+\beta y)\right]d\alpha d\beta$$

$$u(-x,-y,4f) = -4\lambda^2 e^{4ikf}\iint U_0(\alpha,\beta)H\left(\frac{f\alpha}{k}, \frac{f\beta}{f}\right)\exp\left[i(\alpha x+\beta y)\right]d\alpha d\beta$$

$$= -\frac{4\pi^2 k^2}{f^2} e^{4ikf} \iint U_0\left(\frac{kx'}{f}, \frac{ky'}{f}\right) H(x', y') \exp\left[i\left(\frac{kx'}{f}x + \frac{ky'}{f}y\right)\right] dx' dy'$$

According to Convolution theory: $C(t) = \int_{-\infty}^{\infty} A(t') B(t-t') dt' = \int_{-\infty}^{\infty} A(\omega) B(\omega) e^{-i\omega t} d\omega$.

$$\Rightarrow U(-x, -y, 4f) = \iint_{-\infty}^{\infty} h_e(x-x', y-y') U_0(x', y') dx' dy'$$

And $h_e(x, y) = -4\pi^2 e^{4ikf} \iint_{-\infty}^{\infty} H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right) e^{i(\alpha x + \beta y)} d\alpha d\beta = -f^2\lambda^2 e^{4ikf} \iint_{-\infty}^{\infty} H(x', y') \exp\left[i\frac{k}{f}(xx' + yy')\right] dx' dy'$

$$\underline{h_e(x, y) = -\lambda^2 f^2 e^{4ikf} \iint_{-\infty}^{\infty} H(x', y') \exp\left[i\frac{k}{f}(x'x + y'y)\right] dx' dy'} = -\lambda^2 f^2 e^{4ikf} \bar{H}\left(-\frac{kx}{f}, -\frac{ky}{f}\right) \cdot 4\pi^2$$

$$\Rightarrow U(-x, -y, 4f) = -\lambda^2 f^2 e^{4ikf} (2\pi)^2 \iint_{-\infty}^{\infty} \bar{H}\left[\frac{k}{f}(x'-x), \frac{k}{f}(y'-y)\right] U_0(x', y') dx' dy'$$

(b) $A_1(-x, -y) = \iint h_e(x-x', y-y') A_0(x', y') dx dy' = FT^{-1}\left[H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right) \cdot \bar{A}_0(\alpha, \beta)\right] = FT^{-1}\left[H(x, y) \bar{A}_0\left(\frac{kx}{f}, \frac{ky}{f}\right)\right]$

$$\Rightarrow A(-x, -y) = FT^{-1}\left[iqx \bar{A}\left(\frac{kx}{f}, \frac{ky}{f}\right)\right] = iq \iint_{-\infty}^{\infty} x \bar{A}\left(\frac{kx}{f}, \frac{ky}{f}\right) \exp\left[i\left(\frac{kx}{f}x' + \frac{ky}{f}y'\right)\right] d\frac{kx}{f} d\frac{ky}{f}$$

$$= iq\frac{k^2}{f^2} \iint \bar{A}\left(\frac{kx}{f}, \frac{ky}{f}\right) \cdot \frac{f}{ik} \frac{d}{dx'} \exp\left[i\left(\frac{kx}{f}x' + \frac{ky}{f}y'\right)\right] dx dy$$

$$= \frac{qk}{f} \frac{d}{dx'} \iint \bar{A}\left(\frac{kx}{f}, \frac{ky}{f}\right) \exp\left[i\left(\frac{kx}{f}x' + \frac{ky}{f}y'\right)\right] dx dy = \frac{qf}{k}\frac{d}{dx'} A(x', y') = \frac{qf}{k}\frac{d}{dx} A(x, y)$$

$H(x, y) = -q\,(x^2 + y^2)$

$$A(-x, -y) = FT^{-1}\left[-q(x^2+y^2) \bar{A}_0\left(\frac{kx}{f}, \frac{ky}{f}\right)\right] \quad \frac{kx}{f} = \alpha \quad \frac{ky}{f} = \beta \quad \Rightarrow x^2 = \frac{f^2}{k^2}\alpha^2 \quad y^2 = \frac{f^2}{k^2}\beta^2$$

$$\Rightarrow A(-x, -y) = FT^{-1}\left[-q\frac{f^2}{k^2}(\alpha^2 + \beta^2) \bar{A}_0(\alpha, \beta)\right]$$

$$= -q\frac{f^2}{k^2} \iint_{-\infty}^{\infty} (\alpha^2 + \beta^2) \bar{A}_0(\alpha, \beta) \exp[i(\alpha x + \beta y)] d\alpha d\beta$$

$$= q\frac{f^2}{k^2}\left(\iint_{-\infty}^{\infty} \bar{A}_0(\alpha, \beta) \frac{d^2}{dx^2} \exp[i(\alpha x + \beta y)] d\alpha d\beta + \iint_{-\infty}^{\infty} \bar{A}_0(\alpha, \beta) \frac{d^2}{dy^2} \exp[i(\alpha x + \beta y)] d\alpha d\beta\right)$$

$$= q\frac{f^2}{k^2}\left[\frac{d^2}{dx^2} A_0(x, y) + \frac{d^2}{dy^2} A_0(x, y)\right] = \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)\left[\frac{qf^2}{k^2} A_0(x, y)\right]$$

Thus $H(x, y) = iqx$ act like a differentiator. the output is the differential of the input

$H(x, y) = q(x^2 + y^2)$ act like a Laplace operator. the output ist the second-order differential of the input.

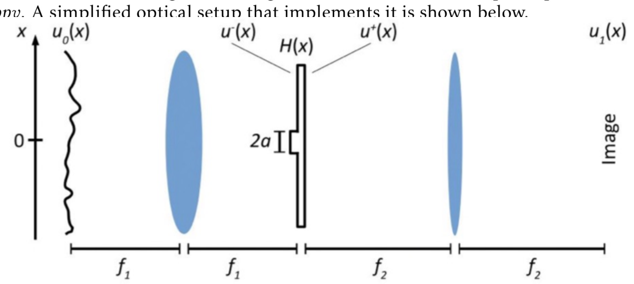
(c) $A_1(-x, -y) = A_0(x, y) \otimes B^*(-x, -y)$

$$= A(-\alpha, -\beta) = A_0(\alpha, \beta) \cdot B^*(-\alpha, -\beta) = A_1\left(-\frac{kx}{f}, \frac{ky}{f}\right) = A_0\left(\frac{kx}{f}, \frac{ky}{f}\right) \cdot B^*\left(-\frac{kx}{f}, \frac{ky}{f}\right)$$

$$A_1(-x, -y) = \iint_{-\infty}^{\infty} h_e(x-x', y-y') A_0(x', y') dx' dy' = FT^{-1}\left[H(x, y) \bar{A}_0\left(\frac{kx}{f}, \frac{ky}{f}\right)\right]$$

$$= A\left(-\frac{kx}{f}, \frac{ky}{f}\right) = H(x, y) \bar{A}_0\left(\frac{kx}{f}, \frac{ky}{f}\right) \Rightarrow \underline{H(x, y) = B^*\left(-\frac{kx}{f}, -\frac{ky}{f}\right)}$$

Biological samples are often almost completely transparent. Consequently, they are very hard to see in a conventional microscope. However, these samples often have inhomogeneities of the refractive index and change the *phase* of the transmitted light. One elegant solution to make those phase profiles visible is the *phase contrast microscopy*. A simplified optical setup that implements it is shown below.



Essential is the phase plate in the center whose effect can be described by the following transmission function

$$H(x) = \begin{cases} \exp(i\varphi_0) & -a \le x \le a \\ 1 & \text{else} \end{cases}.$$

A part of the transmitted light is delayed compared to the rest. We will see in this task that this allows to convert a phase profile into an intensity profile.

a) Derive an expression that describes how $u_1(x)$ depends on $u_0(x)$ and $H(x)$. Consider monochromatic light of wavelength λ. *Hint:* You can simply take the solution from the previous task and adapt it to the one-dimensional setup here.

b) Calculate the image $u_1(x)$ of the initial phase-profile distribution:

$$u_0(x) = e^{iA\cos(\alpha_0 x)} \approx 1 + iA\cos(\alpha_0 x),$$

where $A \ll 1$ and $\alpha_0 = 2\pi/\Lambda$. This field can be thought to be caused approximately by a phase grating with a small amplitude and grating period Λ. Derive the conditions on a and φ_0 so that the setup converts the phase modulation in u_0 into an amplitude modulation in u_1.

(a) $u^-(x) = -i\frac{2\pi}{\lambda f_1} e^{2ikf_1} U_0(\frac{kx}{f_1})$ $\quad U_0(\frac{kx}{f_1}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_0(x') e^{-i\frac{kx}{f_1}x'}dx'$

$u^+(x) = -i\frac{2\pi}{\lambda f_1} e^{2ikf_1} U_0(\frac{kx}{f_1}) H(x)$

$u_1(x) = -i\frac{2\pi}{\lambda f_2} e^{2ikf_2} U^+(\frac{kx}{f_1}) = -\frac{4\pi^2}{\lambda^2 f_1 f_2} e^{2ik(f_1+f_2)} \frac{1}{2\pi}\int_{-\infty}^{\infty} U_0(\frac{kx'}{f_1})H(x')e^{-i\frac{kx}{f_2}x'}dx'$

$= -\frac{2\pi}{\lambda^2 f_1 f_2}\exp[2ik(f_1+f_2)]\int_{-\infty}^{\infty} U_0(\frac{kx'}{f_1})H(x')e^{-i\frac{kx}{f_2}x'}dx'$

(b) $U_0(\frac{kx}{f_1}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[1+iA\cos(\alpha_0 x')]e^{-i\frac{kx}{f_1}x'}dx'$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\frac{kx}{f_1}x'}dx' + \frac{iA}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2}[e^{i\alpha_0 x'} + e^{-i\alpha_0 x'}]dx'$

$= \delta(\frac{kx}{f_1}) + \frac{iA}{2}[\delta(\frac{kx}{f_1}-\alpha_0)+\delta(\frac{kx}{f_1}+\alpha_0)]$

$\Rightarrow u_1(x) = -\frac{2\pi}{\lambda^2 f_1 f_2}\exp[2ik(f_1+f_2)]\int_{-\infty}^{\infty}\{\delta(\frac{kx'}{f_1}) + \frac{iA}{2}[\delta(\frac{kx'}{f_1}-\frac{2\pi}{\Lambda})+\delta(\frac{kx'}{f_1}+\frac{2\pi}{\Lambda})]\}H(x')e^{-i\frac{kx}{f_2}x'}dx'$

only $\delta(\frac{kx'}{f_1})$ is delayed $\Rightarrow \frac{ka}{f_1}\le\alpha_0 \Rightarrow a\le\frac{f_1\alpha_0}{k} = \frac{f_1\lambda}{\Lambda}$

$= -\frac{2\pi}{\lambda^2 f_1 f_2}e^{2ik(f_1+f_2)}\{\int_{-\infty}^{\infty}\delta(\frac{kx'}{f_1})e^{i\varphi_0}e^{-i\frac{kx}{f_2}x'}dx' + \frac{iA}{2}\int_{-\infty}^{\infty}[\delta(\frac{kx'}{f_1}-\frac{2\pi}{\Lambda})+\delta(\frac{kx'}{f_1}+\frac{2\pi}{\Lambda})]e^{-i\frac{kx}{f_2}x'}dx'\}$

Let $\frac{kx'}{f_1}=\xi \Rightarrow dx' = \frac{f_1}{k}d\xi$

$\Rightarrow u_1(x) = -\frac{2\pi}{\lambda^2 f_1 f_2}\cdot\frac{f_1}{k}e^{2ik(f_1+f_2)}\{\int_{-\infty}^{\infty}\delta(\xi)e^{i\varphi_0}e^{-i\frac{f_1}{f_2}x\xi}d\xi + \frac{iA}{2}\int_{-\infty}^{\infty}[\delta(\xi-\frac{2\pi}{\Lambda})+\delta(\xi+\frac{2\pi}{\Lambda})]e^{-i\frac{f_1}{f_2}x\xi}d\xi\}$

$= -\frac{1}{\lambda f_2}e^{2ik(f_1+f_2)}\{e^{i\varphi_0}+\frac{iA}{2}[e^{-i\frac{f_1}{f_2}x(\frac{2\pi}{\Lambda})}+e^{i\frac{f_1}{f_2}x(\frac{2\pi}{\Lambda})}]\}$

$= -\frac{1}{\lambda f_2}e^{2ik(f_1+f_2)}[e^{i\varphi_0}+iA\cos(\frac{f_1}{f_2}\cdot\frac{2\pi}{\Lambda}x)] = \frac{-1}{\lambda f_2}e^{2ik(f_1+f_2)}[e^{i\varphi_0}+iA\cos(\frac{f_1}{f_2}\alpha_0 x)]$

$|u_1(x)|^2 = \frac{1}{\lambda^2 f_2^2}[1+A^2\cos^2(\frac{f_1}{f_2}\alpha_0 x) - iAe^{i\varphi_0}\cos(\frac{f_1}{f_2}\alpha_0 x) + iAe^{-i\varphi_0}\cos(\frac{f_1}{f_2}\alpha_0 x)]$

$u_0(x) = 1+iA\cos(\alpha_0 x)$ $\quad |u_0(x)|^2 = 1+A^2\cos^2(\alpha_0 x)$

Amplitude Modulation $\Rightarrow \varphi_0 = 2k\pi+\frac{\pi}{2} \Rightarrow e^{i\varphi_0}=i,\ e^{-i\varphi_0}=-i$

$$\Rightarrow U_1(x) = \frac{-i}{\lambda f_2} e^{2ik(f_1+f_2)}\left[1 + A\cos\left(\frac{-f_1}{f_2}\alpha_0 x\right)\right] \Rightarrow |U_1(x)|^2 = \frac{-1}{\lambda f_2}\left[1 + A^2\cos^2\left(\frac{-f_1}{f_2}\alpha_0 x\right) + 2A\cos\left(\frac{f_1}{f_2}\alpha_0 x\right)\right]$$