Fundamentals of Modern Optics
series 1
19.10.2015

to be returned on 23.10.2015, at the beginning of the lecture

Initial Remarks

- return the completed assignments by the above mentioned date.
- group work is allowed and encouraged; however each student has to hand in an individual assignment; literal copies will not be accepted.
- problems marked with an asterisk (*) are non-mandatory and can be used to gain extra points.
- assignments will be checked, returned, and discussed in the seminars in the week after the return dates.
- hand in your assignments in hand-writing only; write neatly.
- note the date and time of your seminar (Monday 12-14, Monday 14-16, Wednesday 10-12) on the assignment
- write down all calculations and derivations in a clear and concise manner.

Problem 1 - Fourier Transformations (a=2,b=2+2*,c=2* pts.)

Given is the definition of the Fourier transformation and its inverse, which transforms the time domain representation of a signal $f(t)$ into its frequency domain representation $f(\omega)$ and vice versa:

$$
\begin{aligned}
f(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\exp[i\omega t]\mathrm{d}t \\
f(t) &= \int_{-\infty}^{\infty} f(\omega)\exp[-i\omega t]\mathrm{d}\omega.
\end{aligned}
$$

Use these definitions to find the frequency domain representation of the following signals

a) $f(t) = \begin{cases} 0 & t < 0 \\ A\exp[-\gamma t]\cos(\omega_0 t) & t \geq 0 \end{cases}$

b) $f(t) = A\exp[-\frac{1}{2}\frac{t^2}{t_0^2}]$
This problem involves a complex valued Gaussian integral which you can directly insert its answer to proceed with your solution. The 2 bonus points go to whoever correctly solves that Gaussian integral.

and

c*) Show for the second function that the product of the square root of the second momentum in time domain and in frequency domain is a constant.

Hint: The square root of the second moment $\sqrt{\langle f^2 \rangle}$ of a symmetric function is defined as:

$$\sqrt{\langle f^2 \rangle} = \sqrt{\frac{\int_{-\infty}^{\infty} t^2 \left|f(t)\right|^2 \mathrm{d}t}{\int_{-\infty}^{\infty} \left|f(t)\right|^2 \mathrm{d}t}}$$

Problem 2 - Fourier Transform Properties (a=2,b=2 pts.)

Assume that a signal $f(t)$ is given and its frequency representation $f(\omega)$ is known. Now calculate the frequency domain representation of

a) $f(t-t_0)$, a signal that is translated by t_0

and

b) $\frac{\mathrm{d}}{\mathrm{d}t}f(t)$, the temporal derivative of the signal.

Problem 3 - δ-Functions (a=1*,b=1,c=1,d=1,e=1*,f=1 pts.)

Given is a function $\delta(t)$, with the following properties

i) $\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undefined} & t = 0 \end{cases}$

ii) $\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1.$

With this knowledge, do the followings:

a*) Show that the function $f(t) = \lim_{w\to 0} \frac{1}{\sqrt{\pi w}} \exp[-\frac{t^2}{w^2}]$ fulfils the above mentioned properties and is thus a possible representative of the δ function.

Furthermore, calculate expressions for the following integrals:

b) $\int_{-\infty}^{\infty} \delta(t)f(t)\mathrm{d}t$

c) $\int_{-\infty}^{\infty} \delta(t-t_0)f(t)\mathrm{d}t$

d) $\int_{-\infty}^{\infty} \delta(at)f(t)\mathrm{d}t$

e*) $\int_{-\infty}^{\infty} \delta(g(t))f(t)\mathrm{d}t$, where $g(t)$ is an arbitrary analytic function, with $g(t) = 0 \Leftrightarrow t \in \{t_0^i\} \wedge i \in \{1 \ldots N\}$. The roots of $g(t)$ must be simple roots, meaning that $g(t_0^i) \neq 0$.

Now calculate

f) the Fourier transform of the delta function.

Hint: While the solution of the problems b) to f) is possible with a representative function, we suggest to just use the definition i) and ii), in combination with Taylor expansions or change of variables to find solutions.

Problem 4 - The Convolution Theorem (4 pts.)

Given are two functions $f(t)$ and $g(t)$ and their Fourier transformations $f(\omega)$ and $g(w)$. The convolution $[f \otimes g](t)$ of both functions is defined as

$$[f \otimes g](t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\mathrm{d}\tau.$$

Calculate the fourier transform of the convolution

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} [f \otimes g](t)\exp[i\omega t]\mathrm{d}t.$$

Hint: Replace $f(t)$and $g(t)$ with their respective fourier integrals. Reorder the resulting quadruple integral to generate δ-functions that allow you to solve the integrals.