Optical Metrology and Sensing

Seminar 2

Task1.

a) Prove that the below spherical wave is a solution of the Helmholtz equation in spherical coordinates, where r is the distance from the origin and k is the wave number.

b) and that the wavefronts are concentric spheres separated by radial distance λ, where λ is the wavelength.

Spherical wave : $$U(r) = \frac{A}{r}\, e^{-jkr}$$

Helmholtz equation: $$(\nabla^2 + k^2)U(r) = 0$$

Task2.

Find the Fourier transform of the given circular function:

$$circ(r) = \begin{cases} 1 & r < 1 \\ \frac{1}{2} & r = 1 \\ 0 & otherwise \end{cases}$$

Task 1

$$\nabla\times A = \frac{1}{r^2\sin\theta}\begin{bmatrix} \vec{r} & r\vec{\theta} & r\sin\theta\vec{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{bmatrix}$$

$$(\nabla^2+k^2)U(\vec{r})=0 \qquad \nabla\cdot A=\frac{1}{r^2}\frac{\partial}{\partial r}r^2A_r+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta A_\theta+\frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

$$\Delta f=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$\nabla^2U(\vec{r})=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U(\vec{r})}{\partial r}\right)$$

$$\frac{\partial U(\vec{r})}{\partial r}=\frac{d}{dr}\frac{A}{r}e^{-jkr}=-\frac{A}{r^2}e^{-jkr}-jk\frac{A}{r}e^{-jkr}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial U(\vec{r})}{\partial r}\right)=\frac{\partial}{\partial r}\left(-Ae^{-jkr}-jkAre^{-jkr}\right)=jkAe^{-jkr}-\left(jkAe^{-jkr}+k^2Are^{-jkr}\right)=-k^2Are^{-jkr}$$

$$\Rightarrow\nabla^2U(\vec{r})=\frac{1}{r^2}\left(-k^2Are^{-jkr}\right)=-k^2\frac{A}{r}e^{-jkr}=-k^2U(\vec{r})$$

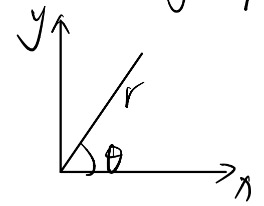
$$\Rightarrow(\nabla^2+k^2)U(\vec{r})=\nabla^2U(\vec{r})+k^2U(\vec{r})=-k^2U(\vec{r})+k^2U(\vec{r})=0$$

b) $U(r_1)=\frac{A}{r_1}e^{-jkr_1}$ $\qquad U(r_2)=\frac{A}{r_2}e^{-jkr_2}$

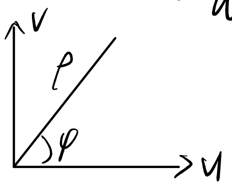
$kr_2-kr_1=T=2\pi$ $\qquad \frac{2\pi}{\lambda}(r_2-r_1)=2\pi\Rightarrow r_2-r_1=\lambda$

Task 2

$$F(u,v)=\mathcal{F}\{f(x,y)\}=\iint_{-\infty}^{\infty}f(x,y)\,e^{i2\pi(ux+vy)}\,dxdy$$

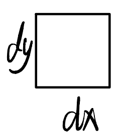


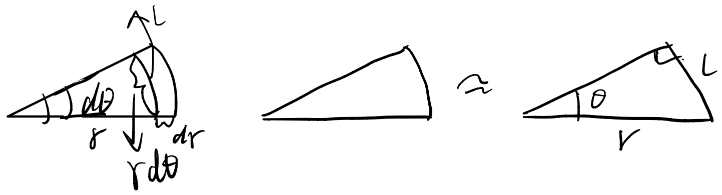
$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$



$\begin{cases}u=\rho\cos\varphi\\ v=\rho\sin\varphi\end{cases}$

$$ux+vy=\rho r\cos\theta\cos\varphi+\rho r\sin\theta\sin\varphi=\rho r\cos(\theta-\varphi)$$





$\sin\theta=\frac{L}{r}\approx d\theta\Rightarrow L=rd\theta$

$\Rightarrow ds=dr\cdot rd\theta=rdrd\theta$

$$\Rightarrow F(\rho,\varphi)=\mathcal{F}\{f(r,\theta)\}=\int_0^{2\pi}\int_0^{\infty}f(r,\theta)e^{i2\pi\rho r\cos(\theta-\varphi)}\,rdrd\theta$$

$$f(r,\theta)=\mathcal{F}^{-1}\{F(\rho,\varphi)\}=\int_0^{2\pi}\int_0^{\infty}F(\rho,\varphi)e^{-i2\pi\rho r\cos(\theta-\varphi)}\,\rho d\rho d\varphi$$

$f(r,\theta)=f_r(r)f_\theta(\theta)$ (Separable in Polar coordination)

$$\mathrm{circ}(r) = \begin{cases} 1 & r<1 \\ \frac{1}{2} & r=0 \\ 0 & r>1 \end{cases} \Rightarrow f_r(r) \Rightarrow f(r,\theta) = f_r(r) f_\theta(\theta) \Rightarrow f_\theta(\theta) = 1$$

$$\underline{F(\rho,\varphi) = \mathcal{F}\{f(r,\theta)\} = 2\pi \int_0^\infty f_r(r) \left(\frac{1}{2\pi} \int_0^{2\pi} e^{i2\pi\rho r\cos(\theta-\varphi)} d\theta \right) r\,dr}$$

$$J_0(2\pi\rho r) = \int_0^{2\pi} e^{i2\pi\rho r\cos(\theta-\varphi)} d\theta \Rightarrow J_0(a) = \int_0^{2\pi} e^{ia\cos(\theta-\varphi)} d\theta$$ (zeroth order Bessel function) (damped cosine

$$\Rightarrow F(\rho,\varphi) = \mathcal{F}\{f(r,\theta)\} = 2\pi \int_0^\infty f_r(r) J_0(2\pi\rho r) r\,dr$$ (Fourier-Bessel transform)

$$F(\rho,\varphi) = \mathcal{F}\{\mathrm{circ}(r)\} = 2\pi \int_0^\infty \mathrm{circ}(r) J_0(2\pi\rho r) r\,dr = 2\pi \int_0^1 J_0(2\pi\rho r) r\,dr$$

$\int_0^\alpha x J_0(x) dx = \alpha J_1(\alpha) \rightarrow$ First order Bessel function

Substitute $r' = 2\pi\rho r \Rightarrow r = \frac{r'}{2\pi\rho} \quad dr = \frac{dr'}{2\pi\rho} \quad r=1 \Rightarrow \underline{r' = 2\pi\rho}$

$$\Rightarrow F(\rho,\varphi) = \frac{1}{2\pi\rho^2} \int_0^{2\pi\rho} J_0(r') r' dr' = \frac{1}{2\pi\rho^2} \cdot 2\pi\rho J_1(2\pi\rho) = \underline{\frac{J_1(2\pi\rho)}{\rho}}$$