Task 1

a)

Solution:

$$I = |\langle \vec{S}(\vec{r},t)\rangle|$$

Expression for time average Poynting vector is missing!

1

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}[\vec{E}(\vec{r})\times\vec{H}^*(\vec{r})]$$

$$= \frac{1}{2}\mathrm{Re}\left[\frac{\alpha A_0^2}{w_0\mu_0}\vec{e}_z + \frac{\beta A_1^2}{w_0\mu_0}\vec{e}_z\right]$$

$$= \frac{\alpha A_0^2+\beta A_1^2}{2w_0\mu_0}\vec{e}_z$$

1

b)

$$\vec{E}(z,t) = A_0\exp[i(\alpha z - w_0 t)]\vec{e}_x + A_1\exp[i(\beta z - w_0 t)]\vec{e}_y$$

$$\therefore \vec{E}(\vec{r},w) = A_0 e^{i\alpha z}\delta(w-w_0)\vec{e}_x + A_1 e^{i\beta z}\delta(w-w_0)\vec{e}_y$$

From Maxwell's equations $\vec{H}(\vec{r},w) = -\frac{i}{w\mu_0}\mathrm{rot}\,\vec{E}(\vec{r},w)$

$$\therefore \vec{H}(\vec{r},w) = -\frac{i}{w\mu_0}[i\alpha A_0 e^{i\alpha z}\delta(w-w_0)\vec{e}_y - i\beta A_1 e^{i\beta z}\delta(w-w_0)\vec{e}_x]$$

$$= \frac{\alpha}{w\mu_0}A_0 e^{i\alpha z}\delta(w-w_0)\vec{e}_y - \frac{\beta}{w\mu_0}A_1 e^{i\beta z}\delta(w-w_0)\vec{e}_x$$

$$\vec{H}(\vec{r},t) = \frac{\alpha A_0}{w_0\mu_0}e^{[i(\alpha z - w_0 t)]}\vec{e}_y - \frac{\beta A_1}{w_0\mu_0}e^{[i(\beta z - w_0 t)]}\vec{e}_x$$

$$I = E(\vec{r})\cdot E^*(\vec{r})$$

$$= A_0^2 + A_1^2$$

0! Use the above expression to calculate the Intensity.
Intensity is proportional to mod sq. of Electric field not equal to.

Task 2

a)

Solution: 2

$$U_0(\alpha,\beta) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} u_0(x,y)\exp[-i(\alpha x+\beta y)]\,dx\,dy$$

$$= \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} A e^{-\frac{x^2+y^2}{w^2}} e^{-i(\alpha x+\beta y)}\,dx\,dy$$

$$= \frac{A}{4\pi^2}\int_{-\infty}^{\infty} e^{-\frac{x^2}{w^2}-i\alpha x}dx\int_{-\infty}^{\infty} e^{-\frac{y^2}{w^2}-i\beta y}dy$$

$$= \frac{A}{4\pi^2}\int_{-\infty}^{\infty} e^{-\frac{1}{w^2}(x+\frac{i\alpha w^2}{2})^2}\cdot e^{-\frac{\alpha^2 w^2}{4}}dx\int_{-\infty}^{\infty} e^{-\frac{1}{w^2}(y+\frac{i\beta w^2}{2})^2}\cdot e^{-\frac{\beta^2 w^2}{4}}dy$$

$$= \frac{A}{4\pi^2}\left[e^{-\frac{\alpha^2 w^2}{4}}\int_{-\infty}^{\infty} e^{-[\frac{1}{w}(x+\frac{i\alpha w^2}{2})]^2}dx \; e^{-\frac{\beta^2 w^2}{4}}\int_{-\infty}^{\infty} e^{-[\frac{1}{w}(y+\frac{i\beta w^2}{2})]^2}dy\right]$$

$$= \frac{A}{4\pi^2}\left[e^{-\frac{\alpha^2 w^2}{4}}\int_{-\infty}^{\infty} w e^{-[\frac{1}{w}(x+\frac{i\alpha w^2}{2})]^2}d[\tfrac{1}{w}(x+\tfrac{i\alpha w^2}{2})]\; e^{-\frac{\beta^2 w^2}{4}}\int_{-\infty}^{\infty} w e^{-[\frac{1}{w}(y+\frac{i\beta w^2}{2})]^2}d[\tfrac{1}{w}(y+\tfrac{i\beta w^2}{2})]\right]$$

From Gaussian integral

$$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$$

$$\therefore U_0(\alpha,\beta) = \frac{A}{4\pi^2}\left(w\sqrt{\pi}e^{-\frac{\alpha^2 w^2}{4}}\cdot w\sqrt{\pi}e^{-\frac{\beta^2 w^2}{4}}\right) = \frac{Aw^2}{4\pi}e^{-\frac{(\alpha^2+\beta^2)w^2}{4}}$$

b) 1

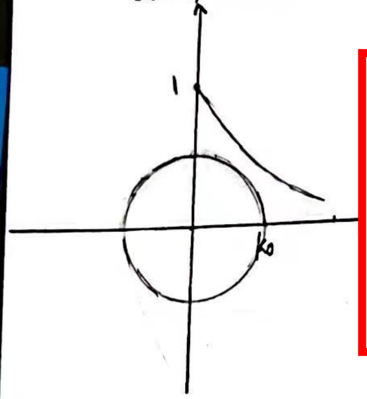
Solution: $\because k_r = \sqrt{k_x^2 + k_y^2}$

$$U_0(k_r = k_0) / U_0(k_r = 0) = \frac{\frac{AW^2}{4\pi} e^{-\frac{k_0^2 W^2}{4}}}{\frac{AW^2}{4\pi}} = e^{-\frac{k_0^2 W^2}{4}} = e^{-\frac{4\pi^2 W^2}{4\lambda^2}} = e^{-\frac{\pi^2 W^2}{\lambda^2}}$$

$W = \lambda / \pi$

$$U_0(k_r) / U_0(k_r = 0) = e^{-\frac{k_r^2 \lambda^2}{4\pi^2}}$$

$$U_0(k_r = k_0) / U_0(k_r = 0) = \frac{1}{e}$$



The curve should be a Gaussian function.
-1

(c) 0.5

Solution:

Draw a circle with radius $= k_0$

The area in the circle ($k_x^2 + k_y^2 \leq k_0^2$) contributes to propagating wave (real part of k_z)

The area beyond the ~~circle~~ circle ($k_x^2 + k_y^2 > k_0^2$) contributes to evanescent wave (imaginary part of k_z)

Plot Kz

Task 3

a)

Solution:

In Talbot Effect, we can get

$$f(x, z=L_T) = f(x, z=0)\exp(ikL_T + i2\pi m_e)$$

$$f(x+a) = f(x)$$

Express $f(x)$ as a Fourier series

$$f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{in\omega_0 x} \quad (n \in Z)$$

$$\because T = a \quad \therefore \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{a}$$

$$f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{in\frac{2\pi}{a}x}$$

$$F\{f(x,z=0)\} = F(\alpha, z=0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} C_n e^{in\frac{2\pi}{a}x} e^{-i\alpha x} dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} C_n e^{-i(\alpha - n\frac{2\pi}{a})x} dx$$

$$= \cancel{\frac{1}{2\pi}} \sum_{n=-\infty}^{+\infty} C_n \delta(\alpha - n\tfrac{2\pi}{a})$$ ✓

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(w-w_0)x} dx = \delta(w-w_0)$$

$$L_T = \frac{2\pi m_e}{\sqrt{k^2 - (\frac{2n\pi}{a})^2} - k}$$

$\because$ The paraxial is valid

$$\therefore \alpha^2 \ll k^2$$

$$\sqrt{k^2 - (\tfrac{2n\pi}{a})^2} \approx k - \frac{(\frac{2n\pi}{a})^2}{2k}$$

$$\therefore L_T \approx \frac{-k a^2 m_e}{n^2 \pi}$$ ✓

$$\min(L_T) = \frac{2a^2}{\lambda}$$

4.5

$$F(\alpha, z) = F(\alpha, z=0)\cdot H(\alpha, z)$$

$$H(\alpha, z) = \exp[i\sqrt{k^2-\alpha^2}\, z]$$

$$\therefore F(\alpha, z) = \cancel{\frac{1}{2\pi}} \sum_{n=-\infty}^{+\infty} C_n \delta(\alpha - n\tfrac{2\pi}{a}) e^{i\sqrt{k^2-\alpha^2} z}$$ ✓

$$f(x,z) = \int_{-\infty}^{+\infty} F(\alpha, z) e^{i\alpha x} d\alpha$$

$$= \int_{-\infty}^{\infty} \cancel{\frac{1}{2\pi}} \sum_{n=-\infty}^{+\infty} C_n \delta(\alpha - n\tfrac{2\pi}{a}) e^{i\sqrt{k^2-\alpha^2} z} e^{i\alpha x} d\alpha$$

$$= \cancel{\frac{1}{2\pi}} \sum_{n=-\infty}^{+\infty} C_n e^{i\sqrt{k^2 - (\frac{2n\pi}{a})^2} z} e^{i\frac{2n\pi}{a}x}$$ ✓

$$\because f(x, z=L_T) = f(x, z=0) e^{i(kL_T + 2\pi m_e)}$$

$$\therefore \cancel{\frac{1}{2\pi}} \sum_{n=-\infty}^{+\infty} e^{i\sqrt{k^2-(\frac{2n\pi}{a})^2} L_T} e^{i\frac{2n\pi}{a}x} = \cancel{\frac{1}{2\pi}} \sum_{n=-\infty}^{+\infty} C_n e^{i\frac{2n\pi}{a}x} e^{i(kL_T + 2\pi m_e)}$$

-0.5

$$\therefore e^{i\sqrt{k^2-(\frac{2n\pi}{a})^2} L_T} = e^{i(kL_T + 2\pi m_e)}$$

$$\sqrt{k^2 - (\tfrac{2n\pi}{a})^2}\, L_T = kL_T + 2\pi m_e$$ ✓

b)

solution:

From (a) we can get

$$L_T = -\frac{k a^2 m_e}{n^2 \pi}$$

$$\because m_e, n \in Z$$

$$\therefore \text{let } m_e = -n^2$$

$$L_T = \frac{ka^2}{\pi} = \frac{2a^2}{\lambda}$$

$$\because L_T = 25.7\,\text{m} \quad a = 3\times 10^{-3}\,\text{m}$$

$$\therefore \lambda = 0.7 \times 10^{-6}\,\text{m}$$ ✓

2

red light

c)

Solution: 3

$$f(x,z=0) = A\cos(x\,2\pi/a_1)\cos(x\,2\pi/a_2)$$

$$= \frac{A}{2}\left[\cos\left(\frac{2\pi}{a_1}x+\frac{2\pi}{a_2}x\right)+\cos\left(\frac{2\pi}{a_1}x-\frac{2\pi}{a_2}x\right)\right] \checkmark$$

$$F(\alpha,z=0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{A}{2}\left[\cos\left(\frac{2\pi}{a_1}x+\frac{2\pi}{a_2}x\right)+\cos\left(\frac{2\pi}{a_1}x-\frac{2\pi}{a_2}x\right)\right]e^{-i\alpha x}dx$$

$$= \frac{A}{4\pi}\int_{-\infty}^{+\infty}\left[\frac{e^{i(\frac{2\pi}{a_1}+\frac{2\pi}{a_2})x}+e^{-i(\frac{2\pi}{a_1}+\frac{2\pi}{a_2})x}}{2}+\frac{e^{i(\frac{2\pi}{a_1}-\frac{2\pi}{a_2})x}+e^{-i(\frac{2\pi}{a_1}-\frac{2\pi}{a_2})x}}{2}\right]e^{-i\alpha x}dx \checkmark$$

suppose $\frac{2\pi}{a_1}+\frac{2\pi}{a_2}=M$ $\quad\frac{2\pi}{a_1}-\frac{2\pi}{a_2}=N$

$$F(\alpha,z=0) = \frac{A}{4\pi}\int_{-\infty}^{+\infty}\frac{1}{2}\left[e^{i(M-\alpha)x}+e^{-i(M+\alpha)x}+e^{i(N-\alpha)x}+e^{-i(N+\alpha)x}\right]dx$$

$$= \frac{2\pi A}{8\pi}\left[\delta(M-\alpha)+\delta(M+\alpha)+\delta(N-\alpha)+\delta(N+\alpha)\right]$$

$$\int_{-\infty}^{\infty}e^{ix\alpha}dx = 2\pi\delta(\alpha)$$

$$F(\alpha,z) = F(\alpha,z=0)e^{i\sqrt{k^2-\alpha^2}z}$$

$$f(x,z) = \int_{-\infty}^{+\infty}F(\alpha,z=0)e^{i\sqrt{k^2-\alpha^2}z}e^{i\alpha x}d\alpha$$

$$= \int_{-\infty}^{+\infty}\frac{A}{8\pi}\left[\delta(M-\alpha)+\delta(M+\alpha)+\delta(N-\alpha)+\delta(N+\alpha)\right]e^{i\sqrt{k^2-\alpha^2}z}e^{i\alpha x}d\alpha$$

$$= \frac{A}{8\pi}\left[e^{i\sqrt{k^2-M^2}z}e^{iMx}+e^{i\sqrt{k^2-M^2}z}e^{-iMx}+e^{i\sqrt{k^2-N^2}z}e^{iNx}+e^{i\sqrt{k^2-N^2}z}e^{-iNx}\right]$$

$$= \frac{A}{\cancel{4\pi}\ 2}\left[e^{i\sqrt{k^2-M^2}z}\cos(Mx)+e^{i\sqrt{k^2-N^2}z}\cos(Nx)\right] \checkmark$$

$\boxed{\cancel{f(x,z=L_T)=f(x,z=0)\exp(ikL_T}}$

$\because f(x,z=L_T)=f(x,z=0)\exp(ikL_T+i2\pi m_\ell)$ the Talbot effect still takes place outside the paraxial regime ✓

then this eq. is not true

$$\therefore \frac{A}{4\pi}\left[e^{i\sqrt{k^2-M^2}L_T}\cos(Mx)+e^{i\sqrt{k^2-N^2}L_T}\cos(Nx)\right] = \frac{A}{4\pi}\left[\cos(Mx)+\cos(Nx)\right]e^{i(kL_T+2\pi m_\ell)}$$

$\therefore$ $\boxed{\cancel{e^{i\sqrt{k^2-M^2}L_T}=e^{i(kL_T+2\pi m_\ell)} \quad e^{i\sqrt{k^2-N^2}L_T}}}$ $\quad L_T =$

d) —