Task 1

Solution:

$$h_F(x,y,z) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{+\infty} H_F(\alpha,\beta;z)\exp[i(\alpha x+\beta y)]\,d\alpha\, d\beta$$

$$= \frac{1}{4\pi^2}\iint_{-\infty}^{+\infty} \exp[ikz]\exp\left[-i\frac{\alpha^2+\beta^2}{2K}z\right]\exp[i(\alpha x+\beta y)]\,d\alpha\, d\beta$$

$$= \frac{e^{ikz}}{4\pi^2}\iint_{-\infty}^{+\infty} e^{-i\frac{\alpha^2}{2k}z+i\alpha x}\cdot e^{-i\frac{\beta^2}{2k}z+i\beta y}\,d\alpha\, d\beta$$

$$= \frac{e^{ikz}}{4\pi^2}\int_{\infty}^{+\infty} e^{-i\frac{\alpha^2}{2k}z+i\alpha x}\,d\alpha\int_{-\infty}^{+\infty} e^{-i\frac{\beta^2}{2k}z+i\beta y}\,d\beta$$

$$= \frac{e^{ikz}}{4\pi^2}\int_{-\infty}^{+\infty} e^{-i\left(\sqrt{\frac{z}{2k}}\alpha-\frac{1}{2}\sqrt{\frac{2k}{z}}x\right)^2}\cdot e^{i\frac{k}{2z}x^2}\,d\alpha\int_{-\infty}^{+\infty} e^{-i\left(\sqrt{\frac{z}{2k}}\beta-\frac{1}{2}\sqrt{\frac{2k}{z}}y\right)^2}\cdot e^{i\frac{k}{2z}y^2}\,d\beta$$

$$= \frac{e^{ikz}}{4\pi^2}e^{i\frac{k}{2z}(x^2+y^2)}\cdot\left(\sqrt{\frac{2k}{z}}\right)^2\cdot\left(\sqrt{\frac{\pi}{i}}\right)^2$$

$$= -i\frac{k}{2\pi z}e^{ikz}e^{i\frac{k}{2z}(x^2+y^2)}$$

$\because k = \frac{2\pi}{\lambda}$

$$\therefore h_F(x,y,z) = -i\frac{1}{\lambda z}e^{ikz}e^{i\frac{k}{2z}(x^2+y^2)}$$

Task 2

a)

Solution:

From Helmholtz' equation

we can get the expression of spherical wave

$$U(\vec{r}) = \frac{A}{r}e^{ikr}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\therefore U(\vec{r}) = \frac{A}{\sqrt{x^2+y^2+z^2}}e^{ik\sqrt{x^2+y^2+z^2}}$$

$$= \frac{A}{z\sqrt{\frac{x^2+y^2}{z^2}+1}}e^{ikz\sqrt{1+\frac{x^2+y^2}{z^2}}}$$

In paraxial approximation $x^2+y^2 \ll z^2$

$$U(\vec{r}) \xrightarrow{\text{Taylor expansion}} \frac{A}{z}\left(1-\frac{x^2+y^2}{2z^2}+\frac{-\frac{1}{2}\cdot(-\frac{1}{2}-1)}{2!}\frac{[(x^2+y^2)]^2}{z^4}+\cdots\right)e^{ikz\left(1+\frac{x^2+y^2}{2z^2}+\frac{\frac{1}{2}\cdot(-\frac{1}{2})}{2!}\frac{(x^2+y^2)^2}{z^4}+\cdots\right)}$$

let $x^2+y^2=\rho^2$

$$\therefore U(\vec{r}) \approx \frac{A}{z}e^{ikz\left(1+\frac{\rho^2}{2z^2}-\frac{\rho^4}{8z^4}\right)} = \frac{A}{z}e^{ik\left(z+\frac{\rho^2}{2z}-\frac{\rho^4}{8z^3}\right)}$$

If the wavefront of a Gaussian beam is the same as a wavefront of the spherical wave

The phase part must be the same

$$\therefore ik\left(z+\frac{x^2+y^2}{2R(z)}\right) = ik\left(z+\frac{x^2+y^2}{2z}-\frac{(x^2+y^2)^2}{8z^3}\right)$$

$$\frac{\rho^2}{2R(z)} = \frac{\rho^2}{2z} - \frac{\rho^4}{8z^3}$$

$$\frac{1}{2R(z)} = \frac{1}{2z} - \frac{\rho^2}{8z^3}$$

$$4z^3 = 4z^2R(z) - \rho^2R(z)$$

$$R(z) = \frac{4z^3}{4z^2-\rho^2} = \cancel{\frac{4z^3}{4-\frac{\rho^2}{z^2}}}\ \frac{4z}{4-\frac{\rho^2}{z^2}}$$

$$\because \rho^2 \ll z^2$$

$$\therefore R(z) \approx z$$

The condition is $R(z) \approx z$

b)

Solution:

when $w = 1.1W_0$

$$\because \begin{cases} W = W_0\sqrt{1+\left(\frac{z}{z_0}\right)^2} \\ z_0 = \frac{\pi}{\lambda_n} W_0^2 \end{cases}$$

$$\therefore z \approx 2.15\times10^4\,m$$

Task 3

a)

Solution:

The ABCD matrix of the lens is $\begin{bmatrix}1 & 0\\ -1/f & 1\end{bmatrix}$

$q(0) = iZ_0 = i\frac{\pi W_0^2}{\lambda} = q_0$ ①

after lens

$$q_L = \frac{Aq_0+B}{Cq_0+D} = \frac{q_0+0}{-\frac{1}{f}q_0+1}$$

$\therefore \frac{1}{q_L} = \frac{1}{q_0} - \frac{1}{f}$ ②

At the waist W_1

$q_1 = q_L + d$ ③

From these three equations, we can get:

$$\frac{1}{q_L} = \frac{1}{i\frac{\pi W_0^2}{\lambda}} - \frac{1}{f} = \frac{\lambda f - i\pi W_0^2}{i\pi W_0^2 f}$$

$$q_1 = \frac{i\pi W_0^2 f}{\lambda f - i\pi W_0^2} + d$$

$$= d - f\frac{\left(\frac{\pi W_0^2}{\lambda}\right)^2}{f^2+\left(\frac{\pi W_0^2}{\lambda}\right)^2} + i\frac{f^2\left(\frac{\pi W_0^2}{\lambda}\right)}{f^2+\left(\frac{\pi W_0^2}{\lambda}\right)^2}$$

$\because q_1$ is the q-parameter of the waist ($Re\left[\frac{1}{q_1}\right]=0$)

$$\therefore d - f\frac{\left(\frac{\pi W_0^2}{\lambda}\right)^2}{f^2+\left(\frac{\pi W_0^2}{\lambda}\right)^2} = 0$$

$$q_1 = i\frac{f^2\left(\frac{\pi W_0^2}{\lambda}\right)}{f^2+\left(\frac{\pi W_0^2}{\lambda}\right)^2}$$

$$\therefore \frac{i}{W_1^2} = -\frac{\pi}{\lambda}\frac{1}{q_1} = \cancel{i\frac{\pi}{\lambda}\frac{f^2\left(\frac{\pi W_0^2}{\lambda}\right)}{f^2+\left(\frac{\pi W_0^2}{\lambda}\right)^2}}\ \frac{i\left[f^2+\left(\frac{\pi W_0^2}{\lambda}\right)^2\right]}{f^2\left(\frac{\pi W_0^2}{\lambda}\right)}\cdot\frac{\pi}{\lambda}$$

$$\therefore \begin{cases} d = f\frac{\left(\frac{\pi W_0^2}{\lambda}\right)^2}{f^2+\left(\frac{\pi W_0^2}{\lambda}\right)^2} & \because \frac{\pi W_0^2}{\lambda} \gg f \quad \therefore d \approx f \\ \frac{1}{W_1^2} = -\cancel{\frac{\pi}{\lambda}\frac{f^2\left(\frac{\pi W_0^2}{\lambda}\right)}{f^2+\left(\frac{\pi W_0^2}{\lambda}\right)^2}}\ \frac{\pi}{\lambda}\frac{f^2+\left(\frac{\pi W_0^2}{\lambda}\right)^2}{f^2\left(\frac{\pi W_0^2}{\lambda}\right)} = \frac{1}{W_0^2} + \frac{1}{f^2}\left(\frac{\pi W_0}{\lambda}\right)^2 & \because \frac{\pi W_0^2}{\lambda} \gg f \end{cases}$$

$\cancel{\therefore \frac{1}{W_1^2} \approx \frac{1}{W_0^2}}$ ✱

b)

Solution:

if $Z_0 \gg f$

$$\frac{1}{W_1^2} = \frac{1}{W_0^2} + \frac{\left(\frac{\pi W_0}{\lambda}\right)^2}{f^2}$$

$\because \frac{\pi W_0^2}{\lambda} \gg f$

$\cancel{\therefore \frac{1}{W_1^2} \approx \frac{1}{W_0^2} + \frac{\pi^2 W_0^2}{\lambda^2}}$