Institute of
Applied Physics
Friedrich-Schiller-Universität Jena

Optical Metrology and Sensing

Lecture 1-2: Introduction

2020-11-03

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Outline

- Errors of measurements
- Sampling theorem

Measurement Errors

- Measurement results:
  Result of measurement = measured value ± uncertainty

- Selection of error types:
  1. material measures
  2. mechanical 'failures' of the system
  3. distortion of Abbe comparator principle
  4. environmental influences
  5. experimenter / observer

- Systematic and random errors:
  Systematic errors: correction of the measured value possible (calibration).
  Can be reproduced and are constant in amount and sign.
  Random errors and systematic errors with unknown sign: uncertainty of measurement

- Propagation of errors:
  1. systematic errors:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

  2. statistical errors:

$$u = \pm\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 (dx)^2 + \left(\frac{\partial f}{\partial y}\right)^2 (dy)^2 + \left(\frac{\partial f}{\partial z}\right)^2 (dz)^2}$$

Ref: R. Kowarschik

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Measurement Errors

- Scattering of values by repeating the measurements
- Distribution of errors:
  Repeatability, width 6σ
- Expected value:
  average for large number of repeated measurements

$$\bar{x} = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} x_j$$

- Variance:

$$\sigma^2 = \frac{1}{N}\sum_{j=1}^{N}\left(x_j - \bar{x}\right)^2$$

- Standard deviation
  root mean square (rms):

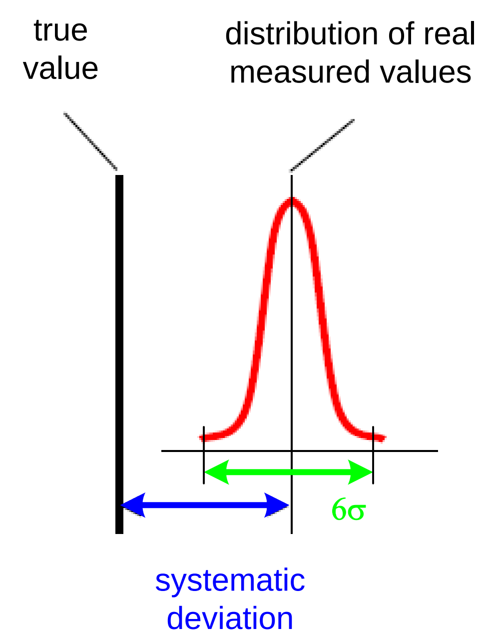
$$\sigma = \sqrt{\frac{1}{N}\sum_{j=1}^{N}\left(x_j - \bar{x}\right)^2}$$

- Higher order moments:
  1. Skewness, kurtosis

$$K = \frac{1}{N}\sum_{j=1}^{N}\left(x_j - \bar{x}\right)^3$$

  2. Peakedness

$$P = \frac{1}{N}\sum_{j=1}^{N}\left(x_j - \bar{x}\right)^4$$

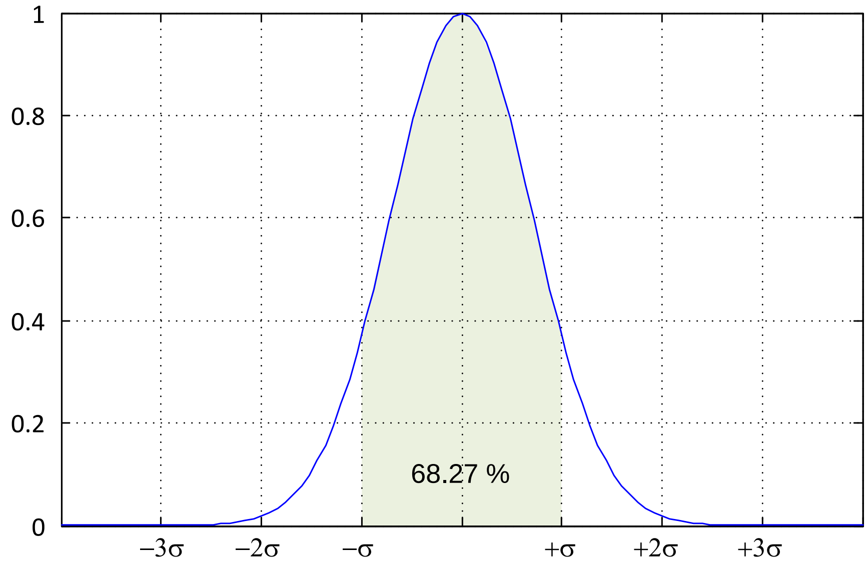


Ref: W. Osten

Distribution of Statistical Errors

- Gaussian or Normal Distribution: $p = e^{-x^2}$

  Within interval $\pm\,\sigma$ are 68.27 % ∀ measured values (statist. certainty: 68.27 %)
  Within interval $\pm\,2\sigma$ are 95.45 % ∀ measured values (statist. certainty: 95.45 %)
  Within interval $\pm\,3\sigma$ are 99.73 % ∀ measured values (statist. certainty: 99.73 %)



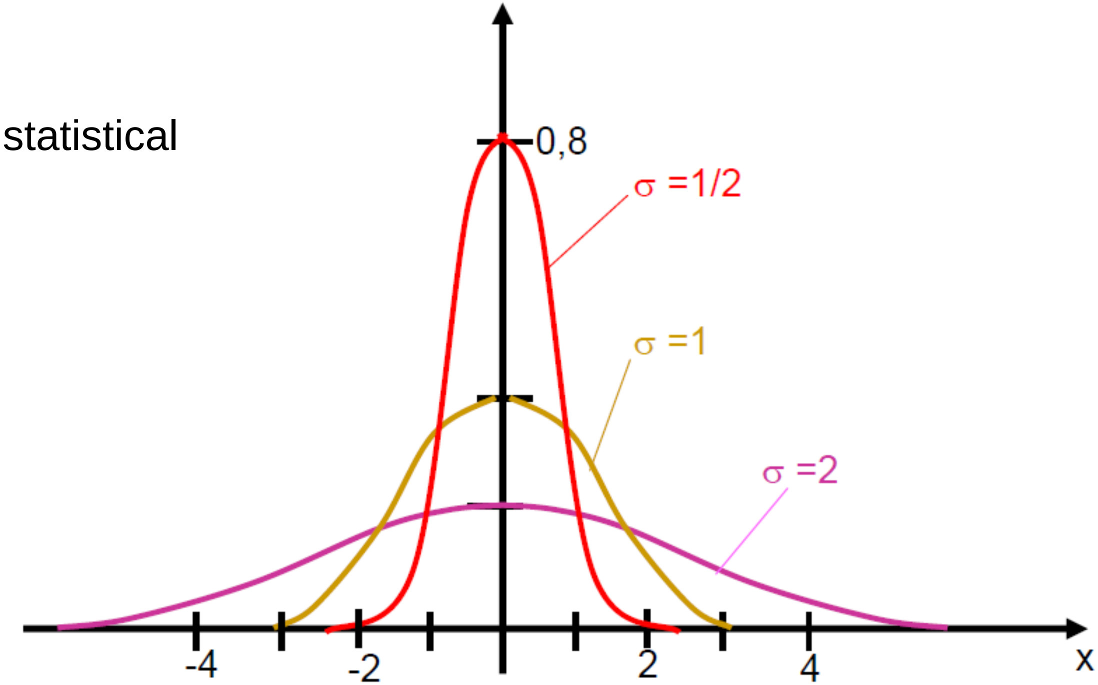
- For a given statistical certainty the corresponding range is called $\pm$ c σ confidence interval (CI)
  The true value lies within the confidence interval for a given statistical certainty if there are no systematic errors

Ref: R. Kowarschik

Distribution of Statistical Errors

- Gaussian or Normal distribution
- Idealized model function for purely statistical influences
- Standardized formulation

$$G(x,\bar{x},\sigma)=\frac{1}{\sqrt{2\pi}\cdot\sigma}\cdot e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$



- Inversion: error function:
  Probability, that the variable t lies within the intervall -z...+z (interval of confidence, integral)
  Examples: p = 0.683 for z=σ
  p = 0.5 gives interval z = 0.6745 σ

$$t=\frac{x-\bar{x}}{\sigma}$$

$$p=erf(z)=\frac{2}{\sqrt{2\pi}}\int_0^z e^{-\frac{t^2}{2}}\,dt$$

Distribution of Statistical Errors

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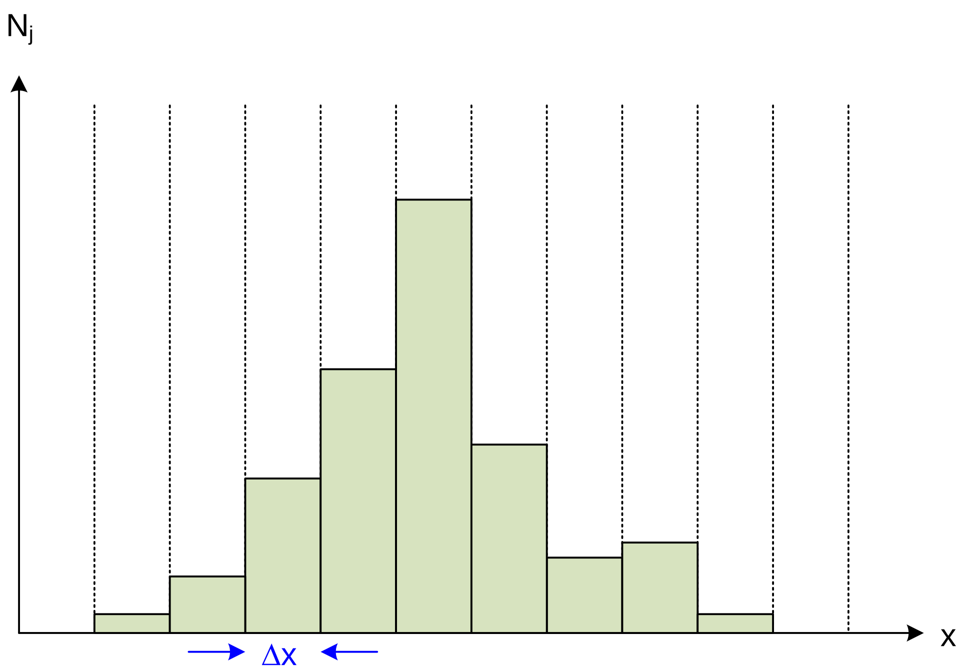
- Probability, that the value is outside the confidence interval (failure):
  a = 1-p
- N measurements:
  Standard deviation of the mean is reduced to $$\bar{\sigma} = \frac{\sigma}{\sqrt{N}}$$
- Confidence range of the mean $$C = K \cdot \frac{\sigma}{\sqrt{N}}$$
  Example: K = 1: confidence +-σ
  a = 0.3174
- Histogram of values for N repeated measurements:
  Number N_j of results inside the same interval



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Linear Trend

- Trend of measurement data as a function of a variable x

$$y_i = m \cdot x_i + b$$

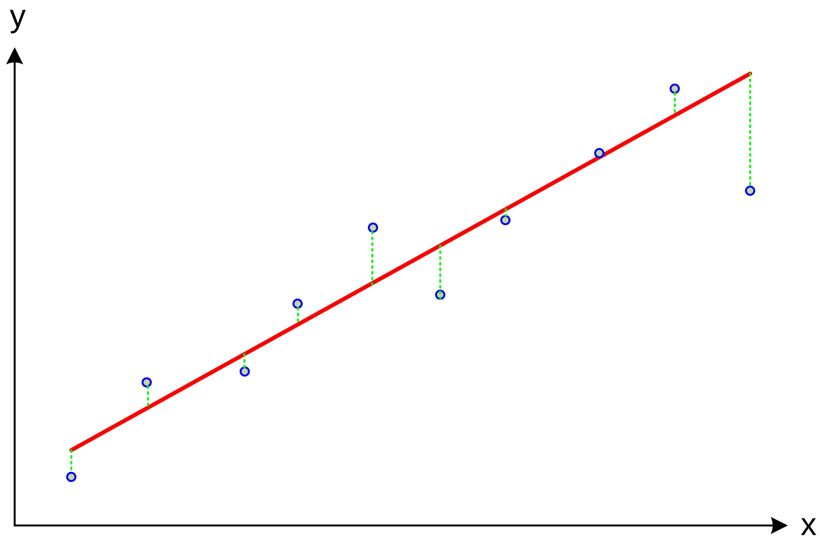
- Calculation of slope (LSQ fit)

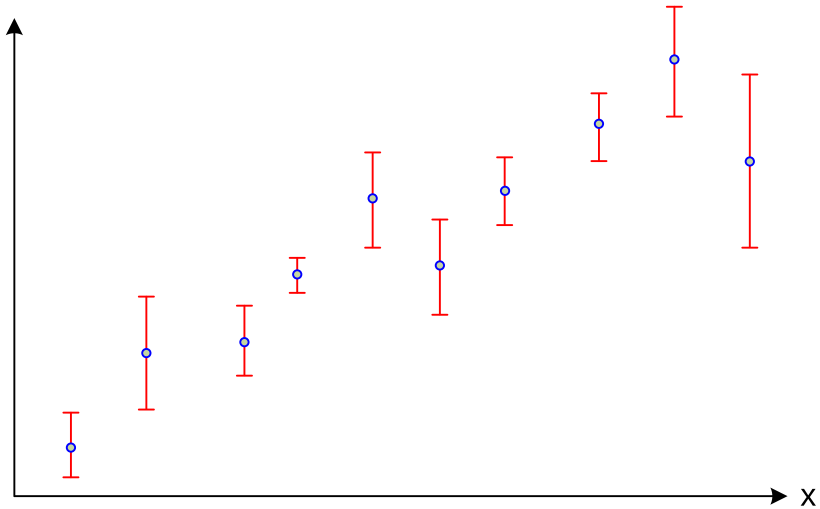
$$m = \frac{\sum_i y_i \cdot (x_i - \overline{x})}{\sum_i (x_i - \overline{x})^2}$$

- Absolute value / constant

$$b = \overline{y} - m \cdot \overline{x}$$

- Special aspects:
  weighting of point inversely to error bars

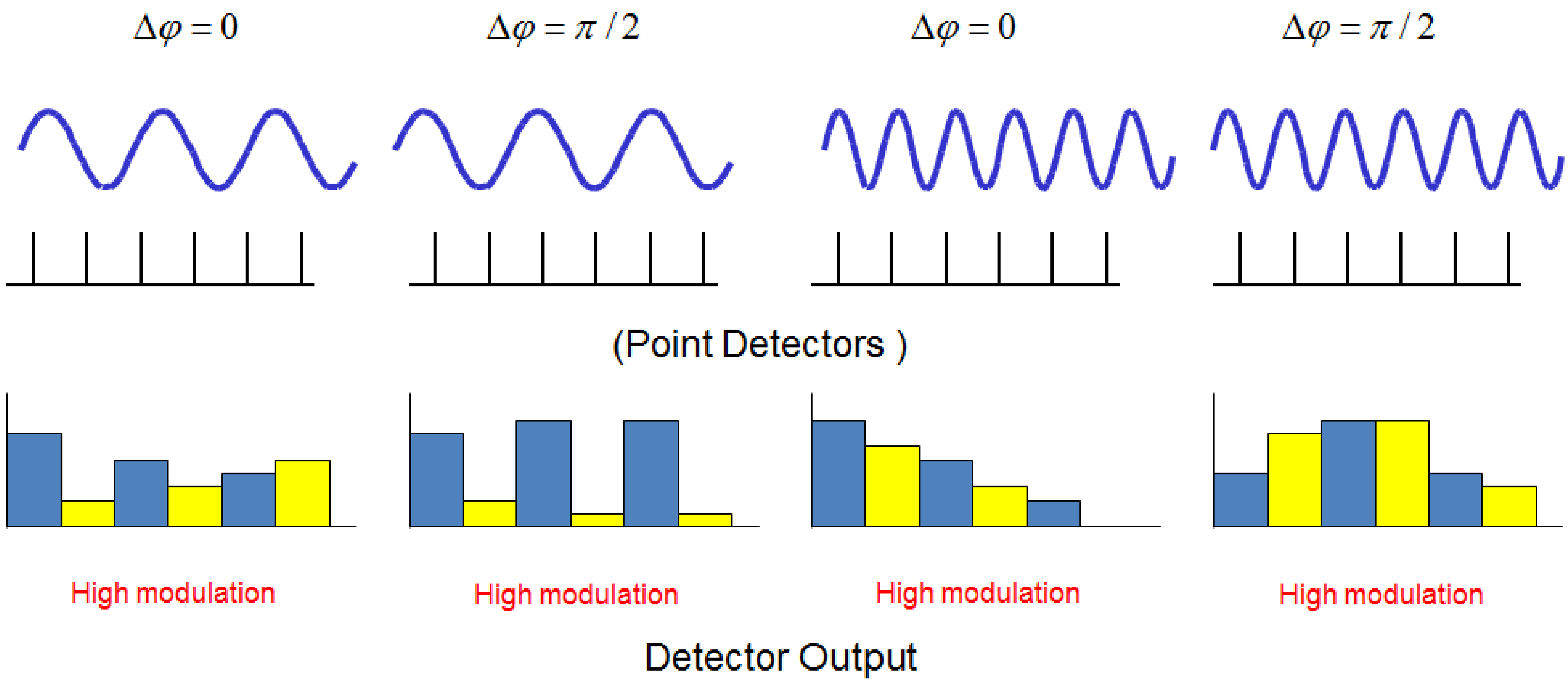




Ref: R. Kowarschik

Basics - Sampling

- Point detector



Sufficient Sampling **Undersampling**

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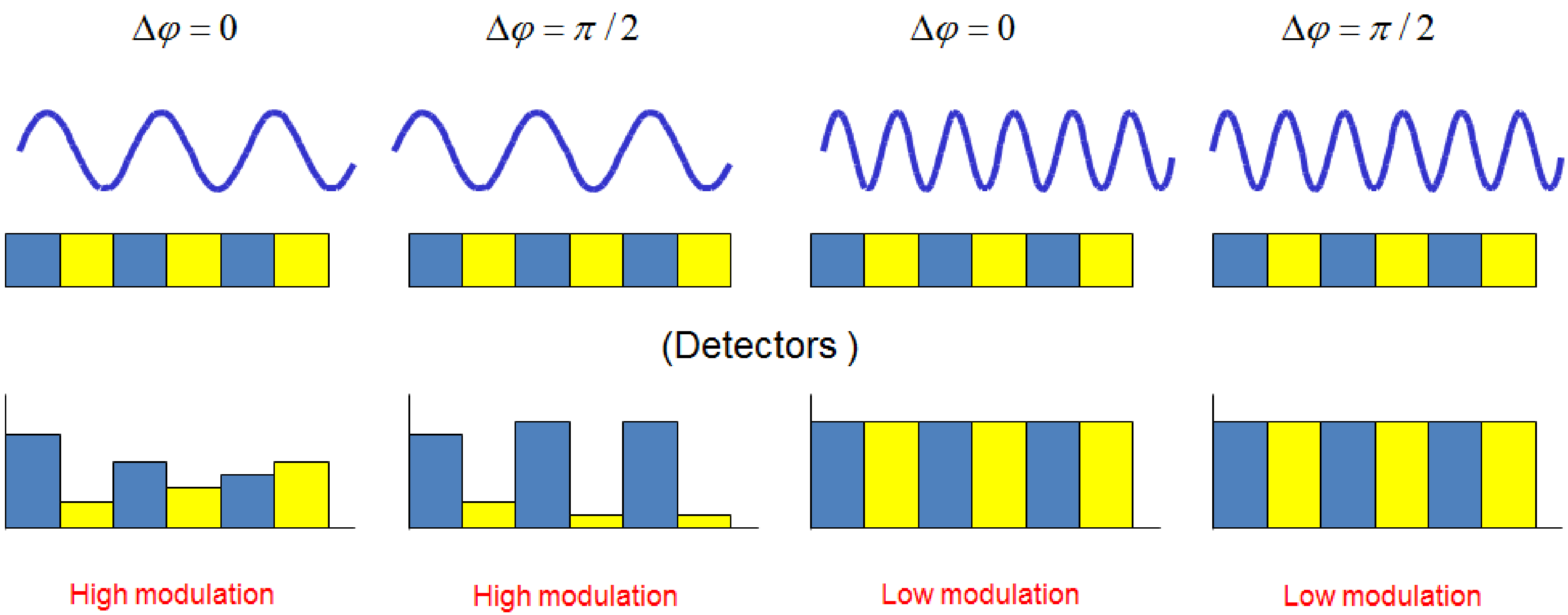
- Detector of finite Size



Detector Output

Sufficient Sampling **Undersampling**

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Sampling Theorem

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- Fourier transform

$$f(\nu) = \int_0^{x_{\max}} F(x) \cdot e^{-2\pi \cdot i \cdot \nu \cdot x} dx$$

- Relation for discrete Fourier transform

$$\Delta x \cdot \Delta \nu = \frac{1}{N}$$

Frequency sampling depends on spatial sampling

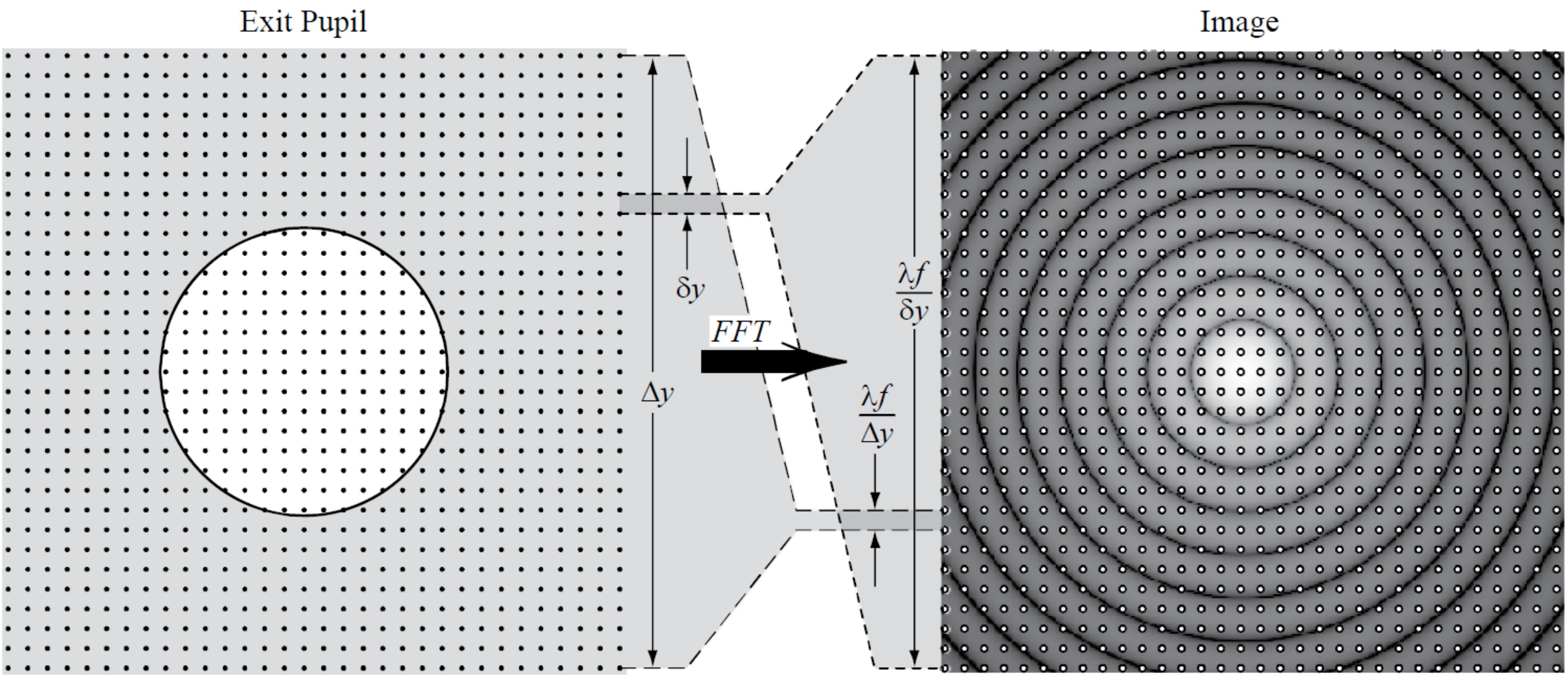
$$\Delta \nu = \frac{2\nu_{\max}}{N} = \frac{1}{x_{\max}}$$

- Discrete sampling:
  - periodicity in frequency space, limits bandwidth at Nyquist frequency
  - 2 points per period necessary to avoid aliasing

$$2\nu_{\max} = 2\nu_{Ny} < \frac{1}{\Delta x}$$

Sampling PSF Calculation

- PSF calculation by FFT:
  - coupling of coordinates in Pupil and image
  - zero-guard band necessary to get enough significant points in the PSF



Ref: B. Stone Proc SPIE

FFT-Sampling of a Gaussian Profile

- Gaussian profile in the spatial domain

$$f(x)=e^{-\frac{x^2}{w_x^2}}$$

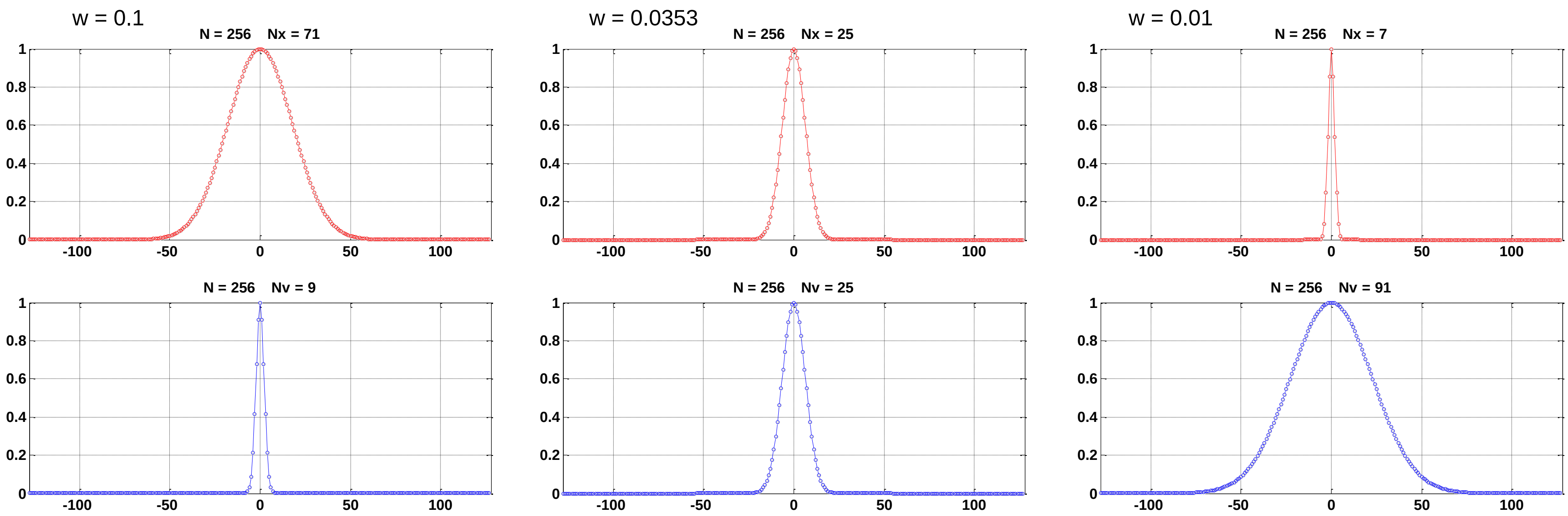
- Fourier transform

$$F(\nu)=w_x\sqrt{\pi}\cdot e^{-\pi^2 w_x^2 \nu^2}=w_x\sqrt{\pi}\cdot e^{-\frac{\nu^2}{w_\nu^2}} \qquad w_\nu=\frac{1}{\pi\cdot w_x}$$

- Sampling theorem
  N: number of discrete points
  D: size of calculation domain

$$D=N\cdot\Delta x \quad , \quad \nu_{\max}=\frac{N}{D}=N\cdot\Delta\nu \quad , \quad \Delta\nu=\frac{1}{D}$$

- Zero padding with large D/w:
  finer pixels in frequency space



Sampling Theorem

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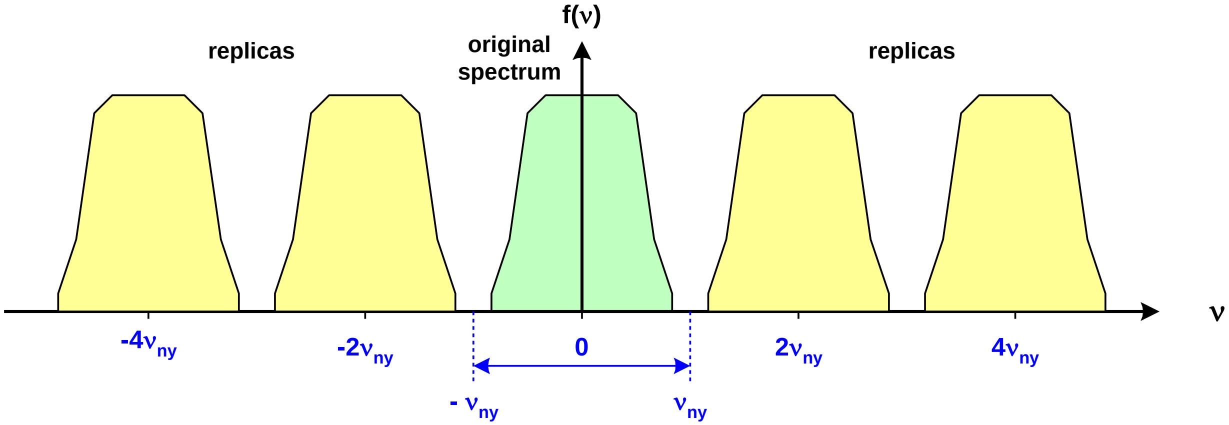
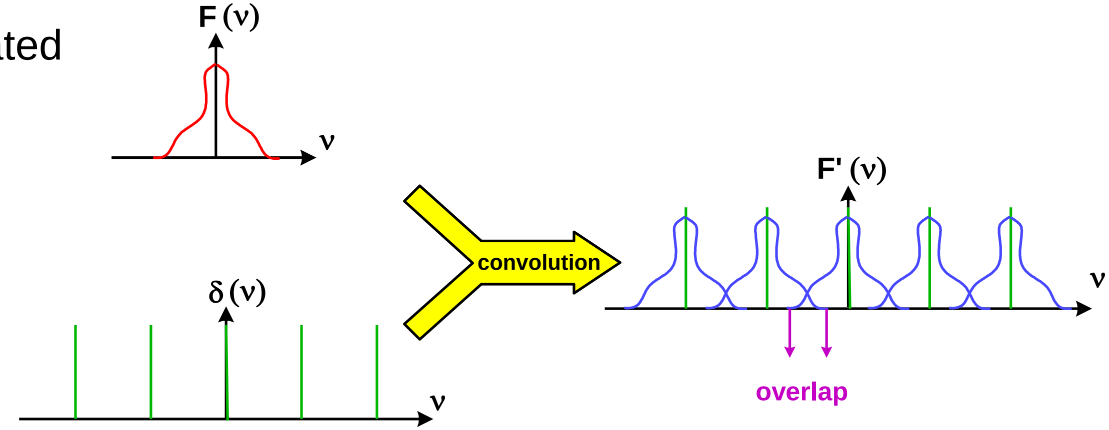
- Periodic spectra must be separated

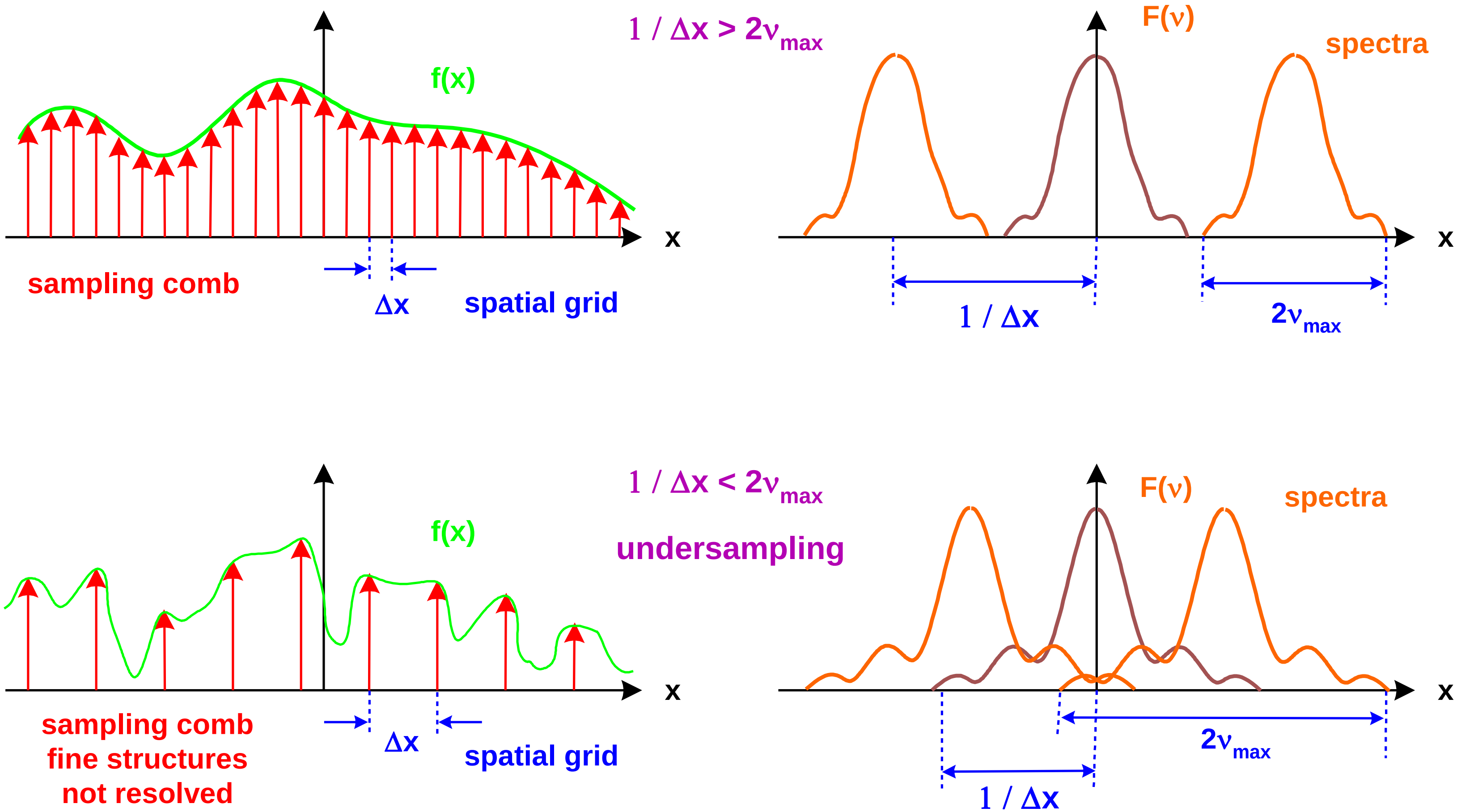


- Overlapp of spectra:
  - aliasing
  - pseudo pattern and Moire generated



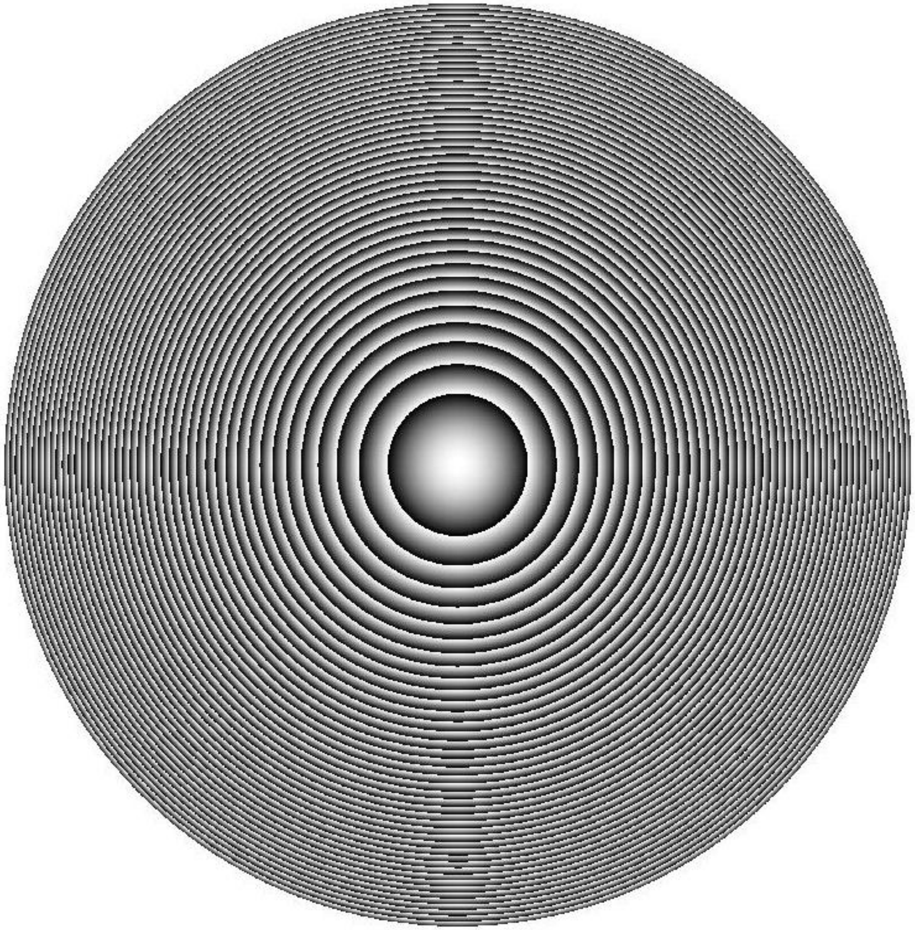
Sampling Theorem

- Necessary sampling in spatial domain to separate spectra in frequency domain
- Comb function creates periodicity



Aliasing Errors

- Discrete ring pattern
- Circular aliasing patterns in outer region



Sampling of Bandlimited Signals

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- Digital discrete signal in spatial domain
  comp function as sampling

$$\tilde{F}(x) = F(x) \cdot comb\left(\frac{x}{\Delta x}\right)$$

- Signal band-limited
  finite extend in spatial domain

$$\tilde{\tilde{F}}(x) = \tilde{F}(x) \cdot rect\left(\frac{x}{x_{\max}}\right) = F(x) \cdot comb\left(\frac{x}{\Delta x}\right) \cdot rect\left(\frac{x}{x_{\max}}\right)$$

- Back-transform
  sampling corresponds to convolution
  with sinc-function

$$F(x) = \tilde{F}(x) \cdot comb\left(\frac{x}{\Delta x}\right) * \frac{1}{\Delta x} \cdot \frac{\sin\left(\pi \cdot \frac{x}{\Delta x}\right)}{\pi \cdot \frac{x}{\Delta x}}$$

$$F(x) = \tilde{F}(x) * R(x)$$

- Ideal reconstructor:
  sinc function

$$R(x) = \frac{\sin\left(\pi \cdot \nu_{ny} \cdot x\right)}{\pi \cdot \nu_{ny} \cdot x} = \sin c\left(\pi \cdot \nu_{ny} \cdot x\right)$$

Sampling of Bandlimited Signals