Task 1

a) Solution:

$$F\{f(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

$$= \cancel{\frac{1}{2\pi}\int_{-\infty}^{\infty}} \frac{1}{2\pi}\int_{0}^{\infty} A e^{-\gamma t} \cos(\omega_0 t) e^{i\omega t}\, dt$$

$$= \frac{A}{2\pi}\boxed{\int_0^{\infty} e^{(-\gamma+i\omega)t}\cos(\omega_0 t)\, dt}$$

$$= \frac{A}{2\pi}\cdot\frac{1}{\omega_0}\int_0^{\infty} e^{(-\gamma+i\omega)t}\, d\sin(\omega_0 t)$$

$$= \frac{A}{2\pi\omega_0}\left(0 - (-\gamma+i\omega)\int_0^{\infty}\sin(\omega_0 t) e^{(-\gamma+i\omega)t}\, dt\right)$$

$$= -\frac{A(-\gamma+i\omega)}{2\pi\omega_0}\cdot\left(-\frac{1}{\omega_0}\int_0^{\infty} e^{(-\gamma+i\omega)t}\, d\cos(\omega_0 t)\right)$$

$$= \frac{A(-\gamma+i\omega)}{2\pi\omega_0^2}\left(1 - (-\gamma+i\omega)\boxed{\int_0^{\infty} e^{(-\gamma+i\omega)t}\cos(\omega_0 t)\, dt}\right)$$

$$\therefore \int_0^{\infty} e^{(-\gamma+i\omega)t}\cos(\omega_0 t)\, dt = \frac{A(-\gamma+i\omega)}{2\pi\omega_0^2 + A(-\gamma+i\omega)^2}$$

b) Solution:

$$F\{f(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} A e^{-\frac{t^2}{2t_0^2}} e^{i\omega t}\, dt$$

$$I = \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{t}{i\sqrt{2}t_0} - \frac{t_0}{\sqrt{2}}\omega\right)^2 - \frac{t_0}{2}\omega^2}\, dt$$

$$I = \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{t}{i\sqrt{2}t_0} - \frac{t_0}{\sqrt{2}}\omega_0\right)^2}\cdot e^{-\frac{t_0}{2}\omega^2}\, dt$$

$$I = \frac{A e^{-\frac{t_0}{2}\omega^2}}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{t}{i\sqrt{2}t_0} - \frac{t_0}{\sqrt{2}}\omega\right)^2}\, dt$$

suppose $z = \frac{t}{i\sqrt{2}t_0} - \frac{t_0}{\sqrt{2}}\omega$

$$I = \frac{A e^{-\frac{t_0}{2}\omega^2}}{2\pi}\int_{-\infty}^{\infty} e^{-z^2}\cdot\frac{1}{i\sqrt{2}t_0}\, dz$$

$$I = -\frac{iA e^{-\frac{t_0}{2}\omega^2}}{2\sqrt{2}\pi t_0}\cdot 2\int_0^{\infty} e^{-z^2}\, dz$$

$$I = -\frac{iA e^{-\frac{t_0}{2}\omega^2}}{2\sqrt{2}\pi t_0}\cdot 2\int_0^{\infty} z^0\cdot e^{-z^2}\, dz$$

$\because$ Gamma function $\Gamma(x) = 2\int_0^{\infty} z^{2x-1} e^{-z^2}\, dz$

when $x = \frac{1}{2}$ $\quad \Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$\therefore I = -\frac{iA e^{-\frac{t_0}{2}\omega^2}}{2\sqrt{2\pi}\, t_0}$$

Task 2

(a)

$f(t-t_0)$

Solution:

$\because \hat{f}(\omega)$ is the frequency representation of $f(t)$

$$\therefore \hat{f}(\omega) = \int_{-\infty}^{\infty}\frac{1}{2\pi} f(t) e^{i\omega t}\, dt$$

Suppose $F(\omega)$ is the frequency representation of $f(t-t_0)$

$$\therefore F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t-t_0) e^{i\omega t}\, dt$$

$$= \frac{e^{i\omega t_0}}{2\pi}\int_{-\infty}^{\infty} f(t-t_0) e^{i\omega(t-t_0)}\, d(t-t_0)$$

$$= e^{i\omega t_0}\tilde{f}(\omega)$$

b) $\frac{d}{dt} f(t)$

Solution:

$\because \hat{f}(\omega)$ is the frequency representation of $f(t)$

$$\therefore \cancel{f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)} \quad \tilde{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt$$

$$f(t) = \int_{-\infty}^{\infty}\hat{f}(\omega) e^{-i\omega t}\, d\omega$$

$$\therefore \frac{d}{dt} f(t) = \int_{-\infty}^{\infty}\frac{d}{dt}\tilde{f}(\omega) e^{-i\omega t}\, d\omega$$

$$\frac{d}{dt} f(t) = -i\omega\int_{-\infty}^{\infty}\hat{f}(\omega) e^{-i\omega t}\, d\omega = F^{-1}\{-i\omega\tilde{f}(\omega)\}$$

$$\therefore F\left\{\frac{d}{dt} f(t)\right\} = -i\omega\tilde{f}(\omega)$$

Task 3

a)

Solution:

$\forall \varepsilon > 0$

$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt$$

$$= \int_{-\infty}^{0-\varepsilon} \delta(t) f(t)\, dt + \int_{0-\varepsilon}^{0+\varepsilon} \delta(t) f(t)\, dt + \int_{0+\varepsilon}^{+\infty} \delta(t) f(t)\, dt$$

$$= \int_{0-\varepsilon}^{0+\varepsilon} \delta(t) f(t)\, dt = \lim_{\varepsilon \to 0} \int_{0-\varepsilon}^{0+\varepsilon} \delta(t) f(t)\, dt$$

$$\therefore \int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0) \int_{0-\varepsilon}^{0+\varepsilon} \delta(t)\, dt$$

$$= \cancel{f(0)}\ f(0)$$

e) Solution:

$$F\{\delta(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(t) e^{i\omega t}\, dt = \frac{1}{2\pi}$$

b)

Solution:

$\forall \varepsilon > 0$

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = \int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t-t_0) f(t)\, dt$$

$$= \lim_{\varepsilon \to 0} \int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t-t_0) f(t)\, dt$$

$$= f(t_0) \int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t-t_0)\, dt = f(t_0)$$

c)

Solution:

$$\because \int_{-\infty}^{\infty} \delta(t)\, dt = 1 \quad \int_{-\infty}^{\infty} \delta(at)\, d at = 1$$

$$\therefore \int_{-\infty}^{\infty} \delta(at)\, d at = \int_{-\infty}^{\infty} \delta(t)\, dt$$

$$|a| \int_{-\infty}^{\infty} \delta(at)\, dt = \int_{-\infty}^{\infty} \delta(t)\, dt$$

$$\therefore |a|\, \delta(at) = \delta(t)$$

$$\delta(at) = \frac{\delta(t)}{|a|}$$

$$\therefore \int_{-\infty}^{\infty} \delta(at) f(t)\, dt = \frac{1}{|a|} \int_{-\infty}^{\infty} \delta(t) f(t)\, dt = \frac{f(0)}{|a|}$$

d)

Solution: suppose when $t = t_i$, $g(t_i) = 0$

$$\because \int_{-\infty}^{\infty} \delta(g(t))\, d g(t) = \int_{-\infty}^{\infty} \delta(t)\, dt$$

$\forall \varepsilon > 0$

$$\therefore \int_{g(t_i-\varepsilon)}^{g(t_i+\varepsilon)} \delta(g(t))\, d g(t) = \int_{t_i-\varepsilon}^{t_i+\varepsilon} \delta(t-t_i)\, dt$$

$$\int_{t_i-\varepsilon}^{t_i+\varepsilon} g'(t)\, \delta(g(t))\, dt = \int_{t_i-\varepsilon}^{t_i+\varepsilon} \delta(t-t_i)\, dt$$

$$\therefore \delta(g(t)) = \boxed{\cancel{g'(t)}} \sum_i \frac{\delta(t-t_i)}{|g'(t_i)|}$$

$$\therefore \int_{-\infty}^{\infty} \delta(g(t)) f(t)\, dt = \sum_i \frac{f(t_i)}{|g'(t_i)|}$$

Task 4:

a)

Proof:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}[f\otimes g](t)e^{i\omega t}dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\tau)g(t-\tau)d\tau\, e^{i\omega t}dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\tau)g(t-\tau)d\tau\, e^{i\omega(t-\tau)}e^{i\omega\tau}dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\tau)e^{i\omega\tau}d\tau\int_{-\infty}^{\infty}g(t-\tau)e^{i\omega(t-\tau)}d(t-\tau)$$

$$=2\pi F\{f\}F\{g\}$$

b)

Solution:

$$F\{\Pi(t)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Pi(t)e^{i\omega t}dt$$

$$=\frac{1}{2\pi}\int_{-\frac{t_0}{2}}^{\frac{t_0}{2}}e^{i\omega t}dt$$

$$=\frac{e^{i\omega\frac{t_0}{2}}-e^{-i\omega\frac{t_0}{2}}}{2\pi i\omega}=\frac{t_0\sin(\omega\frac{t_0}{2})}{2\pi\omega\frac{t_0}{2}}=\frac{t_0}{2\pi}Sa\left(\frac{\omega t_0}{2}\right)$$

$$F\{\cos(\omega_0 t)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(\omega_0 t)e^{i\omega t}dt$$

$$=\frac{1}{4\pi}\int_{-\infty}^{\infty}e^{i(\omega+\omega_0)t}+e^{i(\omega-\omega_0)t}dt$$

$$\because\ \delta(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega t}dt$$

$$\therefore\ F\{\cos(\omega_0 t)\}=\frac{1}{2}[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$$

According to the convolution theorem

$$F\{f\}\otimes F\{g\}=2\pi F\{f\cdot g\}$$

$$F\{f\}\otimes F\{g\}=\frac{t_0}{2\pi}Sa\left(\frac{\omega t_0}{2}\right)\otimes\frac{1}{2}[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$$

$$=\frac{t_0^2}{16\pi^2}\left[Sa\left(\frac{t_0(\omega+\omega_0)}{2}\right)+Sa\left(\frac{t_0(\omega-\omega_0)}{2}\right)\right]$$