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Series 12

FUNDAMENTALS OF MODERN OPTICS

to be returned on **02.02.2023**, at the beginning of the lecture

Task 1: Optical Waveplates (3+2+2+3 points)

A slide of a transparent, uniaxial, anisotropic crystal with the refractive indices n_e and n_o and with thickness d is oriented such that the surface normal is along $\mathbf{e}_z$ and the crystal axis is along $\mathbf{c} = \cos(\alpha)\mathbf{e}_x + \sin(\alpha)\mathbf{e}_y$. An x-polarized plane wave of wavelength λ, with a vacuum wavevector $\mathbf{k} = (2\pi/\lambda)\mathbf{e}_z$, is excited at the beginning of the slide and propagates through the crystal. Consider the lossless propagation and neglect the Fresnel reflection at both interfaces.

a) Decompose the incident field into the normal modes (ordinary and extraordinary waves) of the anisotropic medium and write the dispersion relation for both of them.
Hint: For the decomposition use the crystal coordinate system.

b) Calculate the relation between the wavevector $\boldsymbol{k}$ and the corresponding Poynting vector $\boldsymbol{S}$ for both normal modes. Show that they are parallel.

c) Calculate the electric field in laboratory coordinates after propagating through the slide. What is its polarization state?

d) We choose the thickness d of the slide such that $(n_e - n_o)d = \lambda/2$. Calculate and describe the impact of the crystal onto the polarization state of the plane wave directly after the slide as a function of the crystal rotation angle α. If we place a linear polarizer after this so-called half-wave plate and rotate the crystal, which device do we get?

Solution Task 1:

a) In this task, we are talking about an uniaxial crystal. It has two modes: the ordinary wave, with polarization orthogonal to the optical axis $\mathbf{c}$; and the extraordinary wave, with polarization parallel to the optical axis. We will work in the crystal coordinate system described by the vectors: $\mathbf{c}$, $-(\mathbf{c} \times \mathbf{e}_z)$, $\mathbf{e}_z$. In the crystal coordinate system $\hat{\varepsilon}$-tensor will be written as $\begin{bmatrix} n_e^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_o^2 \end{bmatrix}$. Therefore, decomposition of the electric field $\mathbf{E}(z=0) = E_0\mathbf{e}_x \cdot exp(-ik(\omega)z) = E_0\mathbf{e}_x$ into this coordinate system will look like:

$$\mathbf{E} = E_0 \cdot \mathbf{e}_x = a \cdot \mathbf{c} + b \cdot (-\mathbf{c} \times \mathbf{e}_z) + c \cdot \mathbf{e}_z, \tag{1}$$

there the coefficients a, b, c can be found using the dot product of $\mathbf{E}$ and the radius-vector of the crystal coordinate system. Taking dot product of Eq. (1) with each of $\mathbf{c}$, $-(\mathbf{c} \times \mathbf{e}_z)$, $\mathbf{e}_z$ and remembering $\mathbf{c} = \cos(\alpha)\mathbf{e}_x + \sin(\alpha)\mathbf{e}_y$:

$$\begin{aligned} E_0\mathbf{e}_x \cdot \mathbf{c} &= E_0 \cos(\alpha) = a \\ E_0\mathbf{e}_x \cdot (-\mathbf{c} \times \mathbf{e}_z) &= -E_0\mathbf{c}(\mathbf{e}_z \times \mathbf{e}_x) = -E_0\mathbf{c} \cdot \mathbf{e}_y = -E_0 \sin(\alpha) = b \\ E_0\mathbf{e}_x \cdot \mathbf{e}_z &= 0 = c \end{aligned}$$

Inserting the values of a, b and c in Eq. (1)

$$\mathbf{E} = E_0 [\cos(\alpha)\mathbf{c} - \sin(\alpha)(-\mathbf{c} \times \mathbf{e}_z)]$$

Dispersion relation: for the polarization along $\mathbf{c}$, extraordinary wave: $\mathbf{k} = n_e \frac{\omega}{c}\mathbf{e}_z$. For the polarization along $-(\boldsymbol{c} \times \boldsymbol{e}_z)$, ordinary wave: $\mathbf{k} = n_o \frac{\omega}{c}\mathbf{e}_z$.

b) For the Poynting vector $\mathbf{S}$, we have

$$\mathbf{S} \sim \mathbf{E} \times \mathbf{H}^{\star}$$

And because we have plane waves:

$$\mathbf{H} \sim \mathbf{k} \times \mathbf{E}$$

$$\Rightarrow \mathbf{S} \sim \mathbf{E} \times (\mathbf{k} \times \mathbf{E})^\star = \mathbf{k}^\star \left(\mathbf{E} \cdot \mathbf{E}^\star\right) - \mathbf{E}^\star \left(\mathbf{E} \cdot \mathbf{k}^\star\right)$$

Since $\mathbf{k} \parallel \mathbf{e}_z$ and $\mathbf{k}^\star = \mathbf{k}$ for lossless media, we have

$$\mathbf{E} \cdot \mathbf{k} = \frac{2\pi}{\lambda} \mathbf{E} \cdot \mathbf{e}_z = 0$$

$$\Rightarrow \mathbf{S} \sim \mathbf{k} |\mathbf{E}|^2$$

Therefore, $\mathbf{S}$ is parallel to $\mathbf{k}$.

c) At the input $z = 0$ we have

$$\mathbf{E}(z=0) = E_0 \left[\cos(\alpha)\,\mathbf{c} - \sin(\alpha)(-\mathbf{c} \times \mathbf{e}_z)\right]$$

In an anisotropic medium, these two normal modes, as discussed above, have different phase velocities c/n_o and c/n_e. Therefore, after a distance d they experience different phase-shift: $\varphi_o = n_o \frac{\omega}{c} d$ for the ordinary wave and $\varphi_e = n_e \frac{\omega}{c} d$ for the extraordinary wave. Then, the field after a propagation through distance d:

$$\mathbf{E}(z=d) = E_0 \left[\cos(\alpha) \exp\left(i n_e \frac{\omega}{c} d\right) \mathbf{c} - \sin(\alpha) \exp\left(i n_o \frac{\omega}{c} d\right)(-\mathbf{c} \times \mathbf{e}_z)\right] \tag{2}$$

The crystal axes $\mathbf{c}$ and $(-\mathbf{c} \times \mathbf{e}_z)$ can be expressed in laboratory's coordinate system as

$$\mathbf{c} = \cos(\alpha)\,\mathbf{e}_x + \sin(\alpha)\,\mathbf{e}_y, \quad (-\mathbf{c} \times \mathbf{e}_z) = -\sin(\alpha)\,\mathbf{e}_x + \cos(\alpha)\,\mathbf{e}_y$$

Therefore, Eq. (2) becomes:

$$\begin{aligned}\mathbf{E}(z=d) = E_0 \Bigg[& \left(\cos^2(\alpha) \exp\left(i n_e \frac{\omega}{c} d\right) + \sin^2(\alpha) \exp\left(i n_o \frac{\omega}{c} d\right)\right) \mathbf{e}_x \\ & + \cos(\alpha)\sin(\alpha) \left\{\exp\left(i n_e \frac{\omega}{c} d\right) - \exp\left(i n_o \frac{\omega}{c} d\right)\right\} \mathbf{e}_y \Bigg]\end{aligned}$$

d) Factor out $\exp\left(i n_e \frac{\omega}{c} d\right)$ from the expression above, and use the fact that with $d = \frac{\lambda}{2(n_e - n_o)} = \frac{\pi c}{\omega(n_e - n_o)}$ we will have $\exp\left(i(n_o - n_e)\frac{\omega}{c} d\right) = \exp(-i\pi) = -1$. So we get:

$$\begin{aligned}\mathbf{E}(z=d) &= E_0 \exp\left(i n_e \tfrac{\omega}{c} d\right)\left[\left(\cos^2(\alpha) - \sin^2(\alpha)\right)\mathbf{e}_x + \cos(\alpha)\sin(\alpha)\{1+1\}\mathbf{e}_y\right] \\ &= E_0 \exp\left(i n_e \tfrac{\omega}{c} d\right)\left[\cos(2\alpha)\,\mathbf{e}_x + \sin(2\alpha)\,\mathbf{e}_y\right] \\ &= E_0 \exp\left(i\pi \tfrac{n_e}{n_e - n_0}\right)\left[\cos(2\alpha)\,\mathbf{e}_x + \sin(2\alpha)\,\mathbf{e}_y\right]\end{aligned}$$

Hence, we have linearly polarized output with polarization rotated by angle 2α. If there is an x-polarizer at the output, $\mathbf{E}$ field after it would be

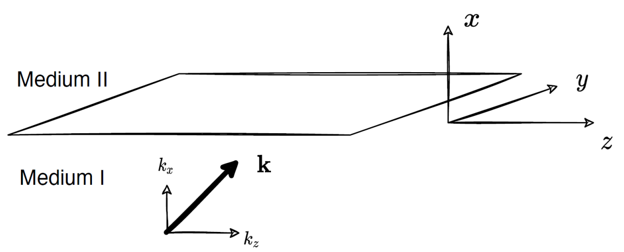
$$\mathbf{E} = E_0 \exp\left(i\pi \frac{n_e}{n_e - n_0}\right) \cos(2\alpha)\,\mathbf{e}_x$$

$$\Rightarrow |\mathbf{E}|^2 = |E_0|^2 \cos^2(2\alpha)$$

Therefore, we get an attenuator by rotating the half-wave plate.

Task 2: Optical interfaces (general) (1 + 2 + 2 = 5 points)

The symmetries and invariances are very useful for the solutions and simplifications of different physical problems. Consider an infinite interface between two media with different relative permittivity ε_r:



a) Consider a general wave-equation:

$$\nabla \times \nabla \times \bar{\mathbf{E}}(\mathbf{r},t) - \frac{\omega^2}{c^2}\bar{\mathbf{E}}(\mathbf{r},\omega) = i\omega\mu_0\bar{\mathbf{j}}(\bar{r},\omega) + \mu_0\omega^2\bar{\mathbf{P}}(\bar{r},\omega)$$

Which simplification of the $\nabla\times\nabla\times\bar{\mathbf{E}}(\mathbf{r},\omega)$ relation can be obtained from translation invariance of the infinite flat interface between two media (1P).

b) Give a short explanation of the $\mathbf{E} = \mathbf{E}_{TM} + \mathbf{E}_{TE}$ decomposition of an arbitrary E-field (1P). What are the advantages of this decomposition for the situation described in a) (1P)?

c) Give a short explanation of the "continuity of field" and the "continuity of wave vector" on an interface. For which fields and polarization components these conditions are satisfied (2P)?

Solution Task 2:

Note to correctors: This task is just a general introduction to optical interfaces and is focused on the general treatments for this problem. The solutions from students have not to be mathematically rigorous, the textual explanation can be enough, but it has to contain the same ideas.

a) Because of translation invariance: $\frac{\partial}{\partial y} \to 0$

$$\nabla\times\nabla\times\bar{\mathbf{E}} = \nabla\nabla.\bar{\mathbf{E}} - \nabla^2\bar{\mathbf{E}} = \begin{bmatrix}\frac{\partial}{\partial x}(\frac{\partial}{\partial x}\bar{E}_x + \frac{\partial}{\partial z}\bar{E}_z)\\ 0 \\ \frac{\partial}{\partial z}(\frac{\partial}{\partial x}\bar{E}_x + \frac{\partial}{\partial z}\bar{E}_z)\end{bmatrix} - \begin{bmatrix}\nabla^2_{x,z}\bar{E}_x\\ \nabla^2_{x,z}\bar{E}_y\\ \nabla^2_{x,z}\bar{E}_z\end{bmatrix}$$

b) $\bar{\mathbf{E}} = \bar{\mathbf{E}}_{TM} + \bar{\mathbf{E}}_{TE}$

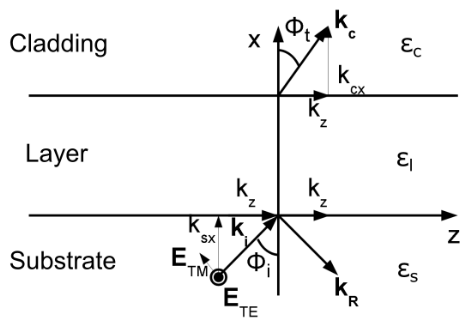
$$\bar{\mathbf{E}}_{TE} = \begin{bmatrix}0\\ \bar{E}_y\\ 0\end{bmatrix} \quad \bar{\mathbf{E}}_{TM} = \begin{bmatrix}\bar{E}_x\\ 0\\ \bar{E}_z\end{bmatrix}$$

(1P) The $\nabla\times\nabla\times\bar{\mathbf{E}}(\mathbf{r},\omega)$ and therewith the wave equation can be split into two independent equations without the"mixing" term and any truncations (1P).

c) - Continuity of fields (see script p. 144)
TE: $\bar{\mathbf{E}} = \bar{\mathbf{E}}_y$ and $\bar{\mathbf{H}}_z$ continuous
TM: $\bar{\mathbf{E}}_z$ and $\bar{\mathbf{H}} = \bar{\mathbf{H}}_y$ continuous
As pointed out before, we can always split any field into its TE and TM polarization components $\mathbf{E} = \mathbf{E}_{TM} + \mathbf{E}_{TE}$ and treat them separately.
- Continuity of wave vectors
Since we have homogeneity in the z-direction, we expect the solution on each side of the interface to vary along the z-direction as $\exp(ik_zZ)$. Since k_z determines the rate at which the field varies along the boundary, the continuity of the transverse field components implies the continuity of the wave vector component k_z across the interface. $\to$ homogeneous in z-direction $\to$ phase $\exp(ik_zZ) \to k_z$ continuous

Task 3: Optical layer (1+3+3*+2* points)

Let us consider a single optical layer with thickness d that is embedded between a substrate and a cladding material as shown below. The refractive indices of the layer, substrate, and cladding materials are $n_\mathrm{l} = \sqrt{\varepsilon_\mathrm{l}}$, $n_\mathrm{s} = \sqrt{\varepsilon_\mathrm{s}}$, and $n_\mathrm{c} = \sqrt{\varepsilon_\mathrm{c}}$, respectively. For simplicity, we consider light in TE polarization only. The incident beam makes an incident angle of φ_i in the substrate.



a) Calculate the angle φ_t of the transmitted beam as a function of the incident angle φ_i.

b) Compute the coefficients of reflection and transmission as functions of the incident angle φ_i.

c) Compute the reflectivity and transmissivity of the single layer, and show that they add up to 1. For simplicity, assume $\varepsilon_l > \varepsilon_s \sin^2(\varphi_i)$ and $\varepsilon_c > \varepsilon_s \sin^2(\varphi_i)$.

d) Consider the special case of a $\lambda/4$-layer, i.e. $k_{l,x}d = d\sqrt{k_l^2 - k_z^2} = \pi/2$, and calculate its reflectivity. Now assume the incident light is perpendicular to the layer ($\varphi_i = 0$) and find the condition for the refractive indices to obtain minimum reflection.

Solution Task 3:

a) Since k_z is continuous,

$$\left.\begin{aligned} k_z &= \sin(\varphi_t)\cdot k_0 \cdot n_c \\ k_z &= \sin(\varphi_i)\cdot k_0 \cdot n_s \end{aligned}\right\} \quad \varphi_t = \sin^{-1}\left(\frac{n_s}{n_c}\sin(\varphi_i)\right).$$

b) Wavevectors in substrate, film, and substrate as a function of incident angle (φ_i):

$$k_z = |\mathbf{k_s}|\sin(\varphi_i) = k_0\sqrt{\varepsilon_s}\sin(\varphi_i) \quad ; \quad k_{s,x} = |\mathbf{k_s}|\cos(\varphi_i) = k_0\sqrt{\varepsilon_s}\cos(\varphi_i)$$

$$k_0^2\varepsilon_l = k_z^2 + k_{l,x}^2 \quad => \quad k_{l,x} = k_0\sqrt{\varepsilon_l - \varepsilon_s\sin^2(\varphi_i)}$$

$$k_0^2\varepsilon_c = k_z^2 + k_{c,x}^2 \quad => \quad k_{c,x} = k_0\sqrt{\varepsilon_c - \varepsilon_s\sin^2(\varphi_i)}$$

Reflection and Transmission coefficients in general multilayer case:

$$R_{TE} = \frac{k_{s,x}M_{22} - k_{c,x}M_{11} - i\left(M_{21} + k_{s,x}k_{c,x}M_{12}\right)}{\underbrace{k_{s,x}M_{22} + k_{c,x}M_{11} + i\left(M_{21} - k_{s,x}k_{c,x}M_{12}\right)}_{N_{TE}}} \quad , \quad T_{TE} = \frac{2k_{s,x}}{N_{TE}}$$

Transfer matrix $\hat{\mathbf{M}}$ for 1 layer:

$$\hat{\mathbf{M}} = \hat{\mathbf{m}}(d) = \begin{Bmatrix} \cos(k_{l,x}d) & \frac{1}{k_{l,x}}\sin(k_{l,x}d) \\ -k_{l,x}\sin(k_{l,x}d) & \cos(k_{l,x}d) \end{Bmatrix}$$

Reflection and Transmission coeficients for film:

$$R_{TE} = \frac{(k_{s,x} - k_{c,x})\cos(\frac{\delta}{2}) + i\left(k_{l,x} - \frac{k_{s,x}k_{c,x}}{k_{l,x}}\right)\sin(\frac{\delta}{2})}{\underbrace{(k_{s,x} + k_{c,x})\cos(\frac{\delta}{2}) - i\left(k_{l,x} + \frac{k_{s,x}k_{c,x}}{k_{l,x}}\right)\sin(\frac{\delta}{2})}_{N_{\mathrm{TE}}}} \quad , \quad T_{TE} = \frac{2k_{s,x}}{N_{\mathrm{TE}}},$$

where $\frac{\delta}{2} = k_{l,x}d$.

c) Reflectivity and transmissivity of the single layer:

$$\rho_{TE} = |R_{TE}|^2 \quad , \quad \tau_{TE} = \frac{\mathrm{Re}[k_{c,x}]}{k_{s,x}}|T_{TE}|^2.$$

Knowing that all k-values are real numbers, we can calculate the absolute-values. After some simplifications we get:

$$\tau_{TE} = \frac{4k_{c,x}k_{s,x}}{(k_{s,x} + k_{c,x})^2\cos^2(\frac{\delta}{2}) + \left(k_{l,x} + \frac{k_{s,x}k_{c,x}}{k_{l,x}}\right)^2\sin^2(\frac{\delta}{2})}$$

$$\begin{aligned} \rho_{TE} &= \frac{(k_{s,x} + k_{c,x})^2\cos^2(\frac{\delta}{2}) - 4k_{c,x}k_{s,x}\cos^2(\frac{\delta}{2}) + \left(k_{l,x} + \frac{k_{s,x}k_{c,x}}{k_{l,x}}\right)^2\sin^2(\frac{\delta}{2}) - 4k_{c,x}k_{s,x}\sin^2(\frac{\delta}{2})}{(k_{s,x} + k_{c,x})^2\cos^2(\frac{\delta}{2}) + \left(k_{l,x} + \frac{k_{s,x}k_{c,x}}{k_{l,x}}\right)^2\sin^2(\frac{\delta}{2})} \\ &= \frac{(k_{s,x} + k_{c,x})^2\cos^2(\frac{\delta}{2}) + \left(k_{l,x} + \frac{k_{s,x}k_{c,x}}{k_{l,x}}\right)^2\sin^2(\frac{\delta}{2}) - 4k_{c,x}k_{s,x}}{(k_{s,x} + k_{c,x})^2\cos^2(\frac{\delta}{2}) + \left(k_{l,x} + \frac{k_{s,x}k_{c,x}}{k_{l,x}}\right)^2\sin^2(\frac{\delta}{2})} \end{aligned}$$

It is clear from these expressions that $\rho_{TE} + \tau_{TE} = 1$.

d)

$$\frac{\delta}{2} = k_{1,x}d = \frac{\pi}{2} \rightarrow \cos(\frac{\delta}{2}) = 0,\ \sin(\frac{\delta}{2}) = 1 \rightarrow \rho_{TE} = \frac{\left(k_{1,x} - \frac{k_{s,x}k_{c,x}}{k_{1,x}}\right)^2}{\left(k_{1,x} + \frac{k_{s,x}k_{c,x}}{k_{1,x}}\right)^2}$$

With $\varphi_i = 0$ we get:

$$k_{1,x} = k_0\sqrt{\varepsilon_1 - \varepsilon_s \sin^2(\varphi_i)} = k_0 n_1,\ k_{c,x} = k_0 n_c,\ k_{s,x} = k_0 n_s$$

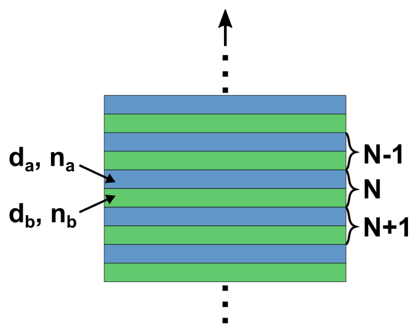
Which results in:

$$\rho_{TE} = \frac{\left(k_{1,x} - \frac{k_{s,x}k_{c,x}}{k_{1,x}}\right)^2}{\left(k_{1,x} + \frac{k_{s,x}k_{c,x}}{k_{1,x}}\right)^2} = \frac{(n_1^2 - n_s n_c)^2}{(n_1^2 + n_s n_c)^2}$$

Zero reflection is achieved under: $n_1 = \sqrt{n_s n_c}$.

Task 4: Stratified Media (4*+2*+2* points)

A beam of light with free-space wave number $k_0 = \omega/c$ is passing through a periodic stack of alternating layers. The odd layers have a refractive index n_a and a thickness d_a, the even layers have a refractive index n_b and a thickness d_b as shown below. For the case of an infinite periodic stack, the fields in the structure follow the Bloch theorem, which states that this structure supports Bloch modes of wave number K. For such a mode the field at the beginning of the double layer N is connected to the field at the beginning of the double layer $N+1$ through $E_{N+1} = \exp(iK\Lambda)E_N$ with $\Lambda = d_a + d_b$. For simplicity consider the case of normal incidence $\varphi_i = 0$.



a) Derive the dispersion relation for K as a function of n_a, n_b, d_a, d_b, and k_0.
b) Determine under which condition we get a propagating or a decaying Bloch mode.
c) For the case of decaying Bloch modes find the frequencies for which the decay becomes strongest. For simplicity assume $n_a d_a = n_b d_b$.

Solution Task 4:

a) The field after one period of propagation satisfies two relations at the same time, one from the Bloch theorem and the other from the matrix method:

$$\begin{pmatrix} E \\ E' \end{pmatrix}_{(N+1)\Lambda} = \exp(iK\Lambda)\begin{pmatrix} E \\ E' \end{pmatrix}_{N\Lambda} = \hat{\mathbf{M}}\begin{pmatrix} E \\ E' \end{pmatrix}_{N\Lambda}$$

where $E' = \partial E/\partial x$. Transfer matrix for TE mode:

$$\hat{\mathbf{M}} = \hat{\mathbf{m}}_a(d_a)\cdot\hat{\mathbf{m}}_b(d_b) = \begin{bmatrix} \cos(k_0 n_a d_a) & \frac{1}{k_0 n_a}\sin(k_0 n_a d_a) \\ -k_0 n_a \sin(k_0 n_a d_a) & \cos(k_0 n_a d_a) \end{bmatrix} \cdot \begin{bmatrix} \cos(k_0 n_b d_b) & \frac{1}{k_0 n_b}\sin(k_0 n_b d_b) \\ -k_0 n_b \sin(k_0 n_b d_b) & \cos(k_0 n_b d_b) \end{bmatrix}$$
$$= \begin{bmatrix} \mathrm{M}_{11} & \mathrm{M}_{12} \\ \mathrm{M}_{21} & \mathrm{M}_{22} \end{bmatrix}$$

Eigenvalue ($\exp(iK\Lambda) = \mu$) problem:

$$\left\{\hat{\mathbf{M}} - \exp(iK\Lambda)\hat{\mathbf{I}}\right\}\begin{pmatrix} E \\ E' \end{pmatrix}_{N\Lambda} = \left\{\hat{\mathbf{M}} - \mu\hat{\mathbf{I}}\right\}\begin{pmatrix} E \\ E' \end{pmatrix}_{N\Lambda} = 0$$

Put the determinant of $\mathbf{M} - \mu\hat{\mathbf{I}}$ to zero:

$$(\mathrm{M}_{11} - \mu)(\mathrm{M}_{22} - \mu) - \underbrace{\mathrm{M}_{12}\mathrm{M}_{21}}_{|\hat{\mathbf{M}}|=1;\quad \mathrm{M}_{12}\mathrm{M}_{21} = \mathrm{M}_{11}\mathrm{M}_{22} - 1} = 0$$

$$\mu^2 - \mu(\mathrm{M}_{11} + \mathrm{M}_{22}) + 1 = 0$$

We assume the layers have no absorption, thus $|\hat{\mathbf{M}}| = 1$. If we solve above quadratic equation for μ:

$$\mu_\pm = \exp(iK_\pm\Lambda) = \frac{\mathrm{M}_{11} + \mathrm{M}_{22}}{2} \pm \sqrt{\left(\frac{\mathrm{M}_{11} + \mathrm{M}_{22}}{2}\right)^2 - 1}$$

$$\exp(iK_\pm\Lambda) = \cos(K\Lambda) \pm i\sin(K\Lambda) = \cos(K\Lambda) \pm \sqrt{\cos^2(K\Lambda) - 1}$$

By comparing above two equation, the dispersion relation is found as

$$\boxed{\cos(K\Lambda) = \frac{\mathrm{M}_{11} + \mathrm{M}_{22}}{2}}.$$

where

$$\mathrm{M}_{11} = \cos(k_0 n_\mathrm{a} d_\mathrm{a})\cos(k_0 n_\mathrm{b} d_\mathrm{b}) - \frac{n_\mathrm{b}}{n_\mathrm{a}}\sin(k_0 n_\mathrm{a} d_\mathrm{a})\sin(k_0 n_\mathrm{b} d_\mathrm{b})$$

$$\mathrm{M}_{22} = \cos(k_0 n_\mathrm{a} d_\mathrm{a})\cos(k_0 n_\mathrm{b} d_\mathrm{b}) - \frac{n_\mathrm{a}}{n_\mathrm{b}}\sin(k_0 n_\mathrm{a} d_\mathrm{a})\sin(k_0 n_\mathrm{b} d_\mathrm{b})$$

b) For propagating we need $\left|\frac{\mathrm{M}_{11}+\mathrm{M}_{22}}{2}\right| < 1$ and for decaying/evanescent $\left|\frac{\mathrm{M}_{11}+\mathrm{M}_{22}}{2}\right| > 1$, which comes from looking at the cosine relation.

c) The larger $\left|\frac{\mathrm{M}_{11}+\mathrm{M}_{22}}{2}\right|$ is, the larger the imaginary part of K becomes. From (a) we have:

$$\cos(K\Lambda) = \cos(k_0 n_\mathrm{a} d_\mathrm{a})\cos(k_0 n_\mathrm{b} d_\mathrm{b}) - \frac{1}{2}\left[\frac{n_\mathrm{b}}{n_\mathrm{a}} + \frac{n_\mathrm{a}}{n_\mathrm{b}}\right]\sin(k_0 n_\mathrm{a} d_\mathrm{a})\sin(k_0 n_\mathrm{b} d_\mathrm{b})$$

To maximize this, we need to minimize $\cos(k_0 n_\mathrm{a} d_\mathrm{a})\cos(k_0 n_\mathrm{b} d_\mathrm{b})$ and maximize $\sin(k_0 n_\mathrm{a} d_\mathrm{a})\sin(k_0 n_\mathrm{b} d_\mathrm{b})$, which happens at $k_0 n_\mathrm{a} d_\mathrm{a} = k_0 n_\mathrm{b} d_\mathrm{b} = (2m+1)\pi/2$ (question assumed $n_\mathrm{a} d_\mathrm{a} = n_\mathrm{b} d_\mathrm{b}$), where m is any integer number. This gives $\cos(K\Lambda) = -\frac{1}{2}\left[\frac{n_\mathrm{b}}{n_\mathrm{a}} + \frac{n_\mathrm{a}}{n_\mathrm{b}}\right]$ which is always smaller than -1. So the frequencies of interest are

$$\omega_m = \frac{(2m+1)\pi c}{2n_\mathrm{a} d_\mathrm{a}} = \frac{(2m+1)\pi c}{2n_\mathrm{b} d_\mathrm{b}}$$