Fundamentals of Modern Optics
Exercise 9
19.12.2014

to be returned: 09.01.2015, at the beginning of the lecture

Problem 1 – Fraunhofer diffraction — 2+3+2* points

Calculate the full fields in the Fraunhofer approximation when shining with a plane wave on

a) a rectangular aperture of width a and height b,

b) a circular aperture of *radius* a and

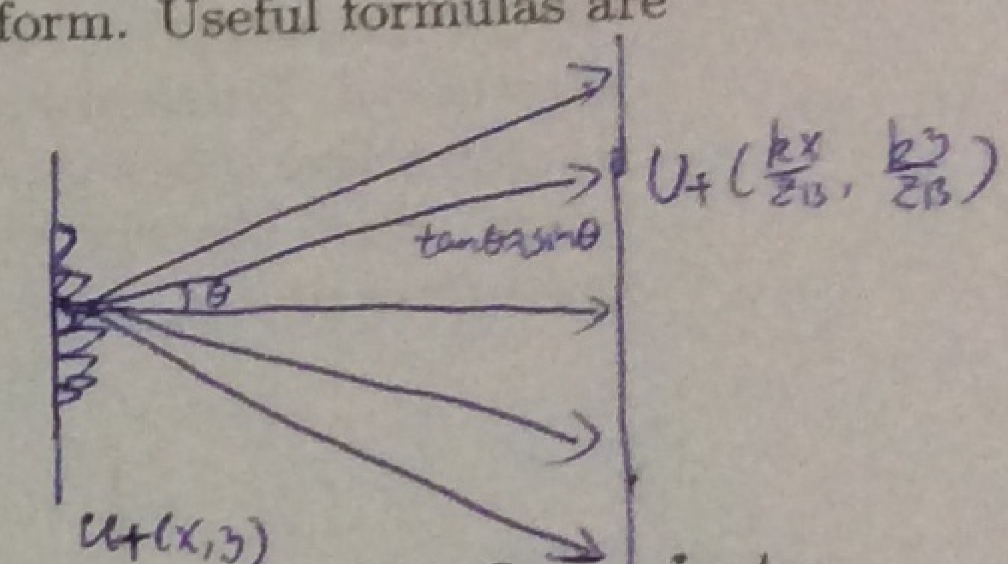
c*) an annular aperture which is bounded by two circles of radius a and εa with $\varepsilon < 1$. Plot the intensity for suitable values of a, ε and z and compare it with the result from b). What do you notice?

Hint for b) and c):* Use polar coordinates for **k** and **r** to solve the Fourier transform. Useful formulas are

$$\frac{\mathrm{i}^{-n}}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}x\cos\alpha}\,\mathrm{e}^{\mathrm{i}n\alpha}\,\mathrm{d}\alpha = J_n(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{n+1}J_{n+1}(x)\right] = x^{n+1}J_n(x)$$

where J_i are the Bessel functions of first kind.



Problem 2 – Diffractive Structures — 2+3 points

Calculate the diffraction patterns in the Fraunhofer approximation for

a) a sequence of N pinholes placed along the x-axis with distances of $b > 2a$ and

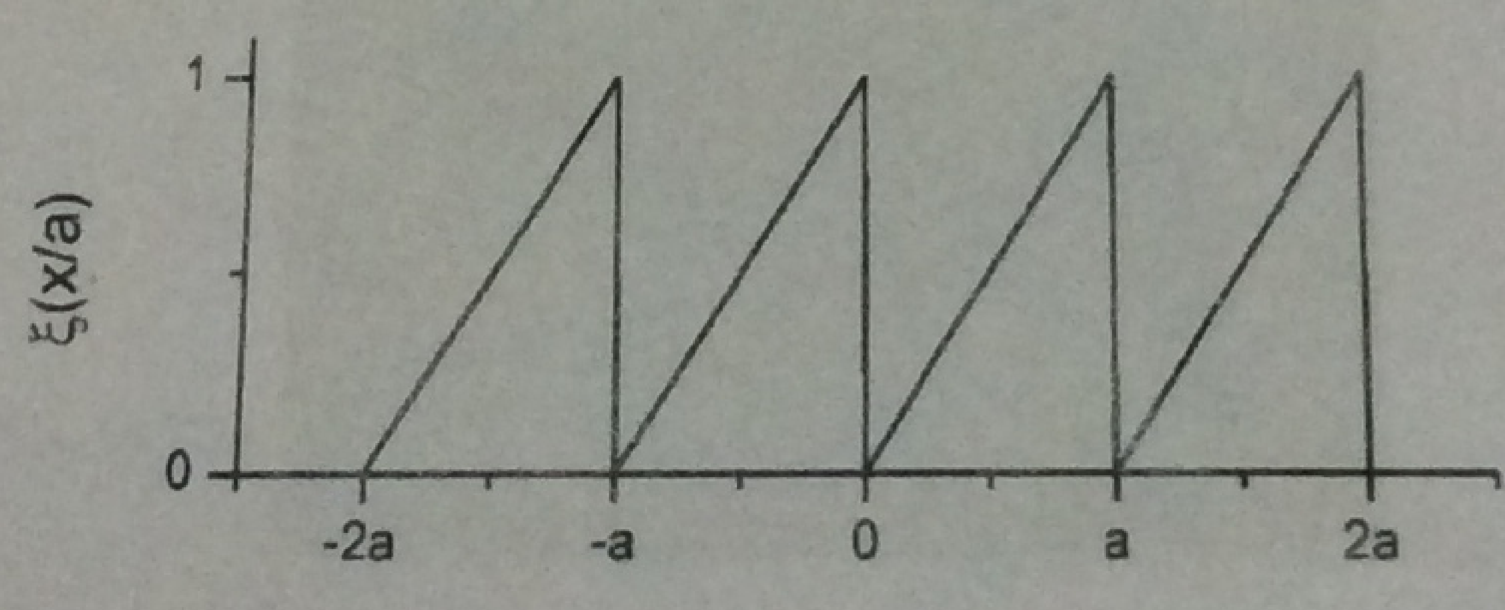
b) a blazed phase grating
闪耀 光栅

$$u_0(x) = \exp\left\{i\varphi_0\,\xi\left(\frac{x}{a}\right)\right\},$$

with N illuminated periods, where

$$\xi\left(\frac{x}{a}\right) = \frac{x}{a} - \left\lfloor\frac{x}{a}\right\rfloor$$

is the saw tooth function below ($\lfloor x \rfloor$ is the largest integer number smaller than x).



Use the results from problem 4 of series 8 and from problem 1. If you did not solve problem 1 you may use the expressions for the diffraction patterns from some other source as long as you cite it.

Problem 3 – (Very) Near field diffraction — 4 points

In the lecture you have learned that the paraxial Fraunhofer approximation is valid provided that the condition $N_F = a^2/(\lambda z) < 0.1$ is satisfied (a ... typical object feature size, λ ... wavelength, z ... distance from object). Assuming feature sizes $a \approx \lambda$, this indicates that diffraction dies out for distances $z \gg \lambda$.
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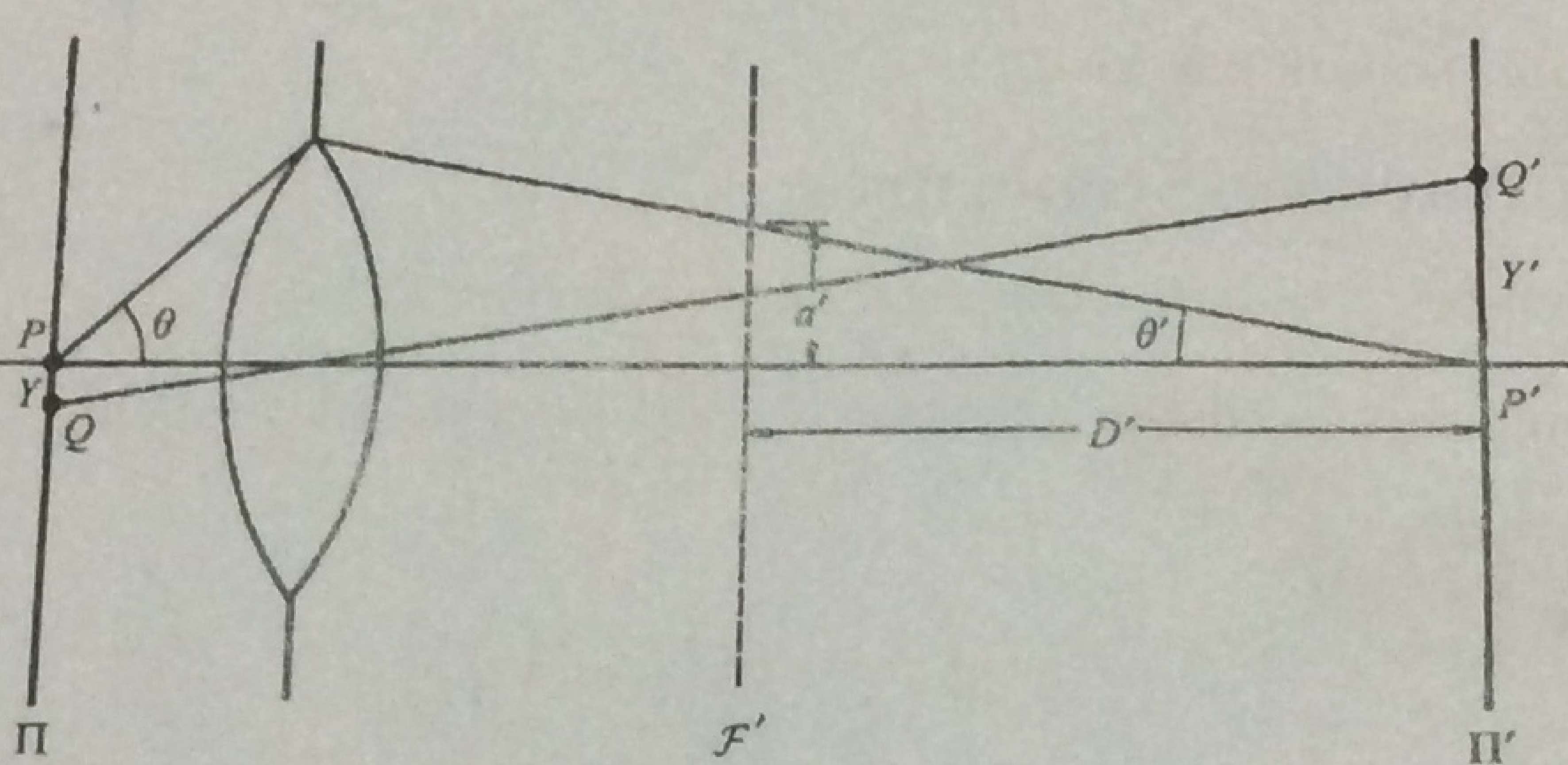
However, we want to have a look at the opposite scenario here, the very near-field. Assume to have a rectangular aperture of characteristic size $a \times a$. Show that under the condition $N_F \gg 10$, diffraction is not yet (in the sense of propagation) observable, such that the propagated light field in Fresnel approximation resembles the light distribution of the aperture.

Problem 4 – Towards Abbe's Diffraction Limit   2+3*+2* points

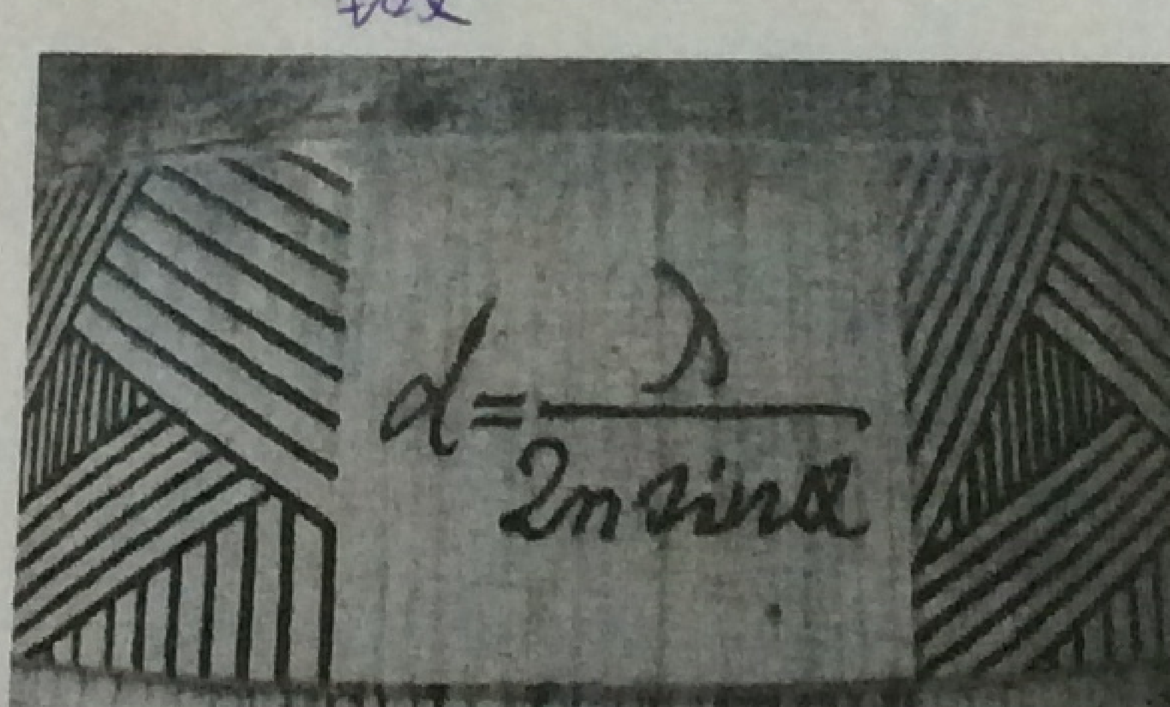
If two objects have a small separation their diffraction patterns will overlap. At some observation distance it will become impossible to detect the presence of two separated objects. This resolution limit depends on the instrument (eye, photographic plate, detector). To have a rough estimate we will use Rayleigh's criterion: *two images are regarded as just resolved when the maximum of one diffraction pattern coincides with the first minimum of the other.*

a) Calculate the angular resolution limit of a telescope with an circular aperture of diameter $D = 20\,\mathrm{cm}$ and of the eye (iris diameter $\approx 8\,\mathrm{mm}$). *Hint:* Assume an object very far away and calculate the first minimum of the diffraction image of the aperture. Relate the position of the minimum to the minimal observation angle $\theta_{\min}$ under which two objects appear separated by Rayleigh's criterion - the angular resolution limit.

b*) Obtain the resolution limit in a microscope. Assume the geometry in the figure below and assume a circular aperture a'. Furthermore $\theta' \ll 1$ and the sine condition has to hold: $nY \sin\theta = -n'Y' \sin\theta'$, where n is the refractive index on the left side of the lens and n' the refractive on the right side of the lens in the figure. *Hint:* the ray from the center of the aperture to Y' is assumed to form the minimal observation angle from a), e.g. $Y' = \theta_{\min} D'$.



c*) Look at the inscription in the figure below. You will find it on a monument in Abbe's honor near the university library (go look for it!). Did you derive a different formula? If you did explain why and under which assumptions the formula in the inscription holds.

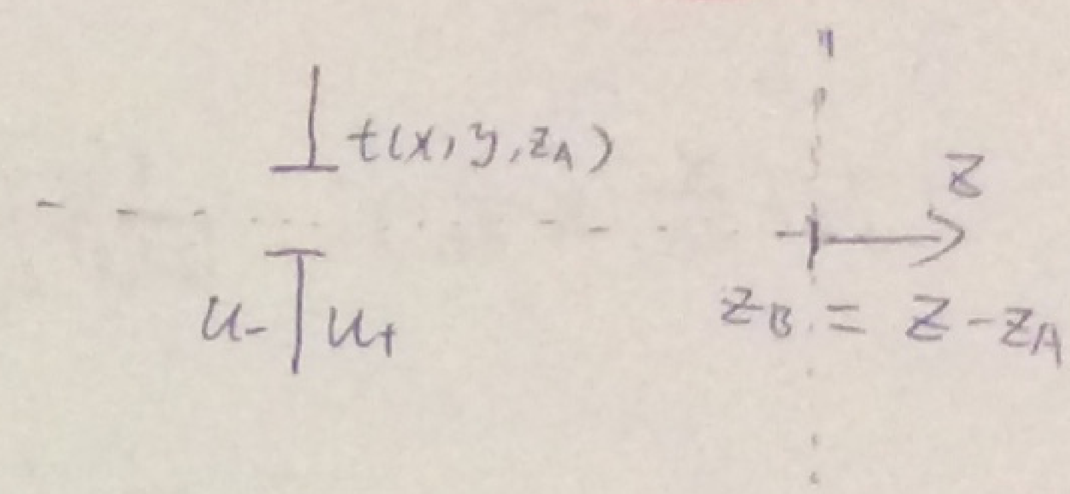


Merry christmas and a happy new year!

Yining Li 149824 FOMO Exercise 9 Monday

15.5/16

Problem 1 - Fraunhofer diffraction



a) $t(x,y) = \begin{cases} 1 & \text{for } |x| \le \frac{a}{2},\ |y| \le \frac{b}{2} \\ 0 & \text{other} \end{cases}$

$$u_-(x, y, z_A) = A e^{i(k_x x + k_y y + k_z z_A)}$$

$$u_+(x, y, z_A) = u_-(x, y, z_A) \cdot t(x,y)$$

The Fourier transform is: (assume $\alpha = k\frac{x}{z_B}$, $\beta = k\frac{y}{z_B}$)

$$U_+\left(k\frac{x}{z_B}, k\frac{y}{z_B}\right) = \frac{A}{(2\pi)^2} e^{ik_z z_A} \iint_{-\infty}^{\infty} t(x',y') e^{-i(k\frac{x}{z_B} - k_x)x' - i(k\frac{y}{z_B} - k_y)y'} dx'dy'$$

$$= A e^{ik_z z_A} T\left(k\frac{x}{z_B} - k_x,\ k\frac{y}{z_B} - k_y\right)$$

Since
$$T\Big(\underbrace{\frac{kx}{z_B}}_{\alpha}, \underbrace{\frac{ky}{z_B}}_{\beta}\Big) = \frac{1}{4\pi^2} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-i\alpha x'} \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{-i\beta y'} dx'dy'$$

...steps missing

$$= \frac{ab}{4\pi^2} \operatorname{sinc}\left(\frac{a}{2}\alpha\right) \operatorname{sinc}\left(\frac{b}{2}\beta\right)$$

$$= \frac{ab}{4\pi^2} \operatorname{sinc}\left(k\frac{xa}{2z_B}, k\frac{yb}{2z_B}\right)$$

With wave vector perpendicular to the mask t, $k_x = k_y = 0$, $k = k_z$

$$T\left(k\frac{x}{z_B} - k_x, k\frac{y}{z_B} - k_y\right) = \frac{ab}{4\pi^2} \operatorname{sinc}\left(k\frac{xa}{2z_B}, k\frac{yb}{2z_B}\right)$$

$$u_{FR}(x,y,z_B) = -\frac{i}{\lambda z_B} e^{ikz_B} e^{i\frac{k}{2z_B}(x^2+y^2)} \iint_{-\infty}^{\infty} u_+(x',y') e^{-i(\frac{kx}{z_B}x' + \frac{ky}{z_B}y')} dx'dy'$$

$$= -i\frac{(2\pi)^2}{\lambda z_B} e^{ikz_B} U_+\left(k\frac{x}{z_B}, k\frac{y}{z_B}\right) e^{i\frac{k}{2z_B}(x^2+y^2)}$$

$$= -i\frac{ab}{\lambda z_B} A e^{ikz_B} \cdot e^{i\frac{k}{2z_B}(x^2+y^2)} \cdot e^{ikz_A} \cdot \operatorname{sinc}\left(\frac{kxa}{2z_B}, k\frac{yb}{2z_B}\right)$$

$$I_{FR}(x,y,z_B) \sim \frac{1}{(\lambda z_B)^2} \left|U_+\left(k\frac{x}{z_B}, k\frac{y}{z_B}\right)\right|^2 \sim \frac{1}{(\lambda z_B)^2} \operatorname{sinc}^2\left(\frac{kxa}{2z_B}, k\frac{yb}{2z_B}\right)$$

1.5/2

Exercise 4

b) $t(x,y) = \begin{cases} 1 & x^2+y^2 \le a^2 \\ 0 & \text{other} \end{cases}$, $\rho^2 = x^2+y^2$

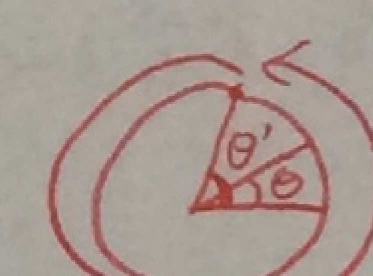
$$T\left(k\frac{x}{z_B}, k\frac{y}{z_B}\right) = \frac{1}{(2\pi)^2}\int_0^{2\pi} d\theta' \int_0^a e^{-i\frac{k}{z_B}x\rho'\cos\theta' - i\frac{k}{z_B}y\rho'\sin\theta'} \rho' d\rho' \quad \begin{pmatrix} x' = \rho'\cos\theta' \\ y' = \rho'\sin\theta' \end{pmatrix}$$

$$= \frac{1}{4\pi^2}\int_0^a \int_0^{2\pi} e^{-i\frac{k}{z_B}\rho\rho'(\cos\theta\cos\theta' + \sin\theta\sin\theta')} d\theta' \rho' d\rho'$$

$$= \frac{1}{4\pi^2}\int_0^a \int_0^{2\pi} e^{-i\frac{k\rho\rho'}{z_B}\cdot\cos(\theta'-\theta)} d\theta' \rho' d\rho'$$

$$= e^{-i\frac{k\rho\rho'}{z_B}\cos\theta'}$$

$$= \frac{1}{2\pi}\int_0^a J_0\Big(\underbrace{\frac{k\rho\rho'}{z_B}}_{x}\Big)\rho' d\rho'$$



integrate from $0 \to 2\pi$.

$$\int_0^{2\pi} e^{-i\cos(\theta'-\theta)} = \int_0^{2\pi} e^{-i\cos\theta'}$$

Since $xJ_1(x) = \int xJ_0(x)\,dx$

$$= \frac{a^2}{2\pi}\cdot\frac{J_1\left(\frac{k\rho}{z_B}a\right)}{\frac{k\rho}{z_B}a}$$

$$= \frac{a^2}{2\pi}\cdot\frac{J_1\left(\frac{ka}{z_B}\sqrt{x^2+y^2}\right)}{\frac{ka}{z_B}\sqrt{x^2+y^2}}$$

From part (a), we know

$$U_{FR}(x,y,z_B) = -i\frac{(2\pi)^2}{\lambda z_B}e^{ikz_B}\,U_+\left(\frac{kx}{z_B}, \frac{ky}{z_B}\right)e^{i\frac{k}{2z_B}(x^2+y^2)}$$

$$= -i\frac{(2\pi)^2}{\lambda z_B}e^{ikz_B}\cdot Ae^{ikz_A}\,T\left(k\frac{x}{z_B}, k\frac{y}{z_B}\right)e^{i\frac{k}{2z_B}(x^2+y^2)}$$

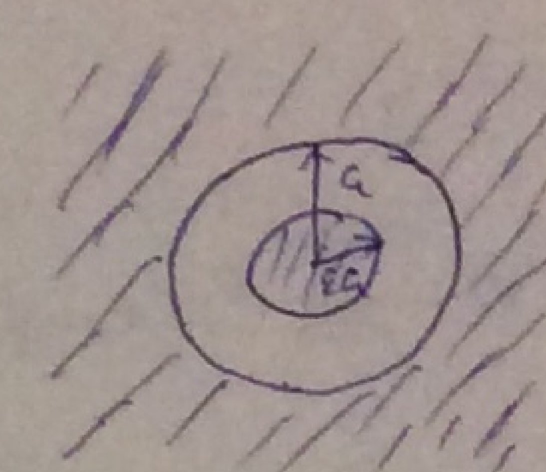
$$= -i\frac{2\pi}{\lambda z_B}\cdot e^{ikz_B}\cdot Ae^{ikz_A}\cdot e^{i\frac{k}{2z_B}(x^2+y^2)}\cdot a^2\cdot\frac{J_1\left(\frac{ka}{z_B}\sqrt{x^2+y^2}\right)}{\frac{ka}{z_B}\sqrt{x^2+y^2}}$$

$$I(x,y,z_B) \propto \left[\frac{J_1\left(\frac{ka}{z_B}\sqrt{x^2+y^2}\right)}{\frac{ka}{z_B}\sqrt{x^2+y^2}}\right]^2 \rightarrow \text{Airy disk.}$$

✓

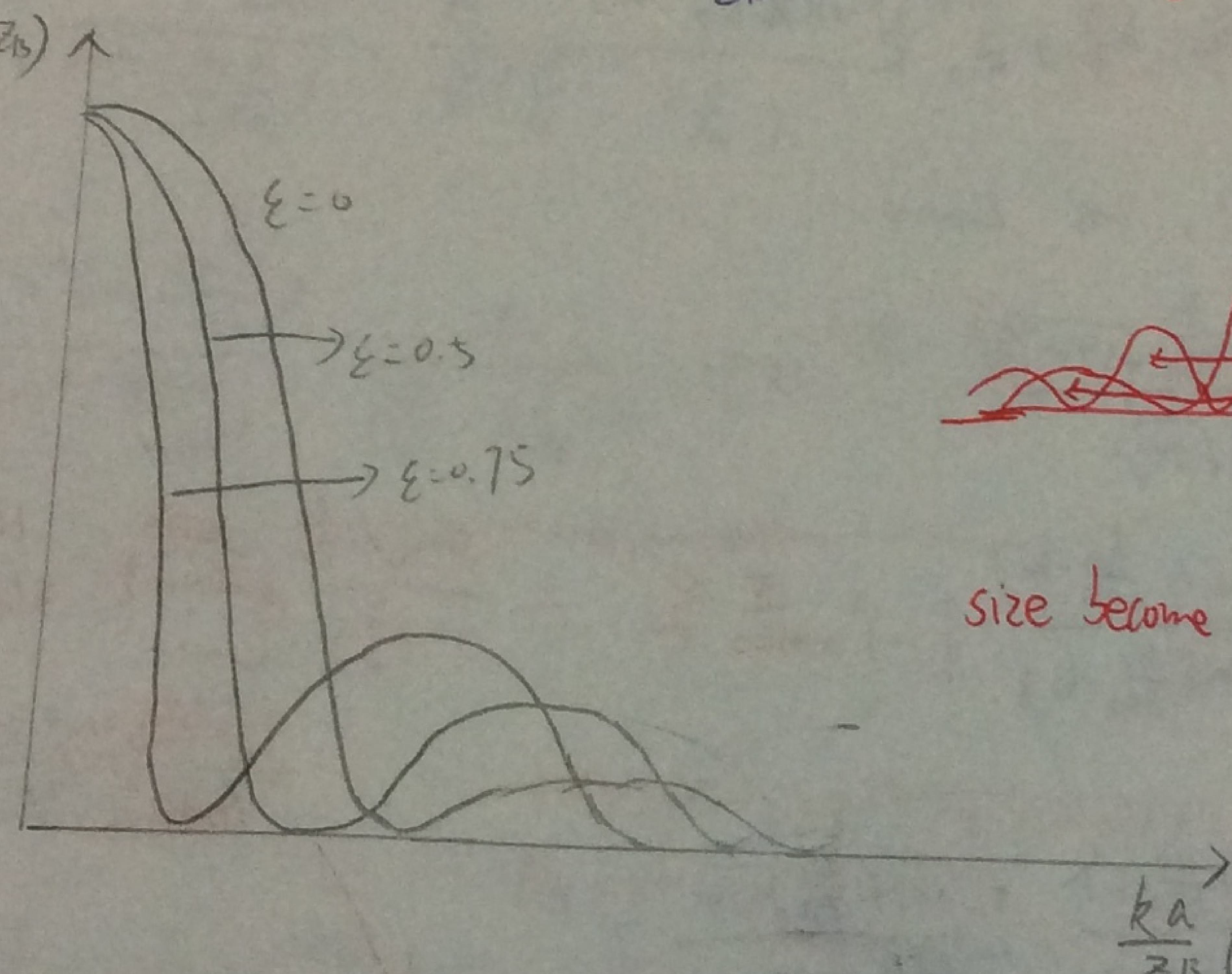
3/3

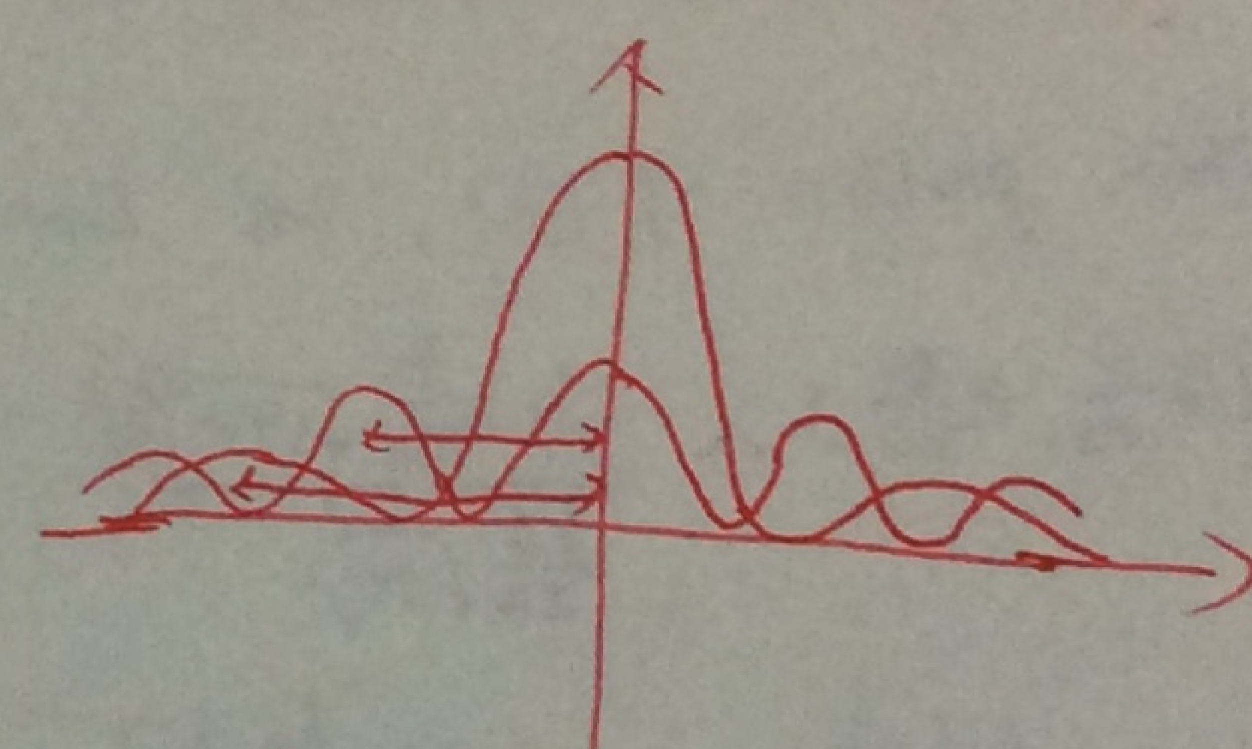
(c) $t(x,y) = \begin{cases} 1 & \varepsilon a < \rho < a \\ 0 \end{cases}$ with $\rho^2 = x^2 + y^2$



$$T\left(k\frac{x}{z_B}, k\frac{y}{z_B}\right) = \frac{a^2}{2\pi} \cdot \frac{J_1\left(\frac{ka}{z_B}\sqrt{x^2+y^2}\right)}{\frac{ka}{z_B}\sqrt{x^2+y^2}} - \frac{(\varepsilon a)^2}{2\pi} \frac{J_1\left(\frac{k\varepsilon a}{z_B}\sqrt{x^2+y^2}\right)}{\frac{k\varepsilon a}{z_B}\sqrt{x^2+y^2}}$$

$$I(x,y,z_B) \sim \left[\frac{J_1\left(\frac{ka}{z_B}\sqrt{x^2+y^2}\right)}{\frac{ka}{z_B}\sqrt{x^2+y^2}} - \varepsilon^2 \frac{J_1\left(\frac{k\varepsilon a}{z_B}\sqrt{x^2+y^2}\right)}{\frac{k\varepsilon a}{z_B}\sqrt{x^2+y^2}} \right]^2$$ ✓





size become larger, radius become smaller.

The pattern are rings.

But what ~~is~~ are the differences with b) case?

Comment missing

$1^*/2^*$

5

Problem 2

a) $t(x)=\sum_{n=0}^{N-1} t_1(x-nb,y)$ with $t_1(x,y)=\begin{cases}1 & x^2+y^2\le a^2\\ 0 & \text{other}\end{cases}$

$$T(k\tfrac{x}{z_B},k\tfrac{y}{z_B})\sim\sum_{n=0}^{N-1}\iint t_1(x'-nb,y')\,e^{-i(k\frac{x}{z_B}x'+k\frac{y}{z_B}y')}dx'dy'$$

set $x'-nb=X'$,

$$T(k\tfrac{x}{z_B},k\tfrac{y}{z_B})\sim\sum_{n=0}^{N-1}\iint t_0(X',y)\,e^{-i(k\frac{x}{z_B}X'+k\frac{y}{z_B}y')}e^{-ik\frac{x}{z_B}nb}dX'dy'$$

$$=T_s(k\tfrac{x}{z_B},k\tfrac{y}{z_B})\sum_{n=0}^{N-1}e^{-ik\frac{x}{z_B}nb}$$

from problem 1 (b), we know

$$T_s(k\tfrac{x}{z_B},k\tfrac{y}{z_B})\sim\frac{J_1(\frac{ka}{z_B}\sqrt{x^2+y^2})}{\frac{ka}{z_B}\sqrt{x^2+y^2}}$$

$$\left|\sum_{n=0}^{N-1}e^{-ik\frac{x}{z_B}nb}\right|=\left|\frac{\sin(N\frac{kx}{2z_B}\cdot b)}{\sin(\frac{kx}{2z_B}\cdot b)}\right|$$

$\sum_{n=0}^{N-1}e^{i\delta n}=\frac{\sin(N\frac{\delta}{2})}{\sin(\frac{\delta}{2})}e^{-i(N+1)\frac{\delta}{2}}$

$1+x+x^2+\dots+x^n=A$

$x(1+x+\dots+x^n)=xA$

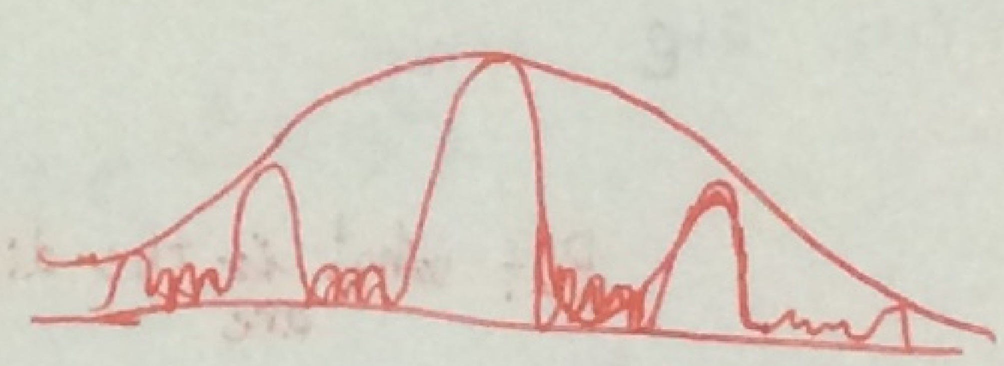
$1-x^{n+1}=A(1-x)$

$A=\frac{1-x^{n+1}}{1-x}$

绝对值为1

So $$I\sim\frac{\sin^2(N\frac{kx}{2z_B}b)}{\sin^2(\frac{kx}{2z_B}b)}\cdot\left[\frac{J_1(\frac{ka}{z_B}\sqrt{x^2+y^2})}{\frac{ka}{z_B}\sqrt{x^2+y^2}}\right]^2$$

✓ 2/2



~~b) $t(x)=u_0(x)=e^{i\varphi_0 s(\frac{x}{a})}$, $s(\frac{x}{a})=\frac{x}{a}-[\frac{x}{a}]$~~

~~b) $t(x)=u_0(x)=e^{i\varphi_0 s(\frac{x}{a})}$, $s(\frac{x}{a})=\frac{x}{a}-[\frac{x}{a}]$~~

~~$\Rightarrow t(x)=\sum_{n=0}^{N-1}t_1(x-na)$ with $t_1(x)=\begin{cases}e^{i\varphi_0\frac{x}{a}} & \text{for } 0<x\le a\\ 0 & \text{other}\end{cases}$~~

~~$T(k\frac{x}{z_B})\sim\sum_{n=0}^{N-1}\int_{-\frac{a}{2}}^{\frac{a}{2}}t_1(x-na)e^{-ik\frac{x}{z_B}x'}dx'$~~

~~$\overset{X'=x-na}{=}\sum_{n=0}^{N-1}e^{ik\frac{x}{z_B}na}\int_{-\frac{a}{2}}^{\frac{a}{2}}e^{-i(\frac{kx}{z_B}-\frac{\varphi_0}{a})X'}dX'\sim\frac{\sin(N\frac{kx}{2z_B}a)}{\sin(\frac{kx}{2z_B}a)}\cdot\mathrm{sinc}(\frac{kx}{2z_B}a-\varphi_0)$~~

1/3

~~$I\sim\frac{\sin^2(N\frac{kx}{2z_B}a)}{\sin^2(\frac{kx}{2z_B}a)}\mathrm{sinc}^2(\frac{kx}{2z_B}a-\varphi_0)$~~

b) $t(x) = \sum_{n=0}^{N-1} t_1(x-na)$ with $t_1(x) = \begin{cases} e^{i\varphi_0 \frac{x}{a}} & \text{for } 0<x\leq a \\ 0 & \text{other} \end{cases}$

$$T(k\tfrac{x}{z_B}) \sim \sum_{n=0}^{N-1} \int_{-\infty}^{\infty} t_1(x-na)\, e^{-ik\frac{x}{z_B}\cdot x'}\, dx'$$

assume $X' = x - na$

$$\therefore T(k\tfrac{x}{z_B}) \sim \sum_{n=0}^{N-1} e^{-ik\frac{x}{z_B}na} \int_0^a e^{i(\varphi_0\frac{x}{a} - \frac{kx}{z_B})X'}\, dX'$$

$$\sim \frac{\sin(N\frac{kxa}{2z_B})}{\sin(\frac{kxa}{2z_B})} \cdot \frac{e^{i(\varphi_0 - \frac{kxa}{z_B})} - 1}{i(\frac{\varphi_0}{a} - k\frac{x}{z_B})}$$

$\frac{a}{(2\pi)^2}\delta(\beta)\sin(\frac{\alpha a - \varphi_0}{2})e^{-i\frac{\alpha a - \varphi_0}{2}}$

$$\therefore I \sim \frac{\sin^2(N\frac{kxa}{2z_B})}{\sin^2(\frac{kxa}{2z_B})} \underbrace{\frac{1}{(\frac{\varphi_0}{a} - k\frac{x}{z_B})^2} \cdot [e^{i(\varphi_0 - \frac{kxa}{z_B})} - 1]^2}_{\cdots \to \mathrm{sinc}^2(\ldots)}$$

2/3

$\frac{a^2}{\lambda^2 z_B^2}\mathrm{sinc}^2(\frac{kxa}{2z_B} - \frac{\varphi_0}{2})\delta^2(\frac{ky}{z_B})$

Problem 3

$N_F = \frac{a^2}{\lambda z_B} \gg 10$

$u(x,y,z_B) = FT^{-1}[H_F(\alpha,\beta,z)\, FT[u(x,y,z)]]$

$H_F = e^{ikz_B} e^{-i\frac{z_B}{2k}(\alpha^2+\beta^2)} \approx 1 \Rightarrow H_F \approx e^{ikz_B} \Rightarrow$ no diffraction

$$u_F(x,y,z_B) = -\frac{i}{\lambda z_B} e^{ikz_B} \iint_{-\infty}^{\infty} u_+(x',y')\, e^{i\frac{k}{2z_B}[(x-x')^2+(y-y')^2]}\, dx'dy'$$

$u_+(x,y) = t(x,y)u_-(x,y)$

$\frac{z_B(\alpha^2+\beta^2)}{2k} = \frac{2z_B(\frac{2\pi}{a})^2}{2\times\frac{2\pi}{\lambda}} = \frac{\lambda z_B \cdot 2\pi}{a^2} \ll 1$

$t(x,y) = \begin{cases} 1 & \text{for } |x|,|y| < \frac{a}{2} \\ 0 & \text{other} \end{cases}$

$\frac{a^2}{\lambda z_B} \gg 2\pi \qquad \frac{a^2}{\lambda z_B} = N_F \gg 10$

$$\Rightarrow u_F(x,y,z_B) = -\frac{i}{\lambda z_B} e^{ikz_B} \iint_{-\frac{a}{2}}^{\frac{a}{2}} u_-(x,y)\, e^{i\frac{k}{2z_B}[(x-x')^2+(y-y')^2]}\, dx'dy'$$

$\frac{k}{2z_B}x'^2 < \frac{k}{2z_B}\cdot(\frac{a}{2})^2 = \frac{\pi}{4}\cdot N_F \gg 2\pi$

if a is large ($a^2 \gg 10\lambda z_B$) it doesn't mean yet that $a \to \infty$

$$\text{So } u_F(x,y,z_B) \approx -\frac{i}{\lambda z_B} e^{ikz_B} \iint_{-\infty}^{\infty} u_-(x,y)\, e^{i\frac{k}{2z_B}[(x-x')^2+(y-y')^2]}\, dx'dy'$$

The beam propagation is in paraxial approximation, which means that there is no diffraction.

0/4

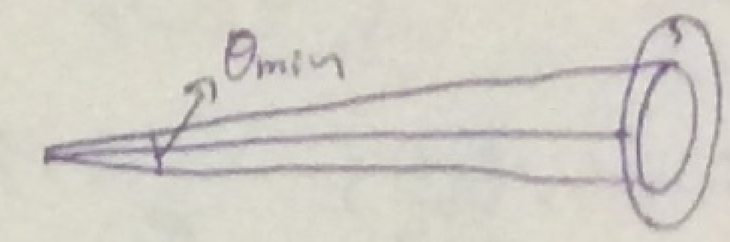
Problem 4

a) From problem 1 (b), we know

$$I \sim \left[\frac{J_1\left(\frac{ka}{z_B}\sqrt{x^2+y^2}\right)}{\frac{ka}{z_B}\sqrt{x^2+y^2}} \right]^2$$

the first zero of the Bessel function.

$\frac{ka}{z_B} \cdot \rho = 1.22\pi$ with $\rho = \sqrt{x^2+y^2}$



$\Rightarrow \frac{\rho}{z_B} = \frac{0.61\lambda}{a}$ $\quad \theta_{min} = \frac{2\rho}{z_B} = \frac{1.22\lambda}{a}$, set $\lambda = 550nm$ ✓

for the telescope : $\theta_{min} = \frac{1.22\lambda}{D} = 3.355\times10^{-6}$ ✓

for the eye : $\theta_{min} = \frac{1.22\lambda}{8mm} = 8.3875\times10^{-5}$ ✓ 2/2

b) since $\frac{\rho}{z_B} = \frac{0.61\lambda}{a}$, in this case, $\theta_{min} = \frac{0.61\lambda'}{a'} = \frac{Y'}{D'}$ (λ': wavelength in medium which refractive index is n')

and $\theta' \ll 1 \Rightarrow \theta' \approx \sin\theta' \approx \tan\theta' = \frac{a'}{D'}$

so $0.61\lambda' = Y' \cdot \frac{a'}{D'} \approx Y'\theta' \approx Y'\sin\theta'$

since $nY\sin\theta = -n'Y'\sin\theta' \Rightarrow |Y| = \left|\frac{n'}{n\sin\theta} \cdot Y'\sin\theta'\right| = \frac{0.61\lambda' n'}{n\sin\theta} = \frac{0.61\lambda_0}{n\sin\theta}$

The resolution limit is $\cancel{\frac{0.61\lambda'}{n\sin\theta}}$ $\frac{0.61\lambda_0}{n\sin\theta}$ (λ_0: wavelength in vacuum)

✓ 3*/3*

c) $d = \frac{\lambda}{2n\sin\alpha}$ is for rectangular aperture

the first zero of rectangular aperture is ✓

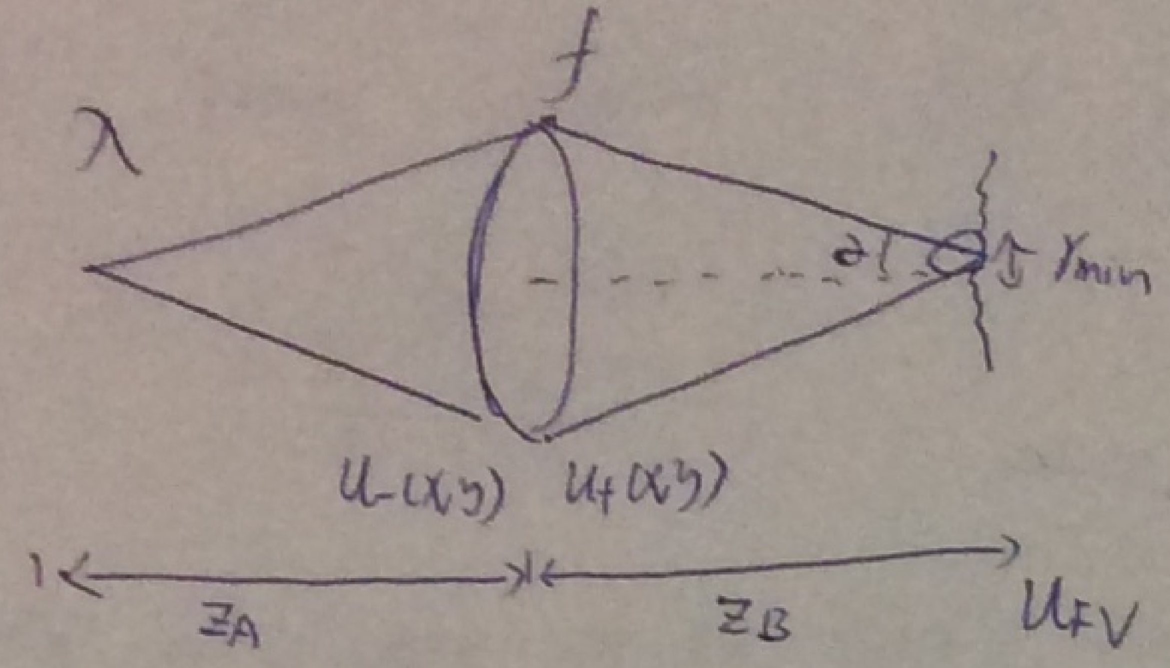
$\frac{ka}{z_B} \cdot x = \pi \xrightarrow{k = \frac{2\pi n}{\lambda' n'}} x = \frac{\lambda' z_B\, n'}{2a\, n}$ ∵ ~~refractive on both side is not the same~~,

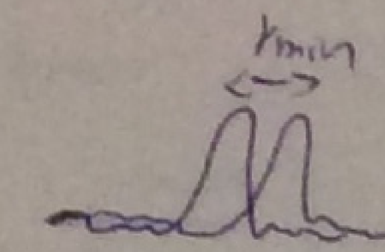
~~we have~~ $\cancel{\frac{n}{n'} \frac{ka}{z_B} x = \pi \Rightarrow x = \frac{\lambda}{2} \frac{z_B}{a} \frac{n'}{n}}$

since θ_{min} is very small, so $\theta_{min} \approx \sin\theta = \frac{a}{z_B}$. (不需要) 1*/2*

$\Rightarrow x = \frac{n'}{n} \frac{\lambda'}{2\sin\theta_{min}} = \frac{\lambda_0}{2n\sin\theta_{min}} \leftrightarrow d = \frac{\lambda}{2n\sin\alpha}$ which is known as Bragg's law.

Exercise 9 Problem 4.





$r_{min} = \frac{\lambda}{2n\sin\alpha}$

$u_-(x,y) = e^{\frac{ik}{2z_A}(x^2+y^2)}$

$u_+(x,y) = u_-(x,y)\, e^{-\frac{ik}{2f}(x^2+y^2)} \times A(r)$ $\quad A(r) = \begin{cases} 1 & r \le a \\ 0 & \text{oder} \end{cases}$ $\quad r=\sqrt{x^2+y^2}$

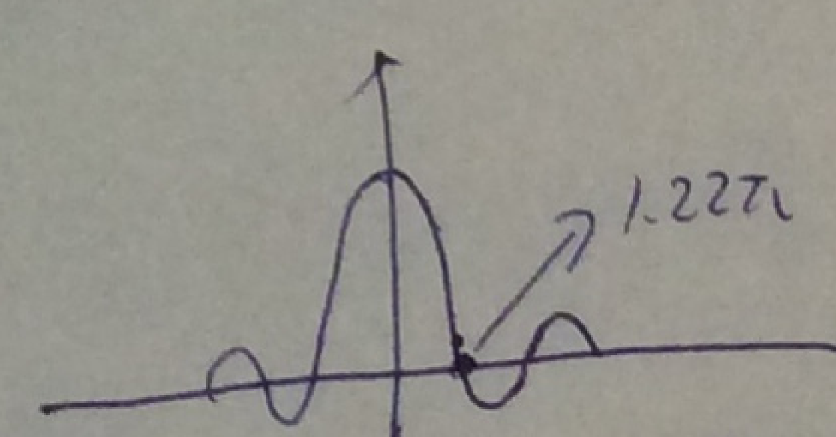
$u_{FV}(x,y) = \iint_{-\infty}^{+\infty} u_+(x',y')\, e^{\frac{ik}{2z_B}\left((x-x')^2+(y-y')^2\right)} dx'dy'$

$\frac{ik}{2z_A}(x'^2+y'^2) - \frac{ik}{2f}(x'^2+y'^2) + \frac{ik}{2z_B}(x^2+x'^2-2xx'+y^2+y'^2-2yy')$

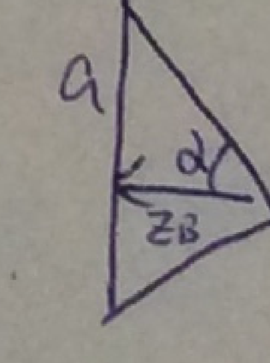
$= \frac{ik}{2}\underbrace{\left(\frac{1}{z_B}+\frac{1}{z_A}-\frac{1}{f}\right)}_{=0}(x'^2+y'^2) + \frac{ik}{2z_B}(x^2+y^2) - \frac{ik}{z_B}(x'x+y'y)$

$\frac{1}{z_A}+\frac{1}{z_B}=\frac{1}{f} \Rightarrow u_{FV}(x,y) \propto e^{\frac{ik}{2z_B}(x^2+y^2)} \iint_{-\infty}^{\infty} A\left(\sqrt{x'^2+y'^2}\right) e^{-i\left(\frac{kx}{z_B}x'+\frac{ky}{z_B}y'\right)} dx'dy'$

$u_{FV} \propto J_1\left(a\sqrt{x^2+y^2}\,\frac{k}{z_B}\right)$, $k=\frac{2\pi n}{\lambda}$



$a\, r_{min} \frac{2\pi n}{\lambda z_B} = 1.22\pi \Rightarrow r_{min} = \frac{0.61\lambda z_B}{an}$



$\sin\alpha \approx \frac{a}{z_B} \Rightarrow r_{min} = \frac{0.61\lambda}{n\sin\alpha}$ Abbe calculation

(under paraxial approximation)

for retta $\quad r_{min} = \frac{\lambda}{2n\sin\alpha} = \frac{\lambda}{2NA}$ $\quad (\sin\alpha = NA)$