Midterm Exam
”Fundamentals of modern optics”
WS 2014/15
solution examples

Problem 1 – Maxwell's equations **3 + 2 + 3 + 1 = 9 points**

1. Write down Maxwell's equations and the the auxiliary fields **D** and **H** in time domain.

   Maxwell's equations:

$$\begin{aligned}
\nabla \times \mathbf{E}(\mathbf{r},t) &= -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} \\
\nabla \times \mathbf{H}(\mathbf{r},t) &= \mathbf{j}(\mathbf{r},t) + \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} \\
\nabla \cdot \mathbf{D}(\mathbf{r},t) &= \rho(\mathbf{r},t) \\
\nabla \cdot \mathbf{B}(\mathbf{r},t) &= 0
\end{aligned}$$

   constitutive equations:

$$\begin{aligned}
\mathbf{D}(\mathbf{r},t) &= \epsilon_0 \mathbf{E}(\mathbf{r},t) + \mathbf{P}(\mathbf{r},t) = \epsilon_0 \int_{-\infty}^{+\infty} \epsilon(\mathbf{r},t-\tau)\mathbf{E}(\mathbf{r},\tau)\mathrm{d}\tau \\
\mathbf{H}(\mathbf{r},t) &= \frac{1}{\mu_0}\mathbf{B}(\mathbf{r},t)
\end{aligned}$$

2. Write down Maxwell's equations in frequency domain in a linear, homogeneous and isotropic dielectric medium in absence of free charges and current density ($\rho = 0$ and $\mathbf{j} = 0$).

$$\begin{aligned}
\nabla \times \mathbf{E}(\mathbf{r},\omega) &= i\omega\mu_0 \mathbf{H}(\mathbf{r},\omega) \\
\nabla \times \mathbf{H}(\mathbf{r},\omega) &= -i\omega \mathbf{D}(\mathbf{r},\omega) \\
\nabla \cdot \mathbf{D} &= 0 \\
\nabla \cdot \mathbf{H} &= 0
\end{aligned}$$

3. Derive the wave equation in Fourier domain for the electric field in a linear, homogeneous and isotropic dielectric medium in absence of free charges and current density ($\rho = 0$ and $\mathbf{j} = 0$).

   From Maxwell's equations:

$$\begin{aligned}
\nabla \times \nabla \times \mathbf{E}(\mathbf{r},\omega) &= i\omega\mu_0 \nabla \times \mathbf{H}(\mathbf{r},\omega) \\
&= i\omega\mu_0(-i\omega \mathbf{D}(\mathbf{r},\omega)) \\
&= \mu_0\omega^2 \mathbf{D}(\mathbf{r},\omega) \\
&= \mu_0\epsilon_0\omega^2 \mathbf{E}(\mathbf{r},\omega) + \mu_0\omega^2 \mathbf{P}(\mathbf{r},\omega) \\
&= \frac{\omega^2}{c^2}\epsilon(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega)
\end{aligned}$$

   and with:

$$\begin{aligned}
&\nabla \times \nabla \times \mathbf{E}(\mathbf{r},\omega) \\
&\qquad = \mathbf{grad}\,\mathrm{div}\,\mathbf{E}(\mathbf{r},\omega) - \Delta \mathbf{E}(\mathbf{r},\omega) \\
&\qquad = -\Delta \mathbf{E}(\mathbf{r},\omega)
\end{aligned}$$

   we can find:

$$\Delta \mathbf{E}(\mathbf{r},\omega) + \frac{\omega^2}{c^2}\epsilon(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega) = 0$$

4. Give the formula of the time averaged Poynting vector.

Time averaged Poynting vector: $\langle \mathbf{S}(\mathbf{r},t)\rangle = \frac{1}{2} Re\left(\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})\right)$

Problem 2 – Poynting Vector and Normal Mode **$2+2+1+3=8$ points**

a) First represent the real valued electric field in its complex form, to identify the wave-vector:

$$\mathbf{E}_r(\mathbf{r},t) = E_0\ \mathbf{e}_x\ e^{-\alpha z}\ \cos(\beta z - \omega t + \phi) = \frac{1}{2}\left[\mathbf{E}_c\ e^{-i\omega t} + \mathbf{E}_c^*\ e^{+i\omega t}\right]$$

with $\mathbf{E}_c = E_0\ e^{i\phi}\mathbf{e}_x\ e^{i(\beta+i\alpha)z}$. We identify the wave-vector as $\mathbf{k} = \mathbf{k}' + i\mathbf{k}'' = (\beta + i\alpha)\ \mathbf{e}_z$. We know the dispersion relation of a plane wave in a homogeneous medium as $\mathbf{k}.\mathbf{k} = \frac{\omega^2}{c^2}\epsilon$. Expanding both sides gives us:

$$k'^2 - k''^2 = \frac{\omega^2}{c^2}\epsilon' \quad , \quad 2k'k'' = \frac{\omega^2}{c^2}\epsilon''$$

From which we find $k' \approx \frac{\omega}{c}\sqrt{\epsilon'}$ and $k'' \approx \frac{\omega}{c}\frac{\epsilon''}{2\sqrt{\epsilon'}}$.

b) Same like electric field, we can present the real valued magnetic field like:

$$\mathbf{H}_r(\mathbf{r},t) = \frac{1}{2}\left[\mathbf{H}_c\ e^{-i\omega t} + \mathbf{H}_c^*\ e^{+i\omega t}\right]$$

Using the time domain Maxwell equation $\nabla \times \mathbf{E}_r = -\mu_0\frac{\partial \mathbf{H}_r}{\partial t}$, we find the realtion between the complex amplitudes to be $\nabla \times \mathbf{E}_c = i\omega\mu_0\mathbf{H}_c$. Followed by:

$$\nabla \times \mathbf{E}_c = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_0\ e^{i\phi}\mathbf{e}_x\ e^{i(\beta+i\alpha)z} & 0 & 0 \end{vmatrix} = i(\beta+i\alpha)E_0\ e^{i\phi}\mathbf{e}_y\ e^{i(\beta+i\alpha)z}$$

Which gives $\mathbf{H}_c = \frac{(\beta+i\alpha)E_0\ e^{i\phi}}{\omega\mu_0}\mathbf{e}_y\ e^{i(\beta+i\alpha)z}$. And we calculate the real valued magnetic field:

$$\mathbf{H}_r = \frac{E_0 e^{-\alpha z}}{2\omega\mu_0}\mathbf{e}_y\left[(\beta+i\alpha)e^{i(\beta z-\omega t+\phi)} + (\beta - i\alpha)e^{-i(\beta z-\omega t+\phi)}\right] = \frac{E_0}{\omega\mu_0}e^{-\alpha z}\mathbf{e}_y\left[\beta\cos(\beta z-\omega t+\phi) - \alpha\sin(\beta z-\omega t+\phi)\right]$$

c) $\mathbf{S}_r(\mathbf{r},t) = \mathbf{E}_r(\mathbf{r},t) \times \mathbf{H}_r(\mathbf{r},t)$

d)

$$\begin{aligned}
\langle \mathbf{S}_r(\mathbf{r},t)\rangle &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\mathbf{S}_r(\mathbf{r},t)dt \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\mathbf{E}_r(\mathbf{r},t)\times\mathbf{H}_r(\mathbf{r},t)dt \\
&= \frac{E_0^2}{\omega\mu_0}e^{-2\alpha z}\mathbf{e}_z\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\cos(\beta z-\omega t+\phi)\left[\beta\cos(\beta z-\omega t+\phi) - \alpha\sin(\beta z-\omega t+\phi)\right]dt \\
&= \frac{E_0^2}{\omega\mu_0}e^{-2\alpha z}\mathbf{e}_z\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\left[\beta\frac{\cos(2(\beta z-\omega t+\phi))+1}{2} - \alpha\frac{\sin(2(\beta z-\omega t+\phi))}{2}\right]dt \\
&= \frac{E_0^2}{\omega\mu_0}\frac{\beta}{2}e^{-2\alpha z}\mathbf{e}_z
\end{aligned}$$

Problem 3 – Beam propagation (Imaging) **3 + 3 + 3 = 9 points**

a) $$U_0(\alpha,\beta;z=0) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} A\exp\left(-i\pi\frac{x^2+y^2}{\lambda\, f} - i(x\alpha+y\beta)\right)\mathrm{d}x\mathrm{d}y$$

$$= \frac{A}{(2\pi)^2}e^{\frac{i(\alpha^2+\beta^2)\lambda f}{4\pi}}\int_{\mathbb{R}^2}\exp\left(\frac{-i\pi}{\lambda\, f}\left(\underbrace{x^2+2\,\frac{\lambda f\alpha}{2\pi}x+\left(\frac{\lambda f\alpha}{2\pi}\right)^2}_{=x'^2}+\underbrace{y^2+2\,\frac{\lambda f\alpha}{2\pi}y-\left(\frac{\lambda f\beta}{2\pi}\right)^2}_{=y'^2}\right)\right)\mathrm{d}x\mathrm{d}y$$

$$= \frac{A}{(2\pi)^2}e^{\frac{i(\alpha^2+\beta^2)\lambda f}{4\pi}}\int_{\mathbb{R}^2}\exp\left(\frac{-i\pi}{\lambda\, f}\underbrace{(x'^2+y'^2)}_{\rho^2}\right)\mathrm{d}x'\mathrm{d}y'$$

$$= \frac{A}{(2\pi)^2}e^{\frac{i(\alpha^2+\beta^2)\lambda f}{4\pi}}\int_0^\infty\int_0^{2\pi}\exp\left(\frac{-i\pi}{\lambda\, f}\rho^2\right)\rho\mathrm{d}\phi\mathrm{d}\rho = \frac{A}{2\pi}e^{\frac{i(\alpha^2+\beta^2)\lambda f}{4\pi}}\int_0^\infty\exp\left(\frac{-i\pi}{\lambda\, f}\rho^2\right)\rho\mathrm{d}\rho$$

$$= -\frac{iA\lambda f}{4\pi^2}\exp\left(i\frac{(\alpha^2+\beta^2)\lambda f}{4\pi}\right) = -\frac{iA\lambda f}{4\pi^2}\exp\left(i\frac{k_\rho^2\lambda f}{4\pi}\right)$$

where $k_\rho^2 = \alpha^2+\beta^2$

b) Free space transfer function:

$$H_F(k_\rho;z) = \exp\left(ik_z z\right) = \exp\left(iz\sqrt{k_0^2-k_\rho^2}\right) = \underbrace{\exp\left(izk_0\left(1-\frac{k_\rho^2}{2k_0^2}\right)\right)}_{\text{paraxial }=>k_\rho/k_0\ll 1} = \exp\left(izk_0 - iz\frac{k_\rho^2\lambda}{4\pi}\right)$$

$H_F(\alpha,\beta;z) = \exp\left(izk_0 - iz(\alpha^2+\beta^2)\lambda/4\pi\right)$
$\Longrightarrow$ Evanescent waves: $k_\rho > k_0$
$\Longrightarrow$ Propagating waves: $k_\rho \leq k_0$

c) $$U(\alpha,\beta;z=f) = U_0(\alpha,\beta;z=0)H_F(\alpha,\beta;f)$$
$$= -\frac{iA\lambda f}{4\pi^2}\exp\left(i\frac{k_\rho^2\lambda f}{4\pi}\right)\exp\left(ik_0 f - i\frac{k_\rho^2\lambda f}{4\pi}\right)$$
$$= -\frac{iA\lambda f}{4\pi^2}e^{ik_0 f}$$
$$u(x,y,z=f) = \int_{\mathbb{R}^2}U(\alpha,\beta;z=f)e^{i(x\alpha+y\beta)}\mathrm{d}\alpha\mathrm{d}\beta$$
$$= -\frac{iA\lambda f}{4\pi^2}e^{ik_0 f}\int_{\mathbb{R}^2}e^{i(x\alpha+y\beta)}\mathrm{d}\alpha\mathrm{d}\beta$$
$$= -\frac{iA\lambda f}{4\pi^2}e^{ik_0 f}\delta(x)\delta(y)$$

Problem 4 - Gaussian beam in a telescope (2+2+2 points)

1. The q parameter at the first lens is given by $q = f_1 + iz_0$. Using the ABCD formalism, we obtain the parameter q' of the beam just after the lens:

$$q' = \frac{f_1+iz_0}{-\frac{1}{f_1}(f_1+iz_0)+1} = \frac{f_1+iz_0}{-\frac{iz_0}{f_1}} = -f_1 + i\frac{f_1^2}{z_0}$$

that means that the waist position is at $z = f_1$ after the lens and the new Rayleigh range is: $z_0' = \frac{f_1^2}{z_0}$.

2. The propagation of the Gaussian beam over a distance of $d = f_1 + f_2$ leads to a parameter $q'' = f_2 + iz_0'$ just in front of the second lens. The lens effect is similar to part a) provided we substitute f_1 with f_2. So we obtain a parameter $q''' = -f_2 + i\frac{f_2^2}{z_0'}$. that means that the waist position is at $z = f_2$ after the lens and the new Rayleigh range is: $z_0'' = \frac{f_2^2}{z_0'}$.

3. Combining part a) and b) we find that $z_0'' = \frac{f_2^2}{f_1^2} z_0$. By substituting in the formula the definition of the Rayleigh range we obtain:

$$\frac{\pi W_0''^2}{\lambda} = \frac{f_2^2}{f_1^2} \frac{\pi W_0^2}{\lambda}$$
$$W_0''^2 = \frac{f_2^2}{f_1^2} W_0^2$$
$$W_0'' = \frac{f_2}{f_1} W_0$$

Problem 5 – Pulse propagation **3 + 3 + 2 = 8 points**

a) The propagation vector $k(\omega)$ is defined as:

$$k(\omega) = \frac{\omega}{c_0} n(\omega)$$

Therefore, the phase and group velocities are:

$$v_{\mathrm{p}} = \frac{\omega}{k(\omega)} = \frac{c_0}{n(\omega_0)} = \frac{c_0}{2 + 4 \times 10^{-2}}$$
$$v_{\mathrm{p}} = \frac{c_0}{2.04}$$
$$v_{\mathrm{g}} = \left[\left. \frac{\partial k(\omega)}{\partial \omega} \right|_{\omega_0} \right]^{-1} = \left. \frac{c_0}{(B + 3C\omega^2)} \right|_{\omega_0} = \frac{c_0}{2 + 0.12}$$
$$v_{\mathrm{g}} = \frac{c_0}{2.12}$$

b) We first calculate the dispersion coefficient

$$D = \left. \frac{\partial^2 k(\omega)}{\partial \omega^2} \right|_{\omega_0} = \frac{6C\omega_0}{c_0} = 4 \times 10^{-25} \frac{\mathrm{s}^2}{\mathrm{m}}$$
$$T_0 = 2/\omega_{\mathrm{S}} = 2\mathrm{ps}$$
$$z_0 = -\frac{T_0^2}{2D} = -5\mathrm{m}$$

Therefore, the pulse width is given as:

$$\text{Width} = T(l) = T_0 \sqrt{1 + \left(\frac{l}{z_0}\right)^2} = \sqrt{17} T_0 \approx 8.246\mathrm{ps}$$

c) Since there is no dispersion in this medium, the group velocity is the same as the phase velocity. Therefore,

$$v_{\mathrm{p2}} = \frac{c_0}{2}$$

and the time difference between the two is given as:

$$\Delta t = L \left| \frac{1}{v_{\mathrm{g1}}} - \frac{1}{v_{\mathrm{g2}}} \right| = \frac{20}{3 \times 10^8} |2.12 - 2| = 8\mathrm{ns}$$

Problem 6 – Fraunhofer diffraction **4 + 2 = 6 points**

1. The *Fresnel approximation* is a paraxial approximation that is used for high Fresnel numbers which is normally associated with the near-field. The *Fraunhofer approximation* is a paraxial approximation that is valid for low Fresnel numbers ($N_F \leq 0.1$) which is normally associated with the far-field.

2. The field in Fraunhofer approximation is proportional to the Fouriertransform of the initial field

$$u(x, z_\mathrm{B}) \propto U_0\left(\frac{kx}{z_\mathrm{B}}\right).$$

$$\begin{aligned} U_0(\alpha) &\propto \int_{-\infty}^{\infty} u_0(x)\mathrm{e}^{-\mathrm{i}\alpha x}\mathrm{d}x \\ &= \int_{-a/2}^{a/2} \mathrm{e}^{-\mathrm{i}\alpha x}\mathrm{d}x \\ &= \frac{1}{-\mathrm{i}\alpha}\mathrm{e}^{-\mathrm{i}\alpha a/2} - \frac{1}{-\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\alpha a/2} \\ &= \frac{\sin(\alpha a/2)}{\alpha/2} \\ &= a\,\mathrm{sinc}(\alpha a/2) \end{aligned}$$

So we get:

$$I(x, z_\mathrm{B}) = |u(x, z_\mathrm{B})|^2 \propto \mathrm{sinc}^2\left(\frac{kx}{2z_\mathrm{B}}a\right)$$

Name: ______________________ Date of birth: ______________ Student ID No. ______________

Final Exam of "Fundamentals of modern optics" WS 2015/16
to be written on February 15, 10:00 - 12:00 am

Problem 1 – Maxwell's equations (MWEs) **$3+2+2+3=10$ points**

a) Write down MWEs in frequency domain, with external sources, in its most general form in a material. Furthermore, write down the constitutive equations for auxiliary fields **D** and **H**, in frequency domain, where the material is inhomogeneous, dispersive, linear, isotropic, and magnetic.

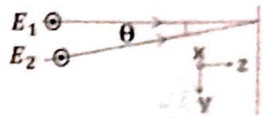
b) Find the wave equation for **E**, from the MWEs of part (a), for an inhomogeneous, dispersive, linear, isotropic, and non-magnetic material, in the presence of external sources.

c) Write down the MWEs of part (a) in their integral form, using the Stoke's and divergence theorems.

d) Derive the continuity equation connecting the charge density and the current density, from MWEs in (a). Write your final answer also in terms of the total charge in a volume and the total current escaping that volume.

Problem 2 – Normal Modes **$2+2+1+3=8$ points**

Consider two monochromatic plane waves with electric field amplitudes E_1 and E_2 polarized along the x direction in free space with wavelength λ. One plane wave is propagating along the z direction and the other in yz plane at an angle θ as shown in the figure.



a) Write down the space and time dependent electric field vector for both plane waves.

b) Find the intensity pattern due to interference of these two waves and plot it along the y direction (assume x=z=0 and E_1=$2E_2$).

c) Calculate the distance between two consecutive maxima (or minima) in y direction. What will be the effect of increasing angle θ on the observed pattern?

d) Define and calculate the time averaged Poynting vector.

Problem 3 – Diffraction theory **$2+1+6=9$ points**

a) Explain the condition shortly and write a mathematical inequality for which you can apply the following approximations: (i) Fresnel approximation. (ii) Fraunhofer approximation.

b) Can the Fraunhofer approximation be applied without the Fresnel approximation. Please explain your answer in two sentences.

c) Give formulas for calculating the observed diffraction patterns of an illumination function $u_0(x, y)$(in scaler approximation) for: (i) non-paraxial case. (ii) Fresnel approximation. (iii) Fraunhofer approximation.

Problem 4 – Pulses **$1+2+3=6$ points**

Consider a fiber of the length $L_1 = 1\,\mathrm{m}$ has the dispersion relation $k_1(\omega) = \alpha_1\omega_0 + \beta_1(\omega-\omega_0) + \frac{\gamma_1}{\omega_0}(\omega-\omega_0)^2$, with the frequency ω and the constants $\omega_0 = 2\cdot 10^{15}\mathrm{rad/s}$, $\alpha_1 = \frac{3}{2c}$, $\beta_1 = \frac{3}{4c}$ and $\gamma_1 = \frac{3}{8c}$, where $c = 3\cdot 10^8\mathrm{m/s}$ is the speed of light in vacuum.

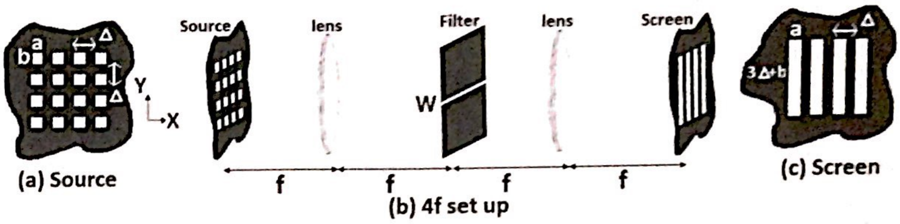
a) What is the phase velocity of monochromatic light of frequency $\omega_1 = 3\cdot 10^{15}\frac{\mathrm{rad}}{\mathrm{s}}$ in this fiber?

b) Now, we couple a Gaussian pulse into the fiber (center frequency $\omega_1 = 3\cdot 10^{15}\frac{\mathrm{rad}}{\mathrm{s}}$ and FWHM 100 fs). Its maximum starts at $t_0 = 0$ sec at the beginning of the fiber. At which time does this maximum arrive at the end of the fiber?

c) Consider a second fiber with the dispersion relation $k_2(\omega) = \alpha_2\omega_0 + \beta_2(\omega-\omega_0) + \frac{\gamma_2}{\omega_0}(\omega-\omega_0)^2$, where $\alpha_2 = \frac{10}{c}$, $\beta_2 = \frac{2}{c}$ and $\gamma_1 = \frac{-4}{9c}$. It is merged to the end of the first fiber. We launch two Gaussian pulses with the center frequencies $\omega_1 = 3\cdot 10^{15}\frac{\mathrm{rad}}{\mathrm{s}}$ and $\omega_2 = 4\cdot 10^{15}\frac{\mathrm{rad}}{\mathrm{s}}$, respectively, and both with the FWHM of 100 fs at the same time into the first fiber. Calculate the length L_2 of the second fiber when both pulses maxima shall arrive at the end of the second fiber at the same time.

Problem 5 – Imaging optics $3 + 4 + 3 = 10$ **points**

Assume, a source with 16 bright illuminating rectangles, each of size $a \times b$. They are arranged in x and y direction with period Δ (shown in figure a).



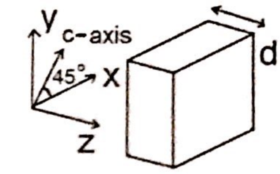
a) Please write the amplitude transmittance function of the source (figure a).

b) Calculate the diffraction pattern at 2f distance just before the filter (shown in figure b).

c) Suppose one wishes to obtain the diffraction pattern on the screen as shown in the figure c. What should the range of aperture width W in the filter be, to realize this pattern?

Problem 6 – Anisotropy $2 + 1 + 3 = 6$ **points**

A layer of a uniaxial crystal of thickness $d = 5\mu m$ is shown in the figure. The extraordinary crystal axis is in the x-y plane and makes a 45° angle with the x and y axis. The ordinary and extraordinary refractive indices are $n_o = 2.2$ and $n_e = 2.15$, respectively. A plane wave with an electric field of $\boldsymbol{E} = E_0 e^{i\frac{2\pi}{\lambda}z}(\hat{x} + i\hat{y})$ is incident on this layer from one side, where $\lambda = 1\mu m$ is the wavelength of the wave in free space.



a) What are the two eigenmodes of the crystal propagating in the z direction. Specify the direction of electric field (in terms of $\hat{x}$ and $\hat{y}$) and the magnitude of k-vector for each eigenmode.

b) Decompose the input electric field polarization into the two eigenmodes of part (a).

c) Calculate the electric field polarization at the other side of this crystal layer. You can ignore the reflections from the crystal surfaces; it is the final polarization state that matters to us.

Problem 7 – Interface $1 + 2 + 2 = 5$ **points**

The reflection coefficient of a TE mode field, with incident angle of θ_1 and refracted angle of θ_2, from a media 1 (n_1) into a media 2 (n_2) is described by the following Fresnel equation,

$$r_{TE} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}.$$

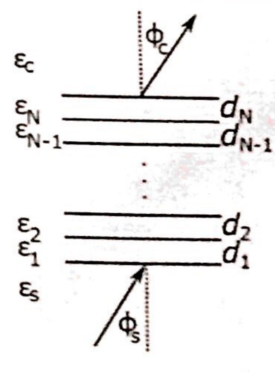
a) Assume $n_1 > n_2$. What is the relation for the critical angle θ_c, after which we have total internal reflection?

b) Show that for $\theta_1 > \theta_c$ we get $|r_{TE}| = 1$.

c) Find the value of the extra phase that the reflected wave acquires for the limiting case of $\theta_1 = \pi/2$.

Problem 8 – Multilayer system $2 + 2 + 4 = 8$ **points**

Consider a multilayer system of N isotropic layers with the permittivities ϵ_i and the thicknesses d_i ($i = 1, 2, ..., N$). In front of and behind this multilayer system, there are the substrate (ϵ_s) and the cladding (ϵ_c), respectively.



a) Let the layer permittivities be real and hold the following relation: $\epsilon_s < \epsilon_1 < \epsilon_2 < ... < \epsilon_i < ... < \epsilon_N < \epsilon_c$. An incoming monochromatic plane wave hits the interface to the multilayer system with an angle of ϕ_s with respect to the interface normal (see figure). Find the the angle ϕ_c of the refracted wave after the multilayer stack.

b) Explain shortly the matrix method and its use for the multilayer system. Give also a short mathematical description.

c) Derive the matrix for a single layer, for a TE-polarized monochromatic plane wave.

Maybe useful formulas: $\nabla \times \nabla \times \mathrm{a} = \nabla(\nabla \cdot \mathrm{a}) - \Delta \mathrm{a}$, $\nabla \cdot (\nabla \times \mathrm{a}) = 0$.

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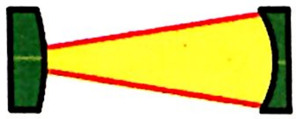
WS 2009/2010

Exam
FUNDAMENTALS OF MODERN OPTICS

February 11, 2010

Exercise 1 **8 Points**

A resonator consists of two spherical mirrors: one of them is a concave mirror with radius $R_1 > 0$, the other one is a convex mirror with radius $R_2 < 0$, and the distance between them is d. Define the conditions for this resonator to be stable, and sketch a stable configuration. Do not forget to mark the positions of the mirrors and their respective centers of curvature in your sketch.



Exercise 2 **10 Points**

In an experiment you have an input Gaussian beam profile with width W_{in} and flat phase (phase curvature $R_{in} = \infty$). Your task is to obtain a beam profile with the same parameters (W_{in} and R_{in}), but at a distance L from the input position. You are allowed to use just a single lens. Calculate the focal length of the lens you need and its position.

Exercise 3 **15 Points**

Illumination of a cross grating produces a light distribution

$$u_0(x,y) = \frac{A}{4}\left(1+\cos\frac{2\pi}{a}x\right)\left(1+\cos\frac{2\pi}{a}y\right),$$

with period length $a = 1\,\mathrm{mm}$. This light field is now imaged by a $4f$-setup, where in the plane $z = 2f$ a slit with the filter function

$$p(x,y) = \begin{cases} 1 & ,|x| < D/2 \\ 0 & ,\text{elsewhere} \end{cases}$$

is applied. The focal length is $f = 1\,\mathrm{m}$ and the wavelength used is $\lambda = 1\,\mu\mathrm{m}$.
Calculate the field $u(x,y,4f)$ at the end of the $4f$-setup for a slit width $D = 1\,\mathrm{mm}$.

Exercise 4 **10+5 Points**

a) Consider a dielectric medium with response function

$$R(t) = \frac{f}{\sqrt{a^2-b^2}}\, e^{-bt} \sin\left(\sqrt{a^2-b^2}\; t\right),$$

where $a > b > 0$. Calculate the electric susceptibility $\chi(\omega)$.

b) A company producing optical instruments is looking for a new homogeneous, isotropic material which should have the optical property

$$\chi(\omega) = Ae^{-\frac{(\omega-\omega_0)^2}{B^2}} + i\,C\delta(\omega-\omega_0),$$

where $A = 0.542$, $B = 1.02\cdot 10^{15}\,s^{-1}$, $C = 3.29$ and $\omega_0 = 4.71\cdot 10^{15}\,s^{-1}$. Do you think their research can be successful? Explain with the help of the Kramers Kronig relations.

Exercise 5 **7 Points**

Consider an uniaxial crystal with refractive indices for the ordinary wave n_o and the extraordinary wave n_e. The crystal's optical axis is parallel to its surface. A monochromatic, circularly polarized wave is normally incident on the crystal. Compute the propagation lengths after which the light is linearly polarized.

Exercise 6 **10 Points**

Two pulses are propagating in a homogeneous plasma. They have a carrier frequency much larger than the plasma frequency. The corresponding wavelengths of the carrier waves are λ_1 and λ_2.
The signals are recorded by a detector that is located at a distance L from the source. Use the dielectric function of the plasma

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

to compute the time delay between the two pulses.

Exercise 7 **5+10 Points**

Consider an interface between two media, $n(x<0) = n_1 = 2$ and $n(x>0) = n_2 = 1$.

a) Compute the angle of incidence ϕ_{Itot} above which total internal reflection occurs.

b) If the second medium (n_2) is not infinite, but forms a layer of thickness d [$n(0<x<d) = n_2 = 1$], and $n(x>d) = n_3 = 2$, do you still expect reflectivity $\rho = 1$ at the interface at $x = 0$? Compute the reflectivity for this layer (TE-polarization) to prove your answer.

Exercise 8 **10+10 Points**

Investigate the propagation of a 1-dimensional initial field distribution $u_0(x) = A\cos^2(x/\Lambda)$ in paraxial (Fresnel) approximation.

a) At which propagation distances z_T do we observe $|u(x,z_T)|^2 = |u_0(x)|^2$?

b) Now consider the above $u_0(x)$ with a finite aperture $a = N\pi\Lambda/2$, so that

$$\tilde{u}_0(x) = \begin{cases} u_0(x) & ,\,|x|<a \\ 0 & ,\text{elsewhere.} \end{cases}$$

Compute the far field intensity distribution of $\tilde{u}_0(x)$, and discuss the necessary propagation distances to apply Fraunhofer approximation with respect to N.

FRIEDRICH-SCHILLER-UNIVERSITÄT JENA
Institute of Condensed Matter Theory and Solid State Optics – PAF
Stefan Skupin
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WS 2010/2011

Exam
FUNDAMENTALS OF MODERN OPTICS

Final exam on February 25

Exercise 1 (Maxwell, linear media, plane waves, TE/TM) **5+2+10+10 Points**
Before you start with this exam, please have a look at the second sheet which provides you with some useful formulas. Good luck!

a) Write down the macroscopic Maxwell equations in temporal Fourier domain. ✓

b) Specify (in Fourier domain) the general relation between $\overline{\mathbf{D}}(\mathbf{r},\omega)$ and $\overline{\mathbf{E}}(\mathbf{r},\omega)$ for a linear, dispersive, inhomogeneous, and anisotropic medium. ✓

c) Let us now consider a linear, dispersive, homogeneous and isotropic dielectric medium in transparent regime: $\overline{\mathbf{D}}(\mathbf{r},\omega) = \epsilon_0\epsilon(\omega)\overline{\mathbf{E}}(\mathbf{r},\omega)$ with $\epsilon(\omega) > 0$, $\overline{\mathbf{B}}(\mathbf{r},\omega) = \mu_0\overline{\mathbf{H}}(\mathbf{r},\omega)$, $\overline{\mathbf{j}}(\mathbf{r},\omega) = 0$, and $\rho = 0$. Show that plane waves of the form $\overline{\mathbf{E}}(\mathbf{r},\omega) = \mathbf{E}(\omega)\exp(i\mathbf{k}(\omega)\cdot\mathbf{r})$ exist which are solutions to Maxwell's equations. Derive the neccessary conditions for $\mathbf{k}(\omega)$ and $\mathbf{E}(\omega)$ [length and direction of $\mathbf{k}(\omega)$ with respect to $\mathbf{E}(\omega)$] and compute the corresponding $\overline{\mathbf{H}}(\mathbf{r},\omega)$. ✓

d) For linear, *inhomogeneous* and isotropic dielectric media things become more complicated, however, in the case of translational invariance in one direction (e.g., y) we can decouple Maxwell's equations into TE and TM polarization. Derive the resulting wave equations for $\overline{\mathbf{E}}_{\mathrm{TE}}(x,z,\omega)$ and $\overline{\mathbf{H}}_{\mathrm{TM}}(x,z,\omega)$, i.e., one equation containing $\overline{E}_y(x,z,\omega)$ and $\epsilon(x,z,\omega)$ only, and one equation containing $\overline{H}_y(x,z,\omega)$ and $\epsilon(x,z,\omega)$ only. ?

Exercise 2 (uniaxial crystal, quarter-wave plate) **7 Points**
Suppose you want to use a thin slice of an uniaxial crystal (quartz, $n_0 \simeq 1.54$, $n_e \simeq 1.55$) to transform circularly polarized light at a wavelength of $\lambda_{vac} = 500$ nm to linear polarization. How do you have to cut the crystal with respect to the optical axis, to make the slice as thin as possible? How thick the resulting "quarter-wave plate" will be? ✓

Exercise 3 (4f-setup, optical integration) **7+10 Points**
A Master of Science in Photonics student wants to build an optical integrator by using a 4f-setup. He/she knows that by placing an appropriate pupil $p(x)$ at distance $z = 2f$, one can perform useful transformations on the entrance field $u_0(x)$.

a) What would be the ideal pupil $p(x)$ to obtain $u(-x, z = 4f) \propto \int\limits_{-\infty}^{x} u_0(x')dx'$? ✓

b) Unfortunately, this ideal pupil function for optical integration has a singularity at $x = 0$. Show that

$$p(x) = \operatorname{sinc}\left(\frac{kb}{f}x\right),$$

✓

which is free of singularities, leads to $u(-x, z = 4f) \propto \int\limits_{x-b}^{x+b} u_0(x')dx'$.

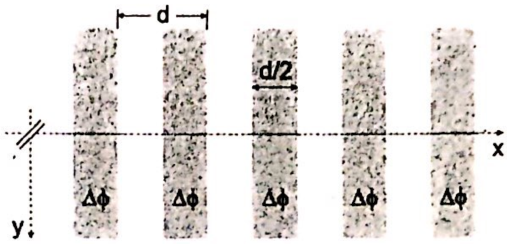
✓

Exercise 4 (dielectric multi-layer systems, reflectivity) **7 Points**
Consider a very complicated (absorption-free in the relevant frequency range) multi-layer system between a substrate ($\epsilon_s = 2$) and cladding ($\epsilon_c = 1$) material. Give the value of the reflectivity for an incident plane wave in the substrate, when the angle of incidence (between $\mathbf{k}$-vector and normal of the layer-system) is $\phi = \pi/3$.

(p.t.o)

Exercise 5 (dielectric function, response function, Kramers-Kronig relations) **7 Points**
A friend of yours claims to have a homogeneous medium characterized by the dielectric function $\epsilon(\omega) = 1 + i\exp(-\omega^2\tau_0^2)$, with $\tau_0 = 42$ fs. Do you believe him? Give reasons for your answer.

Exercise 6 (Fraunhofer approximation, phase grating) **15 Points**
A monochromatic plane wave propagates along $\vec{e}_z$ in vacuum and is diffracted by a thin non-absorbing phase grating of period d, whose N slits (white) and bars (gray) have a width $d/2$. The phase shift of the incident light within the bars amounts to $\Delta\phi = \pi$ relative to the light which passes directly through the slits. Assume an infinite extension along $\vec{e}_y$, i.e. a one-dimensional structure.



Show that the transmitted intensity in paraxial Fraunhofer approximation may be written as

$$I(x,z) \propto \operatorname{sinc}^2\left(\frac{Nkxd}{2z}\right)\tan^2\left(\frac{kxd}{4z}\right).$$

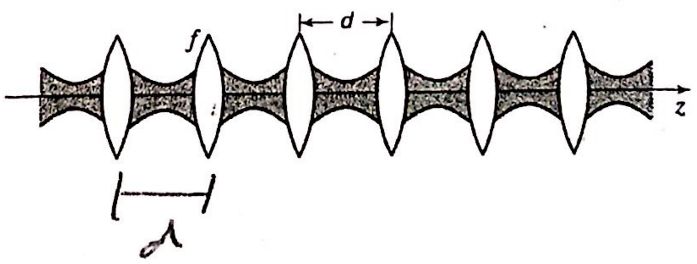
Which diffraction orders m can be observed, which ones vanish? ?

Exercise 7 (pulse propagation in dispersive media) **10 Points**
The group velocity in a homogeneous dielectric used for manufacturing glass fibers can be well approximated in a relatively wide range in the infrared wavelength regime by

$$v_g(\omega) = \frac{c}{n_0}\frac{1}{1+\left(\frac{\omega-\omega_0}{\delta}\right)^2}$$

with the center frequency ω_0, the center group index n_0, a parameter δ, and the speed of light c. Let us consider a plane wave with a Gaussian pulse envelope $u(t, z = 0) = u_0 \exp\{-(t/T_0)^2\}\exp(-i\omega t)$ launched into this material at the plane $z = 0$ and propagating in z-direction. Derive a formula for the propagation length in dependence on the frequency ω, after which the pulse duration has doubled. Consider dispersion up to second order (GVD) only. At which ω the pulse does not broaden upon propagation?

Exercise 8 (propagation of Gaussian beams, beam relaying) **10 Points**
A Gaussian beam is focused by a periodic sequence of identical lenses, each of focal length f and separated by a distance d (see Figure below). The beam is consecutively focused to the same waist and propagates periodically in this sequence of lenses. Show that the condition of periodic transmission ("relaying") can arise only if the inequality $d < 4f$ is satisfied. Where is the beam waist located between the individual lenses?



Formulas which may be useful:

- $\mathrm{rot}\,\mathbf{E}(\mathbf{r}) = \nabla \times \mathbf{E}(\mathbf{r})$, $\mathrm{div}\,\mathbf{E}(\mathbf{r}) = \nabla \cdot \mathbf{E}(\mathbf{r})$, $\mathrm{grad}\,\epsilon(\mathbf{r}) = \nabla\epsilon(\mathbf{r})$, $\nabla^2 = \Delta$
- $\nabla \times [\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r})] = \epsilon(\mathbf{r})\nabla \times \mathbf{E}(\mathbf{r}) - \mathbf{E} \times \nabla\epsilon(\mathbf{r})$
- $\nabla \times [\nabla \times \mathbf{E}(\mathbf{r})] = \nabla[\nabla \cdot \mathbf{E}(\mathbf{r})] - \Delta\mathbf{E}(\mathbf{r})$
- $\nabla \times \left[\frac{1}{\epsilon(\mathbf{r})}\nabla \times \mathbf{H}(\mathbf{r})\right] = \frac{1}{\epsilon(\mathbf{r})}\nabla \times [\nabla \times \mathbf{H}(\mathbf{r})] - [\nabla \times \mathbf{H}(\mathbf{r})] \times \nabla\left[\frac{1}{\epsilon(\mathbf{r})}\right]$
- For a $4f$-system we have
$$u(-x, z = 4f) \propto \int_{-\infty}^{\infty} p\left(\frac{f}{k}\alpha\right) U(\alpha, z = 0)e^{i\alpha x}d\alpha \propto \int_{-\infty}^{\infty} P\left[\frac{k}{f}(x' - x)\right] u(x', z = 0)dx',$$
where $U(\alpha, z = 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x, z = 0)\exp(-i\alpha x)dx$ and $P(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} p(x)\exp(-i\alpha x)dx$ is the Fourier transform of the pupil $p(x)$ situated at distance $z = 2f$.
- $\int_{-\infty}^{\infty} \mathrm{sinc}(x)\exp(-i\alpha x)dx = \pi\theta(\alpha + 1)\theta(\alpha - 1)$ with $\theta(x) = \begin{cases} 0 & x < 0 \\ 1/2 & x = 0 \\ 1 & x > 0 \end{cases}$
- $\sin(\pi/3) = \sqrt{3}/2$
- The linear Kramers-Kronig relations read
$$\Re\chi(\omega) = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} \frac{\Im\chi(\overline{\omega})}{\overline{\omega} - \omega}d\overline{\omega}, \qquad \Im\chi(\omega) = -\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} \frac{\Re\chi(\overline{\omega})}{\overline{\omega} - \omega}d\overline{\omega}.$$
- $\int_{\infty}^{\infty}\exp(-x^2)\exp(-i\alpha x)dx = \sqrt{\pi}\exp(-\alpha^2/4)$
- $\int_{\infty}^{\infty}\exp(-i\alpha x)dx = 2\pi\delta(\alpha)$
- The intensity distribution created by a one-dimensional periodic mask in paraxial Fraunhofer approximation is given by
$$I(x, z) \propto \left|T_S\left(\frac{kx}{z}\right)\right|^2 \frac{\sin^2\left(\frac{Nkxd}{2z}\right)}{\sin^2\left(\frac{kxd}{2z}\right)},$$
where d is the period length, N the number of periods, and $T_S(\alpha) = \frac{1}{2\pi}\int_0^d t(x)\exp(-i\alpha x)dx$ the Fourier transform of one period of the mask.
- $\sin(x) = 2\sin(x/2)\cos(x/2)$
- The propagation of pulse envelopes $v(z, \tau)$ in linear dispersive media is governed by
$$i\partial_z v(z, \tau) = \frac{D}{2}\partial_\tau^2 v(z, \tau),$$
where $\tau = t - z/v_g(\omega_0)$ is the co-moving time, and the wavenumber
$$k(\omega) = \frac{\omega_0 n_0}{c} + \frac{1}{v_g(\omega_0)}(\omega - \omega_0) + \frac{D}{2}(\omega - \omega_0)^2$$
is expanded in a Taylor series around the central frequency ω_0.

(p.t.o)

Institut für Festkörpertheorie und -optik
Friedrich-Schiller-Universität Jena

Klausur zur Vorlesung: Grundkonzepte der Optik - WS 2010/11

Sie müssen nicht alle Aufgaben rechnen.
(67 Punkte sind zu erreichen, 61 Punkte entsprechen 100%)

1. Maxwell-Gleichungen **($\sum$19)**

a) Geben Sie die makroskopischen Maxwell-Gleichungen im Frequenzraum unter der Annahme an, dass keine freien Ströme und keine freien Ladungen vorhanden sind!

(4)

b) Führen Sie die Polarisation **P(r,ω)** sowie die Magnetisierung **M(r,ω)** ein und geben Sie die allgemeinen Relationen zwischen den Feldern **D(r,ω)** bzw. **H(r,ω)** und **E(r,ω)** bzw. **B(r,ω)** an.

Was lässt sich über die Magnetisierung **M(r,ω)** im optischen Spektralbereich aussagen?

(3)

c) Nutzen Sie die Suszeptibilität χ sowie die Responsefunktion R und geben Sie für ein homogenes, lineares, lokales, **isotropes** und **dispersives** Medium die allgemeine Abhängigkeit der Polarisation **P**(ω) von der elektrischen Feldstärke **E**(ω) im Zeit- sowie im Frequenzraum an. Welcher Zusammenhang besteht zwischen χ und R?

(3)

d) Benennen Sie die Normalmoden des homogenen, isotropen Mediums! Wie sind diese polarisiert! Geben Sie deren (vektorielle) Dispersionsrelation (gegenseitige Abhängigkeit der Vektorkomponenten des Wellenzahlvektors und der Frequenz) an!

(4)

e) Zeigen Sie durch explizites Anwenden der Maxwell-Gleichungen, dass für ebene Wellen (Wellenvektor **k**, Frequenz ω) folgende Relation gilt: $\mathbf{E} \perp \mathbf{H} \perp \mathbf{k}$.

(4)

f) Geben Sie einen Ausdruck für den zeitlich gemittelten Poynting-Vektor einer ebenen Welle im isotropen Medium an

(1)

2. Materialgleichungen $(\Sigma 6)$

a) Gegeben sei ein homogenes, isotropes Medium mit einer Einzelresonanz bei der Frequenz ω_0 mit extrem scharfer Absorptionslinie (δ-förmig). Geben Sie einen analytischen Ausdruck für die spektrale Abhängigkeit des Realteils der dielektrischen Funktion $\varepsilon(\omega)$ an! Skizzieren Sie den prinzipiellen Funktionsverlauf!

Hinweis: Nutzen Sie ihr Wissen ***oder*** *wenden Sie die Kramers-Kronig-Relationen explizit an!*

(3)

b) Gegeben sei ein homogenes, isotropes Material mit der Suszeptibilität $\chi(\omega) = -\omega_p^2 / \omega^2$. In welchem Frequenzbereich ist ungedämpfte Ausbreitung möglich? Bei welcher Frequenz können longitudinale Wellen existieren?

(3)

3. Fourieroptik $(\Sigma 8)$

Gegeben sei ein ideales 4f-Setup zur optischen Filterung. In der Eingangsebene z=0 ist ein unendlich ausgedehntes quadratisches Lochgitter der Periode *DxD* plaziert. Die Beleuchtung erfolgt durch eine ebene Welle. In der Brennebene *z=2f* der ersten Linse ist die Lichtverteilung dann durch eine diskrete Verteilung gegeben. Der Abstand der einzelnen Spots sei *FxF*.

In der Brennebene der ersten Linse werden im Folgenden verschiedene Aperturen installiert. Benennen Sie die resultierende Feldverteilung bei *z=4f* und begründen Sie ihre Antwort kurz!

a) Eine Lochblende auf der optischen Achse, welche nur die *(0,0)*-Fourierordnung bzgl. *x*- und *y*-Richtung passieren lässt.

(2)

b) Eine Spaltblende entlang der *x*-Richtung, welche alle *(m,0)*-Fourierordnungen passieren lässt.

(2)

c) Eine Spaltblende entlang der *x-y*-Richtung, welche alle *(m,m)*-Fourierordnungen passieren lässt. Was können Sie über die Periode der resultierenden Bildverteilung aussagen?

(4)

4. Raumzeitliche Lichtausbreitung $(\Sigma 14)$

Die raumzeitliche Ausbreitung eines Lichtpulses mit Feldverteilung $u(x,y,z,t)$ in einem dispersiven, absorptionsfreien Medium ist allgemein gegeben durch

$$u(x,y,z,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} U_0(\alpha,\beta,\omega)\exp\left[i\gamma z\right]\exp[i(\alpha x+\beta y-\omega t)]\; d\alpha\, d\beta\, d\omega$$

wobei $U_0(\alpha,\beta,\omega)$ das raumzeitliche Spektrum des Pulses am Ort z=0 darstellt.

a) Wie ist der Term γ definiert? Geben Sie γ als Funktion der Frequenz ω, sowie der Fourierkomponenten α und β an!

(1)

b) Nehmen Sie nun monochromatische Felder der Frequenz ω_0 an. Führen Sie eine Taylorentwicklung von γ um $(\alpha,\beta)=(0,0)$ bis zur quadratischen Ordnung durch und leiten Sie aus obiger Formel den Ausdruck für die sogenannte Fresnel-Näherung ab!

(3)

c) Nehmen Sie im Folgenden Ebene-Welle-Pulse an $[U_0(\alpha,\beta,\omega)=U_0(\omega)]$, welche sich in z-Richtung ausbreiten! Entwickeln Sie $\gamma(\omega)$ um ω_0 bis zur zweiten Ordnung in ω. Benennen Sie die einzelnen Entwicklungsterme!

(2)

d) Diskutieren Sie **kurz** die physikalische Bedeutung der einzelnen Entwicklungsterme.

(3)

e) Zwei Pulse propagieren in einem Plasma der Länge L deren Permittivität

$$\varepsilon(\omega)=1-\frac{\chi_0}{\omega^2}$$

sei. Die zugehörigen Trägerfrequenzen seien ω_1 und ω_2. Berechnen Sie den Zeitunterschied mit dem beide Signale nach einer Lauflänge L detektiert werden!

(5)

5. Optischer Film (∑6)

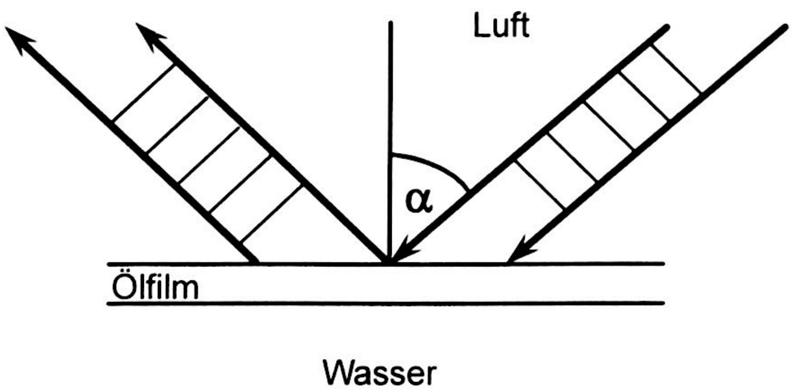
Ein Student blickt auf eine Wasserpfütze, welche mit einem dünnen Ölfilm ($n_f > n_{H20}$) der Dicke d bedeckt ist. Des Weiteren wird die Pfütze mit einer kollimierten Weißlichtlampe schräg beleuchtet. Das Beleuchtungsfeld wird als ebene Wellenfront angenommen!

a) Unter welchem Winkel α ist die Intensität des reflektierten Lichts der Wellenlänge λ maximal?

(4)

b) Der Ölfilm verdunstet langsam. Vergrößert oder verkleinert sich dabei der zuvor berechnete Winkel zur Wellenlänge λ. Begründen Sie Ihre Antwort!

(2)



6. Gaußbündel (∑6)

In einem Experiment ist ein Gaußbündel der Breite W_{in} und unendlichem Phasenkrümmungsradius R gegeben. Ihre Aufgabe ist es dieses Bündel nach einer Länge L exakt zu reproduzieren! Sie können dafür allerdings nur eine Linse einsetzen. Berechnen Sie die Brennweite sowie die Position der Linse!

Hinweis: Die ABCD-Matrix des Freiraums sowie einer dünnen Linse lauten:

$$M_d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \quad M_L = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

7. Oberflächen-Plasmon-Polaritonen (∑8)

Gegeben sei eine einfache Grenzfläche zwischen Luft und einem homogenen, isotropen und dispersiven Metamaterial (charakterisiert durch $\varepsilon(\omega) \neq 1$ und $\mu(\omega) \neq 1$).

a) Leiten Sie eine notwendige Bedingung für die Existenz eines gebundenen Oberflächenzustandes in TE-Polarisation her!

(3)

b) Leiten Sie die Dispersionsrelation $k_z(\omega)$ dieser Zustände ab!

(5)

Hinweis #1: Die Dispersionsrelation ebener Wellen lautet: $k^2 = \frac{\omega^2}{c^2}\varepsilon(\omega)\mu(\omega)$.

Hinweis #2: Der Transmissionskoeffizient an einer einzelnen Grenzfläche in TE-Polarisation ist gegeben durch die nachfolgende Formel.

$$T_{\mathrm{TE}} = \frac{2\alpha_s k_{sx}}{\alpha_s k_{sx} + \alpha_c k_{cx}}$$

$\alpha_{s/c}$ = $1/\mu_{s/c}$

$k_{sx/cx}$ ... Wellenvektorkomponenten senkrecht zur Grenzfläche

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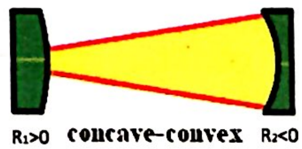
WS 2009/2010

Exam
FUNDAMENTALS OF MODERN OPTICS

February 11, 2010

Exercise 1 **8 Points**

A resonator consists of two spherical mirrors: one of them is a concave mirror with radius $R_1 > 0$, the other one is a convex mirror with radius $R_2 < 0$, and the distance between them is d. Define the conditions for this resonator to be stable, and sketch a stable configuration. Do not forget to mark the positions of the mirrors and their respective centers of curvature in your sketch.



Exercise 2 **10 Points**

In an experiment you have an input Gaussian beam profile with width W_{in} and flat phase (phase curvature $R_{in} = \infty$). Your task is to obtain a beam profile with the same parameters (W_{in} and R_{in}), but at a distance L from the input position. You are allowed to use just a single lens. Calculate the focal length of the lens you need and its position.

Exercise 3 **15 Points**

Illumination of a cross grating produces a light distribution

$$u_0(x, y) = \frac{A}{4}\left(1 + \cos\frac{2\pi}{a}x\right)\left(1 + \cos\frac{2\pi}{a}y\right),$$

with period length $a = 1\,\mathrm{mm}$. This light field is now imaged by a $4f$-setup, where in the plane $z = 2f$ a slit with the filter function

$$p(x, y) = \begin{cases} 1 & , |x| < D/2 \\ 0 & , \text{elsewhere} \end{cases}$$

is applied. The focal length is $f = 1\,\mathrm{m}$ and the wavelength used is $\lambda = 1\,\mu\mathrm{m}$.
Calculate the field $u(x, y, 4f)$ at the end of the $4f$-setup for a slit width $D = 1\,\mathrm{mm}$.

Exercise 4 **10+5 Points**

a) Consider a dielectric medium with response function

$$R(t) = \frac{f}{\sqrt{a^2 - b^2}}\, e^{-bt} \sin\left(\sqrt{a^2 - b^2}\; t\right),$$

where $a > b > 0$. Calculate the electric susceptibility $\chi(\omega)$.

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Exam
FUNDAMENTALS OF MODERN OPTICS

Final exam on February 25

Exercise 1 (Maxwell, linear media, plane waves, TE/TM) **5+2+10+10 Points**
Before you start with this exam, please have a look at the second sheet which provides you with some useful formulas. Good luck!

a) Write down the macroscopic Maxwell equations in temporal Fourier domain.

b) Specify (in Fourier domain) the general relation between $\overline{\mathbf{D}}(\mathbf{r},\omega)$ and $\overline{\mathbf{E}}(\mathbf{r},\omega)$ for a linear, dispersive, inhomogeneous, and anisotropic medium.

c) Let us now consider a linear, dispersive, homogeneous and isotropic dielectric medium in transparent regime: $\overline{\mathbf{D}}(\mathbf{r},\omega) = \epsilon_0\epsilon(\omega)\overline{\mathbf{E}}(\mathbf{r},\omega)$ with $\epsilon(\omega) > 0$, $\overline{\mathbf{B}}(\mathbf{r},\omega) = \mu_0\overline{\mathbf{H}}(\mathbf{r},\omega)$, $\overline{\mathbf{j}}(\mathbf{r},\omega) = 0$, and $\rho = 0$. Show that plane waves of the form $\overline{\mathbf{E}}(\mathbf{r},\omega) = \mathbf{E}(\omega)\exp(i\mathbf{k}(\omega)\cdot\mathbf{r})$ exist which are solutions to Maxwell's equations. Derive the neccessary conditions for $\mathbf{k}(\omega)$ and $\mathbf{E}(\omega)$ [length and direction of $\mathbf{k}(\omega)$ with respect to $\mathbf{E}(\omega)$] and compute the corresponding $\overline{\mathbf{H}}(\mathbf{r},\omega)$.

d) For linear, *inhomogeneous* and isotropic dielectric media things become more complicated, however, in the case of translational invariance in one direction (e.g., y) we can decouple Maxwell's equations into TE and TM polarization. Derive the resulting wave equations for $\overline{\mathbf{E}}_{\mathrm{TE}}(x,z,\omega)$ and $\overline{\mathbf{H}}_{\mathrm{TM}}(x,z,\omega)$, i.e., one equation containing $\overline{E}_y(x,z,\omega)$ and $\epsilon(x,z,\omega)$ only, and one equation containing $\overline{H}_y(x,z,\omega)$ and $\epsilon(x,z,\omega)$ only. ?

Exercise 2 (uniaxial crystal, quarter-wave plate) **7 Points**
Suppose you want to use a thin slice of an uniaxial crystal (quartz, $n_O \simeq 1.54$, $n_e \simeq 1.55$) to transform circularly polarized light at a wavelength of $\lambda_{vac} = 500$ nm to linear polarization. How do you have to cut the crystal with respect to the optical axis, to make the slice as thin as possible? How thick the resulting "quarter-wave plate" will be?

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A Master of Science in Photonics student wants to build an optical integrator by using a $4f$-setup. He/she knows that by placing an appropriate pupil $p(x)$ at distance $z = 2f$, one can perform useful transformations on the entrance field $u_0(x)$.

a) What would be the ideal pupil $p(x)$ to obtain $u(-x, z = 4f) \propto \int\limits_{-\infty}^{x} u_0(x')dx'$?

b) Unfortunately, this ideal pupil function for optical integration has a singularity at $x = 0$. Show that

$$p(x) = \mathrm{sinc}\left(\frac{kb}{f}x\right),$$

which is free of singularities, leads to $u(-x, z = 4f) \propto \int\limits_{x-b}^{x+b} u_0(x')dx'$.

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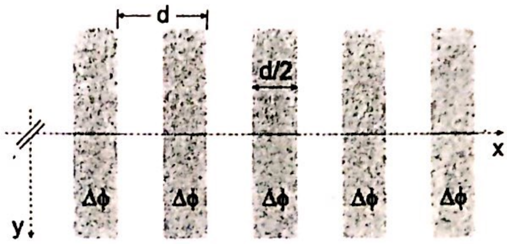
Consider a very complicated (absorption-free in the relevant frequency range) multi-layer system between a substrate ($\epsilon_s = 2$) and cladding ($\epsilon_c = 1$) material. Give the value of the reflectivity for an incident plane wave in the substrate, when the angle of incidence (between k-vector and normal of the layer-system) is $\phi = \pi/3$.

(p.t.o)

Exercise 5 (dielectric function, response function, Kramers-Kronig relations) **7 Points**
A friend of yours claims to have a homogeneous medium characterized by the dielectric function $\epsilon(\omega) = 1 + i\exp(-\omega^2\tau_0^2)$, with $\tau_0 = 42$ fs. Do you believe him? Give reasons for your answer.

Exercise 6 (Fraunhofer approximation, phase grating) **15 Points**
A monochromatic plane wave propagates along $\vec{e}_z$ in vacuum and is diffracted by a thin non-absorbing phase grating of period d, whose N slits (white) and bars (gray) have a width $d/2$. The phase shift of the incident light within the bars amounts to $\Delta\phi = \pi$ relative to the light which passes directly through the slits. Assume an infinite extension along $\vec{e}_y$, i.e. a one-dimensional structure.



Show that the transmitted intensity in paraxial Fraunhofer approximation may be written as

$$I(x,z) \propto \text{sinc}^2\left(\frac{Nkxd}{2z}\right)\tan^2\left(\frac{kxd}{4z}\right).$$

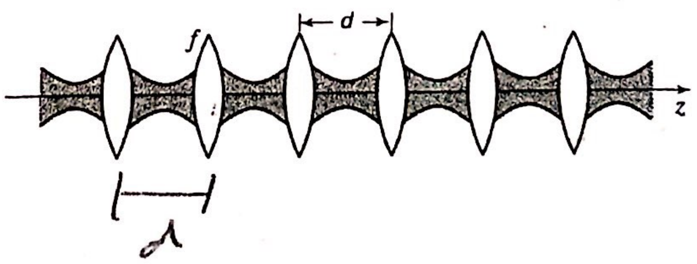
Which diffraction orders m can be observed, which ones vanish? ?

Exercise 7 (pulse propagation in dispersive media) **10 Points**
The group velocity in a homogeneous dielectric used for manufacturing glass fibers can be well approximated in a relatively wide range in the infrared wavelength regime by

$$v_g(\omega) = \frac{c}{n_0}\frac{1}{1+\left(\frac{\omega-\omega_0}{\delta}\right)^2}$$

with the center frequency ω_0, the center group index n_0, a parameter δ, and the speed of light c. Let us consider a plane wave with a Gaussian pulse envelope $u(t, z = 0) = u_0\exp\{-(t/T_0)^2\}\exp(-i\omega t)$ launched into this material at the plane $z = 0$ and propagating in z-direction. Derive a formula for the propagation length in dependence on the frequency ω, after which the pulse duration has doubled. Consider dispersion up to second order (GVD) only. At which ω the pulse does not broaden upon propagation?

Exercise 8 (propagation of Gaussian beams, beam relaying) **10 Points**
A Gaussian beam is focused by a periodic sequence of identical lenses, each of focal length f and separated by a distance d (see Figure below). The beam is consecutively focused to the same waist and propagates periodically in this sequence of lenses. Show that the condition of periodic transmission ("relaying") can arise only if the inequality $d < 4f$ is satisfied. Where is the beam waist located between the individual lenses?



Formulas which may be useful:

- $\mathrm{rot}\,\mathbf{E}(\mathbf{r}) = \nabla \times \mathbf{E}(\mathbf{r})$, $\mathrm{div}\,\mathbf{E}(\mathbf{r}) = \nabla \cdot \mathbf{E}(\mathbf{r})$, $\mathrm{grad}\,\epsilon(\mathbf{r}) = \nabla\epsilon(\mathbf{r})$, $\nabla^2 = \Delta$
- $\nabla \times [\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r})] = \epsilon(\mathbf{r})\nabla \times \mathbf{E}(\mathbf{r}) - \mathbf{E} \times \nabla\epsilon(\mathbf{r})$
- $\nabla \times [\nabla \times \mathbf{E}(\mathbf{r})] = \nabla[\nabla \cdot \mathbf{E}(\mathbf{r})] - \Delta\mathbf{E}(\mathbf{r})$
- $\nabla \times \left[\frac{1}{\epsilon(\mathbf{r})}\nabla \times \mathbf{H}(\mathbf{r})\right] = \frac{1}{\epsilon(\mathbf{r})}\nabla \times [\nabla \times \mathbf{H}(\mathbf{r})] - [\nabla \times \mathbf{H}(\mathbf{r})] \times \nabla\left[\frac{1}{\epsilon(\mathbf{r})}\right]$
- For a $4f$-system we have
$$u(-x, z = 4f) \propto \int_{-\infty}^{\infty} p\left(\frac{f}{k}\alpha\right) U(\alpha, z = 0)e^{i\alpha x}d\alpha \propto \int_{-\infty}^{\infty} P\left[\frac{k}{f}(x' - x)\right] u(x', z = 0)dx',$$
where $U(\alpha, z = 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x, z = 0)\exp(-i\alpha x)dx$ and $P(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} p(x)\exp(-i\alpha x)dx$ is the Fourier transform of the pupil $p(x)$ situated at distance $z = 2f$.
- $\int_{-\infty}^{\infty} \mathrm{sinc}(x)\exp(-i\alpha x)dx = \pi\theta(\alpha + 1)\theta(\alpha - 1)$ with $\theta(x) = \begin{cases} 0 & x < 0 \\ 1/2 & x = 0 \\ 1 & x > 0 \end{cases}$
- $\sin(\pi/3) = \sqrt{3}/2$
- The linear Kramers-Kronig relations read
$$\Re\chi(\omega) = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} \frac{\Im\chi(\overline{\omega})}{\overline{\omega} - \omega}d\overline{\omega}, \qquad \Im\chi(\omega) = -\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} \frac{\Re\chi(\overline{\omega})}{\overline{\omega} - \omega}d\overline{\omega}.$$
- $\int_{\infty}^{\infty} \exp(-x^2)\exp(-i\alpha x)dx = \sqrt{\pi}\exp(-\alpha^2/4)$
- $\int_{\infty}^{\infty} \exp(-i\alpha x)dx = 2\pi\delta(\alpha)$
- The intensity distribution created by a one-dimensional periodic mask in paraxial Fraunhofer approximation is given by
$$I(x, z) \propto \left|T_S\left(\frac{kx}{z}\right)\right|^2 \frac{\sin^2\left(\frac{Nkxd}{2z}\right)}{\sin^2\left(\frac{kxd}{2z}\right)},$$
where d is the period length, N the number of periods, and $T_S(\alpha) = \frac{1}{2\pi}\int_0^d t(x)\exp(-i\alpha x)dx$ the Fourier transform of one period of the mask.
- $\sin(x) = 2\sin(x/2)\cos(x/2)$
- The propagation of pulse envelopes $v(z, \tau)$ in linear dispersive media is governed by
$$i\partial_z v(z, \tau) = \frac{D}{2}\partial_\tau^2 v(z, \tau),$$
where $\tau = t - z/v_g(\omega_0)$ is the co-moving time, and the wavenumber
$$k(\omega) = \frac{\omega_0 n_0}{c} + \frac{1}{v_g(\omega_0)}(\omega - \omega_0) + \frac{D}{2}(\omega - \omega_0)^2$$
is expanded in a Taylor series around the central frequency ω_0.

(p.t.o)

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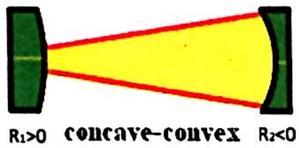
WS 2009/2010

Exam
FUNDAMENTALS OF MODERN OPTICS

February 11, 2010

Exercise 1 **8 Points**

A resonator consists of two spherical mirrors: one of them is a concave mirror with radius $R_1 > 0$, the other one is a convex mirror with radius $R_2 < 0$, and the distance between them is d. Define the conditions for this resonator to be stable, and sketch a stable configuration. Do not forget to mark the positions of the mirrors and their respective centers of curvature in your sketch.



Exercise 2 **10 Points**

In an experiment you have an input Gaussian beam profile with width W_{in} and flat phase (phase curvature $R_{in} = \infty$). Your task is to obtain a beam profile with the same parameters (W_{in} and R_{in}), but at a distance L from the input position. You are allowed to use just a single lens. Calculate the focal length of the lens you need and its position.

Exercise 3 **15 Points**

Illumination of a cross grating produces a light distribution

$$u_0(x,y) = \frac{A}{4}\left(1 + \cos\frac{2\pi}{a}x\right)\left(1 + \cos\frac{2\pi}{a}y\right),$$

with period length $a = 1\,\text{mm}$. This light field is now imaged by a $4f$-setup, where in the plane $z = 2f$ a slit with the filter function

$$p(x,y) = \begin{cases} 1 & , |x| < D/2 \\ 0 & , \text{elsewhere} \end{cases}$$

is applied. The focal length is $f = 1\,\text{m}$ and the wavelength used is $\lambda = 1\,\mu\text{m}$.
Calculate the field $u(x,y,4f)$ at the end of the $4f$-setup for a slit width $D = 1\,\text{mm}$.

Exercise 4 **10+5 Points**

a) Consider a dielectric medium with response function

$$R(t) = \frac{f}{\sqrt{a^2 - b^2}}\, e^{-bt} \sin\left(\sqrt{a^2 - b^2}\; t\right),$$

where $a > b > 0$. Calculate the electric susceptibility $\chi(\omega)$.

b) A company producing optical instruments is looking for a new homogeneous, isotropic material which should have the optical property

$$\chi(\omega) = Ae^{-\frac{(\omega-\omega_0)^2}{B^2}} + i\,C\delta(\omega - \omega_0),$$

where $A = 0.542$, $B = 1.02 \cdot 10^{15}\,s^{-1}$, $C = 3.29$ and $\omega_0 = 4.71 \cdot 10^{15}\,s^{-1}$. Do you think their research can be successful? Explain with the help of the Kramers Kronig relations.

Exercise 5 **7 Points**

Consider an uniaxial crystal with refractive indices for the ordinary wave n_o and the extraordinary wave n_e. The crystal's optical axis is parallel to its surface. A monochromatic, circularly polarized wave is normally incident on the crystal. Compute the propagation lengths after which the light is linearly polarized.

Exercise 6 **10 Points**

Two pulses are propagating in a homogeneous plasma. They have a carrier frequency much larger than the plasma frequency. The corresponding wavelengths of the carrier waves are λ_1 and λ_2.
The signals are recorded by a detector that is located at a distance L from the source. Use the dielectric function of the plasma

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

to compute the time delay between the two pulses.

Exercise 7 **5+10 Points**

Consider an interface between two media, $n(x < 0) = n_1 = 2$ and $n(x > 0) = n_2 = 1$.

a) Compute the angle of incidence ϕ_{Itot} above which total internal reflection occurs.

b) If the second medium (n_2) is not infinite, but forms a layer of thickness d [$n(0 < x < d) = n_2 = 1$], and $n(x > d) = n_3 = 2$, do you still expect reflectivity $\rho = 1$ at the interface at $x = 0$? Compute the reflectivity for this layer (TE-polarization) to prove your answer.

Exercise 8 **10+10 Points**

Investigate the propagation of a 1-dimensional initial field distribution $u_0(x) = A\cos^2(x/\Lambda)$ in paraxial (Fresnel) approximation.

a) At which propagation distances z_T do we observe $|u(x, z_T)|^2 = |u_0(x)|^2$?

b) Now consider the above $u_0(x)$ with a finite aperture $a = N\pi\Lambda/2$, so that

$$\tilde{u}_0(x) = \begin{cases} u_0(x) & , |x| < a \\ 0 & , \text{elsewhere.} \end{cases}$$

Compute the far field intensity distribution of $\tilde{u}_0(x)$, and discuss the necessary propagation distances to apply Fraunhofer approximation with respect to N.

Name: ______________________ Date of birth: ______________ Student ID No. ______________

Final Exam of "Fundamentals of modern optics" WS 2015/16
to be written on February 15, 10:00 - 12:00 am

Problem 1 – Maxwell's equations (MWEs) — $3+2+2+3=10$ points

a) Write down MWEs in frequency domain, with external sources, in its most general form in a material. Furthermore, write down the constitutive equations for auxiliary fields **D** and **H**, in frequency domain, where the material is inhomogeneous, dispersive, linear, isotropic, and magnetic.

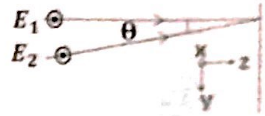
b) Find the wave equation for **E**, from the MWEs of part (a), for an inhomogeneous, dispersive, linear, isotropic, and non-magnetic material, in the presence of external sources.

c) Write down the MWEs of part (a) in their integral form, using the Stoke's and divergence theorems.

d) Derive the continuity equation connecting the charge density and the current density, from MWEs in (a). Write your final answer also in terms of the total charge in a volume and the total current escaping that volume.

Problem 2 – Normal Modes — $2+2+1+3=8$ points

Consider two monochromatic plane waves with electric field amplitudes E_1 and E_2 polarized along the x direction in free space with wavelength λ. One plane wave is propagating along the z direction and the other in yz plane at an angle θ as shown in the figure.



a) Write down the space and time dependent electric field vector for both plane waves.

b) Find the intensity pattern due to interference of these two waves and plot it along the y direction (assume x=z=0 and E_1=$2E_2$).

c) Calculate the distance between two consecutive maxima (or minima) in y direction. What will be the effect of increasing angle θ on the observed pattern?

d) Define and calculate the time averaged Poynting vector.

Problem 3 – Diffraction theory — $2+1+6=9$ points

a) Explain the condition shortly and write a mathematical inequality for which you can apply the following approximations: (i) Fresnel approximation. (ii) Fraunhofer approximation.

b) Can the Fraunhofer approximation be applied without the Fresnel approximation. Please explain your answer in two sentences.

c) Give formulas for calculating the observed diffraction patterns of an illumination function $u_0(x,y)$(in scaler approximation) for: (i) non-paraxial case. (ii) Fresnel approximation. (iii) Fraunhofer approximation.

Problem 4 – Pulses — $1+2+3=6$ points

Consider a fiber of the length $L_1 = 1\,\mathrm{m}$ has the dispersion relation $k_1(\omega) = \alpha_1\omega_0 + \beta_1(\omega-\omega_0) + \frac{\gamma_1}{\omega_0}(\omega-\omega_0)^2$, with the frequency ω and the constants $\omega_0 = 2\cdot 10^{15}\mathrm{rad/s}$, $\alpha_1 = \frac{3}{2c}$, $\beta_1 = \frac{3}{4c}$ and $\gamma_1 = \frac{3}{8c}$, where $c = 3\cdot 10^8\mathrm{m/s}$ is the speed of light in vacuum.

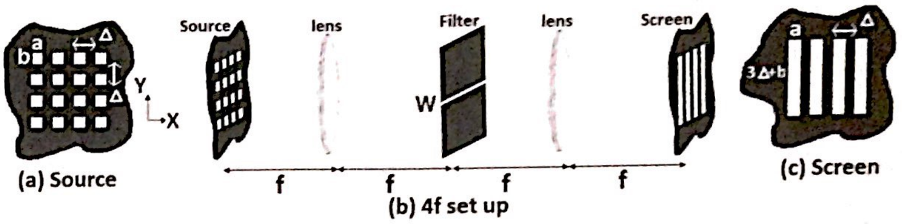
a) What is the phase velocity of monochromatic light of frequency $\omega_1 = 3\cdot 10^{15}\frac{\mathrm{rad}}{\mathrm{s}}$ in this fiber?

b) Now, we couple a Gaussian pulse into the fiber (center frequency $\omega_1 = 3\cdot 10^{15}\frac{\mathrm{rad}}{\mathrm{s}}$ and FWHM 100 fs). Its maximum starts at $t_0 = 0$ sec at the beginning of the fiber. At which time does this maximum arrive at the end of the fiber?

c) Consider a second fiber with the dispersion relation $k_2(\omega) = \alpha_2\omega_0 + \beta_2(\omega-\omega_0) + \frac{\gamma_2}{\omega_0}(\omega-\omega_0)^2$, where $\alpha_2 = \frac{10}{c}$, $\beta_2 = \frac{2}{c}$ and $\gamma_1 = \frac{-4}{9c}$. It is merged to the end of the first fiber. We launch two Gaussian pulses with the center frequencies $\omega_1 = 3\cdot 10^{15}\frac{\mathrm{rad}}{\mathrm{s}}$ and $\omega_2 = 4\cdot 10^{15}\frac{\mathrm{rad}}{\mathrm{s}}$, respectively, and both with the FWHM of 100 fs at the same time into the first fiber. Calculate the length L_2 of the second fiber when both pulses maxima shall arrive at the end of the second fiber at the same time.

Problem 5 – Imaging optics $3 + 4 + 3 = 10$ **points**

Assume, a source with 16 bright illuminating rectangles, each of size $a \times b$. They are arranged in x and y direction with period Δ (shown in figure a).



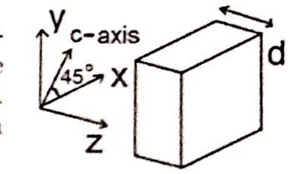
a) Please write the amplitude transmittance function of the source (figure a).

b) Calculate the diffraction pattern at 2f distance just before the filter (shown in figure b).

c) Suppose one wishes to obtain the diffraction pattern on the screen as shown in the figure c. What should the range of aperture width W in the filter be, to realize this pattern?

Problem 6 – Anisotropy $2 + 1 + 3 = 6$ **points**

A layer of a uniaxial crystal of thickness $d = 5\mu m$ is shown in the figure. The extraordinary crystal axis is in the x-y plane and makes a 45° angle with the x and y axis. The ordinary and extraordinary refractive indices are $n_o = 2.2$ and $n_e = 2.15$, respectively. A plane wave with an electric field of $\boldsymbol{E} = E_0 e^{i\frac{2\pi}{\lambda}z}(\hat{x} + i\hat{y})$ is incident on this layer from one side, where $\lambda = 1\mu m$ is the wavelength of the wave in free space.



a) What are the two eigenmodes of the crystal propagating in the z direction. Specify the direction of electric field (in terms of $\hat{x}$ and $\hat{y}$) and the magnitude of k-vector for each eigenmode.

b) Decompose the input electric field polarization into the two eigenmodes of part (a).

c) Calculate the electric field polarization at the other side of this crystal layer. You can ignore the reflections from the crystal surfaces; it is the final polarization state that matters to us.

Problem 7 – Interface $1 + 2 + 2 = 5$ **points**

The reflection coefficient of a TE mode field, with incident angle of θ_1 and refracted angle of θ_2, from a media 1 (n_1) into a media 2 (n_2) is described by the following Fresnel equation,

$$r_{TE} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}.$$

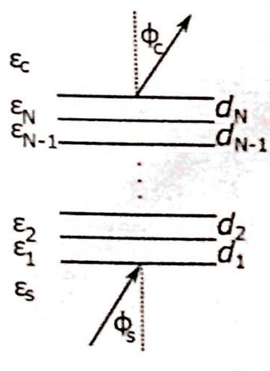
a) Assume $n_1 > n_2$. What is the relation for the critical angle θ_c, after which we have total internal reflection?

b) Show that for $\theta_1 > \theta_c$ we get $|r_{TE}| = 1$.

c) Find the value of the extra phase that the reflected wave acquires for the limiting case of $\theta_1 = \pi/2$.

Problem 8 – Multilayer system $2 + 2 + 4 = 8$ **points**

Consider a multilayer system of N isotropic layers with the permittivities ϵ_i and the thicknesses d_i ($i = 1, 2, ..., N$). In front of and behind this multilayer system, there are the substrate (ϵ_s) and the cladding (ϵ_c), respectively.



a) Let the layer permittivities be real and hold the following relation: $\epsilon_s < \epsilon_1 < \epsilon_2 < ... < \epsilon_i < ... < \epsilon_N < \epsilon_c$. An incoming monochromatic plane wave hits the interface to the multilayer system with an angle of ϕ_s with respect to the interface normal (see figure). Find the the angle ϕ_c of the refracted wave after the multilayer stack.

b) Explain shortly the matrix method and its use for the multilayer system. Give also a short mathematical description.

c) Derive the matrix for a single layer, for a TE-polarized monochromatic plane wave.

Maybe useful formulas: $\nabla \times \nabla \times \mathrm{a} = \nabla(\nabla \cdot \mathrm{a}) - \Delta \mathrm{a}$, $\nabla \cdot (\nabla \times \mathrm{a}) = 0$.

Name: ______________________ Date of birth: ______________ Matrikelnr. ____________

Midterm Exam
"Fundamentals of modern optics"
WS 2016/17
to be written on December 19

Problem 1 – Maxwell's Equations $\quad 4.5 + 2.5 + 3 = 10$ points

a) Write down Maxwell's equations for the electric and magnetic field in the time domain in a material which is non-magnetizable by introducing the external sources $\rho(\mathbf{r}, t)$ and $\mathbf{j}(\mathbf{r}, t)$.

b) Derive wave equation for the electric field from these Maxwell's equations.

c) A homogeneous but dispersive medium cannot respond instantaneously when a time varying electric field is applied to it. Write down the constitutive relation between $\mathbf{D}(\mathbf{r}, \omega)$ and $\mathbf{E}(\mathbf{r}, \omega)$ in this medium and find the corresponding relation in the time domain.

Problem 2 – Poynting Vector and Normal Mode $\quad 3 + 2 + 3 + 2 = 10$ points

Consider a transverse monochromatic plane wave of frequency ω, propagating in a homogeneous isotropic medium, that is an extremely good conductor with conductivity $\sigma >> \omega\varepsilon_0$. The complex representation of the electric field has the form $\mathbf{E}(\mathbf{r}, \omega) = E_0(-\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}})e^{i\beta(1+i)(x+y)}$, where β is a real number and $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors in x and y-direction.

a) Specify the wave-vector $\boldsymbol{k}$ for this plane wave (with its complex amplitude and direction). What is the dispersion relation that this wave-vector satisfies? Find σ as a function of β from this dispersion relation. (You can still fully solve part b and c if you do not manage to solve part a.)

b) Find the magnetic field $\mathbf{H}(\mathbf{r}, \omega)$.

c) Write down the formula for the time-averaged Poynting vector $\langle\mathbf{S}(\mathbf{r}, \mathbf{t})\rangle$, based on the complex representations of the electric and magnetic fields. Find $\langle\mathbf{S}(\mathbf{r}, \mathbf{t})\rangle$ for the field given above.

d) Find the divergence of the time-averaged Poynting vector $\nabla \cdot \langle\mathbf{S}(\mathbf{r}, \mathbf{t})\rangle$ and express it as a function of only the absolute value of the electric field $|\mathbf{E}(\mathbf{r}, \omega)|$ and conductivity σ.

Problem 3 – Beam propagation $\quad 2 + 2 + 1 + 1 = 6$ points

a) Describe an algorithm which makes use of the transfer function $H(\alpha, \beta, z)$ and is capable of calculating an optical field $u(x, y, d)$ at position $z = d$ from a field $u_0(x, y, 0)$ given at a position $z = 0$.

b) What is the explicit mathematical form of $H(\alpha, \beta, z)$ for free space? How can it be approximated for the paraxial case?

c) An initial field $u_0(x, y, 0)$ is given as the superposition of two fields

$$u_0(x, y, 0) = u_0^{(1)}(x, y, 0) + u_0^{(2)}(x, y, 0)$$

How will $u(x, y, z)$ depend on the two input fields and why?

d) Prove that the algorithm in a) corresponds to a convolution operation in real space!

Problem 4 – Propagation of Gaussian Beams $\quad 2 + 2 + 4 = 8$ points

A Gaussian beam with the Rayleigh length $z_0 = kW_0^2/2$, is propagating through a homogeneous medium.

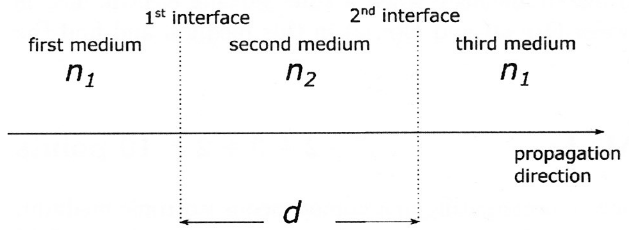
a) What is the q-parameter for Gaussian beam propagation? How can we obtain the radius of the phase curvature and the beam width from it?

Consider a Gaussian beam to propagate from a first homogeneous (refractive index n_1) through a second (n_2) into a third medium (again n_1) with $n_2 > n_1$ (see Figure). Both interfaces can be treated with the matrix method and each of them appears like a spherical interface (first interface radius $R_{int} > 0$, second interface radius $-R_{int}$). The length between the two interfaces is d.

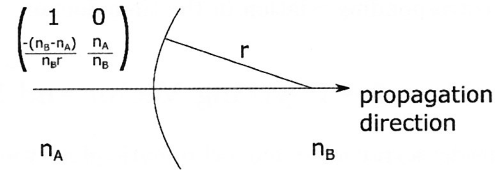
b) Assume that propagation starts directly before the first and ends directly behind the second interface. Calculate the q-parameter of the Gaussian beam after propagation through this system. The q-parameter before the first interface is q_1.

Now, consider d to be small enough to be neglected ($d = 0$). Before reaching the first interface, the beam has propagated the distance L_1 from its waist position.

c) After which distance L_2 from the second interface does the beam exhibit a waist again?



ABCD Matrix for spherical interface:



Problem 5 – Pulses $\quad 1 + 3 + 2 + 2 = 8$ points

Consider a laser source with an output power of 100 mW and a repetition rate of 100 MHz. The output of such a source is a sequence of transform-limited Gaussian pulses with central frequency ω_0. The envelope of each individual pulse in its co-moving frame is defined as $E(t') = E_0 \exp\left[-t'^2/\tau^2\right]$, where the pulse width is $\tau = 8$ ps. This pulse sequence is launched into a fiber characterized by

$$k(\omega) = k_0 + \frac{1.5}{c}(\omega - \omega_0) + \frac{D}{2}(\omega - \omega_0)^2,$$

with $D = 0.08\ \mathrm{ps^2/m}$.

a) Calculate the energy of each individual pulse.

b) Find the dispersion length L_D of each individual pulse. Is the red or the blue part of the spectrum appearing earlier at the end of the fiber?

c) After a fiber length of $L_1 = 4$ km, a second type of fiber is connected to the first. It has a dispersion of $-2000\ \mathrm{fs^2/m}$ and a length of L_2. How long does L_2 have to be in order to fully restore the initial pulse sequence?

d) Now, a pulse sequence with a different frequency $\omega_1 = \omega_0 + \delta\omega$ is launched into the first fiber. Suppose that the detuning $\delta\omega = 1$ THz. Find the group index n_g in that case.

Problem 6 – Fraunhofer diffraction $\quad 4+1=5$ points

A one-dimensional optical field directly behind a specific optical element is given as

$$u_0(x, 0) = \begin{cases} A \exp\left[i\Phi_0\left(\frac{x}{a} + 1\right)\right] & |x| \leq a \\ 0 & \text{otherwise} \end{cases}$$

a) Find the optical intensity $I(x, z)$ in the case that the paraxial approximation holds and the distance $z \gg a$. You may omit possible prefactors.

b) How large does Φ_0 have to be so that the intensity on the optical axis $x = 0$ vanishes?