Problem

a) J

Fundamentals of Modern Optics
Exercise 10
16.01.2014

to be returned: 23.01.2015, at the beginning of the lecture

Problem 1 – Anisotropic Materials — 2+2 points

A slice of a transparent, uniaxial, anisotropic material with the refractive indices n_e and n_o and thickness d is oriented such that the surface normal is oriented along $\mathbf{e}_z$ and the crystal axis is oriented along $\mathbf{c} = \cos(\alpha)\mathbf{e}_x + \sin(\alpha)\mathbf{e}_y$. An x-polarized plane wave of wavelength λ, with a vacuum wavevector $\mathbf{k} = 2\pi/\lambda\mathbf{e}_z$ propagates through the crystal.

$\vec{D}_{lab} = \hat{\varepsilon}_{lab}\vec{E}_{lab}$

$\vec{D}_{cry} = \hat{\varepsilon}_{cry}\vec{E}_{cry}$

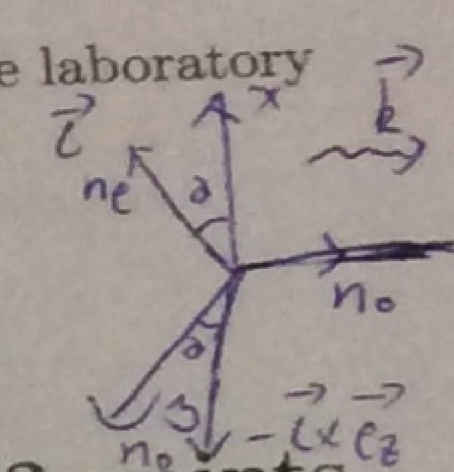
$\hat{R}\vec{D}_{cry} = \hat{R}\hat{\varepsilon}_{cry}\vec{E}_{cry}$

Ex $\vec{D}_{lab} = \hat{R}\hat{\varepsilon}_{cry}\hat{R}^{-1}\vec{E}_{lab}$

a) Derive the permittivity matrix ε_{ij} in the crystal basis $[\mathbf{c}, -(\mathbf{c}\times\mathbf{e}_z), \mathbf{e}_z]$ and in the basis of the laboratory coordinate system $[\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z]$

b) Decompose the incident field into the polarization Eigenmodes of the anisotropic medium.

Hint: You can check your solutions in the next problem.



Problem 2 – Optical Waveplates — 2+2+2+2 points

This problem is the direct continuation of problem 1, therefore, we assume the same geometry. We have lossless propagation and assume the Fresnel reflection at both surfaces to be negligible. The permittivity matrix $\hat{\varepsilon}$ in the crystal basis $[\mathbf{c}, -(\mathbf{c}\times\mathbf{e}_z), \mathbf{e}_z]$ and in the basis of the laboratory coordinate system $[\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z]$ is:

$$\hat{\varepsilon}_{\text{crystal basis}} = \begin{bmatrix} n_e^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_o^2 \end{bmatrix},$$

$$\hat{\varepsilon}_{\text{lab basis}} = \begin{bmatrix} n_e^2\cos^2(\alpha) + n_o^2\sin^2(\alpha) & (n_e^2 - n_o^2)\sin(\alpha)\cos(\alpha) & 0 \\ (n_e^2 - n_o^2)\sin(\alpha)\cos(\alpha) & n_o^2\cos^2(\alpha) + n_e^2\sin^2(\alpha) & 0 \\ 0 & 0 & n_o^2 \end{bmatrix}.$$

The decomposed incident field into the polarization Eigenmodes of the anisotropic medium can be written as: $\boldsymbol{E}_0 = E_0[\cos(\alpha)\boldsymbol{c} - \sin(\alpha)(-\boldsymbol{c}\times\boldsymbol{e}_z)]$.

a) Calculate the dispersion relation and polarization for both eigenmodes.

b) Calculate the relation between the wavevector $\boldsymbol{k}$ and the corresponding Poynting vector $\boldsymbol{S}$ for both eigenmodes. Show that they are in parallel.

c) What is the polarization state of the light after the anisotropic medium?

d) We choose the thickness d of the crystal such that $(n_e - n_o)d = \lambda/2$. Calculate and describe the impact of the crystal onto the polarization state of the plane wave as a function of the crystal rotation angle α. If we place a linear polarizer after this so-called half wave plate and rotate the crystal, what device do we get?

Problem 2 – Jones Formalism — 2+2+2 points

The Jones formalism is a powerful technique for the treatment of polarized light. A monochromatic plane wave in vacuum is of the form $\mathbf{E}(r,t) = \hat{\mathbf{E}}\,e^{i(kz-\omega t)}$, where the electric field vector $\hat{\mathbf{E}} = (\hat{E}_x, \hat{E}_y, 0)$ is polarized in the (x,y)-plane. Light propagation of this plane wave through a polarizing optical element can be written as a linear transformation

$$\mathbf{J}_{\text{out}} = \hat{\mathbf{T}}\cdot\mathbf{J}_{\text{in}} \quad , \text{with} \quad \mathbf{J} = \begin{pmatrix} \hat{E}_x\,e^{i\varphi_x} \\ \hat{E}_y\,e^{i\varphi_y} \end{pmatrix},$$

where $\hat{\mathbf{T}}$ denotes the so-called "Jones matrix" of the element. Accordingly, the polarization state of the electric field is described by the two-dimensional "Jones vector" $\mathbf{J}$.

a) The Jones matrix of a x-polarizer is given by

$$\hat{\mathbf{T}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Define the transmission matrix for a polarizer which is rotated around z by an angle θ.

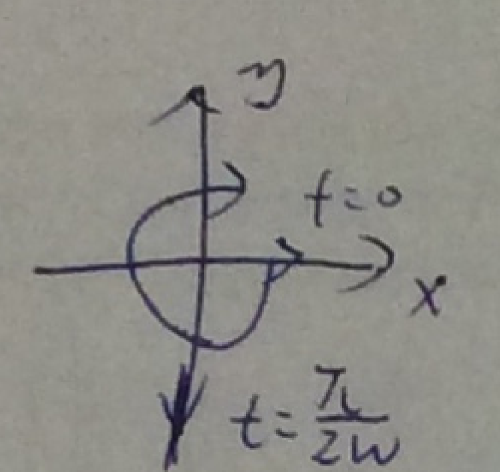
b) Is it possible to rotate the polarization direction of linear polarized light by 90° using two linear polarizers? How large would the total loss (excluding absorption) of such a system be? How large would be the loss if one could use infinitely many linear polarizers?

c) Given is an optical element characterized by

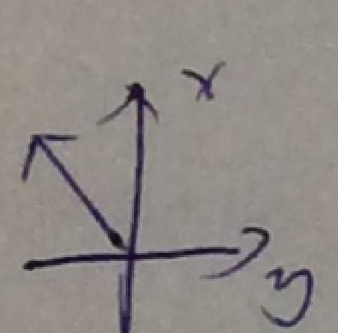
$$\hat{\mathbf{T}} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

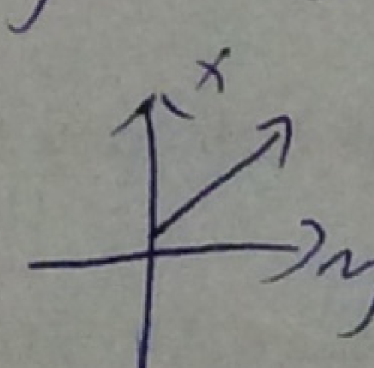
How should the polarization state be to pass the element without change? What optical component would behave like this?

$$Re\left[\frac{E_0}{\sqrt{2}}(x+iy)e^{i\omega t}\right] = \frac{\cos(\omega t)\hat{x} - \sin(\omega t)\hat{y}}{\sqrt{2}} E_0$$



$$\hat{T} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}}_{\text{QWP 1/4波片}} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\text{XP}} \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}}_{\text{QWP}} \xrightarrow{\hat{T}} \text{optical component}$$





$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} \xrightarrow[+45°]{QWP} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow[-45°]{QWP} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

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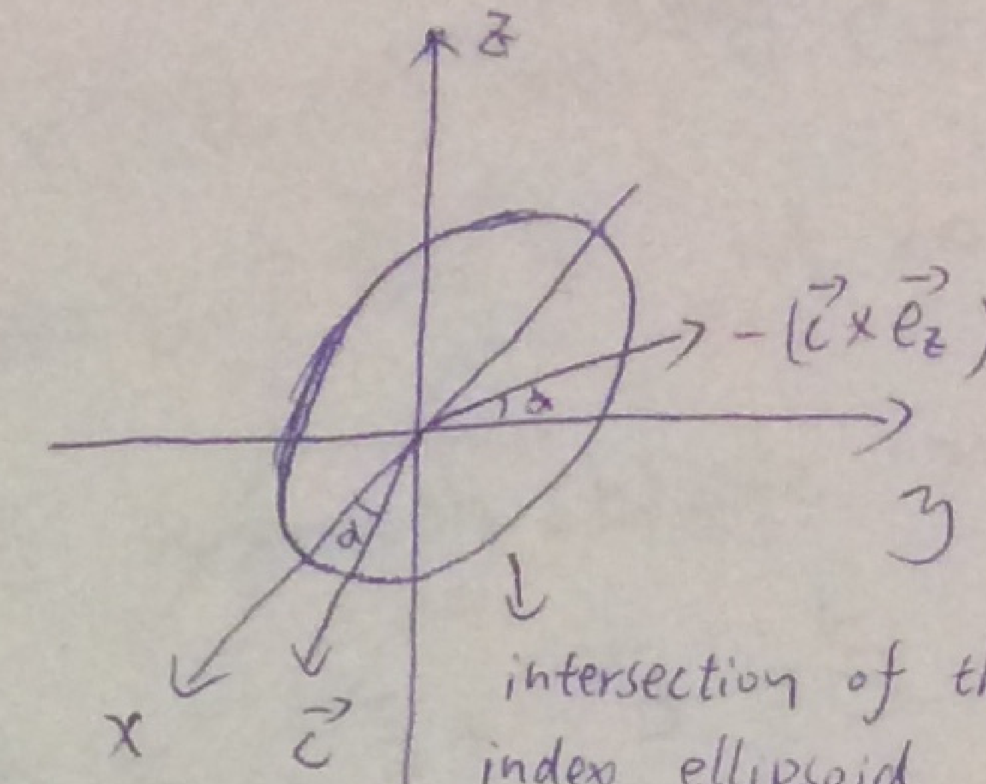
Problem 1 - Anisotropic Materials



intersection of the index ellipsoid in the x.y-plane

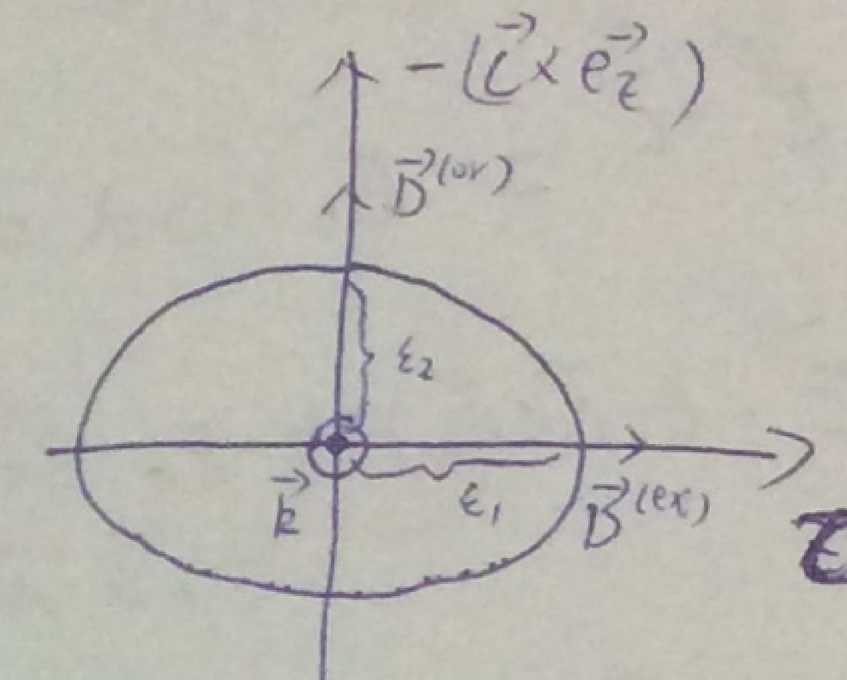
a) In crystal basis, $\hat{\varepsilon}$ has a diagonal form

so $\varepsilon_{ij} = \begin{bmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{bmatrix}$, for uniaxial crystal, $\varepsilon_2 = \varepsilon_3 \neq \varepsilon_1$ ✓

i) ordinary wave $\vec{D}^{(or)}$ is polarized perpendicularly to $\vec{c}$ and $\vec{k}$

$$\left.\begin{matrix} \Rightarrow \vec{D}^{(or)} \perp \vec{c} \\ \vec{D}^{(or)} \perp \vec{k} \end{matrix}\right\} \Rightarrow \vec{D}^{(or)} \parallel -(\vec{c}\times\vec{e_z}) \Rightarrow \varepsilon_2 = \varepsilon_3 = n_o^2$$

(n_o is a constant doesn't depend on $\vec{k}$-direction)



ii) Extra-ordinary wave $\vec{D}^{(ex)}$ is polarized perpendicularly to $\vec{D}^{(or)}$ and $\vec{k}$

$$\left.\begin{matrix} \Rightarrow \vec{D}^{(ex)} \perp \vec{D}^{(or)} \\ \vec{D}^{(ex)} \perp \vec{k} \end{matrix}\right\} \Rightarrow \vec{D}^{(ex)} \parallel \vec{c} \Rightarrow \varepsilon_1 = n_e^2$$

Therefore, $\varepsilon_{ij} = \begin{bmatrix} n_e^2 & & \\ & n_o^2 & \\ & & n_o^2 \end{bmatrix}$ ✓ in crystal basis $[\vec{c}, -(\vec{c}\times\vec{e_z}), \vec{e_z}]$

with $\vec{c} = [\cos\alpha, \sin\alpha, 0]$ ✓, $-(\vec{c}\times\vec{e_z}) = [-\sin\alpha, \cos\alpha, 0]$ ✓, $\vec{e_z} = [0, 0, 1]$

In the basis of the laboratory coordinate system $[\vec{e_x}, \vec{e_y}, \vec{e_z}]$

$$M = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ — rotational matrix around z-axis at the angle α

$$\varepsilon_{ij}' = M^{-1}\varepsilon_{ij}M = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} n_e^2 & & \\ & n_o^2 & \\ & & n_o^2 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} n_e^2\cos^2\alpha + n_o^2\sin^2\alpha & (n_e^2-n_o^2)\sin\alpha\cos\alpha & 0 \\ (n_e^2-n_o^2)\sin\alpha\cos\alpha & n_o^2\cos^2\alpha + n_e^2\sin^2\alpha & 0 \\ 0 & 0 & n_o^2 \end{bmatrix}$$ ✓

$\vec{A}_{lab} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix}$ $\vec{A}_{cry} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\vec{A}_{lab} = \hat{R}\vec{A}_{cry}$

$\hat{R} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R^{-1} = R^T$

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$\vec{D}_{cry} = \hat{\varepsilon}_{cry}\hat{E}_{crys}$

$\vec{D}_{lab} = \hat{\varepsilon}_{lab}\hat{E}_{lab}$

$\hat{R}\vec{D}_{cry} = \hat{R}\hat{\varepsilon}_{cry}\hat{E}_{cry}$

$\Rightarrow \vec{D}_{lab} = \underbrace{\hat{R}\,\cancel{\varepsilon}\,R^{-1}}_{\hat{\varepsilon}_{lab}}\vec{E}_{lab}$

$\hat{\varepsilon}_{lab} = \hat{R}\hat{\varepsilon}_{cry}R^{-1}$

b) because wavevector $\vec{k} = \frac{2\pi}{\lambda}[0,0,1]$, we have ✓

$$\vec{D} = \varepsilon_0 \varepsilon_{ij} \vec{E} = \varepsilon_0 \varepsilon_1 \vec{E}^{(ex)} + \varepsilon_0 \varepsilon_2 \vec{E}^{(or)} = \vec{D}^{(ex)} + \vec{D}^{(or)} \quad (\vec{D}^{(ex)} \parallel \vec{c},\ \vec{D}^{(or)} \parallel (\vec{c} \times \vec{e_z}))$$

$\Rightarrow \vec{D}^{(ex)} \parallel \vec{E}^{(ex)}$, $\vec{D}^{(or)} \parallel \vec{E}^{(or)}$, but $\vec{D} \nparallel \vec{E}$ in general.

as for $\vec{c}$ and $-(\vec{c} \times \vec{e_z})$ eigenvectors.

we should decompose $\vec{E}$ as

$$\vec{E} = \vec{E}^{(or)} + \vec{E}^{ex} = a\vec{c} + b \cdot (-\vec{c} \times \vec{e_z})$$

since $\vec{E} = E_0 \vec{e_x}$

$$\vec{E} \cdot \vec{c} = E_0 \cos\alpha = a$$

$$\vec{E} \cdot (-\vec{c} \times \vec{e_z}) = -E_0 \vec{e_x} \cdot (\vec{c} \times \vec{e_z}) = -E_0 \sin\alpha = b$$

✓

$$\therefore \vec{E} = E_0 [\cos\alpha\, \vec{c} - \sin\alpha (-\vec{c} \times \vec{e_z})] = \underbrace{E_0 \cos\alpha\, \vec{c}}_{\vec{E}^{(ex)}} - \underbrace{E_0 \sin\alpha (-\vec{c} \times \vec{e_z})}_{\vec{E}^{(or)}}$$

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$\nabla\times\vec{E} = i\omega\mu\vec{H}$

$\nabla\times\vec{H} = -i\omega\varepsilon_0\hat{\varepsilon}\vec{E}$

$\nabla\times\nabla\times\vec{E} = \frac{\omega^2}{c^2}\hat{\varepsilon}\vec{E} = \frac{\omega^2}{c^2}\begin{bmatrix} n_e^2 & & \\ & n_o^2 & \\ & & n_o^2\end{bmatrix}\begin{bmatrix}E_0\\0\\0\end{bmatrix} = \frac{\omega^2}{c^2}\begin{bmatrix}n_e^2E_0\\0\\0\end{bmatrix}$

$E = E_0 e^{i\frac{\omega}{c}nz}$

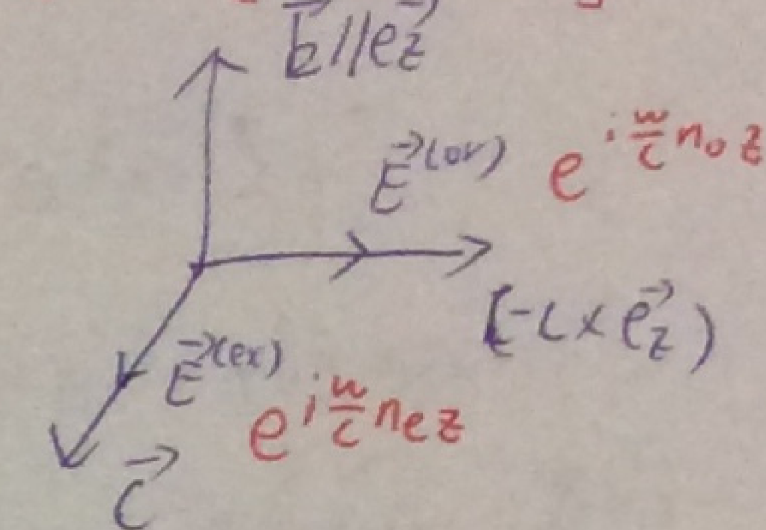
$\nabla\times\nabla\times\begin{bmatrix}E_0e^{i\frac{\omega}{c}nz}\\0\\0\end{bmatrix} = \frac{\omega^2}{c^2}\begin{bmatrix}n_e^2E_0e^{i\frac{\omega}{c}nz}\\0\\0\end{bmatrix}$

Problem 2 Optical Waveplates

a) I have calculated in Problem 1 that

$$\vec{E}^{(ex)} = E_0\cos\alpha\,\vec{c}$$

$$\vec{E}^{(or)} = -E_0\sin\alpha\,(-\vec{c}\times\vec{e}_z)$$



so $\vec{k}^{(ex)} = \frac{\omega}{c}n_e\vec{e}_z = \frac{2\pi}{\lambda}n_e\vec{e}_z$, polarization direction $\vec{c} = [\cos\alpha, \sin\alpha, 0]$

$\vec{k}^{(or)} = \frac{\omega}{c}n_o\vec{e}_z = \frac{2\pi}{\lambda}n_o\vec{e}_z$, polarization direction $-\vec{c}\times\vec{e}_z = [-\sin\alpha, \cos\alpha, 0]$

2

b) $\langle\vec{S}\rangle = \frac{1}{2}\mathrm{Re}[\vec{E}\times\vec{H}^*]$

$\because$ $\nabla\times\vec{E} = i\omega\mu_0\vec{H} \Rightarrow i\vec{k}\times\vec{E} = i\omega\mu_0\vec{H} \Rightarrow \vec{H} = \frac{1}{\omega\mu_0}\vec{k}\times\vec{E}$

$\therefore \langle\vec{S}\rangle = \frac{1}{2}\mathrm{Re}[\vec{E}\times(\frac{1}{\omega\mu_0}\vec{k}\times\vec{E})^*]$

$= \frac{1}{2\omega\mu_0}\mathrm{Re}[\vec{E}\times(\vec{k}\times\vec{E})^*]$

$= \frac{1}{2\omega\mu_0}\mathrm{Re}[\vec{k}^*(\vec{E}\vec{E}^*) - \vec{E}^*(\underbrace{\vec{k}^*\cdot\vec{E}}_{=0})]$ (for $\vec{k}$ is real, so $\vec{k} = \vec{k}^*$)

$= \frac{1}{2\omega\mu_0}\vec{k}|\vec{E}(\omega)|^2$

2

$\Rightarrow \langle\vec{S}\rangle \parallel \vec{k}$

$$\langle\vec{S}\rangle^{(or)} = \frac{1}{2\omega\mu_0}\frac{\omega}{c}n_o|\vec{E}^{(or)}|^2\vec{e}_z = \frac{n_o}{2\mu_0c}|\vec{E}^{(or)}|^2\vec{e}_z$$

$$\langle\vec{S}\rangle^{(ex)} = \frac{1}{2\omega\mu_0}\frac{\omega}{c}n_e|\vec{E}^{(ex)}|^2\vec{e}_z = \frac{n_e}{2\mu_0c}|\vec{E}^{(ex)}|^2\vec{e}_z$$

(c) $\vec{E}(z=0) = E_0(\cos\alpha\,\vec{\imath} - \sin\alpha(-\vec{\imath}\times\vec{e}_z)) = E_0\,\vec{e}_x$

$\vec{E}(z=d) = \vec{E}(z=0)\,e^{ikd}$

$= E_0[\cos\alpha\, e^{i\frac{\omega}{c}n_e d}\,\vec{\imath} - \sin\alpha\, e^{i\frac{\omega}{c}n_o d}(-\vec{\imath}\times\vec{e}_z)]$

since $\vec{\imath} = \cos\alpha\,\vec{e}_x + \sin\alpha\,\vec{e}_y$

$-\vec{\imath}\times\vec{e}_z = -\sin\alpha\,\vec{e}_x + \cos\alpha\,\vec{e}_y$

so, $\vec{E}(z=d) = E_0[\cos^2\alpha\, e^{i\frac{\omega}{c}n_e d}\vec{e}_x + \cos\alpha\sin\alpha\, e^{i\frac{\omega}{c}n_e d}\vec{e}_y + \sin^2\alpha\, e^{i\frac{\omega}{c}n_o d}\vec{e}_x - \cos\alpha\sin\alpha\, e^{i\frac{\omega}{c}n_o d}\vec{e}_y]$

2

$= E_0[(\cos^2\alpha\, e^{i\frac{\omega}{c}n_e d} + \sin^2\alpha\, e^{i\frac{\omega}{c}n_o d})\vec{e}_x + \cos\alpha\sin\alpha(e^{i\frac{\omega}{c}n_e d} - e^{i\frac{\omega}{c}n_o d})\vec{e}_y]$

It means that initially linearly polarized light (in general) elliptically polarized after the anisotropic medium.

(d) $(n_e-n_o)d = \frac{\lambda}{2} = \frac{\pi c}{\omega} \Rightarrow \cancel{d=\frac{\pi c}{\omega(\frac{1}{n_e}-\frac{1}{n_o})}}$ $\quad d = \frac{\pi c}{\omega(n_e-n_o)}$

$\vec{E}(z=d) = E_0[(\cos^2\alpha\, e^{i\pi\frac{n_e}{n_e-n_o}} + \sin^2\alpha\, e^{i\pi\frac{n_o}{n_e-n_o}})\vec{e}_x + \cos\alpha\sin\alpha(e^{i\pi\frac{n_e}{n_e-n_o}} - e^{i\pi\frac{n_o}{n_e-n_o}})\vec{e}_y]$

$= E_0[(\cos^2\alpha\, e^{i\pi\frac{n_e}{n_e-n_o}} + \sin^2\alpha\, e^{i\pi}e^{i\pi\frac{n_e}{n_e-n_o}})\vec{e}_x + \cos\alpha\sin\alpha(e^{i\pi\frac{n_e}{n_e-n_o}} - e^{i\pi}e^{i\pi\frac{n_e}{n_e-n_o}})\vec{e}_y]$

$e^{i\pi}=-1$

$= E_0\, e^{i\pi\frac{n_e}{n_e-n_o}}\cdot[(\cos^2\alpha - \sin^2\alpha)\vec{e}_x + 2\cos\alpha\sin\alpha\,\vec{e}_y]$

$= E_0\, e^{i\pi\frac{n_e}{n_e-n_o}}\cdot[\cos 2\alpha\,\vec{e}_x + \sin 2\alpha\,\vec{e}_y]$

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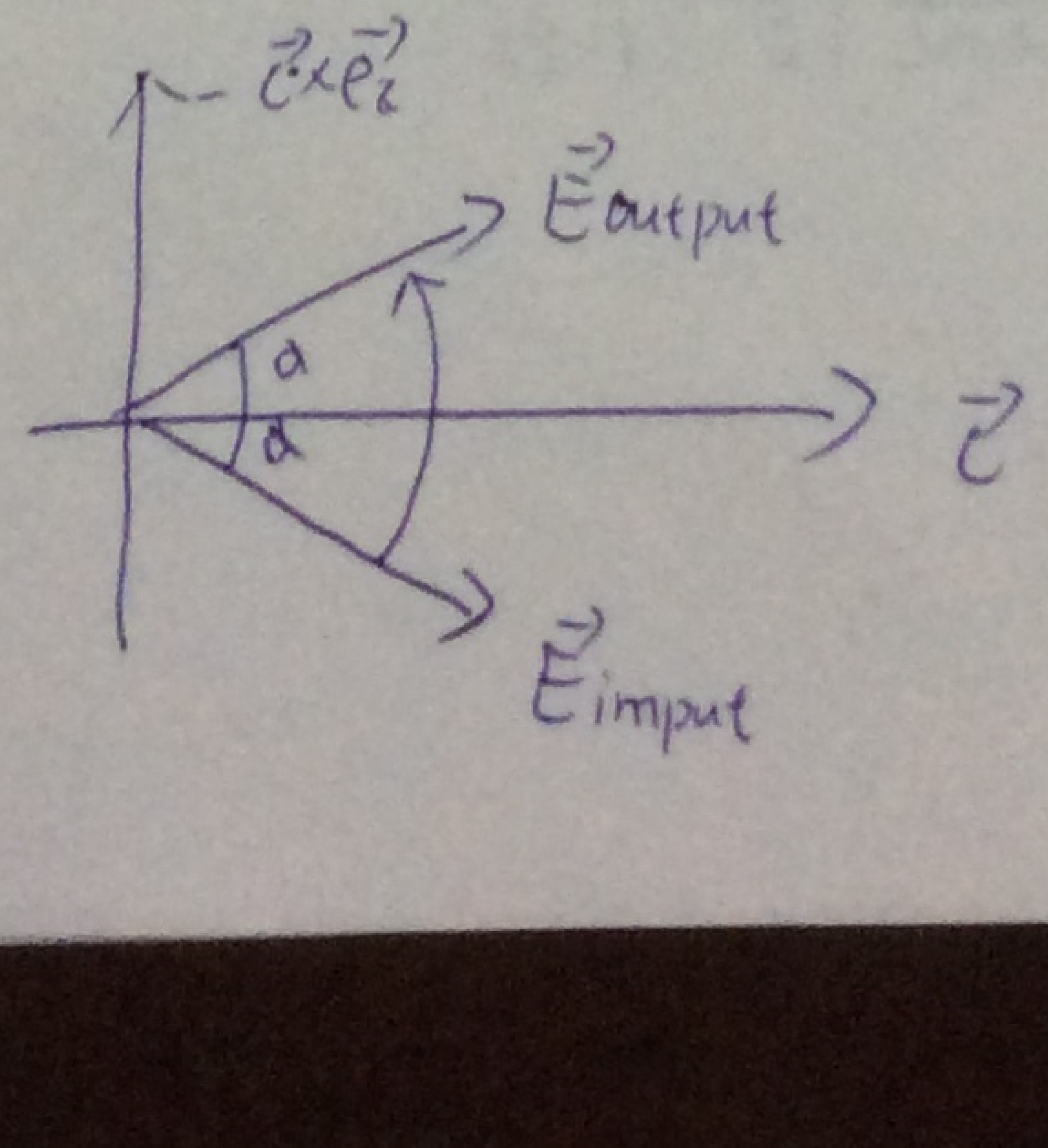
The resulted field is still linealy polarized light, but it rotated angle of 2α compared to the imput field.



Problem 3 - Jones Formalism

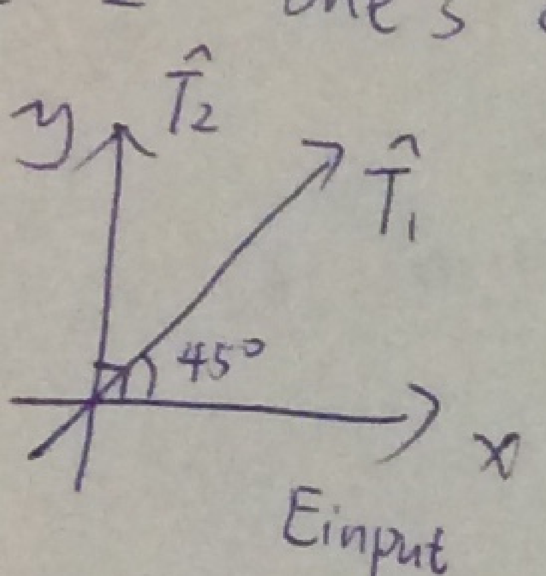
a) $\hat{T} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\hat{T_\theta} = \hat{M}\hat{T}M^{-1}$, where $\hat{M}$ is the change of basis Martix for ratation at angle θ with the x-axis

$\hat{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\hat{M}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ 2

$$\Rightarrow \hat{T_\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$ ✓

b) It is possible if the 1-st polorizer's transmission axis is oriented at 45° and the 2nd one's axis at 90°. with the input field direction. ✓



Let's consider the general case, when angle of $\hat{T_1}$ is θ.

$\hat{T_2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\hat{T_1} = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$, $\vec{E}_{input} = \begin{pmatrix} E_0 \\ 0 \end{pmatrix}$

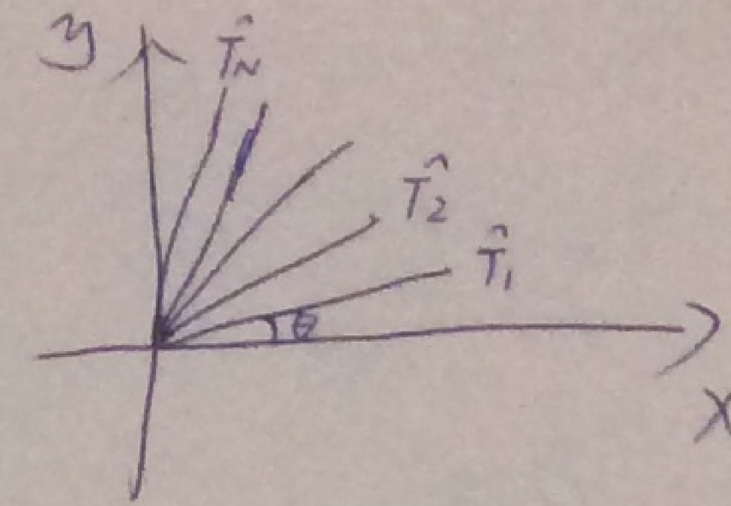
$$\Rightarrow \vec{E}_{output} = \hat{T_2}\hat{T_1}\vec{E}_{input} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}\begin{pmatrix} E_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} E_0\cos^2\theta \\ E_0\sin\theta\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ E_0\sin\theta\cos\theta \end{pmatrix} = E_0\sin\theta\cos\theta\,\vec{e_3} \Rightarrow \vec{E}_{input} \perp \vec{E}_{output}$$

$$\frac{|\vec{E}_{output}|}{|\vec{E}_{input}|} = \frac{E_0\sin\theta\cos\theta}{E_0} = \sin\theta\cos\theta \Rightarrow \text{minimal loss of the system at } \theta = 45°$$ ✓

$$\frac{I_{out}}{I_{in}} = \frac{|\vec{E}_{output}|^2}{|\vec{E}_{input}|^2} = \frac{1}{4}$$ ✓ Loss $= 1 - \frac{1}{4} = 75\%$ ✓

When there are infinitely many linear polarizers.



when angle of $\hat{T}_1$ is θ

$$\vec{E}_1 = \hat{T}_1 \vec{E}_{input} = \begin{pmatrix} E_0 \cos^2\theta \\ E_0 \sin\theta\cos\theta \end{pmatrix}$$

$$\frac{|\vec{E}_1|}{|\vec{E}_{input}|} = \frac{E_0\sqrt{\cos^4\theta + \sin^2\theta\cos^2\theta}}{E_0} = \cos\theta \checkmark$$

$$\Rightarrow \frac{|\vec{E}_N|}{|\vec{E}_{input}|} = \cos^N\theta \quad \text{and} \quad N\theta = \frac{\pi}{2} \Rightarrow N = \frac{\pi}{2\theta} \checkmark$$

$$\Rightarrow \lim_{\theta\to 0} \frac{|\vec{E}_N|}{|\vec{E}_{input}|} = \lim_{\theta\to 0} \cos^{\frac{\pi}{2\theta}}\theta \overset{\theta = x}{=\!=} \lim_{x\to 0} \cos^{\frac{\pi}{2x}} x = \lim_{x\to 0} (1+\cos x - 1)^{\frac{\pi}{2x}} = \lim_{x\to 0} (1+\cos x - 1)^{\frac{1}{\cos x - 1}\cdot\frac{\cos x - 1}{x}\cdot\frac{\pi}{2}}$$

$$= e^{\lim_{x\to 0}\frac{\pi}{2}\frac{\cos x - 1}{x}} \overset{\cos x - 1 \sim -\frac{x^2}{2}}{=\!=\!=} e^{\lim_{x\to 0}\left(\frac{\pi}{2}\left(-\frac{x}{2}\right)\right)} = e^0 = 1 \checkmark$$

(2)

$$\Rightarrow \frac{|\vec{E}_{N\to\infty}|}{|\vec{E}_{input}|} = 1$$ $\Rightarrow$ It means if we use infinitely many polarizers, the loss would be zero. ✓

c) $\vec{\vec{T}} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

$$\vec{\vec{T}}\vec{J} = \lambda\vec{J} \Rightarrow (\vec{\vec{T}} - \lambda I)\vec{J} = 0 \Rightarrow \det|\vec{\vec{T}} - \lambda I| = 0$$

$$\begin{vmatrix} 1-2\lambda & i \\ -i & 1-2\lambda \end{vmatrix} = 0 \Rightarrow (1-2\lambda)^2 - 1 = 0 \Rightarrow \lambda_{1/2} = 0, 1$$

$$\vec{\vec{T}}\vec{J} = \lambda_{1/2}\vec{J}_{1/2} \qquad \vec{J} = \begin{bmatrix} J_x \\ J_y \end{bmatrix}$$

$\lambda = 0$ $\begin{cases} J_x + iJ_y = 0 \\ -iJ_x + J_y = 0 \end{cases}$ $\quad J_x = 1,\ J_y = i \quad$ after normalization $\vec{J}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \rightarrow$ right circular polarizer

$\lambda = 1$ $\begin{cases} -J_x + iJ_y = 0 \\ -iJ_x - J_y = 0 \end{cases}$ $\quad J_x = 1,\ J_y = -i \quad$ after normalization $\vec{J}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \rightarrow$ left circular polarizer ✓

(1)