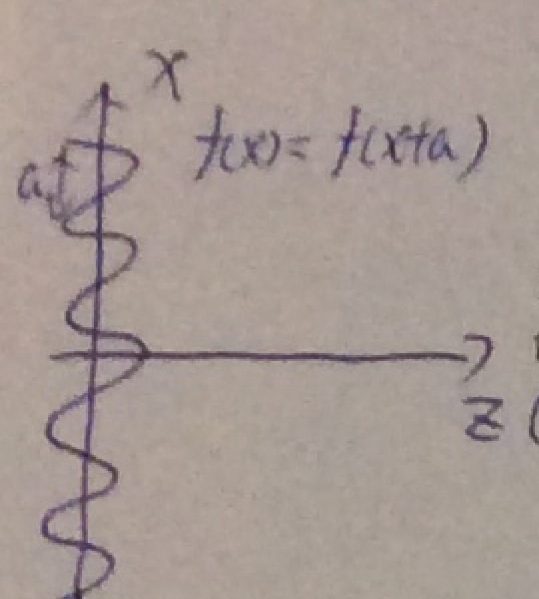
Pro.

Fundamentals of Modern Optics
Exercise 6
24.11.2014

to be returned: 01.12.2014, at the beginning of the lecture



Problem 1 - Talbot Effect (2+1 points)

A linearly polarized beam of wavelength λ is propagating through an isotropic medium with the refractive index 1. In the plane $z = 0$ its field is given by the periodic function $f(x, z = 0)$, with the period a, such that $f(x + a, z = 0) = f(x, z = 0)$. Fresnel's approximation is assumed to be valid.

a) Show that under the action of Fresnel diffraction after a certain length L_T there is a reappearance of the field (except for a constant phase ϕ), such that $f(x, z + L_T) = f(x, z)\exp(i\phi)$.

b) Calculate this, so-called, Talbot-length L_T for a wavelength of 500 nm and a periodicity of 1 mm.

Problem 2 - Fresnel approximation I (2+2+2 points)

Fresnel approximation plays an important role in diffraction theory. We want to understand its physical meaning. Start with the expression for a spherical wave in real space (which is an *exact* solution of Helmholtz' equation)

$$U(\vec{r}) = \frac{A}{r}\exp(-ikr) = \frac{A}{\sqrt{z^2+\rho^2}}\exp\left[-ik\sqrt{z^2+\rho^2}\right].$$

a) Derive the Fresnel approximation $U_F(\vec{r})$ of this spherical wave by expanding $U(\vec{r})$ into a Taylor series for points around the optical axis $\rho \ll z$.

b) What is the geometrical shape of the wavefronts in 0^{th} and 1^{st} order approximation? In which regions are they justified?

c) In practical situations, one prefers to use the angle θ under which a beam propagates with respect to the optical axis to determine if Fresnel approximation is valid; otherwise we already assumed points nearby the optical axis, $\tan\theta \approx \theta$ is valid. Using your expansion, show that the condition therefore is

$$N_F\theta^2 \ll 4,$$

where

$$N_F = \frac{\rho^2}{\lambda z}$$

is the so-called *Fresnel number.*

Problem 3 - Fresnel approximation II (4 points)

Now we consider Fresnel approximation in spatial frequency space. Consider its transfer function

$$H_F(\alpha, \beta, z) = \exp(ikz)\exp\left[-i\frac{\alpha^2+\beta^2}{2k}z\right]$$

to derive the response function $h_F(x, y, z > z_0)$ as given in the lecture.

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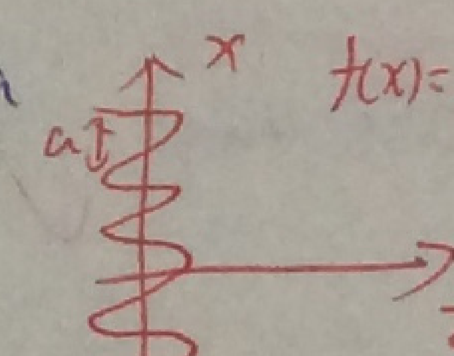
Problem 1 - Talbot Effect

a) $f_0(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f_0(x)e^{-i\alpha x}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_0(x+a)e^{-i\alpha x}dx$

$=\frac{1}{2\pi}\int_{-\infty}^{\infty} f_0(x+a)e^{-i\alpha(x+a)}e^{i\alpha a}dx = \frac{1}{2\pi}\cdot e^{i\alpha a}\int_{-\infty}^{\infty} f_0(x+a)e^{-i\alpha(x+a)}d(x+a)$

$\overset{x'=x+a}{=} \frac{1}{2\pi}\cdot e^{i\alpha a}\int_{-\infty}^{\infty} f(x')e^{-i\alpha x'}dx' = f_0(\alpha)\cdot e^{i\alpha a}$



$t(x)=t(x+a)$

$\Rightarrow f_0(\alpha)\cdot(1-e^{i\alpha a})=0$

So $f_0(\alpha)=\begin{cases}\text{undefined}, & \alpha=\frac{2\pi n}{a},\ n\in \mathbb{Z}.\\ 0 & \alpha\ne\frac{2\pi n}{a}\end{cases}$

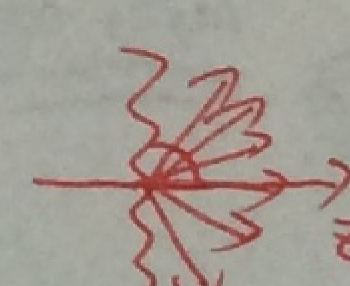
$t(x)=\sum_{l=-\infty}^{+\infty} c_l e^{il\frac{2\pi}{a}x}$

$\Updownarrow$

$\tilde{F}(\alpha)_{z=0}=\sum_{l=-\infty}^{+\infty} c_l\,\delta(l\tfrac{2\pi}{a}-\alpha)$

assume $f_0(\alpha)=\sum_{n=-\infty}^{\infty} f_n\delta(\alpha-\frac{2\pi n}{a})$

$\tilde{F}(\alpha,z)=\tilde{f}(\alpha,z=0)e^{ik_z z}=\tilde{f}(\alpha,z=0)\,e^{i\sqrt{k^2-\alpha^2}\,z}$



if the angle larger than 90°, it will not propagate.

Field in Fresnel approximation

$f(x,z)=\int_{-\infty}^{\infty} e^{i\alpha x}\cdot e^{i(k-\frac{\alpha^2}{2k})z}\cdot f_0(\alpha)\,d\alpha$

$t(x,z)=\int_{-\infty}^{+\infty} f(\alpha,z)e^{i\alpha x}d\alpha$

$=\int_{-\infty}^{\infty} e^{i\alpha x}\cdot e^{i(k-\frac{\alpha^2}{2k})z}\cdot\sum_{n=-\infty}^{\infty} f_n\,\delta(\alpha-\frac{2\pi n}{a})\,d\alpha$

$=\int_{-\infty}^{\infty}\sum_{l=-\infty}^{l=+\infty} c_l\,\delta(l\tfrac{2\pi}{a}x-\alpha)e^{i\sqrt{k^2-\alpha^2}z}\cdot e^{i\alpha x}d\alpha$

$=e^{ikz}\cdot\sum_{n=-\infty}^{\infty} f_n\, e^{i\frac{2\pi n}{a}x}\, e^{-i(\frac{2\pi n}{a})^2\cdot\frac{z}{2k}}$

$=\sum_{l=-\infty}^{+\infty} c_l\, e^{i\sqrt{k^2-(\frac{2\pi}{a}\cdot l)^2}\cdot z}\, e^{i\frac{2\pi l}{a}\cdot x}=t(x)e^{i(kz+\varphi)}$

$\Rightarrow \sum_{l=-\infty}^{+\infty} c_l\,\underbrace{e^{i(\sqrt{k^2-(\frac{2\pi}{a}\cdot l)^2}-k)z}}_{=1}\, e^{i\frac{2\pi l}{a}\cdot x}=t(x)\,e^{i\phi}$

~~for ∀L, we have f(x,z)=~~

~~f(x,z+L) = ∫~~

$\Rightarrow L_T\left(\sqrt{k^2-(\tfrac{2\pi}{a}\cdot l)^2}-k\right)=2\pi m_l$ for every c_l $\quad m_l=0,\pm1,\pm2\dots$

$f(x,z+L)=e^{ikz}\cdot e^{ik\cdot L}\cdot\sum_{n=-\infty}^{\infty} f_n\cdot e^{i\frac{2\pi n}{a}x}\cdot e^{-i(\frac{2\pi n}{a})^2\cdot\frac{z}{2k}}\cdot e^{-i(\frac{2\pi n}{a})^2\cdot\frac{L}{2k}}$

$=e^{ik\cdot L}\cdot e^{-i(\frac{2\pi n}{a})^2\cdot\frac{L}{2k}}\cdot f(x,z)$

$L_T=\frac{2\pi m_l}{\sqrt{k^2-(\frac{2\pi}{a}\cdot l)^2}-k}$ $\quad\alpha^2\ll k^2\Rightarrow\left|\frac{2\pi}{a}\cdot l\right|\ll k$ for $c_l\ne0$

if $\exists L$, $f(x,z)=f(x,z+L)=f(x,z)e^{i\phi}$

$k\sqrt{1-\frac{(2\pi\cdot l)^2}{(a\cdot k)^2}}\approx k\left[1-\frac{1}{2}(\frac{2\pi l}{ak})^2\right]$

Then $e^{i\phi}=e^{ik\cdot L}$ ($\phi=k\cdot L$ is a constant)

$\Rightarrow L_T=-\frac{m_l\cdot a^2}{\frac{\pi}{k}\cdot l^2}$ for $m_l=-l^2$

$e^{-i(\frac{2\pi n}{a})^2\cdot\frac{L}{2k}}=1$

$L_T=\frac{2\cdot a^2}{\lambda}$

$\Rightarrow (\frac{2\pi n}{a})^2\cdot\frac{L}{2k}=2\pi m,\ m\in \mathbb{Z}$

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$$\Rightarrow L_{min} = \frac{2k}{2\pi} \cdot a^2 \overset{k=\frac{2\pi}{\lambda}}{=\!=} \frac{2a^2}{\lambda}$$

So $L_{min} = \frac{2a^2}{\lambda}$ ✓ 2/2

Problem

a)

b) $\lambda = 500nm = 5\times10^{-7}m$

$a = 1mm = 10^{-3}m$

$L = \frac{2\times10^{-6}}{5\times10^{-7}} m = 4m$ ✓ 1/1

Problem 2 - Fresnel approximation I

a) $U(\vec{r}) = \frac{A}{\sqrt{z^2+\rho^2}} e^{-ik\sqrt{z^2+\rho^2}}$

$$\frac{A}{\sqrt{z^2+\rho^2}} = \frac{A}{z\cdot\sqrt{1+\frac{\rho^2}{z^2}}} \overset{\text{Taylor expansion}}{=\!=\!=} \frac{A}{z}\cdot(1-\frac{\rho^2}{2z^2})+\cdots$$

$$-ik\sqrt{z^2+\rho^2} = -ikz\cdot\sqrt{1+\frac{\rho^2}{z^2}} \overset{\text{Taylor expansion}}{=\!=\!=} -ikz\cdot(1+\frac{\rho^2}{2z^2}-\frac{\rho^4}{8z^4}\cdots)$$

$$\approx -ikz - ik\frac{\rho^2}{2z} + ik\cdot\frac{\rho^4}{8z^3}$$

$$\therefore U_F(\vec{r}) = \frac{A}{z}\cdot(1-\frac{\rho^2}{2z^2})\, e^{-ikz-ik\frac{\rho^2}{2z}+ik\cdot\frac{\rho^4}{8z^3}}$$

$$\approx \frac{A}{z}\cdot e^{-ikz}\cdot e^{-ik\frac{\rho^2}{2z}} \quad \rho \ll z.$$

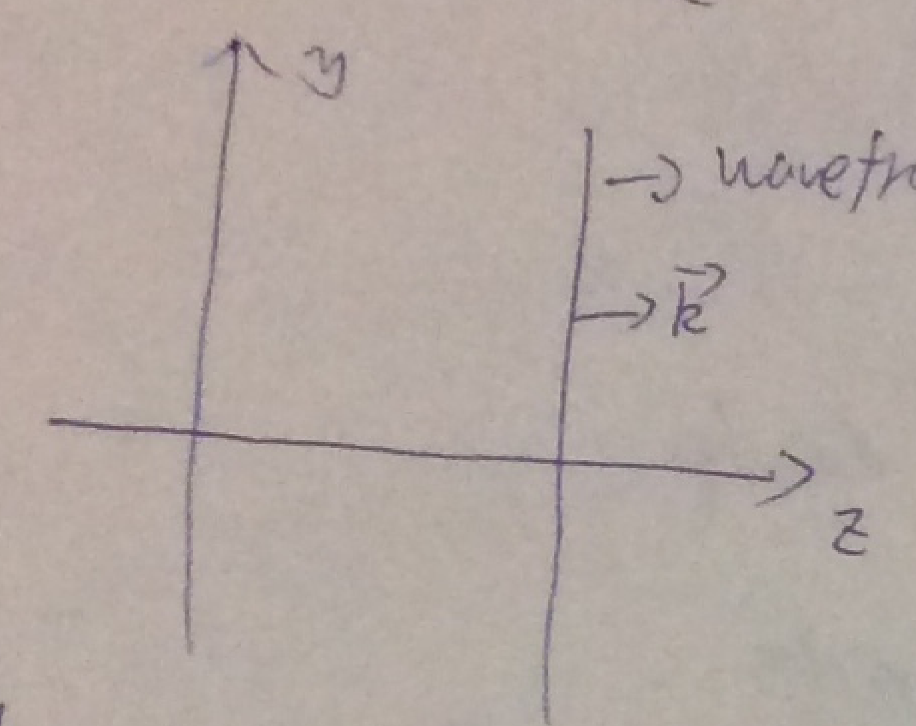
✓ 2/2

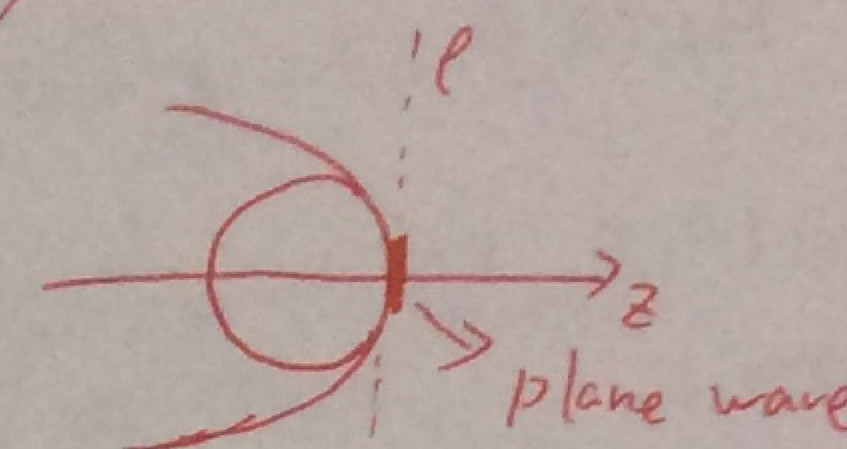
~~b) 0th order e^{-ikz} is the phase, ⇒ plane wave~~

~~1th order $e^{ik\frac{\rho^2}{2z}}$ ⇒ parabola~~

b) 0th order e^{-ikz} $\Rightarrow$ plane wave

$u(\vec{r}) = \frac{A}{r} e^{-ikz(1+\frac{\rho^2}{2z^2})}$





more term of $e^{-ikz(1+\frac{\rho^2}{2z^2}+\cdots)}$

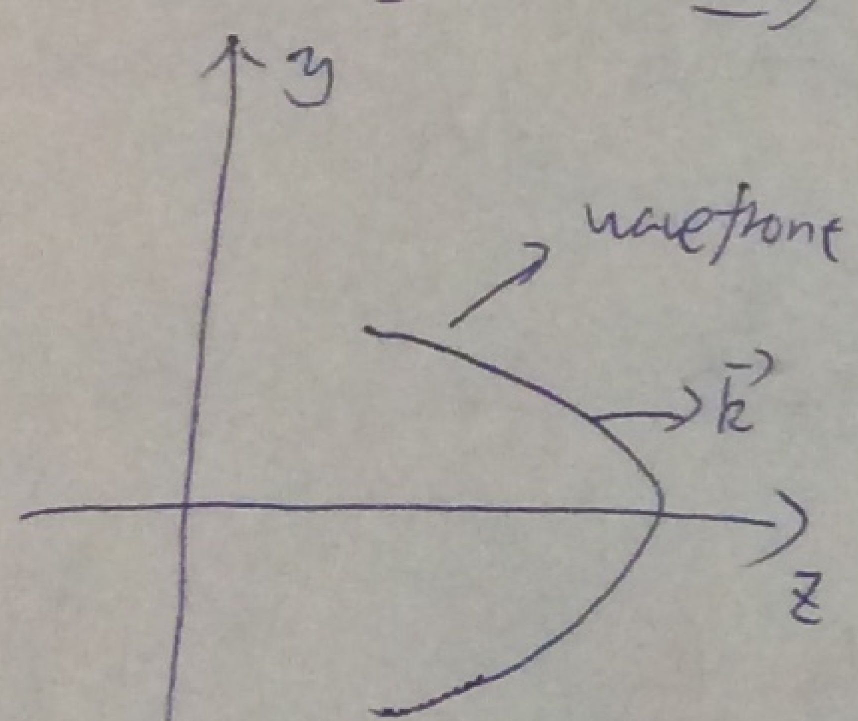
it will more look like a circle.

we should require a slowly variation of the 1th term

i.e. $\frac{\rho^2}{2z} \cdot k \ll \pi \Rightarrow \frac{\rho^2}{\lambda z} \ll 1$

$\left| kz \cdot \frac{1}{2} (\frac{\rho}{z})^2 \right| \ll \pi$

1th order $e^{-ik\frac{\rho^2}{2z}}$ $\Rightarrow$ parabola wave



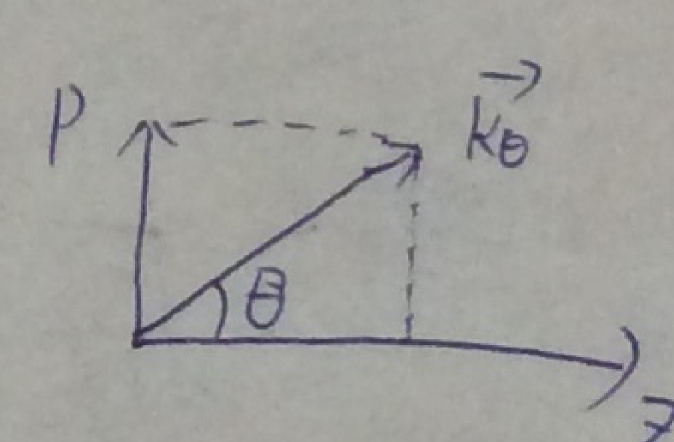
we should require a slowly variation of the 2ed term

i.e. $\frac{\rho^4}{z^3} \cdot \frac{k}{8} \ll \pi \Rightarrow \frac{\rho^4}{4\lambda z^3} \ll 1$

$\left| kz (-\frac{1}{8}) (\frac{\rho}{z})^4 \right| \ll \pi$

2/2

c) assume $\tan\theta = \frac{\rho}{z}$



in Fresnel approximation

$\theta \approx \tan\theta = \frac{\rho}{z}$

from (b) condition: $\frac{\rho^4}{4\lambda z^3} \ll 1$, replace $\theta = \frac{\rho}{z}$ into it, we have

$\theta^2 \cdot \frac{\rho^2}{\lambda z} \ll 4$

$\Rightarrow \theta^2 \cdot N_F \ll 4, \quad N_F = \frac{\rho^2}{\lambda z}$

2/2

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Problem 3 Fresnel approximation II

$$H_F(\alpha,\beta,z) = e^{ikz} e^{-i\frac{\alpha^2+\beta^2}{2k}z}$$

$$h_F(x,y,z>z_0) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} H_F(\alpha,\beta,z)\, e^{i(\alpha x+\beta y)}\, d\alpha\, d\beta$$

$$= \frac{e^{ikz}}{(2\pi)^2}\iint_{-\infty}^{\infty} e^{-\frac{\alpha^2+\beta^2}{2k}z}\, e^{i(\alpha x+\beta y)}\, d\alpha\, d\beta$$

$$= \frac{e^{ikz}}{(2\pi)^2}\cdot\int_{-\infty}^{\infty} e^{-i\left(\frac{\alpha\sqrt{z}}{\sqrt{2k}} - \frac{x}{2}\sqrt{\frac{2k}{z}}\right)^2} e^{+i\frac{x^2k}{2z}}\, d\alpha$$

$$\int_{-\infty}^{\infty} e^{-i\left(\frac{\beta\sqrt{z}}{\sqrt{2k}} - \frac{y}{2}\sqrt{\frac{2k}{z}}\right)^2} e^{+i\frac{y^2k}{2z}}\, d\beta$$

$$= \frac{e^{ikz}}{4\pi^2}\cdot e^{+i\frac{(x^2+y^2)\cdot k}{2z}}\cdot\int_{-\infty}^{\infty} e^{-i\frac{z}{2k}\left(\alpha-\frac{xk}{z}\right)^2}\, d\alpha$$

$$\cdot\int_{-\infty}^{\infty} e^{-i\frac{z}{2k}\left(\beta-\frac{yk}{z}\right)^2}\, d\beta$$

using Gaussian Integral, we have $\int e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$

$$\int_{-\infty}^{\infty} e^{-i\frac{z}{2k}\left(\alpha-\frac{xk}{z}\right)^2}\, d\alpha = \frac{\sqrt{\pi}}{\sqrt{i\frac{z}{2k}}}$$

$$\int_{-\infty}^{\infty} e^{i\frac{z}{2k}\left(\beta-\frac{yk}{z}\right)^2}\, d\beta = \frac{\sqrt{\pi}}{\sqrt{i\frac{z}{2k}}}$$

$$\Rightarrow h_F(x,y,z>z_0) = \frac{e^{ikz}}{4\pi^2}\cdot e^{+i\frac{(x^2+y^2)\cdot k}{2z}}\cdot\sqrt{\frac{\pi}{i\frac{z}{2k}}}\cdot\sqrt{\frac{\pi}{i\frac{z}{2k}}}$$

$$= -\frac{ik}{2\pi z}\, e^{ikz\left(1+\frac{x^2+y^2}{2z^2}\right)}$$

$$\overset{k=\frac{2\pi}{\lambda}}{=} -i\frac{1}{\lambda z}\, e^{ikz\left(1+\frac{x^2+y^2}{2z^2}\right)}$$

4/4

In above, we suppose that $k=\frac{2\pi}{\lambda}\to\infty$, that is Fresnel approximation is valid for large enough region of (α,β), so we can take integral $\iint d\alpha d\beta$ with infinite limits.