Structure of matter: Homework to exercise 5

Electrical and optical properties of continuous media

Due on November 7th 2023 at noon

Please indicate your name on the solution sheets and send it to your seminar leader!

1. Multiple-choice test: Please tick the box(es) with the correct answer(s)!
(correctly ticked box: +1/2 point; wrongly ticked box: -1/2 point)

Anomalous dispersion is	impossible	
	typically observed in spectral regions where no absorption is present	
	typically observed in spectral regions where strong absorption is present	✓
Metals	are good electrical conductors	✓
	are usually good light reflectors in the VIS	✓
	may have refractive indices n that are smaller than 1	✓

2. True or wrong? Make your decision!

(2 points): 1 point per correct decision, 0 points per wrong or no decision

$n = \mathrm{Re}\sqrt{\varepsilon(\omega)}$ $K = \mathrm{Im}\sqrt{\varepsilon(\omega)}$

Assertion	true	wrong
The refractive index n may never be smaller than the extinction coefficient K		✓
For $\omega \to \infty \Rightarrow \mathrm{Im}\,\varepsilon \to 1$ $\quad \mathrm{Im} = \frac{\chi_{stat}\omega\tau}{1+\omega^2\tau^2}$		✓

3. Imagine a fictive isotropic and non-magnetic material with a complex dielectric function ε. At a certain frequency you observe that the real part of the dielectric function is zero, while the imaginary is equal to Imε = 2. Indicate real and imaginary part of the complex index of refraction at that frequency! (4 points)

4. From $\hat{n}$ = 0.1 + 5i, calculate ε = -24.99 + i and vice versa. (4 points)

5. In the EUV and X-ray spectral regions, the complex index of refraction is usually almost real and close to one. Therefore, it may be expressed through the small parameters δ and β (typical EUV-terminology, do not confuse with the polarizability!!) via: $n + iK = 1 - \delta + i\beta$ with δ, β << 1. Assuming an EUV wavelength of 13.5nm, calculate the penetration depth into silicon when $\beta \approx$ 0.002. (2points)

6. Find explicit expressions for the so-called dielectric loss function assuming

- a Drude metal and
- a dielectric material described in terms of the single oscillator model

  The loss function is defined as $-\mathrm{Im}(1/\varepsilon)$. Also, indicate expressions for the resonance angular frequency of the loss function in both situations! (6 points)

3. $\hat{n}(\omega) = \sqrt{\varepsilon(\omega)} = n(\omega) + i\kappa(\omega) = \mathrm{Re}\sqrt{\varepsilon(\omega)} + i\,\mathrm{Im}\sqrt{\varepsilon(\omega)}$

$\varepsilon = a + bi \quad \mathrm{Re}\,\varepsilon = 0 \quad \mathrm{Im}\,\varepsilon = 2 \Rightarrow \varepsilon = 2i \Rightarrow \sqrt{\varepsilon} = \sqrt{2i}$

$e^{ix} = \cos x + i\sin x \quad x = 2k\pi + \frac{\pi}{2} \Rightarrow i = e^{i(2k\pi + \frac{\pi}{2})} \Rightarrow \sqrt{2i} = \sqrt{2}\, e^{i(k\pi + \frac{\pi}{4})}$

$\Rightarrow \sqrt{\varepsilon} = \sqrt{2i} = \sqrt{2}\, e^{i(k\pi + \frac{\pi}{4})} = \sqrt{2}\cos(k\pi + \frac{\pi}{2}) + i\sqrt{2}\sin(k\pi + \frac{\pi}{4})$

$\Rightarrow \underline{\hat{n}(\omega) = \sqrt{2}\cos(k\pi + \frac{\pi}{4}) + i\sqrt{2}\sin(k\pi + \frac{\pi}{4}) \quad \text{and} \quad n(\omega) = \sqrt{2}\cos(k\pi + \frac{\pi}{4}) \quad \kappa(\omega) = \sqrt{2}\sin(k\pi + \frac{\pi}{4})}$

if $k = 2n$ and n is a Integer $\hat{n}(\omega) = \sqrt{2}\cos(2n\pi + \frac{\pi}{4}) + i\sqrt{2}\sin(2n\pi + \frac{\pi}{4}) = 1 + i$ $\quad \underline{n(\omega) = 1 \quad \kappa(\omega) = 1}$

if $k = 2n+1$ and n is a Integer $\hat{n}(\omega) = \sqrt{2}\cos(2n\pi + \frac{5}{4}\pi) + i\sqrt{2}\sin(2n\pi + \frac{5}{4}\pi) = -1 - i$ $\quad \underline{n(\omega) = -1 \quad \kappa(\omega) = -1}$

case $k = 2n+1$ is not physical

4. $\hat{n} = 0.1 + 5i = n(\omega) + i\kappa(\omega) = \mathrm{Re}\sqrt{\varepsilon} + i\,\mathrm{Im}\sqrt{\varepsilon} \Rightarrow \mathrm{Re}\sqrt{\varepsilon} = 0.1 \quad \mathrm{Im}\sqrt{\varepsilon} = 5$

$\Rightarrow \sqrt{\varepsilon} = 0.1 + 5i \quad \underline{\varepsilon = (0.1 + 5i)^2 = 0.01 - 25 + i = -24.99 + i}$

$\varepsilon = -24.99 + i \Rightarrow \sqrt{\varepsilon} = \sqrt{-24.99 + i} = \sqrt{(0.1)^2 + (5i)^2 + i} = \sqrt{(0.1 + 5i)^2} = 0.1 + 5i$

(5) $\vec{D} = \varepsilon_0\varepsilon(\omega)\vec{E} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0\frac{\partial \vec{H}}{\partial t} \quad \nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} = \varepsilon_0\varepsilon(\omega)\frac{\partial \vec{E}}{\partial t}$

$\nabla \times \nabla \times \vec{E} = -\mu_0\frac{\partial \nabla \times \vec{H}}{\partial t} = \mu_0\varepsilon_0\varepsilon(\omega)\frac{\partial^2 \vec{E}}{\partial t} \Rightarrow \nabla(\nabla \cdot \vec{E}) - \Delta\vec{E} = -\frac{\varepsilon(\omega)}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \Leftrightarrow \nabla^2\vec{E} - \frac{\varepsilon(\omega)}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$

Then in frequency domain $\Rightarrow \nabla^2\vec{E} + \frac{\omega^2}{c^2}\varepsilon(\omega)\vec{E} = 0$

$\vec{E} = \vec{E}(\vec{r}, \omega) = \vec{E}_0 e^{-i(\omega t - k\vec{r})} \quad \nabla^2\vec{E} = (ik)^2 \vec{E}_0 e^{-i(\omega t - k\vec{r})} = -k^2\vec{E}$

$\Rightarrow -k^2\vec{E} + \frac{\omega^2}{c^2}\varepsilon(\omega)\vec{E} = 0 \Rightarrow k = \pm\frac{\omega}{c}\sqrt{\varepsilon(\omega)}$

$k = \frac{\omega}{c}\sqrt{\varepsilon(\omega)} \quad \vec{E} = \vec{E}_0 \exp[-i(\omega t - \frac{\omega}{c}\sqrt{\varepsilon(\omega)}\vec{r})]$

$\sqrt{\varepsilon(\omega)} = \mathrm{Re}\sqrt{\varepsilon(\omega)} + i\,\mathrm{Im}\sqrt{\varepsilon(\omega)} = n + i\kappa \Rightarrow \vec{E} = \vec{E}_0 \exp[-i(\omega t - \frac{\omega}{c}(n + i\kappa)\vec{r})] = \vec{E}_0 \exp[-i(\omega t - \frac{\omega}{c}n\vec{r} - i\frac{\omega}{c}\kappa\vec{r})]$

$= \vec{E}_0 e^{-\frac{\omega}{c}\kappa\vec{r}} e^{-i(\omega t - \frac{\omega}{c}n\vec{r})}$

$I = (\vec{E}_0 e^{-\frac{\omega}{c}\kappa\vec{r}})^2 = \vec{E}_0^2 e^{-\frac{2\omega}{c}\kappa\vec{r}}$ The absorption coefficient $\alpha = \frac{2\omega}{c}\kappa$

$\kappa = \beta = 0.002 \quad \omega = \frac{2\pi}{T} = \frac{2\pi c}{\lambda} \Rightarrow \alpha = \frac{4\pi}{\lambda}\beta = \frac{4\pi \times 0.002}{13.5 \times 10^{-7}\,\mathrm{cm}}$

Penetration depth $L = \frac{1}{\alpha} = \frac{13.5 \times 10^{-7}\,\mathrm{cm}}{4\pi \times 0.002} \approx 5.37 \times 10^{-5}\,\mathrm{cm}$

6. dielectric loss function: $-\mathrm{Im}(\frac{1}{\varepsilon})$ $\quad \varepsilon(\omega)=\mathrm{Re}\,\varepsilon(\omega)+i\,\mathrm{Im}\,\varepsilon(\omega)$

Drude mode: $\varepsilon(\omega)=1-\frac{\omega_p^2}{\omega^2+2i\gamma\omega}=\frac{\omega^2-\omega_p^2+2i\gamma\omega}{\omega^2+2i\gamma\omega}$

$$\frac{1}{\varepsilon(\omega)}=\frac{\omega^2+2i\gamma\omega}{\omega^2-\omega_p^2+2i\gamma\omega}=\frac{(\omega^2+2i\gamma\omega)(\omega^2-\omega_p^2-2i\gamma\omega)}{(\omega^2-\omega_p^2)^2+4\gamma^2\omega^2}=\frac{\omega^4-\omega_p^2\omega^2+2i\gamma\omega^3-2i\gamma\omega^3-2i\gamma\omega_p^2\omega+4\gamma^2\omega^2}{(\omega^2-\omega_p^2)^2+4\gamma^2\omega^2}$$

$$=\frac{\omega^4-\omega_p^2\omega^2+4\gamma^2\omega^2}{(\omega^2-\omega_p^2)^2+4\gamma^2\omega^2}+i\frac{-2\gamma\omega_p^2\omega}{(\omega^2-\omega_p^2)^2+4\gamma^2\omega^2}$$

$$\Rightarrow -\mathrm{Im}(\frac{1}{\varepsilon})=\frac{2\gamma\omega_p^2\omega}{(\omega^2-\omega_p^2)^2+4\gamma^2\omega^2}$$

$$[-\mathrm{Im}(\frac{1}{\varepsilon})]'=\frac{2\gamma\omega_p^2[(\omega^2-\omega_p^2)^2+4\gamma^2\omega^2]-2\gamma\omega_p^2\omega[4\omega(\omega^2-\omega_p^2)+8\gamma^2\omega]}{[(\omega^2-\omega_p^2)^2+4\gamma^2\omega^2]^2}=\frac{2\gamma\omega_p^2[(\omega^2-\omega_p^2)^2-4\omega^2(\omega^2-\omega_p^2)-4\gamma^2\omega^2]}{[(\omega^2-\omega_p^2)^2+4\gamma^2\omega^2]^2}$$

$$[-\mathrm{Im}(\frac{1}{\varepsilon})]'=0\Rightarrow(\omega^2-\omega_p^2)^2-4\omega^2(\omega^2-\omega_p^2)-4\gamma^2\omega^2=0\Rightarrow(3\omega^2+\omega_p^2)(\omega^2-\omega_p^2)+4\gamma^2\omega^2=0$$

$$\Rightarrow 3\omega^4+(4\gamma^2-2\omega_p^2)\omega^2-\omega_p^4=0\Rightarrow\omega^2=\frac{2\omega_p^2-4\gamma^2\pm\sqrt{(4\gamma^2-2\omega_p^2)^2+12\omega_p^4}}{6}=\frac{\omega_p^2-2\gamma^2\pm2\sqrt{\gamma^4-\gamma^2\omega_p^2+\omega_p^4}}{3}$$

$\omega^2>0$ thus resonance frequency $\omega^2=\frac{1}{3}(\omega_p^2-2\gamma^2+2\sqrt{\gamma^4-\gamma^2\omega_p^2+\omega_p^4})\Rightarrow\omega=\sqrt{\frac{1}{3}(\omega_p^2-2\gamma^2+2\sqrt{\gamma^4-\gamma^2\omega_p^2+\omega_p^4})}$

Dielectrical material: $\varepsilon(\omega)=1+\frac{\omega_p^2}{\tilde{\omega}_0^2-\omega^2-2i\gamma\omega}=\frac{\omega_p^2+\tilde{\omega}_0^2-\omega^2-2i\gamma\omega}{\tilde{\omega}_0^2-\omega^2-2i\gamma\omega}$

$$\frac{1}{\varepsilon(\omega)}=\frac{\tilde{\omega}_0^2-\omega^2-2i\gamma\omega}{\omega_p^2+\tilde{\omega}_0^2-\omega^2-2i\gamma\omega}=\frac{(\tilde{\omega}_0^2-\omega^2-2i\gamma\omega)(\omega_p^2+\tilde{\omega}_0^2-\omega^2+2i\gamma\omega)}{(\omega_p^2+\tilde{\omega}_0^2-\omega^2)^2+4\gamma^2\omega^2}$$

$$=\frac{(\tilde{\omega}_0^2-\omega^2)(\omega_p^2+\tilde{\omega}_0^2-\omega^2)+2i\gamma\omega(\tilde{\omega}_0^2-\omega^2)-2i\gamma\omega(\omega_p^2+\tilde{\omega}_0^2-\omega^2)+4\gamma^2\omega^2}{[\omega_p^2+\tilde{\omega}_0^2-\omega^2]^2+4\gamma^2\omega^2}$$

$$=\frac{(\tilde{\omega}_0^2-\omega^2)(\omega_p^2+\tilde{\omega}_0^2-\omega^2)+4\gamma^2\omega^2}{(\omega_p^2+\tilde{\omega}_0^2-\omega^2)^2+4\gamma^2\omega^2}+i\frac{-2\gamma\omega\omega_p^2}{(\omega_p^2+\tilde{\omega}_0^2-\omega^2)^2+4\gamma^2\omega^2}$$

$$\Rightarrow-\mathrm{Im}\frac{1}{\varepsilon}=\frac{2\gamma\omega\omega_p^2}{(\omega_p^2+\tilde{\omega}_0^2-\omega^2)^2+4\gamma^2\omega^2}$$

$$[-\mathrm{Im}(\frac{1}{\varepsilon})]'=\frac{2\gamma\omega_p^2[(\omega_p^2+\tilde{\omega}_0^2-\omega^2)^2+4\gamma^2\omega^2]-2\gamma\omega_p^2\omega[-4\omega(\omega_p^2+\tilde{\omega}_0^2-\omega^2)+8\gamma^2\omega]}{[(\omega_p^2+\tilde{\omega}_0^2-\omega^2)^2+4\gamma^2\omega^2]^2}$$

$$=\frac{2\gamma\omega_p^2[(\omega_p^2+\tilde{\omega}_0^2-\omega^2)^2+4\omega^2(\omega_p^2+\tilde{\omega}_0^2-\omega^2)-4\gamma^2\omega^2]}{[(\omega_p^2+\tilde{\omega}_0^2-\omega^2)^2+4\gamma^2\omega^2]^2}=0\Rightarrow(\omega_p^2+\tilde{\omega}_0^2+3\omega^2)(\omega_p^2+\tilde{\omega}_0^2-\omega^2)-4\gamma^2\omega^2=0$$

$$\Rightarrow2\omega^2(\omega_p^2+\tilde{\omega}_0^2)-3\omega^4+(\omega_p^2+\tilde{\omega}_0^2)^2-4\gamma^2\omega^2=0\Rightarrow3\omega^4+[4\gamma^2-2(\omega_p^2+\tilde{\omega}_0^2)]\omega^2-(\omega_p^2+\tilde{\omega}_0^2)^2=0$$

$\Rightarrow$ resonance angular frequency: $\omega=\sqrt{\frac{1}{3}(\omega_p^2+\tilde{\omega}_0^2-2\gamma^2+2\sqrt{\gamma^4-\gamma^2(\omega_p^2+\tilde{\omega}_0^2)+(\omega_p^2+\tilde{\omega}_0^2)^2})}$