Task 1 $\boxed{3.75}$

a) prove:

$\vec{a} \times (\vec{b} \times \vec{c})$

$= \epsilon_{ijk} a_j (\vec{b} \times \vec{c})_k \; \hat{e}_i$

$= \epsilon_{ijk} a_j \epsilon_{klm} b_l c_m \; \hat{e}_i$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_j b_l c_m \hat{e}_i$

$= [\delta_{il}\delta_{jm} a_j b_l c_m - \delta_{im}\delta_{jl} a_j b_l c_m] \hat{e}_i$

$= \hat{e}_i b_i a_j c_j - a_j b_j c_i \hat{e}_i$

$= \vec{b}(\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{b})\vec{c}$

$\vec{b} = b_i \hat{e}_i$

0.5 + 1*

b) prove:

$\vec{\nabla} \times (\alpha \vec{a})$

$= \epsilon_{ijk} \nabla_j (\alpha a_k) \; \hat{e}_i$

$= \epsilon_{ijk} (\alpha \nabla_j a_k + a_k \nabla_j \alpha) \; \hat{e}_i$

$= [\alpha \epsilon_{ijk} \nabla_j a_k + \epsilon_{ijk} a_k \nabla_j \alpha] \hat{e}_i$

$= [\alpha \epsilon_{ijk} \nabla_j a_k - \epsilon_{ikj} a_k \nabla_j \alpha] \hat{e}_i$

$= \alpha(\vec{\nabla} \times \vec{a}) - (\vec{a} \times \vec{\nabla})\alpha$

$= \alpha \vec{\nabla} \times \vec{a} - \vec{a} \times \vec{\nabla}\alpha$

1

c) prove:

$\vec{\nabla} \cdot (\vec{a} \times \vec{b})$

$= \nabla_i (\vec{a} \times \vec{b})_i$

$= \nabla_i \epsilon_{ijk} a_j b_k$

$= \epsilon_{ijk} (b_k \nabla_i a_j + a_j \nabla_i b_k)$

$= \epsilon_{ijk} b_k \nabla_i a_j + \epsilon_{ijk} a_j \nabla_i b_k$

$= b_k \epsilon_{kij} \nabla_i a_j - a_j \epsilon_{jik} \nabla_i b_k$

$= \vec{b}(\vec{\nabla} \times \vec{a}) - \vec{a}(\vec{\nabla} \times \vec{b})$

1

d) prove:

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{a})$

$= \nabla_i (\vec{\nabla} \times \vec{a})_i$

$= \nabla_i \epsilon_{ijk} \nabla_j a_k$

$= \epsilon_{ijk} (\nabla_i \nabla_j a_k + \nabla_j \nabla_i a_k)$

what happend here?

$= \epsilon_{ijk} \nabla_i \nabla_j a_k + \epsilon_{ijk} \nabla_j \nabla_i a_k$

$= \nabla_i \epsilon_{ijk} \nabla_j a_k - \nabla_j \epsilon_{jik} \nabla_i a_k$

$= \vec{\nabla} \cdot (\vec{\nabla} \times \vec{a}) - \vec{\nabla} \cdot (\vec{\nabla} \times \vec{a}) = 0$

$\nabla_i(\nabla_j a_k) \neq \nabla_i \nabla_j a_k + \nabla_j \nabla_i a_k$

$= \nabla_i \nabla_j a_k = \nabla_j \nabla_i a_k$

0.25

4

Task 2:

$\vec{j}(\vec{r},\omega) = j_0 \cosh(\frac{r}{\delta})\vec{e}_z$

$\delta = \delta(\omega)$

$\vec{j}(\vec{r},\omega) = \sigma(\omega)\vec{E}(\vec{r},\omega)$

$\vec{\nabla}\times\vec{H}(\vec{r},\omega) = \vec{j}(\vec{r},\omega) - i\omega\varepsilon_0\varepsilon(\omega)\vec{E}(\vec{r},\omega) = j_0\cosh(\frac{r}{\delta})\vec{e}_z(1-i\omega\varepsilon_0\frac{\varepsilon(\omega)}{\sigma(\omega)})$

According to Stoke's theorem

$$\int \vec{\nabla}\times\vec{H}(\vec{r},\omega)\, d\vec{s} = \int j_0\cosh(\frac{r}{\delta})(1-i\omega\varepsilon_0\frac{\varepsilon(\omega)}{\sigma(\omega)})\vec{e}_z\cdot d\vec{s} = \oint \vec{H}\cdot d\vec{l} = 2\pi r\vec{H}(\vec{r},\omega)$$

$$\therefore j_0(1-i\omega\varepsilon_0\frac{\varepsilon(\omega)}{\sigma(\omega)})\int_0^r r\frac{e^{\frac{r}{\delta}}+e^{-\frac{r}{\delta}}}{2}dr\int_0^{2\pi}d\varphi = \pi j_0(1-i\omega\varepsilon_0\frac{\varepsilon(\omega)}{\sigma(\omega)})\int_0^r(re^{\frac{r}{\delta}}+re^{-\frac{r}{\delta}})dr$$

$$= \pi j_0(1-i\omega\varepsilon_0\frac{\varepsilon(\omega)}{\sigma(\omega)})\Big[(\delta re^{\frac{r}{\delta}}-\delta^2e^{\frac{r}{\delta}})-(\delta re^{-\frac{r}{\delta}}-\delta^2e^{-\frac{r}{\delta}})\Big]_0^r$$

$$= 2\pi j_0(1-i\omega\varepsilon_0\frac{\varepsilon(\omega)}{\sigma(\omega)})[\delta r\sinh(\frac{r}{\delta})-\delta^2\cosh(\frac{r}{\delta})+\delta^2]$$

$$\therefore \vec{H}(\vec{r},\omega) = \frac{j_0}{r}(1-i\omega\varepsilon_0\frac{\varepsilon(\omega)}{\sigma(\omega)})[\delta^2-\delta^2\cosh(\frac{r}{\delta})+\delta r\sinh(\frac{r}{\delta})]$$

Task 3: 3.75

Solution:

Maxwell's equation:

① $\text{rot } \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}$ ③ $\text{div } \vec{D}(\vec{r},t) = \rho$ ✓

② $\text{rot } \vec{H}(\vec{r},t) = \vec{j}(\vec{r},t) + \frac{\partial \vec{D}(\vec{r},t)}{\partial t}$ ✓ ④ $\text{div } \vec{B}(\vec{r},t) = 0$

According to equation ②, ③

we can get:

$\text{div}[\text{rot } \vec{H}(\vec{r},t)] = \text{div } \vec{j}(\vec{r},t) + \frac{\partial \rho}{\partial t}$ ✓

? $\because \text{div}[\text{rot } \vec{H}(\vec{r},t)] = 0$ why? −0.25

$\therefore$ ⑤ $\text{div } \vec{j}(\vec{r},t) + \frac{\partial \rho}{\partial t} = 0$ ✓ (Differential notation)

Explaination of the equation ⑤:

The varying charges is the source of divergence of the current.

Both ⑤ and ⑥ mean the law of charge conservation. ✓

According to the definition of divergence:

$$\text{div } \vec{A} = \lim_{V \to 0} \frac{\oint_S \vec{A} \cdot d\vec{s}}{V}$$

$$\therefore \int_V \text{div } \vec{A} \, dv = \oint_S \vec{A} \cdot d\vec{s}$$ ✓

So the equation ⑤ can be written as:

$$⑥ \quad \oint_S \vec{j}(\vec{r},t) \, d\vec{s} + \int_V \frac{\partial \rho}{\partial t} \, dv = 0$$ ✓

(Integral notation)

Explaination of the equation ⑥:

In the space of volume V enclosed by a closed surface s, the charge flowing into the closed surface S per unit time is the same as the charge lost in space V. ✓

Task 4 :

a) 1.5

1.5

Solution:

Maxwell's equations : (in empty space)

① $\mathrm{rot}\, \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}$ ③ $\mathrm{div}\, \vec{D}(\vec{r},t) = \rho$

② $\mathrm{rot}\, \vec{H}(\vec{r},t) = \vec{j}(\vec{r},t) + \frac{\partial \vec{D}(\vec{r},t)}{\partial t}$ ④ $\mathrm{div}\, \vec{B}(\vec{r},t) = 0$

In empty space ,

applying the curl operator a second time on equation ②

$$\mathrm{rot\,rot}\, \vec{H}(\vec{r},t) = \mathrm{rot}\, \vec{j}(\vec{r},t) + \varepsilon_0\, \mathrm{rot} \frac{\partial \vec{E}(\vec{r},t)}{\partial t}$$

$$⑤ \quad \mathrm{rot\,rot}\, \vec{H}(\vec{r},t) = -\varepsilon_0 \mu_0 \frac{\partial^2 \vec{H}(\vec{r},t)}{\partial t^2} + \mathrm{rot}\, \vec{j}(\vec{r},t)$$

The equation ⑤ is the wave equation for the magnetic field.

$-\nabla^2 H$ -0.5

b) 1.5

Solution:

Fourier domain: $\mathrm{rot\,rot}\,\vec{H}(\vec{r},w) = \frac{w^2}{c^2} \vec{H}(\vec{r},w) + \mathrm{rot}\,\vec{j}(\vec{r},w)$

c) 1.5

Solution:

$$\mathrm{rot\,rot}\, \vec{H} = \mathrm{grad\,div}\, \vec{H} - \Delta \vec{H} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \Delta^{(2)} \vec{H}_x \\ \Delta^{(2)} \vec{H}_y \\ \Delta^{(2)} \vec{H}_z \end{bmatrix}$$

$\Delta^{(2)}$?

$\vec{H} = \vec{H}_\perp + \vec{H}_\parallel$

$$\vec{H}_\perp = \begin{bmatrix} 0 \\ H_y \\ 0 \end{bmatrix} \quad \vec{H}_\parallel = \cancel{\begin{bmatrix} H_x \\ 0 \\ H_z \end{bmatrix}} \quad \begin{bmatrix} \vec{H}_x \\ 0 \\ \vec{H}_z \end{bmatrix}$$

why ? -0.5

Two decoupled equations :

$$\Delta^{(2)} \vec{H}_\perp(\vec{r},w) + \frac{w^2}{c^2} \vec{H}_\perp(\vec{r},w) + \mathrm{rot}\, \vec{j}_\perp(\vec{r},w) = 0$$

$$\Delta^{(2)} \vec{H}_\parallel(\vec{r},w) + \frac{w^2}{c^2} \vec{H}_\parallel(\vec{r},w) + \mathrm{rot}\, \vec{j}_\parallel(\vec{r},w) = 0$$

d)

Solution: 1

Because $\mathrm{div}\, \vec{H}(\vec{r},w) = 0$, the derivation of the decoupled equations for the magnetic field is simpler.

Task 5:

$$F\{\tilde{\theta}(w)\} = \int_{-\infty}^{+\infty} \tilde{\theta}(w) e^{-iwt} dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} P.V.\ \frac{i}{w} e^{-iwt} dw + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \pi\delta(w) e^{-iwt} dw$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0-\varepsilon} \frac{i}{w} e^{-iwt} dw + \int_{0+\varepsilon}^{+\infty} \frac{i}{w} e^{-iwt} dw\right] + \frac{1}{2}$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{-\varepsilon} -\frac{i}{w} e^{i(-w)t} d(-w) + \int_{\varepsilon}^{+\infty} \frac{i}{w} e^{-iwt} dw\right] + \frac{1}{2}$$

$$= \frac{1}{2\pi}\int_{\varepsilon}^{+\infty} \frac{i}{w} e^{-iwt} - \frac{i}{w} e^{iwt} dw + \frac{1}{2}$$

$$= \frac{1}{2\pi}\int_{\varepsilon}^{+\infty} -\frac{i \cdot 2i}{w}\left(\frac{e^{iwt} - e^{-iwt}}{2i}\right) dw + \frac{1}{2}$$

$$= \frac{1}{\pi}\int_{\varepsilon}^{+\infty} \frac{\sin(wt)}{w} dw + \frac{1}{2}$$

when $t > 0$, $F\{\tilde{\theta}(w)\} = \frac{\pi}{2}\cdot\frac{1}{\pi} + \frac{1}{2} = 1$ 4

when $t = 0$, $F\{\tilde{\theta}(w)\} = 0 + \frac{1}{2} = \frac{1}{2}$

when $t < 0$, $F\{\tilde{\theta}(w)\} = -\frac{1}{2} + \frac{1}{2} = 0$