Optical Modeling and Design

Frank Wyrowski

Chapter 1

Electromagnetic fields in homogeneous media

In field tracing light is always represented by electromagnetic fields. First we deal with electromagnetic fields in homogeneous media.

1.1 Maxwell's equations in the time domain

- Light is an electromagnetic field whose physical nature is mathematically governed by Maxwell's equations:

$$\nabla \times \bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r},t) = -\frac{\partial}{\partial t}\bar{\boldsymbol{B}}^{(\mathrm{r})}(\boldsymbol{r},t)\,, \tag{1.1}$$

$$\nabla \times \bar{\boldsymbol{H}}^{(\mathrm{r})}(\boldsymbol{r},t) = \bar{\boldsymbol{j}}^{(\mathrm{r})}(\boldsymbol{r},t) + \frac{\partial}{\partial t}\bar{\boldsymbol{D}}^{(\mathrm{r})}(\boldsymbol{r},t)\,, \tag{1.2}$$

$$\nabla \cdot \bar{\boldsymbol{D}}^{(\mathrm{r})}(\boldsymbol{r},t) = \bar{\rho}^{(\mathrm{r})}(\boldsymbol{r},t)\,, \tag{1.3}$$

$$\nabla \cdot \bar{\boldsymbol{B}}^{(\mathrm{r})}(\boldsymbol{r},t) = 0 \tag{1.4}$$

  with the position vector $\boldsymbol{r} = (x, y, z)$.

- All vectorial field quantities have three components, for instance $\bar{\boldsymbol{E}}^{(\mathrm{r})} = (\bar{E}_x^{(\mathrm{r})}, \bar{E}_y^{(\mathrm{r})}, \bar{E}_z^{(\mathrm{r})})$.
- We introduce the index r to emphasize, that the field quantities stand for the real optical fields, in contrast to the complex valued fields which are introduced later.

The units of the field quantities are:

- Electric field: $[E] = \mathrm{V/m} = \mathrm{m\ kg/s^3A}$
- Magnetic field: $[H] = \mathrm{A/m}$
- Dielectric displacement: $[D] = \mathrm{C/m^2} = \mathrm{A\ s/m^2}$
- Magnetic induction: $[B] = \mathrm{T} = \mathrm{kg/s^2A}$
- Current density: $[j] = \mathrm{A/m^2}$
- Charge density: $[\rho] = \mathrm{C/m^3} = \mathrm{A\ s/m^3}$

1.2 Maxwell's equations in frequency domain

- The time dependency of all field quantities may be transformed into the frequency domain by the Fourier transformation[1]

$$\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r}, \omega) = \mathcal{F}_\omega \bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r}, t) \tag{1.5}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r}, t)\, \mathrm{e}^{+\mathrm{i}\omega t}\, \mathrm{d}t \tag{1.6}$$

[1] Note, that the field units in frequency domain have an extra time dimension factor.

and vice versa by the inverse transformation

$$\bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r},t) = \mathcal{F}_{\omega}^{-1}\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r},\omega) \tag{1.7}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r},\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega \tag{1.8}$$

at the example of the electric field.

- The same may be done for all other electromagnetic quantities, that is for $\boldsymbol{D}$, $\boldsymbol{H}$, $\boldsymbol{B}$, $\boldsymbol{j}$ and ρ.
- Remarks:(1) Fourier transformation can also be defined by the opposite sign in the integral kernel. (2) In optics we typically use the angular frequency ω, which is related to the temporal frequency ν via $\omega = 2\pi\nu$.
- Because the values of $\bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r},t)$ are real-valued, its Fourier transformation is hermitian, that is

$$\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r},\omega) = (\boldsymbol{E}^{(\mathrm{r})})^{*}(\boldsymbol{r},-\omega) \tag{1.9}$$

  with the phase conjugation operation $*$.
- Obviously, the mathematical step of the Fourier transformation formally leads to

negative frequencies. Because of (1.9) they do not carry physical information.

- In order to simplify mathematical procedures the negative frequencies are truncated and the complex field vector in frequency domain

$$\boldsymbol{E}(\boldsymbol{r},\omega) = \begin{cases} 2\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r},\omega) & \text{if } \omega \geq 0 \\ 0 & \text{otherwise} \end{cases} . \tag{1.10}$$

is obtained. That yields the complex field vector

$$\bar{\boldsymbol{E}}(\boldsymbol{r},t) = \boldsymbol{\mathcal{F}}_{\omega}^{-1}\boldsymbol{E}(\boldsymbol{r},\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\boldsymbol{E}(\boldsymbol{r},\omega)\exp\left(-\mathrm{i}\omega t\right)\,\mathrm{d}\omega \tag{1.11}$$

in time domain.

- The function $\boldsymbol{E}(\boldsymbol{r},\omega)$ in (1.10) is not hermitian and therefore $\bar{\boldsymbol{E}}(\boldsymbol{r},t)$ is complex valued. That is the reason to call $\bar{\boldsymbol{E}}(\boldsymbol{r},t)$ the complex electric field which is emphasized by skipping the index r.

- The basic relationship between the real electric field and its complex generalization is given by[2]

$$\bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r},t) = \Re\Big(\bar{\boldsymbol{E}}(\boldsymbol{r},t)\Big)\,. \tag{1.12}$$

- The proof uses Eqs. (1.9)-(1.10).

- In order to formulate the Maxwell's equations in the frequency domain we first replace the real fields, i.e. $\bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r},t)$, by its complex generalization

$$\bar{\boldsymbol{E}}(\boldsymbol{r},t) = \bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r},t) + \mathrm{i}\bar{\boldsymbol{E}}'(\boldsymbol{r},t)\,. \tag{1.13}$$

Here we already used Eq.(1.12).

[2]see Mandel/Wolf, Eq. (3.1-8b)

- Thus, we can reformulate Eq.(1.1) - Eq.(1.4) by

$$\Re\Big(\nabla \times \bar{\boldsymbol{E}}(\boldsymbol{r},t) + \frac{\partial}{\partial t}\bar{\boldsymbol{B}}(\boldsymbol{r},t)\Big) = 0\,, \tag{1.14}$$

$$\Re\Big(\nabla \times \bar{\boldsymbol{H}}(\boldsymbol{r},t) - \bar{\boldsymbol{j}}(\boldsymbol{r},t) - \frac{\partial}{\partial t}\bar{\boldsymbol{D}}(\boldsymbol{r},t)\Big) = 0\,, \tag{1.15}$$

$$\Re\Big(\nabla \cdot \bar{\boldsymbol{D}}(\boldsymbol{r},t) - \bar{\rho}(\boldsymbol{r},t)\Big) = 0\,, \tag{1.16}$$

$$\Re\Big(\nabla \cdot \bar{\boldsymbol{B}}(\boldsymbol{r},t)\Big) = 0\,. \tag{1.17}$$

  Here we benefit from the linear nature of Maxwell's equations.

- Next we can replace all time dependent all quantities by its Fourier transformed version and obtain

$$\Re\left(\mathcal{F}_\omega^{-1}\left[\nabla\times\boldsymbol{E}(\boldsymbol{r},\omega)-\mathrm{i}\omega\boldsymbol{B}(\boldsymbol{r},\omega)\right]\right)=0\,, \tag{1.18}$$

$$\Re\left(\mathcal{F}_\omega^{-1}\left[\nabla\times\boldsymbol{H}(\boldsymbol{r},\omega)-\boldsymbol{j}(\boldsymbol{r},\omega)+\mathrm{i}\omega\boldsymbol{D}(\boldsymbol{r},\omega)\right]\right)=0\,, \tag{1.19}$$

$$\Re\left(\mathcal{F}_\omega^{-1}\left[\nabla\cdot\boldsymbol{D}(\boldsymbol{r},\omega)-\rho(\boldsymbol{r},\omega)\right]\right)=0\,, \tag{1.20}$$

$$\Re\left(\mathcal{F}_\omega^{-1}\left[\nabla\cdot\boldsymbol{B}(\boldsymbol{r},\omega)\right]\right)=0\,. \tag{1.21}$$

- Here we used one big advantage of working in the Fourier domain, that is the replacement of the time derivation by the factor $-\mathrm{i}\omega$.

- Equations (1.18)-(1.21) are solved if the terms in the squared brackets are zero.

Thus we can concentrate on the solution of the equations

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},\omega) = \mathrm{i}\omega \boldsymbol{B}(\boldsymbol{r},\omega) \ , \tag{1.22}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},\omega) = \boldsymbol{j}(\boldsymbol{r},\omega) - \mathrm{i}\omega \boldsymbol{D}(\boldsymbol{r},\omega) \ , \tag{1.23}$$

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r},\omega) = \rho(\boldsymbol{r},\omega) \ , \tag{1.24}$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r},\omega) = 0 \, . \tag{1.25}$$

- Thus in what follows we mainly deal with the complex field expressions and the corresponding field equations. However, we like to give the following urgent warnings when dealing with the complex field quantities:
  - Detector functions, like the energy, are typically defined by the real fields in time domain. Then, Eq.(1.12) is to be used before the detector function can be evaluated. At the example of the electric field that requires ultimately to apply

    $$\bar{\boldsymbol{E}}^{(\mathrm{r})}(\boldsymbol{r},t) = \Re\Big(\mathcal{F}_{\omega}^{-1}\Big[\boldsymbol{E}(\boldsymbol{r},\omega)\Big]\Big) \tag{1.26}$$

    before the detector function can be evaluated.
  - The consistency of conclusions drawn from the complex field equations with conclusions from the real field equations should be checked. To this end keep

in mind, that any complex field equation of the form $\big(\ldots\big) = 0$ in frequency domain is a short version of $\Re\Big(\mathcal{F}_\omega^{-1}\Big[\big(\ldots\big)\Big]\Big) = 0$.

1.3 Linear matter equations

- In macroscopic media we assume specific relations between $\boldsymbol{E}$ and $\boldsymbol{D}$, between $\boldsymbol{E}$ and $\boldsymbol{j}$ and between $\boldsymbol{H}$ and $\boldsymbol{B}$ in the frequency domain. Those relations are called matter or constitutive equations.[3]

- In what follows we assume linear media and obtain linear dependencies of the form:

$$\boldsymbol{j}(\boldsymbol{r},\omega) = \boldsymbol{\sigma}(\boldsymbol{r},\omega)\,\boldsymbol{E}(\boldsymbol{r},\omega)\,, \tag{1.27}$$

$$\boldsymbol{D}(\boldsymbol{r},\omega) = \boldsymbol{\epsilon}(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega) = \epsilon_0\boldsymbol{\epsilon}_{\mathrm{r}}(\boldsymbol{r},\omega)\,\boldsymbol{E}(\boldsymbol{r},\omega)\,, \tag{1.28}$$

$$\boldsymbol{B}(\boldsymbol{r},\omega) = \boldsymbol{\mu}(\boldsymbol{r},\omega)\,\boldsymbol{H}(\boldsymbol{r},\omega)\,. \tag{1.29}$$

- In general the material quantities are tensors (3×3) and called conductivity $\boldsymbol{\sigma}$, electric permittivity $\boldsymbol{\epsilon}$ and magnetic permeability $\boldsymbol{\mu}$.

[3]Often that is also discussed in terms of the material polarization vector $\boldsymbol{P}$. Comprehensive discussion in *Fundamentals* lecture.

- The electric permittivity is often factorized by $\boldsymbol{\epsilon} = \epsilon_0 \boldsymbol{\epsilon}_\mathrm{r}$[4] with the relative permittivity $\boldsymbol{\epsilon}_\mathrm{r}$ and the vacuum electric permittivity

$$\epsilon_0 = 8.8541878176 \times 10^{-12}\,\mathrm{N}^{-1}\,\mathrm{m}^{-2}\,\mathrm{C}^2\,. \tag{1.30}$$

- Analogously a vacuum magnetic permeability is given by

$$\mu_0 = 1.2566370614 \times 10^{-6}\,\mathrm{m}\,\mathrm{kg}\,\mathrm{C}^{-2}\,. \tag{1.31}$$

- Both are related via

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.99792458 \times 10^8\,\mathrm{m}\,\mathrm{s}^{-1}\,. \tag{1.32}$$

  with the speed of light in vacuum c.

- In optics Maxwell's equations are often discussed with at least some of the following restrictions:

  1. In optics free charges are typically of no concern and therefore $\rho = 0$.
  2. Isotropic media: Matter functions are scalar functions instead of tensors.

[4] Index r in $\boldsymbol{\epsilon}_\mathrm{r}$ refers to "‘relative"’ and not "‘real"’!

3. In optics (from IR to UV) natural media are not magnetic and $\boldsymbol{\mu}(\boldsymbol{r},\omega)=\mu_0$.
4. Homogeneous media: Matter functions are not dependent of the location $\boldsymbol{r}$.
5. Non-dispersive media: Matter functions are not dependent of the frequency ω.

- Application of matter equations under consideration of restrictions 1-3 results in the following set of Maxwell's equations:

$$\nabla\times\boldsymbol{E}(\boldsymbol{r},\omega)=\mathrm{i}\omega\mu_0\boldsymbol{H}(\boldsymbol{r},\omega) \tag{1.33}$$
$$\nabla\times\boldsymbol{H}(\boldsymbol{r},\omega)=-\mathrm{i}\omega\epsilon_0\check{\epsilon}_{\mathrm{r}}(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega) \tag{1.34}$$
$$\nabla\cdot\Big(\check{\epsilon}_{\mathrm{r}}(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega)\Big)=0 \tag{1.35}$$
$$\nabla\cdot\boldsymbol{H}(\boldsymbol{r},\omega)=0 \tag{1.36}$$

- Here we introduced the *generalized permittivity*

$$\check{\epsilon}_{\mathrm{r}}(\boldsymbol{r},\omega):=\epsilon_{\mathrm{r}}(\boldsymbol{r},\omega)+\mathrm{i}\frac{\sigma(\boldsymbol{r},\omega)}{\omega\epsilon_0}\,. \tag{1.37}$$

- If we restrict in addition to homogeneous media[5] Maxwell's equations in frequency

[5] Here we exclude the case $\check{\epsilon}_{\mathrm{r}}(\omega)=0$.

domain read

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},\omega) = \mathrm{i}\omega\mu_0 \boldsymbol{H}(\boldsymbol{r},\omega) \tag{1.38}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},\omega) = -\mathrm{i}\omega\epsilon_0 \check{\epsilon}_{\mathrm{r}}(\omega) \boldsymbol{E}(\boldsymbol{r},\omega)\,, \tag{1.39}$$

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r},\omega) = 0\,, \tag{1.40}$$

$$\nabla \cdot \boldsymbol{H}(\boldsymbol{r},\omega) = 0\,. \tag{1.41}$$