Task 1: Gaussian beam (a=1, b=2, c=2 pts.)

In the lecture, we defined the Gaussian beam as

$$v(x,y,z) = A(z)\exp\left[-\frac{x^2+y^2}{w(z)^2}\right]\exp\left[ikz + i\frac{k}{2}\frac{x^2+y^2}{R(z)} + i\varphi(z)\right].$$

a) Write down the expression for the Gaussian beam wavefront and find the constant phase plane at z. *Hint:* Neglect Gouy phase shift of the Gaussian beam.

b) Derive a spherical wave in paraxial approximation and find the condition in which the wavefront of the Gaussian beam (your result from a) is the same as a wavefront of the spherical wave .

c) How far can a Gaussian beam with $\lambda = 800$ nm and $W_0 = 5$ mm stay collimated if we accept 10% broadening of the beam?

(a) Gaussian beam wavefront $k[z+\frac{(x^2+y^2)}{2R(z)}]$

b) $u(\vec{r})=\frac{A}{r}e^{ik\vec{r}}$ $\quad k^2=k_x^2+k_y^2+k_z^2 \Rightarrow k_z=\sqrt{k^2-k_x^2-k_y^2}$ $\quad r=\sqrt{x^2+y^2+z^2}$ $\quad z=\sqrt{r^2-x^2-y^2}$

Paraxial Approximation: $u(\vec{r})=\frac{A}{r}e^{ik\vec{r}} = \frac{A}{\sqrt{x^2+y^2+z^2}}\exp[ik\sqrt{x^2+y^2+z^2}] = \frac{A}{z\sqrt{1+\frac{x^2+y^2}{z^2}}}\exp[ikz\sqrt{1+\frac{x^2+y^2}{z^2}}]$

$= \frac{A}{z(1+\frac{x^2+y^2}{2z^2})}\exp[ikz(1+\frac{x^2+y^2}{2z^2})] = \frac{A}{z}\exp(ikz)\exp[\frac{ik}{2z}(x^2+y^2)]$

$R(z)=z[1+(\frac{z_0}{z})^2]$ $\quad z\gg z_0 \Rightarrow R(z)\approx z$ thus when $z\gg z_0$ Gaussian beam resembles spherical wave

c) $W^2(z)=W_0^2[1+(\frac{z}{z_0})^2]=(1.1W_0)^2 \Rightarrow 1+(\frac{z}{z_0})^2=1.21$ $\quad z_0=\frac{kW_0^2}{2}=\frac{\pi W_0^2}{\lambda}$

$\Rightarrow z=\sqrt{0.21}\,z_0=\frac{\sqrt{0.21}\times 3.14\times 25\,mm^2}{800\times 10^{-6}\,mm}\approx 44966.5\,mm = 44.967\,m$

Task 2: Focusing a Gaussian Beam (a=4, b=2 pts.)

A collimated Gaussian beam of wavelength λ with a waist W_0 (the waist is just before the lens) is focused by a lens with a focal distance f, as shown in Figure 1. The Rayleigh length of the beam before the lens, $z_0 = \frac{\pi W_0^2}{\lambda}$, is much larger than f. The focused Gaussian beam after the lens would have the waist W_1 at the distance d after the lens.

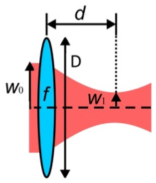

Figure 1: Focusing with a lens.

a) Use the ABCD matrix of the system to calculate W_1 and d exactly. Then use the fact that $z_0 >> f$ to simplify your results.

b) How small can $2W_1$ be? In other words, how small can the focal spot after the lens be? Use the approximate result from a) in the $z_0 >> f$ regime.
Hint: To make a statement about this, you have to make some assumptions. Firstly, you have to notice that for the calculation in a) to be correct, you are assuming that the lens aperture D is large enough to let a substantial part of the Gaussian beam to pass through it. Let us say that $2W_0$ should be smaller than D for a substantial part of the beam to pass through the lens. Moreover, for a thin lens, the size of the aperture is also limited based on its focal length. So let us assume that $D/2$ is smaller than f, such that the ratio $D/2f$ is always smaller than 1. Put all these statements together and find the limit on how small the size of the focused beam can be.

a) $q(z)=z-iz_0$ $\quad q_0(0)=-iz_0=-i\frac{\pi W_0^2}{\lambda}$ $\quad M_1=\begin{bmatrix}1&0\\-1/f&1\end{bmatrix}$ $\quad M_2=\begin{bmatrix}1&d\\0&1\end{bmatrix}$ $\quad \frac{1}{q(z)}=\frac{1}{R(z)}+i\frac{\lambda}{\pi W(z)^2}$

$M=M_2M_1=\begin{bmatrix}1&d\\0&1\end{bmatrix}\begin{bmatrix}1&0\\-1/f&1\end{bmatrix}=\begin{bmatrix}1-\frac{d}{f}&d\\-\frac{1}{f}&1\end{bmatrix}$ $\quad q=\frac{(1-\frac{d}{f})q_0+d}{-\frac{1}{f}q_0+1}=\frac{-i(f-d)z_0+fd}{f+iz_0}$

$q=\frac{fd+i(d-f)z_0}{f+iz_0}=\frac{[fd+i(d-f)z_0](f-iz_0)}{f^2+z_0^2}=\frac{f^2d-iz_0fd+if(d-f)z_0+(d-f)z_0^2}{f^2+z_0^2}$

$\Rightarrow q=\frac{f^2d+(d-f)z_0^2}{f^2+z_0^2}-i\frac{z_0fd+f(f-d)z_0}{f^2+z_0^2}=d-\frac{fz_0^2}{f^2+z_0^2}-i\frac{f^2z_0}{f^2+z_0^2}$

$q=z-iz_1=-iz_1=-i\frac{\pi W_1^2}{\lambda} \Rightarrow d-\frac{fz_0^2}{f^2+z_0^2}=0 \Rightarrow d=\frac{fz_0^2}{f^2+z_0^2}=\frac{f}{\frac{f^2}{z_0^2}+1}$ $\quad z_0\gg f\Rightarrow d\approx f$

$\frac{\pi W_1^2}{\lambda}=\frac{f^2z_0}{f^2+z_0^2}\Rightarrow W_1=\sqrt{\frac{\lambda}{\pi}\frac{f^2z_0}{f^2+z_0^2}}=\sqrt{\frac{\lambda}{\pi}\frac{f^2}{\frac{f^2}{z_0}+z_0}}\propto\sqrt{\frac{\lambda}{\pi}\frac{f^2}{z_0}}\approx\frac{\lambda f}{\pi W_0}$

$f>\frac{D}{2}\Rightarrow W_1=\frac{\lambda f}{\pi W_0}>\frac{D\lambda}{2\pi W_0}$ $\quad 2W_1>\frac{D\lambda}{\pi W_0}$ $\quad W_0<\frac{D}{2}\Rightarrow 2W_1>\frac{2\lambda}{\pi}$

Task 3: Paraxial Optical Cloak (a=6, b=1, c=2, d=4 pts.)

Consider a linear optical system with the total length L and the following ABCD matrix:

光学斗篷 $$\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}$$

Such system can act as an optical cloak in the paraxial limit. This means that the system itself and possibly objects placed inside of it are not visible from outside under a small viewing angle. Hence, the above matrix describes a free space propagation, i.e., light passing through the system acts as if the system was not there and is simply translated by the distance L. In this task, you will derive how such a system could be implemented. For further information, you may take a look at `https://doi.org/10.1364/OE.22.029465` or `https://www.youtube.com/watch?v=vtKBzwKfP8E`.

a) Consider a system of two thin lenses with focal lengths of f_1 and f_2, respectively, and a distance $d = f_1 + f_2$ between them (see Fig. 2). Assume that you have a Gaussian beam of waist W_0 and wavelength λ, where the waist is positioned at a distance f_1 in front of the first lens, as shown Fig. 2. This beam will be focused at a distance g_1 after the first lens, where the beam waist is W_1, and again refocused at a distance g_2 after the second lens, where the waist size is W_2. Use the q-parameter calculation to find g_1, g_2, W_1, and W_2.

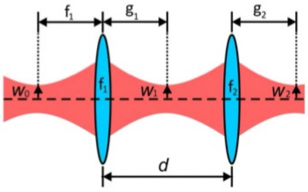

Figure 2: 2 lens system.

b) Estimate W_1 and W_2, when $f_1 = 300\,\text{mm}$, $f_2 = 200\,\text{mm}$, $W_0 = 15\,\text{mm}$, and $\lambda = 633\,\text{nm}$.

c) Calculate the ABCD matrix of the system in a) by only considering the two lenses and the propagation distance between them.

d) Consider a set up with four lenses and a mirror symmetry as illustrated in Figure 3, with $d_1 = f_1 + f_2$. Calculate the ABCD matrix of the system and derive the condition for d_2, as a function of f_1 and f_2, under which this system acts as an optical cloak. Remember that when the optical cloak system has a total ABCD matrix of $\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}$, with the total length of $L = 2d_1 + d_2$. For calculation of the ABCD matrix, consider the 4 lenses and the propagation distances between them.

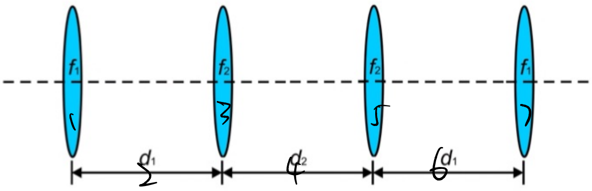

Figure 3: 4 lens system.

(a) $q_0=-iz_0=-i\frac{\pi W_0^2}{\lambda}$

$M_1=\begin{bmatrix}1 & f_1\\ 0 & 1\end{bmatrix}$ $M_2=\begin{bmatrix}1 & 0\\ -1/f_1 & 1\end{bmatrix}$ $M_3=\begin{bmatrix}1 & g_1\\ 0 & 1\end{bmatrix}$

$M_2M_1=\begin{bmatrix}1 & 0\\ -1/f_1 & 1\end{bmatrix}\begin{bmatrix}1 & f_1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & f_1\\ -1/f_1 & 0\end{bmatrix}$

$M=M_3M_2M_1=\begin{bmatrix}1 & g_1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & f_1\\ -1/f_1 & 0\end{bmatrix}=\begin{bmatrix}1-g_1/f_1 & f_1\\ -1/f_1 & 0\end{bmatrix}$

$\Rightarrow q_1=\frac{(1-g_1/f_1)q_0+f_1}{-1/f_1\, q_0}=g_1-f_1-i\frac{f_1^2}{z_0}$

$q_1=z-iz_1=-iz_1=-i\frac{\pi W_1^2}{\lambda}\Rightarrow g_1-f_1=0\Rightarrow g_1=f_1$

$\frac{\pi W_1^2}{\lambda}=\frac{f_1^2}{z_0}\Rightarrow W_1=\sqrt{\frac{\lambda f_1^2}{\pi z_0}}=\frac{\lambda f_1}{\pi W_0}$

$M_1'=\begin{bmatrix}1 & f_2\\ 0 & 1\end{bmatrix}$ $M_2'=\begin{bmatrix}1 & 0\\ -1/f_2 & 1\end{bmatrix}$ $M_3=\begin{bmatrix}1 & g_2\\ 0 & 1\end{bmatrix}$

$M_2'M_1'=\begin{bmatrix}1 & f_2\\ -1/f_2 & 0\end{bmatrix}$ $M'=M_3'M_2'M_1'=\begin{bmatrix}1-g_2/f_2 & f_2\\ -1/f_2 & 0\end{bmatrix}$

$q_2=\frac{(1-g_2/f_2)q_1+f_2}{-1/f_2\, q_1}=g_2-f_2-i\frac{f_2^2}{z_1}=z-iz_2=-iz_2$

$\Rightarrow g_2=f_2$ $\frac{f_2^2}{z_1}=\frac{\pi W_2^2}{\lambda}\Rightarrow W_2=\sqrt{\frac{\lambda f_2^2}{\pi z_1}}=\frac{\lambda f_2}{\pi W_1}=\frac{W_0 f_2}{f_1}$

b) $W_1=\frac{633\,nm\times 300\,mm}{3.14\times 15\,mm}=4.032\,\mu m$ $W_2=W_0\frac{f_2}{f_1}=10\,mm$

(c) $M_1=\begin{bmatrix}1 & 0\\ -1/f_1 & 1\end{bmatrix}$ $M_2=\begin{bmatrix}1 & d\\ 0 & 1\end{bmatrix}$ $M_3=\begin{bmatrix}1 & 0\\ -1/f_2 & 1\end{bmatrix}$

$M_2M_1=\begin{bmatrix}1-\frac{d}{f_1} & d\\ -1/f_1 & 1\end{bmatrix}$ $M=M_3M_2M_1=\begin{bmatrix}1-\frac{d}{f_1} & d\\ 0 & 1-\frac{d}{f_2}\end{bmatrix}=\begin{bmatrix}-\frac{f_2}{f_1} & d\\ 0 & -\frac{f_1}{f_2}\end{bmatrix}$

(d) $M_2M_1=\begin{bmatrix}1 & d_1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1/f_1 & 1\end{bmatrix}=\begin{bmatrix}1-\frac{d_1}{f_1} & d_1\\ -1/f_1 & 1\end{bmatrix}=\begin{bmatrix}-\frac{f_2}{f_1} & d_1\\ -1/f_1 & 1\end{bmatrix}$

$M_3M_2M_1=\begin{bmatrix}1 & 0\\ -1/f_2 & 1\end{bmatrix}\begin{bmatrix}-f_2/f_1 & d_1\\ -1/f_1 & 1\end{bmatrix}=\begin{bmatrix}-f_2/f_1 & d_1\\ 0 & -\frac{f_1}{f_2}\end{bmatrix}$ $M_4M_3M_2M_1=\begin{bmatrix}1 & d_2\\ 0 & 1\end{bmatrix}\begin{bmatrix}-\frac{f_2}{f_1} & d_1\\ 0 & -\frac{f_1}{f_2}\end{bmatrix}=\begin{bmatrix}-\frac{f_2}{f_1} & d_1-\frac{d_2f_1}{f_2}\\ 0 & -\frac{f_1}{f_2}\end{bmatrix}$

$M_6M_5=\begin{bmatrix}1 & d_1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1/f_2 & 1\end{bmatrix}=\begin{bmatrix}-f_1/f_2 & d_1\\ -1/f_2 & 1\end{bmatrix}$ $M_7M_6M_5=\begin{bmatrix}1 & 0\\ -1/f_1 & 1\end{bmatrix}\begin{bmatrix}-f_1/f_2 & d_1\\ -1/f_2 & 1\end{bmatrix}=\begin{bmatrix}-\frac{f_1}{f_2} & d_1\\ 0 & -\frac{f_2}{f_1}\end{bmatrix}$

$M=M_6M_7M_5M_4M_3M_2M_1=\begin{bmatrix}-\frac{f_1}{f_2} & d_1\\ 0 & -\frac{f_2}{f_1}\end{bmatrix}\begin{bmatrix}-\frac{f_2}{f_1} & d_1-\frac{d_2f_1}{f_2}\\ 0 & -\frac{f_1}{f_2}\end{bmatrix}=\begin{bmatrix}1 & -\frac{2d_1f_1}{f_2}+\frac{d_2f_1^2}{f_2^2}\\ 0 & 1\end{bmatrix}$

$\Rightarrow -\frac{2d_1f_1}{f_2}+d_2\frac{f_1^2}{f_2^2}=2(f_1+f_2)+d_2\Rightarrow\left(\frac{f_1^2}{f_2^2}-1\right)d_2=2(f_1+f_2)+\frac{2(f_1+f_2)f_1}{f_2}$

$\Rightarrow\frac{(f_1-f_2)(f_1+f_2)}{f_2^2}d_2=\left(2+2\frac{f_1}{f_2}\right)(f_1+f_2)\Rightarrow d_2=\frac{2f_2^2\left(1+\frac{f_1}{f_2}\right)}{f_1-f_2}=\frac{2f_2(f_2+f_1)}{f_1-f_2}$