Midterm test Solution:

1. Multiple-choice test: Please tick all box(es) with correct answer(s)!

The operator $-\dfrac{d^2}{dx^2}$ is	Self-adjoint	✓
	Not self-adjoint	

- It is the same as kinetic energy operator: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$
- kinetic energy is Hermitian operator
- All Hermition operators are self adjoint

The wavelength of 500nm belongs to the	Microwave spectral range
	Middle infrared spectral range
	X-ray spectral range

- 400nm - 700nm : [visible spectral range]

Vibrations of nuclei in molecules and solids cause light absorption primarily in the	Ultraviolet spectral range	
	Visible spectral range	
	Infrared spectral range	✓

- The nuclei vibration wavelenght is $\lambda_0 \simeq 20\,\mu m$ [Middle infrared spectral range]

The spectral energy density (*u*) of a radiation field may be given in	Js/m^3	✓
	Ws^2/m^3	✓
	$kg/(sm)$	✓

The spectral behavior of the so-called black body radiation could be explained by planck's formula:

$\hbar : [J.s]$

$c : [m/s]$

$\lambda : [m]$

$$u(\omega,T) = \frac{\hbar\omega^3}{c^3\pi^2}\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1} = \frac{J.S\left(\frac{1}{s^3}\right)}{\frac{m^3}{s^3}} = \frac{J.s}{m^3} \checkmark$$

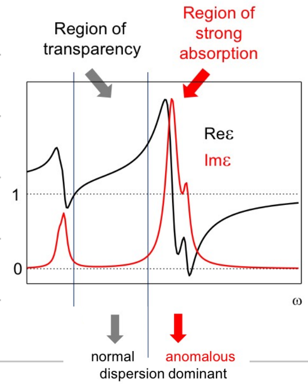
$$J = \frac{kg\,m^2}{s^2} \Rightarrow [u(\omega,t)] = \frac{\frac{kg\,m^2}{s^2}\cdot s}{m^3} = \frac{kg}{m.s} \checkmark$$

$$J = W.s \Rightarrow [u(\omega,t)] = \frac{Ws^2}{m^3} \checkmark$$

The mass moment of inertia of a diatomic molecule is typically of the order	10^{-27} kgm^2
	10^{-47} kgm^2 ✓
	10^{-67} kgm^2

$I = 2.6 \times 10^{-47}$ kgm^2

In spectral regions of strong absorption, the refractive index of a material usually shows	normal dispersion
	no dependence on frequency
	anomalous dispersion ✓



With increasing temperature, the electrical conductivity of semiconductors does typically	increase ✓
	decrease
	remain constant

- But for metals "decrease" is correct.

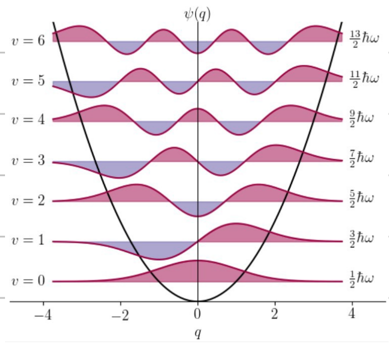
$[\hat{\mathbf{L}}^2, \hat{x}] =$	$\hbar^4$
	$\hbar^5$
	$\hbar^6$

- Since the measurment units do not mach, none of them is correct.

2. Quantum mechanical treatment of the one - dimensional harmonic oscillator with harmonic Potential U=U(x) $= m\omega_0^2 \frac{x^2}{2}$:

Decide whether the assertion is true or wrong and tick the appropriate box):

assertion
All eigenfunctions of the harmonic oscillator are even functions of x. [Wrong]



• every other states are even or odd

In the ground state, $\langle T_{kin} \rangle = 0$ [Wrong]

$\hat{H} = \hat{U} + \hat{T}$ in Harmonic Oscillator $\Rightarrow \hat{U} = \hat{T} = \frac{1}{2} E_n$

$\langle T \rangle|_n = \frac{1}{2} E_n \overset{}{\underset{E_n = \hbar\omega_0 (n + \frac{1}{2})}{=\!=\!=\!=}} \frac{1}{2}\hbar\omega_0 (n + \frac{1}{2}) \Big|_{n=1} = \frac{3}{4}\hbar\omega_0 \neq 0$

In the ground state, $\langle x^2 \rangle = 0$ [Wrong]

$U = m\omega_0^2 \frac{x^2}{2}$

$\langle U \rangle = m\omega_0^2 \frac{\langle x^2 \rangle}{2}$ $\quad \langle U \rangle \neq 0 \Rightarrow \langle x^2 \rangle \neq 0$

In the ground state, $\langle x^3 \rangle = 0$ [True]

$\langle x^3 \rangle = \int_{-\infty}^{\infty} \underbrace{\psi^* x^3 \psi}_{\text{even} \times \text{odd} \times \text{even} = \text{odd function}} \, dx = 0$

Electric dipole-allowed quantum transitions occur only between adjacent energy levels [True] • otherwise it is zero

3. A resting hydrogen atom emits a photon with a wavelength 121.5nm. What is the velocity of the atom after the light emission? Nonrelativistic calculus is sufficient. (4 points)

$\lambda = 121.5\ nm$

Momentum conservation : $\frac{h}{\lambda} = mv \Rightarrow v = \frac{h}{m\lambda}$ $\Big|_{h=6.62 \times 10^{-34}\ Ws^2,\ m = 1.672 \times 10^{-27}\ kg}$ $\approx 3.26\ m/s$

4. Assume an electron confined in a rectangular potential box of length L = 1nm with impermeable walls (one-dimensional case). What is the transition matrix element x_{jk} and the corresponding oscillator strength for the transition from the first to the third excited states ($k = 2 \rightarrow j = 4$)? (4 points)

- Both transition matrix element and the oscillator strength are zero, because states j and k have the same parity

- $\psi_n = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x \quad n = 1, 2, 3, \ldots$

$$x_{jk} = \int_{-\infty}^{\infty} \psi\, x\, \psi^* dx = 2 \int_{-\infty}^{\infty} x \sin\left(\frac{j\pi x}{L}\right) \sin\left(\frac{k\pi x}{L}\right) dx \stackrel{*}{=} 0$$

$$\boxed{* \quad \frac{2}{L} \int_0^L x \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx \underset{n-m\,:\,\text{even}}{=} 0}$$

- Transition is allowed for neighbor states $m = n \pm 1$
$2 \neq 4 \pm 1 \Rightarrow f_{nm} = 0$

5. Consider a two-level system with the excited state populated. Calculate the relaxation time τ assuming that the excited state of the system (2) relaxes to the ground state (1) by spontaneous emission of light only. Calculate the bandwidth $\Delta\lambda$ (FWHM) of the emission line. Assume electric dipole interaction with the following data: q=-e; λ_{21}=250nm; $|x_{21}|$=10^{-10}m

(6 points)

- $\tau = A_{21}^{-1}$

$$A_{21} = \frac{q\,\omega_{21}^3\,|x_{21}|^2}{3\varepsilon_0\pi\hbar c^3} \quad \Bigg| \begin{array}{l} q=-e \\ \lambda_{21}=250nm \\ |x_{21}|=10^{-10}m \end{array}$$

$$\Rightarrow \tau = 0.27\times10^{-8}\,s$$

$$\frac{250\times250\times10^{-9}\times10^{-9}}{2\times3.14\cdot3\cdot\cancel{10^{8}}\cdot0.27\,\cancel{10^{-8}}}$$

- $\frac{\Delta\omega}{\omega} = \frac{\Delta\lambda}{\lambda}$, $\omega = \frac{2\pi c}{\lambda}$, $\Delta\omega \approx \tau^{-1}$ $\Rightarrow \Delta\lambda = \lambda\frac{\Delta\omega}{\omega} \simeq \frac{\lambda^2}{2\pi c\tau} \approx 1.54\times10^{-5}\,nm$

6. By making use of Planck's formula, find an expression for the energy spectral density of radiation in thermodynamic equilibrium *per wavelength interval* $u(\lambda) = \frac{1}{V}\frac{dE}{d\lambda}$. From there, find the temperature dependence of the wavelength λ_{max} where that function has a maximum. (10 points)

$$u(\lambda,T) = \frac{1}{V}\frac{dE}{d\lambda}$$

$$u(\omega,T) = \frac{1}{V}\frac{dE}{d\omega} = \frac{1}{V}\frac{dE}{d\left(\frac{2\pi c}{\lambda}\right)} = -\frac{1}{V}\frac{\lambda^2}{2\pi c}\frac{dE}{d\lambda} = -\frac{\lambda^2}{2\pi c}u(\lambda,T)$$

$$\frac{-\lambda^2}{2\pi c}\times\frac{\hbar\omega^3}{c^3\pi^3}\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1} \Rightarrow \boxed{u(\lambda,T) = -\frac{8hc\pi}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda k_BT}}-1}}$$

for $d\omega>0 \Rightarrow d\lambda<0$

- The reason of negative $u(\lambda,T)$ is the negative corresponding wavelenght interval $\left(\frac{d\omega}{d\lambda} = \frac{-2\pi c}{\lambda^2}\right)$ Returning to a positive wavelength interval is achieved by multiplying by -1, which eliminates the minus sign in the u expression. Another approach to get to positive u expression is:

Total energy must be the same in both descriptions:

$$\int_0^\infty u(\lambda,T)\,d\lambda = \int_0^\infty u(\omega,T)\,d\omega \qquad \omega = 2\pi c/\lambda \qquad \begin{cases} d\omega = d\left(\frac{2\pi c}{\lambda}\right) = -\frac{2\pi c}{\lambda^2}\,d\lambda \\ \omega\to 0 \Rightarrow \lambda\to\infty \\ \omega\to\infty \Rightarrow \lambda\to 0 \end{cases}$$

$$\int_0^\infty u(\lambda,T)\,d\lambda = \int_0^\infty u(\omega,T)\,d\omega = -\int_\infty^0 u(\omega\to\lambda,T)\,\frac{2\pi c}{\lambda^2}\,d\lambda = \int_0^\infty u(\omega\to\lambda,T)\,\frac{2\pi c}{\lambda^2}\,d\lambda$$

$$\Rightarrow\ U(\lambda,T) = U(\omega\to\lambda,T)\times\frac{2\pi c}{\lambda^2} = \frac{\hbar\left(\frac{2\pi c}{\lambda}\right)^3}{c^3\pi^2}\,\frac{1}{e^{\hbar\frac{2\pi c}{\lambda k_B T}}-1}\times\frac{2\pi c}{\lambda^2}$$

$$\boxed{U(\lambda,T) = \frac{8hc\pi}{\lambda^5}\,\frac{1}{e^{\frac{hc}{\lambda k_B T}}-1} \quad \text{if } d\lambda>0}$$

- provided now that $U(\lambda,T)$ has a maximum, then it corresponds to a minimum of $U(\lambda,T)^{-1}$. Let us search the minimum of:

$$f = \lambda^5\left(e^{\frac{hc}{k_B\lambda T}}-1\right) \Rightarrow \frac{df}{d\lambda} = 5\lambda^4\left(e^{\frac{hc}{k_B\lambda T}}-1\right) - \frac{hc}{k_B\lambda^2 T}\,e^{\frac{hc}{k_B\lambda T}}\lambda^5$$

$$\frac{df}{d\lambda}=0 \Rightarrow 5\left(e^{\frac{hc}{k_B\lambda T}}-1\right) - \frac{hc}{k_B\lambda^2 T}\,e^{\frac{hc}{k_B\lambda T}} = 0$$

Define: $x = \dfrac{hc}{\lambda k_B T} \Rightarrow 5(e^x-1) - xe^x = 0$

Let the solution of the equation be x_0, then:

$$x_0 = \frac{hc}{\lambda k_B T} \Rightarrow \boxed{\lambda T = \frac{hc}{k_B x_0} = \text{const}}\text{, i.e. Wien's displacement law}$$

<u>Numerically</u>: $x_0 \simeq 4.97 \quad 4.97 \simeq \dfrac{hc}{\lambda k_B T} \Rightarrow \lambda T \approx \dfrac{hc}{4.9\,k_B T} \approx 2.898\times10^{-3}$ km

i.e. Wiens displacement law with $\lambda T = \text{const} \approx 2.898\times10^{-3}$ km

- Numerical solution was not expected. It was sufficient to indicate that $\lambda T=$const. The constant is not dimensionless