Maxwell Equations

a) Write down Maxwell's equations in the frequency domain for a linear, isotropic, non-magnetizable, inhomogeneous dielectric medium in absence of free charges and current density ($\rho = 0$ and $\mathbf{j} = 0$).

b) Show that for such a medium the wave equation for the electric field can be written as:

$$\Delta \mathbf{E}(\mathbf{r},\omega) + \frac{\omega^2}{c^2}\,\epsilon(\mathbf{r},\omega)\,\mathbf{E}(\mathbf{r},\omega) = -\mathbf{grad}\left\{\frac{\mathbf{grad}\,\epsilon(\mathbf{r},\omega)}{\epsilon(\mathbf{r},\omega)} \cdot \mathbf{E}(\mathbf{r},\omega)\right\}.$$

$\chi = \chi(r,w)$ $\quad \varepsilon(\vec{r},w)$ $\quad$ Homogeneous isotropic dispersive $\chi = \chi(w)$

Time domain:

$$\nabla\times\vec{E}(\vec{r},t) = -\frac{\partial\vec{B}(\vec{r},t)}{\partial t} = -\mu_0\frac{\partial\vec{H}(\vec{r},t)}{\partial t} \qquad \nabla\cdot\vec{D}(\vec{r},t) = \rho(\vec{r},t)$$

$$\nabla\times\vec{H}(\vec{r},t) = \vec{j}(\vec{r},t) + \frac{\partial\vec{D}(\vec{r},t)}{\partial t} = \sigma\vec{E}(\vec{r},t) + \frac{\partial\vec{D}(\vec{r},t)}{\partial t} \qquad \nabla\cdot\vec{H}(\vec{r},t) = 0$$

Frequency domain:

$$\nabla\times\vec{E}(\vec{r},w) = iw\mu_0\vec{H}(\vec{r},w) \qquad \nabla\cdot\vec{D}(\vec{r},w) = \rho(\vec{r},w)$$

$$\nabla\times\vec{H}(\vec{r},w) = \vec{j}(\vec{r},w) - iw\vec{D}(\vec{r},w) = \sigma\vec{E}(\vec{r},w) - iw\vec{D}(\vec{r},w) \qquad \nabla\cdot\vec{H}(\vec{r},w) = 0$$

Wave Equation:

E:

$$\nabla\times\nabla\times\vec{E}(\vec{r},w) = \nabla[\nabla\cdot\vec{E}(\vec{r},w)] - \Delta\vec{E}(\vec{r},w) = iw\mu_0\nabla\times\vec{H}(\vec{r},w)$$

$$\nabla\cdot\vec{D}(\vec{r},w) = \varepsilon_0\nabla\cdot[\varepsilon(r,w)\vec{E}(\vec{r},w)] = \varepsilon_0\nabla\varepsilon(\vec{r},w)\cdot\vec{E}(\vec{r},w) + \varepsilon_0\varepsilon(\vec{r},w)\nabla\cdot\vec{E}(\vec{r},w) = 0$$

$$\Rightarrow \nabla\cdot\vec{E}(\vec{r},w) = -\frac{\nabla\varepsilon(\vec{r},w)}{\varepsilon(\vec{r},w)}\vec{E}(\vec{r},w)$$

$$\Rightarrow -\nabla\left[\frac{\nabla\varepsilon(r,w)}{\varepsilon(r,w)}\vec{E}(\vec{r},w)\right] - \Delta\vec{E}(\vec{r},w) = w^2\mu_0\vec{D}(\vec{r},w) = w^2\mu_0\varepsilon_0\,\varepsilon(\vec{r},w)\vec{E}(\vec{r},w)$$

$$\Rightarrow \underline{\Delta\vec{E}(\vec{r},w) + \frac{w^2}{c^2}\varepsilon(\vec{r},w)\vec{E}(\vec{r},w) = -\nabla\left[\frac{\nabla\varepsilon(\vec{r},w)}{\varepsilon(\vec{r},w)}\vec{E}(\vec{r},w)\right]}$$

$\vec{H}$:

$$\nabla\times\nabla\times\vec{H} = \nabla[\nabla\cdot\vec{H}(\vec{r},w)] - \Delta\vec{H}(\vec{r},w) = -iw\nabla\times\vec{D}(\vec{r},t)$$

$$\Rightarrow \Delta\vec{H}(\vec{r},w) = iw\nabla\times[\varepsilon(\vec{r},w)\vec{E}(\vec{r},w)]$$

$$\nabla\times(\alpha\vec{a}) = \varepsilon_{ijk}\partial_j(\alpha\vec{a}_k) = \varepsilon_{ijk}(\partial_j\alpha\,\vec{a}_k + \alpha\partial_j\vec{a}_k) = \varepsilon_{ijk}a_k\partial_j\alpha + \varepsilon_{ijk}\alpha\,\partial_j\vec{a}_k$$

$$= -a\times\nabla\alpha + \alpha(\nabla\times\vec{a}) = \alpha(\nabla\times\vec{a}) - \vec{a}\times\nabla\alpha$$

$$\Rightarrow \Delta\vec{H}(\vec{r},w) = iw\varepsilon_0[\varepsilon(\vec{r},w)\nabla\times\vec{E}(\vec{r},w) - \vec{E}(\vec{r},w)\times\nabla\varepsilon(\vec{r},w)]$$

$$= iw\varepsilon_0\varepsilon(\vec{r},w)[iw\mu_0\vec{H}(\vec{r},w)] - iw\varepsilon_0\vec{E}(\vec{r},w)\times\nabla\varepsilon(\vec{r},w)]$$

$$\Rightarrow \underline{\Delta\vec{H}(\vec{r},w) + \frac{w^2}{c^2}\varepsilon(\vec{r},w)\vec{H}(\vec{r},w) = -iw\varepsilon_0\vec{E}(\vec{r},w)\times\nabla\varepsilon(\vec{r},w)}$$

a) Write down Maxwell's equations in time domain, in its general form. Furthermore, write down the constitutive equations for auxiliary fields **D** and **H**, in time domain (material is dispersive, linear, isotropic, and non-magnetic).

Time-domain:

$$\vec{D}(\vec{r},t) = \varepsilon_0\vec{E}(\vec{r},t) + P(\vec{r},t) \qquad \vec{H}(\vec{r},t) = \frac{1}{\mu_0}[\vec{B}(\vec{r},t) - \vec{M}(\vec{r},t)]$$

Frequency-domain:

$$\vec{D}(\vec{r},w) = \varepsilon_0\vec{E}(\vec{r},w) + P(\vec{r},w) = \varepsilon_0\vec{E}(\vec{r},w) + \varepsilon_0\chi(\vec{r},w)\vec{E}(\vec{r},w) = \varepsilon_0\varepsilon(\vec{r},w)\vec{E}(\vec{r},w)$$

$$\vec{H}(\vec{r},w) = \frac{1}{\mu_0}[\vec{B}(\vec{r},w) - \vec{M}(\vec{r},w)]$$

d) Give the formula of the time averaged Poynting vector for monochromatic fields.

$$\langle S(\vec r, t)\rangle = \frac{1}{2} Re[\vec E(\vec r) \times \vec H^*(\vec r)]$$

b) Name and write the units (in SI) of all the variables and physical constants you used in part (a).

c) Given the assumptions of part (a), find the wave equation for the **H** field in the time domain.

Dielectric	Polarization	$\vec P(\vec r, t)$	$[As/m^2]$
Magnetic	Polarization	$\vec M(\vec r, t)$	$[Vs/m^2]$
Electric	Constant	$\varepsilon_0 = 8.854 \times 10^{-12}$	$[As/Vm]$
Magnetic	Constant	$\mu_0 = 4\pi \times 10^{-7}$	$[Vs/Am]$
Charge	density	$\rho(\vec r, t)$	$[As/m^3]$
Current	density	$\vec j(\vec r, t)$	$[A/m^2]$
Electrical	field	$\vec E(\vec r, t)$	$[V/m]$
Magnetic	Field	$\vec H(\vec r, t)$	$[A/m]$
Electric	Flux density	$\vec D(\vec r, t)$	$[As/m^2]$
Magnetic	Flux density	$\vec B(\vec r, t)$	$[Vs/m^2]$

Electric	Magnetic
$\vec E$ $[V/m]$	$\vec H$ $[A/m]$
$\vec P$ $[As/m^2]$	$\vec M$ $[Vs/m^2]$
$\vec D$ $[As/m^2]$	$\vec B$ $[Vs/m^2]$
ε_0 $[As/Vm]$	μ_0 $[Vs/Am]$
ρ $[As/m^3]$	j $[A/m^2]$

$$\nabla \times \nabla \times \vec H(\vec r, t) = \nabla \times \vec j(\vec r, t) + \frac{\partial}{\partial t}\nabla \times \vec D(\vec r, t)$$

$$\Rightarrow -\Delta \vec H(\vec r, t) = \sigma \nabla \times \vec E(\vec r, t) + \varepsilon_0 \frac{\partial}{\partial t}\nabla \times \vec E(\vec r, t) + \frac{\partial}{\partial t}\nabla \times \vec P(\vec r, t)$$

$$\Rightarrow \Delta \vec H(\vec r, t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec H(\vec r, t) = \sigma \mu_0 \frac{\partial \vec H(\vec r, t)}{\partial t} - \frac{\partial}{\partial t}\nabla \times \vec P(\vec r, t)$$

$$\nabla \times \nabla \times \vec E(\vec r, t) = -\mu_0 \frac{\partial}{\partial t}\nabla \times \vec H(\vec r, t)$$

$$\nabla \cdot \vec D(\vec r, t) = \varepsilon_0 \nabla \cdot \vec E(\vec r, t) + \nabla \cdot \vec P(\vec r, t) = \rho(\vec r, t)$$

$$\nabla(\nabla \cdot \vec E(\vec r, t)) - \Delta \vec E(\vec r, t) = -\mu_0 \frac{\partial}{\partial t}\left[\vec j(\vec r, t) + \frac{\partial}{\partial t}\vec D(\vec r, t)\right]$$

$$\nabla\left[\frac{\rho(\vec r, t) - \nabla \cdot \vec P(\vec r, t)}{\varepsilon_0}\right] - \Delta \vec E(\vec r, t) = -\mu_0 \sigma \frac{\partial}{\partial t}\vec E(\vec r, t) - \mu_0 \frac{\partial^2}{\partial t^2}[\varepsilon_0 \vec E(\vec r, t) + \vec P(\vec r, t)]$$

$$\Rightarrow \Delta \vec E(\vec r, t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec E(\vec r, t) = \nabla\left[\frac{\rho(\vec r, t) - \nabla \cdot \vec P(\vec r, t)}{\varepsilon_0}\right] + \mu_0 \sigma \frac{\partial}{\partial t}\vec E(\vec r, t) + \mu_0 \frac{\partial^2}{\partial t^2}P(\vec r, t)$$

c) A homogeneous but dispersive medium cannot respond instantaneously when a time varying electric field is applied to it. Write down the constitutive relation between $\mathbf{D}(\mathbf{r}, \omega)$ and $\mathbf{E}(\mathbf{r}, \omega)$ in this medium and find the corresponding relation in the time domain.

transfer function

$$\chi = \chi(\omega) \qquad \vec D(\vec r, \omega) = \varepsilon_0 \vec E(\vec r, \omega) + \vec P(\vec r, \omega) = \varepsilon_0 \vec E(\vec r, \omega) + \varepsilon_0 \chi(\omega) \vec E(\vec r, \omega) = \varepsilon_0 (1 + \chi(\omega)) \vec E(\vec r, \omega) = \varepsilon_0 \varepsilon(\omega) \vec E(\vec r, \omega)$$

$$\vec D(\vec r, t) = \varepsilon_0 \vec E(r, t) + \vec P(\vec r, t) = \varepsilon_0 \vec E(r, t) + \varepsilon_0 \int_{-\infty}^{t} R(\vec r, t - t') \vec E(\vec r, t') dt'$$ response function

$$R(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi(\omega) e^{i\omega t} d\omega \Rightarrow \vec D(\vec r, t) = \varepsilon_0 \vec E(\vec r, t) + \frac{\varepsilon_0}{2\pi}\int_{-\infty}^{t}\int_{-\infty}^{\infty} \chi(\omega) e^{-i\omega t} \vec E(\vec r, t - t') d\omega dt'$$

d) Assume that the external charges are present in the medium in the part 1c. Using MwE, derive the continuity equation that is a relation between the time derivative of the charge density and the current density. Try to explain the meaning of continuity equation in your own words.

$$\vec j = \sigma \vec E(\vec r)$$

$$\nabla \cdot D = \rho \qquad \nabla \times \vec H = \vec j + \frac{\partial \vec D}{\partial t} \qquad \nabla \cdot \nabla \times \vec H = \nabla \vec j + \frac{\partial \nabla D}{\partial t} = 0 \Rightarrow \frac{\partial \rho}{\partial t} = -\nabla \vec j$$

At any spot the loss of charge carries with respect to time causes current.

a) Write down the Maxwell's equations in material in time domain in general form.

b) Write down the material equations relating the vector fields $\boldsymbol{D}, \boldsymbol{E}$, both in time domain and in frequency domain.

c) Derive the wave equation for the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ in a source-free, non-magnetic, isotropic, homogeneous medium with constant real-valued susceptibility χ and constant real-valued conductivity σ (so that the induced electric current density is $\boldsymbol{j}(\boldsymbol{r}, t) = \sigma \boldsymbol{E}(\boldsymbol{r}, t)$).

d) Derive the dispersion relation $k = k(\omega)$ for a plane wave solving the wave equation from part (c) and find how the complex dielectric function $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$ is defined from χ and σ.

e) Derive the wave equation in frequency domain for the electric field $\boldsymbol{E}(\boldsymbol{r}, \omega)$ in source-free, non-magnetic, non-conducting, inhomogeneous medium with spatially-varying dielectric permittivity

$$\varepsilon(\boldsymbol{r}) = \tilde{\varepsilon} + \boldsymbol{\kappa r},$$

where $\tilde{\varepsilon}$ is some constant value and $\boldsymbol{\kappa}$ is a constant vector.

(a) $\nabla \times \vec{E}(\vec{r},t) = -\dfrac{\partial \vec{B}(\vec{r},t)}{\partial t}$ $\quad \nabla \cdot \vec{D}(\vec{r},t) = \rho(\vec{r},t)$

$\nabla \times \vec{H}(\vec{r},t) = \vec{j}(\vec{r},t) + \dfrac{\partial \vec{D}(\vec{r},t)}{\partial t}$ $\quad \nabla \cdot \vec{H}(\vec{r},t) = 0$

(b) $\vec{D}(\vec{r},t) = \varepsilon_0 \vec{E}(\vec{r},t) + P(\vec{r},t) = \varepsilon_0 \vec{E}(\vec{r},t) + \varepsilon_0 \int_{-\infty}^{t} R(\vec{r}, t-t')\vec{E}(\vec{r},t')\,dt'$

$\vec{D}(\vec{r},\omega) = \varepsilon_0 \vec{E}(\vec{r},\omega) + \vec{P}(\vec{r},\omega) = \varepsilon_0 \vec{E}(\vec{r},\omega) + \varepsilon_0 \chi(\vec{r},\omega)\vec{E}(\vec{r},\omega) = \varepsilon_0[1+\chi(\omega)]\vec{E}(\vec{r},\omega) = \varepsilon_0 \varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega)$

(c) $\nabla \times \nabla \times \vec{E}(\vec{r},t) = -\mu_0 \frac{\partial}{\partial t} \nabla \times \vec{H}(\vec{r},t) = -\mu_0 \frac{\partial}{\partial t}\left[\vec{j}(\vec{r},t) + \frac{\partial \vec{D}(\vec{r},t)}{\partial t}\right]$

$\vec{R}(\vec{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi(\vec{r},\omega) e^{-i\omega t}\,d\omega$ $\quad \chi \to \text{constant} \Rightarrow \vec{R}(\vec{r},t) = \frac{\chi}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\,dt = \frac{\chi}{2\pi}\delta(t)$

$\vec{P}(\vec{r},t) = \varepsilon_0 \int_{-\infty}^{t} \frac{\chi}{2\pi}\delta(t-t')\vec{E}(\vec{r},t')\,dt' = \varepsilon_0 \frac{\chi}{2\pi}\vec{E}(\vec{r},t) \Rightarrow \vec{D}(\vec{r},t) = \varepsilon_0 \vec{E}(\vec{r},t) + \varepsilon_0 \frac{\chi}{2\pi}\vec{E}(\vec{r},t)$

$\Rightarrow \nabla \times \nabla \times \vec{E}(\vec{r},t) = \nabla[\nabla \cdot \vec{E}(\vec{r},t)] - \Delta \vec{E}(\vec{r},t) = -\mu_0 \frac{\partial}{\partial t}\left[\sigma \vec{E}(\vec{r},t) + \varepsilon_0(1+\frac{\chi}{2\pi})\frac{\partial}{\partial t}\vec{E}(\vec{r},t)\right]$

$\nabla \cdot \vec{D}(\vec{r},t) = 0 \Rightarrow (1+\frac{\chi}{2\pi})\nabla \cdot \vec{E}(\vec{r},t) = 0 \Rightarrow \nabla \cdot \vec{E}(\vec{r},t) = 0$

$\Rightarrow -\Delta \vec{E}(\vec{r},t) = -\mu_0 \sigma \frac{\partial}{\partial t}\vec{E}(\vec{r},t) - \mu_0 \varepsilon_0 (1+\frac{\chi}{2\pi})\frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t)]$

$$\Rightarrow \underline{\Delta \vec{E}(\vec{r},t) - \frac{1}{c^2}(1+\frac{\chi}{2\pi})\frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t) = \mu_0 \sigma \frac{\partial}{\partial t}\vec{E}(\vec{r},t)}$$

(d) Plane wave $\Rightarrow e^{i(\vec{k}\vec{r}-\omega t)}$

$\Rightarrow \Delta \vec{E}(\vec{r},t) = -k^2 \vec{E}(\vec{r},t)$ $\quad \frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t) = -\omega^2 \vec{E}(\vec{r},t)$ $\quad \frac{\partial}{\partial t}\vec{E}(\vec{r},t) = -i\omega \vec{E}(\vec{r},t)$

$\Rightarrow -k^2 + \frac{\omega^2}{c^2}(1+\frac{\chi}{2\pi}) = -i\omega\mu_0\sigma \Rightarrow k^2(\omega) = \frac{\omega^2}{c^2}(1+\frac{\chi}{2\pi}) + i\omega\mu_0\sigma = \frac{\omega^2}{c^2}(1+\frac{\chi}{2\pi} + i\frac{c^2\mu_0\sigma}{\omega})$

$\Rightarrow k^2(\omega) = \frac{\omega^2}{c^2}(1+\frac{\chi}{2\pi} + i\frac{\sigma}{\omega\varepsilon_0}) \Rightarrow \varepsilon' = 1+\frac{\chi}{2\pi}$ $\quad \varepsilon'' = \frac{\sigma}{\omega\varepsilon_0}$

(e) $\nabla \times \vec{E}(\vec{r},\omega) = i\omega\mu_0 \vec{H}(\vec{r},\omega)$ $\quad \nabla \times \vec{H} = -i\omega \vec{D}(\vec{r},\omega)$ $\quad \vec{D}(\vec{r},\omega) = \varepsilon_0 \varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega)$

$\nabla \cdot \vec{D}(\vec{r},\omega) = \varepsilon_0[\nabla \cdot \varepsilon(\vec{r},\omega)]\vec{E}(\vec{r},\omega) + \varepsilon_0 \varepsilon(\vec{r},\omega)\nabla \cdot \vec{E}(\vec{r},\omega) = 0$

$\Rightarrow [\nabla \cdot (\tilde{\varepsilon} + \vec{\kappa}\vec{r})]\vec{E}(\vec{r},\omega) = -(\tilde{\varepsilon} + \vec{\kappa}\vec{r})\nabla \cdot \vec{E}(\vec{r},\omega)$

$\Rightarrow \nabla \cdot \vec{E}(\vec{r},\omega) = -\dfrac{(\nabla \cdot \vec{\kappa}\vec{r})\vec{E}(\vec{r},\omega)}{\tilde{\varepsilon} + \vec{\kappa}\vec{r}} \Rightarrow \nabla[\nabla \vec{E}(\vec{r},\omega)] = -\nabla\left[\dfrac{(\nabla \cdot \vec{\kappa}\vec{r})\vec{E}(\vec{r},\omega)}{\tilde{\varepsilon} + \vec{\kappa} \cdot \vec{r}}\right]$

$\nabla \times \nabla \times \vec{E}(\vec{r},\omega) = i\omega\mu_0 \nabla \times \vec{H}(\vec{r},\omega) = \omega^2\mu_0 \vec{D}(\vec{r},\omega) = \frac{\omega^2}{c^2}\varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega)$

$\nabla \times \nabla \times \vec{E}(\vec{r},\omega) = \nabla[\nabla \cdot \vec{E}(\vec{r},\omega)] - \Delta \vec{E}(\vec{r},\omega) = -\nabla\left[\dfrac{(\nabla \cdot \vec{\kappa}\vec{r})\vec{E}(\vec{r},\omega)}{\tilde{\varepsilon} + \vec{\kappa} \cdot \vec{r}}\right] - \Delta \vec{E}(\vec{r},\omega)$

$\Rightarrow \Delta \vec{E}(\vec{r},\omega) + \frac{\omega^2}{c^2}(\tilde{\varepsilon} + \vec{\kappa} \cdot \vec{r})\vec{E}(\vec{r},\omega) = -\nabla\left[\dfrac{(\nabla \cdot \vec{\kappa} \cdot \vec{r})\vec{E}(\vec{r},\omega)}{\tilde{\varepsilon} + \vec{\kappa} \cdot \vec{r}}\right] = -\nabla\left[\dfrac{\vec{\kappa} \cdot \vec{E}(\vec{r},\omega)}{\tilde{\varepsilon} + \vec{\kappa} \cdot \vec{r}}\right]$