

# Fundamentals of Modern Optics WS 2013/14

## Midterm Exam

to be written December 9, 8:15 – 9:45 a.m.

### Problem 1 – Maxwell's equations

4 + 5 = 9 points

- Write down Maxwell's equations in the frequency domain for a linear, isotropic, non-magnetizable, inhomogeneous dielectric medium in absence of free charges and current density ( $\rho = 0$  and  $\mathbf{j} = 0$ ).
- Show that for such a medium the wave equation for the electric field can be written as:

$$\Delta \mathbf{E}(\mathbf{r}, \omega) + \frac{\omega^2}{c^2} \epsilon(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega) = -\mathbf{grad} \left\{ \frac{\mathbf{grad} \epsilon(\mathbf{r}, \omega)}{\epsilon(\mathbf{r}, \omega)} \cdot \mathbf{E}(\mathbf{r}, \omega) \right\}.$$

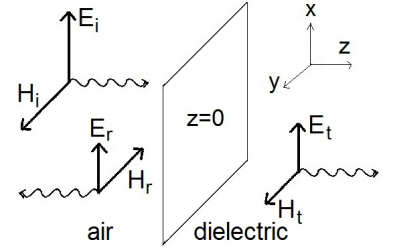
### Problem 2 – Normal modes

3 + 4 + 4 = 11 points

A semi-infinite block of some dielectric (with relative permittivity of  $\epsilon' + i\epsilon''$ ) is illuminated perpendicularly on its surface with a plane monochromatic wave of frequency  $\omega$  from air. The electric field of the incoming and the reflected and the transmitted waves have the form:

$$\mathbf{E}_i = E_i e^{i(k_0 z)} \hat{\mathbf{x}} \quad , \quad \mathbf{E}_r = E_r e^{i(-k_0 z)} \hat{\mathbf{x}} \quad , \quad \mathbf{E}_t = E_t e^{i(k_1 z)} \hat{\mathbf{x}}$$

respectively, where  $k_0 = \omega/c$  and  $k_1 = \frac{\omega}{c} \sqrt{\epsilon' + i\epsilon''} = \frac{\omega}{c} (n + i\kappa)$ .



- Find out the three corresponding magnetic fields using the Maxwell's equations.
- Use the continuity of the tangential components of the electric and magnetic field at the interface between the two media to find  $E_t$  as a function of  $E_i$ .
- Calculate the time averaged Poynting vector in the dielectric medium (transmitted power).  
*Hint:* This energy flux in the dielectric medium should be a function of  $z$ .  
*Hint:* If you failed to find  $E_t$  as a function of  $E_i$  from part b, you can write the transmitted Poynting vector as function of  $E_t$ .

### Problem 3 – Beam propagation (Imaging)

3 + 3 + 3 = 9 points

Given is the field directly behind a two dimensional phase mask

$$u_0(x, y, z = 0) = A \exp \left( -i\pi \frac{x^2 + y^2}{\lambda f} \right),$$

where  $f > 0$ . The field is propagating through vacuum.

- Calculate the spatial frequency spectrum  $U_0(\alpha, \beta; z = 0)$ .
- By introducing the paraxial approximation, derive the free space transfer function ( $H_F(\alpha, \beta; z)$ ). Indicate propagating and evanescent wave regions.
- Calculate the field  $u(x, y, z = f)$ .

**Problem 4 – Gaussian beams****4 + 2 = 6 points**

A laser at  $\lambda_0 = 1000$  nm emits a beam with Gaussian profile and waist radius  $W_0 = 2$  mm.

- Use the matrix approach to calculate the waist position  $z'$  and waist radius  $W'_0$  of the beam after the lens. Approximate the result assuming that the waist of the incoming beam is on the lens and that its Rayleigh range  $z_0 = \frac{\pi W_0^2}{\lambda_0}$  is much longer than the focal length  $f$ .
- Choose a lens such that the waist radius of the focused beam is equal to  $W'_0 = 50 \mu\text{m}$ .

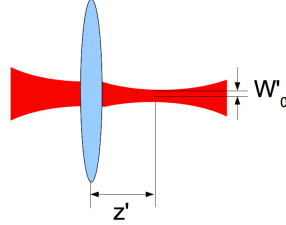


Figure 1: Sketch of the focusing arrangement.

**Problem 5 – Pulse propagation****3 + 6 + 2 = 11 points**

A gaussian pulse travels through a  $l = 2$  mm long medium whose dispersive refractive index is defined as:

$$n(\omega) = B + C\omega^2$$

where  $B = 1.5$  and  $C = 8 \times 10^{-33} \text{s}^2$ . Before entering the medium, the pulse has a bandwidth of  $\omega_s = 100 \times 10^{12} \text{Hz}$  and is centered around the carrier frequency  $\omega_0 = 2 \times 10^{15} \text{Hz}$ .

- What are the phase and group velocities of the pulse? You may leave your answers in terms of the velocity of light  $c_0$ .
- Calculate the pulse width after propagating through  $z = l$ .
- Another pulse was simultaneously launched in a different medium whose  $n(\omega)$  is the same as before with a small difference that  $C = 0$  now. Calculate the difference between the time it takes for the two pulses to reach  $z = l$ .

**Problem 6 – Fraunhofer diffraction****4 + 2 = 6 points**

- Calculate the intensity of the diffracted field pattern  $I(x, z_B) = |u(x, z_B)|^2$  at  $z = z_B$  in paraxial Fraunhofer approximation for two slits illuminated with a normally incident plane wave (just the diffraction pattern, no prefactors). The width of each slit is  $a$  ( $a > \lambda$ ) and separated by a distance  $d$  ( $d > a$ ):

$$u_0(x, z = 0) = \begin{cases} 1, & \text{for } |x \pm d/2| \leq a/2 \\ 0, & \text{elsewhere.} \end{cases}$$

- Try to roughly sketch a figure of the intensity and identify the factor due to interference from the one due to slit diffraction.

*Hint: The Fouriertransform of a single slit of width  $a$  is  $\propto \text{sinc}(\alpha a)$ .*

**— Useful formulas —**

$$\nabla \times \nabla \times \mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) - \Delta \mathbf{a}$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

$$\nabla \cdot (c\mathbf{a}) = \mathbf{a} \cdot \nabla c + c \nabla \cdot \mathbf{a}$$

Gaussian  $q$ -parameter transform law:

$$q' = \frac{Aq + B}{Cq + D}$$

ABCD matrix for a thin lens:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$