Task 1: Normal modes in low loss materials (a=2, b=3, c=3 pts.)

Assume an isotropic and homogeneous medium with the complex-valued relative permittivity $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$. A plane wave is propagating in this medium with the frequency-domain electric-field expression of $\mathbf{E}(\mathbf{r},\omega) = \mathbf{E}_0(\omega)\exp(i\mathbf{k}(\omega)\cdot\mathbf{r})$, where the wave vector $\mathbf{k} = \mathbf{k}' + i\mathbf{k}''$ is complex-valued. The electric field satisfies the Helmholtz equation in the frequency domain $\left[\Delta + \frac{\omega^2}{c^2}\varepsilon(\omega)\right]\mathbf{E}(\mathbf{r},\omega) = 0$.

a) Derive the set of equations which connect $\mathbf{k}'$ and $\mathbf{k}''$ with ε' and ε''.

Now assume $\mathbf{k}' \parallel \mathbf{k}'' \parallel \hat{k}$, which gives $\mathbf{k}' + i\mathbf{k}'' = \hat{k}\frac{\omega}{c}(n + i\kappa)$. We would like to use the 扰动理论 *perturbation theory* to find n and κ as functions of ε' and ε'', in other words $n(\varepsilon',\varepsilon'')$ and $\kappa(\varepsilon',\varepsilon'')$. Perturbative approaches are applied to many different physics problems, and the idea is as follows: Assume that you have a mathematical problem that is hard to solve analytically (although our particular problem has an analytical solution given in the lecture note, which is derived by solving a second order polynomials, but let's assume we do know how to do that and we are stuck at the two equations derived in part (a)). However, it turns out that if you set a certain parameter in the problem equal to zero, then it becomes much simpler to find an exact solution. In our case that parameter is ε''. By putting $\varepsilon'' = 0$, you can find the *zeroth-order* approximation solutions, let us call them $n_0(\varepsilon',\varepsilon'')$ and $\kappa_0(\varepsilon',\varepsilon'')$. If ε'' is very small (in this problem it means $|\varepsilon'| \gg \varepsilon''$), this could already be a good enough approximation. But say you want to improve the accuracy of your answer. You can calculate what is called a corrected *first-order* answer. To find this, you take the original problem with $\varepsilon'' \neq 0$, and if $\kappa_0 = 0$, then substitute n with n_0 and then try to find κ. The κ you find in this way is your corrected first-order answer, call it κ_1. Now go back to the original problem again and this time substitute κ with κ_1 and then try to find n. The n you find is your corrected first-order answer n_1. For the case of $n_0 = 0$, follow the same mentioned steps just with n and κ exchanged in the instructions. You can do this repeatedly to find the higher order corrections to $n(\varepsilon',\varepsilon'')$ and $\kappa(\varepsilon',\varepsilon'')$.

Find $n_1(\varepsilon',\varepsilon'')$ and $\kappa_1(\varepsilon',\varepsilon'')$ for the two practical situations below:

b) The case of a low loss dielectric with $\varepsilon'(\omega) > 0$ and $\varepsilon''(\omega) > 0$ and $\varepsilon'(\omega) \gg \varepsilon''(\omega)$.

c) The case of a low loss metal with $\varepsilon'(\omega) < 0$ and $\varepsilon''(\omega) > 0$ and $|\varepsilon'(\omega)| \gg \varepsilon''(\omega)$.

$\sqrt{\varepsilon} = \frac{c}{\omega}k = \frac{c}{\omega}k' + i\frac{c}{\omega}k''$

$\varepsilon = \frac{c^2}{\omega^2}k^2$

(a) $[\Delta + \frac{\omega^2}{c^2}\varepsilon(\omega)]\vec{E}(\vec{r},\omega) \Rightarrow \Delta E_0(\omega)\exp[i\vec{k}(\omega)\cdot\vec{r}] + \frac{\omega^2}{c^2}\varepsilon(\omega)\vec{E}(\vec{r},\omega)$

$\Rightarrow -k^2(\omega)E_0(\omega)\exp[i\vec{k}(\omega)\cdot\vec{r}] + \frac{\omega^2}{c^2}\varepsilon(\omega)\vec{E}(\vec{r},\omega) \Rightarrow k^2(\omega) = \frac{\omega^2}{c^2}\varepsilon(\omega)$

$\vec{k} = \vec{k}' + i\vec{k}'' \Rightarrow k^2(\omega) = k'^2 - k''^2 + i2\vec{k}'\cdot\vec{k}'' = \frac{\omega^2}{c^2}[\varepsilon'(\omega) + i\varepsilon''(\omega)] \Rightarrow \begin{cases} k'^2 - k''^2 = \frac{\omega^2}{c^2}\varepsilon'(\omega) \\ 2\vec{k}'\cdot\vec{k}'' = \frac{\omega^2}{c^2}\varepsilon''(\omega) \end{cases}$

(b) $k' \parallel k'' \parallel \hat{k} \Rightarrow k' + ik'' = \frac{\omega}{c}(n + i\kappa) = k'^2 - k''^2 + 2ik'k'' = \frac{\omega^2}{c^2}(n^2 - \kappa^2 + 2in\kappa) \Rightarrow \varepsilon' + i\varepsilon'' = n^2 - \kappa^2 + i2n\kappa$

$k = \frac{\omega}{c}\sqrt{\varepsilon(\omega)}$ $\quad k' + ik'' = \frac{\omega}{c}(\mathrm{Re}\sqrt{\varepsilon} + i\,\mathrm{Im}\sqrt{\varepsilon}) = \frac{\omega}{c}(n + i\kappa) \Rightarrow n = \mathrm{Re}\sqrt{\varepsilon} = \frac{c}{\omega}k'$ $\quad \kappa = \mathrm{Im}\sqrt{\varepsilon} = \frac{c}{\omega}k''$

$\Rightarrow k^2 = \frac{\omega^2}{c^2}(n^2 - \kappa^2 + 2in\kappa) \Rightarrow \underline{\varepsilon' = n^2 - \kappa^2 \quad \varepsilon'' = 2n\kappa}$

$\varepsilon''^2 = 4n^2\kappa^2 \quad n^2 = \kappa^2 + \varepsilon' \Rightarrow \varepsilon''^2 = 4(\kappa^2 + \varepsilon')\kappa^2 = 4\kappa^4 + 4\varepsilon'\kappa^2 - \varepsilon''^2 = 0$

$\Rightarrow \kappa = \frac{\sqrt{2}}{2}(\sqrt{\varepsilon'^2 + \varepsilon''^2} - \varepsilon')^{\frac{1}{2}}$

$\kappa^2 = n^2 - \varepsilon' \Rightarrow \varepsilon''^2 = 4n^2(n^2 - \varepsilon') = 4n^4 - 4\varepsilon' n^2 - \varepsilon''^2 \Rightarrow n^2 = \frac{4\varepsilon' \pm \sqrt{16\varepsilon'^2 + 16\varepsilon''^2}}{8}$

$\Rightarrow n = \frac{\sqrt{2}}{2}[\varepsilon' + \sqrt{\varepsilon'^2 + \varepsilon''^2}]^{\frac{1}{2}}$ $\quad \varepsilon'\sqrt{1 + \frac{\varepsilon''^2}{\varepsilon'^2}} \approx \varepsilon'(1 + \frac{\varepsilon''^2}{2\varepsilon'^2})$

$\varepsilon'\sqrt{1 + \frac{\varepsilon''^2}{\varepsilon'^2}} \approx \varepsilon'(1 + \frac{\varepsilon''^2}{2\varepsilon'^2})$

$\underline{n(\varepsilon', \varepsilon'') = \frac{\sqrt{2}}{2}(\varepsilon' + \sqrt{\varepsilon'^2 + \varepsilon''^2})^{\frac{1}{2}}}$; $\underline{\kappa(\varepsilon', \varepsilon'') = \frac{\sqrt{2}}{2}(\sqrt{\varepsilon'^2 + \varepsilon''^2} - \varepsilon')^{\frac{1}{2}}}$ $\quad (\varepsilon' > 0, \varepsilon'' > 0)$

$\sqrt{\frac{\varepsilon''^2}{2\varepsilon'}} = \quad \frac{\varepsilon''}{\sqrt{2}\,\varepsilon'} = \frac{\varepsilon''}{2\sqrt{\varepsilon'}}$

zeroth-order approximation: $\varepsilon'' = 0$

$n(\varepsilon', \varepsilon'') = n_0(\varepsilon', 0) = \sqrt{\varepsilon'}$ $\quad \kappa(\varepsilon', \varepsilon'') = \kappa_0(\varepsilon', 0) = 0$

correct first-order answer: $\varepsilon'' \neq 0$ $\kappa_0 = 0$

$n = n_0 = \sqrt{\varepsilon'} \Rightarrow \varepsilon' = n_0^2 - \kappa_1^2 \Rightarrow \kappa_1 = 0$ $\quad$ or $\varepsilon'' = 2n_0\kappa_1 \Rightarrow \kappa_1 = \frac{\varepsilon''}{2\sqrt{\varepsilon'}}$

$\kappa = \kappa_1 = 0$ $\quad \varepsilon'' \neq 0 \Rightarrow \varepsilon' = n_1^2 - \kappa_1^2 \Rightarrow n_1 = \sqrt{\varepsilon'}$

(c) $\varepsilon' < 0$; $\varepsilon' = n^2 - \kappa^2$ $\quad \varepsilon'' = 2n\kappa$ $\quad \varepsilon'' = 0 \Rightarrow n = 0$ or $\kappa = 0$

if $\kappa = 0$, $\varepsilon' = n^2 \geq 0$ so it should be $n_0 = 0 \Rightarrow \varepsilon' = -\kappa^2 \Rightarrow \kappa_0 = \sqrt{\varepsilon'}$

$k_0 = \sqrt{-\varepsilon'} \Rightarrow \varepsilon'' = 2n_1 k_0 \Rightarrow n_1 = \dfrac{\varepsilon''}{2\sqrt{-\varepsilon'}}$

$n_1 = 0 \quad \varepsilon' = n_1^2 - k_1^2 \Rightarrow k_1 = \sqrt{-\varepsilon'}$

(b) $\varepsilon' = n^2 - k^2 \quad \varepsilon'' = 2nk \quad k = \dfrac{\varepsilon''}{2n} \Rightarrow \varepsilon' = n^2 - \dfrac{\varepsilon''^2}{4n^2}$

$\Rightarrow 4n^4 - 4\varepsilon' n^2 - \varepsilon''^2 = 0$ Similarly $\varepsilon' = \dfrac{\varepsilon''}{4k^2} - k^2 \Rightarrow 4k^4 + 4\varepsilon' k^2 - \varepsilon''^2 = 0$

$\varepsilon'' = 0 \Rightarrow 4n^4 - 4\varepsilon' n^2 = 0 \Rightarrow n^2 = 0$ or $n^2 = \varepsilon' \Rightarrow n_0 = \sqrt{\varepsilon'}$

$k_0 = \sqrt{-\varepsilon'} \Rightarrow \varepsilon' = n_1^2 - k_0^2 \Rightarrow n_1 = 0$ or $\varepsilon'' = 2 n_1 k_0 \Rightarrow n_1 = \frac{\varepsilon''}{2\sqrt{-\varepsilon'}}$

$n_1 = 0 \quad \varepsilon' = n_1^2 - k_1^2 \Rightarrow k_1 = \sqrt{-\varepsilon'}$

Task 2: Plane waves in a conducting medium (a=2, b=2, c=2, d=2, e=2, f=2 pts.)

Consider an electromagnetic wave in a uniform isotropic **conductive** medium, where the current ($\mathbf{j}$) and free charge (ρ) densities are connected by the continuity equation as

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0.$$

a) For $\mathbf{j} = \sigma \mathbf{E}$ and $\operatorname{div} \mathbf{D} = \rho$, where σ is the conductivity, use the continuity equation and show that in the conductive medium, the density of free charges decreases with time.

Consider the case where $\operatorname{div} \mathbf{D} = 0$ so that positive and negative charges are balanced.

b) Write down the Maxwell's equations and derive the wave equation for the electric field $\mathbf{E}(\mathbf{r}, t)$ in time domain.

c) Derive the dispersion relation $k = k(\omega)$ in this medium by solving the wave equation for the case of a plane wave propagating along $\mathbf{e}_x$: $\mathbf{E}(x,t) = \mathbf{E}_0(k,\omega)e^{i(kx-\omega t)}$. Simplify your solution by using the fact that the wavenumber takes the form $k^2 = \frac{\omega^2}{c^2}\varepsilon^c$, where $\varepsilon^c = \varepsilon' + i\varepsilon''$ is the permittivity of the medium, and by introducing a new variable $\delta = \arctan \frac{\varepsilon''}{\varepsilon'}$.
Hint: The complex number ε^c can be decomposed into an amplitude and a phase as $\varepsilon^c = |\varepsilon^c|e^{i\delta}$, where δ is so-called **electric loss angle**.

d) Find formulas for $k'(\omega)$ and $k''(\omega)$, which are the real and imaginary parts of the wavenumber as $k(\omega) = k'(\omega) + ik''(\omega)$.

e) In the limit of $\omega \to 0$, will the losses be low or high? What can you infer from that about the propagation length of the plane wave in the conductive medium?

f) In the limit of $\omega \to \infty$, illustrate graphically the behaviour of k' vs. ω and k'' vs. ω. Explain your answer.

ε^c

$k^2 = \frac{w^2}{c^2}\left(\varepsilon + i\frac{\sigma}{w\varepsilon_0}\right)$

$= \frac{w^2}{c^2}\left[\underbrace{\varepsilon' + i\left(\varepsilon'' + \frac{\sigma}{w\varepsilon_0}\right)}_{\varepsilon^c}\right]$

$=$

a) $\operatorname{div}\vec{j} = \sigma \operatorname{div}\vec{E} \qquad \operatorname{div}\vec{D} = \varepsilon_0\varepsilon \operatorname{div}\vec{E} = \rho \Rightarrow \operatorname{div}\vec{E} = \frac{\rho}{\varepsilon_0\varepsilon} \Rightarrow \operatorname{div}\vec{j} = \frac{\sigma\rho}{\varepsilon_0\varepsilon}$

$\Rightarrow \frac{\partial \rho}{\partial t} = -\operatorname{div}\vec{j} = -\frac{\sigma\rho}{\varepsilon_0\varepsilon} \Rightarrow \frac{d\rho}{\rho} = -\frac{\sigma}{\varepsilon_0\varepsilon}dt \Rightarrow \ln\rho = -\frac{\sigma}{\varepsilon_0\varepsilon}t \Rightarrow \rho = \exp\left(-\frac{\sigma}{\varepsilon_0\varepsilon}t\right)$

Thus, the density of free charges decrease with time.

b) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \nabla \times \vec{D} = -\varepsilon_0\mu_0\varepsilon\frac{\partial \vec{H}}{\partial t} = -\frac{\varepsilon}{c^2}\frac{\partial \vec{H}}{\partial t} \qquad \nabla\cdot\vec{D} = 0$

$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} = \sigma\vec{E} + \frac{\partial \vec{D}}{\partial t} \qquad \nabla\cdot\vec{H} = 0$

$\nabla \times \nabla \times \vec{E} = -\mu_0\frac{\partial \nabla\times\vec{H}}{\partial t} = -\mu_0\frac{\partial \vec{j}}{\partial t} - \varepsilon_0\mu_0\varepsilon\frac{\partial^2 E}{\partial t^2}$

$\nabla\times\nabla\times\vec{E} = \nabla(\nabla\cdot\vec{E}) - \Delta\vec{E} \qquad \nabla\cdot\vec{E} = \frac{1}{\varepsilon_0\varepsilon}\nabla\cdot\vec{D} = 0 \Rightarrow \Delta\vec{E} = \mu_0\frac{\partial \vec{j}}{\partial t} + \frac{\varepsilon}{c^2}\frac{\partial^2 E}{\partial t^2}$

$\Rightarrow \Delta\vec{E} - \mu_0\sigma\frac{\partial \vec{E}}{\partial t} - \frac{\varepsilon}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$

Frequency domain $\varepsilon(w) = \varepsilon'(w) + i\varepsilon''(w)$

c) $\Delta\vec{E} = -k^2\vec{E} \qquad \frac{\partial \vec{E}}{\partial t} = -iw\vec{E} \qquad \frac{\partial^2 \vec{E}}{\partial t^2} = -w^2\vec{E}$

$\Rightarrow -k^2 + iw\mu_0\sigma + \frac{w^2\varepsilon}{c^2} = 0 \Rightarrow k^2 = \frac{w^2}{c^2}\left(\varepsilon + i\frac{\sigma}{w\varepsilon_0}\right) = \frac{w^2}{c^2}\left(\varepsilon + i\frac{\sigma}{w\varepsilon_0}\right) \qquad \tan\delta = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{w\varepsilon_0\varepsilon}$

d) $(k' + ik'')^2 = k'^2 - k''^2 + i2k'k'' \Rightarrow \begin{cases} k'^2 - k''^2 = \frac{w^2}{c^2}\varepsilon \\ 2k'k'' = \frac{w^2}{c^2}\frac{\sigma}{w\varepsilon_0} = \frac{w\sigma}{\varepsilon_0 c^2} \end{cases}$

$k' + ik'' = \frac{w}{c}\sqrt{\varepsilon^c}\left(\cos\frac{\delta}{2} + i\sin\left(\frac{\delta}{2}\right)\right) = \frac{w}{c}\sqrt{\varepsilon^c}\cos\frac{\delta}{2} + i\frac{w}{c}\sqrt{\varepsilon^c}\sin\frac{\delta}{2} \Rightarrow k' = \frac{w}{c}\sqrt{\varepsilon^c}\cos\frac{\delta}{2}\ ; \quad k'' = \frac{w}{c}\sqrt{\varepsilon^c}\sin\frac{\delta}{2}$

e) $k' = \frac{w\sigma}{2k''\varepsilon_0 c^2} \Rightarrow \frac{w^2\sigma^2}{4k''^2\varepsilon_0^2 c^4} - k''^2 = \frac{w^2}{c^2}\varepsilon \Rightarrow 4k''^4 + 4\frac{w^2}{c^2}\varepsilon k''^2 - \frac{w^2\sigma^2}{\varepsilon_0^2 c^4} = 0$

$k''^2 = \frac{-4\frac{w^2}{c^2}\varepsilon \pm \sqrt{16\frac{w^4}{c^4}\varepsilon^2 + 16\frac{w^2\sigma^2}{\varepsilon_0^2 c^4}}}{8} = \frac{1}{2}\left(\frac{w^2}{c^2}\sqrt{\varepsilon^2 + \frac{\sigma^2}{\varepsilon_0^2 w^2}} - \frac{w^2}{c^2}\varepsilon\right) = \frac{1}{2}\frac{w^2}{c^2}\left(\sqrt{\varepsilon^2 + \frac{\sigma^2}{\varepsilon_0^2 w^2}} - \varepsilon\right)$

$k'' = \frac{\sqrt{2}}{2}\frac{w}{c}\sqrt{\sqrt{\varepsilon^2+\frac{\sigma^2}{\varepsilon_0^2 w^2}}-\varepsilon} = \frac{\sqrt{2}}{2c}\sqrt{\sqrt{w^4\varepsilon^2+\frac{w^2\sigma^2}{\varepsilon_0^2}}-\varepsilon w^2}$ $\quad f(w)=\sqrt{w^4\varepsilon^2+\frac{w^2\sigma^2}{\varepsilon_0^2}}-\varepsilon w^2$

$f'(w) = -2w\varepsilon + \frac{2w^3\varepsilon^2+\frac{w\sigma^2}{\varepsilon_0^2}}{\sqrt{w^4\varepsilon^2+\frac{w^2\sigma^2}{\varepsilon_0^2}}} = \frac{2w^3\varepsilon^2+\frac{w\sigma^2}{\varepsilon_0^2}-2w\varepsilon\sqrt{w^4\varepsilon^2+\frac{w^2\sigma^2}{\varepsilon_0^2}}}{\sqrt{w^4\varepsilon^2+\frac{w^2\sigma^2}{\varepsilon_0^2}}}$

$(2w^3\varepsilon^2+\frac{w\sigma^2}{\varepsilon_0^2})^2 = 4w^6\varepsilon^4+\frac{w^2\sigma^4}{\varepsilon_0^4}+\frac{4w^4\sigma^2\varepsilon^2}{\varepsilon_0^2}$ $\quad (2w\varepsilon\sqrt{w^4\varepsilon^2+\frac{w^2\sigma^2}{\varepsilon_0^2}})^2 = 4w^6\varepsilon^2+\frac{4w^4\sigma^2\varepsilon^2}{\varepsilon_0^2}$

$(2w^3\varepsilon^2+\frac{w\sigma^2}{\varepsilon_0^2})^2-(2w\varepsilon\sqrt{w^4\varepsilon^2+\frac{w^2\sigma^2}{\varepsilon_0^2}})^2 = \frac{w^2\sigma^4}{\varepsilon_0^4}>0$ thus $f'(w)$ is always bigger than zero

$f'(w)>0 \Rightarrow$ k'' increase with w increase.

$E(x,t)=E_0(k,w)e^{i(kx-wt)} = E_0(k,w)e^{i[(k'+ik'')x-wt]} = E_0(k,w)e^{-k''x}e^{i(k'x-wt)}$

when w decreases k'' decreases and thus the propagation length will increase

If $k''=\frac{w\sigma}{2k'\varepsilon_0c^2} \Rightarrow k'^2-\frac{w^2\sigma^2}{4k'^2\varepsilon_0^2c^4}=\frac{w^2}{c^2}\varepsilon \Rightarrow 4k'^4-\frac{4w^2}{c^2}\varepsilon k'^2-\frac{w^2\sigma^2}{\varepsilon_0^2c^4}=0$

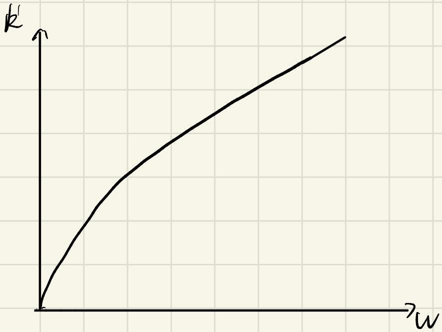
$\Rightarrow k'^4-\frac{w^2}{c^2}\varepsilon k'^2-\frac{w^2\sigma^2}{4\varepsilon_0^2c^4}$ $\quad k'^2=\frac{1}{2}\left(\frac{w^2}{c^2}\varepsilon\pm\sqrt{\frac{w^4}{c^4}\varepsilon^2+\frac{w^2\sigma^2}{\varepsilon_0^2c^4}}\right)=\frac{1}{2c^2}\left(w^2\varepsilon\pm\sqrt{w^4\varepsilon^2+\frac{w^2\sigma^2}{\varepsilon_0^2}}\right)$

$k'^2\geq 0$ Thus $k'=\frac{\sqrt{2}}{2c}\sqrt{\sqrt{w^4\varepsilon^2+\frac{w^2\sigma^2}{\varepsilon_0^2}}+w^2\varepsilon}$

$f(w)=\sqrt{w^4\varepsilon^2+\frac{w^2\sigma^2}{\varepsilon_0^2}}+w^2\varepsilon$ $\quad f'(w)=\frac{2w\varepsilon\sqrt{w^4\varepsilon^2+\frac{w^2\sigma^2}{\varepsilon_0^2}}+2w^3\varepsilon^2+\frac{w\sigma^2}{\varepsilon_0^2}}{\sqrt{w^4\varepsilon^2+\frac{w^2\sigma^2}{\varepsilon_0^2}}}>0$

Thus k' increase with w increase.

$w=0$ $k''=0$ and $k'=0$ $\quad w\to\infty$ $k'\to\infty$ $k''\to$



$\lim_{w\to\infty}\sqrt{w^4\varepsilon^2+\frac{w^2\sigma^2}{\varepsilon_0^2}}-\varepsilon w^2 = \lim_{w\to\infty} w^2\varepsilon\left(\sqrt{1+\frac{\sigma^2}{w^2\varepsilon^2\varepsilon_0^2}}-1\right) = \lim_{w\to\infty}\frac{\sqrt{1+\frac{\sigma^2}{w^2\varepsilon^2\varepsilon_0^2}}-1}{\frac{1}{w^2\varepsilon}}$

$=\lim_{w\to\infty}\frac{\left(\sqrt{1+\frac{\sigma^2}{w^2\varepsilon^2\varepsilon_0^2}}-1\right)'}{\left(\frac{1}{w^2\varepsilon}\right)'} = \lim_{w\to\infty}\frac{\frac{2\sigma^2}{\varepsilon\varepsilon_0^2}}{\sqrt{1+\frac{\sigma^2}{w^2\varepsilon^2\varepsilon_0^2}}} = \frac{2\sigma^2}{\varepsilon\varepsilon_0^2}$ $\quad \frac{-2\frac{\sigma^2}{w^3\varepsilon^2\varepsilon_0^2}}{2\sqrt{1+\frac{\sigma^2}{w^2\varepsilon^2\varepsilon_0^2}}} \times -\frac{w^3\varepsilon}{2}$

Task 3: Evanescent and propagating waves in diffraction theory (a=2, b=2, c=2 pts.)

Consider an external source of an electric field in vacuum, oscillating with a frequency ω, with a Gaussian amplitude distribution of $u_0(x,y)$ at the $z=0$ plane in Cartesian coordinates (neglect the polarization):

$$u_0(x,y) = A\exp\left[-(x^2+y^2)/W^2\right] \quad \text{with} \quad W>0.$$

a) Compute its spatial frequency spectrum $U_0(k_x,k_y)$.

b) Consider $U_0(k_x,k_y)$ as a function of $k_r=\sqrt{k_x^2+k_y^2}$. What is the ratio $U_0(k_r=k_0)/U_0(k_r=0)$? This should only be a function of W and λ. Now consider the specific case of $W=\lambda/\pi$ and plot $U_0(k_r)/U_0(k_r=0)$ as a function of k_r. Have your horizontal axis in units of $k_0=2\pi/\lambda$, where λ is the vacuum wavelength. What is the value of $U_0(k_r=k_0)/U_0(k_r=0)$ on this plot?

c) We know from the diffraction theory that the spatial frequency spectrum at the plane of $z=z_0$ is $U(k_x,k_y;z_0) = U_0(k_x,k_y)\exp\left[ik_z(k_x,k_y)z_0\right]$, with $k_z(k_x,k_y)=\sqrt{k_0^2-k_x^2-k_y^2}$. Plot the real and imaginary part of $k_z(k_x,k_y)$ in the same plot as part (b). Specify the (k_x,k_y) ranges of $U_0(k_x,k_y)$ that contribute to propagating and evanescent waves along the z-direction.

(a) $U_0(k_x,k_y) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} u_0(x,y)\exp[-i(k_x x+k_y y)]\,dx\,dy$

$= \frac{A}{4\pi^2}\iint_{-\infty}^{\infty}\exp[-\frac{(x^2+y^2)}{W^2}-i(k_x x+k_y y)]\,dx\,dy$

$= \frac{A}{4\pi^2}\iint_{-\infty}^{\infty}\exp[-(\frac{x^2}{W^2}+ik_x x)]\exp[-(\frac{y^2}{W^2}+ik_y y)]\,dy\,dx$

$= \frac{A}{4\pi^2}\int_{-\infty}^{\infty}\exp[-\frac{1}{W^2}(x^2+ik_xW^2x)]\,dx\int_{-\infty}^{\infty}\exp[-\frac{1}{W^2}(y^2+ik_yW^2y)]\,dy$

$= \frac{A}{4\pi^2}\int_{-\infty}^{\infty}\exp[-\frac{1}{W^2}(x+\frac{ik_xW^2}{2})^2-\frac{W^2k_x^2}{4}]\,dx\int_{-\infty}^{\infty}\exp[-\frac{1}{W^2}(x+\frac{ik_yW^2}{2})^2-\frac{W^2k_y^2}{4}]\,dy$

$= \frac{A}{4\pi^2}\exp[-\frac{W^2}{4}(k_x^2+k_y^2)]\int_{-\infty}^{\infty}\exp(-\frac{a^2}{W^2})\,da\int_{-\infty}^{\infty}\exp(-\frac{b^2}{W^2})\,db$

$= \frac{AW^2}{4\pi^2}\exp[-\frac{W^2}{4}(k_x^2+k_y^2)]\int_{-\infty}^{\infty}e^{-\eta^2}\,d\eta\int_{-\infty}^{\infty}e^{-\xi^2}\,d\xi = \frac{AW^2}{4\pi}\exp[-\frac{W^2}{4}(k_x^2+k_y^2)]$

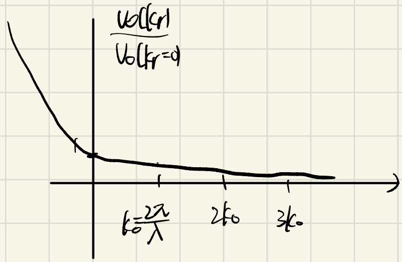
(b) $U_0(k_r=k_0) \Rightarrow k_0=\sqrt{k_x^2+k_y^2} \Rightarrow k_0^2=k_x^2+k_y^2 \Rightarrow U_0=\frac{AW^2}{4\pi}\exp(-\frac{W^2k_0^2}{4})$

$U_0(k_r=0) \Rightarrow 0=\sqrt{k_x^2+k_y^2} \Rightarrow k_x^2+k_y^2=0 \Rightarrow U_0=\frac{AW^2}{4\pi}$

$\Rightarrow \frac{U_0(k_r=k_0)}{U_0(k_r=0)} = \exp(-\frac{W^2k_0^2}{4}) = \exp(-\frac{W^2\pi^2}{\lambda^2})$

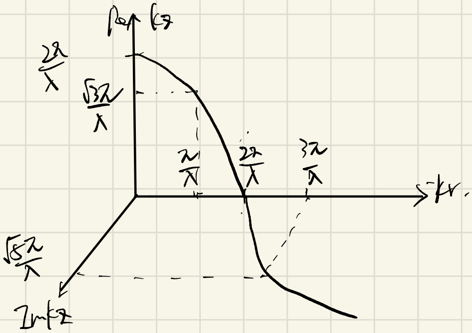
$\frac{U_0(k_r)}{U_0(k_r=0)} = \exp(-\frac{W^2k_r^2}{4}) = \exp(-\frac{\lambda^2k_r^2}{4\pi^2})$

$k_r=k_0=\frac{2\pi}{\lambda} \Rightarrow \frac{U_0(k_r)}{U_0(k_r=0)} = \frac{1}{e}$



(c) $k_z=\sqrt{k_0^2-k_x^2-k_y^2} = \sqrt{k_0^2-(k_x^2+k_y^2)} = \sqrt{\frac{4\pi^2}{\lambda^2}-(k_x^2+k_y^2)} = i\sqrt{(k_x^2+k_y^2)-\frac{4\pi^2}{\lambda^2}}$ 3

$= \sqrt{\frac{4\pi^2}{\lambda^2}-k_r^2} = i\sqrt{k_r^2-\frac{4\pi^2}{\lambda^2}}$ $\frac{\pi^2}{\lambda^2}$ $\frac{\sqrt{5}\pi}{\lambda}$ $\sqrt{2}$


Re kz
2π/λ
√3π/λ
π/λ
2π/λ
3π/λ
kr
√5π/λ
Im kz