One of the most fascinating phenomena related to the light propagation in anisotropic media is birefringence when the ordinary and extraordinary waves can become spatially separated upon propagation in the crystal. The reason for this effect is the fact that the beam propagation direction (i.e. the direction of Poynting vector) and the direction of wavevector (i.e. the direction of normal to the wavefront) are in general different.

Consider that we have a transparent, uniaxial crystal with the ordinary refractive index n_o and the extraordinary index n_e, the optical axis of the crystal is oriented along $\mathbf{c}$. The angle between the optical axis and wavevector $\mathbf{k}$ is α.

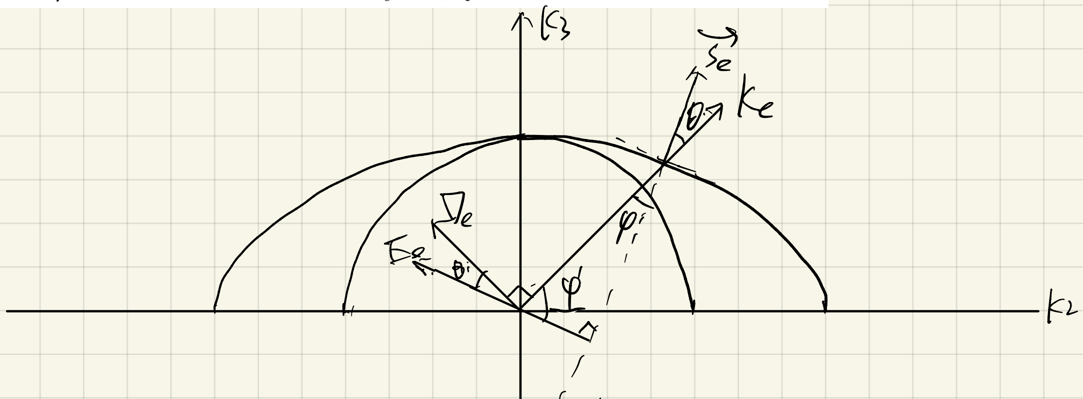
a) Derive and explain why the angle between Poynting vector $\mathbf{S}$ and wavevector $\mathbf{k}$ is equal to the angle between electric flux density vector $\mathbf{D}$ and electric field vector $\mathbf{E}$.

b) What is the angle between Poynting vector $\mathbf{S}_o$ and wavevector $\mathbf{k}_o$ for the ordinary wave? Sketch $\mathbf{S}_o$, $\mathbf{k}_o$ and the field vectors $\mathbf{E}_o$, $\mathbf{D}_o$.

c) What is the angle between the Poynting vector $\mathbf{S}_e$ and the wavevector $\mathbf{k}_e$ for the extraordinary wave. Sketch $\mathbf{S}_e$, $\mathbf{k}_e$ and the field vectors $\mathbf{E}_e$, $\mathbf{D}_e$.

d*) Find the expression for α that gives the maximum angle between the Poynting vector $\mathbf{S}_e$ and wavevector $\mathbf{k}_e$ for the extraordinary wave. Calculate the value of it for $n_o = 2.4$, $n_e = 2.7$.

(a)



Let's assume that $\vec{k}$ is in $k_2 k_3$ plane.

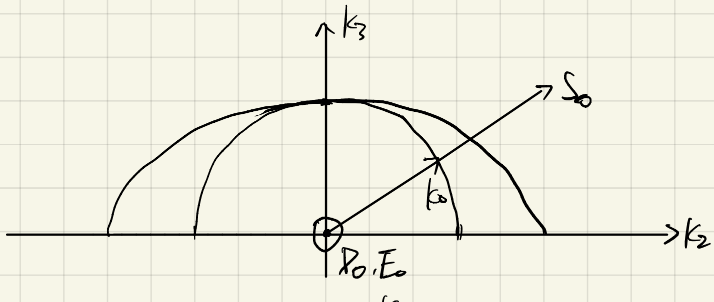
Because $\vec{S} = \vec{E}_t \times \vec{H}_r$. thus $\vec{S} \perp \vec{E}$ and $\vec{D} \perp \vec{k}$ $\vec{D} \nparallel \vec{E}$

Therefore, we can draw $\vec{k}, \vec{S}_e, \vec{E}_e, \vec{D}_e$ in the normal surfaces

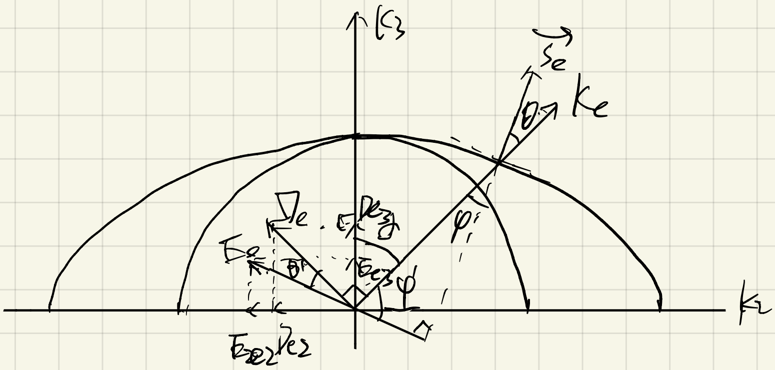
According to basic geometrical rules. it's clear that $\theta = \varphi$. $\theta' + \varphi' = \frac{\pi}{2}$ and $\varphi + \varphi' = \frac{\pi}{2}$

Thus $\begin{cases} \theta + \varphi' = \frac{\pi}{2} \\ \theta' + \varphi' = \frac{\pi}{2} \end{cases} \Rightarrow \alpha = \alpha'$. Therefore, the angle between $\vec{S}_e$ and $\vec{k}_e$ is equal to the angle between D_e and E_e

(b) $S_o \parallel k_o$ thus the angle is zero.



(c)



Decompose D_e into D_{e2} parallel to k_2 and D_{e3} parallel to k_3 and E_e into E_{e2} parallel to k_2 and E_{e3} parallel to k_3

Thus $D_{e2} = \varepsilon_0 \varepsilon_o E_{e2}$ $D_{e3} = \varepsilon_0 \varepsilon_e E_{e3}$

$$\tan\left(\frac{\pi}{2}-\alpha\right)=\frac{D_{e2}}{D_{e3}}=\frac{\varepsilon_o}{\varepsilon_e}\cdot\frac{E_{e2}}{E_{e3}}=\cot\alpha \qquad \tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$$

$$\Rightarrow \tan\delta=\frac{E_{e2}}{E_{e3}}=\frac{\varepsilon_e}{\varepsilon_o}\cot\alpha \qquad \tan\theta'=\tan\left[\delta-\left(\frac{\pi}{2}-\alpha\right)\right]=\frac{\tan\delta-\tan(\frac{\pi}{2}-\alpha)}{1+\tan\delta\tan(\frac{\pi}{2}-\alpha)}$$

$$\Rightarrow \tan\theta'=\frac{\frac{\varepsilon_e}{\varepsilon_o}\cot\alpha-\cot\alpha}{1+\frac{\varepsilon_e}{\varepsilon_o}\cot^2\alpha} \Rightarrow \theta'=\arctan\left[\frac{(\varepsilon_e-\varepsilon_o)\cot\alpha}{\varepsilon_o+\varepsilon_e\cot^2\alpha}\right]$$

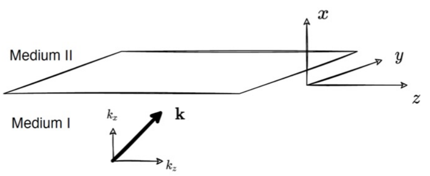
d) assume $\cot\alpha=x \Rightarrow f(x)=\frac{(\varepsilon_e-\varepsilon_o)x}{\varepsilon_o+\varepsilon_e x^2}$

$$f'(x)=\frac{(\varepsilon_e-\varepsilon_o)(\varepsilon_o+\varepsilon_e x^2)-(\varepsilon_e-\varepsilon_o)x\cdot 2x\varepsilon_e}{(\varepsilon_o+\varepsilon_e x^2)^2}=0 \Rightarrow (\varepsilon_e-\varepsilon_o)(\varepsilon_o+\varepsilon_e x^2)=2x^2\varepsilon_e(\varepsilon_e-\varepsilon_o)$$

$$\Rightarrow x=\frac{\sqrt{\varepsilon_o}}{\sqrt{\varepsilon_e}} \quad \text{Thus } \cot\alpha_m=\sqrt{\frac{\varepsilon_o}{\varepsilon_e}} \text{ and } \theta'_m=\arctan\left[\frac{(\varepsilon_e-\varepsilon_o)\sqrt{\frac{\varepsilon_o}{\varepsilon_e}}}{2\varepsilon_o}\right]=\arctan\frac{(n_e^2-n_o^2)}{2n_o n_e}$$

$$n_o=2.4.\ n_e=2.7 \Rightarrow \theta'_m=\arctan\left(\frac{2.7^2-2.4^2}{2\times2.7\times2.4}\right)\approx 6.73^\circ \qquad \alpha_m=\operatorname{arccot}\left(\sqrt{\frac{\varepsilon_o}{\varepsilon_e}}\right)=48.37^\circ$$

Consider a plane wave with wavevector $\mathbf{k}$ incident on an infinite interface between two media with different permittivity:



Consider a general wave-equation:

$$\nabla\times\nabla\times\bar{\mathbf{E}}(\mathbf{r},\omega)-\frac{\omega^2}{c^2}\bar{\mathbf{E}}(\mathbf{r},\omega)=i\omega\mu_0\bar{\mathbf{j}}(\bar{r},\omega)+\mu_0\omega^2\bar{\mathbf{P}}(\bar{r},\omega)$$

a) Simplify $\nabla\times\nabla\times\bar{\mathbf{E}}(\mathbf{r},\omega)$ using translation invariance of the system in the direction, perpendicular to the plane of light incidence.

b) Decompose the wave equation into separate equations for $\mathbf{E}_{TE}$ and $\mathbf{E}_{TM}$, where $\mathbf{E}_{TE}$ is polarized perpendicular to the plane of light incidence, and $\mathbf{E}_{TM}$ is polarized parallel to this plane.

c) Give a short explanation of the "continuity of field" and the "continuity of wave vector" on an interface. For which fields and polarization components these conditions are satisfied?

d) Use the continuity of the tangential components of the $\mathbf{E}$ and $\mathbf{H}$ fields at the interface between two different, homogeneous, isotropic media with refractive indices n_1 and n_2, respectively (see Series 3), to derive laws of refraction and reflection, that connect the angle of reflection and the angle of refraction with the angle of incidence.

(a) $k_y=0 \quad \frac{\partial}{\partial y}=0 \quad \nabla\times\nabla\times\vec{E}=\nabla(\nabla\cdot\vec{E})-\nabla^2E=\begin{bmatrix}\partial_x(\partial_x\vec{E}_x+\partial_z\vec{E}_z)\\0\\\partial_z(\partial_x\vec{E}_x+\partial_z\vec{E}_z)\end{bmatrix}-\begin{bmatrix}\nabla^2_{x,z}\vec{E}_x\\\nabla^2_{xz}\vec{E}_y\\\nabla^2_{xz}\vec{E}_z\end{bmatrix}$

(b) $\vec{E}=\vec{E}_{TE}+\vec{E}_{TM}=\begin{bmatrix}0\\E_y\\0\end{bmatrix}+\begin{bmatrix}E_x\\0\\E_z\end{bmatrix}$

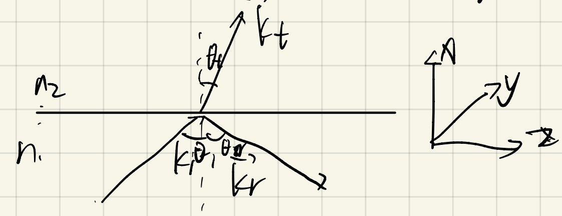
(c) $\vec{H}=\vec{H}_{TE}+\vec{H}_{TM}=\begin{bmatrix}H_x\\0\\H_z\end{bmatrix}+\begin{bmatrix}0\\H_y\\0\end{bmatrix}$

For TE: H_z // surface and E_y // surface thus H_z and E_y are continuous

For TM: H_y // surface. and E_z // surface thus H_y and E_z are continuous.

For $\vec{k}$: $k_y=0$. k_z // surface thus k_z is continuous.

(d)



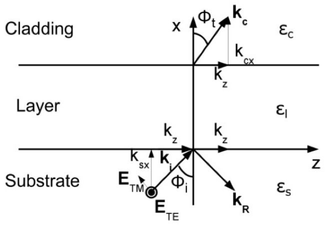
Because of the continuity of k_z.

$\Rightarrow k_{iz}=k_{rz}=k_{tz} \qquad k_i=k_r=\frac{\omega}{c}n_1 \quad k_t=\frac{\omega}{c}n_2$

$\Rightarrow k_i\sin\theta_i=k_r\sin\theta_r \Rightarrow \frac{\omega}{c}n_1\sin\theta_i=\frac{\omega}{c}n_1\sin\theta_r \Rightarrow \theta_i=\theta_r$ (reflection Law

$k_{tz} = k_{iz} \Rightarrow k_t \sin\theta_t = k_i \sin\theta_i \Rightarrow \frac{\omega}{c} n_1 \sin\theta_i = \frac{\omega}{c} n_2 \sin\theta_t \Rightarrow n_1 \sin\theta_i = n_2 \sin\theta_t$ (refraction law

Let us consider a single optical layer with thickness d that is embedded between a substrate and a cladding material as shown below. The refractive indices of the layer, substrate, and cladding materials are $n_l = \sqrt{\varepsilon_l}$, $n_s = \sqrt{\varepsilon_s}$, and $n_c = \sqrt{\varepsilon_c}$, respectively. For simplicity, we consider light in TE polarization only.



a) Calculate the angle φ_t of the transmitted beam as a function of the incident angle φ_i.
b) Compute the coefficients of reflection and transmission as functions of the incident angle φ_i. You can use the formulas derived on the lecture for a general multlayer case.
c) Compute the reflectivity and transmissivity of the single layer, and show that they add up to 1. For simplicity, assume $\varepsilon_l > \varepsilon_s \sin^2(\varphi_i)$ and $\varepsilon_c > \varepsilon_s \sin^2(\varphi_i)$.
d) Consider the special case of a $\lambda/4$-layer, i.e. $k_{l,x}d = d\sqrt{k_l^2 - k_z^2} = \pi/2$, and calculate its reflectivity. Now assume the incident light is perpendicular to the layer ($\varphi_i = 0$) and find the condition for the refractive indices to obtain minimum reflection.

a) $k_i \sin\phi_i = k_c \sin\phi_t \qquad k_i = \frac{\omega}{c} n_s \quad k_c = \frac{\omega}{c} n_c \Rightarrow n_s \sin\phi_i = n_c \sin\phi_t$

$\Rightarrow \sin\phi_t = \frac{n_s}{n_c}\sin\phi_i \Rightarrow \phi_t = \arcsin\left(\frac{n_s}{n_c}\sin\phi_i\right)$

Reflection coefficient: $R_{TE} = \frac{k_{sx} M_{22}^{TE} - k_{cx} M_{11}^{TE} - i(M_{21}^{TE} + k_{sx}k_{cx}M_{12}^{TE})}{k_{sx}M_{22}^{TE} + k_{cx}M_{11}^{TE} + i(M_{21}^{TE} - k_{sx}k_{cx}M_{12}^{TE})}$ $\qquad \hat{M} = \prod \hat{m} = \hat{m}(x)$

Transmission coefficient: $T_{TE} = \frac{2k_{sx}}{k_{sx}M_{22}^{TE} + k_{cx}M_{11}^{TE} + i(M_{21}^{TE} - k_{sx}k_{cx}M_{12}^{TE})} = \frac{2k_{sx}}{N_{TE}}$

$$\hat{m}(x) = \begin{bmatrix} \cos(k_{ex}x) & \frac{1}{k_{ex}\alpha_e}\sin(k_{ex}x) \\ -k_{ex}\alpha_e\sin(k_{ex}x) & \cos(k_{ex}x) \end{bmatrix} = \begin{bmatrix} \cos(k_{ex}x) & \frac{1}{k_{ex}}\sin(k_{ex}x) \\ -k_{ex}\sin(k_{ex}x) & \cos(k_{ex}x) \end{bmatrix}$$

$$R_{TE} = \frac{k_{sx}\cos(k_{ex}x) - k_{cx}\cos(k_{ex}x) - i[k_{ex}\sin(k_{ex}x) + k_{sx}k_{cx}\cdot\frac{1}{k_{ex}}\sin(k_{ex}x)]}{k_{sx}\cos(k_{ex}x) + k_{cx}\cos(k_{ex}x) + i[-k_{ex}\sin(k_{ex}x) - \frac{k_{sx}k_{cx}}{k_{ex}}\sin(k_{ex}x)]} = \frac{(k_{sx}-k_{cx})\cos(k_{ex}x) + i\left(k_{ex} - \frac{k_{sx}k_{cx}}{k_{ex}}\right)\sin(k_{ex}x)}{(k_{sx}+k_{cx})\cos(k_{ex}x) - i\left(k_{ex} + \frac{k_{sx}k_{cx}}{k_{ex}}\right)\sin(k_{ex}x)}$$

$k_{sx} = k_i\cos\varphi_i = \frac{\omega}{c}n_s\cos\varphi_i \qquad k_{sz} = k_i\sin\varphi_i = \frac{\omega}{c}n_s\sin\varphi_i \qquad k_{sz} = k_{ez} = k_{cz}$

$k_{ex}^2 + k_{ez}^2 = k_e^2 = \frac{\omega^2}{c^2}n_e^2 \Rightarrow k_{ex}^2 = \frac{\omega^2}{c^2}n_e^2 - k_{ez}^2 = \frac{\omega^2}{c^2}n_e^2 - \frac{\omega^2}{c^2}n_s^2\sin^2\varphi_i \Rightarrow k_{ex} = \frac{\omega}{c}\sqrt{n_e^2 - n_s^2\sin^2\varphi_i}$

Similarly $k_{cx} = \frac{\omega}{c}\sqrt{n_c^2 - n_s^2\sin^2\varphi_i}$

$$T_{TE} = \frac{2\frac{\omega}{c}n_s\cos\varphi_i}{(k_{sx}+k_{cx})\cos(k_{ex}x) - i\left[k_{ex}\sin(k_{ex}x) - \frac{k_{sx}k_{cx}}{k_{ex}}\sin(k_{ex}x)\right]}$$

c) $\rho_{TE,TM} = |R_{TE,TM}|^2 \qquad T_{TE,TM} = \frac{Re[k_{cx}]}{k_{sx}}|T_{TE,TM}|^2$

$$R_{TE} = \frac{(k_{sx}-k_{cx})\cos(k_{ex}x) + i\left(k_{ex} - \frac{k_{sx}k_{cx}}{k_{ex}}\right)\sin(k_{ex}x)}{(k_{sx}+k_{cx})\cos(k_{ex}x) - i\left(k_{ex} + \frac{k_{sx}k_{cx}}{k_{ex}}\right)\sin(k_{ex}x)}$$

$$|R_{TE}|^2 = \frac{(k_{sx}-k_{cx})^2\cos^2(k_{ex}x) + \left(k_{ex} - \frac{k_{sx}k_{cx}}{k_{ex}}\right)^2\sin^2(k_{ex}x)}{(k_{sx}+k_{cx})^2\cos^2(k_{ex}x) + \left(k_{ex} + \frac{k_{sx}k_{cx}}{k_{ex}}\right)^2\sin^2(k_{ex}x)} = \rho_{TE}$$

$$|T_{TE}|^2 = \frac{4k_{sx}^2}{(k_{sx}+k_{cx})^2\cos^2(k_{ex}x) + \left(k_{ex} + \frac{k_{sx}k_{cx}}{k_{ex}}\right)^2\sin^2(k_{ex}x)} \Rightarrow \mathcal{T}_{TE} = \frac{4k_{sx}k_{cx}}{(k_{sx}+k_{cx})^2\cos^2(k_{ex}x) + \left(k_{ex} + \frac{k_{sx}k_{cx}}{k_{ex}}\right)^2\sin^2(k_{ex}x)}$$

$$\rho_{TE} + \mathcal{T}_{TE} = \frac{(k_{sx}-k_{cx})^2\cos^2(k_{ex}x) + \left(k_{ex} - \frac{k_{sx}k_{cx}}{k_{ex}}\right)^2\sin^2(k_{ex}x) + 4k_{sx}k_{cx}}{(k_{sx}+k_{cx})^2\cos^2(k_{ex}x) + \left(k_{ex} + \frac{k_{sx}k_{cx}}{k_{ex}}\right)^2\sin^2(k_{ex}x)}$$

$$= \frac{(k_{sx}^2+k_{cx}^2)\cos^2(k_{ex}x) + \left[k_{ex}^2 + \left(\frac{k_{sx}k_{cx}}{k_{ex}}\right)^2\right]\sin^2(k_{ex}x) + 2k_{sx}k_{cx}}{(k_{sx}+k_{cx})^2\cos^2(k_{ex}x) + \left(k_{ex} + \frac{k_{sx}k_{cx}}{k_{ex}}\right)^2\sin^2(k_{ex}x)}$$

$$= \frac{(k_{sx}+k_{cx})^2\cos^2(k_{ex}x) + \left(k_{ex} + \frac{k_{sx}k_{cx}}{k_{ex}}\right)^2\sin^2(k_{ex}x)}{(k_{sx}+k_{cx})^2\cos^2(k_{ex}x) + \left(k_{ex} + \frac{k_{sx}k_{cx}}{k_{ex}}\right)^2\sin^2(k_{ex}x)} = 1$$

$$k_{ex}d = \frac{\pi}{2} \Rightarrow \cos(k_{ex}d) = 0 \quad \sin(k_{ex}d) = 1 \Rightarrow \rho_{TE} = |R_{TE}|^2 = \frac{\left(k_{ex} - \frac{k_{sx}k_{cx}}{k_{ex}}\right)^2}{\left(k_{ex} + \frac{k_{sx}k_{cx}}{k_{ex}}\right)^2} = \frac{(k_{ex}^2 - k_{sx}k_{cx})^2}{(k_{ex}^2 + k_{sx}k_{cx})^2}$$

$$\varphi_i = 0 \Rightarrow k_{sx} = \frac{w}{c}n_s\cos\varphi_i = \frac{w}{c}n_s \quad k_{ex} = \frac{w}{c}\sqrt{n_e^2 - n_s^2\sin^2\varphi_i} = \frac{w}{c}n_e \quad k_{cx} = \frac{w}{c}\sqrt{n_c^2 - n_s^2\sin^2\varphi_i} = \frac{w}{c}n_c$$

$$\Rightarrow \rho_{TE} = \frac{\left(\frac{w^2}{c^2}n_e^2 - \frac{w^2}{c^2}n_sn_c\right)^2}{\left(\frac{w^2}{c^2}n_e^2 + \frac{w^2}{c^2}n_sn_c\right)^2} = \left(\frac{n_e^2 - n_sn_c}{n_e^2 + n_sn_c}\right)^2$$

Minimum Reflection: $\rho_{TE} = 0 \Rightarrow n_e^2 - n_sn_c = 0 \Rightarrow n_e = \sqrt{n_sn_c}$.

a*) A plane wave is incident on a planar interface to a *lower* refractive index medium under a fixed angle $\theta > \theta_c$, so that it undergoes total internal reflection (i.e. no light is transmitted to the lower refractive index medium). A solar application engineer tries to make the light pass through by adding multilayer stacks of arbitrary dielectric materials on top of the interface. Show that his effort is hopeless.

b*) Light-guiding in step-index fibers can be seen as total internal reflection at the core-cladding interface. Assume a fiber with refractive indices n_c and n_{cl} of the core and cladding material, respectively, and calculate the numerical aperture NA = $\sin(\theta_a)$ of such a fiber in air. (*Hint*: θ_a is the maximum acceptance angle for which light can be guided inside the fiber by total internal reflection.)

$$\hat{m}(x) = \begin{bmatrix} \cos(k_{fx}x) & \frac{1}{k_{fx}\alpha_f}\sin(k_{fx}x) \\ -k_{fx}\alpha_f\sin(k_{fx}x) & \cos(k_{fx}x) \end{bmatrix}$$

TE $\alpha_f = 1$

TM $\alpha_f = \frac{1}{\varepsilon_f}$

$$\hat{M}(x) = \prod_{i=1}^{N}\hat{m}(x)$$

a) $$T_{TE} = \frac{2k_{sx}}{(k_{sx}M_{22} + k_{cx}M_{11}) + i(M_{21} - k_{sx}k_{cx}M_{12})}$$

$$\mathcal{T}_{TE} = \frac{\mathrm{Re}(k_{cx})}{k_{sx}}|T_{TE}|^2 = \frac{4k_{sx}k_{cx}}{(k_{sx}M_{22} + k_{cx}M_{11})^2 + (M_{21} - k_{sx}k_{cx}M_{12})^2}$$

$$k_{sz} = k_s\sin\theta = k_{cz} \qquad k_c^2 = k_{cz}^2 + k_{cx}^2 \Rightarrow k_{cx}^2 = \left(\frac{w^2}{c^2}n_c^2 - \frac{w^2}{c^2}n_s^2\sin^2\theta\right)$$

$$\Rightarrow k_{cx} = \frac{w}{c}\sqrt{n_c^2 - n_s^2\sin^2\theta}$$

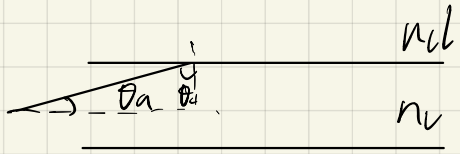
In the condition of no stacks and $\theta > \theta_c$

$$k_s\sin\theta = k_c \Rightarrow \frac{w}{c}n_s\sin\theta = \frac{w}{c}n_c \Rightarrow n_c = n_s\sin\theta$$

Thus, when he adding multilayer stacks

$k_{cx} = \frac{w}{c}\sqrt{n_c^2 - n_s^2\sin^2\theta} = 0 \rightarrow \mathcal{T}_{TE} = 0$ which means that there will be no beam in the medium with lower refractive index.

b)



$k_c \sin\theta_c = k_{cl} \Rightarrow n_c \sin\theta_c = n_{cl} \Rightarrow \sin\theta_c = \frac{n_{cl}}{n_c}$

$\sin\theta_a = \sin(\frac{\pi}{2} - \theta_c) = \cos\theta_c = \sqrt{1-\sin^2\theta_c} = \sqrt{1-\frac{n_{cl}^2}{n_c^2}}$

Thus $NA = \sin\theta_a = \sqrt{1-\frac{n_{cl}^2}{n_c^2}}$