Task 1

a)

Solution:

$$\vec{E}(\vec{r}) = e^{i\beta x}[A_1 e^{ikz} + A_2 \cos(kz)]\,\vec{e_y}$$

$$\vec{E}(\vec{r},t) = e^{i\beta x}[A_1 e^{ikz} + A_2 \cos(kz)]\,e^{-i\omega t}\,\vec{e_y}$$

$$\vec{\vec{E}}(\vec{r},\omega) = e^{i\beta x}[A_1 e^{ikz} + A_2 \cos(kz)]\,\delta(\omega'-\omega)\,\vec{e_y}$$

From Maxwell's equations

We can get : $\boxed{\cancel{\vec{\vec{H}}(\vec{r},\omega) = -\frac{i}{\omega\mu_0}\vec{\vec{E}}(\vec{r},\omega)}}$

$$\vec{\vec{H}}(\vec{r},\omega) = -\frac{i}{\omega\mu_0}\,\mathrm{rot}\,\vec{\vec{E}}(\vec{r},\omega)$$

$$\therefore \vec{\vec{H}}(\vec{r},\omega) = \frac{\beta}{\omega\mu_0} e^{i\beta x}[A_1 e^{ikz} + A_2\cos(kz)]\,\delta(\omega'-\omega)\,(\vec{e_x}\times\vec{e_y})$$

$$- \frac{i\cdot(ik)}{\omega\mu_0} e^{i\beta x}[A_1 e^{ikz} + iA_2\sin(kz)]\,\delta(\omega'-\omega)\,(\vec{e_z}\times\vec{e_y})$$

$$= \frac{\beta}{\omega\mu_0} e^{i\beta x}[A_1 e^{ikz} + A_2\cos(kz)]\,\delta(\omega'-\omega)\,\vec{e_z} + \frac{k}{\omega\mu_0} e^{i\beta x}[A_1 e^{ikz} + iA_2\sin(kz)]\,\delta(\omega'-\omega)(-\vec{e_x})$$

$$\vec{H}(\vec{r},t) = \frac{\beta}{\omega\mu_0} e^{i\beta x}[A_1 e^{ikz} + A_2\cos(kz)]\,e^{-i\omega t}\,\vec{e_z} + \frac{k}{\omega\mu_0} e^{i\beta x}[A_1 e^{ikz} + iA_2\sin(kz)]\,e^{-i\omega t}(-\vec{e_x})$$

$$\therefore \vec{H}(\vec{r}) = \frac{\beta}{\omega\mu_0} e^{i\beta x}[A_1 e^{ikz} + A_2\cos(kz)]\,\vec{e_z} - \frac{k}{\omega\mu_0} e^{i\beta x}[A_1 e^{ikz} + iA_2\sin(kz)]\,\vec{e_x}$$

b)

Solution:

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}[\vec{E}(\vec{r})\times\vec{H}(\vec{r})^*]$$

$$= \frac{1}{2}\mathrm{Re}\left\{\frac{\beta}{\omega\mu_0} E(\vec{r})\, e^{-i\beta x}[A_1 e^{-ikz} + A_2\cos(kz)](-\vec{e_x}) + \frac{k}{\omega\mu_0} E(\vec{r})\, e^{-i\beta x}[A_1 e^{-ikz} - iA_2\sin(kz)]\,(\vec{e_y}\times\vec{e_z})\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\frac{\beta}{\omega\mu_0}(A_1^2 + 2A_1A_2\cos^2(kz) + A_2^2\cos^2(kz))(-\vec{e_x}) + \frac{k}{\omega\mu_0}[A_1^2 - iA_1A_2 e^{ikz}\sin(kz) + A_1A_2 e^{-ikz}\cos(kz) - iA_2^2\sin(kz)\cos(kz)]\,(\vec{e_y}\times\vec{e_z})\right\}$$

$$= \frac{\beta}{2\omega\mu_0}[A_1^2 + 2A_1A_2\cos^2(kz) + A_2^2\cos^2(kz)](-\vec{e_x}) + \frac{k}{2\omega\mu_0}[A_1^2 + A_1A_2\sin^2(kz) + A_1A_2\cos^2(kz)]\,(\vec{e_y}\times\vec{e_z})$$

$$= \frac{\beta}{2\omega\mu_0}[A_1^2 + A_2(A_2+2A_1)\cos^2(kz)](-\vec{e_x}) + \frac{k}{2\omega\mu_0}(A_1^2 + A_1A_2)\,(\vec{e_y}\times\vec{e_z})$$

$$= \frac{k}{2\omega\mu_0}(A_1^2 + A_1A_2)\,\vec{e_x} - \frac{\beta}{2\omega\mu_0}[A_1^2 + A_2(A_2+2A_1)\cos^2(kz)]\,\vec{e_z}$$

c)

prove:

Use the Helmholtz equation in vacuum:

$$\Delta \vec{E}(\vec{r},\omega) + \frac{\omega^2}{c^2}\vec{E}(\vec{r},\omega) = 0$$

$$\cancel{\therefore (i\beta)^2 \vec{E}(\vec{r}) + (ik)^2 \vec{E}(\vec{r}) + \frac{\omega^2}{c^2}\vec{E}(\vec{r}) = 0}$$

$$(i\beta)^2 e^{i\beta x}[A_1 e^{ikz} + A_2\cos(kz)]\vec{e}_y + (ik)^2 e^{i\beta x}[A_1 e^{ikz} + A_2\cos(kz)]\vec{e}_y + \frac{\omega^2}{c^2} e^{i\beta x}[A_1 e^{ikz} + A_2\cos(kz)]\vec{e}_y = 0$$

$$(i\beta)^2\vec{E}(\vec{r}) + (ik)^2\vec{E}(\vec{r}) + \frac{\omega^2}{c^2}\vec{E}(\vec{r}) = 0$$

$$\therefore \frac{\omega^2}{c^2} = \beta^2 + k^2$$

$$\because \omega = \frac{2\pi c}{\lambda_0}$$

$$\therefore \frac{4\pi^2}{\lambda_0^2} = \beta^2 + k^2$$

$$\lambda_0^2 = \frac{4\pi^2}{\beta^2 + k^2}$$

$$\lambda_0 = \frac{2\pi}{\sqrt{\beta^2 + k^2}}$$

Task 2

a) Solution:

From the Helmholtz equation $\Delta \vec{H}(\vec{r},\omega) + \frac{w^2}{c^2}\varepsilon \vec{H}(\vec{r},\omega) = 0$

We can get:

$x<0$ $\quad \varepsilon = \varepsilon_M$

$$-k_x^2 e^{[i(k_x x + k_z z)]} - k_z^2 e^{[i(k_x x + k_z z)]} + \frac{w^2}{c^2}\varepsilon_M e^{[i(k_x x + k_z z)]} = 0 \quad ①$$

$x>0$ $\quad \varepsilon = 1$

$$-k_x^2 e^{[i(k_x x + k_z z)]} - k_z^2 e^{[i(k_x x + k_z z)]} + \frac{w^2}{c^2} e^{[i(k_x x + k_z z)]} = 0 \quad ②$$

From equation ①

we can get $(x<0)$

$$k_x^2 = -(k_z^2 - \frac{w^2}{c^2}\varepsilon_M)$$

$$k_x = -i\sqrt{k_z^2 - \frac{w^2}{c^2}\varepsilon_M} \quad (x\to -\infty, e^{ik_x x}\to 0)$$

From equation ②

we can get $(x>0)$

$$k_x^2 = -(k_z^2 - \frac{w^2}{c^2})$$

$$k_x = i\sqrt{k_z^2 - \frac{w^2}{c^2}} \leftarrow (x\to +\infty, e^{ik_x x}\to 0)$$

$\therefore$ $\cancel{H_y = H_0 e^{ik_z z}}$ $H_y = H_0 e^{ik_z z}\begin{cases} \exp\{-\sqrt{k_z^2 - \frac{w^2}{c^2}}x\}, & x>0 \\ \exp\{+\sqrt{k_z^2 - \frac{w^2}{c^2}\varepsilon_M}x\}, & x<0 \end{cases}$

b)

Solution:

From Maxwell's equations

$$\mathrm{rot}\,\vec{H}(\vec{r},\omega) = -\varepsilon_0\varepsilon i w \vec{E}(\vec{r},\omega)$$

$(\vec{e}_x \times \vec{e}_y = \vec{e}_z \quad \vec{e}_z \times \vec{e}_y = -\vec{e}_x)$

for $x>0$

$$-\sqrt{k_z^2 - \frac{w^2}{c^2}} H_0 e^{ik_z z} e^{-\sqrt{k_z^2 - \frac{w^2}{c^2}}x}\vec{e}_z + ik_z H_0 e^{ik_z z} e^{-\sqrt{k_z^2 - \frac{w^2}{c^2}}x}(-\vec{e}_x) = -i\varepsilon_0 w \vec{E}(\vec{r})$$

$$\vec{E}(\vec{r}) = \frac{-i}{w\varepsilon_0}\sqrt{k_z^2 - \frac{w^2}{c^2}} H_0 e^{ik_z z} e^{-\sqrt{k_z^2 - \frac{w^2}{c^2}}x}\vec{e}_z + \frac{k_z}{w\varepsilon_0} H_0 e^{ik_z z} e^{-\sqrt{k_z^2 - \frac{w^2}{c^2}}x}(\vec{e}_x) = \frac{-i}{w\varepsilon_0}\sqrt{k_z^2 - \frac{w^2}{c^2}} H_0 e^{ik_z z} e^{-\sqrt{k_z^2 - \frac{w^2}{c^2}}x}\vec{e}_z + \frac{k_z}{w\varepsilon_0} H_0 e^{ik_z z} e^{-\sqrt{k_z^2 - \frac{w^2}{c^2}}x}\vec{e}_x$$

for $x<0$

$$\sqrt{k_z^2 - \frac{w^2}{c^2}\varepsilon_M} H_0 e^{ik_z z} e^{\sqrt{k_z^2 - \frac{w^2}{c^2}\varepsilon_M}x}\vec{e}_z + ik_z H_0 e^{ik_z z} e^{\sqrt{k_z^2 - \frac{w^2}{c^2}\varepsilon_M}x}(-\vec{e}_x) = \cancel{-iw\varepsilon_0\vec{E}(\vec{r})} - iw\varepsilon_0\varepsilon_M\vec{E}(\vec{r})$$

$$\vec{E}(\vec{r}) = \frac{k_z}{w\varepsilon_0\varepsilon_M} H_0 e^{ik_z z} e^{\sqrt{k_z^2 - \frac{w^2}{c^2}\varepsilon_M}x}\vec{e}_x + \frac{i}{w\varepsilon_0\varepsilon_M}\sqrt{k_z^2 - \frac{w^2}{c^2}\varepsilon_M} H_0 e^{ik_z z} e^{\sqrt{k_z^2 - \frac{w^2}{c^2}\varepsilon_M}x}\vec{e}_z$$

c)

From the continuity of E_z

we can get:

$$-\frac{i}{w\varepsilon_0}\sqrt{k_z^2-\frac{w^2}{c^2}}\,H_0 e^{ik_z z}e^{-\sqrt{k_z^2-\frac{w^2}{c^2}}x} = \frac{i}{w\varepsilon_0\varepsilon_M}\sqrt{k_z^2-\frac{w^2}{c^2}\varepsilon_M}\,H_0 e^{ik_z z}e^{\sqrt{k_z^2-\frac{w^2}{c^2}\varepsilon_M}x}$$

(In this equation $x=0$)

$$\therefore\ -\frac{1}{w\varepsilon_0}\sqrt{k_z^2-\frac{w^2}{c^2}} = \frac{1}{w\varepsilon_0\varepsilon_M}\sqrt{k_z^2-\frac{w^2}{c^2}\varepsilon_M}$$

$$\therefore\ -\varepsilon_M\sqrt{k_z^2-\frac{w^2}{c^2}} = \sqrt{k_z^2-\frac{w^2}{c^2}\varepsilon_M} \quad ①$$

From equation ①

we can get:

$$\varepsilon_M^2\left(k_z^2-\frac{w^2}{c^2}\right) = k_z^2-\frac{w^2}{c^2}\varepsilon_M$$

$$\therefore\ (\varepsilon_M^2-1)k_z^2 = \frac{w^2}{c^2}(\varepsilon_M^2-\varepsilon_M)$$

$$k_z^2 = \frac{w^2}{c^2}\frac{\varepsilon_M^2-\varepsilon_M}{\varepsilon_M^2-1} = \frac{w^2}{c^2}\frac{\varepsilon_M}{(\varepsilon_M+1)}$$

$\therefore$

$$k_z = \frac{w}{c}\sqrt{\frac{\varepsilon_M}{\varepsilon_M+1}}$$

$\because \varepsilon_M < 0$

$\therefore$ if we want to have a real valued k_z

(ε_M+1) must be smaller than 0

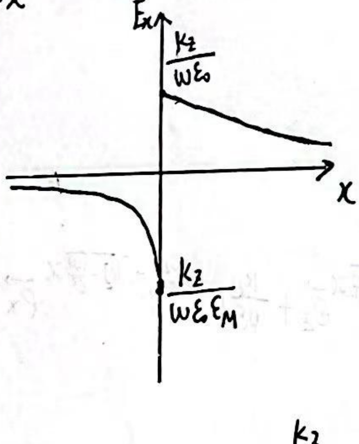
$\therefore$ The condition on ε_M is

$$\varepsilon_M + 1 < 0$$

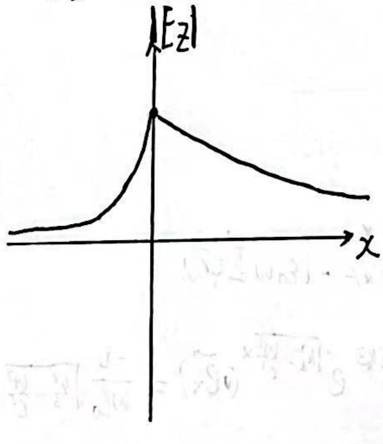
$$\varepsilon_M < -1$$

d)

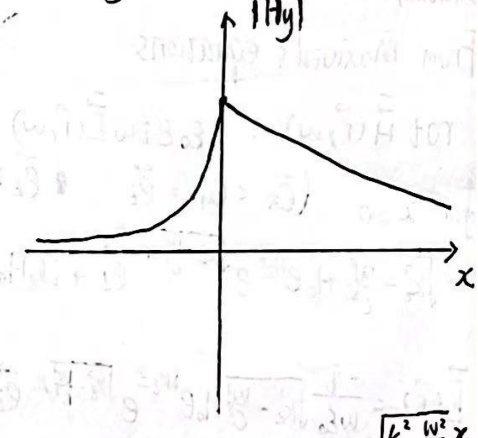
E_x at $z=0$



E_z at $z=0$



H_y at $z=0$



$x>0$, $E_{x+} = \frac{k_z}{w\varepsilon_0}H_0 e^{-\sqrt{k_z^2-\frac{w^2}{c^2}}x}$

$x<0$ $E_{x-} = \frac{k_z}{w\varepsilon_0\varepsilon_M}H_0 e^{\sqrt{k_z^2-\frac{w^2}{c^2}\varepsilon_M}x}$

$x>0$, $E_{z+} = -\frac{i}{w\varepsilon_0}\sqrt{k_z^2-\frac{w^2}{c^2}}H_0 e^{-\sqrt{k_z^2-\frac{w^2}{c^2}}x}$

$x<0$, $E_{z-} = \frac{i}{w\varepsilon_0\varepsilon_M}\sqrt{k_z^2-\frac{w^2}{c^2}\varepsilon_M}H_0 e^{\sqrt{k_z^2-\frac{w^2}{c^2}\varepsilon_M}x}$

$x=0$ $E_{z+} = E_{z-}$

$x>0$, $H_{y+} = H_0 e^{-\sqrt{k_z^2-\frac{w^2}{c^2}}x}$

$x<0$, $H_{y-} = H_0 e^{\sqrt{k_z^2-\frac{w^2}{c^2}\varepsilon_M}x}$

$x=0$, $H_{y+} = H_{y-}$

e)

Solution:

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}\left[\vec{E}(\vec{r})\times\vec{H}^{*}(\vec{r})\right]$$

for $x>0$

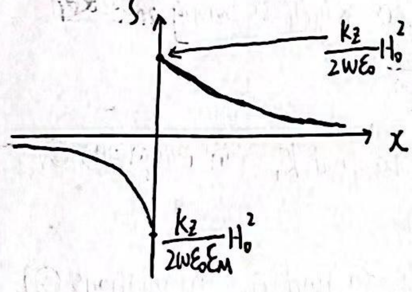
$$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}\left[\frac{k_z}{w\varepsilon_0}H_0^2 e^{-2\sqrt{k_z^2-\frac{w^2}{c^2}}x}\,\vec{e_z} + \frac{i}{w\varepsilon_0}\sqrt{k_z^2-\frac{w^2}{c^2}}H_0^2 e^{-2\sqrt{k_z^2-\frac{w^2}{c^2}}x}\,\vec{e_x}\right]$$

$$= \frac{k_z}{2w\varepsilon_0}H_0^2 e^{-2\sqrt{k_z^2-\frac{w^2}{c^2}}x}\,\vec{e_z}$$

for $x<0$

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}\left[\frac{k_z}{w\varepsilon_0\varepsilon_M}H_0^2 e^{2\sqrt{k_z^2-\frac{w^2}{c^2}\varepsilon_M}x}\,\vec{e_z} - \frac{i}{w\varepsilon_0\varepsilon_M}\sqrt{k_z^2-\frac{w^2}{c^2}\varepsilon_M}H_0^2 e^{2\sqrt{k_z^2-\frac{w^2}{c^2}\varepsilon_M}x}\,\vec{e_x}\right]$$

$$= \frac{k_z}{2w\varepsilon_0\varepsilon_M}H_0^2 e^{2\sqrt{k_z^2-\frac{w^2}{c^2}\varepsilon_M}x}\,\vec{e_z}$$



f)

Solution:

$$\int_{-\infty}^{\infty}\langle\vec{S}\rangle\,dx$$

$$= \int_{-\infty}^{0}\frac{k_z}{2w\varepsilon_0\varepsilon_M}H_0^2 e^{2\sqrt{k_z^2-\frac{w^2}{c^2}\varepsilon_M}x} + \int_{0}^{+\infty}\frac{k_z}{2w\varepsilon_0}H_0^2 e^{-2\sqrt{k_z^2-\frac{w^2}{c^2}}x}\,\vec{e_z}$$

$$= \frac{k_z}{2w\varepsilon_0\varepsilon_M}H_0^2\left(\frac{1}{2\sqrt{k_z^2-\frac{w^2}{c^2}\varepsilon_M}}\right) + \frac{k_z}{2w\varepsilon_0}H_0^2\left(\frac{1}{2\sqrt{k_z^2-\frac{w^2}{c^2}}}\right)\vec{e_z}$$

$$= \frac{H_0^2 k_z}{2w\varepsilon_0}\left(\frac{1}{2\varepsilon_M\sqrt{k_z^2-\frac{w^2}{c^2}\varepsilon_M}} + \frac{1}{2\sqrt{k_z^2-\frac{w^2}{c^2}}}\right)\vec{e_z}$$

From the continuity of E_z

we can get:

$$-\varepsilon_M\sqrt{k_z^2-\frac{w^2}{c^2}} = \sqrt{k_z^2-\frac{w^2}{c^2}\varepsilon_M} \quad (\varepsilon_M<-1)$$

$$\therefore \int_{-\infty}^{\infty}\langle\vec{S}\rangle\,dx = \frac{H_0^2 k_z}{4w\varepsilon_0\sqrt{k_z^2-\frac{w^2}{c^2}}}\left(1-\frac{1}{\varepsilon_M^2}\right)\vec{e_z}$$

$$\because \varepsilon_M<-1 \quad \therefore 1-\frac{1}{\varepsilon_M^2}>0$$

$\therefore$ The net energy flow has the same direction with $\vec{e_z}$

Task 3:

a) Solution:

From the Helmholz equation in the frequency domain

$$\left[\Delta + \frac{w^2}{c^2}\varepsilon(w)\right]\vec{E}(\vec{r},w) = 0$$

We can get:

$$\left[-k^2 + \frac{w^2}{c^2}\varepsilon(w)\right]\vec{E}(\vec{r},w) = 0$$

$$-k^2 + \frac{w^2}{c^2}\varepsilon(w) = 0$$

$\because k = k' + ik''$

$$\therefore k'^2 - k''^2 + 2ik'k'' = \frac{w^2}{c^2}\varepsilon' + i\frac{w^2}{c^2}\varepsilon''$$

$$k'^2 - k''^2 = \frac{w^2}{c^2}\varepsilon' \quad ①$$

$$2k'k'' = \frac{w^2}{c^2}\varepsilon'' \quad ②$$

b) Solution:

$\because \vec{k}' \parallel \vec{k}'' \parallel \hat{k}$

$\cancel{\vec{k}' + i\vec{k}'' = \hat{k}\frac{w}{c}(n+ik)}$ $\vec{k}' + i\vec{k}'' = \hat{k}\frac{w}{c}(n+iK)$

$\therefore k' = \frac{w}{c}n \quad k'' = \frac{w}{c}K$

$$③\begin{cases} n^2 - K^2 = \varepsilon' \\ 2nK = \varepsilon'' \end{cases}$$

1. put ~~putting~~ $\varepsilon'' = 0$

$2n_0K_0 = 0$

$n_0^2 - K_0^2 = \varepsilon'$

if $n_0 = 0$ $-K_0^2 = \varepsilon' > 0$ which is wrong

$\therefore K_0 = 0 \quad n_0 = \sqrt{\varepsilon'}$

2. Substitute n with n_0 to find K_1 (in equations ③)

$K_1 = 0 \quad (\varepsilon'' \neq 0)$

3. substitute K with K_1 to find n_1 (in equations ③)

$n_1 = \sqrt{\varepsilon'}$

c) Solution:

$$④\begin{cases} n^2 - K^2 = \varepsilon' \\ 2nK = \varepsilon'' \end{cases}$$

1. put $\varepsilon'' = 0$

$2n_0K_0 = 0$

$n_0^2 - K_0^2 = \varepsilon'$

$\therefore n_0 = 0$ (if $K_0 = 0$, $n_0^2 = \varepsilon' < 0$ which is wrong)

$K_0 = \sqrt{-\varepsilon'}$

2. $(\varepsilon'' \neq 0)$ substitute K with K_0 to find n_1

$\therefore n_1 = 0$

3. substitute n with n_1 to find K_1

$\therefore K_1 = \sqrt{-\varepsilon'}$