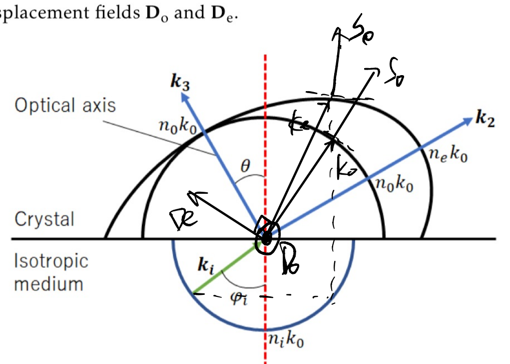
Consider the interface between an isotropic medium (refractive index n_i) and a uniaxial crystal (ordinary and extraordinary refractive indices n_o and n_e, respectively) as shown in the figure below. A plane wave is incident on the crystal at an angle φ_i to the surface normal. The optical axis of the crystal ($\mathbf{k_3}$) and the surface normal form an angle θ.

a) Draw in the figure the wave vectors of the refracted ordinary wave $\mathbf{k_o}$ and the refracted extraordinary wave $\mathbf{k_e}$ for unpolarized incident light.

b) Draw the corresponding Poyinting vectors $\mathbf{S_o}$ and $\mathbf{S_e}$. Are they parallel?

c) Draw the electric displacement fields $\mathbf{D_o}$ and $\mathbf{D_e}$.

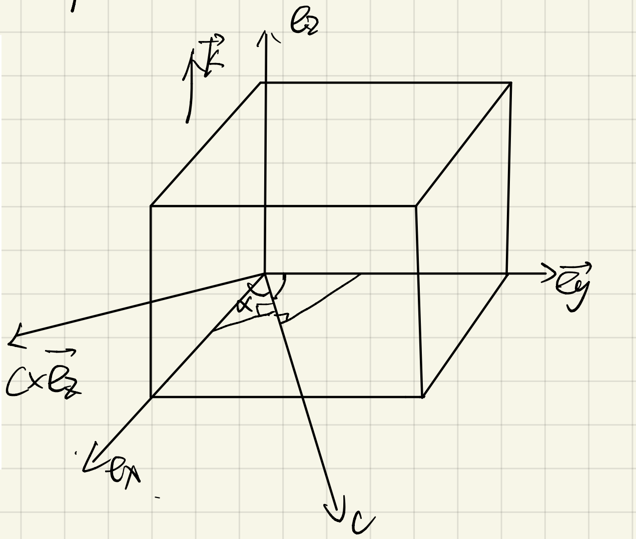


(a) $n_e > n_o$ $\quad n\sin\varphi_i = n'\sin\theta \quad \sin\theta = \frac{n}{n'}\sin\varphi_i \Rightarrow \theta_e < \theta_o$

(b) $\langle \vec{S} \rangle \perp \vec{E}$ $\quad$ ordinary $\vec{E} \parallel \vec{D}$ $\quad$ extraordinary $\vec{E} \nparallel \vec{D}$

(c) D_o polarized in x direction $\quad D_e$ polarized in y-z plane

A slide of a transparent, uniaxial, anisotropic crystal with the refractive indices n_e and n_o and with a thickness d is oriented such that the surface normal is along $\mathbf{e}_z$, and the crystal axis is along $\mathbf{c} = \cos(\alpha)\mathbf{e}_x + \sin(\alpha)\mathbf{e}_y$. An x-polarized plane wave of wavelength λ, with a vacuum wavevector $\mathbf{k} = 2\pi/\lambda\mathbf{e}_z$, is excited at the beginning of the slide and propagates through the crystal. Consider the lossless propagation and neglect the Fresnel reflection at both interfaces.



a) Decompose the incident electric field into the normal modes (ordinary and extraordinary waves) of the anisotropic medium and write the dispersion relation for both of them. Find the angle between the normal modes of the displacement $\mathbf{D}^{(a,b)}$ and electric fields $\mathbf{E}^{(a,b)}$.
Hint: For the decomposition, use the crystal coordinate system and notice the special case of thepropagation direction.

b) Calculate the relation between the wavevector $\boldsymbol{k}$ and the corresponding Poynting vector $\boldsymbol{S}$ for both normal modes. Show that they are parallel.

c) Calculate the electric field in laboratory coordinates after propagating through the slide. What is its polarization state?

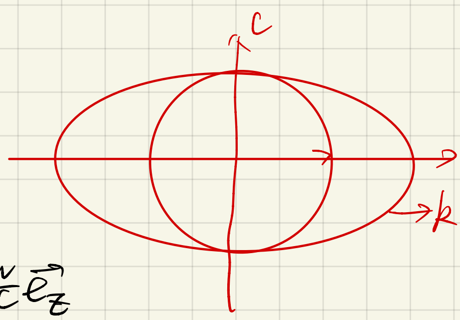
d) We choose the thickness d of the slide such that $(n_e - n_o)d = \lambda/2$. Calculate and describe the impact of the crystal onto the polarization state of the plane wave directly after the slide as a function of the crystal rotation angle α. If we place a linear polarizer after this so-called half-wave plate and rotate the crystal, which device do we get?

(a) crystal coordinate system: $\vec{c},\ (\vec{c} \times \vec{e_z}),\ \vec{e_z}$

$\vec{E} = E_0 \exp[-ik(\omega)z]\,\vec{e_x} \quad z=0 \quad \vec{E} = E_0\vec{e_x} = a\cdot\vec{c} + b\cdot(\vec{c}\times\vec{e_z}) + c\cdot\vec{e_z}$

$E_0\vec{e_x}\cdot\vec{c} = E_0\vec{e_x}\cdot[\cos(\alpha)\vec{e_x} + \sin(\alpha)\vec{e_y}] = E_0\cos(\alpha) = a$

$E_0\vec{e_x}\cdot(c\times\vec{e_z}) = E_0\vec{c}\cdot(\vec{e_z}\times\vec{e_x}) = E_0\vec{c}\cdot\vec{e_y} = E_0\sin(\alpha) = b$



$E_0\vec{e_x}\cdot\vec{e_z} = 0 = c \quad \Rightarrow \vec{E} = E_0[\cos(\alpha)\vec{c} + \sin(\alpha)(\vec{c}\times\vec{e_z})]$

extraordinary wave $\vec{k} = n_e\frac{\omega}{c}\vec{e_z}$ $\quad$ ordinary wave $\vec{k} = n_o\frac{\omega}{c}\vec{e_z}$

$$\frac{1}{n^2(\theta)} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2} \quad \theta = \frac{\pi}{2} \Rightarrow \frac{1}{n^2(\frac{\pi}{2})} = \frac{\cos^2\frac{\pi}{2}}{n_o^2} + \frac{\sin^2\frac{\pi}{2}}{n_e^2} \Rightarrow n(\tfrac{\pi}{2}) = n_e.$$

$\Rightarrow n_a = n_o \quad n_b = n_e$ $\quad$ hence $\quad D^a \parallel E^a;\ D^b \parallel E^b.$

(b) $\langle \vec{S} \rangle = \frac{1}{2}Re[E\times H^*] \quad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \frac{d}{dz}\times\vec{E} = i\omega\mu_0\vec{H} \Rightarrow \vec{H} = \frac{1}{i\omega\mu_0}\vec{k}\times\vec{E}$

$\Rightarrow \langle\vec{S}\rangle \sim \vec{E}\times H^* = \vec{E}\times(\vec{k}\times\vec{E})^* = \vec{k}^*(\vec{E}\cdot\vec{E}^*) - E^*(\vec{E}\cdot\vec{k}^*)$

In laboratory coordinate $\vec{E} = E_x \exp[-i(\vec{k}\cdot\vec{z})]\,\vec{e}_x$ $\qquad \vec{k}^* = k\vec{e}_z$

$\Rightarrow \vec{E}\cdot\vec{k}^* = 0 \Rightarrow \langle\vec{S}\rangle \sim \vec{k}^*(\vec{E}\cdot\vec{E}^*) = \vec{k}^*|\vec{E}|^2 \Rightarrow \vec{S} \parallel \vec{k}$

(c) $\vec{E} = E_0 \exp(ikz)\,\vec{e}_x = E_0[\cos(\alpha)\exp(-ik_b z)\,\vec{c} + \sin(\alpha)\exp(-ik_a z)(\vec{c}\times\vec{e}_z)]$

$k_b = \frac{w}{c}n_e \qquad k_a = \frac{w}{c}n_o$

$\Rightarrow \vec{E} = E_0[\cos(\alpha)\exp(-i\frac{w}{c}n_e d)\,\vec{c} + \sin(\alpha)\exp(-i\frac{w}{c}n_o d)(\vec{c}\times\vec{e}_z)]$

$\vec{c} = \cos(\alpha)\vec{e}_x + \sin(\alpha)\vec{e}_y \qquad \vec{c}\times\vec{e}_z = -\cos(\alpha)\vec{e}_y + \sin(\alpha)\vec{e}_x$

$\Rightarrow \vec{E} = E_0[\cos^2\alpha\exp(-i\frac{w}{c}n_e d)\vec{e}_x + \cos\alpha\sin\alpha\exp(-i\frac{w}{c}n_e d)\vec{e}_y - \sin(\alpha)\cos(\alpha)\exp(-i\frac{w}{c}n_o d)\vec{e}_y$
$\quad + \sin^2\alpha\exp(-i\frac{w}{c}n_o d)\vec{e}_x]$

$= E_0[\cos^2\alpha\exp(-i\frac{w}{c}n_e d) + \sin^2\alpha\exp(-i\frac{w}{c}n_o d)]\vec{e}_x$
$+ E_0[\sin\alpha\cos\alpha\exp(-i\frac{w}{c}n_e d) - \sin\alpha\cos\alpha\exp(-i\frac{w}{c}n_o d)]\vec{e}_y$

$\delta = \varphi_c - \varphi_{\vec{c}\times\vec{e}_z} = \frac{w}{c}(n_e - n_o)d = \frac{2\pi}{\lambda}(n_e - n_o)d$

if $\frac{2\pi}{\lambda}(n_e - n_o)d = n\pi \Rightarrow (n_e - n_o)d = \frac{n\lambda}{2}$ $\quad n = 1, 2\cdots \Rightarrow$ Linear Polarization

if $\frac{2\pi}{\lambda}(n_e - n_o)d = \frac{\pi}{2}(2n+1) \Rightarrow (n_e - n_o)d = \frac{(2n+1)\lambda}{4}$ $\quad n = 0, 1, 2 \cdots \Rightarrow$ circular Polarization

other cases $\Rightarrow$ elliptical polarization

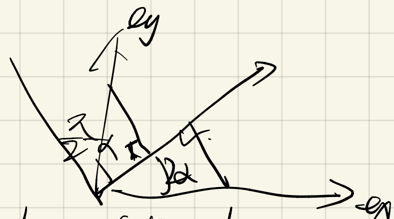
(d) $\vec{E}(d) = E_0\exp(-i\frac{w}{c}n_e d)[\cos^2\alpha + \sin^2\alpha\exp[i\frac{w}{c}d(n_e - n_o)]]\vec{e}_x$
$+ E_0\exp(-i\frac{w}{c}n_e d)[\sin\alpha\cos\alpha - \sin\alpha\cos\alpha\exp[i\frac{w}{c}d(n_e - n_o)]]\vec{e}_y$

$(n_e - n_o)d = \frac{\lambda}{2} \qquad \frac{w}{c} = k = \frac{2\pi}{\lambda} \Rightarrow \frac{w}{c}d(n_e - n_o) = \pi$

$\Rightarrow \vec{E}(d) = E_0\exp(-i\frac{w}{c}n_e d)\{[\cos^2\alpha + \sin^2\alpha\, e^{i\pi}]\vec{e}_x + (\sin\alpha\cos\alpha - \sin\alpha\cos\alpha\, e^{i\pi})\vec{e}_y\}$

$= E_0\exp(-i\frac{w}{c}n_e d)[[\cos^2\alpha - \sin^2\alpha]\vec{e}_x + 2\sin\alpha\cos\alpha\,\vec{e}_y]$

$= E_0\exp(-i\frac{w}{c}n_e d)[\cos(2\alpha)\vec{e}_x + \sin 2\alpha\,\vec{e}_y]$



As mentioned in (c) it's a linear polarized light and the rotational angle is 2α.

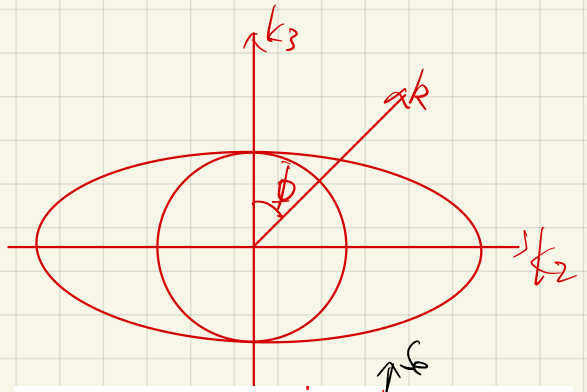
If we place an x-polarizer after the output

$E_{out} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\vec{E}(d) = E_0\exp(-i\frac{w}{c}n_o d)\cos(2\alpha)\vec{e}_x$

$I_0 = |\vec{E}(d)|^2 = E_0^2[\cos(2\alpha)\vec{e}_x + \sin 2\alpha\,\vec{e}_y][\cos(2\alpha)\vec{e}_x + \sin 2\alpha\,\vec{e}_y] = E_0^2(\cos^2 2\alpha + \sin^2 2\alpha) = E_0^2$

$I_{out} = |E_{out}|^2 = E_0^2 \cos^2(2\alpha)$ Thus we get an attenuator

Let us consider a uniaxial crystal with $\varepsilon_1 = \varepsilon_2 \neq \varepsilon_3$, where we define $n_o \equiv \sqrt{\varepsilon_1}$ and $n_e \equiv \sqrt{\varepsilon_3}$. For a plane-wave to propagate with a $\boldsymbol{k}$-vector in the k_2k_3 plane, there are two possible normal modes (eigenmodes): an ordinary wave with a refractive index $n_a(\Phi) = n_o$, and an extraordinary wave with refractive index $n_b(\Phi) = [\varepsilon_1\varepsilon_3/(\varepsilon_1 \sin^2\Phi + \varepsilon_3\cos^2\Phi)]^{1/2}$, where Φ is the angle between the optical axis of the crystal and the $\boldsymbol{k}$-vector of the wave inside the crystal.



a) Consider the interface between an isotropic medium (refractive index n_i) and the uniaxial crystal described above. A plane wave is incident at an angle φ_i to the surface normal. The optical axis of the crystal and the surface normal form an angle θ (see Fig. 2, the optical axis is called 'crystal axis' there). Show that there will be two refracted waves with two different directions, $\boldsymbol{k}_o$ and $\boldsymbol{k}_e$:
   1. the *ordinary* one, which obeys the ordinary Snell's law $n_i \sin\varphi_i = n_o \sin\varphi_o$,
   2. and the *extraordinary* one, which obeys $n_i \sin\varphi_i = n_b(\theta + \varphi_e)\sin\varphi_e$.

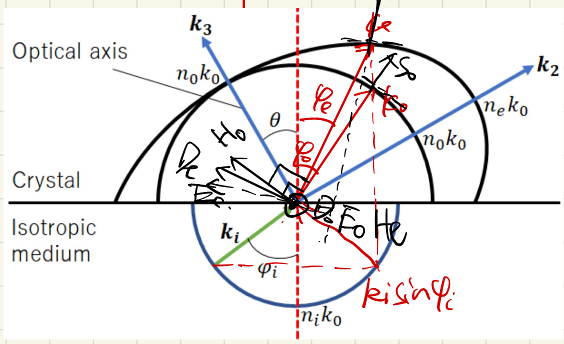
   Hint: Use the fact that the tangential component of the wave-vector $\mathbf{k}$ needs to be continuous at the interface.

b) Draw the electric displacement fields $\mathbf{D}_o$ and $\mathbf{D}_e$ and state the reasonings for your choice. *Hint:* Use the ratio of the individual field components.

c) Draw the corresponding electric and magnetic fields knowing that the Poyinting vectors $\mathbf{S}_o$ and $\mathbf{S}_e$ must be parallel with respect to the tangent vector of the index ellipse. Are they parallel? If not, what are the consequences?

d*) Prove that the Poynting vector $\boldsymbol{S}$ is perpendicular to its corresponding index-ellipsoid surfaces.

Note: This task and the lecture use the word 'optical axis' or 'crystal axis' to refer to the 'optic axis of the crystal'. The reason is probably that both are called 'optische Achse' in German. In any case all of these terms refer to the direction in a uniaxial crystal in which a traveling ray does not experience birefringence. Make sure the students understand the physical meaning of this direction.



a) 1. $k_i \sin\varphi_i = k_o \sin\varphi_o$ $\quad k_i = \frac{\omega}{c} n_i \quad k_o = \frac{\omega}{c} n_o$

$\Rightarrow \frac{\omega}{c} n_i \sin\varphi_i = \frac{\omega}{c} n_o \sin\varphi_o \Rightarrow n_i \sin\varphi_i = n_o \sin\varphi_o$

$k_i \sin\varphi_i = k_e \sin\varphi_e \quad k_e = \frac{\omega}{c} n(\Phi)$

$\Rightarrow \frac{\omega}{c} n_i \sin\varphi_i = \frac{\omega}{c} n(\Phi)\sin\varphi_e \quad \Phi = \varphi_e + \theta \Rightarrow n_i \sin\varphi_i = n(\varphi_e + \theta)\sin\varphi_e$.

(b) $D_1 : D_2 : D_3 = \frac{\varepsilon_{or} k_1}{\frac{\omega^2}{c^2}\varepsilon_{or} - k^2} : \frac{\varepsilon_{or} k_2}{\frac{\omega^2}{c^2}\varepsilon_{or} - k^2} : \frac{\varepsilon_e k_3}{\frac{\omega^2}{c^2}\varepsilon_e - k^2}$

For extraordinary wave $k_1 = 0$ thus $D_{1e} = 0$ hence D_e is in k_2-k_3 plane. and $D_e \perp D_o$, thus D_o is along k_1-direction.

(c) They are parallel, if not $\vec{S} \not\perp \vec{E}$ and $\vec{S} \not\perp \vec{H}$

(d) Ellipsoid: $\frac{x^2}{n_e^2} + \frac{y^2}{n_o^2} = 1$