Midterm Exam
"Fundamentals of modern optics"
WS 2014/15
solution examples

Problem 1 – Maxwell's equations **3 + 2 + 3 + 1 = 9 points**

1. Write down Maxwell's equations and the the auxiliary fields **D** and **H** in time domain.

   Maxwell's equations:

$$
\begin{aligned}
\nabla \times \mathbf{E}(\mathbf{r}, t) &= -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \\
\nabla \times \mathbf{H}(\mathbf{r}, t) &= \mathbf{j}(\mathbf{r}, t) + \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} \\
\nabla \cdot \mathbf{D}(\mathbf{r}, t) &= \rho(\mathbf{r}, t) \\
\nabla \cdot \mathbf{B}(\mathbf{r}, t) &= 0
\end{aligned}
$$

   constitutive equations:

$$
\begin{aligned}
\mathbf{D}(\mathbf{r}, t) &= \epsilon_0 \mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r}, t) = \epsilon_0 \int_{-\infty}^{+\infty} \epsilon(\mathbf{r}, t - \tau) \mathbf{E}(\mathbf{r}, \tau) \mathrm{d}\tau \\
\mathbf{H}(\mathbf{r}, t) &= \frac{1}{\mu_0} \mathbf{B}(\mathbf{r}, t)
\end{aligned}
$$

2. Write down Maxwell's equations in frequency domain in a linear, homogeneous and isotropic dielectric medium in absence of free charges and current density ($\rho = 0$ and $\mathbf{j} = 0$).

$$
\begin{aligned}
\nabla \times \mathbf{E}(\mathbf{r}, \omega) &= i\omega\mu_0 \mathbf{H}(\mathbf{r}, \omega) \\
\nabla \times \mathbf{H}(\mathbf{r}, \omega) &= -i\omega \mathbf{D}(\mathbf{r}, \omega) \\
\nabla \cdot \mathbf{D} &= 0 \\
\nabla \cdot \mathbf{H} &= 0
\end{aligned}
$$

3. Derive the wave equation in Fourier domain for the electric field in a linear, homogeneous and isotropic dielectric medium in absence of free charges and current density ($\rho = 0$ and $\mathbf{j} = 0$).

   From Maxwell's equations:

$$
\begin{aligned}
\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, \omega) &= i\omega\mu_0 \nabla \times \mathbf{H}(\mathbf{r}, \omega) \\
&= i\omega\mu_0(-i\omega \mathbf{D}(\mathbf{r}, \omega)) \\
&= \mu_0 \omega^2 \mathbf{D}(\mathbf{r}, \omega) \\
&= \mu_0 \epsilon_0 \omega^2 \mathbf{E}(\mathbf{r}, \omega) + \mu_0 \omega^2 \mathbf{P}(\mathbf{r}, \omega) \\
&= \frac{\omega^2}{c^2} \epsilon(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega)
\end{aligned}
$$

   and with:

$$
\begin{aligned}
\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, \omega) & \\
&= \mathbf{grad}\,\mathrm{div}\,\mathbf{E}(\mathbf{r}, \omega) - \Delta \mathbf{E}(\mathbf{r}, \omega) \\
&= -\Delta \mathbf{E}(\mathbf{r}, \omega)
\end{aligned}
$$

   we can find:

$$
\Delta \mathbf{E}(\mathbf{r}, \omega) + \frac{\omega^2}{c^2} \epsilon(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega) = 0
$$

4. Give the formula of the time averaged Poynting vector.

   Time averaged Poynting vector: $\langle \mathbf{S}(\mathbf{r},t)\rangle = \frac{1}{2} Re\left(\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})\right)$

Problem 2 – Poynting Vector and Normal Mode **$2+2+1+3=8$ points**

a) First represent the real valued electric field in its complex form, to identify the wave-vector:

$$\mathbf{E}_r(\mathbf{r},t) = E_0 \; \mathbf{e}_x \; e^{-\alpha z} \; \cos(\beta z - \omega t + \phi) = \; \frac{1}{2}\left[\mathbf{E}_c \; e^{-i\omega t} + \mathbf{E}_c^* \; e^{+i\omega t}\right]$$

with $\mathbf{E}_c = E_0 \; e^{i\phi} \mathbf{e}_x \; e^{i(\beta+i\alpha)z}$. We identify the wave-vector as $\mathbf{k} = \mathbf{k}' + i\mathbf{k}'' = (\beta + i\alpha) \; \mathbf{e}_z$. We know the dispersion relation of a plane wave in a homogeneous medium as $\mathbf{k}.\mathbf{k} = \frac{\omega^2}{c^2}\epsilon$. Expanding both sides gives us:

$$k'^2 - k''^2 = \frac{\omega^2}{c^2}\epsilon' \;\; , \;\; 2k'k'' = \frac{\omega^2}{c^2}\epsilon''$$

From which we find $k' \approx \frac{\omega}{c}\sqrt{\epsilon'}$ and $k'' \approx \frac{\omega}{c}\frac{\epsilon''}{2\sqrt{\epsilon'}}$.

b) Same like electric field, we can present the real valued magnetic field like:

$$\mathbf{H}_r(\mathbf{r},t) = \; \frac{1}{2}\left[\mathbf{H}_c \; e^{-i\omega t} + \mathbf{H}_c^* \; e^{+i\omega t}\right]$$

Using the time domain Maxwell equation $\nabla \times \mathbf{E}_r = -\mu_0 \frac{\partial \mathbf{H}_r}{\partial t}$, we find the realtion between the complex amplitudes to be $\nabla \times \mathbf{E}_c = i\omega\mu_0 \mathbf{H}_c$. Followed by:

$$\nabla \times \mathbf{E}_c = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_0 \; e^{i\phi}\mathbf{e}_x \; e^{i(\beta+i\alpha)z} & 0 & 0 \end{vmatrix} = i(\beta+i\alpha)E_0 \; e^{i\phi}\mathbf{e}_y \; e^{i(\beta+i\alpha)z}$$

Which gives $\mathbf{H}_c = \frac{(\beta+i\alpha)E_0 \; e^{i\phi}}{\omega\mu_0}\mathbf{e}_y \; e^{i(\beta+i\alpha)z}$. And we calculate the real valued magnetic field:

$$\mathbf{H}_r = \frac{E_0 e^{-\alpha z}}{2\omega\mu_0}\mathbf{e}_y\left[(\beta+i\alpha)e^{i(\beta z-\omega t+\phi)} + (\beta-i\alpha)e^{-i(\beta z-\omega t+\phi)}\right] = \frac{E_0}{\omega\mu_0}e^{-\alpha z}\mathbf{e}_y\left[\beta\cos(\beta z-\omega t+\phi) - \alpha\sin(\beta z-\omega t+\phi)\right]$$

c) $\mathbf{S}_r(\mathbf{r},t) = \mathbf{E}_r(\mathbf{r},t) \times \mathbf{H}_r(\mathbf{r},t)$

d)

$$\begin{aligned}
\langle \mathbf{S}_r(\mathbf{r},t)\rangle &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\mathbf{S}_r(\mathbf{r},t)dt \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\mathbf{E}_r(\mathbf{r},t)\times\mathbf{H}_r(\mathbf{r},t)dt \\
&= \frac{E_0^2}{\omega\mu_0}e^{-2\alpha z}\mathbf{e}_z\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\cos(\beta z-\omega t+\phi)\left[\beta\cos(\beta z-\omega t+\phi)-\alpha\sin(\beta z-\omega t+\phi)\right]dt \\
&= \frac{E_0^2}{\omega\mu_0}e^{-2\alpha z}\mathbf{e}_z\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\left[\beta\frac{\cos(2(\beta z-\omega t+\phi))+1}{2} - \alpha\frac{\sin(2(\beta z-\omega t+\phi))}{2}\right]dt \\
&= \frac{E_0^2}{\omega\mu_0}\frac{\beta}{2}e^{-2\alpha z}\mathbf{e}_z
\end{aligned}$$

Problem 3 – Beam propagation (Imaging) **3 + 3 + 3 = 9 points**

a) $$U_0(\alpha,\beta;z=0) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} A\exp\left(-i\pi\frac{x^2+y^2}{\lambda\ f} - i(x\alpha+y\beta)\right)\mathrm{d}x\mathrm{d}y$$

$$= \frac{A}{(2\pi)^2}e^{\frac{i(\alpha^2+\beta^2)\lambda f}{4\pi}}\int_{\mathbb{R}^2}\exp\left(\frac{-i\pi}{\lambda\ f}\left(\underbrace{x^2+2\ \frac{\lambda f\alpha}{2\pi}x+\left(\frac{\lambda f\alpha}{2\pi}\right)^2}_{=x'^2}+\underbrace{y^2+2\ \frac{\lambda f\alpha}{2\pi}y-\left(\frac{\lambda f\beta}{2\pi}\right)^2}_{=y'^2}\right)\right)\mathrm{d}x\mathrm{d}y$$

$$= \frac{A}{(2\pi)^2}e^{\frac{i(\alpha^2+\beta^2)\lambda f}{4\pi}}\int_{\mathbb{R}^2}\exp\left(\frac{-i\pi}{\lambda\ f}\underbrace{(x'^2+y'^2)}_{\rho^2}\right)\mathrm{d}x'\mathrm{d}y'$$

$$= \frac{A}{(2\pi)^2}e^{\frac{i(\alpha^2+\beta^2)\lambda f}{4\pi}}\int_0^\infty\int_0^{2\pi}\exp\left(\frac{-i\pi}{\lambda\ f}\rho^2\right)\rho\mathrm{d}\phi\mathrm{d}\rho = \frac{A}{2\pi}e^{\frac{i(\alpha^2+\beta^2)\lambda f}{4\pi}}\int_0^\infty\exp\left(\frac{-i\pi}{\lambda\ f}\rho^2\right)\rho\mathrm{d}\rho$$

$$= -\frac{iA\lambda f}{4\pi^2}\exp\left(i\frac{(\alpha^2+\beta^2)\lambda f}{4\pi}\right) = -\frac{iA\lambda f}{4\pi^2}\exp\left(i\frac{k_\rho^2\lambda f}{4\pi}\right)$$

where $k_\rho^2 = \alpha^2+\beta^2$

b) Free space transfer function:

$$H_F(k_\rho;z) = \exp(ik_z z) = \exp\left(iz\sqrt{k_0^2-k_\rho^2}\right) = \underbrace{\exp\left(izk_0\left(1-\frac{k_\rho^2}{2k_0^2}\right)\right)}_{\text{paraxial} => k_\rho/k_0 \ll 1} = \exp\left(izk_0 - iz\frac{k_\rho^2\lambda}{4\pi}\right)$$

$H_F(\alpha,\beta;z) = \exp\left(izk_0 - iz(\alpha^2+\beta^2)\lambda/4\pi\right)$
$\Longrightarrow$ Evanescent waves: $k_\rho > k_0$
$\Longrightarrow$ Propagating waves: $k_\rho \leq k_0$

c) $$U(\alpha,\beta;z=f) = U_0(\alpha,\beta;z=0)H_F(\alpha,\beta;f)$$

$$= -\frac{iA\lambda f}{4\pi^2}\exp\left(i\frac{k_\rho^2\lambda f}{4\pi}\right)\exp\left(ik_0 f - i\frac{k_\rho^2\lambda f}{4\pi}\right)$$

$$= -\frac{iA\lambda f}{4\pi^2}e^{ik_0 f}$$

$$u(x,y,z=f) = \int_{\mathbb{R}^2} U(\alpha,\beta;z=f)e^{i(x\alpha+y\beta)}\mathrm{d}\alpha\mathrm{d}\beta$$

$$= -\frac{iA\lambda f}{4\pi^2}e^{ik_0 f}\int_{\mathbb{R}^2} e^{i(x\alpha+y\beta)}\mathrm{d}\alpha\mathrm{d}\beta$$

$$= -\frac{iA\lambda f}{4\pi^2}e^{ik_0 f}\delta(x)\delta(y)$$

Problem 4 - Gaussian beam in a telescope (2+2+2 points)

1. The q parameter at the first lens is given by $q = f_1 + iz_0$. Using the ABCD formalism, we obtain the parameter q' of the beam just after the lens:

$$q' = \frac{f_1+iz_0}{-\frac{1}{f_1}(f_1+iz_0)+1} = \frac{f_1+iz_0}{-\frac{iz_0}{f_1}} = -f_1 + i\frac{f_1^2}{z_0}$$

   that means that the waist position is at $z = f_1$ after the lens and the new Rayleigh range is: $z_0' = \frac{f_1^2}{z_0}$.

2. The propagation of the Gaussian beam over a distance of $d = f_1 + f_2$ leads to a parameter $q'' = f_2 + iz_0'$ just in front of the second lens. The lens effect is similar to part a) provided we substitute f_1 with f_2. So we obtain a parameter $q''' = -f_2 + i\frac{f_2^2}{z_0'}$. that means that the waist position is at $z = f_2$ after the lens and the new Rayleigh range is: $z_0'' = \frac{f_2^2}{z_0'}$.

3. Combining part a) and b) we find that $z_0'' = \frac{f_2^2}{f_1^2} z_0$. By substituting in the formula the definition of the Rayleigh range we obtain:

$$\frac{\pi W_0''^2}{\lambda} = \frac{f_2^2}{f_1^2} \frac{\pi W_0^2}{\lambda}$$
$$W_0''^2 = \frac{f_2^2}{f_1^2} W_0^2$$
$$W_0'' = \frac{f_2}{f_1} W_0$$

Problem 5 – Pulse propagation **3 + 3 + 2 = 8 points**

a) The propagation vector $k(\omega)$ is defined as:

$$k(\omega) = \frac{\omega}{c_0} n(\omega)$$

Therefore, the phase and group velocities are:

$$v_\mathrm{p} = \frac{\omega}{k(\omega)} = \frac{c_0}{n(\omega_0)} = \frac{c_0}{2 + 4 \times 10^{-2}}$$
$$v_\mathrm{p} = \frac{c_0}{2.04}$$
$$v_\mathrm{g} = \left[\left. \frac{\partial k(\omega)}{\partial \omega} \right|_{\omega_0} \right]^{-1} = \left. \frac{c_0}{(B + 3C\omega^2)} \right|_{\omega_0} = \frac{c_0}{2 + 0.12}$$
$$v_\mathrm{g} = \frac{c_0}{2.12}$$

b) We first calculate the dispersion coefficient

$$D = \left. \frac{\partial^2 k(\omega)}{\partial \omega^2} \right|_{\omega_0} = \frac{6C\omega_0}{c_0} = 4 \times 10^{-25} \frac{\mathrm{s}^2}{\mathrm{m}}$$
$$T_0 = 2/\omega_\mathrm{S} = 2\mathrm{ps}$$
$$z_0 = -\frac{T_0^2}{2D} = -5\mathrm{m}$$

Therefore, the pulse width is given as:

$$\text{Width} = T(l) = T_0 \sqrt{1 + \left(\frac{l}{z_0}\right)^2} = \sqrt{17} T_0 \approx 8.246\mathrm{ps}$$

c) Since there is no dispersion in this medium, the group velocity is the same as the phase velocity. Therefore,

$$v_{\mathrm{p}2} = \frac{c_0}{2}$$

and the time difference between the two is given as:

$$\Delta t = L \left| \frac{1}{v_{\mathrm{g}1}} - \frac{1}{v_{\mathrm{g}2}} \right| = \frac{20}{3 \times 10^8} |2.12 - 2| = 8\mathrm{ns}$$

Problem 6 – Fraunhofer diffraction **4 + 2 = 6 points**

1. The *Fresnel approximation* is a paraxial approximation that is used for high Fresnel numbers which is normally associated with the near-field. The *Fraunhofer approximation* is a paraxial approximation that is valid for low Fresnel numbers ($N_F \leq 0.1$) which is normally associated with the far-field.

2. The field in Fraunhofer approximation is proportional to the Fouriertransform of the initial field

$$u(x, z_{\mathrm{B}}) \propto U_0\left(\frac{kx}{z_{\mathrm{B}}}\right).$$

$$\begin{aligned}
U_0(\alpha) &\propto \int_{-\infty}^{\infty} u_0(x)\mathrm{e}^{-\mathrm{i}\alpha x}\mathrm{d}x \\
&= \int_{-a/2}^{a/2} \mathrm{e}^{-\mathrm{i}\alpha x}\mathrm{d}x \\
&= \frac{1}{-\mathrm{i}\alpha}\mathrm{e}^{-\mathrm{i}\alpha a/2} - \frac{1}{-\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\alpha a/2} \\
&= \frac{\sin(\alpha a/2)}{\alpha/2} \\
&= a\,\mathrm{sinc}(\alpha a/2)
\end{aligned}$$

So we get:

$$I(x, z_{\mathrm{B}}) = |u(x, z_{\mathrm{B}})|^2 \propto \mathrm{sinc}^2\left(\frac{kx}{2z_{\mathrm{B}}}a\right)$$