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Midterm Exam

FUNDAMENTALS OF MODERN OPTICS

to be written on December 13, 8:15 am – 9:45 am

Problem 1: Maxwell's Equations — 4 + 2 + 4 = 10 points

a) Write down the macroscopic Maxwell's equations in the time domain and in the frequency domain for a linear, isotropic, dispersive, non-magnetizable, and inhomogeneous dielectric medium without any free charges, i.e., current density $\mathbf{j}(\mathbf{r},t) = 0$ and charge density $\rho(\mathbf{r},t) = 0$.

b) Write down the relation between $\mathbf{D}$ and $\mathbf{E}$ in the time and frequency domain in this medium. Name additional functions used in this relation.

c) Derive the wave equations for the electric field $\mathbf{E}(\mathbf{r},\omega)$ and $\mathbf{H}(\mathbf{r},\omega)$ for that case. Note that the equation for $\mathbf{H}(\mathbf{r},\omega)$ will still be coupled to $\mathbf{E}(\mathbf{r},\omega)$.

Hint: $\nabla\cdot(f\mathbf{F}) = f(\nabla\cdot\mathbf{F}) + (\nabla f)\mathbf{F}$

Problem 2: Poynting Vector — 3 + 2 + 1 + 3 = 9 points

A monochromatic electromagnetic plane wave in a homogeneous, linear, isotropic, and non-magnetic medium without external charges and currents is given as

$$\mathbf{E}(x,y,z,t) = E_0\cos\left[(y+z)k - \omega t\right]\hat{\mathrm{x}},$$

where k is the wave number and $\hat{\mathrm{x}}$ is the unit vector in the Cartesian coordinate system.

a) Calculate the magnetic field $\mathbf{H}(x,y,z,t)$ corresponding to the given electric field.

b) Write down the general formula which defines the time averaged Poynting vector $\langle\mathbf{S}(\mathbf{r},t)\rangle$ and the optical intensity $I(\mathbf{r})$.

c) Describe the general relation between medium properties and $\nabla\cdot\langle\mathbf{S}(\mathbf{r},t)\rangle$ in three cases: $\nabla\cdot\langle\mathbf{S}(\mathbf{r},t)\rangle$ is positive, negative, and zero.

d) Calculate $\langle\mathbf{S}(\mathbf{r},t)\rangle$ and $\nabla\cdot\langle\mathbf{S}(\mathbf{r},t)\rangle$ for the electromagnetic wave defined in a). Hint: $\cos^2(a) = \frac{1+\cos(2a)}{2}$

Problem 3: Normal modes in a medium — 1 + 3 + 4 = 8 points

With a good approximation, a medium can be modeled by an ensemble of damped harmonic oscillators, known as the Lorentz model. The response function of this medium is given as:

$$\hat{R}_{mn}(\mathbf{r},t) = \delta_{mn}R(t) \qquad R(t) = \begin{cases} 0 & \text{for } t \le 0 \\ \dfrac{f}{\Omega}\, e^{-\Gamma t}\sin\Omega t & \text{for } t > 0 \end{cases}, \qquad \Omega = \sqrt{\omega_0^2 - \Gamma^2}.$$

a) Based on the given response function, specify the type of the medium by choosing one or more items from the following list: inhomogeneous, homogeneous, anisotropic, isotropic, dispersive, non-dispersive.

b) Calculate the susceptibility $\chi(\omega)$ and the relative permittivity $\varepsilon_r(\omega)$ of the medium. Find its real part $\varepsilon'(\omega)$ and its imaginary part $\varepsilon''(\omega)$.

Consider a superposition of two monochromatic plane waves at frequency ω propagating in this medium with the electric field expression of

$$\mathbf{E}(\mathbf{r}) = E_0\cos\left[(g+ih)x\right]e^{(\mathrm{i}g-h)z}\hat{\mathbf{y}} \quad ,$$

where g and h are real values with the unit 1/m.

c) Derive the two wave vectors $\mathbf{k}_1 = \mathbf{k}_1' + i\mathbf{k}_1''$ and $\mathbf{k}_2 = \mathbf{k}_2' + i\mathbf{k}_2''$ of the two plane waves from the electric field expression. Find the relations which connect g and h with the real and imaginary parts of $\varepsilon_r(\omega)$.

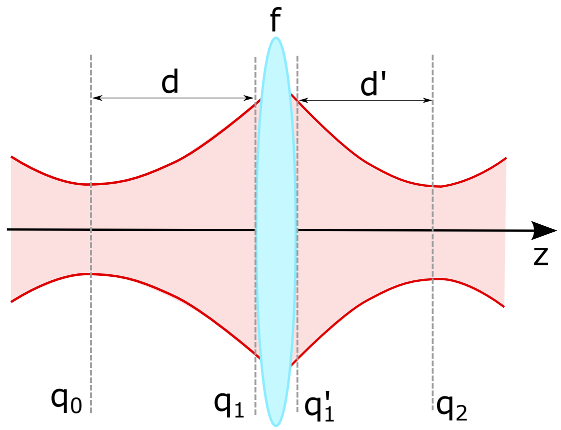
Problem 4: Gaussian Beams **3 + 2 + 2 + 2 = 9 points**

a) The q-parameter of a Gaussian beam describes the evolution of the beam width $W(z)$ and curvature $R(z)$ under propagation with:

$$\frac{1}{q(z)} = \frac{1}{R(z)} + i\frac{\lambda}{\pi W(z)^2}$$

Sketch $W(z)$, make sure to include the waist w_0, Rayleigh length z_0 and its corresponding value $W(z = z_0)$. How do $W(z)$ and $R(z)$ evolve in the limit for large z?

b) Consider a Gaussian beam with Rayleigh length z_0. A focusing thin lens of focal length f is placed at distance d from the beam waist w_0. Write down the ABCD-Matrix connecting q_0 to q_1' and express q_1' in terms of z_0, d, and f.

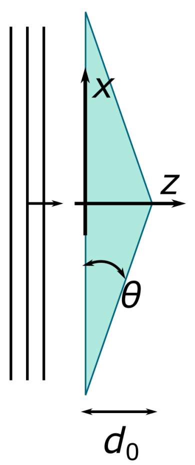


c) Find the distance d' from the lens to the beam waist w_0' and new Rayleigh length z_0' expressed in terms of z_0, d, and f.

d) Consider you move the lens such that $d = f$, find the magnification $M = w_0'/w_0$.

Problem 5: Beam Propagation **3 + 2 + 3 + 2 = 10 points**

Consider the propagation of a monochromatic, scalar field $u(x, y, z)$ at a wavelength λ along the z-direction starting from a given initial field distribution $u(x, y, z = 0) = u_0(x, y)$.



a) Describe the steps necessary to calculate the field distribution $u(x, y, z)$ for any $z > 0$ by using the free space transfer function. Name and define all functions and quantities that you use and give the necessary formulas for this calculation.

b) Sketch the amplitude and phase of the free space transfer function $\mathrm{H}(\alpha, \beta = 0, z)$ for a finite distance z. Interpret your results when $\alpha > k_0$ and $\alpha \leq k_0$.

Now, we consider a 2 dimensional space for the sake of simplicity. As illustrated in the figure, we have a thin phase mask with an edge angle of $\theta \ll 1°$ (so that it is sufficiently large along x-axis), and a height of d_0, and made of a material with a refractive index of n. The phase mask can be described in some approximation as:

$$t(x) = 2h_0 \cos\left[(n-1)\theta k_0 x\right],$$

where $h_0 = e^{ik_0 d_0}$.

c) Assuming that a monochromatic plane wave is incident on the phase mask, calculate the Fourier transform $U_0(\alpha)$ of the initial field right after the phase mask. Describe the fields after the phase mask: What do you get after the phase mask? What will happen upon propagation?

d) Now, imagine the phase mask in the figure is rotated around z-axis to create a 3D phase mask which has a conical shape. Based on your result in c), what conclusion can you draw for the case of this 3D phase mask? What shape of electromagnetic field will you get after the phase mask, if a plane wave is incident on it?

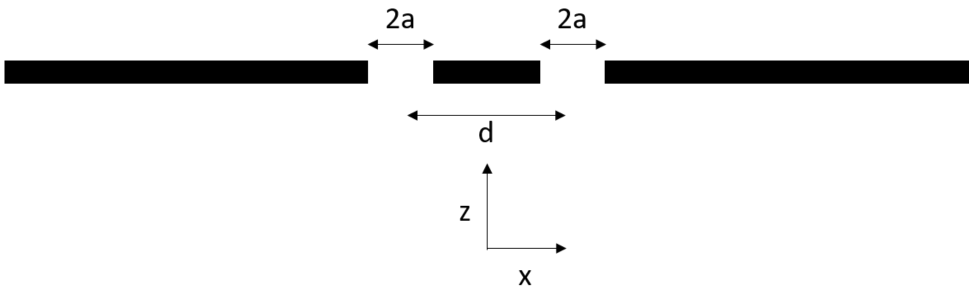
Problem 6: Diffraction **2 + 1 + 2 = 5 points**

a) State the condition for the paraxial approximation and derive the free-space transfer function in spatial frequency domain using this approximation of the general case.

b) Sketch the real part of the argument of the paraxially approximated free space transfer function for $\beta = 0$ as a function of α.

c) Consider a plane wave of the shape $u(x,z) = \exp(ikz)$ in vacuum, incident on a double slit at z=0, having the transmission function

$$t(x) = \begin{cases} 1 & \text{if } \frac{d}{2} - a < |x| < \frac{d}{2} + a \\ 0 & \text{otherwise} \end{cases}$$



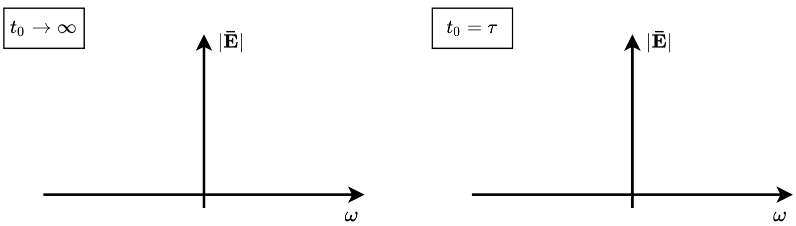
Derive the spatial frequency spectrum of the field transmitted through the double slit.

Problem 7: Pulse Propagation **2 + 1 + 2 = 5 points**

The given plane wave is modulated in the time domain by the Gauss-function:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E_0} \exp\left(-\frac{t^2}{t_0^2}\right) \cos(\mathbf{k}(\omega_0) \cdot \mathbf{r} + \omega_0 t)$$

a) Sketch the magnitude of the Fourier representation (with respect to time) $|\bar{\mathbf{E}}(\mathbf{r},\omega)| = |FT(\mathbf{E}(\mathbf{r},t))|$ of the given wave schematically for the case $t_0 \to \infty$ **and** $t_0 = \tau$, where τ is a finite constant (calculation of Fourier representation is not necessary).



b) Would you expect changes of the given pulse-shapes when propagating through dispersive **and** non-dispersive media? Please, give a **short** explanation.

c) The polynomial dispersion term of the medium is assumed as: $n(\omega) = l + m\omega^2 + s\omega^3$, where $l, m, s \in \mathbb{R}$. Calculate the phase velocity, group velocity and group velocity dispersion in this medium. Which condition (relation between l, m, s and ω_0, **and** $T_0 = 2\pi/\omega_0$ and $t_0 = \tau$, see task a)) has to be fulfilled so that the first two quantities (phase and group velocity) remain applicable for the pulse propagation description?