FoMO in a nutshell

Important things to memorize for the course »Fundamentals of Modern Optics«*

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Here, we collect important relations, facts, formulas and other things that one should memorize throughout the course to give you a summary of the most important results. Of course, we cannot list every required formula and we assume that you already have a solid math background. Hence, we cannot guarantee completeness of the list.

FOMO topics

Maxwell's equations

- Macroscopic Maxwell's equations (time domain):

$$\nabla \cdot \mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t) \qquad \nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0 \qquad \nabla \times \mathbf{H}(\mathbf{r},t) = \mathbf{j}(\mathbf{r},t) + \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t}$$

- Macroscopic Maxwell's equations (frequency domain):

$$\nabla \cdot \mathbf{D}(\mathbf{r},\omega) = \rho(\mathbf{r},\omega) \qquad \nabla \times \mathbf{E}(\mathbf{r},\omega) = i\omega \mathbf{B}(\mathbf{r},\omega)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},\omega) = 0 \qquad \nabla \times \mathbf{H}(\mathbf{r},\omega) = \mathbf{j}(\mathbf{r},\omega) - i\omega \mathbf{D}(\mathbf{r},\omega)$$

- Constitutive relations (linear material response; in optics usually $\mu(\mathbf{r},\omega) \equiv 1$ (non-magnetizable):

$$\mathbf{D}(\mathbf{r},\omega) = \varepsilon_0 \mathbf{E}(\mathbf{r},\omega) + \mathbf{P}(\mathbf{r},\omega) = \varepsilon_0 \varepsilon(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega) \qquad \mathbf{B}(\mathbf{r},\omega) = \mu_0 \mu(\mathbf{r},\omega)\mathbf{H}(\mathbf{r},\omega)$$

$$\mathbf{P}(\mathbf{r},\omega) = \varepsilon_0 \chi(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega)$$

- Time domain material response (response function):

$$\mathbf{P}(\mathbf{r},t) = \varepsilon_0 \int_{-\infty}^{t} R(\mathbf{r},t-t')\mathbf{E}(\mathbf{r},t)\,\mathrm{d}t, \text{ where } R(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi(\mathbf{r},\omega)e^{-i\omega t}\,\mathrm{d}\omega$$

- Complex permitivity:

conductivity

$$\varepsilon(\mathbf{r},\omega) = 1 + \chi(\mathbf{r},\omega) + i\,\frac{\sigma(\mathbf{r},\omega)}{\varepsilon_0 \omega}$$

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- Continuity equation (conservation of charge):

$$\nabla \cdot \mathbf{j}(\mathbf{r},t) + \frac{\partial \rho(\mathbf{r},t)}{\partial t} = 0 \qquad \underbrace{\oiint_{\partial V} \mathbf{j}\, d\mathbf{S}}_{I} = -\frac{\partial}{\partial t} \underbrace{\iiint_{V} \rho\, dV}_{Q}$$

- Time averaged Poynting vector, loss:

$$\langle \mathbf{S}(\mathbf{r}) \rangle = \frac{1}{2} \mathrm{Re}\left[\mathbf{E}(\mathbf{r},\omega) \times \mathbf{H}^*(\mathbf{r},\omega)\right] \qquad \nabla \cdot \langle \mathbf{S}(\mathbf{r}) \rangle < 0 \Leftrightarrow \text{system is lossy}$$

- Wave equations (vacuum):

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}(\mathbf{r},t)}{\partial t^2} = 0 \qquad \nabla \times \nabla \times \mathbf{H}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2 \mathbf{H}(\mathbf{r},t)}{\partial t^2} = 0$$

Normal modes in homogeneous, isotropic, non-magnetizable matter

- Helmholtz equation (wave equation in temporal Fourier domain; homogeneous, isotropic matter):

$$\Delta \mathbf{E}(\mathbf{r},\omega) + \frac{\omega^2}{c^2}\varepsilon(\omega)\mathbf{E}(\mathbf{r},t) = 0$$

- Plane waves are the eigenmodes of free space. They take the form

$$\mathbf{E}(\mathbf{r},\omega) = \mathbf{E}_0 e^{i\mathbf{kr}}$$

  and their dispersion relation reads as:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}\varepsilon(\omega)$$

- Refractive index:

$$n(\omega) = \frac{k(\omega)}{k_0} = \frac{k(\omega)}{\omega} c$$

- Maxwell relation:

  $n(\omega) = \sqrt{\varepsilon(\omega)}$ which implies for real and imaginary parts: $(n' + in'')^2 = n'^2 - n''^2 + 2in'n'' = \varepsilon' + i\varepsilon''$

- Finding electric or magnetic field from each other in frequency domain in regions without source:

$$\mathbf{H}(\mathbf{r},\omega) = -\frac{i}{\omega\mu_0}\nabla \times \mathbf{E}(\mathbf{r},\omega) \qquad \mathbf{E}(\mathbf{r},\omega) = \frac{i}{\omega\varepsilon_0\varepsilon(\mathbf{r},\omega)}\nabla \times \mathbf{H}(\mathbf{r},\omega)$$

- Propagating, lossy and evanescent waves for direction vector **u**:

$$\mathbf{u} \cdot \langle \mathbf{S} \rangle \neq 0, \nabla \cdot \langle \mathbf{S} \rangle = 0 \Leftrightarrow \text{lossless propagation}$$
$$\mathbf{u} \cdot \langle \mathbf{S} \rangle \neq 0, \nabla \cdot \langle \mathbf{S} \rangle < 0 \Leftrightarrow \text{lossy propagation}$$
$$\mathbf{u} \cdot \langle \mathbf{S} \rangle = 0, \nabla \cdot \langle \mathbf{S} \rangle = 0 \Leftrightarrow \text{evanescence}$$

Beam propagation ($k_x = \alpha, k_y = \beta$)

- Beam propagation scheme:

real space

z_0 z

$$u_0(x,y,z_0) \xrightarrow{\text{convolution with response function}} u(x,y,z)$$

$$u(x,y,z) = \iint h(x-x', y-y', z)\, u_0(x', y', z_0)\, \mathrm{d}x'\mathrm{d}y'$$

Fourier transform

$$U_0(\alpha,\beta,z_0) = \frac{1}{(2\pi)^2} \iint u_0(x,y,z_0)\, e^{-i\alpha x - i\beta y}\, \mathrm{d}x\mathrm{d}y$$

inverse Fourier transform

$$u(x,y,z) = \iint U(\alpha,\beta,z)\, e^{i\alpha x + i\beta y}\, \mathrm{d}\alpha\mathrm{d}\beta$$

z_0 z

$$U(\alpha,\beta,z) = U_0(\alpha,\beta,z_0)\cdot H(\alpha,\beta,z)$$

$$U_0(\alpha,\beta,z_0) \xrightarrow[\text{multiplication with transfer function}]{} U(\alpha,\beta,z)$$

transverse
Fourier space

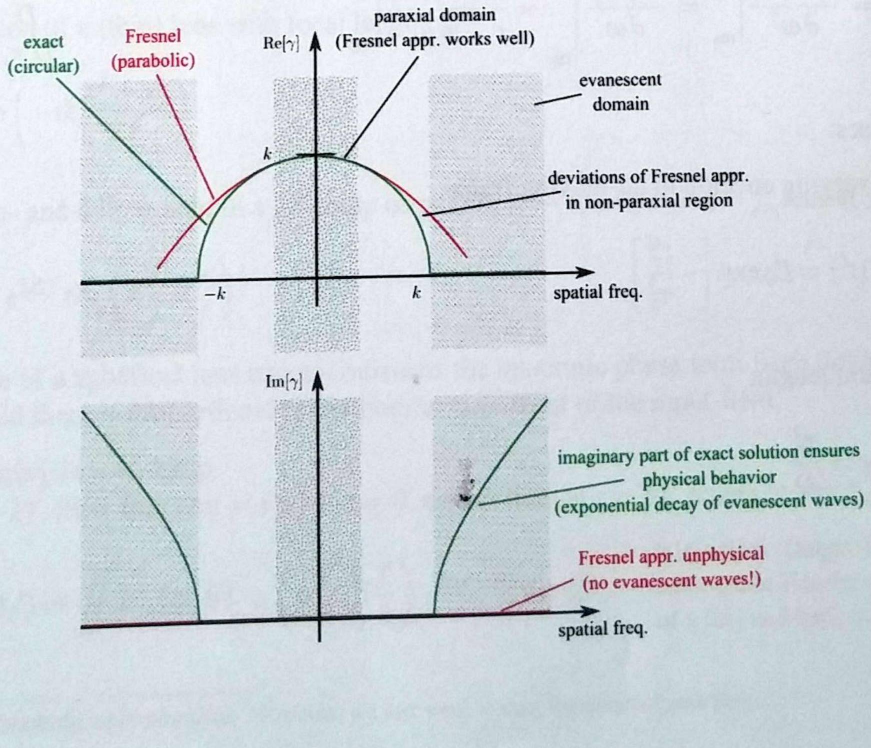
- Relation between transfer- and response function (comes with an extra $1/4\pi^2$ factor):

$$h(x,y,z) = \frac{1}{(2\pi)^2} \iint H(\alpha,\beta,z)\, e^{i\alpha x + i\beta y}\, \mathrm{d}\alpha\, \mathrm{d}\beta$$

- Transfer functions for homogeneous space:

$$H(\alpha,\beta,z) = \exp\left[i\gamma(\alpha,\beta)z\right] = \exp\left[i\sqrt{k^2-\alpha^2-\beta^2}\,z\right] \qquad \text{exact solution}$$

$$H_F(\alpha,\beta,z) = \exp\left[ikz\right]\exp\left[-i\frac{\alpha^2+\beta^2}{2k}z\right] \qquad \text{Fresnel approximation}$$



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Gaussian beams

- Rayleigh length:

$$z_0 = \frac{k}{2} w_0^2$$

- q-parameter definition:

$$\frac{1}{q(z)} = \frac{1}{R(z)} + i\frac{2}{kw^2(z)}$$

- q-parameter at waist position:

$$q_0 = -iz_0$$

- q-parameter after propagation:

$$q(d) = q(0) + d$$

- Condition for finding waist position:

$$\mathrm{Re}[q(z)] = 0$$

- ABCD matrix formalism:

$$\hat{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \hat{M}_{\text{tot}} = \hat{M}_n \cdot \ldots \hat{M}_3 \cdot \hat{M}_2 \cdot \hat{M}_1$$

$$q_{n+1} = \frac{Aq_n + B}{Cq_n + D}$$

Optical pulses

- Taylor expansion of the dispersion relation:

$$\begin{aligned} k(\omega) &= k_0 + \left.\frac{\partial k(\omega)}{\partial \omega}\right|_{\omega_0} (\omega - \omega_0) + \frac{1}{2} \left.\frac{\partial^2 k(\omega)}{\partial \omega^2}\right|_{\omega_0} (\omega - \omega_0)^2 \\ &= \frac{\omega_0}{c} + \frac{1}{v_g}(\omega - \omega_0) + \frac{D}{2}(\omega - \omega_0)^2 \end{aligned}$$

- Group velocity, group index, group velocity dispersion:

$$v_g(\omega) = \left[\left.\frac{\partial k(\omega)}{\partial \omega}\right|_{\omega_0} \right]^{-1}$$

(velocity, at which the pulse travels)

$$n_g(\omega) = \frac{c}{v_g} = c \cdot \left.\frac{\partial k(\omega)}{\partial \omega}\right|_{\omega_0}$$

$$D(\omega) = \left.\frac{\partial^2 k(\omega)}{\partial \omega^2}\right|_{\omega_0} = \left.\frac{\partial\left(\frac{1}{v_g}\right)}{\partial \omega}\right|_{\omega_0} = \frac{1}{c} \left.\frac{\partial n_g(\omega)}{\partial \omega}\right|_{\omega_0}$$

(measures change in group index,
$D > 0 \rightarrow$ red is faster,
$D < 0 \rightarrow$ blue is faster)

- Gaussian pulses:
  - Slowly varying envelop in co-moving frame:

$$E(t') = E_0 \exp\left[-\frac{t'^2}{\tau_0^2}\right]$$

  - Dispersion length:

$$L_D = \frac{\tau_0^2}{D}$$

Fraunhofer diffraction, Fourier Optics

Fraunhofer approximation, Far-field diffraction pattern

- Condition for validity: Fresnel approx. valid (paraxial)[1] + propagation to the far-field:

$$z^2 \gg x^2 + y^2 \qquad \text{or if you use the Fresnel number:} \qquad N_f = \frac{a^2}{\lambda z} < 0.1,$$

  where a is the aperture size or the beam diameter.

- Interaction with plane aperture masks in thin element approximation, complex transmission function $t(x,y)$:

$$u_+(x,y,z_0) = u_0(x,y,z_0) \cdot t(x,y)$$

- Far-field:

$$u(x,y,z) = \frac{(2\pi)^2}{i\lambda z} \exp\left[ik\left(z + \frac{x^2+y^2}{2z}\right)\right] U_+\left(k\frac{x}{z}, k\frac{y}{z}\right)$$

- Far-field intensity:

$$I(x,y,z) \sim \left|U_+\left(k\frac{x}{z}, k\frac{y}{z}\right)\right|^2$$

  The far-field intensity is proportional to the squared absolute value of the Fourier spectrum behind the aperture, evaluated at the spatial frequencies $\alpha = kx/z, \beta = ky/z$.

- Scheme for calculating a Fraunhofer diffraction pattern:
  1.) Multiply incident illumination $u_-(x,y,0)$ with transmission function $t(x,y)$.
  2.) Compute the Fourier transform $U_+(\alpha,\beta) = 1/(2\pi)^2 \iint u_+(x,y,0)\, e^{-i\alpha x - i\beta y}\, \mathrm{d}x\, \mathrm{d}y$.
  3.) Substitute the angular frequencies by $\alpha \to kx/z, \beta \to ky/z$ to get a function of (x,y).
  4.) Either take the absolute value squared $|U_+(kx/z, ky/z)|^2$ to get the principal intensity pattern (if asked) or multiply the prefactors from above to get the compelx field.

Lenses, 2f & 4f-setups

- Transmission function of a (thin) lens with focal length f:

$$t_L(x,y) = \exp\left(-ik\frac{x^2+y^2}{2f}\right)$$

- Relation between in- and output field in a $2f$-setup (input at $z=-f$, output at $z=f$, lens at $z=0$):

$$u_{\text{out}} = \frac{(2\pi)^2}{i\lambda f} e^{2ikf}\, U_{\text{in}}\left(k\frac{x}{f}, k\frac{y}{f}\right)$$

  The phase curvature of a spherical lens exactly balances the quadratic phase term from diffraction in the back-focal plane. The field there is proportional to the Fourier transfrom of the input-field.

- Fourier Optical filtering in a 4f-setup
  (input field at $z=-2f$, filter function $p(x,y)$ at $z=0$, output field at $z=2f$, lenses at $z=\pm f$):

$$u_{\text{out}}(-x,-y,4f) = \iint U_{\text{in}}(\alpha,\beta) \cdot p\left(\alpha\frac{f}{k}, \beta\frac{f}{k}\right) e^{i\alpha x + i\beta y}\, \mathrm{d}\alpha\, \mathrm{d}\beta$$

  Attention: Output is inverted due to 2 subsequent Fourier transforms (instead of a for- and backward transform)!

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[1] There exists a non-paraxial Fraunhofer approximation. However, we just want to treat the paraxial case here.

- Scheme for computing Fourier Optical Filtering (formulated using a backward transform):

  1.) Compute the Fourier Transform of the input field as function of (α,β):

$$U_{\text{in}}(\alpha,\beta)=\frac{1}{(2\pi)^2}\iint u_{\text{in}}(x,y)\,e^{-i\alpha x-i\alpha y}\,\mathrm{d}x\,\mathrm{d}y$$

  2.) Make a change of variables (**no** transform!) in the filter function $p(x,y)$, to make it a function of (α,β):

$$x\to\alpha\frac{f}{k},\quad y\to\beta\frac{f}{k}$$

  3.) Multiply them and compute the Fourier back-transform as given above

$$\iint U_{\text{in}}(\alpha,\beta)\cdot p\left(\alpha\frac{f}{k},\beta\frac{f}{k}\right)\,e^{i\alpha x+i\beta y}\,\mathrm{d}\alpha\,\mathrm{d}\beta$$

  4.) Change the sign of every x and y to get the resulting output field:

$$x\to-x,\quad y\to-y$$

Polarization, anisotropic media

Jones formalism

- For plane waves, every possible polarization state can be described as the superposition of two orthogonal modes. The Jones formalism uses a linear basis. The strength and relative phase of the two pol. components is described by a Jones vector

$$\mathbf{J}=\begin{pmatrix}E_x\\E_y\end{pmatrix}$$

- Possible Jones vectors:

  $\begin{pmatrix}1\\0\end{pmatrix}$ (hor. linear) $\quad$ $\begin{pmatrix}0\\1\end{pmatrix}$ (vert. linear) $\quad$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ (diagonal linear)

  $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$ (left-h. circular) $\quad$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$ (right-h. circular)

  (otherwise) $\to$ left- or right-h. elliptical

- By passing through a polarization-sensitive device, the state of polarization may change. This can be described with a matrix formalism which acts on the Jones vector:

$$\mathbf{J}_{\text{out}}=\hat{\mathbf{M}}\cdot\mathbf{J}_{\text{in}}$$

- The action of many elements is described by matrix multiplication (first element *last*!):

$$\hat{\mathbf{M}}=\hat{\mathbf{M}}_n\cdot\ldots\hat{\mathbf{M}}_2\cdot\hat{\mathbf{M}}_1$$

- Some simple Jones matrices:

  $\begin{pmatrix}1&0\\0&0\end{pmatrix}$ hor. linear polarizer $\quad$ $\begin{pmatrix}0&0\\0&1\end{pmatrix}$ vert. linear polarizer

- Find polarization eigenstates of system $\hat{\mathbf{M}}\to$ solve eigenvalue problem $\hat{\mathbf{M}}\cdot\mathbf{J}_{\text{eig}}=\lambda\mathbf{J}_{\text{eig}}$ for λ and $\mathbf{J}_{\text{eig}}$.

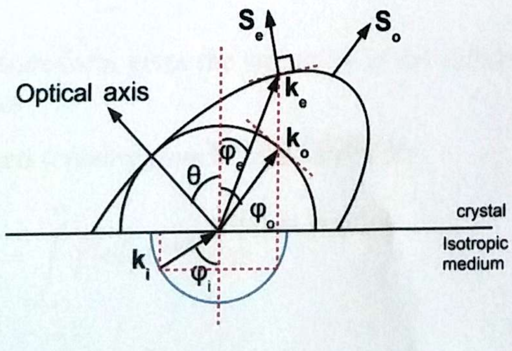
  → Pass through the system without change.

Optics of crystals

- In a crystal, optical properties are not isotropic any more → tensor replaces dielectric function

$$\mathbf{D}(\mathbf{r},\omega) = \varepsilon_0 \hat{\boldsymbol{\varepsilon}}(\mathbf{r},\omega) \cdot \mathbf{E}(\mathbf{r},\omega) \qquad \hat{\boldsymbol{\varepsilon}} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$$

- In dependence of the propagation direction, **D** and **E** are thus not automatically parallel.
- Consequences for Poynting vector, since always: $\mathbf{D} \perp \mathbf{k}$ and $\mathbf{S} \perp \mathbf{E}$, i.e. **S** will not always be parallel to **k** now!
- Most important, the dispersion relation will become dependent on direction and polarization now. There are always two linear independent eigenmodes for every direction vector **u** → need to find their polarization direction and refractive indices.
- Classification:
  - $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 \Longrightarrow$ isotropic
  - $\varepsilon_1 = \varepsilon_2 = \varepsilon_o \neq \varepsilon_3 = \varepsilon_e \Longrightarrow$ uniaxial ("optical axis" is the direction of ε_e in the index ellipsoid, since the dispersion relation in the plane normal to that direction is always circle, irrespective of the direction of **E**), "ordinary" and "extraordinary", possible cases:
    - ∗ $\mathbf{u} \parallel$ opt. ax. $\Longrightarrow$ "boring case"
      Both eigenmodes experience n_o (degeneracy) and their polarization direction is arbitrary in a plane with $\mathbf{n} \parallel \mathbf{u}$. The crystal behaves like an "ordinary" isotropic medium with refractive index n_0. Nothing happens to any arbitrary input state of polarization.
    - ∗ $\mathbf{u} \perp$ opt. ax. $\Longrightarrow$ "waveplate case"
      There are two non-degenerate eigenmodes – one with refractive index n_o (polarized $\perp$ to the opt. ax. and $\perp \mathbf{u}$) and one with n_e (polarized $\parallel$ to the opt. ax. and $\perp \mathbf{u}$). For both, $\mathbf{S} \parallel \mathbf{k}$ holds (*no* double refraction). An arbitrary linear input polarization **E** will have an angle α between electrical field vector and optical axis and must be decomposed into the two eigenpolarizations. After a propagation length d, the two components will have a phase shift which determines the polarization state (Jones vector) at the output. Important cases are:
      - · $d = \dfrac{\lambda}{2(n_e - n_o)}$ → "half-wave plate", rotates linear polarization by an angle 2α
      - · $d = \dfrac{\lambda}{4(n_e - n_o)}$ and $\alpha = 45°$ → if not we will get elliptical pol → "quarter-wave plate", transfers linear to circular polarization
      - · all other cases lead to elliptical polarization, except for $\alpha = 0$ or $\alpha = 90°$ (only eigenmode excited), in which the polarization stays unchanged irrespective of d.
    - ∗ $\angle(\mathbf{u}$, opt. ax.$) = \theta \neq 0°, 90°$ → "double refraction case"



  - $\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3 \Longrightarrow$ biaxial (the two optical axes do not coincide with one of the crystal axes → very complicated!)

Interfaces

- Continuity of tangential field components:

$$\iint_A \nabla \times \mathbf{E}(\mathbf{r},\omega)\,\mathrm{d}\mathbf{A} \overset{\text{Stokes}}{=} \oint_{\partial A} \mathbf{E}\,\mathrm{d}\mathbf{r} \overset{A\to 0}{=} \mathbf{n}\times(\mathbf{E}_2-\mathbf{E}_1)=0 \qquad \Rightarrow \mathbf{n}\times\mathbf{E}_1=\mathbf{n}\times\mathbf{E}_2$$

  (analog for **H**)

- Continuity of tangential **k**-components:
  follows directly from reflection/transmission problem (all fields need to be in phase at every point)
- Law of reflection: $\theta=\theta'$
  follows directly from continuity of $\mathbf{n}\times\mathbf{k}$
- Law of refraction: $n_1\sin\theta_1=n_2\sin\theta_2$
  follows also directly from continuity of $\mathbf{n}\times\mathbf{k}$
- Fresnel equations (reflection and transmission coefficients at interface, follow from continuity of field components):

$$r_{\mathrm{TE}}=\frac{n_1\cos\theta_1-n_2\cos\theta_2}{n_1\cos\theta_1+n_2\cos\theta_2} \qquad t_{\mathrm{TE}}=\frac{2n_1\cos\theta_1}{n_1\cos\theta_1+n_2\cos\theta_2}$$

$$r_{\mathrm{TM}}=\frac{n_2\cos\theta_1-n_1\cos\theta_2}{n_2\cos\theta_1+n_1\cos\theta_2} \qquad t_{\mathrm{TM}}=\frac{2n_1\cos\theta_1}{n_2\cos\theta_1+n_1\cos\theta_2}$$

- Critical angle (total internal reflection): $\sin\theta_c=n_1/n_2$.
- Brewster angle: $\tan\theta_b=n_2/n_1$
- Reflectivity and Transmissivity (energy):

$$R=|r|^2 \qquad T=\frac{\mathrm{Re}(\mathbf{n}\cdot\mathbf{k}_2)}{\mathbf{n}\cdot\mathbf{k}_1}|t|^2$$

Mathematical tools

Miscellaneous math formulas

- Complex exponentials, trigonometric and hyperbolic functions:

$$e^{ix}=\cos x+i\sin x$$

$$\cos x=\frac{1}{2}\left(e^{ix}+e^{-ix}\right)$$

$$\sin x=\frac{1}{2i}\left(e^{ix}-e^{-ix}\right)$$

$$\cosh x=\frac{1}{2}\left(e^{x}+e^{-x}\right) \qquad \cosh(ix)=\cos x$$

$$\sinh x=\frac{1}{2}\left(e^{x}-e^{-x}\right) \qquad \sinh(ix)=i\sin x$$

- Integration:

$$\int u(x)\frac{\mathrm{d}v(x)}{\mathrm{d}x}\,\mathrm{d}x=u(x)v(x)-\int\frac{\mathrm{d}u(x)}{\mathrm{d}x}v(x)\,\mathrm{d}x \qquad \text{(partial integration)}$$

$$\int f(x)\,\mathrm{d}x=\int f(\xi)\frac{\mathrm{d}x}{\mathrm{d}\xi}\,\mathrm{d}\xi \qquad \text{(substitution)}$$

- Gaussian functions:

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$$

$$\mathrm{FT}\left\{A\exp\left[-\frac{1}{2}\frac{t^2}{t_0^2}\right]\right\} = \frac{At_0}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\frac{\omega^2}{(1/t_0)^2}\right]$$

The Fourier transform of a Gaussian function is a Gaussian function.

- Area element in radial coordinates:

$$\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\varphi$$

Field theory

- Vector identities:

$$\nabla\times\nabla\times\mathbf{a} = \nabla\cdot(\nabla\mathbf{a}) - \Delta\mathbf{a}$$
$$\Delta a = \nabla(\nabla a),\ \text{but}\ \Delta\mathbf{a} = (\nabla\cdot\nabla)\mathbf{a} = \nabla^2\mathbf{a} \neq \nabla(\nabla\mathbf{a})$$
$$\mathbf{a}\times\mathbf{b}\times\mathbf{c} = \mathbf{b}(\mathbf{ac}) - \mathbf{c}(\mathbf{ab})$$

- Integral theorems:

$$\iiint_V \nabla\cdot\mathbf{a}\,\mathrm{d}V = \oiint_{\partial V} \mathbf{a}\,\mathrm{d}\mathbf{S} \qquad \text{(Gauss)}$$
$$\iint_A \nabla\times\mathbf{a}\,\mathrm{d}\mathbf{S} = \oint_{\partial A} \mathbf{a}\,\mathrm{d}\mathbf{r} \qquad \text{(Stokes)}$$

Fourier transform, δ-function

- In the course, we define the one-dimensional Fourier transform as (these definitions regarding sign and prefactor conventions influence nearly every expression in this document that contains Fourier transforms):

Forward (*going to* Fourier domain):

$$\tilde{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,e^{i\omega t}\,\mathrm{d}t \qquad \text{(temporal Fourier domain)}$$

$$\tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,\mathrm{d}x \qquad \text{(spatial Fourier domain)}$$

A Fourier transform gives the strengths of the different plane wave (space) or time harmonic (time) frequency components.

Backward (*coming from* Fourier domain):

$$f(t) = \int_{-\infty}^{\infty} \tilde{f}(\omega)\,e^{-i\omega t}\,\mathrm{d}\omega \qquad \text{(temporal Fourier domain)}$$

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(k)\,e^{ikx}\,\mathrm{d}k \qquad \text{(spatial Fourier domain)}$$

An inverse Fourier transform represents a decomposition of a function into plane wave components (space) or harmonic oscillations (time).

- Fourier shifting theorem:

$$\mathrm{FT}\{f(t-t_0)\} = e^{i\omega t_0}\tilde{f}(\omega) \qquad \mathrm{FT}^{-1}\{\tilde{f}(\omega-\omega_0)\} = e^{-i\omega_0 t}f(t)$$
$$\mathrm{FT}\{f(x-x_0)\} = e^{-ikx_0}\tilde{f}(k) \qquad \mathrm{FT}^{-1}\{\tilde{f}(k-k_0)\} = e^{ik_0 x}f(x)$$

The shifting of a function corresponds to a harmonic modulation in Fourier domain.

- Fourier transform of a derivative (I cannot recall an occasion where the backward relations would be needed, so I leave them out):

$$\mathrm{FT}\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right\} = -i\omega\tilde{f}(\omega)$$
$$\mathrm{FT}\left\{\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right\} = ik\tilde{f}(k)$$

A derivative in the real domain corresponds to a simple multiplication in Fourier domain.

- δ-function:

$$\delta(x-x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_0)}\,\mathrm{d}k$$

The δ-function is the Fourier transform of a plane wave (space) or harmonic oscillation (time).

$$f(x_0) = \int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,\mathrm{d}x$$

The δ-function »picks out« the value of a function at a particular point. It is just meaningful under an integral.

- Fourier transform of the δ-function:

$$\mathrm{FT}\{\delta(t-t_0)\} = \frac{1}{2\pi}e^{i\omega t_0} \qquad \mathrm{FT}^{-1}\{\delta(\omega-\omega_0)\} = e^{-i\omega_0 t}$$
$$\mathrm{FT}\{\delta(x-x_0)\} = \frac{1}{2\pi}e^{-ikx_0} \qquad \mathrm{FT}^{-1}\{\delta(k-k_0)\} = e^{ik_0 x}$$

The Fourier transform of a δ-function is a plane wave (space) of harmonic oscillation (time)

- Convolution:

$$[f\otimes g](t) = \int f(\tau)g(t-\tau)\,\mathrm{d}t \qquad [f\otimes g](x) = \int f(x')g(x-x')\,\mathrm{d}x$$
$$\mathrm{FT}\{[f\otimes g](t)\} = 2\pi\tilde{f}(\omega)\tilde{g}(\omega) \qquad \mathrm{FT}\{[f\otimes g](x)\} = 2\pi\tilde{f}(k)\tilde{g}(k)$$

Convolution in the real domain corresponds to a simple multiplication in the Fourier domain.