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Series 11

Fundamentals of Modern Optics

to be returned on **26.01.2023**, at the beginning of the lecture

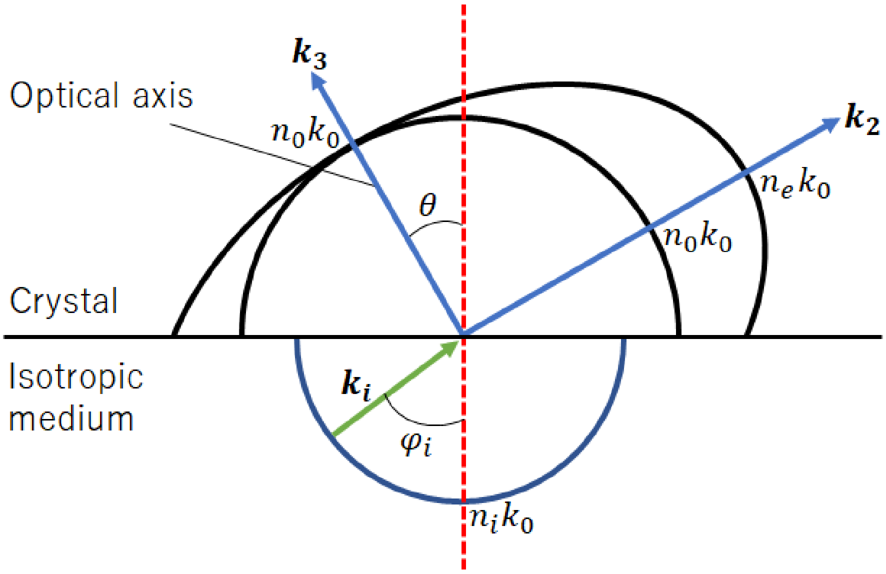
Task 1: Ewald construction (2+2+2 points)

Consider the interface between an isotropic medium (refractive index n_i) and a uniaxial crystal (ordinary and extraordinary refractive indices n_o and n_e, respectively) as shown in the figure below. A plane wave is incident on the crystal at an angle φ_i to the surface normal. The optical axis of the crystal ($\mathbf{k_3}$) and the surface normal form an angle θ.

a) Draw in the figure the wave vectors of the refracted ordinary wave $\mathbf{k_o}$ and the refracted extraordinary wave $\mathbf{k_e}$ for unpolarized incident light.

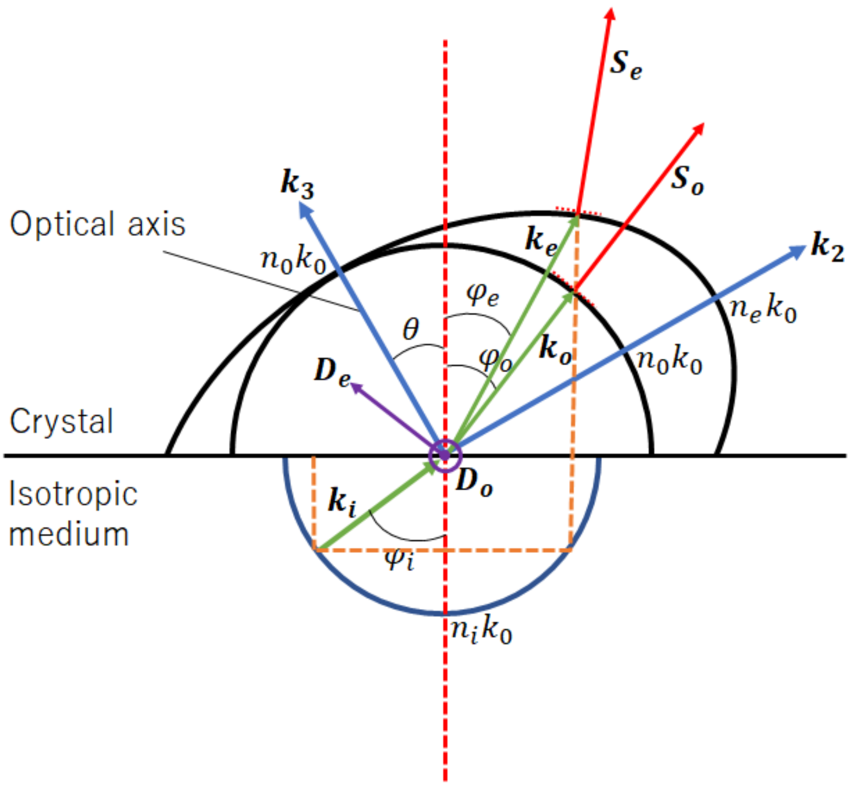
b) Draw the corresponding Poynting vectors $\mathbf{S}_\mathrm{o}$ and $\mathbf{S}_\mathrm{e}$. Are they parallel to their respective wave vectors? Are they parallel to each other? If not, what are the consequences?

c) Draw the electric displacement fields $\mathbf{D}_\mathrm{o}$ and $\mathbf{D}_\mathrm{e}$.



Solution Task 1:

a) See below.



b) The Poynting vectors are not parallel to each other. The Poynting vector of ordinary wave is parallel to its wave vector, but not in the case of extraordinary wave.
The extraordinary and ordinary waves are orthogonally polarized to each other and they are spatially separated in the crystal due to the different directions of the Poynting vectors. Therefore, one can spatially split the two orthogonal polarizations. Therefore, one see double images after the crystal.

Task 2: Birefringence in anisotropic media (2+2+3+2* points)

One of the most fascinating phenomena related to the light propagation in anisotropic media is birefringence when the ordinary and extraordinary waves can become spatially separated upon propagation in the crystal. The reason for this effect is the fact that the beam propagation direction (i.e. the direction of Poynting vector) and the direction of wavevector (i.e. the direction of normal to the wavefront) are in general different.

Consider that we have a transparent, uniaxial crystal with the ordinary refractive index n_o and the extraordinary index n_e, the optical axis of the crystal is oriented along $\mathbf{c}$. The angle between the optical axis and wavevector $\mathbf{k}$ is α.

a) Derive and explain why the angle between Poynting vector $\mathbf{S}$ and wavevector $\mathbf{k}$ is equal to the angle between electric flux density vector $\mathbf{D}$ and electric field vector $\mathbf{E}$.

b) What is the angle between Poynting vector $\mathbf{S}_\mathrm{o}$ and wavevector $\mathbf{k}_\mathrm{o}$ for the ordinary wave? Sketch $\mathbf{S}_\mathrm{o}$, $\mathbf{k}_\mathrm{o}$ and the field vectors $\mathbf{E}_\mathrm{o}$, $\mathbf{D}_\mathrm{o}$.

c) What is the angle between the Poynting vector $\mathbf{S}_\mathrm{e}$ and the wavevector $\mathbf{k}_\mathrm{e}$ for the extraordinary wave. Sketch $\mathbf{S}_\mathrm{e}$, $\mathbf{k}_\mathrm{e}$ and the field vectors $\mathbf{E}_\mathrm{e}$, $\mathbf{D}_\mathrm{e}$.

d*) Find the expression for α that gives the maximum angle between the Poynting vector $\mathbf{S}_\mathrm{e}$ and wavevector $\mathbf{k}_\mathrm{e}$ for the extraordinary wave. Calculate the value of it for $n_\mathrm{o} = 2.4$, $n_\mathrm{e} = 2.7$.

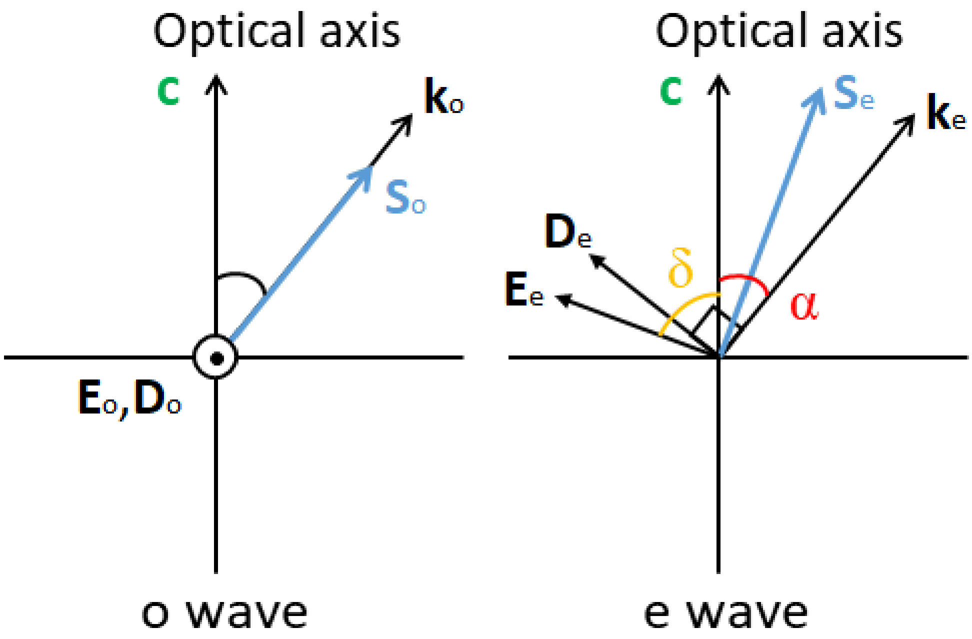
Solution Task 2:

a) From the Maxwell's equations:

$$\nabla \times \mathbf{E} = -\mu_0 \frac{d\mathbf{H}}{dt} \Rightarrow \mathbf{k} \times \mathbf{E} = \mu_0 \omega \mathbf{H}$$

$$\nabla \times \mathbf{H} = \frac{d\mathbf{D}}{dt} \Rightarrow \mathbf{k} \times \mathbf{H} = -\omega \mathbf{D},$$

there $\mathbf{D} = \varepsilon_0 \hat{\varepsilon} \mathbf{E}$, so $\mathbf{E}$ is not parallel to $\mathbf{D}$! But from the Maxwell's equation for a plane-wave and from $\mathbf{S} \sim \mathbf{E} \times \mathbf{H}^\star$ we know that all the vectors $\mathbf{E}, \mathbf{D}, \mathbf{S}, \mathbf{k}$ lie in the plane orthogonal to the vector $\mathbf{H}$. From the Poynting vector formula we know that $\mathbf{E} \perp \mathbf{S}$, and for a normal mode we also know that $\mathbf{D} \perp \mathbf{k}$. For this reason it is clear that the angle between Poynting vector $\mathbf{S}$ and wavevector $\mathbf{k}$ is equal to the angle between electric flux density vector $\mathbf{D}$ and electric field vector $\mathbf{E}$.



b) For the ordinary wave we have $\mathbf{D}_\mathrm{o} \perp \mathbf{c}$ and $\mathbf{D}_\mathrm{o} \parallel \mathbf{E}_\mathrm{o}$, what means $\mathbf{S}_\mathrm{o} \parallel \mathbf{k}_\mathrm{o}$, i.e. the angle between Poynting vector $\mathbf{S}_\mathrm{o}$ and wavevector $\mathbf{k}_\mathrm{o}$ equals zero.

c) To find the angle between $\mathbf{S}_\mathrm{e}$ and $\mathbf{k}_\mathrm{e}$ we can find the angle between $\mathbf{E}_\mathrm{e}$ and $\mathbf{D}_\mathrm{e}$ instead. To do so, we have to decompose the electric field vector into two components, $\mathbf{E}_{\mathrm{e}\perp}$ and $\mathbf{E}_{\mathrm{e}\parallel}$, that are orthogonal and parallel to the crystal axis respectively.
Since $\mathbf{D}_\mathrm{e} \perp \mathbf{k}_\mathrm{e}$, the angle between the optical axis and electric flux density vector $\mathbf{D}_\mathrm{e}$ is $\frac{\pi}{2} - \alpha$.
Then we get: $D_{\mathrm{e}\perp} = D_\mathrm{e} \sin(\frac{\pi}{2} - \alpha) = D_\mathrm{e} \cos\alpha = \varepsilon_0 \varepsilon_{or} E_{\mathrm{e}\perp}$ and $D_{\mathrm{e}\parallel} = D_\mathrm{e} \cos(\frac{\pi}{2} - \alpha) = D_\mathrm{e} \sin\alpha = \varepsilon_0 \varepsilon_e E_{\mathrm{e}\parallel}$
If δ is the angle between the optical axis and electric field vector $\mathbf{E}_\mathrm{e}$, then we get:

$$\tan\delta = \frac{E_{\mathrm{e}\perp}}{E_{\mathrm{e}\parallel}} = \frac{\varepsilon_e}{\varepsilon_{or}} \cot\alpha. \tag{1}$$

The angle β between electric flux density vector $\mathbf{D}_\mathrm{e}$ and electric field vector $\mathbf{E}_\mathrm{e}$ equals $\delta - (\frac{\pi}{2} - \alpha)$. Then we obtain:

$$\tan\beta = \tan(\delta - (\frac{\pi}{2} - \alpha)) = \frac{\tan\delta - \tan(\frac{\pi}{2} - \alpha)}{1 + \tan(\frac{\pi}{2} - \alpha)\tan\delta} =$$
$$\frac{\cot\alpha\left(\frac{\varepsilon_e}{\varepsilon_{or}} - 1\right)}{1 + \cot^2\alpha\,\frac{\varepsilon_e}{\varepsilon_{or}}} = \frac{\cot\alpha(n_e^2 - n_o^2)}{n_o^2 + n_e^2\cot^2\alpha}.$$

d) Since $\tan\beta$ is monotonic function of β, we can simply find the maximum of $\tan\beta$ as the function of $\cot\alpha$. Then from the expression in previous part we easily obtain the maximum for:

$$\cot\alpha_m = \frac{n_o}{n_e}. \tag{2}$$

The maximum angle is determined by:

$$\tan\beta_m = \frac{n_e^2 - n_o^2}{2 n_o n_e}. \tag{3}$$

For $n_\mathrm{o} = 2.4$, $n_\mathrm{e} = 2.7$, $\alpha_m = 48°$, $\beta_m = 6.7°$.