Institute of Applied Physics
Friedrich-Schiller-Universität Jena

Laser Physics problem sheet 10

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Problem 2 (6 points)

A graduate student is commended with the task of designing a stable optical cavity for a laser. Unfortunately, due to the global financial crisis his university can only afford to buy a very small active crystal, two identical concave mirrors (nominal radius of curvature R and tolerance $\pm\Delta R$) and a plane mirror. He feels frustrated by the scarcity of elements and does not really find a way to set-up the different elements into a working system. Can you help him?

a) Design a simple stable optical resonator able to make the most of the very small active crystal. (1 point)

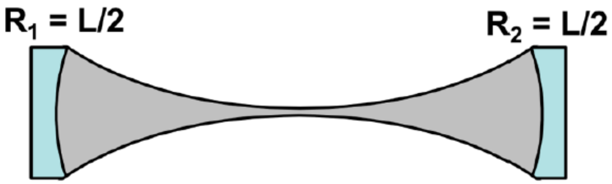
Inventory List:

- **1 small active crystal**
- **2 curved mirrors with $R_1 \pm \Delta R_1$**
- **1 plane mirror**

Since we want a small focus we want to choose an according cavity design..

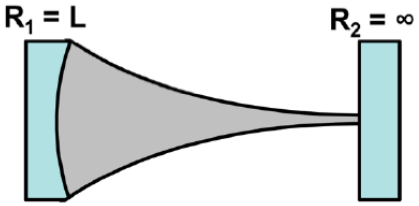
From script we know:

D) *Concentric resonator*: $g_1 = g_2 = -1$ (it is also at the edge of stability)



It is worth mentioning that the concentric resonator creates the smallest waist

A prominent example for non-symmetric resonators is the hemi-spherical resonator illustrated below.



In terms of focal spot size they are identical ..

We choose this one

b) Taking into account the tolerance of the curvature radius of the concave mirrors, make a stability analysis of your cavity and decide what the student has to do, in case that his cavity results unstable. (2 points)

Now we have to do a stability analysis:

The resonator is stable if:

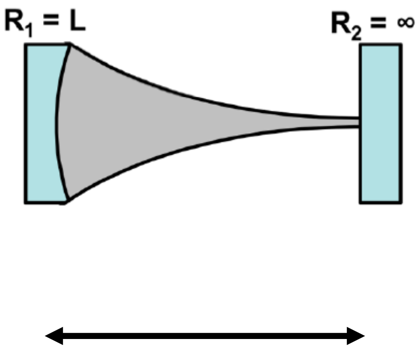
$$0 \le g_1 g_2 \le 1$$

So we can choose our coordinates as:

$$g_1 = 1 - \frac{L'}{R_1} = 1 - \frac{2L}{R_1}$$

$$g_2 = 1 - \frac{L'}{R_1} = 1 - \frac{2L}{R_1}$$

Here we do a little trick:

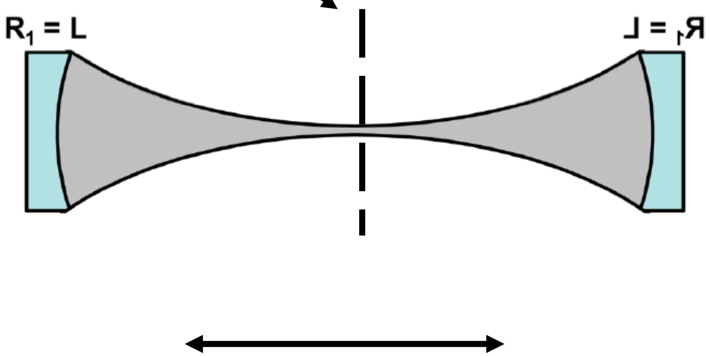


$L = R_1$

equivalent

$\Leftrightarrow$

Folded around this axis



$L' = 2L = 2R_1$

b) Taking into account the tolerance of the curvature radius of the concave mirrors, make a stability analysis of your cavity and decide what the student has to do, in case that his cavity results unstable. (2 points)

Now we have to do a stability analysis:

The resonator is stable if:

$$0 \leq g_1 g_2 \leq 1, \qquad \text{with } g_1 = g_2 = 1 - \frac{2L}{R_1}$$

$$\Rightarrow 0 \leq \left(1 - \frac{2L}{R_1 \pm \Delta R_1}\right)^2 \leq 1$$

This term is always ≥ 0

$$\Rightarrow \left(1 - \frac{2L}{R_1 \pm \Delta R_1}\right)^2 \leq 1$$

$$\Rightarrow -1 \leq 1 - \frac{2L}{R_1 \pm \Delta R_1} \leq 1$$

Evaluate the two cases:

$$1 - \frac{2L}{R_1 \pm \Delta R_1} \leq 1 \Rightarrow \frac{2L}{R_1 \pm \Delta R_1} \geq 0 \Rightarrow R < \infty$$

$$1 - \frac{2L}{R_1 \pm \Delta R_1} \geq -1 \Rightarrow \frac{2L}{R_1 \pm \Delta R_1} \leq 2 \Rightarrow 2L \leq 2R_1 \pm 2\Delta R_1$$

If $+\Delta R \Rightarrow$ cavity is stable

If $-\Delta R \Rightarrow$ cavity is unstable
$\Rightarrow$ make $L \leq R_1 - \Delta R_1$

Problem 2 (6 points)

A graduate student is commended with the task of designing a stable optical cavity for a laser. Unfortunately, due to the global financial crisis his university can only afford to buy a very small active crystal, two identical concave mirrors (nominal radius of curvature R and tolerance $\pm\Delta R$) and a plane mirror. He feels frustrated by the scarcity of elements and does not really find a way to set-up the different elements into a working system. Can you help him?

c) The student has very bad luck and, while transporting his laser to another laboratory, he tripped over and the laser fell to the floor. As a result of this accident one mirror broke. According to Murphy's law the broken mirror will be the one that forces him to change the cavity design if he has to rebuild the laser with the remaining components. Can you help him again and design a new cavity? (1 point)

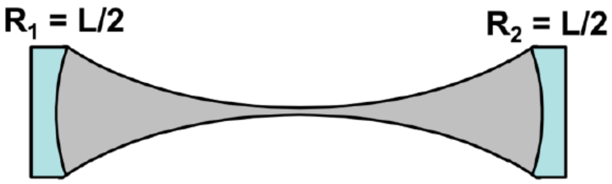
Inventory List:

- **1 small active crystal**
- **2 curved mirrors with $R_1 \pm \Delta R_1$**
- ~~**1 plane mirror**~~

Since we want a small focus we want to choose an according cavity design..

From script we know:

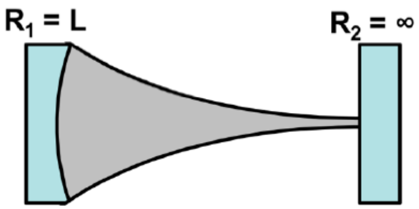
D) *Concentric resonator*: $g_1 = g_2 = -1$ (it is also at the edge of stability)



It is worth mentioning that the concentric resonator creates the smallest waist

We choose this one

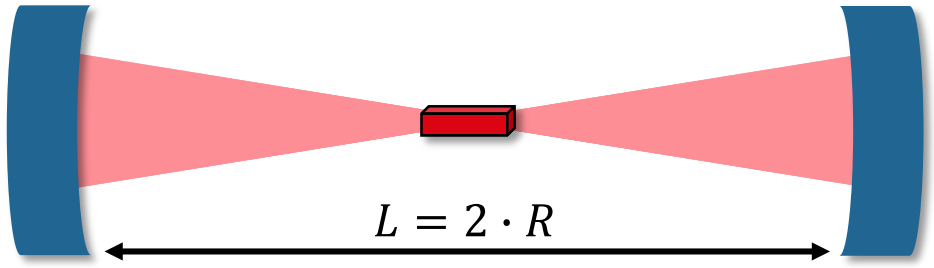
A prominent example for non-symmetric resonators is the hemi-spherical resonator illustrated below.



d) As before, please make a stability analysis of the new cavity and give advice to the student what to do if his cavity is unstable when he builds it. (2 points)

concentric configuration:

$R_1' = R \pm \Delta R_1$ $\qquad$ $R_2' = R \pm \Delta R_2$



Stability condition (from script):

$$0 \leq g_1 \cdot g_2 \leq 1 \qquad\qquad 0 \leq \left(1 - \frac{L}{R_1'}\right) \cdot \left(1 - \frac{L}{R_2'}\right) \leq 1$$

$$0 \leq 1 - \frac{L \cdot (R_1' + R_2' - L)}{R_1' \cdot R_2'} \leq 1$$

(1)

$$0 \leq 1 - \frac{L \cdot (R_1' + R_2' - L)}{R_1' \cdot R_2'}$$

$$L \cdot (2R \pm \Delta R_1 \pm \Delta R_2 - L) \leq R^2 + (\pm\Delta R_1 \pm \Delta R_2) R \pm \Delta R_1 \, \Delta R_2$$

$$\underbrace{L \cdot 2R - L^2}_{=0} + (\pm\Delta R_1 \pm \Delta R_2) L \leq R^2 + (\pm\Delta R_1 \pm \Delta R_2) R \pm \Delta R_1 \, \Delta R_2$$

with $L = 2 \cdot R$

$$0 \leq R^2 - (\pm\Delta R_1 \pm \Delta R_2) R \pm \Delta R_1 \, \Delta R_2$$

necessarily true as $R \gg \Delta R_1, \Delta R_2$

(2)

$$1 - \frac{L \cdot (R_1' + R_2' - L)}{R_1' \cdot R_2'} \leq 1$$

$$\frac{\cancel{L} \cdot (R_1' + R_2' - L)}{\cancel{R_1' \cdot R_2'}} \geq 0$$

$$R_1' + R_2' \geq L$$

$$2R \pm \Delta R_1 \pm \Delta R_2 \geq 2 \cdot R$$

if $\pm\Delta R_1 \pm \Delta R_2 \geq 0$, stable

if $\pm\Delta R_1 \pm \Delta R_2 < 0$, unstable $\Longrightarrow$ shorten the cavity length to stabilize it

Problem 2 (6 Points)

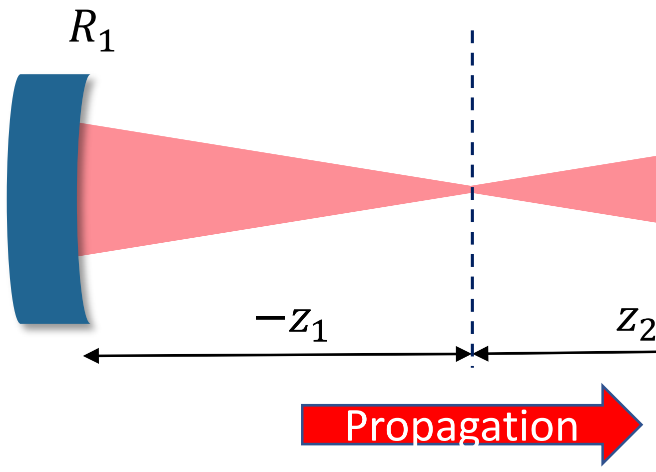
a) Calculate the position of the beam waist in a general stable linear resonator as a function of the g-parameters. (2 points)

b) Calculate the beam waist radius as a function of the g-parameters. (1 point)

c) Demonstrate that the concentric symmetric resonator has the thinnest waist of all symmetric resonators. On the contrary, which resonator has the largest waist? (1 point)

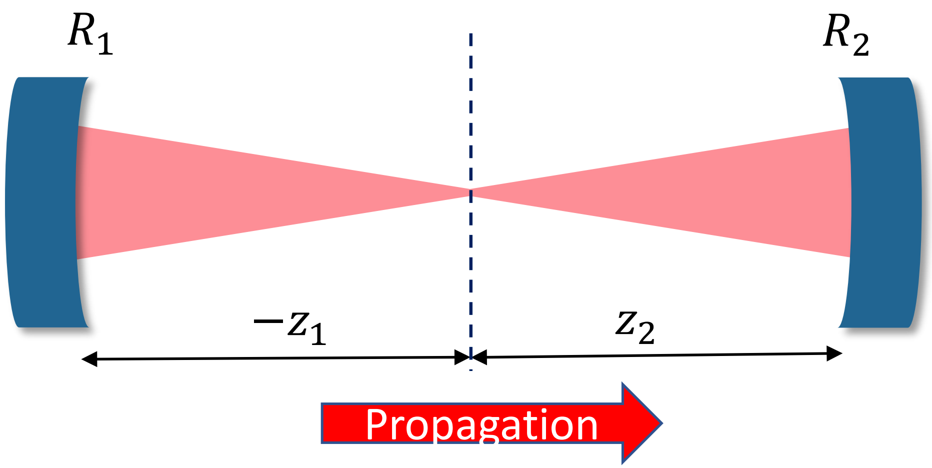
d) Calculate the beam size at the resonator mirrors as a function of the g-parameters. (1 point)

e) It is known that the stability condition of a resonator is independent of the wavelength. Therefore, a stable resonator designed for an Ar-laser will also be stable for a CO_2 laser. Which other parameters have to be taken into account that might degrade the performance of the laser when changing the wavelength? (1 point)



- At the waist, the wavefront is flat
- At the mirror, the wavefront matches the curvature of the mirror

a) Calculate the position of the beam waist in a general stable linear resonator as a function of the g-parameters. (2 points)



Goal: express z_1 and/or z_2 as the function of g_i

(1) At the waist (z=0), the wavefront is flat

(2) At the mirror, the wavefront curvature matches the curvature of the mirror

$$R(z) = z + \frac{z_R^2}{z} \Rightarrow \begin{cases} -R_1 = z_1 + \frac{z_R^2}{z_1} \\ R_2 = z_2 + \frac{z_R^2}{z_2} \end{cases}$$

$$g_i = 1 - \frac{L}{R_i} \Rightarrow R_i = \frac{L}{1-g_i}$$

$$L = -z_1 + z_2$$

$$-\frac{L}{1-g_1} = z_1 + \frac{z_R^2}{z_1}$$

$$\frac{L}{1-g_2} = z_2 + \frac{z_R^2}{z_2}$$

$$\Rightarrow \frac{Lz_2}{1-g_2} + \frac{Lz_1}{1-g_1} = z_2^2 - z_1^2$$

$$\frac{L(L+z_1)}{1-g_2} + \frac{Lz_1}{1-g_1} = L(L+2z_1)$$

$$\boxed{z_1 = \frac{-g_2(1-g_1)L}{g_1+g_2-2g_1g_2}} \qquad \boxed{z_2 = L + z_1 = \frac{g_1(1-g_2)L}{g_1+g_2-2g_1g_2}}$$

b) Calculate the beam waist radius as a function of the g-parameters. (1 point)

$$\omega_0^2 = \frac{z_R \lambda}{\pi}$$

$$\frac{L}{1-g_2} = z_2 + \frac{z_R^2}{z_2} \qquad z_R^2 = \frac{L z_2}{1-g_2} - z_2^2 = \frac{g_1 L^2}{g_1 + g_2 - 2g_1 g_2} - \frac{g_1^2 (1-g_2)^2 L^2}{(g_1+g_2-2g_1g_2)^2} = \frac{g_1 g_2 (1-g_1g_2) L^2}{(g_1+g_2-2g_1g_2)^2}$$

$$\omega_0^2 = \frac{z_R \lambda}{\pi} = \frac{\lambda L}{\pi}\sqrt{\frac{g_1g_2(1-g_1g_2)}{(g_1+g_2-2g_1g_2)^2}} \qquad \boxed{\omega_0 = \sqrt{\frac{\lambda L}{\pi}}\left(\frac{g_1g_2(1-g_1g_2)}{(g_1+g_2-2g_1g_2)^2}\right)^{\frac{1}{4}}}$$

c) Demonstrate that the concentric symmetric resonator has the thinnest waist of all symmetric resonators. On the contrary, which resonator has the largest waist? (1 point)

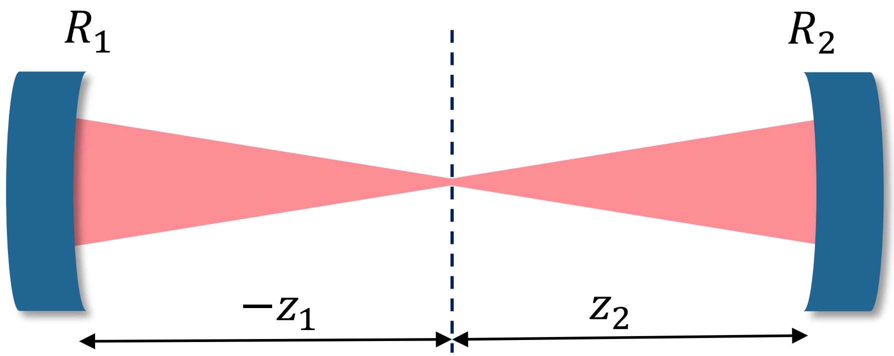
For symmetric cavity: $g_1 = g_2 = g$

$$\omega_0^2 = \frac{\lambda L}{\pi}\sqrt{\frac{g^2(1-g^2)}{(2g-2g^2)^2}} = \frac{\lambda L}{2\pi}\sqrt{\frac{1+g}{1-g}}$$

For concentric symmetric cavity: $g = -1$ $\qquad \omega_0 = 0$

On the contrary, if $g = 1$ $\qquad \omega_0 = \infty$ $\qquad$ Plane-plane cavity

d) Calculate the beam size at the resonator mirrors as a function of the g-parameters. (1 point)



$$\omega(z) = \omega_0 \sqrt{1 + \frac{z^2}{z_R^2}}$$

$$\boxed{\omega_0 = \sqrt{\frac{\lambda L}{\pi}\left(\frac{g_1 g_2 (1 - g_1 g_2)}{(g_1 + g_2 - 2 g_1 g_2)^2}\right)^{\frac{1}{4}}}}$$

$$\boxed{z_1 = \frac{-g_2(1-g_1)L}{g_1 + g_2 - 2g_1 g_2}} \quad \boxed{z_2 = \frac{g_1(1-g_2)L}{g_1 + g_2 - 2g_1 g_2}} \quad \boxed{z_R^2 = \frac{g_1 g_2 (1 - g_1 g_2) L^2}{(g_1 + g_2 - 2 g_1 g_2)^2}}$$

$$\omega(z_1) = \omega_0 \sqrt{1 + \frac{z_1^2}{z_R^2}} = \omega_0 \sqrt{1 + \frac{g_2^2 (1-g_1)^2 L^2}{g_1 g_2 (1 - g_1 g_2) L^2}} = \omega_0 \sqrt{\frac{g_1 + g_2 - 2 g_1 g_2}{g_1 (1 - g_1 g_2)}} = \cdots = \sqrt{\frac{\lambda L}{\pi}\left(\frac{g_2}{g_1 (1 - g_1 g_2)}\right)^{\frac{1}{4}}}$$

$$\omega(z_2) = \cdots = \sqrt{\frac{\lambda L}{\pi}\left(\frac{g_1}{g_2 (1 - g_1 g_2)}\right)^{\frac{1}{4}}}$$

e) It is known that the stability condition of a resonator is independent of the wavelength. Therefore, a stable resonator designed for an Ar-laser will also be stable for a CO_2 laser. Which other parameters have to be taken into account that might degrade the performance of the laser when changing the wavelength? (1 point)

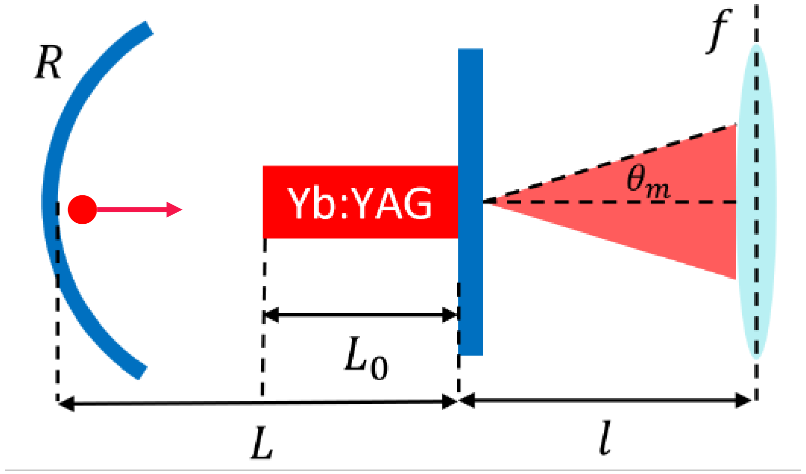
$$\omega(z_2) = \cdots = \sqrt{\frac{\lambda L}{\pi}\left(\frac{g_1}{g_2 (1 - g_1 g_2)}\right)^{\frac{1}{4}}}$$

Ar-laser: 514 nm

CO2-laser: 10.6 µm

Switching from 514 nm to 10.6 µm results in the increase of beam size. If the mirror size stays unchanged, extra loss will be introduced.

A Yb:YAG crystal, which length and refractive index are L_0 and n_0, is placed in a cavity. The cavity, as the following figure shows, is formed by a thin concave mirror having radius of curvature R and a thin plane mirror. The distance between the mirrors is L.



1) Free space $L-L_0$: $\begin{pmatrix}1 & L-L_0\\ 0 & 1\end{pmatrix}$
2) Plane interface: $\begin{pmatrix}1 & 0\\ 0 & 1/n_0\end{pmatrix}$
3) Propagation in Yb:YAG: $\begin{pmatrix}1 & L_0\\ 0 & 1\end{pmatrix}$
4) Plane mirror: $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$
5) Propagation in Yb:YAG: $\begin{pmatrix}1 & L_0\\ 0 & 1\end{pmatrix}$
6) Plane interface: $\begin{pmatrix}1 & 0\\ 0 & n_0/1\end{pmatrix}$
7) Free space $L-L_0$: $\begin{pmatrix}1 & L-L_0\\ 0 & 1\end{pmatrix}$
8) Concave mirror: $\begin{pmatrix}1 & 0\\ -2/R & 1\end{pmatrix}$

a) Write the ABCD matrix for one round trip in this cavity. (1 point)

$$M=\begin{pmatrix}1 & 0\\ -\frac{2}{R} & 1\end{pmatrix}\begin{pmatrix}1 & L-L_0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \frac{n_0}{1}\end{pmatrix}\begin{pmatrix}1 & L_0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & L_0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \frac{1}{n_0}\end{pmatrix}\begin{pmatrix}1 & L-L_0\\ 0 & 1\end{pmatrix}$$

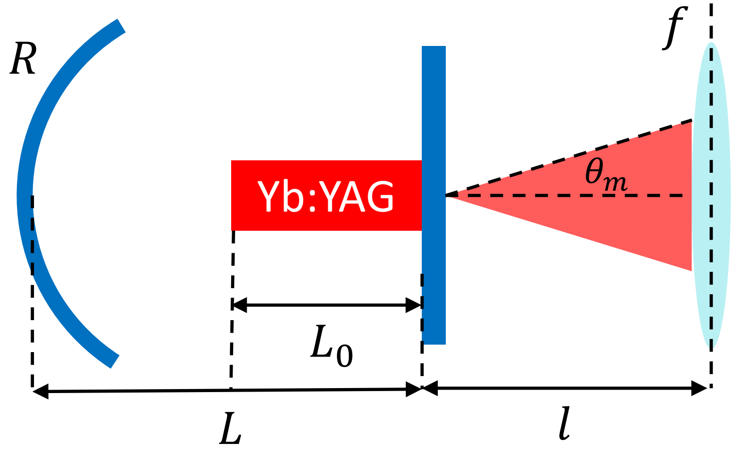
$$\boxed{=\begin{pmatrix}1 & \frac{2(L_0+n_0L-n_0L_0)}{n_0}\\ -\frac{2}{R} & -\frac{4L_0+4n_0L-4n_0L_0-n_0R}{n_0R}\end{pmatrix}}$$

b) For light at $\lambda = 1\mu m$ the refracive index of the active material is $n_0 = 1.82$. Given the following geometric parameters: $L_0 = 50mm$, $R = 200mm$, $L = 109mm$, justify whether the cavity is stable or not. (1 point)

$$g_1 = 1 - \frac{L - L_0 + n_0 L_0}{R} = 1 - \frac{150}{200} = 0.25 \;, g_2 = 1 \qquad g_1 g_2 = 0.25 < 1$$

$\boxed{\text{Stable cavity}}$

c) Assume that the output beam from the cavity is a point source with a numerical aperture $NA = sin\theta_m = 0.22$. A lens with a focal length of f is placed on the propagation path of the beam at a distance l. Try to use the ABCD matrix method to prove that the condition to obtain a collimated beam after the lens is $l = f$ and calculate the diameter of the collimated beam. (2 points)



The ABCD-matrix in this case (free space propagation + lens):

$$M_1 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & l \\ -\frac{1}{f} & 1 - \frac{l}{f} \end{pmatrix}$$

The input ray vector: $\begin{pmatrix} 0 \\ \theta \end{pmatrix}$

The output ray vector:

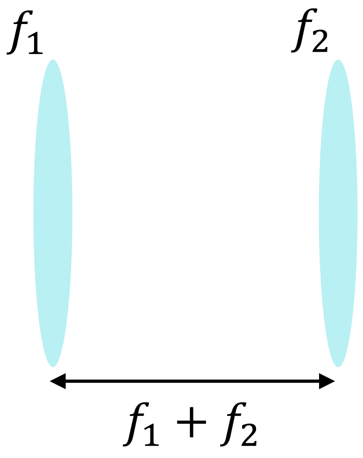
$$\begin{pmatrix} r \\ r' \end{pmatrix} = \begin{pmatrix} 1 & l \\ -\frac{1}{f} & 1 - \frac{l}{f} \end{pmatrix} \begin{pmatrix} 0 \\ \theta \end{pmatrix} = \begin{pmatrix} l\theta \\ \theta\left(1 - \frac{l}{f}\right) \end{pmatrix}$$

For a collimated beam, its angle r' should not vary with θ $\boxed{l = f}$

The beam at the edge, whose ray vector is $\begin{pmatrix} 0 \\ \theta_m \end{pmatrix}$

The output ray vector: $\begin{pmatrix} l\theta_m \\ 0 \end{pmatrix}$, the beam diameter is $\boxed{2l\theta_m}$

d) In experiments, people often use telescope systems to magnify or demagnify a collimated beam. A telescope is formed by 2 lenses placed in such a way that their distance is $f_1 + f_2$, where f_1 is the focal length of the first lens and f_2 the focal length of the second lens. Write the ABCD-Matrix of such a telescope system and calculate its magnification as a function of f_1 and f_2. (2 points)



The ABCD-matrix of a telescope system:

$$M_2 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & f_1 + f_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \begin{pmatrix} -\frac{f_2}{f_1} & f_1 + f_2 \\ 0 & -\frac{f_1}{f_2} \end{pmatrix}$$

Consider the edge ray of a collimated beam $\begin{pmatrix} r \\ 0 \end{pmatrix}$, the output ray vector is

$$\begin{pmatrix} -\frac{f_2}{f_1} & f_1 + f_2 \\ 0 & -\frac{f_1}{f_2} \end{pmatrix} \begin{pmatrix} r \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{f_2}{f_1} r \\ 0 \end{pmatrix}$$

Magnification: $-\frac{f_2}{f_1}$

e) In our laboratory we have the 6 lenses listed on the table:

Number	Aperture diameter/inch	Focal length/mm
1	1	30
2	1	60
3	1	80
4	2	100
5	2	300
6	2	500

1 inch = 25.40 mm

Point source: $NA = \sin(\theta_m) = 0.22$

$\theta_m \approx \sin(\theta_m) = 0.22$

Using these lenses and the light source in c), what is the diameter of the lartgest collimated beam that can be obtained without clipping (i.e. this occurs when the beam becomes larger than any of the lenses)? (2 points)

- Due to the aperture limitation, any value larger than 2 inch (50.80 mm) is not possible

1) Using lens 1 for collimating gives us a collimated beam of $2f_1\theta_m = 13.52$ mm
2) Combining lens 3 and lens 5 as a telescope gives a magnification of 300/80, this in the end magnifies the collimated beam to 13.52 mm *300/80 = 50.70 mm