Name: ______________________ Date of birth: ______________ Student ID No.: __________

FRIEDRICH SCHILLER UNIVERSITY JENA WS 2020/2021
Institute of Applied Physics
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Midterm Exam

FUNDAMENTALS OF MODERN OPTICS

to be written on December 14, 8:15 am – 9:45 am

Problem 1: Maxwell's Equations 2 + 2 + 3 + 2 = 9 points

a) Write down Maxwell's equations (MwE) in the time domain and in the frequency domain in their general form in the presence of external charges and current densities.

b) For a medium that is linear, homogeneous, non-magnetic, dispersive and isotropic, derive the relation between $\mathbf{D}(\mathbf{r},\omega)$ and $\mathbf{E}(\mathbf{r},\omega)$ from MwE by using the relation between the polarization and the electric field.

c) Derive the Helmholtz equation for the magnetic field $\mathbf{H}(\mathbf{r},\omega)$ from MwE in this medium. Assume that $\mathbf{j}(\mathbf{r},\omega) = \sigma(\omega)\,\mathbf{E}(\mathbf{r},\omega)$ and $\rho(\mathbf{r},\omega) = 0$.

d) Assume that the external charges are present in the medium in the part 1c. Using MwE, derive the continuity equation that is a relation between the time derivative of the charge density and the current density. Try to explain the meaning of continuity equation in your own words.

Problem 2: Poynting Vector 3 + 1 + 2 = 6 points

A plane electromagnetic wave in a homogeneous, linear, isotropic, and non-magnetic medium without external charges and currents is $\mathbf{H}$ given as

$$\mathbf{H}(x,y,z,t) = H_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} - \omega t\right]\hat{\mathbf{z}},$$

where k is the wave number and $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are the unit vectors in the Cartesian coordinate system.

a) Calculate the electric field $\mathbf{E}(x,y,z,t)$ corresponding to the above magnetic field.

b) Write down the formula for the time averaged Poynting vector $\langle \mathbf{S}(\mathbf{r},t)\rangle$.

c) Find the time averaged Poynting vector for this electromagnetic wave.

Problem 3: Material model 1 + 1 + 3 + 3 + 2 = 10 points

With a good approximation, a medium can be modeled by an ensemble of damped harmonic oscillators, known as the Lorentz model. The response function of this medium is given as:

$$\hat{R}_{mn}(\mathbf{r},t) = \delta_{mn}R(t) \qquad R(t) = \begin{cases} 0 & \text{for } t \le 0 \\ \dfrac{f}{\Omega}\, e^{-\gamma t}\sin\Omega t & \text{for } t > 0 \end{cases}, \qquad \Omega = \sqrt{\omega_0^2 - \gamma^2}.$$

a) Based on the given response function, specify the type of the medium by ticking in the table below.

Inhomogeneous	Homogeneous
Anisotropic	Isotropic
Dispersive	Non-dispersive

b) Write down the relation between the polarization $\mathbf{P}(\mathbf{r},t)$, the response function $R(t)$, and the electric field $\mathbf{E}(\mathbf{r},t)$.

c) Calculate the susceptibility $\chi(\omega)$ of the medium.

d) Compute the polarization $\mathbf{P}(\mathbf{r},t)$ for the above medium with an electric field excitation of

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r})\exp(-i\omega_{\text{cw}}t).$$

Explain how the complex susceptibility influences on the relation between the polarization $\mathbf{P}(\mathbf{r},t)$ and the electric field $\mathbf{E}(\mathbf{r},t)$. What happens if the damping factor $\gamma = 0$?

e) Compute the polarization $\mathbf{P}(\mathbf{r},t)$ for the above medium with an electric field excitation of

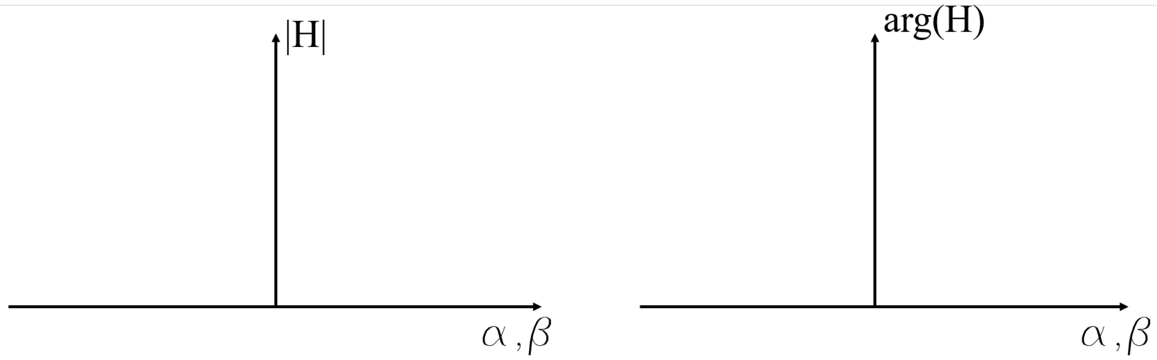
$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r})\delta(t - t_0).$$

and explain the result.

Problem 4: Beam Propagation **2 + 2 + 2 = 6 points**

a) What are the properties of homogeneous and evanescent waves in terms of amplitude and energy transfer? Write down the complex transfer function $H(\alpha, \beta; z)$ in a homogeneous space and define the evanescent and homogeneous wave regions depending on the spatial frequencies α and β.

b) Plot the amplitude and phase of the transfer function on the below graphs. Indicate characteristic points or dimensions of the drawn transfer function (such as the amplitudes, radius *etc.*) on the graphs.



c) The beam propagation in free-space can be formulated as a superposition of plane waves with different phase evolution due to the propagation. For a given initial field $u_0(x, y)$, we want to calculate the field at a distance z. By considering $U_0(\alpha, \beta) = \mathrm{FT}\{u_0(x, y)\}$, fill the blanks in the following formula.

$$u(x,y,z) = \underbrace{\iint_{-\infty}^{\infty} \mathrm{d}\,\alpha \mathrm{d}\beta}_{\substack{\text{interference of}\\ \text{eigenstates to}\\ \text{form}\\ \text{the field pattern}\\ \text{after}\\ \text{propagation}}} \times \underbrace{\qquad\qquad}_{\substack{\text{Amplitude of the}\\ \text{excited}\\ \text{eigenstates}}} \times \underbrace{\qquad\qquad\qquad}_{\substack{\text{phase factor which is}\\ \text{accumulated by the}\\ \text{eigenstates during}\\ \text{propagation}}} \times \underbrace{\qquad\qquad\qquad}_{\substack{\text{Shape of eigenstates}\\ \text{(plane waves)}}}$$

Problem 5: Gaussian Beams **2+2+2+2=8 points**

a) Write down a Gaussian beam profile $v_0(x, y, z = 0)$ with a beam waist of w_0 in the stationary state. Explain in few sentences how one can propagate it in free-space with paraxial approximation.

b) When this Gaussian beam $v_0(x, y)$ is propagated at a distance of z in paraxial approximation, we get the following Gaussian beam

$$v(x,y,z) = A_0 \frac{1}{1 + iz/z_0} \exp\left[-\frac{x^2 + y^2}{w_0^2(1 + iz/z_0)}\right],$$

where z_0 is the Rayleigh length and A_0 is the initial amplitude. Restructure the provided equation and give explicit expressions for the following terms as functions of z: i) amplitude evolution, ii) beam width evolution, iii) radius of phase curvature, and iv) The Gouy phase.

c) Sketch diagrams on xz-plane for the evolution of the normalized beam intensity, the beam width, the radius of phase curvature, and the Gouy phase over the propagation distance z. Indicate the values of the quantities at the propagation distances $z = \pm z_0$.

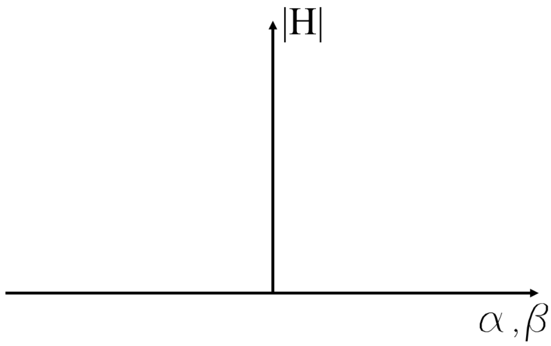
d) If a continuous wave Gaussian beam with a wavelength of $\lambda = 1\ \mu$m is propagating in air, what conclusions can be drawn about the validity of the equation in b) when the beam waist is $w_0 = 1\ \mu$m, $w_0 = 10\ \mu$m, or $w_0 = 1$ mm? What can you tell about the divergence of the beam in each case and why?

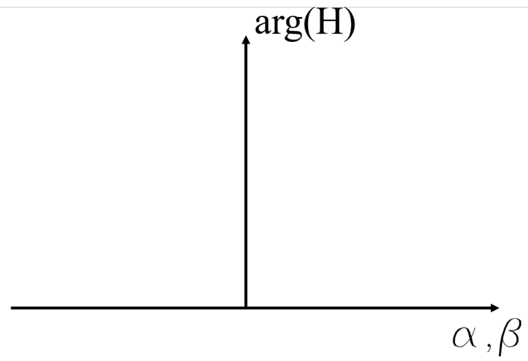
Problem 6: Diffraction and Fresnel approximation **3 + 2 + 2 + 2 = 9 points**

a) Consider an initial field of wavelength λ (with translation symmetry in the y-direction) of the form $u(x, z=0) = A\cos(\alpha_0 x)$ that propagates along the z direction. A and $\alpha_0 < 2\pi/\lambda$ are positive real numbers.
   (i) Find the exact diffracted pattern after it propagates a distance d (without approximations).
   (ii) What particular property can you identify from the diffracted pattern? Give also some physical reasoning on your result.

b) Using the Taylor expansion, derive the Fresnel transfer function (paraxial approximation) from the general transfer function in spatial frequency domain for free-space while considering a propagation along the z-direction. What condition needs to be satisfied regarding the spatial frequency components of the field?

c) Sketch the amplitude and phase of the Fresnel transfer function on the below graphs. Indicate the characteristic points or dimensions on the graphs.





d) From the Fresnel condition, one can find a relation between the smallest feature size of an initial monochromatic beam that propagates in vacuum and its wavelength.
   (i) Find this relation.
   (ii) Consider a monochromatic beam of wavelength 1 μm that contain features of sizes 100 nm, 1 μm, and 10 μm. Which features will be resolved after propagation under the Fresnel diffraction condition?