Fundamentals of Modern Optics

Series 12

10+8+2=20

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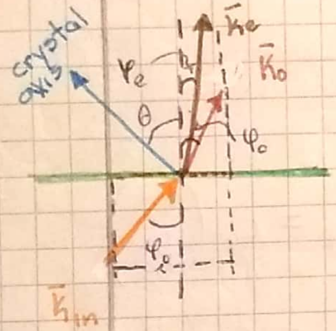
Problem 1

Uniaxial crystal $\epsilon_1 = \epsilon_2 \neq \epsilon_3$ $\therefore$ $n_o \equiv \sqrt{\epsilon_1}$; $n_e \equiv \sqrt{\epsilon_3}$
$\bar{k}$ $\therefore$ $\in k_2 k_3$ plane $\Rightarrow n_a(\phi) = n_o$ & $n_b(\phi) = \sqrt{\dfrac{\epsilon_1 \epsilon_3}{\epsilon_1 \sin^2\phi + \epsilon_3 \cos^2\phi}}$

where $\phi \equiv \measuredangle(\text{optic axis}, \bar{k})$

a) Interface n_i & Crystal



i) Since the $\bar{k}_{\parallel}$ has to be continues at the interface

$$\Rightarrow |\bar{k}_{in}| \sin\varphi_{in} = |\bar{k}_o| \sin\varphi_o$$

Furthermore, ~~since the each is~~ ~~are constant~~, the dispersion relations for these $\bar{k}$'s are:

$$|\bar{k}_{in}| = \frac{\omega}{c} n_i \quad \& \quad |\bar{k}_o| = \frac{\omega}{c} n_o$$

$$\Rightarrow \cancel{\frac{\omega}{c}} n_i \sin\varphi_{in} = \cancel{\frac{\omega}{c}} n_o \sin\varphi_o \quad \Rightarrow n_i \sin\varphi_{in} = n_o \sin\varphi_o ,$$

ii) Similarly, the extraordinary wave has to be continues at the interface:

$$\Rightarrow |\bar{k}_{in}| \sin\varphi_i = |\bar{k}_e| \sin\varphi_e$$

Now, the refractive index for the extraordinary component depends on the angle between that one and the crystal axis, from the figure is easy to see that is $\phi = \theta + \varphi_e$ $\Rightarrow n_e = n_b(\theta+\varphi_e) = \sqrt{\dfrac{\epsilon_1\epsilon_3}{\epsilon_1 \sin^2(\theta+\varphi_e) + \epsilon_3 \cos^2(\theta+\varphi_e)}}$

Hence, the dispersion relation for that component is: $|\bar{k}_e| = \frac{\omega}{c} n_b(\theta+\varphi_e)$

$$\Rightarrow \cancel{\frac{\omega}{c}} n_i \sin\varphi_i = \cancel{\frac{\omega}{c}} n_b(\theta+\varphi_e) \sin\varphi_e$$

$$\Rightarrow n_i \sin\varphi_i = n_b(\theta+\varphi_e) \sin\varphi_e$$

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b) By the definition of u_i, we know that:

$$k_i^2 = k_0^2 n^2 u_i^2 \quad (*)$$ where $k_0 = |\bar{k}_0| \equiv$ the incident ~~kog~~ -vector

$$\Rightarrow \begin{cases} k^2 = k_2^2 + k_3^2 = k_0^2 n_{or}^2 (\cancel{u_2^2 + u_3^2})^{1} \\ \dfrac{k_2^2}{n_e^2} + \dfrac{k_3^2}{n_{or}^2} = k_0^2 (\cancel{u_2^2 + u_3^2})^{1} \end{cases}$$

$$\Rightarrow \begin{cases} k_2^2 + k_3^2 = n_o^2 k_o^2 \Rightarrow \text{circle} \Rightarrow \text{ordinary wave} \\ \dfrac{k_2^2}{n_e^2} + \dfrac{k_3^2}{n_o^2} = k_0^2 \quad (*) \Rightarrow \text{ellipse} \Rightarrow \text{extraordinary wave} \end{cases}$$

To find the tangent of each surface:

$$\begin{cases} \cancel{2} k_2 \, dk_2 + \cancel{2} k_3 \, dk_3 = 0 \\ \cancel{2} \dfrac{k_2}{n_e^2} dk_2 + \cancel{2} \dfrac{k_3^2}{n_o^2} dk_3 = 0 \end{cases} \Rightarrow \begin{cases} \bar{n}_o = C_o \begin{pmatrix} k_2 \\ k_3 \end{pmatrix} \\ \bar{n}_e = C_e \begin{pmatrix} k_2/n_e^2 \\ k_3/n_o^2 \end{pmatrix} \end{cases}$$

where $C_o, C_e \in \mathbb{R}$

Now, respect to $\bar{E}$, we know that:

$$E_2 : E_3 = \frac{k_2}{k_0^2 n_o^2 - \underbrace{(k_2^2 + k_3^2)}_{k^2}} : \frac{k_3}{k_0^2 n_e^2 - \underbrace{(k_2^2 + k_3^2)}_{k^2}} \Rightarrow \bar{E} = c \begin{pmatrix} E_2 \\ E_3 \end{pmatrix}; \; c \in \mathbb{R}$$

~~and $\bar{E}$ ~~ ~~tangent~~

$$\Rightarrow \bar{n}_e \cdot \bar{E} = c C_e \left[\frac{k_2}{n_e^2} \left(\frac{k_2}{k_0^2 n_o^2 - (k_2^2 + k_3^2)} \right) + \frac{k_3}{n_o^2} \left(\frac{k_3}{k_0^2 n_e^2 - (k_2^2 + k_3^2)} \right) \right]$$

$$= c C_e \left[\frac{k_2^2 (n_o^2 n_e^2 k_0^2 - n_o^2 (k_2^2 + k_3^2)) + k_3^2 (n_o^2 n_e^2 k_0^2 - n_e^2 (k_2^2 + k_3^2))}{n_o^2 n_e^2 (k_0^2 n_e^2 - k^2)(k_0^2 n_o^2 - k^2)} \right]$$

$$= c C_e \left[\frac{(k_2^2 + k_3^2) k_0^2 n_o^2 n_e^2 - (k_2^2 n_o^2 + k_3^2 n_e^2)(k_2^2 + k_3^2)}{n_o^2 n_e^2 (k_0^2 n_e^2 - k^2)(k_0^2 n_o^2 - k^2)} \right]$$

$$= c C_e (k^2) \left[\frac{\overbrace{k_0^2 n_o^2 n_e^2 - (k_2^2 n_o^2 + k_3^2 n_e^2)}^{(*) \to 0}}{n_o^2 n_e^2 (k_0^2 n_e^2 - k^2)(k_0^2 n_o^2 - k^2)} \right] = 0$$

$$\& \; \bar{n}_{or} \cdot \bar{E} = c C_o \left[\left(\frac{k_2^2}{n_o^2 k_0^2 - k^2} \right) + \left(\frac{k_3^2}{n_e^2 k_0^2 - k^2} \right) \right] = \frac{k_2^2 (n_e^2 k_0^2 - k^2) + k_3^2 (n_o^2 k_0^2 - k^2)}{(n_o^2 k_0^2 - k^2)(n_e^2 k_0^2 - k^2)}$$

$$= \frac{-k^2 (k_2^2 + k_3^2) + k_0^2 (n_e^2 k_2^2 + k_3^2 n_o^2)}{(n_o^2 k_0^2 - k^2)(n_e^2 k_0^2 - k^2)} \overset{(*)}{=} \frac{-k^4 + k^4}{(n_o^2 k_0^2 - k^2)(n_e^2 k_0^2 - k^2)} = 0$$

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$\therefore \bar{n} \perp \bar{E}$ and obviously $\bar{n} \perp \bar{H}$

Now, by definition $\bar{S} = \bar{E} \times \bar{H} \Rightarrow \bar{S} \perp \bar{E}$ & $\bar{S} \perp \bar{H}$

$\Rightarrow \bar{S} \parallel \bar{n}$

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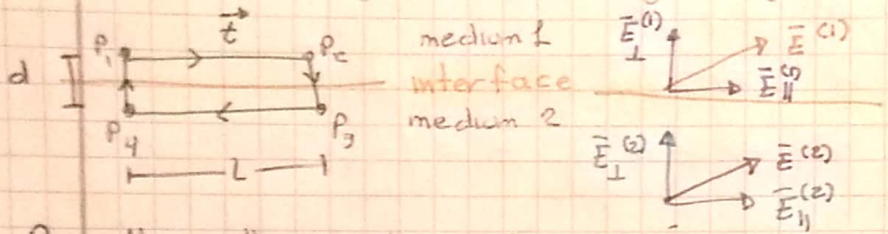
$\therefore \bar{S} \perp$ to the normal surface of both the ordinary & the extraordinary waves

Problem 2.

a) Prove $\bar{E}_{\parallel}, \bar{H}_{\parallel}, \bar{K}_{\parallel}$ have to be continuous at the interface bet. two diff. homo, iso. media by using Maxwell's eqs.

~~Gotts~~ Considering the following trayectory $(\bar{t})$:



NOTE: In gral. you have 4 eqs. from boundary conditions but only one pair together are independent

By Maxwell equations:

* $$\oint_t \bar{E}(\bar{r}, t')\, d\bar{l} = -\frac{\partial}{\partial t} \int_{S_t} \bar{B}(\bar{r}, t')\, d\bar{S}$$; $t' \equiv$ temporal variable

$$\left.\begin{array}{l} \Rightarrow d \to 0 \\ \Rightarrow S \to 0 \end{array}\right\} \Rightarrow \left[E_{\parallel}^{(1)}(\bar{r}, t') - E_{\parallel}^{(2)}(\bar{r}, t')\right] L = 0$$

$$\therefore E_{\parallel}^{(1)}(\bar{r}, t') = E_{\parallel}^{(2)}(\bar{r}, t') \quad (*)$$

* $$\oint_t \bar{H}(\bar{r}, t) \cdot d\bar{l} = \int_{S_t} \left[\cancel{\bar{J}(\bar{r}, t)}^{\,0} + \frac{\partial \bar{D}}{\partial t}\right] \cdot d\bar{S}$$

$$\left.\begin{array}{l} \Rightarrow d \to 0 \\ \Rightarrow S \to 0 \end{array}\right\} \Rightarrow \left[H_{\parallel}^{(1)}(\bar{r}, t) - H_{\parallel}^{(2)}(\bar{r}, t)\right] L = 0$$

$$\therefore H_{\parallel}^{(1)}(\bar{r}, t) = H_{\parallel}^{(2)}(\bar{r}, t)$$

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Now, in general $\bar{E}_{\parallel}^{(i)} = \bar{E}_{\parallel}^{(i)} e^{i \bar{K}_{\parallel}^{(i)} \cdot \bar{r}}$

$$* \Rightarrow e^{i \bar{K}_{\parallel}^{(1)} \cdot \bar{r}} = e^{i \bar{K}_{\parallel}^{(2)} \cdot \bar{r}} \Rightarrow \bar{K}_{\parallel}^{(1)} = \bar{K}_{\parallel}^{(2)}$$

b) Prove $\theta_1 = \theta_1'$ & $n_1 \sin\theta_1 = n_2 \sin\theta_2$

Since the tangential component of the k vector is continuous:

~~[illegible]~~ $|\bar{K}^{(1)}| \sin\theta_1 = |\bar{K}^{(2)}| \sin\theta_2$; $|\bar{K}_{Incident}| = |\bar{K}^{(1)}|$ & $|\bar{K}_{Reflected}| = |\bar{K}^{(1)}|$ & $|\bar{K}_{Transmited}| = |\bar{K}^{(2)}|$

And from the dispersion relation

$$|\bar{K}^{(1)}| = K_0 n_1 \quad \& \quad |\bar{K}^{(2)}| = K_0 n_2$$

$$\Rightarrow K_0 n_1 \sin\theta_I = K_0 n_0 \sin\theta_R = K_0 n_2 \sin\theta_2$$

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$$\Rightarrow \begin{cases} \theta_I = \theta_R & \Rightarrow \text{Reflection's Law} \\ \& \ n_1 \sin\theta_I = n_2 \sin\theta_T & \Rightarrow \text{Refraction's Law} \end{cases}$$

c) ~~Same~~ TM polarization $\Rightarrow \bar{E} = \begin{pmatrix} 0 \\ E_y \\ 0 \end{pmatrix}$ & $\bar{H} = \begin{pmatrix} H_x \\ 0 \\ H_z \end{pmatrix}$

The continuity of the tangential components of E & H implys:

$$(*) \quad E_i + E_r = E_t \quad \& \quad -H_i \cos\theta_i + H_R \cos\underbrace{\theta_i}_{\theta_r} = -H_t \cos\theta_t \quad (\circledS)$$

Now, from Maxwell's Eqs.:

$$\nabla \times \bar{E} = i\omega\mu_0 \bar{H} \quad \Rightarrow \quad i\bar{k} \times \bar{E} = i\omega\mu_0 \bar{H}$$

$$\Rightarrow H = \frac{k}{\omega\mu_0} E = \sqrt{\frac{\epsilon_j}{\mu_0}} E$$

~~where~~ for each medium j

$$* \ \& \ \circledS \Rightarrow -\sqrt{\frac{\epsilon_1}{\cancel{\mu_0}}} E_i \cos\theta_i + \sqrt{\frac{\epsilon_1}{\cancel{\mu_0}}} E_r \cos\theta_i = -\sqrt{\frac{\epsilon_2}{\cancel{\mu_0}}} E_t \cos\theta_t$$

By definition $\sqrt{\epsilon_l} = n_l$

$$\Rightarrow n_1 (E_r - E_i) \cos\theta_i = n_2 E_t \cos\theta_t$$

$$\Rightarrow r_{TM} = \frac{E_r}{E} = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$

$$\& \quad t_{TM} = \frac{E_t}{E} = \frac{2 n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$

From before ~~we~~ : $n_1 \sin\theta_1 = n_2 \sin\theta_2 \Rightarrow \cos\theta_t = \sqrt{1 - \frac{n_1^2}{n_2^2} \sin^2\theta_i}$

Let $\theta_i = \theta$

$$\Rightarrow r_{TM} = \frac{n_1 \cos\theta - \sqrt{n_2^2 - n_1^2 \sin^2\theta}}{n_1 \cos\theta + \sqrt{n_2^2 - n_1^2 \sin^2\theta}}$$

$$\& \quad t_{TM} = \frac{2 n_1 \cos\theta}{n_1 \cos\theta + \sqrt{n_2^2 - n_1^2 \sin^2\theta}}$$

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Problem 3

On the One hand

Total internal reflection occurss when the incident angle is greater than a critical angle θ_c when the output angle is exactly $\pi/2 \Rightarrow n_1 \sin\theta_c = n_2$

$$\Rightarrow \theta > \theta_c \Rightarrow \sin\theta > \sin\theta_c$$

$$\Rightarrow n_1 \sin\theta > n_2$$

On the other hand

The tangential components of $\bar{k}$ are continuos :

$$\Rightarrow k_{i\parallel} = k_{t\parallel} = n_1 k_0 \sin\theta > n_2 k_0 = |\bar{k}_t|$$

$$\Rightarrow k_{t\perp} = \pm\sqrt{|\bar{k}_t|^2 - k_{t\parallel}^2} = \pm\sqrt{(n_2 k_0)^2 - (n_1 k_0 \sin\theta)^2}$$

$$= \pm i k_0 \sqrt{(n_1 \sin\theta)^2 - n_2^2}$$

$$\Rightarrow \bar{E}_t(\bar{r}, t) = \bar{E}_{0t} \exp[i(\bar{k}\cdot\bar{r} - \omega t)]$$

$$= \bar{E}_{0t} \exp[i(k_{t\parallel} r_\parallel + k_{t\perp} r_\perp - \omega t)]$$

$$= \bar{E}_{0t} \exp\left[-k_0 r_\perp \sqrt{(n_1 \sin\theta)^2 - n_2^2}\right]$$

$$\cdot \exp[i(n_1 k_0 r_\parallel \sin\theta - \omega t)]$$

$$\Rightarrow |E|_t \propto \exp\left[-k_0 r_\perp \sqrt{(n_1 \sin\theta)^2 - n_2^2}\right]$$

$\Rightarrow |E|_t$ decays exponentially during propagation

$\therefore$ light doesn't pass through the medium

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