Task 1: Traveling and standing waves (a=2, b=2, c=2 pts.)

Consider a monochromatic field of frequency ω with the complex amplitude vector

$$\mathbf{E}(\mathbf{r}) = \exp(i\beta x)\left[A_1\exp(i\kappa z) + A_2\cos(\kappa z)\right]\mathbf{e}_y,$$

where β, κ, A_1, and A_2 are real-valued numbers.

a) Derive an expression for the complex amplitude vector of the magnetic field $\mathbf{H}(\mathbf{r})$.
Hint: All calculations become easier if you first try to reformulate the given electric field as a sum of plane waves.

b) Calculate the time-averaged Poynting vector.

c) Use the Helmholtz equation in vacuum to find the relation between the parameters β, κ and λ_0 (the vacuum wavelength).

$\nabla\times\vec E = -\mu_0\frac{\partial \vec H}{\partial t}$ $\quad H = H_0 e^{-i\omega t}$

$\Rightarrow i\vec k\times\vec E = i\omega\mu_0\vec H$ $\quad H = H_0 e^{i(\vec k\vec r-\omega t)}$

$\vec H = \frac{1}{\omega\mu_0}\vec k\times\vec E$

$(A_1+A_2)e^{i\beta x} + (A_1+\frac{A_2}{2})e^{ikz} + \frac{A_2}{2}e^{-ikz}$

(a) $\vec E(\vec r) = A_1 e^{i(\beta x+kz)}\vec e_y + \frac{A_2}{2}e^{i(\beta x+kz)}\vec e_y + \frac{A_2}{2}e^{i(\beta x-kz)}\vec e_y$ $\qquad \vec E(\vec r,t) = \vec E(\vec r)e^{-i\omega t}$

$= \left[(A_1+\frac{A_2}{2})e^{i(\beta x+kz)} + \frac{A_2}{2}e^{i(\beta x-kz)}\right]\vec e_y = E_y\vec e_y$ $\qquad \vec H(\vec r,t) = \vec H(\vec r)e^{-i\omega t}$

$\nabla\times\vec E(\vec r,t) = -\frac{\partial\vec B(\vec r,t)}{\partial t} = -\mu_0\frac{\partial\vec H(\vec r,t)}{\partial t} = i\omega\mu_0\vec H(\vec r)e^{-i\omega t}$

$\nabla\times\vec E(\vec r,t) = e^{-i\omega t}\cdot\nabla\times\vec E(\vec r) = e^{-i\omega t}\cdot\nabla\times\vec E(x,y,z) = i\omega\mu_0\vec H(\vec r)e^{-i\omega t} \Rightarrow \vec H(\vec r) = \frac{\nabla\times\vec E(x,y,z)}{i\omega\mu_0} = \frac{-i}{\omega\mu_0}\nabla\times\vec E(x,y,z)$

$$\nabla\times\vec E = \begin{vmatrix}\vec e_x & \vec e_y & \vec e_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_x & E_y & E_z\end{vmatrix} = \left(\cancel{\frac{\partial E_z}{\partial y}} - \frac{\partial E_y}{\partial z}\right)\vec e_x + \left(\cancel{\frac{\partial E_x}{\partial z}} - \cancel{\frac{\partial E_z}{\partial x}}\right)\vec e_y + \left(\frac{\partial E_y}{\partial x} - \cancel{\frac{\partial E_x}{\partial y}}\right)\vec e_z$$

$E_x = E_z = 0 \Rightarrow \nabla\times\vec E = -\frac{\partial E_y}{\partial z}\vec e_x + \frac{\partial E_y}{\partial x}\vec e_z = -\left[ik(A_1+\frac{A_2}{2})e^{i(\beta x+kz)} - \frac{iA_2k}{2}e^{i(\beta x-kz)}\right]\vec e_x + i\beta E_y\vec e_z$

$= ik\left[\frac{A_2}{2}e^{i(\beta x-kz)} - (A_1+\frac{A_2}{2})e^{i(\beta x+kz)}\right]\vec e_x + i\beta\left[(A_1+\frac{A_2}{2})e^{i(\beta x+kz)} + \frac{A_2}{2}e^{i(\beta x-kz)}\right]\vec e_z$

$\Rightarrow \vec H(\vec r) = \frac{-i}{\omega\mu_0}(\nabla\times\vec E) = \frac{k}{\omega\mu_0}\left[\frac{A_2}{2}e^{i(\beta x-kz)} - (A_1+\frac{A_2}{2})e^{i(\beta x+kz)}\right]\vec e_x + \frac{\beta}{\omega\mu_0}\left[(A_1+\frac{A_2}{2})e^{i(\beta x+kz)} + \frac{A_2}{2}e^{i(\beta x-kz)}\right]\vec e_z$

(b) $\langle\vec S(\vec r,t)\rangle = \frac12 Re\left[\vec E(\vec r)\times\vec H^*(\vec r)\right]$

$\vec H^*(\vec r) = \frac{k}{\omega\mu_0}\left[\frac{A_2}{2}e^{-i(\beta x-kz)} - (A_1+\frac{A_2}{2})e^{-i(\beta x+kz)}\right]\vec e_x + \frac{\beta}{\omega\mu_0}\left[(A_1+\frac{A_2}{2})e^{-i(\beta x+kz)} + \frac{A_2}{2}e^{-i(\beta x-kz)}\right]\vec e_z$

$$\vec E(\vec r)\times\vec H^*(\vec r) = \begin{vmatrix}\vec e_x & \vec e_y & \vec e_z\\ E_x & E_y & E_z\\ H_x^* & H_y^* & H_z^*\end{vmatrix} = (E_yH_z^* - \cancel{E_zH_y^*})\vec e_x + (\cancel{E_zH_x^*} - \cancel{E_xH_z^*})\vec e_y + (\cancel{E_xH_y^*} - E_yH_x^*)\vec e_z$$

$E_z = E_x = H_y^* = 0 \Rightarrow \vec E(\vec r)\times\vec H^*(\vec r) = E_yH_z^*\vec e_x - E_yH_x^*\vec e_z$

$E_y\cdot H_z^* = \left[(A_1+\frac{A_2}{2})e^{i(\beta x+kz)} + \frac{A_2}{2}e^{i(\beta x-kz)}\right]\times\frac{\beta}{\omega\mu_0}\left[(A_1+\frac{A_2}{2})e^{-i(\beta x+kz)} + \frac{A_2}{2}e^{-i(\beta x-kz)}\right]$

$= \frac{\beta}{\omega\mu_0}\left[(A_1+\frac{A_2}{2})^2 + \frac{A_2}{2}(A_1+\frac{A_2}{2})e^{2ikz} + \frac{A_2}{2}(A_1+\frac{A_2}{2})e^{-2ikz} + \frac{A_2^2}{4}\right] = \frac{\beta}{\omega\mu_0}\left[\frac{A_2^2}{4} + (A_1+\frac{A_2}{2})^2 + A_2(A_1+\frac{A_2}{2})\cos(2kz)\right]$

$E_y\cdot H_x^* = \left[(A_1+\frac{A_2}{2})e^{i(\beta x+kz)} + \frac{A_2}{2}e^{i(\beta x-kz)}\right]\times\frac{k}{\omega\mu_0}\left[\frac{A_2}{2}e^{-i(\beta x-kz)} - (A_1+\frac{A_2}{2})e^{-i(\beta x+kz)}\right]$

$= \frac{k}{\omega\mu_0}\left[\frac{A_2}{2}(A_1+\frac{A_2}{2})e^{2ikz} - (A_1+\frac{A_2}{2})^2 + \frac{A_2^2}{4} - \frac{A_2}{2}(A_1+\frac{A_2}{2})e^{-2ikz}\right] = \frac{k}{\omega\mu_0}\left[\frac{A_2^2}{4} - (A_1+\frac{A_2}{2})^2 + iA_2(A_1+\frac{A_2}{2})\sin(2kz)\right]$

$= \frac{k}{\omega\mu_0}\left[\frac{A_2^2}{4} - (A_1+\frac{A_2}{2})^2\right] + \frac{ikA_2(A_1+\frac{A_2}{2})}{\omega\mu_0}\sin(2kz)$

$\Rightarrow \langle\vec S(\vec r,t)\rangle = \frac12 Re\left[\vec E(\vec r)\times\vec H^*(\vec r)\right] = \frac{\beta}{2\omega\mu_0}\left[\frac{A_2^2}{4} + (A_1+\frac{A_2}{2})^2 + A_2(A_1+\frac{A_2}{2})\cos(2kz)\right]\vec e_x + \frac{k}{2\omega\mu_0}\left[A_1(A_1+A_2)\right]\vec e_z$

c) Helmholtz Equation: $\Delta\vec{E}(\vec{r},\omega) + \frac{\omega^2}{c^2}\varepsilon(\omega)\vec{E}(\vec{r},\omega) = 0$

In vacuum $\Rightarrow \Delta\vec{E}(\vec{r},\omega) + \frac{\omega^2}{c^2}\vec{E}(\vec{r},\omega) = 0 \Rightarrow \Delta\vec{E}(\vec{r}) + \frac{\omega^2}{c^2}\vec{E}(\vec{r}) = 0$

$$\text{grad}\,\vec{E} = \frac{\partial E(x,y,z)}{\partial x}\vec{e}_x + \frac{\partial E(x,y,z)}{\partial y}\vec{e}_y + \frac{\partial E(x,y,z)}{\partial z}\vec{e}_z$$

$$= i\beta\left[\left(A_1+\frac{A_2}{2}\right)e^{i(\beta x+kz)} + \frac{A_2}{2}e^{i(\beta x-kz)}\right]\vec{e}_x + ik\left[\left(A_1+\frac{A_2}{2}\right)e^{i(\beta x+kz)} - \frac{A_2}{2}e^{i(\beta x-kz)}\right]\vec{e}_z$$

$$\text{div}\,(\text{grad}\,\vec{E}) = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = -\beta^2\left[\left(A_1+\frac{A_2}{2}\right)e^{i(\beta x+kz)} + \frac{A_2}{2}e^{i(\beta x-kz)}\right] - k^2\left[\left(A_1+\frac{A_2}{2}\right)e^{i(\beta x+kz)} + \frac{A_2}{2}e^{i(\beta x-kz)}\right]$$

$$-(\beta^2+k^2)\left[\left(A_1+\frac{A_2}{2}\right)e^{i(\beta x+kz)} + \frac{A_2}{2}e^{i(\beta x-kz)}\right]$$

$$\Delta\vec{E} = \text{div}\,(\text{grad}\,\vec{E}) = -(\beta^2+k^2)\left[\left(A_1+\frac{A_2}{2}\right)e^{i(\beta x+kz)} + \frac{A_2}{2}e^{i(\beta x-kz)}\right] = -\frac{\omega^2}{c^2}\left[\left(A_1+\frac{A_2}{2}\right)e^{i(\beta x+kz)} + \frac{A_2}{2}e^{i(\beta x-kz)}\right]$$

$-(\beta^2+k^2) = -\frac{\omega^2}{c^2}$ $\quad \omega = \frac{2\pi}{T} = \frac{2\pi c}{\lambda} \Rightarrow \beta^2+k^2 = \frac{4\pi^2}{\lambda_0^2}$

Task 2: Poynting vector of a spherical wave (a=2, b=1, c=2 pts.)

Consider the simplest expression for an electromagnetic spherical wave in vacuum, written in the spherical coordinates (r,θ,φ) as

$$\mathbf{E}(r,\theta,\varphi,t) = E_0\frac{\sin(\theta)}{r}\left[\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right]\hat{\varphi},$$

where $k = \omega/c$.

a) Find the expression for the magnetic field. Show that $\mathbf{E}$ satisfies the Maxwell's equations in vacuum, by checking all 4 equations.
b) Calculate the Poynting vector.
c) Find the expression for the total radiated power through a spherical surface A, given by $\int \mathbf{I}\cdot d\mathbf{A}$, with $\mathbf{I}$ the intensity vector.

In Spherical coordinate

$$\text{grad} f = \nabla f = \frac{\partial f}{\partial r}\vec{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\vec{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\vec{\varphi}$$

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin(\theta)A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}A_\varphi$$

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\vec{e}_r & r\vec{e}_\theta & r\sin\theta\,\vec{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi\end{vmatrix}$$

$$\text{div}\,(\text{grad} f) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{\partial f}{\partial\varphi}\right)$$

$\frac{\partial}{\partial r}\sin(kr-\omega t) = k\cos(kr-\omega t)$

$-\frac{1}{kr^2}\cos(kr-\omega t) - \frac{1}{r}\sin(kr-\omega t)$

$\frac{2E_0\cos\theta k}{\omega\mu_0 r^2}\left[\left(1-\frac{1}{k^2r^2}\right)\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right]$

$\frac{2E_0\sin\theta\cos\theta}{r\sin\theta\,\mu_0\omega r^2}$

$$\nabla\times\vec{E} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\vec{r} & r\vec{\theta} & r\sin\theta\,\vec{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ E_r & rE_\theta & r\sin\theta E_\varphi\end{vmatrix} = \frac{1}{r^2\sin\theta}\left[\left(\frac{\partial}{\partial\theta}r\sin\theta E_\varphi - \frac{\partial}{\partial\varphi}rE_\theta\right)\vec{r} + r\left(\frac{\partial}{\partial\varphi}E_r - \frac{\partial}{\partial r}(r\sin\theta E_\varphi)\right)\vec{\theta} + r\sin\theta\left(\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial}{\partial\theta}E_r\right)\vec{\varphi}\right]$$

$$= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta E_\varphi - \frac{\partial}{\partial\varphi}E_\theta\right)\vec{r} + \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\varphi}E_r - \frac{\partial}{\partial r}r\sin\theta E_\varphi\right)\vec{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}rE_\theta - \frac{\partial}{\partial\theta}E_r\right)\vec{\varphi}$$

$$E(r,\theta,\varphi,t) = E_0\frac{\sin\theta}{r}\left[\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right]\vec{\varphi} = E_\varphi\vec{\varphi} \Rightarrow E_r = E_\theta = 0$$

$$\Rightarrow \nabla\times\vec{E} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta E_\varphi\vec{r} - \frac{1}{r\sin\theta}\frac{\partial}{\partial r}r\sin\theta E_\varphi\vec{\theta} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta E_\varphi\vec{r} - \frac{\partial}{\partial r}r\sin\theta E_\varphi\vec{\theta}\right)$$

$$\frac{\partial}{\partial\theta}\sin\theta E_\varphi = \frac{E_0}{r}\left[\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right]\frac{d\sin^2\theta}{d\theta} = \frac{2E_0}{r}\sin\theta\cos\theta\left[\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right]$$

$$\frac{\partial}{\partial r}r\sin\theta E_\varphi = E_0\sin^2\theta\left[\frac{\partial}{\partial r}\cos(kr-\omega t) - \frac{\partial}{\partial r}\frac{1}{kr}\sin(kr-\omega t)\right]$$

$$= E_0 \sin\theta \left[-k\sin(kr-\omega t) - \left[-\frac{1}{kr^2}\sin(kr-\omega t) + \frac{1}{r}\cos(kr-\omega t)\right]\right]$$

$$= E_0\sin\theta \left[\left(\frac{1}{kr^2}-k\right)\sin(kr-\omega t) - \frac{1}{r}\cos(kr-\omega t)\right]$$

$$\Rightarrow \nabla\times\vec{E} = \frac{2E_0\cos\theta}{r^2}\left[\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right]\vec{r} + \frac{E_0\sin\theta}{r}\left[\frac{1}{r}\cos(kr-\omega t) - \left(\frac{1}{kr^2}-k\right)\sin(kr-\omega t)\right]\vec{\theta}$$

$$\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0\frac{\partial \vec{H}}{\partial t} \Rightarrow \partial\vec{H} = -\frac{1}{\mu_0}(\nabla\times\vec{E})\partial t \Rightarrow \vec{H}(r,\theta,\varphi,t) = -\frac{1}{\mu_0}\int \nabla\times\vec{E}\, dt$$

$$\int(\nabla\times\vec{E})dt = \frac{2E_0\cos\theta}{r^2}\left[\int\cos(kr-\omega t)dt - \frac{1}{kr}\int\sin(kr-\omega t)dt\right]\vec{r} + \frac{E_0\sin\theta}{r}\left[\frac{1}{r}\int\cos(kr-\omega t)dt - \left(\frac{1}{kr^2}-k\right)\int\sin(kr-\omega t)dt\right]\vec{\theta}$$

$$= \frac{2E_0\cos\theta}{r^2}\left[-\frac{1}{\omega}\sin(kr-\omega t) - \frac{1}{\omega kr}\cos(kr-\omega t)\right]\vec{r} + \frac{E_0\sin\theta}{r}\left[-\frac{1}{\omega r}\sin(kr-\omega t) - \frac{1}{\omega}\left(\frac{1}{kr^2}-k\right)\cos(kr-\omega t)\right]\vec{\theta}$$

$$\Rightarrow \vec{H}(r,\theta,\varphi,t) = \frac{2E_0\cos\theta}{\omega\mu_0 r^2}\left[\sin(kr-\omega t) + \frac{1}{kr}\cos(kr-\omega t)\right]\vec{r} + \frac{E_0\sin\theta}{\mu_0\omega r^2}\left[\sin(kr-\omega t) + \left(\frac{1}{kr}-kr\right)\cos(kr-\omega t)\right]\vec{\theta}$$

Maxwell' equations in Vacanm: ① $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ ② $\nabla\cdot\vec{D} = 0$ ③ $\nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t}$ ④ $\nabla\cdot\vec{H} = 0$

As I have found the expression for the magnetic field by equation ①, $\vec{E}$ satisfies the first equation.

for ②: $\nabla\cdot\vec{D} = \nabla\cdot\varepsilon_0\vec{E} = 0 \Rightarrow \nabla\cdot\vec{E} = 0 \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}r^2E_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta E_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}E_\varphi = 0$

since $E_r = E_\theta = 0$, if $\frac{\partial}{\partial\varphi}E_\varphi = 0$, $\nabla\cdot\vec{E} = 0$

$$\frac{\partial}{\partial\varphi}E_\varphi = \frac{\partial}{\partial\varphi}E_0\frac{\sin\theta}{r}\left[\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right] = 0$$, thus $\nabla\cdot\vec{D} = 0$

for ③: $\frac{\partial\vec{D}}{\partial t} = \varepsilon_0\frac{\partial\vec{E}}{\partial t} = \frac{\varepsilon_0E_0\sin\theta}{r}\left[\frac{\partial}{\partial t}\cos(kr-\omega t) - \frac{1}{kr}\frac{\partial}{\partial t}\sin(kr-\omega t)\right]\vec{\varphi} = \frac{\varepsilon_0E_0\sin\theta\,\omega}{r}\left[\sin(kr-\omega t) + \frac{1}{kr}\cos(kr-\omega t)\right]\vec{\varphi}$

$$\nabla\times\vec{H} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\vec{r} & r\vec{\theta} & r\sin\theta\vec{\varphi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi}\\ H_r & rH_\theta & r\sin\theta H_\varphi\end{vmatrix} \xRightarrow{H_\varphi = 0} \begin{vmatrix}\vec{r} & r\vec{\theta} & r\sin\theta\vec{\varphi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi}\\ H_r & rH_\theta & 0\end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\left(-\frac{\partial}{\partial\varphi}rH_\theta\right)\vec{r} + r\frac{\partial}{\partial\varphi}H_r\vec{\theta} + r\sin\theta\left(\frac{\partial}{\partial r}rH_\theta - \frac{\partial}{\partial\theta}H_r\right)\vec{\varphi}\right]$$

$$= \frac{-1}{r\sin\theta}\frac{\partial}{\partial\varphi}H_\theta\vec{r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}H_r\vec{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}rH_\theta - \frac{\partial}{\partial\theta}H_r\right)\vec{\varphi}$$

$\frac{\partial}{\partial\varphi}H_\theta = \frac{\partial}{\partial\varphi}\left\{\frac{E_0\sin\theta}{\mu_0\omega r^2}\left[\sin(kr-\omega t) + \left(\frac{1}{kr}-kr\right)\cos(kr-\omega t)\right]\right\} = 0$ similarly $\frac{\partial}{\partial\varphi}H_r = 0$

$$\frac{\partial}{\partial\theta}H_r = \frac{-2E_0\sin\theta}{\omega\mu_0 r^2}\left[\sin(kr-\omega t) + \frac{1}{kr}\cos(kr-\omega t)\right]$$

$$\frac{\partial}{\partial r}rH_\theta = \frac{E_0\sin\theta}{\mu_0\omega}\left[\frac{\partial}{\partial r}\frac{1}{r}\sin(kr-\omega t) + \frac{\partial}{\partial r}\left(\frac{1}{kr^2}-k\right)\cos(kr-\omega t)\right]$$

$$= \frac{E_0\sin\theta}{\mu_0\omega}\left[-\frac{1}{r^2}\sin(kr-\omega t) + \frac{k}{r}\cos(kr-\omega t) - \frac{2}{kr^3}\cos(kr-\omega t) - \left(\frac{1}{r^2}-k^2\right)\sin(kr-\omega t)\right]$$

$$= \frac{E_0\sin\theta}{\mu_0\omega}\left[\left(k^2-\frac{2}{r^2}\right)\sin(kr-\omega t) + \left(\frac{k}{r}-\frac{2}{kr^3}\right)\cos(kr-\omega t)\right] = \frac{E_0\sin\theta}{\mu_0\omega r^2}\left[(r^2k^2-2)\sin(kr-\omega t) + \left(rk-\frac{2}{kr}\right)\cos(kr-\omega t)\right]$$

$$\frac{\partial}{\partial r}rH_\theta - \frac{\partial}{\partial\theta}H_r = \frac{E_0\sin\theta}{\mu_0\omega r^2}\left[r^2k^2\sin(kr-\omega t) + rk\cos(kr-\omega t)\right]$$

$$\nabla\times\vec{H} = \frac{1}{r}\left(\frac{\partial}{\partial r}rH_\theta - \frac{\partial}{\partial\theta}H_r\right)\vec{\varphi} = \frac{E_0\sin\theta}{\mu_0\omega r}\left[k^2\sin(kr-\omega t) + \frac{k}{r}\cos(kr-\omega t)\right]$$ $k = \frac{\omega}{c} \Rightarrow k^2 = \frac{\omega^2}{c^2}$ $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \Rightarrow k = \omega^2\varepsilon_0\mu_0$

$$= \frac{E_0\sin\theta k^2}{\mu_0\omega r}\left[\sin(kr-\omega t) + \frac{1}{kr}\cos(kr-\omega t)\right] = \underline{\frac{\varepsilon_0E_0\sin\theta\,\omega}{r}\left[\sin(kr-\omega t) + \frac{1}{kr}\cos(kr-\omega t)\right] = \frac{\partial\vec{D}}{\partial t}}$$

(b) $\vec{E}(r,\theta,\varphi,t) = E_0\frac{\sin\theta}{2r}\left[e^{i(kr-\omega t)} + e^{-i(kr-\omega t)} - \frac{1}{ikr}\left[e^{i(kr-\omega t)} - e^{-i(kr-\omega t)}\right]\right]\vec{\varphi}$

$$= E_0\frac{\sin\theta}{2r}\left[e^{i(kr-\omega t)} - \frac{1}{ikr}e^{i(kr-\omega t)}\right]\vec{\varphi} + E_0\frac{\sin\theta}{2r}\left[e^{-i(kr-\omega t)} - \frac{1}{ikr}e^{-i(kr-\omega t)}\right]\vec{\varphi}$$

$$= \frac{1}{2}\left[E_0\frac{\sin\theta}{r}\left(e^{ikr}-\frac{1}{ikr}e^{ikr}\right)e^{-i\omega t}+E_0\frac{\sin\theta}{r}\left(e^{-ikr}-\frac{1}{ikr}e^{-ikr}\right)e^{i\omega t}\right]\vec{\varphi}$$

$$= \frac{1}{2}\left[\tilde{\vec{E}}(\vec{r},t)e^{i\omega t}+c.c\right] = \vec{E}_r(\vec{r},t) \Rightarrow \text{Similarly } \vec{H}(r,\varphi,\varphi,t) = \vec{H}_r(\vec{r},t)$$

$$\vec{S}(\vec{r},t) = \vec{E}_r(\vec{r},t)\times\vec{H}_r(\vec{r},t)$$

$$= \begin{vmatrix} \vec{r} & \vec{\theta} & \vec{\varphi} \\ E_r & E_\theta & E_\varphi \\ H_r & H_\theta & H_\varphi \end{vmatrix} = (E_\theta H_\varphi - E_\varphi H_\theta)\vec{r} + (E_\varphi H_r - E_r H_\varphi)\vec{\theta} + (E_r H_\theta - E_\theta H_r)\vec{\varphi} \xrightarrow{E_\theta = E_r = H_\varphi = 0} -E_\varphi H_\theta \vec{r} + E_\varphi H_r \vec{\theta}$$

$\cos 2x = \cos^2 x - \sin^2 x$
$= 1-2\sin^2 x = 2\cos^2 x - 1$
$\sin(2x) = 2\sin x\cos x$

$$E_\varphi H_\theta = E_0\frac{\sin\theta}{r}\left[\cos(kr-\omega t)-\frac{1}{kr}\sin(kr-\omega t)\right]\times\frac{E_0\sin\theta}{\mu_0\omega r^2}\left[\sin(kr-\omega t)+\left(\frac{1}{kr}-kr\right)\cos(kr-\omega t)\right]$$

$$= \frac{E_0^2\sin^2\theta}{\mu_0\omega r^3}\left[\sin(kr-\omega t)\cos(kr-\omega t)+\left(\frac{1}{kr}-kr\right)\cos^2(kr-\omega t)-\frac{1}{kr}\sin^2(kr-\omega t)+\left(1-\frac{1}{k^2r^2}\right)\sin(kr-\omega t)\cos(kr-\omega t)\right]$$

$$= \frac{E_0^2\sin^2\theta}{\mu_0\omega r^3}\left[\frac{1}{2}\left(2-\frac{1}{k^2r^2}\right)\sin[2(kr-\omega t)]+\frac{1}{kr}\cos[2(kr-\omega t)]-\frac{kr}{2}\left(1+\cos[2(kr-\omega t)]\right)\right]$$

$$= \frac{E_0^2\sin^2\theta}{\mu_0\omega r^3}\left[\left(1-\frac{1}{2k^2r^2}\right)\sin[2(kr-\omega t)]+\left(\frac{1}{kr}-\frac{kr}{2}\right)\cos[2(kr-\omega t)]-\frac{kr}{2}\right]$$

$$E_\varphi H_r = \frac{E_0\sin\theta}{r}\left[\cos(kr-\omega t)-\frac{1}{kr}\sin(kr-\omega t)\right]\times\frac{2E_0\cos\theta}{\mu_0\omega r^2}\left[\sin(kr-\omega t)+\frac{1}{kr}\cos(kr-\omega t)\right]$$

$$= \frac{E_0^2\sin(2\theta)}{\omega\mu_0 r^3}\left[\sin(kr-\omega t)\cos(kr-\omega t)+\frac{1}{kr}\cos^2(kr-\omega t)-\frac{1}{kr}\sin^2(kr-\omega t)-\frac{1}{k^2r^2}\sin(kr-\omega t)\cos(kr-\omega t)\right]$$

$$= \frac{E_0^2\sin(2\theta)}{\omega\mu_0 r^3}\left[\frac{1}{2}\left(1-\frac{1}{k^2r^2}\right)\sin[2(kr-\omega t)]+\frac{1}{kr}\cos[2(kr-\omega t)]\right]$$

$$\Rightarrow \vec{S}(\vec{r},t) = \frac{E_0^2\sin^2\theta}{\mu_0\omega r^3}\left[\left(\frac{1}{2k^2r^2}-1\right)\sin[2(kr-\omega t)]+\left(\frac{kr}{2}-\frac{1}{kr}\right)\cos[2(kr-\omega t)]+\frac{kr}{2}\right]\vec{r}+\frac{E_0^2\sin(2\theta)}{\omega\mu_0 r^3}\left[\frac{1}{2}\left(1-\frac{1}{k^2r^2}\right)\sin[2(kr-\omega t)]+\frac{1}{kr}\cos[2(kr-\omega t)]\right]\vec{\theta}$$

(c) $I = |\langle\vec{S}(\vec{r},t)\rangle|$ $\quad\langle\vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}[\vec{E}(\vec{r})\times\vec{H}^*(\vec{r})]$

$$\vec{E}(r,\theta,\varphi,t) = \frac{1}{2}\left[E_0\frac{\sin\theta}{r}\left(e^{ikr}-\frac{1}{ikr}e^{ikr}\right)e^{-i\omega t}+E_0\frac{\sin\theta}{r}\left(e^{-ikr}-\frac{1}{ikr}e^{-ikr}\right)e^{i\omega t}\right] = \frac{1}{2}\left[\vec{E}(\vec{r})e^{-i\omega t}+\vec{E}^*(\vec{r})e^{i\omega t}\right]$$

$$\Rightarrow \vec{E}(\vec{r}) = E_0\frac{\sin\theta}{r}\left(e^{ikr}-\frac{1}{ikr}e^{ikr}\right) = E_0\frac{\sin\theta}{r}\left(1-\frac{1}{ikr}\right)e^{ikr}\vec{\varphi}$$

$$H_\theta = \frac{E_0\sin\theta}{\omega\mu_0 r^2}\left[\sin(kr-\omega t)+\left(\frac{1}{kr}-kr\right)\cos(kr-\omega t)\right] = \frac{E_0\sin\theta}{\omega\mu_0 r^2}\left\{\frac{1}{2i}\left[e^{i(kr-\omega t)}-e^{-i(kr-\omega t)}\right]+\frac{1}{2}\left(\frac{1}{kr}-kr\right)\left[e^{i(kr-\omega t)}+e^{-i(kr-\omega t)}\right]\right\}$$

$$= \frac{E_0\sin\theta}{2\omega\mu_0 r^2}\left[\left(\frac{1}{i}+\frac{1}{kr}-kr\right)e^{ikr}e^{-i\omega t}+\left(\frac{1}{kr}-kr-\frac{1}{i}\right)e^{-ikr}e^{i\omega t}\right]$$

$$H_r = \frac{2E_0\cos\theta}{\mu_0\omega r^2}\left[\sin(kr-\omega t)+\frac{1}{kr}\cos(kr-\omega t)\right] = \frac{E_0\cos\theta}{\mu_0\omega r^2}\left\{\frac{1}{i}\left[e^{i(kr-\omega t)}-e^{-i(kr-\omega t)}\right]+\frac{1}{kr}\left[e^{i(kr-\omega t)}+e^{-i(kr-\omega t)}\right]\right\}$$

$$= \frac{E_0\cos\theta}{\mu_0\omega r^2}\left[\left(\frac{1}{kr}+\frac{1}{i}\right)e^{ikr}e^{-i\omega t}+\left(\frac{1}{kr}-\frac{1}{i}\right)e^{-ikr}e^{i\omega t}\right]$$

$$\Rightarrow \vec{H}(\vec{r}) = \frac{2E_0\cos\theta}{\mu_0\omega r^2}\left(\frac{1}{kr}-i\right)e^{ikr}\vec{r}+\frac{E_0\sin\theta}{\omega\mu_0 r^2}\left(\frac{1}{kr}-kr-i\right)e^{ikr}\vec{\theta},\quad \vec{H}^*(\vec{r}) = \frac{2E_0\cos\theta}{\mu_0\omega r^2}\left(\frac{1}{kr}+i\right)e^{-ikr}\vec{r}+\frac{E_0\sin\theta}{\omega\mu_0 r^2}\left(\frac{1}{kr}-kr+i\right)e^{-ikr}\vec{\theta}$$

$$\vec{E}(\vec{r})\times\vec{H}^*(\vec{r}) = \begin{vmatrix} \vec{r} & \vec{\theta} & \vec{\varphi} \\ E_r & E_\theta & E_\varphi \\ H_r^* & H_\theta^* & H_\varphi^* \end{vmatrix} = (E_\theta H_\varphi^* - E_\varphi H_\theta^*)\vec{r} + (E_\varphi H_r^* - E_r H_\varphi^*)\vec{\theta} + (E_r H_\theta^* - E_\theta H_r^*)\vec{\varphi} \xrightarrow{E_r = E_\theta = H_\varphi^* = 0} -E_\varphi H_\theta^*\vec{r} + E_\varphi H_r^*\vec{\theta}$$

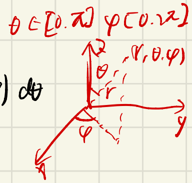
$$E_\varphi H_\theta^* = E_0\frac{\sin\theta}{r}\left(1-\frac{1}{ikr}\right)e^{ikr}\times\frac{E_0\sin\theta}{\omega\mu_0 r^2}\left(\frac{1}{kr}-kr+i\right)e^{-ikr} = \frac{E_0^2\sin^2\theta}{\omega\mu_0 r^3}\left(\frac{i}{k^2r^2}-kr\right)$$

$$E_\varphi H_r^* = E_0\frac{\sin\theta}{r}\left(1-\frac{1}{ikr}\right)e^{ikr}\times\frac{2E_0\cos\theta}{\mu_0\omega r^2}\left(\frac{1}{kr}+i\right)e^{-ikr} = \frac{iE_0^2\sin 2\theta}{\mu_0\omega r^3}\left(\frac{1}{k^2r^2}+1\right)$$

$$\vec E(\vec r)\times \vec H^*(\vec r) = \frac{E_0^2\sin^2\theta}{\omega\mu_0 r^3}\left(kr - \frac{1}{k^2r^2}\right)\vec r + i\frac{E_0^2\sin 2\theta}{\mu_0\omega r^3}\left(\frac{1}{k^2r^2}+1\right) \quad \langle \vec S(\vec r,t)\rangle = \frac12 Re[\vec E(\vec r)\times\vec H^*(\vec r)] = \frac12 \frac{E_0^2\sin^2\theta k}{\omega\mu_0 r^2}\vec r$$

$$I = |\langle\vec S(\vec r,t)\rangle| = \sqrt{\langle \vec S(\vec r,t)\rangle^2} = \frac{E_0^2\sin^2\theta k}{2\omega\mu_0 r^2}$$

$dA = r^2\sin\theta d\theta d\varphi$ $\quad dv = r^2\sin\theta dr d\theta d\varphi$ $\quad \theta\in[0,\pi]$ $\varphi\in[0,2\pi]$



$$\int I\cdot dA = \int_0^{2\pi}\int_0^\pi \frac{E_0^2\sin^2\theta k}{2\omega\mu_0 r^2} r^2\sin\theta d\theta d\varphi = \frac{E_0^2 k}{2\omega\mu_0}\int_0^{2\pi}d\varphi\int_0^\pi \sin^3\theta d\theta = \frac{\pi E_0^2 k}{\omega\mu_0}\int_0^\pi \sin\theta(1-\cos^2\theta)\,d\theta$$

$$= \frac{\pi E_0^2 k}{\omega\mu_0}\int_0^\pi \cos^2\theta - 1\, d\cos\theta$$

$$= \frac{\pi E_0^2 k}{\omega\mu_0}\left(\int_0^\pi \cos^2\theta\, d\cos\theta - \int_0^\pi d\cos\theta\right)$$

$$= \frac{\pi E_0^2 k}{\omega\mu_0}\left(\frac13\cos^3\theta\Big|_0^\pi - \cos\theta\Big|_0^\pi\right)$$

$$= \frac{4\pi E_0^2 k}{3\omega\mu_0}$$

Task 3: Poynting Vector for a Surface Guided Wave (a=2*, b=2, c=2, d=2, e=2, f=2 pts.)

We have learned in the previous exercise series about the separation of EM modes in TE and TM for the cases that we have invariance of the structure in one direction. Let us investigate one such structure with invariance in y direction: a metal-to-air interface at $z = 0$ with $\varepsilon(z < 0) = \varepsilon_M < 0$ and $\varepsilon(z > 0) = 1$. It turns out that this geometry supports a TM surface guided mode (propagating in the x-direction and confined to the interface along the z-direction from both sides) with frequency ω with the following magnetic field:

$$\mathbf{H} = \begin{bmatrix} 0 \\ H_y \\ 0 \end{bmatrix}; \quad H_y = H_0 e^{ik_x x}\begin{cases}\exp\{-\sqrt{k_x^2 - \frac{\omega^2}{c^2}}\, z\} & , z > 0 \\ \exp\{+\sqrt{k_x^2 - \frac{\omega^2}{c^2}\varepsilon_M}\, z\} & , z < 0\end{cases}$$

with $k_x > 0$ and $k_x^2 > \frac{\omega^2}{c^2}$.

a*) Show that if we want to have such a guided/confined wave with the above mentioned characteristics, H_y would indeed have the form given above.
Hint: The eigenmodes of each of these half spaces are still plane waves of the form $H_y \propto e^{i(k_x x + k_z z)}$, where the wave numbers satisfy the dispersion relations of their corresponding media. The key to have a mode that coexists in both media is that both waves in metal and air should have the same k_x (satisfying the boundary condition for tangential magnetic field across the interface).

b) Use Maxwell's equations to find the expression for the electric field.

c) Make a drawing of both fields and their behaviours close to the metal-air interface.

d) Use the continuity of E_x at the interface to show that

$$-\varepsilon_M\sqrt{k_x^2 - \frac{\omega^2}{c^2}} = \sqrt{k_x^2 - \frac{\omega^2}{c^2}\varepsilon_M}$$

From this relation it can be seen that having $\varepsilon_M < 0$ is essential for having a confined wave in the z-direction. But is it enough for having it propagating in the x-direction? Find k_x from this equation and find the condition on ε_M for having a real valued k_x.

e) Calculate the time-averaged Poynting vector as a function of distance z from the metal-air interface. Make a schematic drawing of it.

f) Calculate the net flow of energy per unit length (per unit length of the y-direction) by integrating the time-averaged Poynting vector from e) over $z \in [-\infty, \infty]$. In which direction does the net energy flow?

$$H_0\exp\{-\sqrt{k_x^2-\frac{\omega^2}{c^2}}z \cos k_x x$$

$$\nabla\times\vec H = j + \frac{\partial D}{\partial t} = j + \varepsilon_0\varepsilon(\omega)\frac{\partial\vec E}{\partial t} \quad j=0 \Rightarrow \nabla\times\vec H = \varepsilon_0\varepsilon(\omega)\frac{\partial \vec E}{\partial t}$$

$$\nabla\times\nabla\times\vec H = \nabla(\nabla\cdot\vec H) - \Delta\vec H = -\Delta\vec H = \varepsilon_0\varepsilon(\omega)\frac{\partial\nabla\times\vec E}{\partial t} = -\mu_0\varepsilon_0\varepsilon\frac{\partial^2 H}{\partial t^2} \Rightarrow \Delta\vec H - \frac{\varepsilon}{c^2}\frac{\partial^2 H}{\partial t^2}=0$$

Invariance in y direction $\Rightarrow H_x=H_z=0$ $\quad \Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}$ and $k_y=0$

$$\Rightarrow \frac{\partial^2}{\partial x^2}H_y + \frac{\partial^2}{\partial z^2}H_y = \frac{\varepsilon}{c^2}\frac{\partial^2 H_y(\vec r,t)}{\partial t^2} \Rightarrow -(k_x^2+k_z^2) + \frac{\omega^2}{c^2}\varepsilon = 0 \Rightarrow k_z = \pm i\sqrt{k_x^2 - \frac{\omega^2}{c^2}\varepsilon}$$

Plane wave function: $H = H_0 e^{i\vec k\vec r} = H_0 e^{i(k_x x + k_y y + k_z z)} \xrightarrow{k_y=0} H_0 e^{i(k_x x + k_z z)}$

$z>0$, $\varepsilon=1$ $\quad k_z = i\sqrt{k_x^2-\frac{\omega^2}{c^2}} \Rightarrow H_y = H_0e^{ik_xx}\exp[i(i\sqrt{k_x^2-\frac{\omega^2}{c^2}})z] = H_0e^{ik_xx}\exp(-\sqrt{k_x^2-\frac{\omega^2}{c^2}}z)$

$z<0$, $\varepsilon=\varepsilon_M$ $\quad k_z=-i\sqrt{k_x^2-\frac{\omega^2}{c^2}\varepsilon_M} \Rightarrow H_y = H_0e^{ik_xx}\exp[i(-i\sqrt{k_x^2-\frac{\omega^2}{c^2}\varepsilon_M})z] = H_0e^{ik_xx}\exp(\sqrt{k_x^2-\frac{\omega^2}{c^2}\varepsilon_M}z)$

Thus $H_y = H_0e^{ik_xx}\begin{cases}\exp(-\sqrt{k_x^2-\frac{\omega^2}{c^2}}z) & z>0\\ \exp(\sqrt{k_x^2-\frac{\omega^2}{c^2}\varepsilon_M}z) & z<0\end{cases}$

(b) $$\nabla\times\vec H = \begin{vmatrix}\vec e_x & \vec e_y & \vec e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z\end{vmatrix} = \left(\frac{\partial}{\partial y}H_z - \frac{\partial}{\partial z}H_y\right)\vec e_x + \left(\frac{\partial}{\partial z}H_x - \frac{\partial}{\partial x}H_z\right)\vec e_y + \left(\frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x\right)\vec e_z$$

$$\xrightarrow{H_x=H_z=0} -\frac{\partial}{\partial z}H_y\vec e_x + \frac{\partial}{\partial x}H_y\vec e_z = -i\omega\varepsilon_0\varepsilon(\omega)\vec E \Rightarrow \vec E = \frac{1}{-i\omega\varepsilon_0\varepsilon(\omega)}\left(\frac{\partial}{\partial x}H_y\vec e_z - \frac{\partial}{\partial z}H_y\vec e_x\right)$$

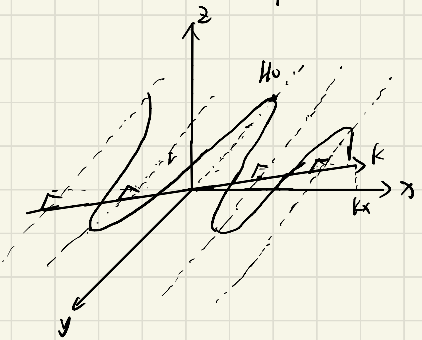
$z>0$ $\quad \frac{\partial}{\partial z}H_y = -H_0\sqrt{k_x^2-\frac{\omega^2}{c^2}}\,e^{ik_xx}\exp(-\sqrt{k_x^2-\frac{\omega^2}{c^2}}z)$ $\xrightarrow{\varepsilon=1} \vec E = \begin{bmatrix} -\frac{H_0\sqrt{k_x^2-\frac{\omega^2}{c^2}}}{i\omega\varepsilon_0}e^{ik_xx}\exp(-\sqrt{k_x^2-\frac{\omega^2}{c^2}}z) \\ 0 \\ -\frac{H_0k_x}{\omega\varepsilon_0}e^{ik_xx}\exp(-\sqrt{k_x^2-\frac{\omega^2}{c^2}}z)\end{bmatrix}$

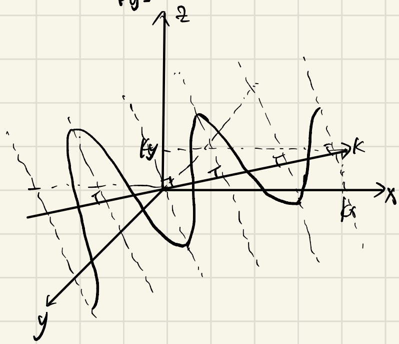
$\frac{\partial}{\partial x}H_y = iH_0k_xe^{ik_xx}\exp(-\sqrt{k_x^2-\frac{\omega^2}{c^2}}z)$

$z<0$ $\quad \frac{\partial}{\partial z}H_y = H_0\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_m}\, e^{ik_x x}\exp\left(\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_m}\, z\right)\vec{e}_x$

$\frac{\partial}{\partial x}H_y = iH_0k_x e^{ik_x x}\exp\left(\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}\, z\right)\vec{e}_z \xrightarrow[\varepsilon=\varepsilon_M]{} \vec{E} = \begin{bmatrix} -\frac{iH_0\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}}{w\varepsilon_0\varepsilon_M} e^{ik_x x}\exp\left(\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}\, z\right) \\ 0 \\ \frac{-H_0k_x}{w\varepsilon_0\varepsilon_M} e^{ik_x x}\exp\left(\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}\, z\right)\end{bmatrix}$

(c) $\vec{H}=H_y$ $\quad \mathrm{Re}H_y = H_0\exp\left(-\sqrt{k_x^2-\frac{w^2}{c^2}}\, z\right)\cos k_x x$ $\qquad \vec{E} = \begin{bmatrix} E_x \\ 0 \\ E_y \end{bmatrix}$





(d) $E_{x_1} = E_{x_2} \Rightarrow \frac{H_0\sqrt{k_x^2-\frac{w^2}{c^2}}}{iw\varepsilon_0} e^{ik_x x}\exp\left(-\sqrt{k_x^2-\frac{w^2}{c^2}}\, z\right) = \frac{iH_0\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}}{w\varepsilon_0\varepsilon_M} e^{ik_x x}\exp\left(\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}\, z\right)$

at the surface $z=0 \Rightarrow \frac{\sqrt{k_x^2-\frac{w^2}{c^2}}}{i} = \frac{i\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}}{\varepsilon_M} \Rightarrow -\varepsilon_M\sqrt{k_x^2-\frac{w^2}{c^2}} = \sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}$

$\Rightarrow \varepsilon_M^2\left(k_x^2-\frac{w^2}{c^2}\right) = k_x^2-\frac{w^2}{c^2}\varepsilon_M = (\varepsilon_M^2-1)k_x^2 = \frac{w^2}{c^2}(\varepsilon_M^2-\varepsilon_M) \Rightarrow (\varepsilon_M-1)(\varepsilon_M+1)k_x^2 = \frac{w^2}{c^2}\varepsilon_M(\varepsilon_M-1)$

$\varepsilon_M \neq 1 \Rightarrow k_x^2 = \frac{w^2\varepsilon_M}{c^2(\varepsilon_M+1)} \Rightarrow k_x = \frac{w}{c}\sqrt{\frac{\varepsilon_M}{\varepsilon_M+1}}$

if k_x has a real value, then $\varepsilon_M<-1$

(e) $\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}\left[\vec{E}(\vec{r})\times\vec{H}^*(\vec{r})\right]$ $\qquad \vec{E}(\vec{r})\times\vec{H}^*(\vec{r}) = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ E_x & E_y & E_z \\ H_x^* & H_y^* & H_z^* \end{vmatrix} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ E_x & 0 & E_z \\ 0 & H_y^* & 0 \end{vmatrix}$

$z>0$ $\quad H_y^* = H_0 e^{-ik_x x}\exp\left(-\sqrt{k_x^2-\frac{w^2}{c^2}}\, z\right)$

$E_z H_y^* = \frac{-H_0k_x}{w\varepsilon_0} e^{ik_x x}\exp\left(-\sqrt{k_x^2-\frac{w^2}{c^2}}\, z\right)\cdot H_0 e^{-ik_x x}\exp\left(-\sqrt{k_x^2-\frac{w^2}{c^2}}\, z\right) = \frac{-H_0^2k_x}{w\varepsilon_0}\exp\left(-2\sqrt{k_x^2-\frac{w^2}{c^2}}\, z\right)$

$E_x H_y^* = \frac{-H_0\sqrt{k_x^2-\frac{w^2}{c^2}}}{iw\varepsilon_0} e^{ik_x x}\exp\left(-\sqrt{k_x^2-\frac{w^2}{c^2}}\, z\right)\cdot H_0 e^{-ik_x x}\exp\left(-\sqrt{k_x^2-\frac{w^2}{c^2}}\, z\right) = \frac{iH_0^2\sqrt{k_x^2-\frac{w^2}{c^2}}}{w\varepsilon_0}\exp\left(-2\sqrt{k_x^2-\frac{w^2}{c^2}}\, z\right)$

$\vec{E}(\vec{r})\times\vec{H}^*(\vec{r}) = \frac{H_0^2k_x}{w\varepsilon_0}\exp\left(-2\sqrt{k_x^2-\frac{w^2}{c^2}}\, z\right)\vec{e}_x + \frac{iH_0^2\sqrt{k_x^2-\frac{w^2}{c^2}}}{w\varepsilon_0}\exp\left(-2\sqrt{k_x^2-\frac{w^2}{c^2}}\, z\right)\vec{e}_z$

$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}\left[\vec{E}(\vec{r})\times\vec{H}^*(\vec{r})\right] = \frac{1}{2}\frac{H_0^2k_x}{w\varepsilon_0}\exp\left(-2\sqrt{k_x^2-\frac{w^2}{c^2}}\, z\right)\vec{e}_x$

$z<0$ $\quad H_y^* = H_0 e^{-ik_x x}\exp\left(\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}\, z\right)$

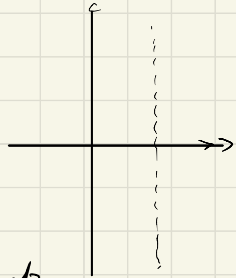
$E_z\cdot H_y^* = \frac{-H_0k_x}{w\varepsilon_0\varepsilon_M} e^{ik_x x}\exp\left(\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}\, z\right)\cdot H_0 e^{-ik_x x}\exp\left(\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}\, z\right) = \frac{-H_0^2k_x}{w\varepsilon_0\varepsilon_{M_2}}\exp\left(2\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}\, z\right)$

$E_x\cdot H_y^* = \frac{-iH_0\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}}{w\varepsilon_0\varepsilon_M} e^{ik_x x}\exp\left(\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}\, z\right)\cdot H_0 e^{-ik_x x}\exp\left(\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}\, z\right) = \frac{-iH_0^2\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}}{\varepsilon_0\varepsilon_M w}\exp\left(2\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}\, z\right)$

$\Rightarrow \vec{E}(\vec{r})\times\vec{H}^*(\vec{r}) = \frac{H_0^2k_x}{w\varepsilon_0\varepsilon_M}\exp\left(2\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}\, z\right)\vec{e}_x - \frac{iH_0^2\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}}{\varepsilon_0\varepsilon_M w}\exp\left(2\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}\,\right)\vec{e}_z$

$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}\left[\vec{E}(\vec{r})\times\vec{H}^*(\vec{r})\right] = \frac{1}{2}\frac{H_0^2k_x}{w\varepsilon_0\varepsilon_M}\exp\left(2\sqrt{k_x^2-\frac{w^2}{c^2}\varepsilon_M}\, z\right)\vec{e}_x$

Thus $\langle \vec{S}(\vec{r},t)\rangle = \begin{cases} \frac{1}{2}\frac{H_0^2 k_x}{\omega\varepsilon_0}\exp\left(-2\sqrt{k_x^2-\frac{\omega^2}{c^2}}\,z\right)\vec{e}_x & z>0 \\ \frac{1}{2}\frac{H_0^2 k_x}{\omega\varepsilon_0\varepsilon_m}\exp\left(2\sqrt{k_x^2-\frac{\omega^2}{c^2}\varepsilon_m}\,z\right)\vec{e}_x & z<0 \end{cases}$



f) net flow of energy per unit lenght $u=\int_{-\infty}^{\infty}\langle\vec{S}(\vec{r},t)\rangle\,dz$

$$\Rightarrow u=\int_0^{\infty}\frac{1}{2}\frac{H_0^2k_x}{\omega\varepsilon_0}\exp\left(-2\sqrt{k_x^2-\frac{\omega^2}{c^2}}\,z\right)dz+\int_{-\infty}^{0}\frac{1}{2}\frac{H_0^2k_x}{\omega\varepsilon_0\varepsilon_m}\exp\left(2\sqrt{k_x^2-\frac{\omega^2}{c^2}\varepsilon_m}\,z\right)dz$$

$$=\frac{1}{2}\frac{H_0^2k_x}{\omega\varepsilon_0}\left.\frac{\exp\left(-2\sqrt{k_x^2-\frac{\omega^2}{c^2}}z\right)}{-2\sqrt{k_x^2-\frac{\omega^2}{c^2}}}\right|_0^{\infty}+\frac{1}{2}\frac{H_0^2k_x}{\omega\varepsilon_0\varepsilon_m}\left.\frac{\exp\left(2\sqrt{k_x^2-\frac{\omega^2}{c^2}\varepsilon_m}\,z\right)}{2\sqrt{k_x^2-\frac{\omega^2}{c^2}\varepsilon_m}}\right|_{-\infty}^{0}$$

$$=\frac{H_0^2k_x}{4\omega\varepsilon_0\sqrt{k_x^2-\frac{\omega^2}{c^2}}}+\frac{H_0^2k_x}{4\omega\varepsilon_0\varepsilon_m\sqrt{k_x^2-\frac{\omega^2}{c^2}\varepsilon_m}}=\frac{H_0^2k_x}{4\varepsilon\varepsilon_0}\left(\frac{1}{\sqrt{k_x^2-\frac{\omega^2}{c^2}}}+\frac{1}{\varepsilon_m\sqrt{k_x^2-\frac{\omega^2}{c^2}\varepsilon_m}}\right)$$

$$\sqrt{k_x^2-\frac{\omega^2}{c^2}\varepsilon_m}=-\varepsilon_m\sqrt{k_x^2-\frac{\omega^2}{c^2}}\Rightarrow u=\frac{H_0^2k_x}{4\varepsilon\varepsilon_0}\frac{1}{\sqrt{k_x^2-\frac{\omega^2}{c^2}}}\left(1-\frac{1}{\varepsilon_m^2}\right)$$

$$\vec{E} = (A_1+A_2)e^{i\beta x} + (A_1+\frac{A_2}{2})e^{ikz} + \frac{A_2}{2}e^{-ikz} \qquad \nabla\times\vec{E} = \nabla\times E_0 e^{ikr}e^{-i\omega t} = -\mu_0 \frac{\partial H_0 e^{ikr}e^{-i\omega t}}{dt}$$

$$= i\vec{k}\times\vec{E} = i\omega\mu_0\vec{H} \Rightarrow \vec{H} = \frac{1}{\omega\mu_0}\vec{k}\times\vec{E}$$

$$\vec{k}\times\vec{E} = \begin{vmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ k_x & k_y & k_z \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ \beta & 0 & k \\ E_x & 0 & E_{z1} \end{vmatrix} + \begin{vmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ \beta & 0 & -k \\ E_x & 0 & E_{z2} \end{vmatrix}$$

$$\Rightarrow kE_x - \beta E_{z1} - kE_x$$

① $\vec{E}(\vec{r}) = e^{i\beta x}[E_1 e^{ikz} + E_2 e^{-ikz}]$ $E_1 = A_1 + \frac{A_2}{2}$ $E_2 = \frac{A_2}{2}$