$V_{g,0} = \frac{\quad}{1.5}$ m/s

1.5 , $\frac{2\times 1\times 10}{\quad} \times (1\times 10 - 1.5\times 10$

Name: ______________________ Date of birth: ______________ Student ID No. __________

10

Final Exam of "Fundamentals of modern optics" WS 2014/15 to be written on February 20, 10:00 - 11:30 am

Problem 1 – Maxwell's Equations

4 + 3 + 4 points

a) Write down the macroscopic Maxwell's equations in time and frequency domain in its general form, including $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{B}$, and $\boldsymbol{D}$ fields. The external charge density is zero, and only a conductive current $\boldsymbol{J}$ is present.

b) Below we give equations that connect the induced electric polarization $\boldsymbol{P}(\boldsymbol{r},t)$ to the electric field $\boldsymbol{E}(\boldsymbol{r},t)$, in time domain. For each case specify the type of the material that induces this polarization in terms of four characteristics: Dispersive or non-dispersive, isotropic or anisotropic, homogeneous or inhomogeneous, and linear or nonlinear:

i) $\boldsymbol{P} = \varepsilon_0(A + Be^{-z^2})\boldsymbol{E}$

ii) $A\frac{\partial^2 \boldsymbol{P}}{\partial^2 t} + B\frac{\partial \boldsymbol{P}}{\partial t} + \boldsymbol{P} = \varepsilon_0 C\boldsymbol{E}$

iii) $P_x = \varepsilon_0(AE_x + BE_y)$

where $A \neq B \neq C \neq 0$ are some constants.

	dispersive	anisotropic	inhomogeneous	nonlinear
i)				
ii)				
iii)				

c) For the case of a monochromatic field of frequency ω_0 we have the constitutive relations $\boldsymbol{B}(\boldsymbol{r},t) = \mu_0\boldsymbol{H}(\boldsymbol{r},t)$, $\boldsymbol{D}(\boldsymbol{r},t) = \varepsilon_0\varepsilon(\boldsymbol{r},\omega_0)\boldsymbol{E}(\boldsymbol{r},t)$, and $\boldsymbol{J}(\boldsymbol{r},t) = \sigma(\boldsymbol{r},\omega_0)\boldsymbol{E}(\boldsymbol{r},t)$. Here the relative permittivity $\varepsilon = \varepsilon' + i\varepsilon''$ is a complex number and the conductivity σ is a real number. Start with the MWEs in the frequency domain, and find the relation between $\nabla \cdot \langle \boldsymbol{S} \rangle$ and the parameters σ and ε'', or the so called divergence theorem, where $\langle \boldsymbol{S} \rangle$ is the time averaged Poynting vector. Comment on the connection between the sign of σ and ε'' and the gain and loss of the material.

Problem 2 – Normal Modes

2 + 3 points

We have the x-polarization of the electric field of a 2D system with the form:

$$E_x(x,y) = \begin{cases} E_1 e^{iAx - Bx - Cy}, & y > 0 \\ E_2 e^{iAx - Bx + Dy - iFy} & y < 0 \end{cases}, \quad x > 0$$

Where A, B, C, D, F are all real and positive numbers, and E_1, E_2 are some complex numbers.

a) What can you deduce from this field regarding the geometry of the system that can support such a mode? Give a simple sketch including the type of materials in each part of your geometry. The type of materials that you can use can be divided into lossless dielectric, lossy dielectric, lossless metal, and lossy metal.

b) What can you say about the nature of this mode in the context of Normal modes. In each of the directions, a mode can be evanescent, propagating, and or decaying. Correspondingly, fill out the tables below.

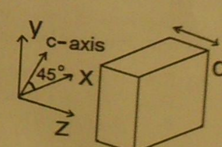
$y > 0$	evanescent	decaying	propagating
x			
y			

$y < 0$	evanescent	decaying	propagating
x			
y			

Problem 3 – Anisotropy

6 points

An infinitely broad (in x and y direction) layer of a uniaxial crystal of thickness $d = 7.5\mu$m is shown in figure. The extraordinary crystal axis is in the x-y plane and makes a 45 degrees angle with the x and y axis.The ordinary refractive index is $n_o = 2.2$ and the extraordinary refractive index is $n_e = 2.1$. A plane wave with an electric field of $\boldsymbol{E} = \frac{E_0}{\sqrt{2}} e^{i\frac{2\pi}{\lambda}z}\left(\hat{x} + e^{i\pi/2}\hat{y}\right)$ is incident on this layer from one side, where $\lambda = 1\mu m$ is the wavelength of the wave in free space. Evaluate the electric field polarization at the other side of this crystal layer.
You can ignore the reflections from the crystal surfaces; it is the final polarization state that matters to us. You can use any means of calculation, or explaining, or sketching, to find the final answer, as long as you explain it fully.



Problem 4 – Pulses

3 + 4 + 3 points

A fiber is characterized by the following wavenumber dependence

$$k(\omega) = \frac{\omega_0}{c} + \frac{\alpha}{c}(\omega - \omega_0) + \frac{\beta}{c}(\omega - \omega_0)^2,$$

where $\omega_0 = 1.5 \times 10^{14}$ rad/s, $\alpha = 1.5$, $\beta = 1 \times 10^{-14}$ s and $c = 3 \times 10^8$ m/s is the speed of light in vacuum. The length of the fiber is $L = 300$ m. continue to next page

$V_{g,0} = \frac{3\cdot 10^8\ m/s}{1.5}$

$2\beta(\omega_1-\omega_0) = \frac{1.5}{c} + \frac{}{c}$

a) You shine light of two different frequencies $\omega_1 = 1 \times 10^{14}$ rad/s and $\omega_2 = 2 \times 10^{14}$ rad/s into the fiber. At the beginning of the fiber the phase difference between both components is $\Delta\phi = 0$. What is the phase difference at the end of the fiber? *Hint:* It is not neccessary to reduce it to $[0, 2\pi]$.

b) Now you send two Gaussian pulses with a FWHM of $\Delta t = 10$ ps of the intensity profile into the fiber. One has the center frequency ω_1, the other one has ω_2 (values from part a)). Suppose both pulses start at the same time, which pulse maximum arrives first at the end of the fiber? What is the difference between the arrival times?

c) Suppose you modify the dispersion relation by adding a fourth term $\tilde{k}(\omega) = k(\omega) + \gamma/c \cdot (\omega - \omega_0)^3$ with $\gamma = -2.2 \times 10^{-28}\,\mathrm{s}^2$. What is the optimal center frequency ω to have the shortest possible pulses at the end of the fiber?

Problem 5 – Diffraction

3 + 4 points

a) Name the two most prominent approximations made in diffraction theory. Explain their validity im mathematical terms and also explain for what they are normally used (use a maximum of three sentences each!).

b) Calculate the diffraction pattern in the far-field at a distance z_B behind an aperture with the following transmission function using the convolution theorem

$$t(x) = \left[\frac{\sin(x)}{x}\right]^2.$$

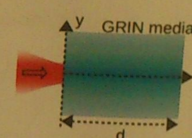
Hint: you are interested in the intensity pattern only so drop any prefactors or phase terms!

Problem 6 – Gaussian Optic

6 points

A Gaussian beam with a waist of W_0 hits a media that is described by the following matrix

$$\mathbf{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos(\alpha d) & \frac{1}{\alpha}\sin(\alpha d) \\ -\alpha\sin(\alpha d) & \cos(\alpha d) \end{pmatrix},$$



where $\frac{2\pi}{\alpha}$ is the period in which the beam propagation shape repeats itself. The incoming beam is focused at the interface. What is the minimum and the maximum width of the Gaussian beam within the media? What can you tell about the phase curvature at these two points?

This media is called graded index (GRIN) media and it has varying refractive index profile, highest at the center. This type of media is used as a fiber or a lens.

Problem 7 – Interface

3 + 4 + 1 points

The reflection coefficient of a TE mode field, with incident angle of θ_1 and refracted angle of θ_2, from a media 1 (n_1) into a media 2 (n_2) is described by the following Fresnel equation,

$$r_{TE} = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2}.$$

The reflection coefficient ($r_{TE} = |r_{TE}|e^{i\,\phi_{TE}}$) is in general complex; thus, it not only gives the amount of field that is reflected ($|r_{TE}|$) from the interface but also the phase delay (ϕ_{TE}).

a) If $n_1 < n_2$, what can you tell about the magnitude ($|r_{TE}|$) and the phase (ϕ_{TE}) of the reflection coefficient? Draw diagrams of $|r_{TE}|$ and ϕ_{TE} versus incidence angle θ_1. *Hint: Consider it at two limiting cases $\theta_1 = 0$ and $\theta_1 = \pi/2$, and draw conclusion.*

b) If $n_1 > n_2$, the critical angle θ_c, at which the total internal reflection occurs, plays important role in $|r_{TE}|$ and ϕ_{TE}. Find ϕ_{TE} as a function of θ_c and θ_1. What can you tell about $|r_{TE}|$ and ϕ_{TE} in the range between $\theta_1 = [0, \theta_c)$ and $\theta_1 = [\theta_c, \pi/2]$? Draw diagrams of $|r_{TE}|$ and ϕ_{TE} versus incidence angle θ_1.

c) What is different for a TM incident field?

— Useful formulas —

$\nabla \times \nabla \times \vec{a} = \nabla(\nabla \cdot \vec{a}) - \Delta\vec{a}$

$\nabla \cdot (c\vec{a}) = \vec{a} \cdot \nabla c + c\,\nabla \cdot \vec{a}$

Gaussian q-parameter transform law:

$q' = \frac{Aq + B}{Cq + D}$

$\nabla \cdot (\nabla \times \vec{a}) = 0$

$\nabla \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\nabla \times \vec{a}) - \vec{a} \cdot (\nabla \times \vec{b})$

ABCD matrix for a thin lens:

$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$