1. (a) $f(\omega) = \frac{1}{2\pi}\int^{\infty} f(t)\, e^{i\omega t}\, dt$ 149824 Mon exercise 1

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Fundamentals of Modern Optics

series 1

20.10.2014

to be returned on 27.10.2014, at the beginning of the lecture

Initial Remarks

- return the completed assignments by the above mentioned date.
- group work is allowed and encouraged; however each student has to hand in an individual assignment; literal copies will not be accepted.
- problems marked with an asterisk (*) are non-mandatory and can be used to gain extra points.
- assignments will be checked, returned, and discussed in the seminars in the week after the return dates.
- hand in your assignments in hand-writing only; write neatly.
- note your seminar day (Mon, Tue, Thu) on the assignment
- write down all calculations and derivations in a clear and concise manner.

Problem 1 - Fourier Transformations (a=2,b=2+2*,c=2* pts.)

Given is the definition of the Fourier transformation and its inverse, which transforms the time domain representation of a signal $f(t)$ into its frequency domain representation $f(\omega)$ and vice versa:

$$
\begin{aligned}
f(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\exp[i\omega t]\mathrm{d}t \\
f(t) &= \int_{-\infty}^{\infty} f(\omega)\exp[-i\omega t]\mathrm{d}\omega.
\end{aligned}
$$

Use these definitions to find the frequency domain representation of the following signals

a) $f(t) = \begin{cases} 0 & t < 0 \\ A\exp[-\gamma t]\cos(\omega_0 t) & t \geq 0 \end{cases}$

b) $f(t) = A\exp[-\frac{1}{2}\frac{t^2}{t_0^2}]$
This problem involves a complex valued Gaussian integral which you can directly insert its answer to proceed with your solution. The 2 bonus points go to whoever correctly solves that Gaussian integral.

and

c*) Show for the second function that the product of the square root of the second momentum in time domain and in frequency domain is a constant.

Hint: The square root of the second moment $\sqrt{\langle f^2\rangle}$ of a symmetric function is defined as:

$$
\sqrt{\langle f^2\rangle} = \sqrt{\frac{\int_{-\infty}^{\infty} t^2\left|f(t)\right|^2\mathrm{d}t}{\int_{-\infty}^{\infty}\left|f(t)\right|^2\mathrm{d}t}}
$$

Problem 2 - Fourier Transform Properties (a=2,b=2 pts.)

Assume that a signal $f(t)$ is given and its frequency representation $f(\omega)$ is known. Now calculate the frequency domain representation of

a) $f(t-t_0)$, a signal that is translated by t_0

and

b) $\frac{\mathrm{d}}{\mathrm{d}t}f(t)$, the temporal derivative of the signal.

Problem 3 - δ-Functions (a=1*,b=1,c=1,d=1,e=1*,f=1 pts.)

Given is a function $\delta(t)$, with the following properties

i) $\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undefined} & t = 0 \end{cases}$

ii) $\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1$.

With this knowledge, do the followings:

a*) Show that the function $f(t) = \lim_{w\to 0} \frac{1}{\sqrt{\pi}w} \exp[-\frac{t^2}{w^2}]$ fulfils the above mentioned properties and is thus a possible representative of the δ function.

Furthermore, calculate expressions for the following integrals:

b) $\int_{-\infty}^{\infty} \delta(t)f(t)\mathrm{d}t$

c) $\int_{-\infty}^{\infty} \delta(t-t_0)f(t)\mathrm{d}t$

d) $\int_{-\infty}^{\infty} \delta(at)f(t)\mathrm{d}t$

e*) $\int_{-\infty}^{\infty} \delta(g(t))f(t)\mathrm{d}t$, where $g(t)$ is an arbitrary analytic function, with $g(t) = 0 \Leftrightarrow t \in \{t_0^i\} \wedge i \in \{1 \ldots N\}$.
任意的

Now calculate

f) the Fourier transform of the delta function.

Hint: While the solution of the problems b) to f) is possible with a representative function, we suggest to just use the definition i) and ii), in combination with Taylor expansions or change of variables to find solutions.

Problem 4 - The Convolution Theorem (4 pts.)

卷积

Given are two functions $f(t)$ and $g(t)$ and their Fourier transformations $f(\omega)$ and $g(w)$. The convolution $[f \otimes g](t)$ of both functions is defined as

$$[f \otimes g](t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\mathrm{d}\tau.$$

Calculate the fourier transform of the convolution

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} [f \otimes g](t) \exp[i\omega t]\mathrm{d}t.$$

Hint: Replace $f(t)$and $g(t)$ with their respective fourier integrals. Reorder the resulting quadruple integral to generate δ-functions that allow you to solve the integrals.

Yining Li 149824 Mon exercise 1

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1. (a) 6

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

$$= \frac{1}{2\pi}\int_0^{\infty} A e^{-rt} \cdot \cos(\omega_0 t) \cdot e^{i\omega t} dt$$

$$= \frac{A}{2\pi}\int_0^{\infty} e^{-rt} \cdot \frac{e^{i\omega_0 t}+e^{-i\omega_0 t}}{2} \cdot e^{i\omega t} dt$$

$$= \frac{A}{4\pi}\int_0^{\infty} (e^{-rt} \cdot e^{i\omega_0 t} \cdot e^{i\omega t} + e^{-rt} \cdot e^{-i\omega_0 t} \cdot e^{i\omega t}) dt$$

$$= \frac{A}{4\pi}\int_0^{\infty} (e^{(-r+i\omega_0+i\omega)t} + e^{-(r-i\omega_0+i\omega)t}) dt$$

$$= \frac{A}{4\pi}\left[\int_0^{\infty} e^{(-r+i\omega_0+i\omega)t} dt + \int_0^{\infty} e^{(-r-i\omega_0+i\omega)t} dt\right]$$

$$= \frac{A}{4\pi}\left[\frac{e^{(-r+i\omega_0+i\omega)t}}{-r+i\omega_0+i\omega}\Big|_0^{\infty} + \frac{e^{(-r-i\omega_0+i\omega)t}}{-r-i\omega_0+i\omega}\Big|_0^{\infty}\right]$$

$$= \frac{A}{4\pi}\left[\frac{e^{[-r+i(\omega_0+\omega)]t}}{-r+i(\omega_0+\omega)}\Big|_0^{\infty} + \frac{e^{[-r+i(\omega-\omega_0)]t}}{-r+i(\omega-\omega_0)}\Big|_0^{\infty}\right]$$

$$= \frac{A}{4\pi}\left[\frac{0-1}{-r+i(\omega_0+\omega)} + \frac{0-1}{-r+i(\omega-\omega_0)}\right]$$

$$= \frac{A}{4\pi}\left[\frac{1}{r-i(\omega_0+\omega)} + \frac{1}{r-i(\omega-\omega_0)}\right]$$

$$= \frac{A}{4\pi} \cdot \frac{r-i(\omega-\omega_0)+r-i(\omega+\omega_0)}{r^2-i(\omega-\omega_0)r-i(\omega+\omega_0)r+i^2(\omega^2-\omega_0^2)}$$

$$= \frac{A}{4\pi} \cdot \frac{2r-2i\omega}{r^2-2i\omega r+\omega_0^2-\omega^2}$$

$$= \frac{A}{2\pi} \cdot \frac{r-i\omega}{r^2-2i\omega r+\omega_0^2-\omega^2}$$

(b) $$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A \cdot e^{-\frac{1}{2}\frac{t^2}{t_0^2}} e^{i\omega t} dt$$

$$\left(-\frac{1}{2}\frac{t^2}{t_0^2} + i\omega t = b^2-(at+b)^2 = -a^2t^2-2abt\right.$$

$$\Rightarrow a = \frac{1}{\sqrt{2}t_0} \quad b = \frac{-i\omega t_0}{\sqrt{2}}$$

$\frac{1}{2t_0^2}=k$
$$= \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-kt^2} \cdot e^{i\omega t} dt$$

$$= \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-k[t^2-2\cdot\frac{i\omega}{2k}\cdot t+(\frac{i\omega}{2k})^2]} \cdot e^{\frac{i^2\omega^2}{4k}} dt$$

$$du = \frac{dt}{\sqrt{2}t_0}$$

$$= \frac{A}{2\pi} \cdot e^{\frac{-\omega^2}{4k}}\int_{-\infty}^{\infty} e^{-k(t-\frac{i\omega}{2k})^2} dt$$

$$-\infty < t < +\infty$$

$\underline{u = t-\frac{i\omega}{2k},\ dt=du}$ $$\frac{A}{2\pi} \cdot e^{-\frac{\omega^2}{4k}} \cdot \int_{-\infty}^{\infty} e^{-ku^2} du$$

$$\int_{-\infty}^{\infty} e^{-(\underbrace{\frac{t}{\sqrt{2}t_0}-\frac{i\omega t_0}{\sqrt{2}}}_{u})^2} dt \quad -\infty-\frac{i\omega t_0}{\sqrt{2}} < u < +\infty-\frac{i\omega t_0}{\sqrt{2}}$$

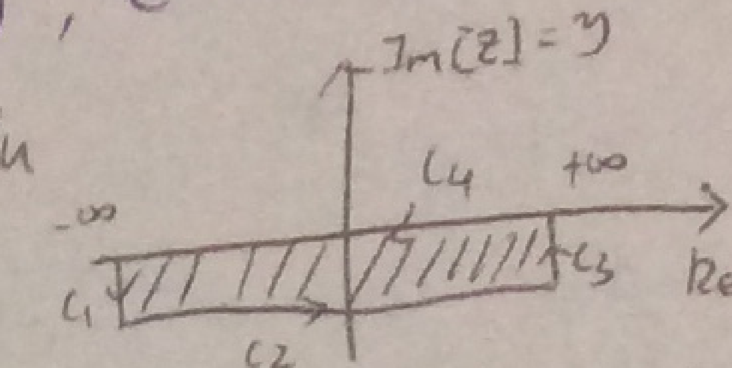
$$= \sqrt{2}t_0\int_{-\infty-\frac{i\omega t_0}{\sqrt{2}}}^{+\infty-\frac{i\omega t_0}{\sqrt{2}}} e^{-u^2} du$$

$\int_{-\infty}^{\infty} e^{-ku^2}du$ is a Gaussian integral.

so $\int_{-\infty}^{\infty} e^{-ku^2}du = \sqrt{\frac{\pi}{k}} = \sqrt{2\pi}\, t_0$ $\left(\because k=\frac{1}{2t_0^2}\right)$, $e^{\frac{-w^2}{4k}} = e^{-\frac{w^2 t_0^2}{2}}$

$\therefore f(w) = \frac{A}{2\pi} e^{-\frac{w^2t_0^2}{2}} \cdot \sqrt{2\pi}\, t_0 \qquad \sqrt{2}t_0 \int_{-\infty-\frac{iwt_0}{\sqrt{2}}}^{+\infty-\frac{iwt_0}{\sqrt{2}}} e^{-u^2}du$

$= \frac{A}{\sqrt{2\pi}} t_0 \cdot e^{-\frac{w^2 t_0^2}{2}}$



$C_4 = \int_{+\infty}^{-\infty} e^{-x^2}dx = -\sqrt{\pi} \qquad \oint e^{-z^2}dz = 0$

$C_3 = \int_{+\infty-\frac{iwt_0}{\sqrt{2}}}^{+\infty} e^{-z^2}dy = 0 \qquad \int_{C_1}+\int_{C_2}+\int_{C_3}+C_4 = 0$

$(z = x+iy) \to \infty,\ e^{-z^2} \to 0$

$\therefore C_2 = -C_4 = \sqrt{\pi}$

(c) Now, in time domain

$\int_{-\infty}^{\infty} t^2 |f(t)|^2 dt = \int_{-\infty}^{\infty} t^2 \cdot A^2 \cdot e^{-\frac{t^2}{t_0^2}} dt$

$= A^2 t_0^3 \int_{-\infty}^{\infty} \left(\frac{t}{t_0}\right)^2 \cdot e^{-\left(\frac{t}{t_0}\right)^2} d\left(\frac{t}{t_0}\right)$

$\overset{x=\frac{t}{t_0}}{=} -\frac{A^2 t_0^3}{2} \int_{-\infty}^{\infty} x \cdot (-2x \cdot e^{-x^2})dx$

$= -\frac{A^2 t_0^3}{2} \cdot \left[x \cdot e^{-x^2}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (x)' \cdot e^{-x^2}dx \right]$ (integration by parts)

$= -\frac{A^2 t_0^3}{2} \cdot \left[x \cdot e^{-x^2}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 1 \cdot e^{-x^2}dx \right]$

$= -\frac{A^2 t_0^3}{2} \cdot [(0-0) - \sqrt{\pi}]$

$= \frac{1}{2} \cdot A^2 \cdot t_0^3 \sqrt{\pi}$

Another way: $B = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} A^2 e^{-\frac{t^2}{t_0^2}} dt = A^2 \cdot t_0 \sqrt{\pi}$

$\frac{\partial B}{\partial t_0} = \frac{2t^2}{t_0^3} \cdot \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{2t^2}{t_0^3} B = A^2 \cdot \sqrt{\pi}$

$\Rightarrow t^2 \cdot B = \frac{t_0^3}{2} \cdot A^2 \cdot \sqrt{\pi}$

$\sqrt{f^2(t)} = \sqrt{\frac{t^2 B}{B}} = \sqrt{\frac{t_0^3 \cdot A^2 \cdot \sqrt{\pi}}{2A^2 \cdot \sqrt{\pi} \cdot t_0}} = \sqrt{\frac{1}{2}}\, t_0$

$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} A^2 \cdot e^{-\frac{t^2}{t_0^2}} dt$

$= A^2 \cdot t_0 \cdot \int_{-\infty}^{\infty} e^{-\frac{t^2}{t_0^2}} d\left(\frac{t}{t_0}\right)$

$\overset{\frac{t}{t_0}=x}{=} A^2 \cdot t_0 \cdot \int_{-\infty}^{\infty} e^{-x^2} dx$

$= A^2 \cdot t_0 \cdot \sqrt{\pi}$

$\therefore \sqrt{\langle f^2(t) \rangle} = \sqrt{\frac{\int_{-\infty}^{\infty} t^2 |f(t)|^2 dt}{\int_{-\infty}^{\infty} |f(t)|^2 dt}} = \sqrt{\frac{\frac{1}{2}A^2 \cdot t_0^3 \sqrt{\pi}}{A^2 \cdot t_0 \cdot \sqrt{\pi}}} = \sqrt{\frac{1}{2}}\, t_0$

Now, in frequency domain,

according to (b) $f(w) = \frac{A}{\sqrt{2\pi}} t_0 \cdot e^{-\frac{w^2 t_0^2}{2}}$

$\therefore \int_{-\infty}^{\infty} w^2 \cdot |f(w)|^2 dw = \int_{-\infty}^{\infty} w^2 \cdot \frac{A^2}{2\pi} \cdot t_0^2 \cdot e^{-w^2 t_0^2} dw$

$= \frac{A^2}{2\pi} \int_{-\infty}^{\infty} w^2 t_0^2 \cdot e^{-w^2 t_0^2} dw.$

just like the previous one. $\int_{-\infty}^{\infty} w^2 \cdot |f(w)|^2 dw = \frac{A^2}{2\pi t_0} \sqrt{\pi} \times \frac{1}{2}$

$$\int_{-\infty}^{\infty} |f(\omega)| \, d\omega = \int_{-\infty}^{\infty} \frac{A^2}{2\pi} \cdot t_0^2 \cdot e^{-\omega^2 t_0^2} d\omega$$

$$= \frac{A^2}{2\pi} t_0 \cdot \int_{-\infty}^{\infty} e^{-\omega^2 t_0^2} d(\omega t_0)$$

$$= \frac{A^2}{2\pi} t_0 \cdot \sqrt{\pi}$$

$$\therefore \sqrt{\langle f^2(\omega) \rangle} = \sqrt{\frac{\int_{-\infty}^{\infty} \omega^2 \cdot |f(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |f(\omega)|^2 d\omega}}$$

$$= \sqrt{\frac{\frac{A^2}{2\pi t_0} \sqrt{\pi} \times \frac{1}{2}}{\frac{A^2}{2\pi} t_0 \sqrt{\pi}}}$$

$$= \sqrt{\frac{1}{2}} \times \frac{1}{t_0}$$

$$\sqrt{\langle f^2(t) \rangle} \times \sqrt{\langle f^2(\omega) \rangle} = \sqrt{\frac{1}{2}} t_0 \times \sqrt{\frac{1}{2}} \times \frac{1}{t_0} = \frac{1}{2}$$

So, the product of the square root of the 2^{nd} momentum in time and in frequency domain is a constant

2、(a) Fourier transform of $f(t-t_0)$ is

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$$F[f(t-t_0)] = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} f(t-t_0) e^{i\omega t} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{i\omega(u+t_0)} du \quad (u = t - t_0 \Rightarrow dt = du, \ t = u + t_0)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{i\omega u} du \, e^{i\omega t_0}$$

$$= f(\omega) e^{i\omega t_0}$$

(b) Fourier transform of $\frac{d}{dt} f(t)$

$$\frac{d}{dt} f(t) = \frac{d}{dt} \left[\int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} d\omega \right]$$

$$= \int_{-\infty}^{\infty} f(\omega) \cdot \frac{d}{dt} [e^{-i\omega t}] d\omega$$

$$= \int_{-\infty}^{\infty} f(\omega) \cdot (-i\omega) \cdot e^{-i\omega t} d\omega$$

$$= -i\omega \cdot f(t)$$

$$\therefore F\left[\frac{d}{dt} f(t)\right] = -i\omega \cdot f(\omega)$$

3、(a) ~~$\int_{-\infty}^{\infty} f(t)\,dt = \int \lim_{w\to 0} e^{-\frac{t^2}{w^2}} dt$~~

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~~$= \lim_{w\to 0} \int_{-\infty}^{\infty} e^{-(\frac{t}{w})^2} dt$~~

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{\infty} \lim_{w\to 0} \frac{1}{\sqrt{\pi} w} e^{-\frac{t^2}{w^2}} dt$$

$$\lim_{w\to 0}\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-(\frac{t}{w})^2} \cdot d(\tfrac{t}{w})$$

$$\overset{\frac{t}{w}=x}{=} \lim_{x\to\infty} \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi}$$

$$= 1$$

$$\therefore \int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

when $t=0$, $f(t) = \lim_{w\to 0} \frac{1}{\sqrt{\pi} w} \cdot 1 \to \infty$

when $t \neq 0$ $w \to 0$, $-\frac{t^2}{w^2} \to -\infty$ $\therefore e^{-\frac{t^2}{w^2}} \to 0$ Yes... but what about t^{-2} factor? Use L'Hopital's rule

$\therefore f(t) = \lim_{w\to 0} \frac{1}{\sqrt{\pi}\cdot w} \to 0.$ → Forgot to prove 2nd property!

So, $f(t) = \lim_{w\to 0} \frac{1}{\sqrt{\pi}\cdot w} e^{-\frac{t^2}{w^2}}$ is a possible repesentative of the δ function.

(b) $\int_{-\infty}^{\infty} \delta(t) f(t)\,dt = f(0)$ 1/2

because $\delta(t)$ exists only at $t=0$ and everything else is 0 when $t \neq 0$.

(c) $\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = f(t_0)$ 1/2

because $\delta(t-t_0)$ exists only at $t-t_0=0 \Rightarrow t=t_0$ → DO THE SAME MATHEMATICALLY WITH A VARIABLE SUB

(d) $\int_{-\infty}^{\infty} \delta(at) f(t)\,dt \overset{u=at}{=} \int_{-\infty}^{\infty} \delta(u) f(t)\, d(\tfrac{u}{a}) = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(u) f(t)\,du = \frac{1}{|a|} f(0)$ ✓

because $\delta(u)$ exists only at $u=0$ (that is $t=0$)

(f) $F[\delta(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(t)\cdot e^{iwt}\,dt = \frac{1}{2\pi} \times e^{iw\cdot 0} = \frac{1}{2\pi}$ ✓

4. $\frac{1}{2\pi}\int_{-\infty}^{\infty} [f \otimes g](t)\, e^{iwt}\,dt$

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$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z)\, g(t-z)\, e^{iwt}\,dz\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z) \cdot \int_{-\infty}^{\infty} g(t-z)\cdot e^{iw(t-z)} e^{iwz}\,dz\,dt$ ✓

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)\cdot e^{iwz}\,dz \cdot \int_{-\infty}^{\infty} g(t-z)\cdot e^{iw(t-z)}\,dt$ ✓

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)\cdot e^{iwz}\,dz \cdot \int_{-\infty}^{\infty} g(u)\, e^{iwu}\,du$ ($u=t-z$, $du=dt$) ✓

$= 2\pi\cdot[f(w)\cdot g(w)]$ ✓

just like the previou

3\. (a) $\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undefined} & t = 0 \end{cases}$ $\qquad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$

$$f(t) = \lim_{w\to 0} \frac{1}{\sqrt{\pi}\,w} e^{-\frac{t^2}{w^2}} \qquad f(t=0) = \lim_{w\to 0} \frac{1}{\sqrt{\pi}\,w} \to \infty$$

$$f(t\neq 0) = \lim_{w\to 0} \frac{g(w)}{h(w)} \qquad g(w) = \frac{1}{\sqrt{\pi}\,w} \to \infty$$

$$h(w) = e^{\frac{t^2}{w^2}} \to \infty$$

$$= \lim_{w\to 0} \frac{g'(w)}{h'(w)} = \frac{\frac{-1}{\sqrt{\pi}\,w^2}}{\frac{-2t^2}{w^3} e^{\frac{t^2}{w^2}}} = \underbrace{\frac{w}{2\sqrt{\pi}\,t^2}}_{=0} \underbrace{e^{-\frac{t^2}{w^2}}}_{=0} = 0$$

$$\int_{-\infty}^{\infty} \lim_{w\to 0} \frac{1}{\sqrt{\pi}\,w} e^{-\frac{t^2}{w^2}} dt = \lim_{w\to 0} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du = \lim_{w\to 0} \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

(b) $\int_{-\infty}^{\infty} \delta(t) f(t)\,dt = \int_{-\infty}^{\infty} \delta(t)\left[f(0) + t\cdot f'(0) + \frac{t^2}{2!} f''(0) + \cdots\right] dt = f(0)$

$$\int_{-\infty}^{\infty} \delta(t)\cdot t\,dt = 0$$

(c) $\int_{-\infty}^{\infty} \delta(\underbrace{t-t_0}_{t'}) \underbrace{f(t)}_{(t'+t_0)}\,dt = f(t_0)$

(d) $\int_{-\infty}^{\infty} \delta(at) f(t)\,dt$

$t' = at$

$dt' = a\,dt$

$$\Rightarrow \begin{cases} \int_{-\infty}^{\infty} \delta(t') f\left(\frac{t'}{a}\right) \frac{dt'}{a} = \frac{f(0)}{a}, & a > 0 \\ -\int_{-\infty}^{\infty} \delta(t') f\left(\frac{t'}{a}\right) \frac{dt'}{a} = \frac{-f(0)}{a}, & a < 0 \end{cases}$$

$$= \frac{f(0)}{|a|}$$

~~$\int_{-\infty}^{\infty} \delta(g(t))$~~

e) $\int_{-\infty}^{\infty} \delta(g(t)) f(t)\,dt = \sum_{i=1}^{N} \lim_{\varepsilon\to 0} \int_{t_0^i-\varepsilon}^{t_0^i+\varepsilon} \delta(g(t)) f(t)\,dt$

$$= \sum_{i=1}^{N} \lim_{\varepsilon\to 0} \int_{t_0^i-\varepsilon}^{t_0^i+\varepsilon} \delta\left(g'(t_0^i)(t-t_0^i) + g''(t_0^i)(t-t_0^i)^2 + \cdots\right) f(t)\,dt$$

$$= \sum_{i=1}^{H} \lim \int_{t_0^i-\varepsilon}^{t_0^i+\varepsilon} \delta\left(g'(t_0^i)(t-t_0^i)\right) f(t)\,dt = \sum_{i=1}^{H} \frac{1}{|g'(t_0^i)|} f(t_0^i)$$

f) $F[\delta(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(t) e^{i\omega t}\,dt = \frac{1}{2\pi}$

$$FT^{-1}\left[\frac{1}{2\pi}\right] = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-i\omega t}\, d\omega = \delta(t) \underset{\omega \to t}{\overset{t \to \omega-\omega'}{\Rightarrow}} \int_{-\infty}^{\infty} e^{i(\omega-\omega')t}\, dt = 2\pi\,\delta(\omega-\omega')$$