

FoMO in a nutshell

Important things to memorize for the course »Fundamentals of Modern Optics«*

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Here, we collect important relations, facts, formulas and other things that one should memorize throughout the course to give you a summary of the most important results. Of course, we cannot list every required formula and we assume that you already have a solid math background. Hence, we cannot guarantee completeness of the list.

FOMO topics

Maxwell's equations

- Macroscopic Maxwell's equations (time domain):

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho(\mathbf{r}, t)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{j}(\mathbf{r}, t) + \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t}$$

- Macroscopic Maxwell's equations (frequency domain):

$$\nabla \cdot \mathbf{D}(\mathbf{r}, \omega) = \rho(\mathbf{r}, \omega)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, \omega) = i\omega \mathbf{B}(\mathbf{r}, \omega)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, \omega) = 0$$

$$\nabla \times \mathbf{H}(\mathbf{r}, \omega) = \mathbf{j}(\mathbf{r}, \omega) - i\omega \mathbf{D}(\mathbf{r}, \omega)$$

- Constitutive relations (linear material response; in optics usually $\mu(\mathbf{r}, \omega) \equiv 1$ (non-magnetizable):

$$\mathbf{D}(\mathbf{r}, \omega) = \epsilon_0 \mathbf{E}(\mathbf{r}, \omega) + \mathbf{P}(\mathbf{r}, \omega) = \epsilon_0 \epsilon(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega)$$

$$\mathbf{B}(\mathbf{r}, \omega) = \mu_0 \mu(\mathbf{r}, \omega) \mathbf{H}(\mathbf{r}, \omega)$$

$$\mathbf{P}(\mathbf{r}, \omega) = \epsilon_0 \chi(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega)$$

- Time domain material response (response function):

$$\mathbf{P}(\mathbf{r}, t) = \epsilon_0 \int_{-\infty}^t R(\mathbf{r}, t-t') \mathbf{E}(\mathbf{r}, t') dt', \text{ where } R(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi(\mathbf{r}, \omega) e^{-i\omega t} d\omega$$

- Complex permittivity:

$$\epsilon(\mathbf{r}, \omega) = 1 + \chi(\mathbf{r}, \omega) + i \frac{\sigma(\mathbf{r}, \omega)}{\epsilon_0 \omega}$$

conductivity ↓

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- Continuity equation (conservation of charge):

$$\nabla \cdot \mathbf{j}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = 0$$

$$\underbrace{\oint_{\partial V} \mathbf{j} \cdot d\mathbf{S}}_I = - \frac{\partial}{\partial t} \underbrace{\iiint_V \rho \, dV}_Q$$

- Time averaged Poynting vector, loss:

$$\langle \mathbf{S}(\mathbf{r}) \rangle = \frac{1}{2} \operatorname{Re} [\mathbf{E}(\mathbf{r}, \omega) \times \mathbf{H}^*(\mathbf{r}, \omega)] \quad \nabla \cdot \langle \mathbf{S}(\mathbf{r}) \rangle < 0 \Leftrightarrow \text{system is lossy}$$

- Wave equations (vacuum):

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}(\mathbf{r}, t)}{\partial t^2} = 0$$

$$\nabla \times \nabla \times \mathbf{H}(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \mathbf{H}(\mathbf{r}, t)}{\partial t^2} = 0$$

Normal modes in homogeneous, isotropic, non-magnetizable matter

- Helmholtz equation (wave equation in temporal Fourier domain; homogeneous, isotropic matter):

$$\Delta \mathbf{E}(\mathbf{r}, \omega) + \frac{\omega^2}{c^2} \epsilon(\omega) \mathbf{E}(\mathbf{r}, \omega) = 0$$

- Plane waves are the eigenmodes of free space. They take the form

$$\mathbf{E}(\mathbf{r}, \omega) = \mathbf{E}_0 e^{i\mathbf{k}\mathbf{r}}$$

and their dispersion relation reads as:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} \epsilon(\omega)$$

- Refractive index:

$$n(\omega) = \frac{k(\omega)}{k_0} = \frac{k(\omega)}{\omega} c$$

- Maxwell relation:

$$n(\omega) = \sqrt{\epsilon(\omega)} \text{ which implies for real and imaginary parts: } (n' + in'')^2 = n'^2 - n''^2 + 2in'n'' = \epsilon' + i\epsilon''$$

- Finding electric or magnetic field from each other in frequency domain in regions without source:

$$\mathbf{H}(\mathbf{r}, \omega) = -\frac{i}{\omega\mu_0} \nabla \times \mathbf{E}(\mathbf{r}, \omega)$$

$$\mathbf{E}(\mathbf{r}, \omega) = \frac{i}{\omega\epsilon_0\epsilon(\mathbf{r}, \omega)} \nabla \times \mathbf{H}(\mathbf{r}, \omega)$$

- Propagating, lossy and evanescent waves for direction vector \mathbf{u} :

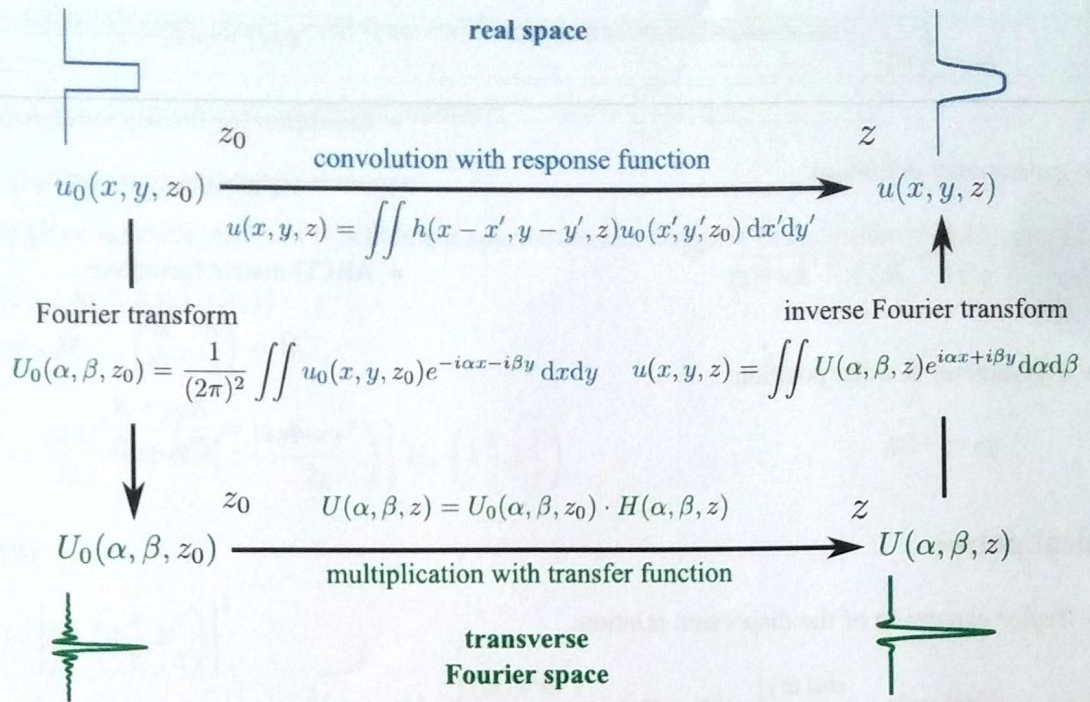
$$\mathbf{u} \cdot \langle \mathbf{S} \rangle \neq 0, \nabla \cdot \langle \mathbf{S} \rangle = 0 \Leftrightarrow \text{lossless propagation}$$

$$\mathbf{u} \cdot \langle \mathbf{S} \rangle \neq 0, \nabla \cdot \langle \mathbf{S} \rangle < 0 \Leftrightarrow \text{lossy propagation}$$

$$\mathbf{u} \cdot \langle \mathbf{S} \rangle = 0, \nabla \cdot \langle \mathbf{S} \rangle = 0 \Leftrightarrow \text{evanescence}$$

Beam propagation ($k_x = \alpha, k_y = \beta$)

- Beam propagation scheme:



- Relation between transfer- and response function (comes with an extra $1/4\pi^2$ factor):

$$h(x, y, z) = \frac{1}{(2\pi)^2} \iint H(\alpha, \beta, z) e^{i\alpha x + i\beta y} d\alpha d\beta$$

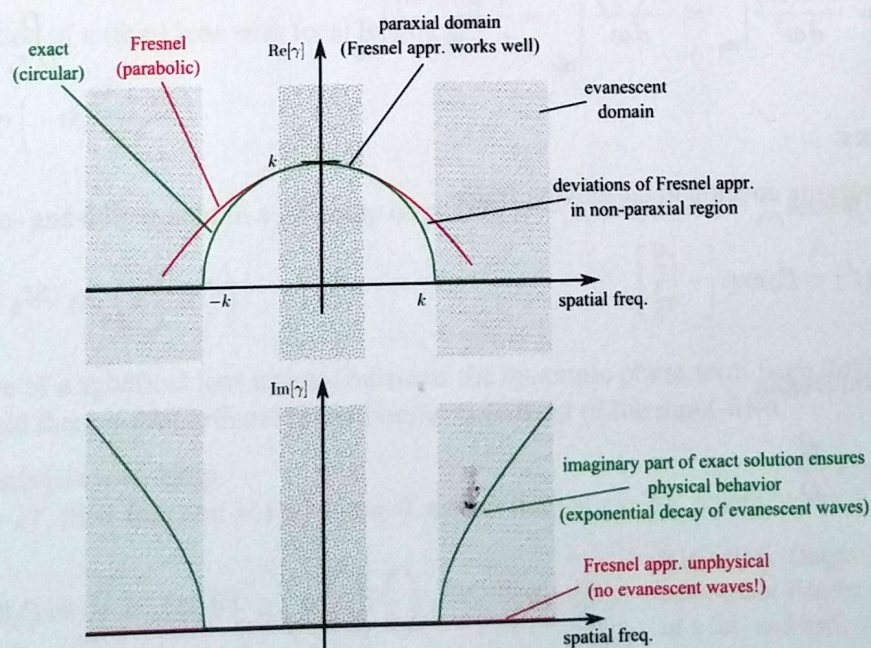
- Transfer functions for homogeneous space:

$$H(\alpha, \beta, z) = \exp[i\gamma(\alpha, \beta)z] = \exp\left[i\sqrt{k^2 - \alpha^2 - \beta^2}z\right]$$

exact solution

$$H_F(\alpha, \beta, z) = \exp[ikz] \exp\left[-i\frac{\alpha^2 + \beta^2}{2k}z\right]$$

Fresnel approximation



Gaussian beams

- Rayleigh length:

$$z_0 = \frac{k}{2} w_0^2$$

- q -parameter definition:

$$\frac{1}{q(z)} = \frac{1}{R(z)} + i \frac{2}{k w^2(z)}$$

- q -parameter at waist position:

$$q_0 = -iz_0$$

- q -parameter after propagation:

$$q(d) = q(0) + d$$

- Condition for finding waist position:

$$\text{Re}[q(z)] = 0$$

- ABCD matrix formalism:

$$\hat{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \hat{M}_{\text{tot}} = \hat{M}_n \cdot \dots \cdot \hat{M}_3 \cdot \hat{M}_2 \cdot \hat{M}_1$$

$$q_{n+1} = \frac{Aq_n + B}{Cq_n + D}$$

Optical pulses

- Taylor expansion of the dispersion relation:

$$\begin{aligned} k(\omega) &= k_0 + \left. \frac{\partial k(\omega)}{\partial \omega} \right|_{\omega_0} (\omega - \omega_0) + \frac{1}{2} \left. \frac{\partial^2 k(\omega)}{\partial \omega^2} \right|_{\omega_0} (\omega - \omega_0)^2 \\ &= \frac{\omega_0}{c} + \frac{1}{v_g} (\omega - \omega_0) + \frac{D}{2} (\omega - \omega_0)^2 \end{aligned}$$

- Group velocity, group index, group velocity dispersion:

$$v_g(\omega) = \left[\left. \frac{\partial k(\omega)}{\partial \omega} \right|_{\omega_0} \right]^{-1}$$

(velocity, at which the pulse travels)

$$n_g(\omega) = \frac{c}{v_g} = c \cdot \left. \frac{\partial k(\omega)}{\partial \omega} \right|_{\omega_0}$$

$$D(\omega) = \left. \frac{\partial^2 k(\omega)}{\partial \omega^2} \right|_{\omega_0} = \left. \frac{\partial \left(\frac{1}{v_g} \right)}{\partial \omega} \right|_{\omega_0} = \frac{1}{c} \left. \frac{\partial n_g(\omega)}{\partial \omega} \right|_{\omega_0}$$

(measures change in group index,
 $D > 0 \rightarrow$ red is faster,
 $D < 0 \rightarrow$ blue is faster)

- Gaussian pulses:

- Slowly varying envelop in co-moving frame:

$$E(t') = E_0 \exp \left[-\frac{t'^2}{\tau_0^2} \right]$$

- Dispersion length:

$$L_D = \frac{\tau_0^2}{D}$$

Fraunhofer diffraction, Fourier Optics

Fraunhofer approximation, Far-field diffraction pattern

- Condition for validity: Fresnel approx. valid (paraxial)¹ + propagation to the far-field:

$$z^2 \gg x^2 + y^2 \quad \text{or if you use the Fresnel number:} \quad N_f = \frac{a^2}{\lambda z} < 0.1,$$

where a is the aperture size or the beam diameter.

- Interaction with plane aperture masks in thin element approximation, complex transmission function $t(x, y)$:

$$u_+(x, y, z_0) = u_0(x, y, z_0) \cdot t(x, y)$$

- Far-field:

$$u(x, y, z) = \frac{(2\pi)^2}{i\lambda z} \exp\left[ik\left(z + \frac{x^2 + y^2}{2z}\right)\right] U_+\left(k\frac{x}{z}, k\frac{y}{z}\right)$$

- Far-field intensity:

$$I(x, y, z) \sim \left| U_+\left(k\frac{x}{z}, k\frac{y}{z}\right) \right|^2$$

The far-field intensity is proportional to the squared absolute value of the Fourier spectrum behind the aperture, evaluated at the spatial frequencies $\alpha = kx/z, \beta = ky/z$.

- Scheme for calculating a Fraunhofer diffraction pattern:

- 1.) Multiply incident illumination $u_-(x, y, 0)$ with transmission function $t(x, y)$.
- 2.) Compute the Fourier transform $U_+(\alpha, \beta) = 1/(2\pi)^2 \iint u_+(x, y, 0) e^{-i\alpha x - i\beta y} dx dy$.
- 3.) Substitute the angular frequencies by $\alpha \rightarrow kx/z, \beta \rightarrow ky/z$ to get a function of (x, y) .
- 4.) Either take the absolute value squared $|U_+(kx/z, ky/z)|^2$ to get the principal intensity pattern (if asked) or multiply the prefactors from above to get the complex field.

Lenses, 2f & 4f-setups

- Transmission function of a (thin) lens with focal length f :

$$t_L(x, y) = \exp\left(-ik\frac{x^2 + y^2}{2f}\right)$$

- Relation between in- and output field in a 2f-setup (input at $z = -f$, output at $z = f$, lens at $z = 0$):

$$u_{\text{out}} = \frac{(2\pi)^2}{i\lambda f} e^{2ikf} U_{\text{in}}\left(k\frac{x}{f}, k\frac{y}{f}\right)$$

The phase curvature of a spherical lens exactly balances the quadratic phase term from diffraction in the back-focal plane. The field there is proportional to the Fourier transform of the input-field.

- Fourier Optical filtering in a 4f-setup
(input field at $z = -2f$, filter function $p(x, y)$ at $z = 0$, output field at $z = 2f$, lenses at $z = \pm f$):

$$u_{\text{out}}(-x, -y, 4f) = \iint U_{\text{in}}(\alpha, \beta) \cdot p\left(\alpha\frac{f}{k}, \beta\frac{f}{k}\right) e^{i\alpha x + i\beta y} d\alpha d\beta$$

Attention: Output is inverted due to 2 subsequent Fourier transforms (instead of a forward and backward transform)!

¹There exists a non-paraxial Fraunhofer approximation. However, we just want to treat the paraxial case here.

- Scheme for computing Fourier Optical Filtering (formulated using a backward transform):

1.) Compute the Fourier Transform of the input field as function of (α, β) :

$$U_{in}(\alpha, \beta) = \frac{1}{(2\pi)^2} \iint u_{in}(x, y) e^{-i\alpha x - i\beta y} dx dy$$

2.) Make a change of variables (**no transform!**) in the filter function $p(x, y)$, to make it a function of (α, β) :

$$x \rightarrow \alpha \frac{f}{k}, \quad y \rightarrow \beta \frac{f}{k}$$

3.) Multiply them and compute the Fourier back-transform as given above

$$\iint U_{in}(\alpha, \beta) \cdot p\left(\alpha \frac{f}{k}, \beta \frac{f}{k}\right) e^{i\alpha x + i\beta y} d\alpha d\beta$$

4.) Change the sign of every x and y to get the resulting output field:

$$x \rightarrow -x, \quad y \rightarrow -y$$

Polarization, anisotropic media

Jones formalism

- For plane waves, every possible polarization state can be described as the superposition of two orthogonal modes. The Jones formalism uses a linear basis. The strength and relative phase of the two pol. components is described by a Jones vector

$$\mathbf{J} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

- Possible Jones vectors:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ (hor. linear)}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ (vert. linear)}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ (diagonal linear)}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \text{ (left-h. circular)}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \text{ (right-h. circular)}$$

(otherwise) \rightarrow left- or right-h. elliptical

- By passing through a polarization-sensitive device, the state of polarization may change. This can be described with a matrix formalism which acts on the Jones vector:

$$\mathbf{J}_{out} = \hat{\mathbf{M}} \cdot \mathbf{J}_{in}$$

- The action of many elements is described by matrix multiplication (first element *last!*):

$$\hat{\mathbf{M}} = \hat{\mathbf{M}}_n \cdot \dots \cdot \hat{\mathbf{M}}_2 \cdot \hat{\mathbf{M}}_1$$

- Some simple Jones matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ hor. linear polarizer}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ vert. linear polarizer}$$

- Find polarization eigenstates of system $\hat{\mathbf{M}} \rightarrow$ solve eigenvalue problem $\hat{\mathbf{M}} \cdot \mathbf{J}_{eig} = \lambda \mathbf{J}_{eig}$ for λ and \mathbf{J}_{eig} .
 \rightarrow Pass through the system without change.

Optics of crystals

- In a crystal, optical properties are not isotropic any more \rightarrow tensor replaces dielectric function

$$\mathbf{D}(\mathbf{r}, \omega) = \epsilon_0 \hat{\epsilon}(\mathbf{r}, \omega) \cdot \mathbf{E}(\mathbf{r}, \omega) \quad \hat{\epsilon} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}$$

- In dependence of the propagation direction, \mathbf{D} and \mathbf{E} are thus not automatically parallel.
- Consequences for Poynting vector, since always: $\mathbf{D} \perp \mathbf{k}$ and $\mathbf{S} \perp \mathbf{E}$, i.e. \mathbf{S} will not always be parallel to \mathbf{k} now!
- Most important, the dispersion relation will become dependent on direction and polarization now. There are always two linear independent eigenmodes for every direction vector $\mathbf{u} \rightarrow$ need to find their polarization direction and refractive indices.

Classification:

- $\epsilon_1 = \epsilon_2 = \epsilon_3 \Rightarrow$ isotropic
- $\epsilon_1 = \epsilon_2 = \epsilon_o \neq \epsilon_3 = \epsilon_e \Rightarrow$ uniaxial ("optical axis" is the direction of ϵ_e in the index ellipsoid, since the dispersion relation in the plane normal to that direction is always circle, irrespective of the direction of \mathbf{E}), "ordinary" and "extraordinary", possible cases:

* $\mathbf{u} \parallel$ opt. ax. \Rightarrow "boring case"

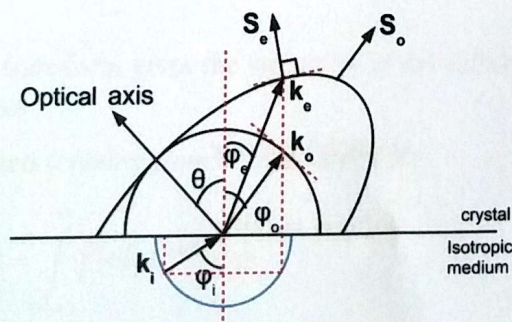
Both eigenmodes experience n_o (degeneracy) and their polarization direction is arbitrary in a plane with $\mathbf{n} \parallel \mathbf{u}$. The crystal behaves like an "ordinary" isotropic medium with refractive index n_o . Nothing happens to any arbitrary input state of polarization.

* $\mathbf{u} \perp$ opt. ax. \Rightarrow "waveplate case"

There are two non-degenerate eigenmodes – one with refractive index n_o (polarized \perp to the opt. ax. and $\perp \mathbf{u}$) and one with n_e (polarized \parallel to the opt. ax. and $\perp \mathbf{u}$). For both, $\mathbf{S} \parallel \mathbf{k}$ holds (no double refraction). An arbitrary linear input polarization \mathbf{E} will have an angle α between electrical field vector and optical axis and must be decomposed into the two eigenpolarizations. After a propagation length d , the two components will have a phase shift which determines the polarization state (Jones vector) at the output. Important cases are:

- $d = \frac{\lambda}{2(n_e - n_o)} \rightarrow$ "half-wave plate", rotates linear polarization by an angle 2α
- $d = \frac{\lambda}{4(n_e - n_o)}$ and $\alpha = 45^\circ \rightarrow$ "quarter-wave plate", transfers linear to circular polarization *if not we will get elliptical pol.*
- all other cases lead to elliptical polarization, except for $\alpha = 0$ or $\alpha = 90^\circ$ (only eigenmode excited), in which the polarization stays unchanged irrespective of d .

* $\angle(\mathbf{u}, \text{opt. ax.}) = \theta \neq 0^\circ, 90^\circ \rightarrow$ "double refraction case"



- $\epsilon_1 \neq \epsilon_2 \neq \epsilon_3 \Rightarrow$ biaxial (the two optical axes do not coincide with one of the crystal axes \rightarrow very complicated!)

Interfaces

- Continuity of tangential field components:

$$\iint_A \nabla \times \mathbf{E}(\mathbf{r}, \omega) \, d\mathbf{A} \stackrel{\text{Stokes}}{=} \oint_{\partial A} \mathbf{E} \, d\mathbf{r} \stackrel{A \rightarrow 0}{=} \mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \quad \Rightarrow \mathbf{n} \times \mathbf{E}_1 = \mathbf{n} \times \mathbf{E}_2$$

(analog for \mathbf{H})

- Continuity of tangential \mathbf{k} -components:
follows directly from reflection/transmission problem (all fields need to be in phase at every point)
- Law of reflection: $\theta = \theta'$
follows directly from continuity of $\mathbf{n} \times \mathbf{k}$
- Law of refraction: $n_1 \sin \theta_1 = n_2 \sin \theta_2$
follows also directly from continuity of $\mathbf{n} \times \mathbf{k}$
- Fresnel equations (reflection and transmission coefficients at interface, follow from continuity of field components):

$$r_{\text{TE}} = \frac{n_1 \cos \theta_1 - n_2 \cos \theta_2}{n_1 \cos \theta_1 + n_2 \cos \theta_2}$$

$$t_{\text{TE}} = \frac{2n_1 \cos \theta_1}{n_1 \cos \theta_1 + n_2 \cos \theta_2}$$

$$r_{\text{TM}} = \frac{n_2 \cos \theta_1 - n_1 \cos \theta_2}{n_2 \cos \theta_1 + n_1 \cos \theta_2}$$

$$t_{\text{TM}} = \frac{2n_1 \cos \theta_1}{n_2 \cos \theta_1 + n_1 \cos \theta_2}$$

- Critical angle (total internal reflection): $\sin \theta_c = n_1/n_2$.
- Brewster angle: $\tan \theta_b = n_2/n_1$
- Reflectivity and Transmissivity (energy):

$$R = |r|^2$$

$$T = \frac{\text{Re}(\mathbf{n} \cdot \mathbf{k}_2)}{\mathbf{n} \cdot \mathbf{k}_1} |t|^2$$

Mathematical tools

Miscellaneous math formulas

- Complex exponentials, trigonometric and hyperbolic functions:

$$e^{ix} = \cos x + i \sin x$$

$$\cos x = \frac{1}{2} (e^{ix} + e^{-ix})$$

$$\sin x = \frac{1}{2i} (e^{ix} - e^{-ix})$$

$$\cosh x = \frac{1}{2} (e^x + e^{-x})$$

$$\cosh(ix) = \cos x$$

$$\sinh x = \frac{1}{2} (e^x - e^{-x})$$

$$\sinh(ix) = i \sin x$$

- Integration:

$$\int u(x) \frac{dv(x)}{dx} dx = u(x)v(x) - \int \frac{du(x)}{dx} v(x) dx \quad (\text{partial integration})$$

$$\int f(x) dx = \int f(\xi) \frac{dx}{d\xi} d\xi \quad (\text{substitution})$$

- Gaussian functions:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\text{FT} \left\{ A \exp \left[-\frac{1}{2} \frac{t^2}{t_0^2} \right] \right\} = \frac{A t_0}{\sqrt{2\pi}} \exp \left[-\frac{1}{2} \frac{\omega^2}{(1/t_0)^2} \right]$$

The Fourier transform of a Gaussian function is a Gaussian function.

- Area element in radial coordinates:

$$dA = r dr d\varphi$$

Field theory

- Vector identities:

$$\nabla \times \nabla \times \mathbf{a} = \nabla \cdot (\nabla \mathbf{a}) - \Delta \mathbf{a}$$

$$\Delta a = \nabla(\nabla a), \text{ but } \Delta \mathbf{a} = (\nabla \cdot \nabla) \mathbf{a} = \nabla^2 \mathbf{a} \neq \nabla(\nabla \mathbf{a})$$

$$\mathbf{a} \times \mathbf{b} \times \mathbf{c} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

- Integral theorems:

$$\iiint_V \nabla \cdot \mathbf{a} dV = \oint_{\partial V} \mathbf{a} d\mathbf{S} \quad (\text{Gauss})$$

$$\iint_A \nabla \times \mathbf{a} d\mathbf{S} = \oint_{\partial A} \mathbf{a} d\mathbf{r} \quad (\text{Stokes})$$

Fourier transform, δ -function

- In the course, we define the one-dimensional Fourier transform as (these definitions regarding sign and prefactor conventions influence nearly every expression in this document that contains Fourier transforms):

Forward (going to Fourier domain):

$$\tilde{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \quad (\text{temporal Fourier domain})$$

$$\tilde{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \quad (\text{spatial Fourier domain})$$

A Fourier transform gives the strengths of the different plane wave (space) or time harmonic (time) frequency components.

Backward (coming from Fourier domain):

$$f(t) = \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t} d\omega \quad (\text{temporal Fourier domain})$$

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk \quad (\text{spatial Fourier domain})$$

An inverse Fourier transform represents a decomposition of a function into plane wave components (space) or harmonic oscillations (time).

- Fourier shifting theorem:

$$\text{FT}\{f(t-t_0)\} = e^{i\omega t_0} \tilde{f}(\omega)$$

$$\text{FT}\{f(x-x_0)\} = e^{-ikx_0} \tilde{f}(k)$$

$$\text{FT}^{-1}\{\tilde{f}(\omega-\omega_0)\} = e^{-i\omega_0 t} f(t)$$

$$\text{FT}^{-1}\{\tilde{f}(k-k_0)\} = e^{ik_0 x} f(x)$$

The shifting of a function corresponds to a harmonic modulation in Fourier domain.

- Fourier transform of a derivative (I cannot recall an occasion where the backward relations would be needed, so I leave them out):

$$\text{FT}\left\{\frac{df(t)}{dt}\right\} = -i\omega \tilde{f}(\omega)$$

$$\text{FT}\left\{\frac{df(x)}{dx}\right\} = ik \tilde{f}(k)$$

A derivative in the real domain corresponds to a simple multiplication in Fourier domain.

- δ -function:

$$\delta(x-x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x_0)} dk$$

The δ -function is the Fourier transform of a plane wave (space) or harmonic oscillation (time).

$$f(x_0) = \int_{-\infty}^{\infty} f(x) \delta(x-x_0) dx$$

The δ -function »picks out« the value of a function at a particular point. It is just meaningful under an integral.

- Fourier transform of the δ -function:

$$\text{FT}\{\delta(t-t_0)\} = \frac{1}{2\pi} e^{i\omega t_0}$$

$$\text{FT}\{\delta(x-x_0)\} = \frac{1}{2\pi} e^{-ikx_0}$$

$$\text{FT}^{-1}\{\delta(\omega-\omega_0)\} = e^{-i\omega_0 t}$$

$$\text{FT}^{-1}\{\delta(k-k_0)\} = e^{ik_0 x}$$

The Fourier transform of a δ -function is a plane wave (space) or harmonic oscillation (time)

- Convolution:

$$[f \otimes g](t) = \int f(\tau) g(t-\tau) d\tau$$

$$\text{FT}\{[f \otimes g](t)\} = 2\pi \tilde{f}(\omega) \tilde{g}(\omega)$$

$$[f \otimes g](x) = \int f(x') g(x-x') dx$$

$$\text{FT}\{[f \otimes g](x)\} = 2\pi \tilde{f}(k) \tilde{g}(k)$$

Convolution in the real domain corresponds to a simple multiplication in the Fourier domain.

