Fundamentals of Modern Optics
series 2
27.10.2014

to be returned on 03.11.2014, at the beginning of the lecture

Problem 1 - Vector analysis (3+1* points)

Prove the following vector identities (all quantities are functions of $\mathbf{r}$, **Bold** variables are vectors, non-bold variables are scalars.):

a) $\nabla \times \nabla \times \mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$
(Some people also use the notation $\triangle \mathbf{a}$ instead of $\nabla^2 \mathbf{a}$, for the so called Laplace operator.)

b) $\nabla \times (\alpha \mathbf{a}) = \alpha \nabla \times \mathbf{a} - \mathbf{a} \times \nabla \alpha$

c) $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$

(*Hint:* As it might have occurred to you already, you can obviously prove each relation by brute-force, simply writing each side using all the Cartesian vector components and crossing out similar terms. While that is perfectly acceptable here, we encourage you to use the so called *Einstein summation notation*, that makes use of the *Levi-Civita symbol*, ϵ_{ijk}, to simplify complicated summations. Look it up online! and try to make use of the identities of the Levi-Civita symbol (which you do not need to prove). To encourage you even more, one bonus point goes to whoever proves at least one of these relations in this way.)

Problem 2 - Stokes' theorem (4 Points)

Inside a cylindrically symmetric conducting wire of radius R the current density $\bar{\mathbf{j}}(\mathbf{r},\omega)$ depends just on the radial coordinate r and may be approximated as

$$\bar{\mathbf{j}}(\mathbf{r},\omega) = \bar{j}(r,\omega)\mathbf{e}_z = j_0 \cosh\left(\frac{r}{\delta}\right)\mathbf{e}_z,$$

with a frequency dependent parameter $\delta = \delta(\omega)$, called the *skin depth*. 趋肤深度 Assume Ohm's law $\bar{\mathbf{j}} = \sigma(\omega)\overline{\mathbf{E}}$ is valid inside the conductor and use Stokes' theorem on Maxwell's equation

$$\nabla \times \overline{\mathbf{H}}(\mathbf{r},\omega) = \bar{\mathbf{j}}(\mathbf{r},\omega) - i\omega\varepsilon_0\varepsilon(\omega)\overline{\mathbf{E}}(\mathbf{r},\omega)$$

to find the magnetic field $\overline{\mathbf{H}}(\mathbf{r},\omega)$ *inside* ($r < R$) the wire.
Something to think about: Try to convince yourself why this method cannot be used to find the magnetic field outside the wire. Would this change for a static field?

Problem 3 - Maxwell's equations (2+1 points)

Consider Maxwell's equations in vacuum for the electric field $\mathbf{E}(\mathbf{r},t)$ and the magnetic field $\mathbf{H}(\mathbf{r},t)$ with sources $\rho(\mathbf{r},t), \mathbf{j}(\mathbf{r},t) \neq 0$.

a) Derive the wave equation for the electric field $\mathbf{E}(\mathbf{r},t)$ for that case.
导出

b) Now assume you want to solve this equation for a monochromatic field $\overline{\mathbf{E}}(\mathbf{r},\omega)$. What equation does this field have to fulfil?

Problem 4 - Polarization and Symmetry (3 points)

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In the source-free case, the general wave equation for the electric field in a dispersionless dielectric medium of relative permitivity $\epsilon(\mathbf{r})$ reads as

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$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, t) + \frac{\epsilon(\mathbf{r})}{c^2}\frac{\partial^2 \mathbf{E}(\mathbf{r}, t)}{\partial t^2} = 0.$$

Show that this equation splits into two independent orthogonal polarization contributions *if* there is a translational symmetry, e.g. in y-direction, or in other words, if we are dealing with a 2D system. (*Hint:* Think of what happens to the y-derivatives in that case!)

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Problem 5 - Heaviside Function (4+2* points)

The Heaviside function,

单位阶跃函数

$$\Theta(t) = \begin{cases} 0 & ,t < 0 \\ 1 & ,t > 0 \end{cases}$$

is of huge interest in physics and engineering, and knowing its Fourier transform can sometimes become handy (e.g. when you try to prove the properties of the Kramers-Kronig relation in your "Structure of Matter" course). Attempting to directly calculate the Fourier transform

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$$\overline{\Theta}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Theta(t) e^{i\omega t} dt$$

fails, as the anti-derivative $e^{i\omega t}/i\omega$ does not converge at the upper integration limit. However, the Fourier transform does exist and is given by

$$\overline{\Theta}(\omega) = \frac{1}{2\pi}\left(\mathrm{P}\frac{i}{\omega} + \pi\delta(\omega)\right).$$

Here, P stands for Cauchy's principal value. That is a function $f(x)$ may be undefined or unbounded in one point, but the integral across this point may nevertheless exist. Its value is defined by

$$\mathrm{P}\int_a^b f(x)dx = \lim_{\varepsilon \to 0}\left(\int_a^{c-\varepsilon} f(x)dx + \int_{c-\varepsilon}^{b} f(x)dx\right),$$

where c is the point at which f is not defined.

a) Show that the inverse Fourier transform of $\overline{\Theta}(\omega)$ as given above indeed leads to the Heaviside function. Make use of the relation $\int_0^\infty \frac{\sin(ax)}{x}dx = \frac{\pi}{2}$ for $a > 0$.

b*) Try to prove $\int_0^\infty \frac{\sin(ax)}{x}dx = \frac{\pi}{2}$ for $a > 0$. You will need complex integration, involving Cauchy's integral theorem and Jordan's lemma. Look it up online!

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Problem 1. - Vector analysis

a) $\vec{\nabla}\times\vec{\nabla}\times\vec{a} = \vec{\nabla}(\nabla\cdot\vec{a}) - \vec{\nabla}^2\vec{a}$

proof: $\vec{b} = \vec{\nabla}\times\vec{a} = \begin{vmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ \partial_x & \partial_y & \partial_z \\ a_x & a_y & a_z \end{vmatrix} = (\partial_y a_z - \partial_z a_y)\vec{e_x} + (\partial_z a_x - \partial_x a_z)\vec{e_y} + (\partial_x a_y - \partial_y a_x)\vec{e_z}$

$(\vec{\nabla}\times\vec{b})_x = \partial_y b_z - \partial_z b_y$

$= \partial_y\partial_x a_y - \partial_y^2 a_x + \partial_z\partial_x a_z - \partial_z^2 a_x + \partial_x\partial_x a_x - \partial_x^2 a_x$

$= \underbrace{\partial_x(\partial_x a_x + \partial_y a_y + \partial_z a_z)}_{\nabla\vec{a}} \underbrace{- \partial_x^2 a_x - \partial_y^2 a_x - \partial_z^2 a_x}_{-\Delta a_x}$

$= [\vec{\nabla}(\vec{\nabla}\cdot\vec{a}) - \Delta\vec{a}]_x$ (x- components)

for the y. z components, we have.

$(\vec{\nabla}\times\vec{b})_y = [\vec{\nabla}(\vec{\nabla}\cdot\vec{a}) - \Delta\vec{a}]_y$ $(\vec{\nabla}\times\vec{b})_z = [\vec{\nabla}(\vec{\nabla}\cdot\vec{a}) - \Delta\vec{a}]_z$

So, $\vec{\nabla}\times\vec{\nabla}\times\vec{a} = \vec{\nabla}(\vec{\nabla}\cdot\vec{a}) - \vec{\nabla}^2\vec{a}$

b) $\vec{\nabla}\times(\alpha\vec{a}) = \alpha\vec{\nabla}\times\vec{a} - \vec{a}\times\nabla\alpha$

proof: $(\vec{\nabla}\times\vec{a})^i \overset{det}{=\!=} \varepsilon^{ijk}\frac{\partial}{\partial x_j}a_k$ where ε^{ijk} - Levi-Civita symbol.

$\vec{\nabla}\times(\alpha\vec{a})^i \overset{det}{=\!=} \varepsilon^{ijk}\frac{\partial}{\partial x_j}(\alpha a_k) = \varepsilon^{ijk}(\alpha\frac{\partial a_k}{\partial x_j} + a_k\frac{\partial\alpha}{\partial x_j})$

$= \alpha(\varepsilon^{ijk}\frac{\partial a_k}{\partial x_j}) + \varepsilon^{ijk}\frac{\partial\alpha}{\partial x_j}a_k = \alpha(\vec{\nabla}\times\vec{a})^i + \varepsilon^{ijk}\frac{\partial\alpha}{\partial x_j}a_k = \alpha(\vec{\nabla}\times\vec{a})^i + (\vec{\nabla}\alpha\times\vec{a})^i$

$= \alpha(\vec{\nabla}\times\vec{a}) - (\vec{a}\times\vec{\nabla}\alpha)^i$ $i = 1, 2, 3$

So, $\vec{\nabla}\times(\alpha\vec{a}) = \alpha(\vec{\nabla}\times\vec{a}) - (\vec{a}\times\vec{\nabla}\alpha)$

another method: $\nabla\times(\alpha\vec{a})_x = \frac{\partial(\alpha\vec{a})_z}{\partial y} - \frac{\partial(\alpha\vec{a})_y}{\partial z} = \frac{\partial\alpha}{\partial y}\vec{a_z} + \alpha\cdot\frac{\partial\vec{a_z}}{\partial y} - \frac{\partial\alpha}{\partial z}\vec{a_y} - \alpha\cdot\frac{\partial\vec{a_y}}{\partial z}$

$= \alpha\cdot(\frac{\partial\vec{a_z}}{\partial y} - \frac{\partial\vec{a_y}}{\partial z}) + (\frac{\partial\alpha}{\partial y}\vec{a_z} - \frac{\partial\alpha}{\partial z}\vec{a_y})$

$= \alpha(\nabla\times\vec{a})_x + (\nabla\alpha\times\vec{a})_x = \alpha(\nabla\times\vec{a})_x - (\vec{a}\times\nabla\alpha)_x$

c) $\vec{\nabla}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{\nabla}\times\vec{a}) - \vec{a}\cdot(\vec{\nabla}\times\vec{b})$

proof: $\vec{\nabla}\cdot(\vec{a}\times\vec{b}) = \frac{\partial}{\partial x^i}\varepsilon^{ijk}a_j b_k = \varepsilon^{ijk}a_j\frac{\partial b_k}{\partial x^i} + \varepsilon^{ijk}b_k\frac{\partial a_j}{\partial x_i}$

$= -\varepsilon^{jik}a_j\frac{\partial b_k}{\partial x_i} + b_k\,\varepsilon^{kij}\frac{\partial}{\partial x_i}a_j = -a_j\left(\varepsilon^{jik}\frac{\partial}{\partial x_i}b_k\right) + b_k\left(\varepsilon^{kij}\frac{\partial a_j}{\partial x_i}\right)$

$= \vec{b}\cdot(\vec{\nabla}\times\vec{a}) - \vec{a}\cdot(\vec{\nabla}\times\vec{b})$ ✓

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Problem 2 Stokes' theorem

$j(\vec{r},w) = \vec{j}(r,w)\,\vec{e_z} = j_0\cosh\left(\frac{r}{\delta}\right)e_z$

$\delta = \delta(w)$ —skin depth.

$\vec{j} = \sigma(w)\vec{E}$

$\nabla\times\vec{H}(\vec{r},w) = \vec{j}(\vec{r},w) - iw\varepsilon_0\varepsilon(w)\vec{E}(\vec{r},w)$

$= j_0\,ch\left(\frac{r}{\delta}\right)\vec{e_z} - iw\varepsilon_0\varepsilon\frac{1}{\sigma}j_0\,ch\left(\frac{r}{\delta}\right)\vec{e_z}$

$= j_0\,ch\left(\frac{r}{\delta}\right)\vec{e_z}\left(1 - iw\varepsilon_0\frac{\varepsilon(w)}{\sigma(w)}\right)$ ✓

Due to Stokes' theorem

$\int(\vec{\nabla}\times\vec{H}(\vec{r},w))\cdot d\vec{S} = \int\left(1 - iw\varepsilon_0\frac{\varepsilon(w)}{\sigma(w)}\right)j_0\,ch\left(\frac{r}{\delta}\right)\vec{e_z}\cdot d\vec{S} = \oint\vec{H}\cdot d\vec{l} = 2\pi r\vec{H}(\vec{r},w)$ ✓

$\cancel{\Rightarrow\left(1 - iw\varepsilon_0\frac{\varepsilon(w)}{\sigma(w)}\right)j_0\int_0^r ch\left(\frac{r'}{\delta}\right)r'dr'\int_0^{2\pi}d\theta}$

$\vec{e_z}\cdot d\vec{S} = r\,dr\,d\varphi$ ✓

$\Rightarrow \left(1 - iw\varepsilon_0\frac{\varepsilon(w)}{\sigma(w)}\right)j_0\int_0^r ch\left(\frac{r'}{\delta}\right)\cdot r'dr'\int_0^{2\pi}d\theta = \pi\left(1 - iw\varepsilon_0\frac{\varepsilon(w)}{\sigma(w)}\right)j_0\left(\delta r'e^{\frac{r'}{\delta}}\Big|_0^r - \delta^2 e^{\frac{r'}{\delta}}\Big|_0^r - \delta r'e^{-\frac{r'}{\delta}}\Big|_0^r - \delta e\right.$

$= 2\pi\left(1 - \frac{iw\varepsilon_0\varepsilon}{\sigma}\right)j_0\left[\delta r\sinh\left(\frac{r}{\delta}\right) - \delta^2\cosh\left(\frac{r}{\delta}\right) + \delta^2\right] = 2\pi r\cdot\vec{H}(\vec{r},w)$

Then, $H(\vec{r},w) = \frac{j_0}{r}\left(1 - iw\varepsilon_0\frac{\varepsilon(w)}{\sigma(w)}\right)\left(\delta^2 - \delta^2 ch\left(\frac{r}{\delta}\right) + \delta r\,sh\left(\frac{r}{\delta}\right)\right)$, for $r<R$ ✓

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roblem 3 Maxwell's equations

a) $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ $\qquad$ $\vec{\nabla}\times\vec{H} = \vec{J}(\vec{r},t) + \frac{\partial\vec{D}}{\partial t}$

$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ $\qquad$ $\nabla\cdot\vec{H} = 0$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}\times\left(\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\frac{\partial}{\partial t}\left[\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + \mu_0\vec{J}\right] = -\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} - \mu_0\frac{\partial\vec{J}}{\partial t}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\nabla\cdot\vec{E}) - (\vec{\nabla}\cdot\vec{\nabla})\vec{E}$$

$$= \frac{\vec{\nabla}\rho}{\varepsilon_0} - \Delta\vec{E}$$

$$\Longrightarrow \quad -\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} - \mu_0\frac{\partial\vec{J}}{\partial t} = \frac{\vec{\nabla}\rho}{\varepsilon_0} - \Delta\vec{E}$$

$\Rightarrow$ wave equation: $\boxed{\Delta\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \frac{\vec{\nabla}\rho}{\varepsilon_0} + \mu_0\frac{\partial\vec{J}}{\partial t}}$ ✓

b) for monochromatic wave: $\vec{E}(\vec{r},t) = \vec{E}(\vec{r})\,e^{-i\omega t}$

$\vec{J}(\vec{r},t) = \vec{J}(\vec{r})\,e^{-i\omega t}$

$\vec{\nabla}\times\vec{E} = i\omega\vec{B}$ $\qquad$ $\vec{\nabla}\times\vec{B} = -\frac{i\omega}{c^2}\vec{E} + \mu_0\vec{J}$

$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ $\qquad$ $\vec{\nabla}\cdot\vec{B} = 0$

$\Rightarrow$ wave equation $\boxed{\left(\Delta + \frac{\omega^2}{c^2}\right)\vec{E} = \frac{\vec{\nabla}\rho}{\varepsilon_0} + i\omega\mu_0\vec{J}}$ $-$

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Problem 4   Polarization and Symmetry

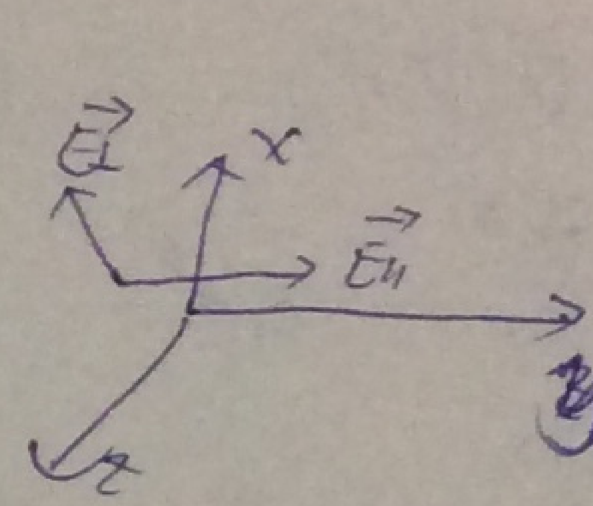
In the source-free case, $\vec{\nabla}\cdot\vec{E}=0$ $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E}(\vec{r},t)+\frac{\varepsilon(\vec{r})}{c^2}\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}=0$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\underbrace{\vec{\nabla}^2\vec{E}}_{\Delta\vec{E}}=\vec{\nabla}\left(\frac{\partial\vec{E}_x}{\partial x}+\frac{\partial\vec{E}_y}{\partial y}+\frac{\partial\vec{E}_z}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\vec{E}$$

if there is a translational symmetry in y-direction.

$$\frac{\partial}{\partial y}\vec{E}(\vec{r},t)=0\ ,\quad \vec{E}=\begin{bmatrix}\vec{E}_\perp=E_x\vec{e}_x+E_z\vec{e}_z\\ \vec{E}_\parallel=E_y\vec{e}_y\end{bmatrix}$$



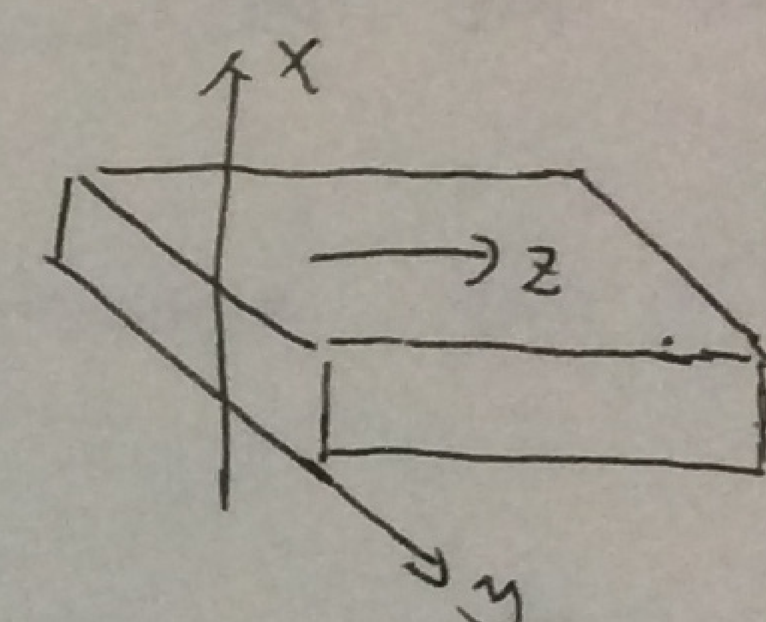
$\vec{E}_\perp\perp\vec{E}_\parallel$

$\vec{E}_\parallel // oy$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\left[\frac{\partial}{\partial x}(\vec{\nabla}_\perp\cdot\vec{E}_\perp)\ ;\ 0;\ \frac{\partial}{\partial z}(\vec{\nabla}_\perp\cdot\vec{E}_\perp)\right]-\Delta_\perp(\vec{E}_\perp+\vec{E}_\parallel),$$ ✓

where $\Delta_\perp=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}$ $\vec{\nabla}_\perp=\left(\frac{\partial}{\partial x},0,\frac{\partial}{\partial z}\right)$ ✓

So, $\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\left(\vec{\nabla}_\perp(\vec{\nabla}_\perp\cdot\vec{E}_\perp)-\Delta_\perp\vec{E}_\perp\right)-\Delta_\perp\vec{E}_\parallel$ ✓



$$\Rightarrow \vec{\nabla}\times\vec{\nabla}\times\vec{E}(\vec{r},t)+\frac{\varepsilon(\vec{r})}{c^2}\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}$$

$$=\left(\left(\vec{\nabla}_\perp(\vec{\nabla}_\perp\cdot\vec{E}_\perp)-\Delta_\perp\vec{E}_\perp\right)-\Delta_\perp\vec{E}_\parallel\right)+\frac{\varepsilon}{c^2}\frac{\partial^2\vec{E}_\perp}{\partial t^2}+\frac{\varepsilon}{c^2}\frac{\partial^2\vec{E}_\parallel}{\partial t^2}=0$$

$\frac{\partial}{\partial y}=0$

$$\Rightarrow \begin{cases}\vec{\nabla}_\perp(\vec{\nabla}_\perp\cdot\vec{E}_\perp)+\frac{\varepsilon(\vec{r})}{c^2}\frac{\partial^2\vec{E}_\perp}{\partial t^2}-\Delta_\perp\vec{E}_\perp=0 \ ✓\\ \frac{\varepsilon(\vec{r})}{c^2}\frac{\partial^2\vec{E}_\parallel}{\partial t^2}-\Delta_\perp\vec{E}_\parallel=0 \ ✓\end{cases}$$

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Problem 5 Heaviside Function

(a) $\theta(\tilde{w}) = \frac{1}{2\pi} \left(i P \frac{i}{w} + \pi \delta(w) \right)$

$$\theta(t) = \int_{-\infty}^{\infty} \theta(w) e^{-iwt} dw = \frac{1}{2\pi} P \int_{-\infty}^{\infty} \frac{1}{w} e^{-iwt} dw + \frac{1}{2} \int_{-\infty}^{\infty} \delta(w) e^{-iwt} dw$$

$$= \frac{1}{2\pi} \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-\varepsilon} \frac{1}{w} e^{-iwt} dw + \int_{\varepsilon}^{+\infty} \frac{1}{w} e^{-iwt} dw \right) + \frac{1}{2}$$

$$= \frac{1}{2\pi} \lim_{\varepsilon \to 0} \left[\int_{\varepsilon}^{\infty} \frac{1}{w} \left(e^{-iwt} - e^{iwt} \right) dw \right] + \frac{1}{2}$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{\sin(wt)}{w} dw + \frac{1}{2}$$

$$= \begin{cases} \frac{1}{\pi} \cdot \frac{\pi}{2} + \frac{1}{2} = 1, & t > 0 \\ \frac{1}{\pi} \cdot 0 + \frac{1}{2} = \frac{1}{2}, & t = 0 \\ \frac{1}{\pi} \cdot \left(- \int_0^{\infty} \frac{\sin w|t|}{w} dw \right) + \frac{1}{2} = -\frac{1}{2} + \frac{1}{2} = 0, & t < 0 \end{cases}$$

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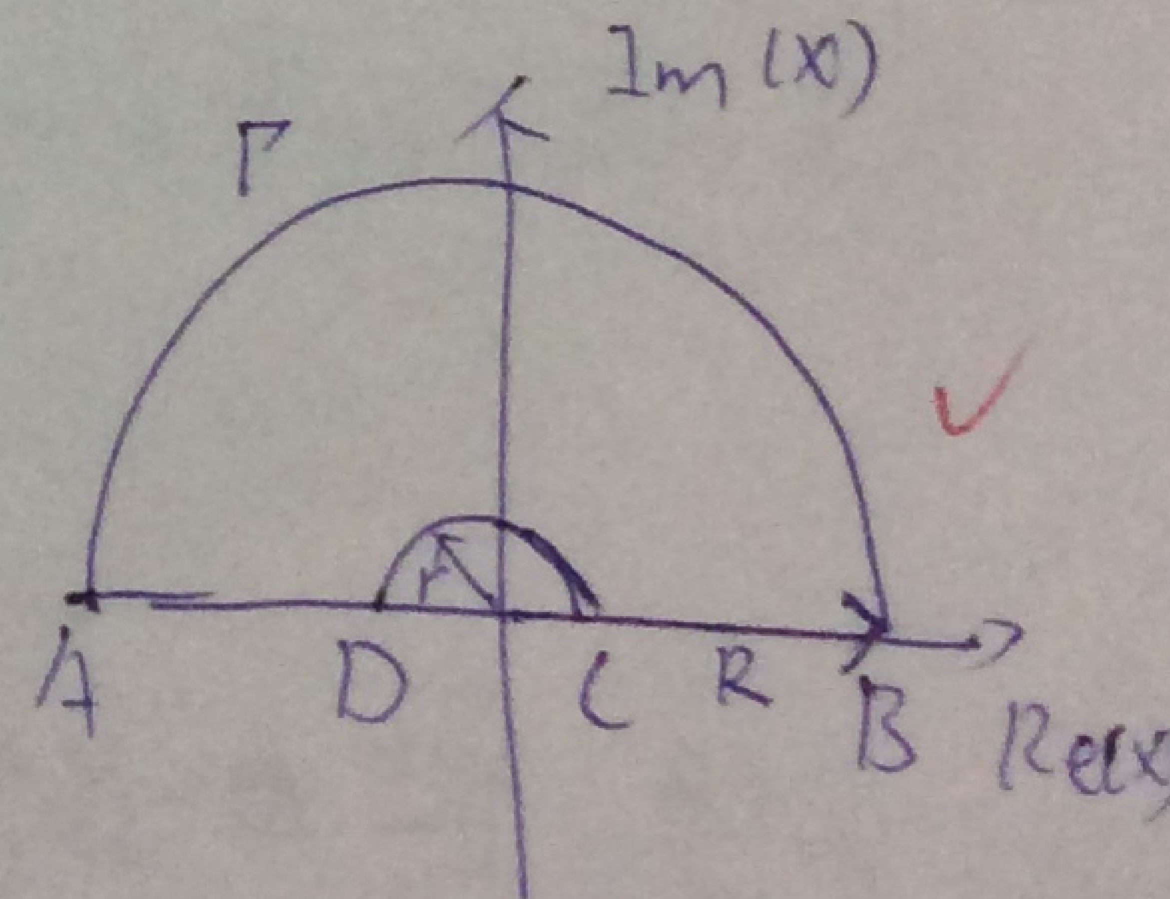
(b) $\int_0^{\infty} \frac{\sin(ax)}{x} dx = \int_{+0}^{\infty} \frac{e^{iax} - e^{-iax}}{2ix} dx$

$$= \int_{+0}^{\infty} \frac{e^{iax}}{2ix} dx + \int_{\infty}^{+0} \frac{e^{-iax}}{2ix} dx$$

$$= \int_{+0}^{\infty} \frac{e^{iax}}{2ix} dx + \int_{-\infty}^{0-} \frac{e^{iax}}{2ix} dx$$

$$= P \int_{-\infty}^{\infty} \frac{e^{iax}}{2ix} dx = I$$



As the right picture, we have

$$\lim_{\substack{r \to 0 \\ R \to \infty}} \oint_{\Gamma} \frac{e^{iax}}{x} dx = \lim_{R \to \infty} \int_{AB} \frac{e^{iax}}{x} dx + \lim_{r \to 0} \int_{\widehat{DC}} \frac{e^{iax}}{x} dx$$

$$+ \lim_{\substack{r \to 0 \\ R \to \infty}} \left(\int_{-R}^{-r} \frac{e^{iax}}{x} dx + \int_{r}^{R} \frac{e^{iax}}{x} dx \right)$$

$$= \lim_{R \to \infty} \int \frac{e^{iax}}{x} dx + \lim_{r \to 0} \int_{\widehat{DC}} \frac{e^{iax}}{x} dx + 2iI = 0$$

$\lim_{R \to \infty} \int \frac{e^{iax}}{x} dx = 0$ (Jordan's lemma)

$$\Rightarrow I = \frac{i}{2} \lim_{r\to 0} \int_{\partial C} \frac{e^{iax}}{x}\,dx = \frac{i}{2} \lim_{r\to 0} \int_{\partial C} \left(\frac{1}{x} + \sum_{n=1}^{\infty} \frac{(ia)^n \cdot x^n}{n!\,x} \right) dx$$

$$\overset{x = re^{i\varphi}}{=} \frac{i}{2} \left[\lim_{r\to 0} \int_{\pi}^{0} \left(\frac{r}{re^{i\varphi}} + \sum_{n=1}^{\infty} \frac{(iar)^n e^{i\varphi n}}{n!\,re^{i\varphi}}\, r \right) de^{i\varphi} \right]$$

$$= \frac{i}{2} \left[\lim_{r\to 0} \left(i \int_{\pi}^{0} d\varphi \right) + \underbrace{\lim_{r\to 0} \sum_{n=1}^{\infty} \frac{(iar)^n\, i}{n!} \int_{\pi}^{0} e^{i\varphi n}\, d\varphi}_{=0} \right]$$

$$= \frac{\pi}{2}$$

So, $I = \int_0^{\infty} \frac{\sin(ax)}{x}\,dx = \frac{\pi}{2}, \quad a>0$ ✓

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