A laser at $\lambda_0 = 1000$ nm emits a beam with Gaussian profile and waist radius $W_0 = 2$ mm.

a) Use the matrix approach to calculate the waist position z' and waist radius W'_0 of the beam after the lens. Approximate the result assuming that the waist of the incoming beam is on the lens and that its Rayleigh range $z_0 = \frac{\pi W_0^2}{\lambda_0}$ is much longer than the focal length f.

b) Choose a lens such that the waist radius of the focused beam is equal to $W'_0 = 50\mu m$.

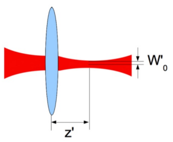


Figure 1: Sketch of the focusing arrangement.

$q = z - iz_0$ $\quad q = \frac{Aq_0+B}{Cq_0+C}$ $\quad z_0 = \frac{\pi W_0^2}{\lambda}$

(a) $M_1 = \begin{bmatrix}1&0\\-1/f&1\end{bmatrix}$ $M_2 = \begin{bmatrix}1&z'\\0&1\end{bmatrix}$ $\Rightarrow q = \frac{q_0}{-\frac{1}{f}q_0+1} + z' = \frac{fq_0}{-q_0+f} = \frac{-ifz_0}{iz_0+f}$

$\Rightarrow q = \frac{if(z_0)(iz_0-f)}{f^2+z_0^2} + z' = z' - \frac{fz_0^2}{f^2+z_0^2} - i\frac{f^2z_0}{f^2+z_0^2} \approx z' - f - i\frac{f^2}{z_0}$

Thus $z' = f$ $\quad z_1 = \frac{f^2}{z_0} = \frac{\lambda f^2}{\pi W_0^2} = \frac{\pi W_0'^2}{\lambda} \Rightarrow W_0' = \frac{\lambda f}{\pi W_0}$

(b) $W_0' = \frac{1\mu m \cdot f}{3.14 \cdot 2000\mu m} = 50\mu m \Rightarrow f = \frac{100000}{3.14}\mu m \approx 31847\ \mu m$

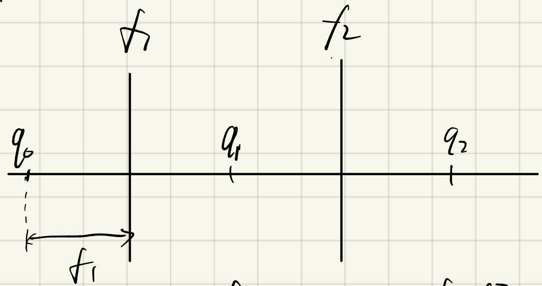
A lens of focal length f_1 is placed at a distance $d = f_1$ from the waist of a Gaussian beam.

a) Use the ABCD formalism to find the position of the waist and the Rayleigh range of the gaussian beam after the lens.

A second lens of focal length f_2 is placed after the first one at a distance $d_2 = f_1 + f_2$.

b) calculate the position of the waist of the Gaussian beam after the second lens.

c) calculate the waist radius after the second lens as a function of the waist radius W_0 of the initial beam and the focal lengths f_1 and f_2.



$q_0 + f_1$ $\quad \begin{bmatrix}1&0\\-1/f_1&1\end{bmatrix} \Rightarrow \frac{q_0+f_1}{-\frac{1}{f_1}(q_0+f_1)+1} \Rightarrow z_1 + \frac{q_0+f_1}{-\frac{1}{f_1}(q_0+f_1)+1} = z_1 + \frac{f_1q_0+f_1^2}{-q_0-f_1+f_1} = z_1 + \frac{-iz_0f_1+f_1^2}{iz_0}$

$\Rightarrow q_1 = z_1 - f_1 - i\frac{f_1^2}{z_0} \Rightarrow z_1 = f_1 \quad z_0' = \frac{f_1^2}{z_0} = \frac{\lambda f_1^2}{\pi W_0^2}$

$M_1 = \begin{bmatrix}1&f_1\\0&1\end{bmatrix}$ $M_2 = \begin{bmatrix}1&0\\-\frac{1}{f_1}&1\end{bmatrix}$ $M_3 = \begin{bmatrix}1&z\\0&1\end{bmatrix}$ $M_2M_1 = \begin{bmatrix}1&0\\-\frac{1}{f_1}&1\end{bmatrix}\begin{bmatrix}1&f_1\\0&1\end{bmatrix} = \begin{bmatrix}1&f_1\\-\frac{1}{f_1}&0\end{bmatrix}$

$M_3M_2M_1 = \begin{bmatrix}1&z\\0&1\end{bmatrix}\begin{bmatrix}1&f_1\\-\frac{1}{f_1}&0\end{bmatrix} = \begin{bmatrix}1-\frac{z}{f_1}&f_1\\-\frac{1}{f_1}&0\end{bmatrix} \Rightarrow q_1 = \frac{(1-\frac{z}{f_1})q_0+f_1}{-\frac{1}{f_1}q_0} = \frac{(f_1-z)q_0+f_1^2}{-q_0}$

$\Rightarrow q_1 = \frac{(z-f_1)iz_0+f_1^2}{iz_0} = z - f_1 - \frac{if_1^2}{z_0} = -iz_1 \Rightarrow z_1 = \frac{f_1^2}{z_0}$

(b) $z = f_1 \Rightarrow M_3M_2M_1 = \begin{bmatrix}0&f_1\\-\frac{1}{f_1}&0\end{bmatrix}$ $M_4 = \begin{bmatrix}1&f_2\\0&1\end{bmatrix}$ $M_5 = \begin{bmatrix}1&0\\-\frac{1}{f_2}&1\end{bmatrix}$ $M_6 = \begin{bmatrix}1&z_2\\0&1\end{bmatrix}$

$M_6M_5M_4 = \begin{bmatrix}1-\frac{z_2}{f_2}&f_2\\-\frac{1}{f_2}&0\end{bmatrix}$ $\quad M = M_6M_5M_4M_3M_2M_1 = \begin{bmatrix}1-\frac{z_2}{f_2}&f_2\\-\frac{1}{f_2}&0\end{bmatrix}\begin{bmatrix}0&f_1\\-\frac{1}{f_1}&0\end{bmatrix} = \begin{bmatrix}-\frac{f_2}{f_1}&f_1(1-\frac{z_2}{f_2})\\0&-\frac{f_1}{f_2}\end{bmatrix}$

$q_2 = \frac{-\frac{f_2}{f_1}q_0 + f_1(1-\frac{z_2}{f_2})}{-\frac{f_1}{f_2}} = -\frac{f_2^2}{f_1^2}iz_0 - f_2(1-\frac{z_2}{f_2}) = z_2 - f_2 - i\frac{f_2^2}{f_1^2}z_0 \Rightarrow z_2 = f_2$

$\frac{f_2^2}{f_1^2}\frac{\pi W_0^2}{\lambda} = \frac{\pi W^2}{\lambda} \Rightarrow W = \frac{f_2}{f_1}W_0$

$q_2 = z_2 + \frac{q_1+f_2}{-\frac{1}{f_2}(q_1+f_2)+1} = z_2 + \frac{f_2(f_2-iz_1)}{iz_1} = z_2 - f_2 - i\frac{f_2^2}{z_1} \Rightarrow z_2 = f_2$

$z'' = \frac{\pi W_0''^2}{\lambda} = \frac{f_2^2}{f_1^2}z_0 = \frac{f_2^2}{f_1^2}\frac{\pi W_0^2}{\lambda} \Rightarrow W_0'' = \frac{f_2}{f_1}W_0$

b) Show that the distances of the waists (focii) of the incident and transmitted beams, D_1 and D_2, respectively, to the lens are related by

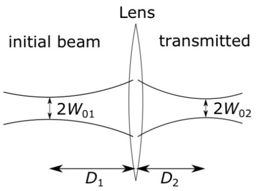
$$\frac{D_2}{f} = \frac{D_1/f - 1}{(D_1/f-1)^2 + (z_1/f)^2} + 1 \quad .$$

Use the q-parameter method.

c) Find the dependency of the waist W_{02} on λ, f, z_1, and D_1.

d) We now want to make the location of the new waist W_{02} as distant as possible from the lens, i.e., we want to maximize D_2. For a given ratio z_1/f, show that the optimal value of D_1 is $D_1 = f + z_1$.

e) For this optimal case, determine the values of the distances D_2, the waist W_{02} of the transmitted beam and the corresponding magnification $M = W_{02}/W_{01}$ depending on λ, f, and z_1.



Gaussian Beam

$$V(x,y,z) = A(z)\exp\left(-\frac{x^2+y^2}{W(z)^2}\right)\exp\left[\frac{ik}{2}\frac{x^2+y^2}{R(z)}\right]e^{i\varphi(z)}$$

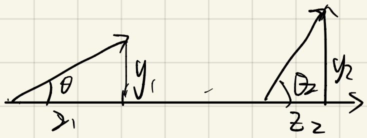
$$W^2(z) = W_0^2\left(1+\frac{z^2}{z_0^2}\right) \qquad R(z) = z\left(1+\frac{z_0^2}{z^2}\right)$$

$$q = z - iz_0 \qquad \frac{1}{q} = \frac{1}{z-iz_0} = \frac{z+iz_0}{z^2+z_0^2}$$

$$\Rightarrow \frac{1}{q} = \frac{z}{z^2+z_0^2} + i\frac{z_0}{z^2+z_0^2} = \frac{1}{z\left(1+\frac{z_0^2}{z^2}\right)} + i\frac{1}{z_0\left(1+\frac{z^2}{z_0^2}\right)}$$

$$= \frac{1}{R(z)} + i\frac{\lambda}{\pi W(z)^2}$$

If $M = \begin{bmatrix}A & B\\ C & D\end{bmatrix}$ $\quad q_1 = \frac{Aq_0+B}{Cq_0+D}$



$y_1 = z_1\tan\theta_1$ $\quad y_2 = z_2\tan\theta_2$ $\quad$ Paraxial Approximation: $z_1 = \frac{y_1}{\theta_1}$ $\quad z_2 = \frac{y_2}{\theta_2}$

$$\frac{y_2}{\theta_2} = \begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}y_1\\ \theta_1\end{bmatrix} = \frac{Ay_1+B\theta_1}{Cy_1+D\theta_1} = \frac{A\frac{y_1}{\theta_1}+B}{C\frac{y_1}{\theta_1}+D} = \frac{Az_1+B}{Cz_1+D} = z_2$$

(b) $M = \begin{bmatrix}1 & 0\\ -\frac{1}{f} & 1\end{bmatrix}$

$$q = D_2 + \frac{q_0+D_1}{-\frac{1}{f}(q_0+D_1)+1} = D_2 + \frac{f(D_1-iz_1)}{f-D_1+iz_1} = D_2 + \frac{f(D_1-iz_1)(f-D_1-iz_1)}{(f-D_1)^2+z_1^2}$$

$$\Rightarrow q = D_2 + \frac{fD_1(f-D_1) - ifD_1z_1 - i(f-D_1)z_1 - fz_1^2}{(f-D_1)^2+z_1^2} = D_2 + \frac{fD_1(f-D_1)-fz_1^2}{(f-D_1)^2+z_1^2} - i\frac{fD_1z_1+(f-D_1)z_1}{(f-D_1)^2+z_1^2}$$

$$D_2 + \frac{fD_1(f-D_1)-fz_1^2}{(f-D_1)^2+z_1^2} = 0 \Rightarrow \frac{D_2}{f} = \frac{z_1^2-D_1(f-D_1)}{(f-D_1)^2+z_1^2} = 1 + \frac{D_1(D_1-f)-(D_1-f)^2}{(f-D_1)^2+z_1^2} = \frac{(D_1-f)f}{(f-D_1)^2+z_1^2}+1$$

$$\Rightarrow \frac{D_2}{f} = \frac{\left(\frac{D_1}{f}-1\right)}{\left(\frac{D_1}{f}-1\right)^2+\left(\frac{z_1}{f}\right)^2}$$

(c)

$$\frac{fD_1z_1+(f-D_1)z_1}{(f-D_1)^2+z_1^2} = z_2 \Rightarrow \frac{fD_1+f-D_1}{(f-D_1)^2+z_1^2}\cdot\frac{\pi W_{01}^2}{\lambda} = \frac{\pi W_{02}^2}{\lambda} \Rightarrow W_{02} = W_{01}\sqrt{\frac{fD_1+f-D_1}{(f-D_1)^2+z_1^2}}$$

$$\frac{(fD_1+f-D_1)z_1}{(f-D_1)^2+z_1^2} = \frac{\pi W_{02}^2}{\lambda} \Rightarrow W_{02} = \sqrt{\frac{(fD_1+f-D_1)z_1}{(f-D_1)^2+z_1^2}\cdot\frac{\lambda}{\pi}}$$

(d)

$$D_2 = \frac{f\left(\frac{D_1}{f}-1\right)}{\left(\frac{D_1}{f}-1\right)^2+\left(\frac{z_1}{f}\right)^2} \qquad \left(\frac{D_1}{f}-1\right) = x \qquad \left(\frac{z_1}{f}\right)^2 = a \Rightarrow D_2 = \frac{fx}{x^2+a} \qquad D_2' = \frac{f(x^2+a)-fx\cdot(2x)}{(x^2+a)^2}$$

$$D_2' = 0 \Rightarrow f(x^2+a)-2fx^2 = 0 \Rightarrow x^2 = a \Rightarrow x = \pm\sqrt{a} = \pm\frac{z_1}{f}$$

Maximal $D_2 \Rightarrow x = \frac{z_1}{f}$

$$\frac{D_1}{f}-1 = \frac{z_1}{f} \Rightarrow \underline{D_1 = f+z_1}$$

(e)

$$D_2 = \frac{D_1-f}{\frac{1}{f^2}(D_1-f)^2+\left(\frac{z_1}{f}\right)^2} = \frac{z_1}{\frac{z_1^2}{f^2}+\frac{z_1^2}{f^2}} = \frac{f^2}{2z_1}$$

$$W_{02} = \sqrt{\frac{(fD_1+f-D_1)z_1}{(f-D_1)^2+z_1^2}\cdot\frac{\lambda}{\pi}} = \sqrt{\frac{f(f+z_1)-z_1}{2z_1}\cdot\frac{\lambda}{\pi}}$$

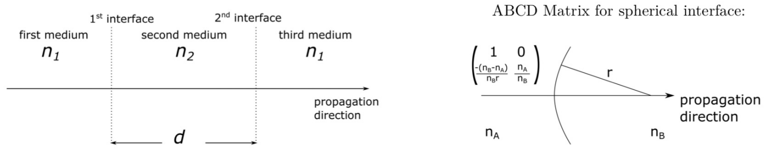
$$M = \frac{W_{02}}{W_{01}} = \sqrt{\frac{fD_1+f-D_1}{(f-D_1)^2+z_1^2}} = \sqrt{\frac{f(f+z_1)-z_1}{2z_1^2}}$$

Consider a Gaussian beam to propagate from a first homogeneous (refractive index n_1) through a second (n_2) into a third medium (again n_1) with $n_2 > n_1$ (see Figure). Both interfaces can be treated with the matrix method and each of them appears like a spherical interface (first interface radius $R_{int} > 0$, second interface radius $-R_{int}$). The length between the two interfaces is d.

b) Assume that propagation starts directly before the first and ends directly behind the second interface. Calculate the q-parameter of the Gaussian beam after propagation through this system. The q-parameter before the first interface is q_1.

Now, consider d to be small enough to be neglected ($d = 0$). Before reaching the first interface, the beam has propagated the distance L_1 from its waist position.

c) After which distance L_2 from the second interface does the beam exhibit a waist again?



$$M_1 = \begin{bmatrix} 1 & 0 \\ -\frac{(n_2-n_1)}{n_2 r} & \frac{n_1}{n_2} \end{bmatrix} \quad M_2 = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \quad M_3 = \begin{bmatrix} 1 & 0 \\ \frac{(n_1-n_2)}{n_1 r} & \frac{n_2}{n_1} \end{bmatrix}$$

$$M_2 M_1 = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{(n_2-n_1)}{n_2 r} & \frac{n_1}{n_2} \end{bmatrix} = \begin{bmatrix} 1-\frac{d(n_2-n_1)}{n_2 r} & \frac{d n_1}{n_2} \\ -\frac{(n_2-n_1)}{n_2 r} & \frac{n_1}{n_2} \end{bmatrix}$$

$$M_3 M_2 M_1 = \begin{bmatrix} 1 & 0 \\ -\frac{(n_2-n_1)}{n_1 r} & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1-\frac{d(n_2-n_1)}{n_2 r} & \frac{d n_1}{n_2} \\ -\frac{(n_2-n_1)}{n_2 r} & \frac{n_1}{n_2} \end{bmatrix} = \begin{bmatrix} 1-\frac{d(n_2-n_1)}{n_2 r} & \frac{d n_1}{n_2} \\ \frac{d(n_2-n_1)^2}{n_1 n_2 r^2} - \frac{2(n_2-n_1)}{n_1 r} & 1-\frac{(n_2-n_1)d}{n_2 r} \end{bmatrix}$$

$$q = \frac{\left[1-\frac{d(n_2-n_1)}{n_2 r}\right] q_1 + \frac{d n_1}{n_2}}{\left[\frac{d(n_2-n_1)^2}{n_1 n_2 r^2} - \frac{2(n_2-n_1)}{n_1 r}\right] q_1 + 1 - \frac{(n_2-n_1)d}{n_2 r}}$$

(c) $$M = M_3 M_1 = \begin{bmatrix} 1 & 0 \\ \frac{(n_1-n_2)}{n_1 r} & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{n_2-n_1}{n_2 r} & \frac{n_1}{n_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2(n_1-n_2)}{n_1 r} & 1 \end{bmatrix}$$

$$q = \frac{q_0}{\frac{2(n_1-n_2)}{n_1 r} q_0 + 1} + L_2 = \frac{i z_0}{1 - 2i\frac{(n_1-n_2)}{n_1 r} z_0} + L_2 = \frac{i z_0\left[1 + 2i\frac{(n_1-n_2)}{n_1 r} z_0\right]}{1 + 4\frac{(n_1-n_2)^2 z_0^2}{n_1^2 r^2}} + L_2$$

$$= L_2 - \frac{\frac{2(n_1-n_2) z_0^2}{n_1 r}}{1 + 4\frac{(n_1-n_2)^2 z_0^2}{n_1^2 r^2}} = L_2 - \frac{2 n_1 r (n_1-n_2) z_0^2}{n_1^2 r^2 + 4(n_1-n_2)^2 z_0^2} \Rightarrow L_2 = \frac{2 n_1 r (n_1-n_2) z_0^2}{n_1^2 r^2 + 4(n_1-n_2)^2 z_0^2}$$

The evolution of a Gaussian beam in paraxial approximation is given as

$$v(x, y, z) = A_0 \frac{1}{1 + iz/z_0} exp\left(-\frac{x^2+y^2}{w_0^2(1 + iz/z_0)}\right).$$

a) Restructure the provided equation and give explicit expressions for the following terms as functions of z:
   i) Amplitude evolution
   ii) Beam width evolution
   iii) Radius of phase curvature
   iiii) The Gouy phase

b) Sketch three diagrams for the evolution of the normalized beam intensity, beam width and radius of phase curvature over propagation distance z and indicate the values of the quantities at the propagation distances $z = -z_0$, $z = 0$ and $z = z_0$.

c) Plot the lines of constant phase of a Gaussian beam in a diagram over x and z. The diagram only needs to cover the region $-z_0 \le z \le z_0$.
   What happens for $z \gg z_0$?

d) Explain in your own words how to derive the stability condition for the fundamental gaussian mode in a resonator.

$$V(x,y,z) = A(z) \exp\left[-\frac{x^2+y^2}{w(z)^2}\right) \exp\left[\frac{ik}{2}\frac{x^2+y^2}{R(z)}\right] e^{i\varphi(z)}$$

$$z_0 = \frac{\pi w_0^2}{\lambda} = \frac{k w_0^2}{2}$$

(a) $$\frac{1}{1+i\frac{z}{z_0}} = \frac{1 - i\frac{z}{z_0}}{1+\frac{z^2}{z_0^2}} = \frac{1}{1+\frac{z^2}{z_0^2}} - i\frac{1}{\frac{z_0}{z} + \frac{z}{z_0}} = \frac{1}{1+\frac{z^2}{z_0^2}} - i\frac{1}{\frac{z}{z_0}\left(1+\frac{z_0^2}{z^2}\right)}$$

$$= \frac{1}{1+\frac{z^2}{z_0^2}} - i\frac{k w_0^2}{2z\left(1+\frac{z_0^2}{z^2}\right)}$$

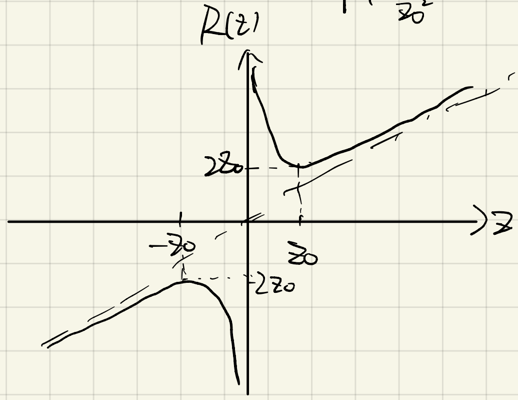
$$-\frac{(x^2+y^2)}{w_0^2(1+iz/z_0)} = -\frac{x^2+y^2}{w_0^2\left(1+\frac{z^2}{z_0^2}\right)} + i\frac{k(x^2+y^2)}{2z\left(1+\frac{z_0^2}{z^2}\right)}$$

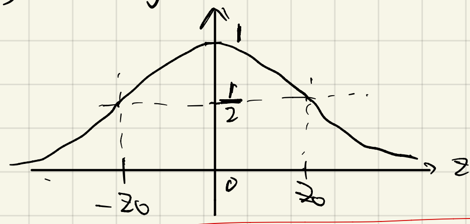
$$\Rightarrow \exp\left[-\frac{x^2+y^2}{w_0^2[1+iz/z_0]}\right] = \exp\left[-\frac{x^2+y^2}{w_0^2(1+\frac{z^2}{z_0^2})}\right]\exp\left[\frac{ik}{2}\frac{x^2+y^2}{z(1+\frac{z_0^2}{z^2})}\right] \qquad A(z)^2 = \frac{A^2}{1+\frac{z^2}{z_0^2}} \Rightarrow A(z) = \frac{A_0}{\sqrt{1+\frac{z^2}{z_0^2}}}$$

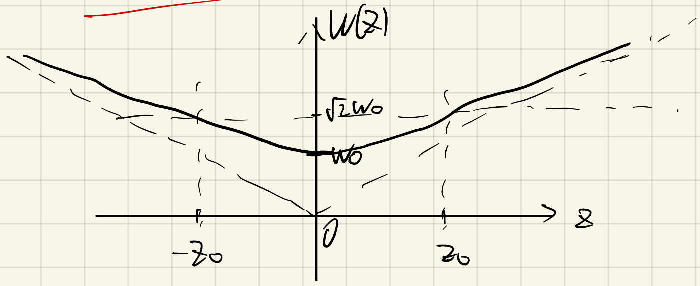
$$\cancel{\Rightarrow A(z) = \frac{A_0}{1+i\frac{z}{z_0}}} \qquad W(z) = w_0\sqrt{1+\frac{z^2}{z_0^2}} \qquad R(z) = z\left(1+\frac{z_0^2}{z^2}\right) \qquad \text{Gouy phase } e^{i\varphi(z)} = 1$$

Beam Intensity $I = \frac{A_0^2}{1+\frac{z^2}{z_0^2}} \cancel{\exp\left[-\frac{2(x^2+y^2)}{W^2(z)}\right]}$

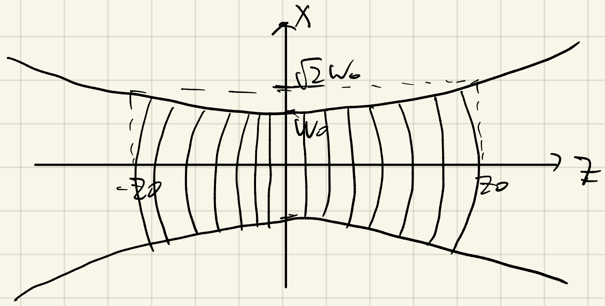
$I \sim A(z)^2$







(C)



$z \gg z_0$ the wavefronts have the same shape of wavefronts of a spherical wave.

d) The radius of the two sides of a resonator should be the same as two wavefronts of a Gaussian Beam

The mirrors and wavefronts should coincide

According to $R(z) = z\left(1+\frac{z_0^2}{z^2}\right)$, the mirror at any z, the radius of the mirrors have to be $z\left(1+\frac{z_0^2}{z^2}\right)$

If the two mirrors are at z_1 and z_2 The $R_1 = z_1 + \frac{z_0^2}{z_1}$ $R_2 = -z_2 - \frac{z_0^2}{z_2}$

and the distance between the two mirror is $d = z_2 - z_1$

$$z_0^2 = z_1(R_1 - z_1) = -z_2(R_2+z_2) \Rightarrow z_1(R_1-z_1) = -(d+z_1)(R_2+d+z_1)$$

$$\Rightarrow z_1R_1 - z_1^2 = -d(R_2+d) - dz_1 - z_1(R_2+d) - z_1^2$$

$$= z_1(R_1+d+R_2+d) = -d(R_2+d) \Rightarrow z_1 = \frac{-d(R_2+d)}{2d+R_1+R_2}$$

$$\Rightarrow z_0^2 = z_1(R_1-z_1) = \frac{-d(R_2+d)}{2d+R_1+R_2}\left[\frac{R_1(2d+R_1+R_2)+d(R_2+d)}{2d+R_1+R_2}\right]$$

$$= \frac{-d(R_2+d)[R_1(2d+R_1+R_2)+d(R_2+d)]}{(2d+R_1+R_2)^2} \geq 0$$

$$\Rightarrow d(R_2+d)[R_1(R_1+d+R_2+d)+d(R_2+d)] \leq 0$$

$$\Rightarrow d(R_2+d)[R_1(R_1+d)+R_1(R_2+d)+d(R_2+d)] = d(R_2+d)[R_1(R_1+d)+(R_1+d)(R_2+d)]$$

$$= d[R_2+d][R_1+d](R_1+R_2+d) \le 0 \qquad g_1 = 1+\frac{d}{R_1} \quad g_2 = 1+\frac{d}{R_2}$$

$$\Rightarrow dR_2R_1\left[1+\frac{d}{R_2}\right]\left(1+\frac{d}{R_1}\right)(R_1+R_2+d) = dR_2R_1\left(1+\frac{d}{R_2}\right)\left(1+\frac{d}{R_1}\right)R_1R_2\left(\frac{1}{R_2}+\frac{1}{R_1}+\frac{d}{R_1R_2}\right)$$

$$= (R_1R_2)^2\left(1+\frac{d}{R_1}\right)\left(1+\frac{d}{R_2}\right)\left(\frac{d}{R_2}+\frac{d}{R_1}+\frac{d^2}{R_1R_2}+1-1\right)$$

$$= (R_1R_2)^2\left(1+\frac{d}{R_1}\right)\left(1+\frac{d}{R_2}\right)\left[\left(1+\frac{d}{R_2}\right)\left(1+\frac{d}{R_1}\right)-1\right] = (R_1R_2)^2 g_1g_2(g_1g_2-1) \le 0$$

$$\Rightarrow g_1g_2 < 0 \quad g_1g_2-1>0 \quad \text{or} \quad g_1g_2>0 \quad g_1g_2-1<0$$

$$\Rightarrow 0<g_1g_2<1 \Rightarrow 0<\left(1+\frac{d}{R_1}\right)\left(1+\frac{d}{R_2}\right)<1$$

a) Write down a Gaussian beam profile $v_0(x, y, z = 0)$ with a beam waist of w_0 in the stationary state. Explain in few sentences how one can propagate it in free-space with paraxial approximation.

$$v(x,y,z) = \frac{A_0}{\sqrt{1+\frac{z^2}{z_0^2}}}\exp\left[-\frac{x^2+y^2}{w_0^2\left(1+\frac{z^2}{z_0^2}\right)}\right]\exp\left[\frac{ik}{2}\frac{(x^2+y^2)}{z\left(1+\frac{z_0^2}{z^2}\right)}\right]e^{i\varphi(z)}$$

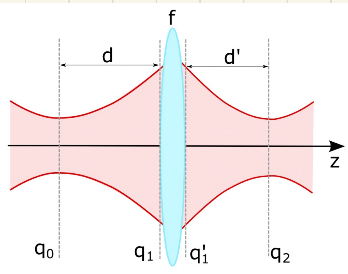
$$z=0 \quad v_0(x,y,z=0) = A_0\exp\left(-\frac{x^2+y^2}{w_0^2}\right)$$

One can first do Fourier transfor to transfor $v_0(x,y,z=0)$ to $V_0(\alpha,\beta,z=0)$ in frequeny domain and then multiply $V_0(\alpha,\beta,z=0)$ with $\exp\left[\frac{-iz}{2k}(\alpha^2+\beta^2)\right]$ to get the profile at z: $V_0(\alpha,\beta,z) = V_0(\alpha,\beta,z=0)\exp\left[\frac{-iz}{2k}(\alpha^2+\beta^2)\right]$. Finally do inverse Fourier transform to get the profile in space domain $v_0(x,y,z) = \int_{-\infty}^{\infty} V_0(\alpha,\beta,z=0)\exp\left[\frac{-iz}{2k}(\alpha^2+\beta^2)\right]e^{i\alpha x}d\alpha$

d) If a continuous wave Gaussian beam with a wavelength of $\lambda = 1\ \mu m$ is propagating in air, what conclusions can be drawn about the validity of the equation in b) when the beam waist is $w_0 = 1\ \mu m$, $w_0 = 10\ \mu m$, or $w_0 = 1$ mm? What can you tell about the divergence of the beam in each case and why?

$$w_0 = 1\ \mu m \Rightarrow z_0 = \frac{\pi w_0^2}{\lambda} = 1\pi(\mu m) \quad w_0 = 10\ \mu m \Rightarrow 100\pi\ (\mu m) \quad w_0 = 1000\mu m \Rightarrow z_0 = 10^6\pi(\mu m)$$

Thus the smaller w_0 is the faster gaussian beam broadens



$$d \quad q_2 = z + \frac{q_0+f}{-\frac{1}{f}(q_0+f)+1}$$

$$= z + \frac{(q_0+f)f}{-q_0}$$

$$= z + \frac{f(f-iz_0)}{iz_0} = z - f - i\frac{f^2}{z_0}$$

$$\frac{f^2}{z_0^2} = \frac{\lambda f^2}{\pi w_0^2} = \frac{\pi w_0'^2}{\lambda} \quad w_0' = \frac{\lambda f}{\pi w_0}$$

$$\Rightarrow \frac{w_0'}{w_0} = \frac{\lambda f}{\pi w_0^2} = \frac{f}{z_0}$$

c) Find the distance d' from the lens to the beam waist w_0' and new Rayleigh length z_0' expressed in terms of z_0, d, and f.

d) Consider you move the lens such that $d = f$, find the magnification $M = w_0'/w_0$.

$$c) \quad q_2 = d' + \frac{q_0+d}{-\frac{1}{f}(q_0+d)+1} = d' + \frac{f(d-iz_0)}{(iz_0-d)+f}$$

$$= d' + \frac{fd - ifz_0}{f-d+iz_0} = d' + \frac{f(d-iz_0)(f-d-iz_0)}{(f-d)^2+z_0^2} = d' + \frac{fd(f-d) - ifdz_0 - if(f-d)z_0 - fz_0^2}{(f-d)^2+z_0^2}$$

$$= d' + \frac{fd(f-d)-fz_0^2}{(f-d)^2+z_0^2} - i\frac{[fd+f(f-d)]z_0}{(f-d)^2+z_0^2} = d' - \frac{fd(d-f)+fz_0^2}{(f-d)^2+z_0^2} - i\frac{f^2z_0}{(f-d)^2+z_0^2}$$

$$\frac{f^2z_0}{(f-d)^2+z_0^2} = \frac{\pi w_0'^2}{\lambda} \quad d=f \Rightarrow \frac{f^2z_0}{z_0^2} = \frac{f^2}{z_0} = \frac{\pi w_0'^2}{\lambda} \Rightarrow \frac{f^2\lambda}{\pi w_0^2} = \frac{\pi w_0'^2}{\lambda} \Rightarrow w_0' = \frac{f\lambda}{\pi w_0}$$

$$\Rightarrow \frac{W_0'}{W_0} = \frac{f\lambda}{\pi W_0^2} = \frac{f}{z_0}$$