Name: ______________________ Date of birth: ______________ Student ID No.: ____________

FRIEDRICH SCHILLER UNIVERSITY JENA
Institute of Applied Physics
Prof. Dr. Thomas Pertsch

WS 2020/2021

Final Exam

FUNDAMENTALS OF MODERN OPTICS

to be written on February 15, 10:00 – 12:00

Problem 1: Maxwell's Equations — 2 + 3 + 2 = 7 Points

a) Given is a medium that is linear, *inhomogeneous*, non-magnetic, dispersive, isotropic, and without sources and currents. Write down the Maxwell's equations (MWE) in *frequency domain* using only the following fields $\mathbf{E}(\mathbf{r},\omega)$, $\mathbf{H}(\mathbf{r},\omega)$, $\varepsilon(\mathbf{r},\omega)$.

b) From the MWE in a), derive the wave equation for the electric field $\mathbf{E}(\mathbf{r},\omega)$ in this medium.

c) Under which conditions will the electric field $\mathbf{E}(\mathbf{r},\omega)$ be divergence-free such that:

$$\nabla \cdot \mathbf{E}(\mathbf{r},\omega) = 0?$$

Rewrite the wave equation in the case of a divergence-free electric field. What is the physical consequence for light propagation of $\nabla\varepsilon(\mathbf{r},\omega) \neq 0$ in the above wave equations in part b?

Problem 2: Normal modes and Poynting vector — 2 + 2 + 3 + 2 = 9 Points

A monochromatic plane wave of frequency ω_0 is travelling in a homogeneous and isotropic medium, and it is described by the following complex representation

$$\mathbf{E}(\mathbf{r},\mathbf{t}) = E_0 \exp\left(-\alpha y + \mathrm{i}\beta z\right)\exp\left(-i\omega t\right)\mathbf{e}_x \quad \text{for } y > 0,$$

where α and β describe the propagation of the wave, and E_0 is a real-valued field amplitude.

a) What are the general definitions of homogeneous and evanescent waves based on the wave vector as $\mathbf{k} = \mathbf{k}' + i\mathbf{k}''$? Under what conditions of α and β, is the wave evanescent? Consider that the wave is propagating in the z direction.

b) Calculate the complex magnetic field $\mathbf{H}(\mathbf{r},\mathbf{t})$ corresponding to the given electric field.

c) Write the general expression for the *time-averaged* Poynting vector, and calculate it for the given electromagnetic wave.

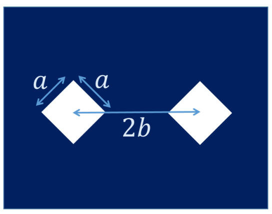
d) Find the real valued magnetic field $\mathbf{H}_r(\mathbf{r},t)$ and the real valued electric field $\mathbf{E}_r(\mathbf{r},t)$ of the above given evanescent wave.

Problem 3: Diffraction — 2 + 2 + 4 = 8 Points

a) Describe the algorithm to propagate an initial field $u(x,y,z=0)$ for a distance of z in homogeneous space by means of a general i) transfer function in Fourier space $H(\alpha,\beta;z)$ and ii) response function in real space $h(x,y,z)$. Give the general formulas in both cases for calculating the propagated field $u(x,y,z)$.

b) Write down the general transfer function of free space beam propagation for a distance of z. From it, derive the paraxial transfer function by using the Taylor expansion. Describe mathematically and physically when this approximation is valid.

c) Consider the below amplitude mask with two identical square holes that are 45°-rotated, have a side length $a = 10\ \mu$m, and are horizontally separated by a distance of $2b$ from their centers where $b = 50\ \mu$m. The mask extends infinitely in the transverse direction. A plane wave of wavelength $\lambda = 500$ nm illuminates the mask with normal incidence.



i) Show that the features of the mask satisfy the paraxiality condition (Fresnel diffraction).

ii) Estimate the minimum propagation distance z from the mask to the observation plane for the paraxial Fraunhofer approximation to be valid.

iii) Calculate the propagated intensity pattern in the paraxial Fraunhofer approximation. *Hint: Remember you can define the coordinate system at your will!*

Problem 4: Pulses **2 + 2 + 2 + 2 + 2 = 10 Points**

a) Write down the Taylor expansion of the frequency dependence of the wavenumber around the central frequency ω_0 under the parabolic approximation, and explain under which condition it is applicable.

b) Explain the physical meaning of coefficients in the above Taylor expansion of the wavenumber. Which part of the spectrum is faster, when $\left.\frac{\partial^2 k}{\partial \omega^2}\right|_{\omega_0} < 0$?

The diffraction of a beam and the dispersion of a pulse during a propagation are analogous so that by substituting respective analogous parameters in beam diffraction equations, one can obtain useful equations that describe the behaviour of a pulse dispersion. For example, spatial frequencies α and β are analogous to the spectral frequency $\overline{\omega}$.

c) By applying analogy to the paraxial Fraunhofer diffraction, write a formula to obtain a pulse's temporal intensity profile after a very large propagation distance of z.

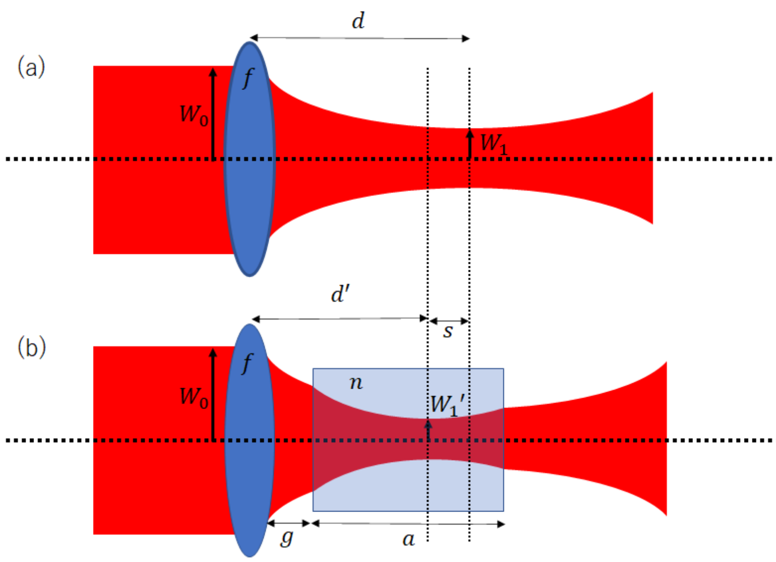
d) Now assume that a dispersive medium can be modeled with a dispersion parameter D_ω. Calculate the pulse's temporal intensity profile after a propagation of very large z through this medium for the following rectangular pulse with a temporal width of T_0

$$\tilde{v}_0(\tau) = \begin{cases} A_0 & 0 \le \tau \le T_0 \\ 0 & \text{elsewhere} \end{cases}.$$

e) Estimate the temporal width of the pulse after the propagation by using the first node of the pulse's temporal intensity profile $|\tilde{v}(\tau, z)|^2$.

Problem 5: Gaussian Beams **1 + 3 + 3 = 7 Points**

A collimated Gaussian beam of a wavelength λ with a waist W_0 (the waist is just in front of the lens) is focused by a lens, as shown in Figure (a). The lens has a focal length f. The Rayleigh length of the beam before the lens, $z_0 = \frac{\pi W_0^2}{\lambda}$, is much larger than f. The focused Gaussian beam after the lens has a waist W_1 at a distance d after the lens.



a) Write down the q-parameter of the input Gaussian beam just before the lens.

b) Consider the lens focusing system shown in Figure (a) and do the following calculations.

   i) Calculate the q-parameter just after the lens and at the beam waist W_1.

   ii) From the above result, calculate W_1 and d exactly.

   iii) Simplify your results by using the fact that $z_0 \gg f$.

c) Now assume that we insert a medium with refractive index n as shown in Figure (b). The length of the medium a is long enough such that the focal spot is in the medium, and we will ignore the backward reflection from the medium surface. When the distance between the lens and the medium is g, calculate the new beam waist W_1' and new waist position d' exactly. Then obtain the shift of the waist position s when $z_0 \gg f$.

Hint: You may use the ABCD matrix of refraction from air to the medium given by $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{pmatrix}$.

Problem 6: Fourier Optics — 1 + 1 + 1 + 2 + 2 = 7 Points

We will consider a 4f-setup in 1 dimension as illustrated in below image, and we want to use it for spatial filtering of the initial field $f(x)$ by placing an amplitude mask $p(x)$ at the Fourier plane. The resulting field $g(x)$ is formed on the image plane.

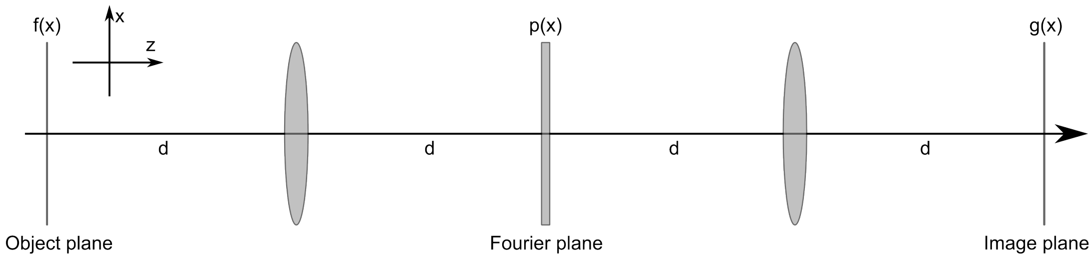
a) If the focal length of the lenses is d, what is the field distribution $(u_-(x, 2d))$ right in front of the amplitude mask $p(x)$.

b) What is the field $(u_+(x, 2d))$ right after the amplitude mask $p(x)$.

c) If $g(x) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$, $\mathrm{G}(\alpha) = \mathrm{FT}[g(x)]$, and $\mathrm{F}(\alpha) = \mathrm{FT}[f(x)]$, what should be the relation between $\mathrm{G}(\alpha)$ and $\mathrm{F}(\alpha)$?

d) To have $g(x) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$, what should be the amplitude mask $p(x)$? Remember, you can use the results from c).

e) If the numerical aperture of the lens is given by $\mathrm{NA} = \sin\theta$ where θ is the half opening angle of the lens, argue how it will affect the formed image on Image plane. What would be the restriction on the maximum spatial frequency of the incident field to be fully resolved with the 4f-setup.

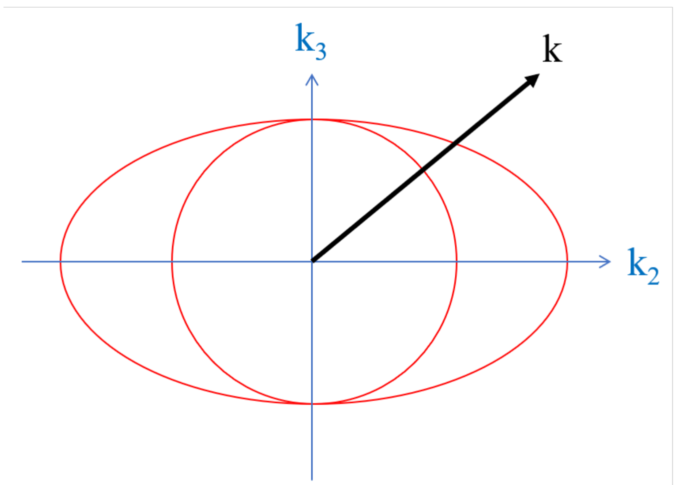


Problem 7: Anisotropy — 1 + 3 + 3 + 2 = 9 Points

a) Describe the optical axis of a crystal. How many optical axis do uniaxial and biaxial crystals have?

b) For a uniaxial crystal and a wave propagating in an arbitrary direction, describe the ordinary and extra ordinary normal modes (polarization and phase velocity).

c) Here you can see a cross section of normal surfaces of a uniaxial crystal. Please draw the direction of **E**, **D**, and **S**, **k**, **B**, and **H** for ordinary and extra ordinary normal modes. Note: If two vectors are perpendicular or parallel to each other, make sure it is clearly indicated in the figures.



d) A wave is propagating in the direction of (1,1,1) in an imaginary biaxial crystal with $\varepsilon_1 = 1$, $\varepsilon_2 = 2$, and $\varepsilon_3 = 3$. Notice that the direction vector is not yet normalized! Using the dispersion relation of the crystals $\left(\sum_i \frac{u_i^2}{[n^2 - \varepsilon_i]} = \frac{1}{n^2}\right)$, find the refractive indices of normal modes n_a and n_b.

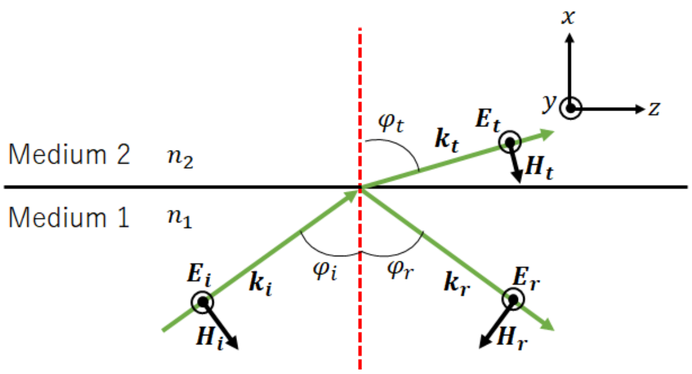
Problem 8: Interfaces **2 + 2 + 1 = 5 Points**

We consider an interface between two different media (non-magnetic, homogeneous, non-dispersive, and isotropic) with refractive indices of n_1 and n_2 where $n_1 > n_2$. A TE-polarized monochromatic plane wave travels from medium 1 to medium 2 at an incident angle of φ_i while the reflected and transmitted angles are φ_r and φ_t, respectively as shown in the figure below.

a) Write down all the existing continuity equations for the electric and magnetic field components of the incident, reflected and transmitted waves.

b) Which component (tangential k_z or normal k_x) of the wave vector is continuous? Use this continuity of the wave vector, and find the relation between φ_t, n_1, n_2, and φ_i.

c) Calculate the critical angle condition for the total internal reflection to occur. What is the relation between n_2, n_1, and the critical angle φ_i?



Problem 1: Maxwell's Equations (v) **2 + 3 + 2 = 7 Points**

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b) From the MWE in a), derive the wave equation for the electric field $\mathbf{E}(\mathbf{r},\omega)$ in this medium.

c) Under which conditions will the electric field $\mathbf{E}(\mathbf{r},\omega)$ be divergence-free such that:

$$\nabla \cdot \mathbf{E}(\mathbf{r},\omega) = 0?$$

$\nabla \cdot D = 0 \rightarrow \nabla \cdot (\varepsilon_0 \varepsilon_r E(r,\omega)) = 0.$

Rewrite the wave equation in the case of a divergence-free electric field. What is the physical consequence for light propagation of $\nabla\varepsilon(\mathbf{r},\omega) \neq 0$ in the above wave equations in part b?

Solution (a)

$\nabla \times E = -\mu_0 \frac{\partial H(r,t)}{\partial t}$ $\qquad \nabla \cdot E = 0$

$\nabla \times H = j(r,t) + \frac{\partial D(r,t)}{\partial t}$ $\qquad \nabla \cdot H = 0$

$= j^0 + \frac{\partial P(r,t)}{\partial t} + \varepsilon_0 \frac{\partial E(r,t)}{\partial t}$

0

$\nabla \times E = i\omega\mu_0 H(r,\omega)$

$\nabla \times H(r,\omega) = -i\omega \varepsilon_0 E(r,\omega)$

$\varepsilon_0 \nabla \cdot (\varepsilon \cdot E) = 0,$

$\nabla \cdot H = 0$

(b)

$\nabla \times \nabla \times E = -\mu_0 \frac{\partial}{\partial t} (\nabla \times H)$

$= -\mu_0 \frac{\partial}{\partial t} \left(\frac{\partial P}{\partial t} + \varepsilon_0 \varepsilon \frac{\partial E}{\partial t} \right) = -\mu_0 \frac{\partial^2}{\partial t^2} P - \varepsilon_0 \mu_0 \varepsilon \frac{\partial^2}{\partial t^2} E.$

$\nabla \times \nabla \times E + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E(r,t) = -\mu_0 \frac{\partial^2}{\partial t^2} P(r,t) \Rightarrow 0$

$\nabla \times \nabla \times E(r,\omega) - \frac{\omega^2}{c^2} E(r,\omega) = \mu_0 \omega^2 P(r,\omega) \Rightarrow 0$

$\nabla \times \nabla \times E = \Delta E(r,\omega) + \frac{\omega^2}{c^2} \vec{E}(r,\omega)\ \vec{E}(r,\omega),$

$\Delta \vec{E}(r,\omega) + \frac{\omega^2}{c^2} \varepsilon(r,\omega)\ E(r,\omega) = 0.$

(c)

$\nabla \cdot E(r,\omega) = 0 \rightarrow \text{grad}\ \varepsilon(r,\omega) = 0 \rightarrow \varepsilon(r,\omega) = \varepsilon(\omega)$

wave eq: $\Delta E(r,\omega) + \frac{\omega^2}{c^2} \varepsilon(r,\omega)\ \vec{E}(r,\omega) = 0.$

Problem 2: Normal modes and Poynting vector **2 + 2 + 3 + 2 = 9 Points**

A monochromatic plane wave of frequency ω_0 is travelling in a homogeneous and isotropic medium, and it is described by the following complex representation

$$\mathbf{E}_c(\mathbf{r},\mathbf{t}) = E_0 \exp(-\alpha y + i\beta z)\exp(-i\omega t)\mathbf{e}_x \quad \text{for } y > 0,$$

where α and β describe the propagation of the wave, and E_0 is a real-valued field amplitude.

a) What are the general definitions of homogeneous and evanescent waves based on the wave vector as $\mathbf{k} = \mathbf{k}' + i\mathbf{k}''$? Under what conditions of α and β, is the wave evanescent? Consider that the wave is propagating in the z direction.

b) Calculate the complex magnetic field $\mathbf{H}(\mathbf{r},\mathbf{t})$ corresponding to the given electric field.

c) Write the general expression for the *time-averaged* Poynting vector, and calculate it for the given electromagnetic wave.

d) Find the real valued magnetic field $\mathbf{H}_r(\mathbf{r},t)$ and the real valued electric field $\mathbf{E}_r(\mathbf{r},t)$ of the above given evanescent wave.

(a) $k = k' + ik''$ $\quad k'r = \text{constant}$: plane of constant phase

$k''r = \text{constant}$: plane of constant amplitude

for ① homogeneous: $k' \parallel k''$

② evanescent : $k' \perp k''$

③ inhomogeneous: other case

$E(r,t) = E(\omega)\exp i(kr - \omega t)$

$E(r,t) = E_0 \exp i(i\alpha\frac{y}{z} + \beta)z \exp(-i\omega t)e_x,$

$\alpha, \beta \neq 0$, is the wave equation.

we identify: $k = k' + ik'' = (\beta + i\alpha\frac{y}{z})e_z,$

In condition: $\beta \perp \alpha\frac{y}{z}$, the wave is evanescent.

(b) $E(r,t) = E_0(-\alpha y + i\beta z)\exp(-i\omega t)e_x \quad y>0.$

$\nabla\times E_r = -\mu_0\frac{\partial H_r}{\partial t} \qquad \nabla\times E_c = i\omega\mu_0 H_c$

$$\nabla\times E_r = \begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_0\exp(-\alpha y + i\beta z)\exp(-i\omega t)e_x & 0 & 0 \end{vmatrix}$$

$= E_0 \cdot (i\beta)\, e^{(-\alpha y + i\beta z)} e^{-i\omega t}\, \hat{y} - E_0\alpha\, e^{(-\alpha y + i\beta z)} e^{-i\omega t}\, \hat{z}$

$\int \nabla\times E\, dt = \int E_0 \cdot (i\beta)\, e^{(-\alpha y + i\beta z)} e^{-i\omega t}\, dt - \int E_0(\alpha)\, e^{(-\alpha y + i\beta z)} e^{-i\omega t}\, dt$

$= -\frac{E_0}{i\omega} i\beta\, e^{(-\alpha y + i\beta z)} e^{-i\omega t}\, \hat{y} + \frac{E_0}{i\omega}\alpha\, e^{(-\alpha y + i\beta z)} e^{-i\omega t}\, \hat{z}$

$= -E_0\frac{\beta}{\omega} e^{(-\alpha y + i\beta z)} e^{-i\omega t}\, \hat{y} + \frac{E_0\alpha}{i\omega} e^{(-\alpha y + i\beta z)} e^{-i\omega t}\, \hat{z}$

$H_r = -\frac{1}{\mu_0}\int \nabla\times E_r\, dt$

$= \frac{\beta}{\mu_0\omega} E_0\, e^{(-\alpha y + i\beta z)} e^{-i\omega t}\, \hat{y} - \frac{\alpha}{i\omega\mu_0} E_0\, e^{(-\alpha y + i\beta z)} e^{-i\omega t}\, \hat{z}$

$\cos(\beta z - \omega t) = \frac{e^{i(\beta z - \omega t)} + e^{i(\omega t - \beta z)}}{2}$

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$

$E_r = E_c + E_c^{*}$

(c) General expression:

$$S_r(r,t) = E_r(r,t) \times H_r(r,t)$$

$$\langle S_r(r,t)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} S_r(r,t)\,dt$$

$$S_r(x,t) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_x & 0 & 0 \\ 0 & H_y & H_z \end{vmatrix} = \hat{x}(0) - \hat{y}(E_x H_z) + \hat{z}(E_x H_y)$$

$$= -\hat{y}(E_x H_z) + \hat{z}(E_x H_y)$$

① $\nabla \times E_r = -\mu_0 \frac{dH_r}{dt}$

$$\bar{E}_r = E_0\, e^{(-\alpha y + i\beta z)}\, e^{(-i\omega t)}\, \hat{x}$$

$$H_{ry} = \frac{\beta}{\mu_0 \omega} E_0\, e^{(-\alpha y - i\beta z)}\, e^{+i\omega t}\, \hat{y}$$

② $\nabla \times E_c = i\omega\mu_0 H_c$

$$H_{rz} = -\frac{\alpha}{i\omega\mu_0} E_0\, e^{(-\alpha y - i\beta z)}\, e^{+i\omega t}\, \hat{z}$$

$$\mathrm{Re}\langle S \rangle = \hat{y}\left(E_0^2 \left(-\frac{\alpha}{i\omega\mu_0}\right) e^{(-2\alpha y)} \right) - \hat{z}\left(E_0^2 \left(\frac{\beta}{\mu_0\omega}\right) e^{(-2\alpha y)} \right)$$

$$= \cancel{-\hat{y}\, E_0^2 \frac{\alpha}{i\omega\mu_0} e^{-2\alpha y}} - \hat{z}\, \frac{1}{2} E_0^2 \frac{\beta}{\mu_0\omega} e^{-2\alpha y}.$$

(d) complex magnetic field

$$\vec{E}_r(r,t) = \frac{1}{2}\left[E_c(r) e^{-i\omega t} + E_c^*(r) e^{i\omega t}\right]$$

$$\vec{H}_r(r,t) = \frac{1}{2}\left[H_c(r) e^{-i\omega t} + H_c^*(r) e^{i\omega t}\right]$$

$$\nabla \times \vec{E}_c = i\omega\mu_0 \vec{H}_c$$

$$E_c(r,t) = E_0 \exp(-\alpha y + i\beta z)\, e^{-i\omega t}\, \hat{e}_x$$

$$\vec{E}_r(r,t) = \frac{1}{2}\left[E_0 \exp(-iqy + i\beta z)\, e^{-i\omega t} + E_0 \exp(iqy - i\beta z)\, e^{i\omega t} \right] e_x$$

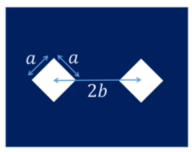
$$\boxed{\nabla \times \vec{E}_r = -\mu_0 \frac{\partial H_r}{\partial t}}$$

Problem 3: Diffraction **2 + 2 + 4 = 8 Points**

a) Describe the algorithm to propagate an initial field $u(x, y, z = 0)$ for a distance of z in homogeneous space by means of a general i) transfer function in Fourier space $H(\alpha, \beta; z)$ and ii) response function in real space $h(x, y, z)$. Give the general formulas in both cases for calculating the propagated field $u(x, y, z)$.

b) Write down the general transfer function of free space beam propagation for a distance of z. From it, derive the paraxial transfer function by using the Taylor expansion. Describe mathematically and physically when this approximation is valid.

c) Consider the below amplitude mask with two identical square holes that are 45°-rotated, have a side length $a = 10\ \mu$m, and are horizontally separated by a distance of $2b$ from their centers where $b = 50\ \mu$m. The mask extends infinitely in the transverse direction. A plane wave of wavelength $\lambda = 500$ nm illuminates the mask with normal incidence.



i) Show that the features of the mask satisfy the paraxiality condition (Fresnel diffraction).

ii) Estimate the minimum propagation distance z from the mask to the observation plane for the paraxial Fraunhofer approximation to be valid.

iii) Calculate the propagated intensity pattern in the paraxial Fraunhofer approximation. *Hint: Remember you can define the coordinate system at your will!*

COP ① For transfer function in Fourier space $H(\alpha,\beta,z)$

$$H(\alpha,\beta,z) = e^{i\gamma z} = \exp\left[i\sqrt{k^2-\alpha^2-\beta^2}\, z\right]$$

$$u(x,y,z) = \iint_{-\infty}^{\infty} \exp\left(i\sqrt{k^2-\alpha^2-\beta^2}\, z\right) U_0(\alpha,\beta) \exp[i(\alpha x+\beta y)]\, d\alpha\, d\beta.$$

② real space response function $u(x,y,z) = \iint_{-\infty}^{\infty} h(x-x', y-y', z)\, u(x', y', z=0)\, dx'\, dy'$

$$h(x,y,z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} H(\alpha,\beta,z) \exp[i(\alpha x+\beta y)]\, d\alpha\, d\beta.$$

1b) $\alpha^2+\beta^2 \ll k^2$ p58-59.

$$\gamma = \sqrt{k^2-\alpha^2-\beta^2} = k\sqrt{1-\frac{\alpha^2+\beta^2}{k^2}}$$

$\to$ Taylor expansion $\to k\left(1-\frac{\alpha^2+\beta^2}{2k^2}\right) = k - \frac{\alpha^2+\beta^2}{2k}.$

$$H_F(\alpha,\beta,z) = \exp(ikz)\exp\left(-i\frac{\alpha^2+\beta^2}{2k} z\right)$$

valid condition: mathematically: $U_0(\alpha,\beta) \neq 0$ only for $\alpha^2+\beta^2 \ll k^2$

physically: spatial frequency spectrum is narrow

(c) $|\Delta x|, |\Delta y| > 10\frac{\lambda}{n}$ $\qquad |\Delta x| = |\Delta y| \approx 10\,\mu m > 10\lambda = 5\,\mu m$

$$\Delta x = \Delta y = \frac{2a}{\sqrt{2}} = \frac{2\sqrt{2}}{2}a = \sqrt{2}a = 10\sqrt{2}\,\mu m$$

$$\lambda = 500\times10^{-9}\,m = 500\times10^{-3}\,\mu m$$

$\rightarrow \Delta x, \Delta y > 10\frac{\lambda}{n}$, satisfy the condition.

(d) $N_F = \frac{2b}{\lambda}\cdot\frac{2b}{z_B} \le 0.1$ $\left(N_F = \frac{a}{\lambda}\,\frac{a}{z_B}\right)$

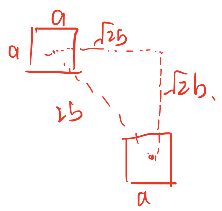
a is the largest size of the aperture.

$b = 50\,\mu m$:

$$\frac{100\,\mu m\cdot 100\,\mu m}{500\,nm\cdot z_B} \le 0.1 \qquad z_B \ge \frac{4b^2}{0.1\lambda} = 2\ cm$$

(e) $I = |u(x,y,z_B)|^2$ $\qquad u_{FR} = -\frac{2\pi i}{\lambda z_B}\exp(ikz_B)\,U_0\!\left(k\frac{x}{z_B}, k\frac{y}{z_B}\right)\exp\left[\frac{ik}{2z_B}(x^2+y^2)\right]$

$$u(x,y) = u_0(x,y) + u_0(x-\sqrt{2}b,\, y+\sqrt{2}b)$$



$$u_0(x,y) = \begin{cases} 1 & |x|<\frac{a}{2},\ |y|<\frac{a}{2} \\ 0 & \text{elsewhere.} \end{cases}$$

$$U_0\!\left(\frac{kx}{z_B},\frac{ky}{z_B}\right) \sim \mathrm{sinc}\!\left(\frac{kx}{z_B}\cdot\frac{a}{2}\right)\cdot \mathrm{sinc}\!\left(\frac{ky}{z_B}\cdot\frac{a}{2}\right)$$

$$FT[u_0(x-\sqrt{2}b,\, y-\sqrt{2}b)] = U_0\!\left(\frac{kx}{z_B},\frac{ky}{z_B}\right)\cdot\exp\left(-i\frac{kx}{z_B}\sqrt{2}b\right)\exp\left(i\frac{ky}{z_B}\sqrt{2}b\right)$$

$$\sim \mathrm{sinc}\!\left(\frac{kx}{z_B}\,\frac{a}{2}\right)\mathrm{sinc}\!\left(\frac{ky}{z_B}\cdot\frac{a}{2}\right)\cdot\exp\left(-i\frac{kx}{z_B}\sqrt{2}b\right)\exp\left(i\frac{ky}{z_B}\sqrt{2}b\right)$$

$$U_{FR} = \mathrm{sinc}\!\left(\frac{kax}{2z_B}\right)\mathrm{sinc}\!\left(\frac{kay}{2z_B}\right)\left(1+\exp\left(-i\frac{(x-y)}{z_B}\sqrt{2}kb\right)\right)$$

$$I = U_{FR}\cdot U_{FR}^*$$

Problem 4: Pulses **2 + 2 + 2 + 2 + 2 = 10 Points**

a) Write down the Taylor expansion of the frequency dependence of the wavenumber around the central frequency ω_0 under the parabolic approximation, and explain under which condition it is applicable.

b) Explain the physical meaning of coefficients in the above Taylor expansion of the wavenumber. Which part of the spectrum is faster, when $\left.\frac{\partial^2 k}{\partial \omega^2}\right|_{\omega_0} < 0$?

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c) By applying analogy to the paraxial Fraunhofer diffraction, write a formula to obtain a pulse's temporal intensity profile after a very large propagation distance of z.

d) Now assume that a dispersive medium can be modeled with a dispersion parameter D_ω. Calculate the pulse's temporal intensity profile after a propagation of very large z through this medium for the following rectangular pulse with a temporal width of T_0

$$\tilde{v}_0(\tau) = \begin{cases} A_0 & 0 \le \tau \le T_0 \\ 0 & \text{elsewhere} \end{cases}.$$

e) Estimate the $\boxed{\text{temporal width}}$ of the pulse after the propagation by using the first node of the pulse's temporal intensity profile $|\tilde{v}(\tau, z)|^2$.

(a) l_p 3- P85: $k(\omega) = k(\omega_0) + \frac{dk}{d\omega}|(\omega-\omega_0) + \frac{1}{2}\left.\frac{d^2k}{d\omega^2}\right|_{\omega_0}(\omega-\omega_0)^2$

condition: For narrowly excited spectrum
(long pulse respect to optical cycle)

(b) ① $k(\omega_0) = k_0 \to \frac{1}{v_{ph}} = \frac{k_0}{\omega_0} = \frac{n(\omega_0)}{c}$ ② $\frac{dk}{d\omega} = \frac{1}{v_g}$

when $\frac{d^2k}{d\omega^2} < 0$, $D < 0$ ↓ phase velocity. Group velocity

$D = \frac{d}{d\omega}\left(\frac{1}{v_g}\right) = -\frac{1}{v_g^2}\frac{dv_g}{d\omega} \to \frac{dv_g}{d\omega} > 0$ when $\omega = \omega_0$ → spectrum with higher frequency is faster.

③ $\left.\frac{d^2k}{d\omega^2}\right|_{\omega_0} = D$ Group velocity dispersion (GVD)

(c) $H_p = \exp\left[iz\frac{D}{2}\bar{\omega}^2\right]$

P 90,

$$v(z,\tau) = \int_{-\infty}^{\infty} V_0(\bar{\omega}) \exp\left[iz\frac{D}{2}\bar{\omega}^2\right]\exp[-i\bar{\omega}\tau]\,d\bar{\omega}$$

(d) D_ω T_0 $\hat{v}_0(\tau) = \begin{cases} A_0 & 0 \le \tau \le T_0 \\ 0 & \text{elsewhere} \end{cases}$

$$\bar{V}_0(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{v}_0(\tau)\cdot\exp(i\omega\tau)\,d\tau$$

$$= \frac{1}{2\pi}\int_0^{T_0} A_0 \exp(i\omega\tau)\,d\tau$$

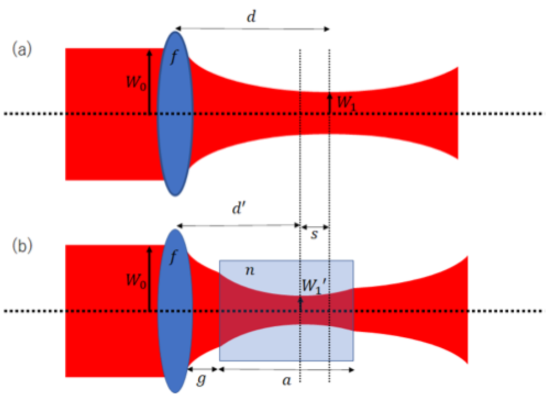
$$= \frac{T_0 A_0}{\pi}\exp\left(i\omega\frac{T_0}{2}\right)\mathrm{sinc}\left(\frac{\omega T_0}{2}\right)$$

$$|\bar{v}(\tau z)|^2 = \frac{2\pi}{Dz}\,\frac{A T_0}{\pi}\,\mathrm{sinc}^2\left(\frac{\tau T_0}{2Dz}\right)$$

(e)

Problem 5: Gaussian Beams **1 + 3 + 3 = 7 Points**

A collimated Gaussian beam of a wavelength λ with a waist W_0 (the waist is just in front of the lens) is focused by a lens, as shown in Figure (a). The lens has a focal length f. The Rayleigh length of the beam before the lens, $z_0 = \frac{\pi W_0^2}{\lambda}$ is much larger than f. The focused Gaussian beam after the lens has a waist W_1 at a distance d after the lens.



a) Write down the q-parameter of the input Gaussian beam just before the lens.
b) Consider the lens focusing system shown in Figure (a) and do the following calculations.
   i) Calculate the q-parameter just after the lens and at the beam waist W_1.
   ii) From the above result, calculate W_1 and d exactly.
   iii) Simplify your results by using the fact that $z_0 \gg f$.

$q(z) = z - iz_0$

c) Now assume that we insert a medium with refractive index n as shown in Figure (b). The length of the medium a is long enough such that the focal spot is in the medium, and we will ignore the backward reflection from the medium surface. When the distance between the lens and the medium is g, calculate the new beam waist W_1' and new waist position d' exactly. Then obtain the shift of the waist position s when $z_0 \gg f$.

Hint: You may use the ABCD matrix of refraction from air to the medium given by $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{pmatrix}$.

(a) $z_0 = -\frac{\pi W_0^2}{\lambda}$ $\quad \frac{1}{q(z)} = \frac{1}{R(z)} + i\frac{\lambda}{\pi W^2} = \frac{1}{z - iz_0}$

$q_0 = i\frac{\pi W_0^2}{\lambda} = -iz_0$

(b) ① For q just after the lens

$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$ so $q_1 = \frac{Aq_0 + B}{Cq_0 + D} = \frac{q_0}{-\frac{q_0}{f} + 1}$

$= \frac{-iz_0}{\frac{iz_0}{f} + 1} = \frac{-iz_0\left(i\frac{z_0}{f} - 1\right)}{\left(\frac{iz_0}{f} + 1\right)\left(\frac{iz_0}{f} - 1\right)} = \frac{\frac{z_0^2}{f} + iz_0}{-\frac{z_0^2}{f^2} - 1}$

$= -\frac{\frac{z_0^2}{f}}{\frac{z_0^2}{f^2} + 1} - i\frac{z_0}{\frac{z_0^2}{f^2} + 1}$

For q_2 at waist at w_1 $= -\frac{z_0^2 f}{z_0^2+f^2} - i\frac{z_0 f^2}{z_0+f^2}$

$q_2 = q_1 + d$

If we want to let it at waist, it should be purely imaginary part, therefore $q_2 = -iz_1 = -i\frac{z_0 f^2}{z_0+f^2}$

② $d = \frac{z_0^2 f}{z_0^2+f^2}$ $\quad q_2 = -iz_1 = -i\frac{\pi w_0^2}{\lambda} = -i\frac{z_0 f^2}{z_0+f^2}$

$$\frac{\pi w_0^2}{\lambda} = \frac{z_0 f^2}{z_0^2+f^2} \qquad W_0 = \sqrt{\frac{f^2 z_0 \lambda}{(z_0+f^2)\pi}}$$

③ For $z_0 \gg f$, $d \approx f \rightarrow W_1 = \sqrt{\frac{\lambda}{\pi}\frac{f^2}{z_0}}$

$$= \sqrt{\frac{\lambda}{\pi}\frac{f^2\lambda}{\pi w_0^2}} = \frac{\lambda f}{\pi w_0}$$

(C) The beam enter the medium

$$M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}\begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} 1, & g \\ -\frac{1}{f}, & -\frac{g}{f}+1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{bmatrix}$$

$$q_3 = \frac{Aq_0+B}{Cq_0+D} = \frac{q_0 + \frac{g}{n}}{-\frac{q_0}{f}+\frac{1}{n}-\frac{g}{fn}} \qquad = \begin{bmatrix} 1, & \frac{g}{n} \\ -\frac{1}{f}, & -\frac{g}{fn}+\frac{1}{n} \end{bmatrix}$$

$q_0 = -iz_0$

$$= \frac{\frac{q_0 n+g}{n}}{\frac{f-nq_0-g}{fn}} = \frac{fq_0 n+fg}{f-nq_0-g}.$$

$$q_3 = \frac{-iz_0 fn+fg}{f+inz_0-g} = \frac{(-iz_0 fn+fg)(f-g-inz_0)}{(f-g)^2-(inz_0)^2}$$

$$= \frac{-iz_0 f^2 n + iz_0 fgn - z_0^2 fn^2 + f^2 g - fg^2 - inz_0 fg}{f^2-2fg+g^2+n^2z_0^2}$$

$$= i\frac{z_0 fgn - z_0 f^2 n - nz_0 fg}{\quad} + \frac{f^2 g - fg^2 - z_0^2 fn^2}{\quad}$$

$$f^2-2fg+g^2+n^2z_0^2 \qquad \overline{f^2-2fg+g^2+n^2z_0^2}$$

$$= i\frac{-z_0 f^2 n}{f^2-2fg+g^2+n^2z_0^2} + \frac{f^2g-fg^2-z_0^2fn^2}{f^2-2fg+g^2+n^2z_0^2}$$

In order to let it at waist, → pure imaginary part,

$$q_4 = q_3 + a \quad \rightarrow \quad a = -\frac{f^2g-fg^2-z_0^2fn^2}{f^2-2fg+g^2+n^2z_0^2}$$

$$d' = a+g = \frac{z_0^2fn^2+fg^2-f^2g+f^2g-2fg^2+g^3+n^2z_0^2g}{f^2-2fg+g^2+n^2z_0^2}$$

$$= \frac{z_0^2n^2(f+g)-fg^2+g^3}{f^2-2fg+g^2+n^2z_0^2} \qquad s = d-d'$$

$$q_4 = -iz_3 = -i\frac{z_0f^2n}{f^2-2fg+g^2+n^2z_0^2} = -i\frac{\pi w_0^2}{\lambda}$$

$$\frac{z_0f^2n}{f^2-2fg+g^2+n^2z_0^2} = \frac{\pi w_0^2}{\lambda} \qquad w_0 = \sqrt{\frac{z_0f^2n\lambda}{\pi(f^2-2fg+g^2+n^2z_0^2)}}$$

Problem 6: Fourier Optics **1 + 1 + 1 + 2 + 2 = 7 Points**

We will consider a 4f-setup in 1 dimension as illustrated in below image, and we want to use it for spatial filtering of the initial field $f(x)$ by placing an amplitude mask $p(x)$ at the Fourier plane. The resulting field $g(x)$ is formed on the image plane.

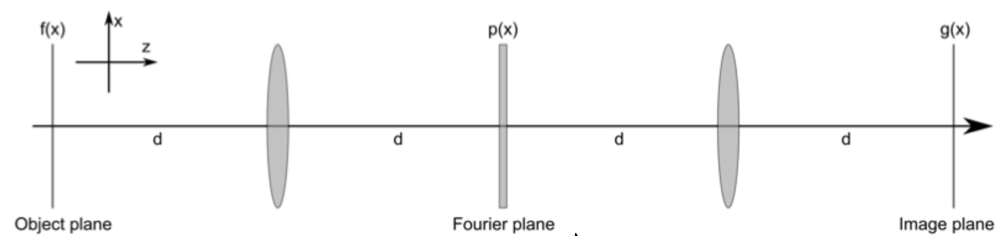
a) If the focal length of the lenses is d, what is the field distribution ($u_{-}(x, 2d)$) right in front of the amplitude mask $p(x)$.

b) What is the field ($u_{+}(x, 2d)$) right after the amplitude mask $p(x)$.

c) If $g(x) = \frac{d^2 f}{dx^2}$, $G(\alpha) = FT[g(x)]$, and $F(\alpha) = FT[f(x)]$, what should be the relation between $G(\alpha)$ and $F(\alpha)$?

d) To have $g(x) = \frac{d^2 f}{dx^2}$, what should be the amplitude mask $p(x)$? Remember, you can use the results from c).

e) If the numerical aperture of the lens is given by $NA = \sin\theta$ where θ is the half opening angle of the lens, argue how it will affect the formed image on Image plane. What would be the restriction on the maximum spatial frequency of the incident field to be fully resolved with the 4f-setup.



(a) $u^{-}(x, 2d) = -i\frac{(2\pi)}{\lambda f_1} e^{2} xp(2ikf_1) U_0(\frac{-kx}{d})$

(b) $u^{+}(x) = u^{-}(x)\cdot p(x) = -i\frac{(2\pi)^2}{\lambda d}\exp(2ikd) U_0(\frac{kx}{d})\cdot p(x)$

(c) $g(x) = -\frac{(2\pi)^2}{\lambda^2 f_1 f_2}\exp(2ik(f_1+f_2))\, FT[F(\frac{kx}{f_1})p(x)](\frac{-kx}{f_2})$

$G(\alpha) = FT(\frac{d^2 f}{dx^2}) = \frac{1}{2\pi}\int \exp(-i\alpha x)\frac{d^2 f}{dx^2}dx$ $\qquad \frac{d}{dx} = -i\alpha$

$= -\alpha^2\, FT(f(x)) = -\alpha^2 F(\alpha)$? $\qquad \frac{d^2}{dx^2} = -\alpha^2$,

(d) $G(\alpha) = -\alpha^2 F(\alpha)$

$g(x) \sim \int_{-\infty}^{\infty} p(\frac{f}{k}\alpha, \frac{f}{k}\beta) U_0(\alpha,\beta)\exp[i(\alpha x+\beta y)]\,d\alpha\, d\beta$,

$= \int_{-\infty}^{\infty} p(\frac{f}{k}\alpha)\, U_0(\alpha)\exp(i\alpha\frac{d\alpha}{k})\,d\alpha$.

$G(\alpha) \sim p(\frac{d}{k}\alpha)\, F(\alpha)$,

$G(\alpha) = -\alpha^2 F(\alpha) \rightarrow p(\frac{d}{k}\alpha) = -\alpha^2$

$x = \frac{d\alpha}{k}$ $\qquad p(x) = -\frac{k^2}{d^2}x^2$.

(e) $\Delta r = \dfrac{1.22\lambda f}{D}$, $\dfrac{f}{D} = \dfrac{1}{2\sin\theta}$ $2f\sin\theta = D$.

$NA = \sin\theta$.

$= \dfrac{1.22\lambda}{2\sin\theta} = 0.61\dfrac{\lambda}{\sin\theta}$

$f = \dfrac{1}{\Delta r} = \dfrac{\sin\theta}{0.61\lambda}$

It will filter high frequency spectrum.

1 2

Problem 7: Anisotropy

7 ? 1 + 3 + 3 + 2 = 9 Points

a) Describe the optical axis of a crystal. How many optical axis do uniaxial and biaxial crystals have?

b) For a uniaxial crystal and a wave propagating in an arbitrary direction, describe the ordinary and extra ordinary normal modes (polarization and phase velocity).

c) Here you can see a cross section of normal surfaces of ~~a~~ uniaxial crystal. Please draw the direction of **E**, **D**, and **S**, **k**, **B**, and **H** for ordinary and extra ordinary normal modes. Note: If two vectors are perpendicular or parallel to each other, make sure it is clearly indicated in the figures.

① For ordinary

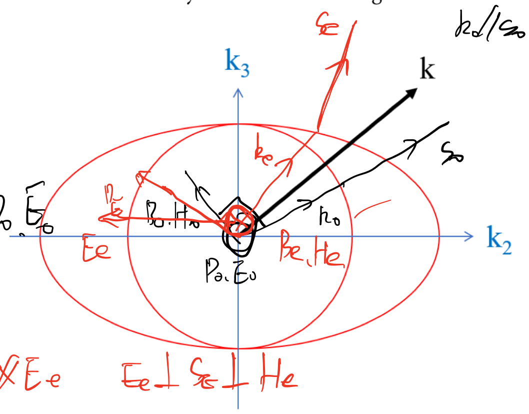
$B_0 \parallel H_0$

$D_0, E_0 \perp k_0, S_0 \perp D_0, E_0$

② For extraordinary

$k_e \not\parallel S_e$

$D_E \perp k_E$



$D_E \not\parallel E_e$ $E_e \perp S_E \perp H_e$

d) A wave is propagating in the direction of (1,1,1) in an imaginary biaxial crystal with $\varepsilon_1 = 1$, $\varepsilon_2 = 2$, and $\varepsilon_3 = 3$. Notice that the direction vector is not yet normalized! Using the dispersion relation of the crystals $\left(\sum_i \frac{u_i^2}{[n^2-\varepsilon_i]} = \frac{1}{n^2}\right)$, find the refractive indices of normal modes n_a and n_b.

Solution: (a) uniaxial have 1 optical axis. Optical axis is the

biaxial have 2 optical axis, direction along which $n_a = n_b$,

(b)

① ordinary wave: n_o independent of propagation direction.

$D_{(or)}$ is $\perp$ z axis and k, $k_{or} = k_0 \cdot n_0$

and do not interact with E_e $D \perp k$. $D \parallel E$.

$V_{ph} = \frac{c}{n_0} = \frac{c \cdot \omega}{c \cdot k} = \frac{\omega}{k}$ $\frac{1}{V_{ph}} = \frac{k}{\omega}$ $D, E \perp$ optical axis.

② extraordinary wave

k_n, n_b depends on propagation direction, $D \perp k$ $D \not\parallel E$.

$D_{(eo)}$ polarized $\perp$ k and $D_{(or)}$) velocity depends on direction

(d) $\sum_i \frac{u_i^2}{[n^2 - \varepsilon_i]} = \frac{1}{n^2}$ $\varepsilon(1, 2, 3)$ $\bar{u} = \frac{u}{\sqrt{3}} = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$

$$\frac{\frac{1}{3}}{n^2 - 1} + \frac{\frac{1}{3}}{n^2 - 2} + \frac{\frac{1}{3}}{n^2 - 3} = \frac{1}{n^2}$$

$$n^2\left[(n^2-2)(n^2-3)+(n^2-1)(n^2-3)+(n^2-1)(n^2-2)\right]=3(n^2-1)(n^2-2)(n^2-3),$$

$$x(x-2)(x-3)+x(x-1)(x-3)+x(x-1)(x-2)=3(x-1)(x-2)(x-3)$$

$$x(x^2-2x-3x+6)+x(x^2-x-3x+3)+x(x^2-x-2x+2)=3(x-1)(x^2-2x-3x+6)$$

$$x(x^2-5x+6)+x(x^2-4x+3)+x(x^2-3x+2)=3(x-1)(x^2-5x+6)$$

$$6x^2-22x+18=0$$

$$x=\frac{-b\pm\sqrt{D}}{2a}=\frac{22\pm\sqrt{484-432}}{12},=\frac{22\pm\sqrt{52}}{12}.$$

$$n_a=\sqrt{\frac{11+\sqrt{13}}{6}}\qquad n_b=\sqrt{\frac{11-\sqrt{13}}{6}}=\frac{11\pm\sqrt{13}}{6}.$$

Problem 8: Interfaces **2 + 2 + 1 = 5 Points**

We consider an interface between two different media (non-magnetic, homogeneous, non-dispersive, and isotropic) with refractive indices of n_1 and n_2 where $n_1 > n_2$. A TE-polarized monochromatic plane wave travels from medium 1 to medium 2 at an incident angle of φ_i while the reflected and transmitted angles are φ_r and φ_t, respectively as shown in the figure below.

a) Write down all the existing continuity equations for the electric and magnetic field components of the incident, reflected and transmitted waves.

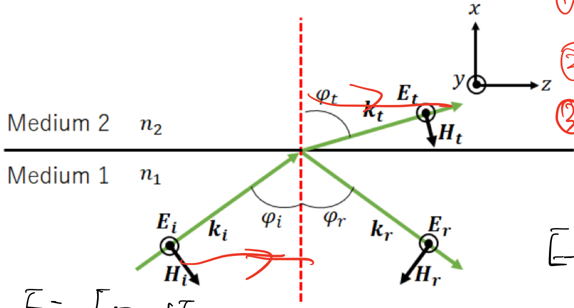
b) Which component (tangential k_z or normal k_x) of the wave vector is continuous? Use this continuity of the wave vector, and find the relation between φ_t, n_1, n_2, and φ_i.

c) Calculate the critical angle condition for the total internal reflection to occur. What is the relation between n_2, n_1, and the critical angle φ_i?

① $k_i \sin\varphi_i = k_r \sin\varphi_r = k_t \sin\varphi_t$.

② $H_i \cos\varphi_i = H_r \cos\varphi_r = H_t \cos\varphi_t$.

③.



$E = E_{TM} + E_{TE}$.

$E_{TE} = \begin{pmatrix} 0 \\ E_y \\ 0 \end{pmatrix}$ $H_{TE} = \begin{pmatrix} H_x \\ 0 \\ H_z \end{pmatrix}$

$E_{TM} = \begin{pmatrix} E_x \\ 0 \\ E_z \end{pmatrix}$ $H_{TM} = \begin{pmatrix} 0 \\ H_y \\ 0 \end{pmatrix}$

Solution: (a)

For TE: E_y, H_z continuous

For TM E_z, H_y continuous.

$E(x,z,t) = E_{TE}(x)\exp[i(k_z z - \omega t)] + E_{TM}(x)\exp[i(k_z z - \omega t)]$

(b) k_z,

k_z is continuous

$k_z = n_1 k_0 \sin\varphi_i$

$k_z = n_2 k_0 \sin\varphi_t$

$n_1 \sin\varphi_i = n_2 \sin\varphi_t$

(c) Total internal reflection

$\sin\varphi_i = \frac{n_2}{n_1}$