12.5

Y

Fundamentals of Modern Optics
Exercise 10
09.12.2014

to be returned: 16.01.2015, at the beginning of the lecture

Problem 1 – Fourier Optical Filtering — 2+2+2 points

Consider a 4f-setup consisting of two identical lenses for focal length f placed at a distance of $2f$ illuminated with light of wavelength λ. Halfway between the two lenses, centered on the optical axis, we place a filter $H(x, y)$. We shall now consider this "device" as a black box, which takes a certain input field $A_0(x, y)$ incident at the focal plane before the first lens and transforms it into an output field $A_1(x, y)$ at a distance of f after the second lens.

a) The action of the device can be described by a convolution with a response function $h(x, y)$ such that $A_1(-x, -y) = \iint h(x - x', y - y')A_0(x', y')dx'dy'$. Why is that so? How are $h(x, y)$ and $H(x, y)$ related?

b) Assuming that the filter could also include optical gain (tranmission > 1) one could implement (实现) the following two filters: $H(x, y) = iqx$ and $H(x, y) = -q(x^2 + y^2)$. What effect do they have on the optical field?

c) The above mentioned $4f$ setup can be used to build an optical correlator, which measures the similarity between two wave fields $A_0(x, y)$ and a reference wave field $B(x, y)$. The correlation $A_1^{\mathrm{corr}}(x, y)$ between these two fields is calculated by $A_1^{\mathrm{corr}}(x, y) = A_0(x, y) \otimes B^*(-x, -y)$, where $\otimes$ denotes a convolution filtering operation. How does the aperture function have to look like in order to implement this special filtering operation? How could it be implemented practically?

Problem 2 – Phase Contrast Microscopy — 3 points

In biology one often has to deal with samples which are mostly transparent. Those samples introduce a phase modulation only and it is hard to see them in the microscope by normal means. By measuring the light intensity the phase information gets lost. We will now look at a solution to this problem which was introduced by Zernike: the phase contrast microscopy.

Suppose you shine with a plane wave onto a transparent sample which introduces a phase modulation $u_0(x, y) = e^{i\varphi(x,y)}$. Now you pass this light into a $4f$-setup with the following transfer function

$$h(x, y) = \begin{cases} 1 & , x \approx 0 \text{ and } y \approx 0 \\ e^{i\frac{\pi}{2}} = i & , \text{else} \end{cases}$$

real

This means that a part of the spectrum is retarded by $\pi/2$. Assume small phase variations $|\varphi(x, y)| \ll 1$ and show that such a setup introduces an intensity modulation which depends on the local phase $\varphi(x, y)$ of the sample.

Problem 3 – Prison Break — 2+2+2 points

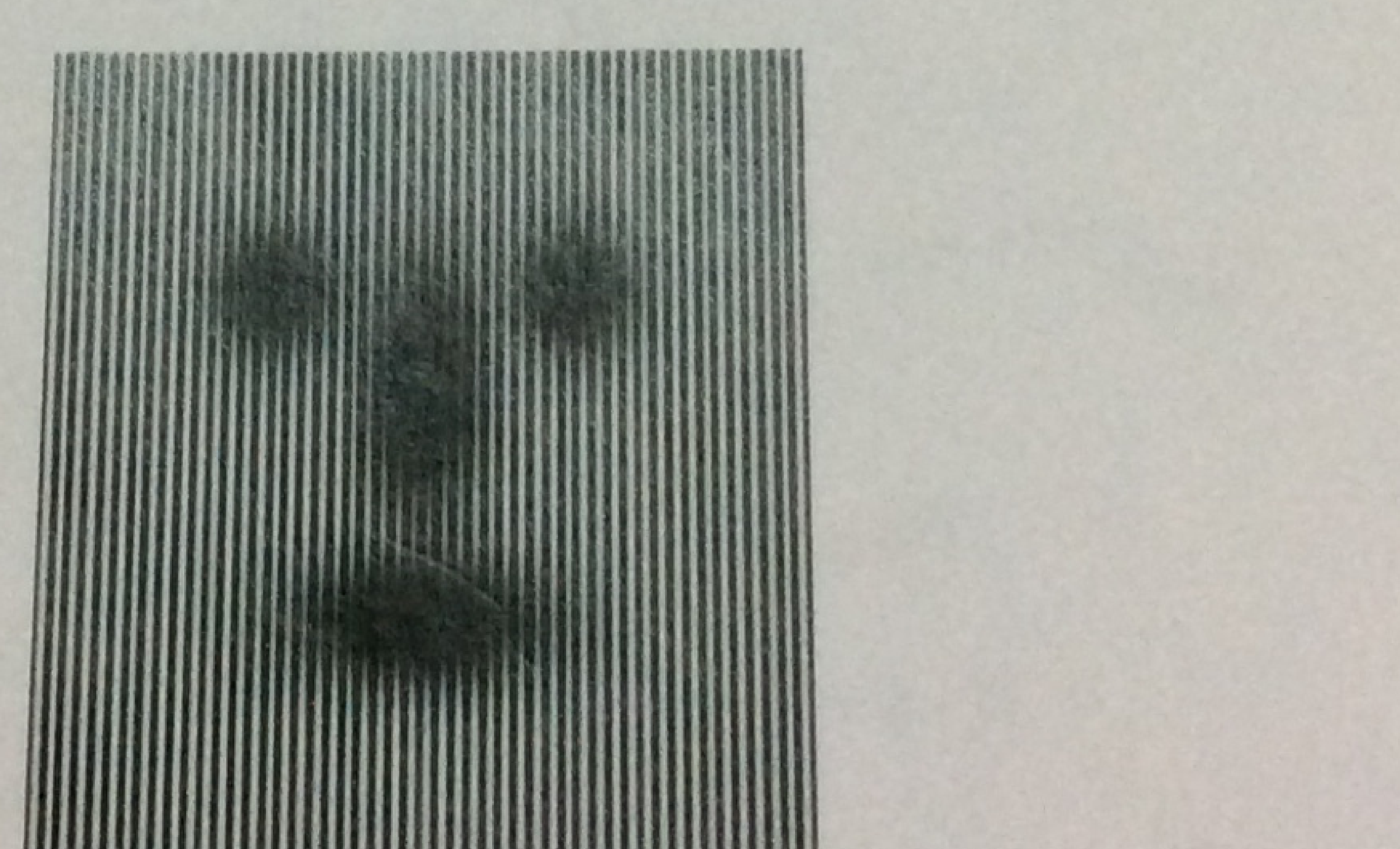

Figure 1: Hello! I am Günther! Please help me to get out of jail! I am innocent. I swear!

Fourier Optical Filtering

Your German friend Günther Gaußig was sent to prison for admitting that it is possible to overcome the resolution limit. He was locked by sheriff Peter Periodic using a grating in x-direction which has the following form:

警长

$$\text{PRISON} = g(x) = \frac{1}{2}\left[1 + \cos\left(\frac{2\pi}{d}x\right)\right]$$

Luckily, Günther is a very "Gaussian" person in real space, where he usually lives

$$\text{GÜNTHER} = f(x,y) = m(x,y) + n(x,y) + e_l(x,y) + e_r(x,y),$$

with

$$\text{Nose} = n(x,y) = \exp\left[-\frac{x^2}{w^2} - \frac{\left(y-\frac{R}{5}\right)^2}{l^2}\right]$$

$$\text{Mouth} = m(x,y) = \exp\left[-\frac{x^2}{l^2} - \frac{\left(y+\frac{R}{2}\right)^2}{w^2}\right]$$

$$\text{LeftEye} = e_l(x,y) = \exp\left[-\frac{\left(x-\frac{R}{2}\right)^2}{w^2} - \frac{\left(y-\frac{R}{2}\right)^2}{w^2}\right]$$

$$\text{RightEye} = e_r(x,y) = \exp\left[-\frac{\left(x+\frac{R}{2}\right)^2}{w^2} - \frac{\left(y-\frac{R}{2}\right)^2}{w^2}\right]$$

and

$$R > l > w \gg d$$

As an Abbe School of Photonics student you want to free your friend by optical means. You find Günther superimposed by the prison in the spatial domain as a starting point, i.e. the function $u_0(x,y) = f(x,y) + g(x)$ (see Fig. 1.).

a) Develop a plan to free Günther, i.e. to remove the prison, by Fourier optical filtering.

b) Construct and sketch an optical setup which is able to perform that task.

c) Why does Günther necessarily get battered (at least a little bit) by the escape out of prison? How can you minimize that effect?

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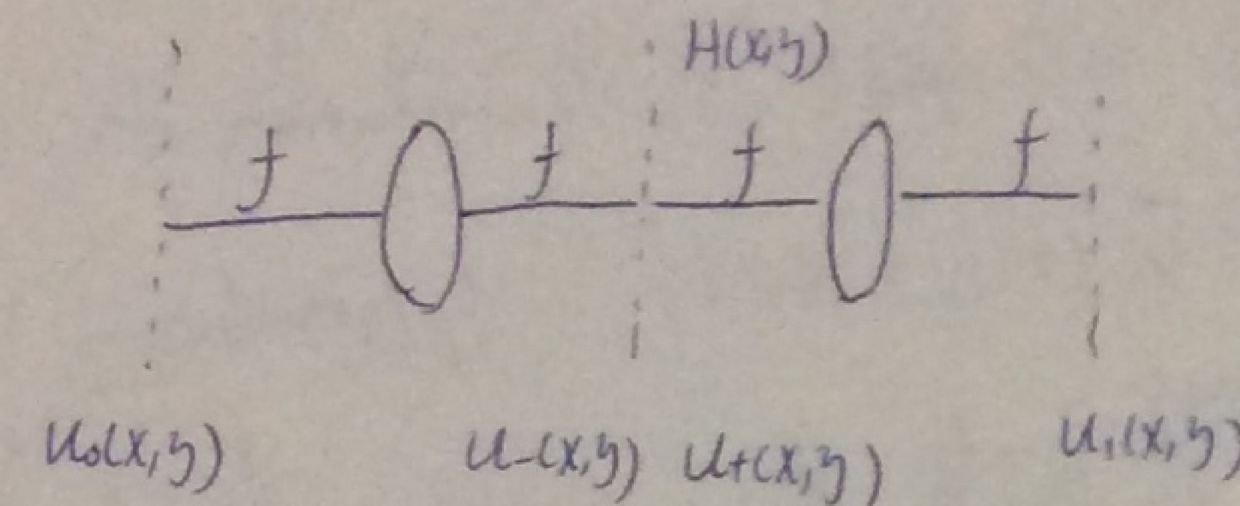
Problem 1 Fourier Optical Filtering



a) $U_-(x,y) = -i\frac{(2\pi)^2}{\lambda f} e^{i2kf} U_0\left(\frac{kx}{f}, \frac{ky}{f}\right)$ ✓

$U_0\left(\frac{kx}{f}, \frac{ky}{f}\right) = FT\{u_0(x,y)\}$ set $\alpha = \frac{kx}{f}$, $\beta = \frac{ky}{f}$ $\Rightarrow$ $x = \frac{f\alpha}{k}$, $y = \frac{f\beta}{k}$

so $H(x,y) = H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right)$

$$u_+\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right) = u_-\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right) \cdot H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right)$$

$$= -i\frac{(2\pi)^2}{\lambda f} e^{i2kf} FT\{u_0(x,y)\} \cdot H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right)$$

The second lens serves as Fourier back transform

$$u_1(x,y) = -i\frac{(2\pi)^2}{\lambda f} e^{i2kf} \cdot U_+\left(\frac{k}{f}x, \frac{k}{f}y\right)$$

$$U_+\left(\frac{kx}{f}, \frac{ky}{f}\right) = FT\left\{u_+\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right)\right\}$$

$$\Rightarrow u_1(x,y) = -i\frac{(2\pi)^2}{\lambda f} e^{i2kf} FT\left\{u_-\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right) \cdot H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right)\right\}$$

$$= -i\frac{(2\pi)^2}{\lambda f} e^{i2kf} \cdot \frac{1}{(2\pi)^2} \iint u_-\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right) \cdot H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right) e^{-i(\alpha x + \beta y)} d\alpha d\beta$$

$$= \frac{(2\pi)^2}{(\lambda f)^2} e^{i4kf} \iint_{-\infty}^{+\infty} U_0(\alpha, \beta) H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right) e^{-i(\alpha x + \beta y)} d\alpha d\beta$$

By passing to mirrored coordinates $x \to -x$, $y \to -y$, we get

$$u_1(-x,-y) = \frac{(2\pi)^2}{(\lambda f)^2} e^{i4kf} \iint_{-\infty}^{+\infty} U_0(\alpha, \beta) H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right) e^{i(\alpha x + \beta y)} d\alpha d\beta$$

we can see that $u_1(-x,-y)$ is Fourier back transform of $U_0(\alpha,\beta)$ multiplied by function $H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right) \cdot \frac{(2\pi)^2}{(\lambda f)^2} e^{i4kf}$

$$\Rightarrow U_1(-\alpha,-\beta) = \frac{(2\pi)^2}{(\lambda f)^2} e^{i4fk}\, U_0(\alpha,\beta)\cdot H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right)$$

Multiplication in Fourier space is equal to convolution in (x,y) space.

$$u_1(-x,-y) = \iint_{-\infty}^{\infty} h(x-x', y-y')\, u_0(x',y')\, dx'dy' \quad (\text{convolution theorem})$$

where $h(x,y) = FT^{-1}\left[\frac{(2\pi)^2}{(\lambda f)^2} e^{i4kf} H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right)\right]$ ✓

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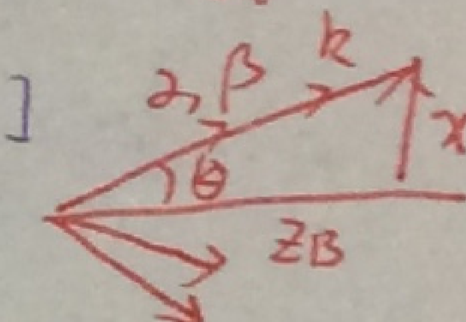
Therefore, response function $h(x,y)$ is a Fourier-back transform of $H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right)$ multiplied by some coefficient.

$h(x,y) = iqx \quad h(x,y) = q\frac{\partial}{\partial x}\delta\left(-\frac{kx}{f}\right)\delta\left(-\frac{ky}{f}\right) \Leftarrow \delta(x) = \int e^{ikx}dx$

$u(x,y) = \int_{-\infty}^{\infty} h(x-x', y-y')\, u_0(x',y')\,dx'dy' = \frac{\partial u_0}{\partial x} \qquad \frac{\partial \delta}{\partial x} = \int ike^{ikx}dx$

b) $$\frac{d^n f(x)}{dx^n} = FT^{-1}\left[(i\alpha)^n f(\alpha)\right] \quad \text{where} \quad f(x) = FT^{-1}\left[f(\alpha)\right]$$



$\frac{kx}{zB} = k\sin\theta \to \alpha$

$\frac{ky}{zB} = k\cos\theta \to \beta$

$$\Rightarrow H_1(x,y)\, U_0\left(\frac{kx}{f}, \frac{ky}{f}\right) = iqx\, U_0\left(\frac{kx}{f}, \frac{ky}{f}\right) = iq\frac{f\alpha}{k} U_0(\alpha,\beta) \xrightarrow{FT^{-1}} q\frac{f}{k}\frac{\partial}{\partial x} u_0(x,y)$$ ✓

$$\Rightarrow H_2(x,y)\, U_0\left(\frac{kx}{f}, \frac{ky}{f}\right) = -q(x^2+y^2)\, U_0\left(\frac{kx}{f}, \frac{ky}{f}\right) = q\frac{f^2}{k^2}\left((i\alpha)^2 + (i\beta)^2\right) U_0(\alpha,\beta) \xrightarrow{FT^{-1}} q\frac{f^2}{k^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u_0(x,y)$$ ✓

in case 1, we will get derivation of input field with respect to x

in case 2, we will get Laplacian of input field with respect to (x,y)

2

The output fields are follows:

$$u_1(x,y) \sim q\frac{\partial u_0(x,y)}{\partial x}$$

$$u_2(x,y) \sim q\Delta^{(2)} u_0(x,y)$$

What's more, in case 2, the amplitude of output field doesn't depend on f and λ, because of optical gain. since

$$\frac{(2\pi)^2}{(\lambda f)^2}\cdot q\cdot\frac{f^2}{k^2} = q \Rightarrow u_2(x,y) = e^{i4fk}\, q\,\Delta^{(2)} u_0(x,y)$$

c) $A_1^{corr}(x,y) = u_0(x,y) \otimes B^*(-x,-y)$

$$= \iint_{-\infty}^{\infty} U_0(\alpha,\beta)\, B^*(\alpha,\beta)\, e^{-i(\alpha x+\beta y)}\, d\alpha\, d\beta$$

from part (a), we get

$$u_1(x,y) = \frac{(2\pi)^2}{(\lambda f)^2} e^{i4kf} \iint_{-\infty}^{\infty} U_0(\alpha,\beta)\, H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right) e^{-i(\alpha x+\beta y)}\, d\alpha\, d\beta$$

$$\sim \iint_{-\infty}^{\infty} U_0(\alpha,\beta)\, H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right) e^{-i(\alpha x+\beta y)}\, d\alpha\, d\beta$$

$$\Rightarrow H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right) = B^*(\alpha,-\beta)$$

1.5

$H(x,y) = \frac{-k^2\lambda^2}{2\pi} e^{-i4kf} \tilde{B}^*\left(\frac{kx}{f}, \frac{ky}{f}\right)$

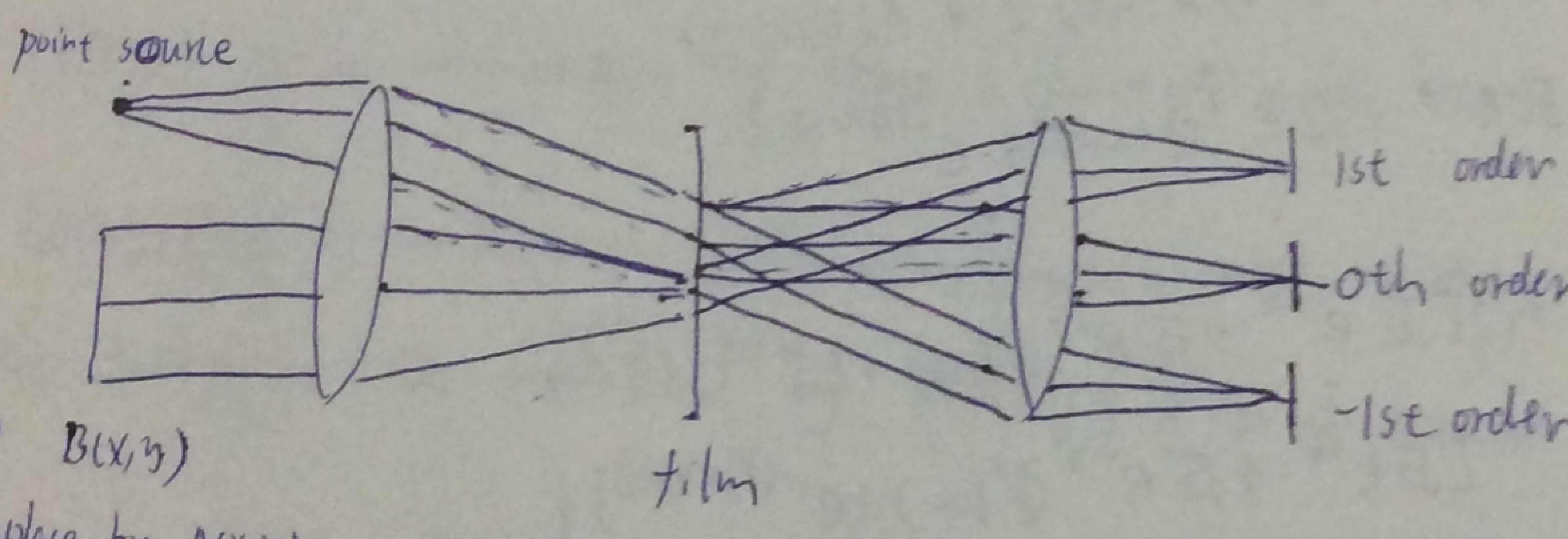
~~FT domain~~

FT domain

$H(x,y) \propto F\{B^*(-\frac{kx}{f}, -\frac{ky}{f})\}$ with $x = \frac{f\alpha}{k}$, $y = \frac{f\beta}{k}$

$H(x,y) \propto F\{B^*\}\big|_{\alpha=\frac{k}{f}x,\ \beta=\frac{k}{f}y}$

one of the methods to get correlator using recordable film have been proposed by Vander Lugt in 1963. As follow:



① Recording of Fourier-transformed field on a light-sensitive film. The imput field is $B(x,y)$ superimposed with point light source.

② After recording, the $B(x,y)$ is replaced by $A_0(x,y)$, the point source is still remained. The output field is exactly the correlation of $A_0(x,y)$ and $B(x,y)$ in 1st diffraction order.

Problem 2 - Phase Contrast Microscopy

From problem 1 (a), we know

$$u(-x,-y) = A \iint_{-\infty}^{\infty} U_0(\alpha,\beta)\, H\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right) e^{i(\alpha x+\beta y)}\, d\alpha\, d\beta \quad \text{with } A = \frac{(2\pi)^2}{(\lambda f)^2} e^{i4kf}$$

since $h(x,y) = \begin{cases} 1 & x \approx 0,\ y \approx 0 \\ e^{i\frac{\pi}{2}} = i & \text{else} \end{cases}$

so $h(x,y) = (1-i)\delta(x,y) + i$ X

$$\Rightarrow u(-x,-y) = A \iint_{-\infty}^{\infty} (1-i)\,\delta\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right) U_0(\alpha,\beta)\, e^{i(\alpha x+\beta y)}\, d\alpha\, d\beta$$

$$+ A i \iint_{-\infty}^{\infty} U_0(\alpha,\beta)\, e^{i(\alpha x+\beta y)}\, d\alpha\, d\beta$$

1

since $u_0(x,y) = e^{i\varphi(x,y)}$, and $|\varphi(x,y)| \ll 1$,

so $u_0(x,y) \approx 1 + i\varphi(x,y) \Rightarrow U_0(\alpha,\beta) \approx \delta(\alpha,\beta) + i\,\Phi(\alpha,\beta)$

So $u(-x,-y) = A[(1-i) + i(1-i)\Phi(0,0)] + A i\, u_0(x,y)$

$$= A[\sqrt{2}e^{-i\frac{\pi}{4}} + \sqrt{2}e^{i\frac{\pi}{4}}\Phi(0,0) + e^{i[\varphi(x,y)+\frac{\pi}{2}]}]$$ X

$$I \sim |u(-x,-y)|^2 = A^2\{[\sqrt{2}e^{-i\frac{\pi}{4}} + \sqrt{2}e^{i\frac{\pi}{4}}\Phi(0,0) + e^{i[\varphi(x,y)+\frac{\pi}{2}]}]$$

$$\cdot [\sqrt{2}e^{i\frac{\pi}{4}} + \sqrt{2}e^{-i\frac{\pi}{4}}\Phi^*(0,0) + e^{-i[\varphi(x,y)+\frac{\pi}{2}]}]\}$$

$$= A^2[2 + 2e^{-i\frac{\pi}{2}}\Phi^*(0,0) + \sqrt{2}e^{-i[\varphi(x,y)+\frac{3}{4}\pi]} + 2e^{i\frac{\pi}{2}}\Phi(0,0)$$

$$+ 2|\Phi(0,0)|^2 + \sqrt{2}e^{-i[\varphi(x,y)+\frac{\pi}{4}]}\Phi(0,0) + \sqrt{2}e^{i[\varphi(x,y)+\frac{3}{4}\pi]}$$

$$+ \sqrt{2}e^{i[\varphi(x,y)+\frac{\pi}{4}]}\Phi^*(0,0) + 1]$$

$$= A^2\{[3 + 2|\Phi(0,0)|^2 + 2\sqrt{2}\cos[\varphi(x,y)+\frac{3}{4}\pi] + 2[e^{i\frac{\pi}{2}}\Phi(0,0) + e^{-i\frac{\pi}{2}}\Phi^*(0,0)]$$

$$+ \sqrt{2}[e^{-i[\varphi(x,y)+\frac{\pi}{4}]}\Phi(0,0) + e^{i[\varphi(x,y)+\frac{\pi}{4}]}\Phi^*(0,0)]]\}$$

The Intensity modulation depends on the local phase $\varphi(x,y)$ of the sample.

Problem 3 Prison Break

a) $FT[f(x,y) + g(x)] = FT[f(x,y)] + FT[g(x)]$

$FT[g(x)] = FT[\frac{1}{2}(1+\cos\frac{2\pi}{d}x)] = \frac{1}{2}\delta(\frac{\alpha}{2\pi})\delta(\frac{\beta}{2\pi}) + \frac{1}{2}\delta(\frac{\alpha}{2\pi} - \frac{1}{d})\delta(\frac{\beta}{2\pi}) + \frac{1}{2}\delta(\frac{\alpha}{2\pi} + \frac{1}{d})\delta(\frac{\beta}{2\pi})$

$= \frac{\cancel{4\pi^2}}{2}\delta(\beta)[\delta(\alpha) + \frac{\delta(\alpha - \frac{2\pi}{d})}{2} + \frac{\delta(\alpha + \frac{2\pi}{d})}{2}]$ ✓

$FT[f(x,y)] = \sum_i^4 e^{-\frac{\alpha^2}{4p_i^2}} e^{-\frac{\beta^2}{4q_i^2}} \cdot \frac{1}{4\pi p_i q_i} e^{-i(\Delta_i^x \alpha + \Delta_i^y \beta)}$

where $\{p_i\} = (\frac{1}{w}, \frac{1}{l}, \frac{1}{w}, \frac{1}{w})$

$\{q_i\} = (\frac{1}{l}, \frac{1}{w}, \frac{1}{w}, \frac{1}{w})$

$\{\Delta_i^x\} = (0, 0, \frac{k}{2}, -\frac{k}{2})$

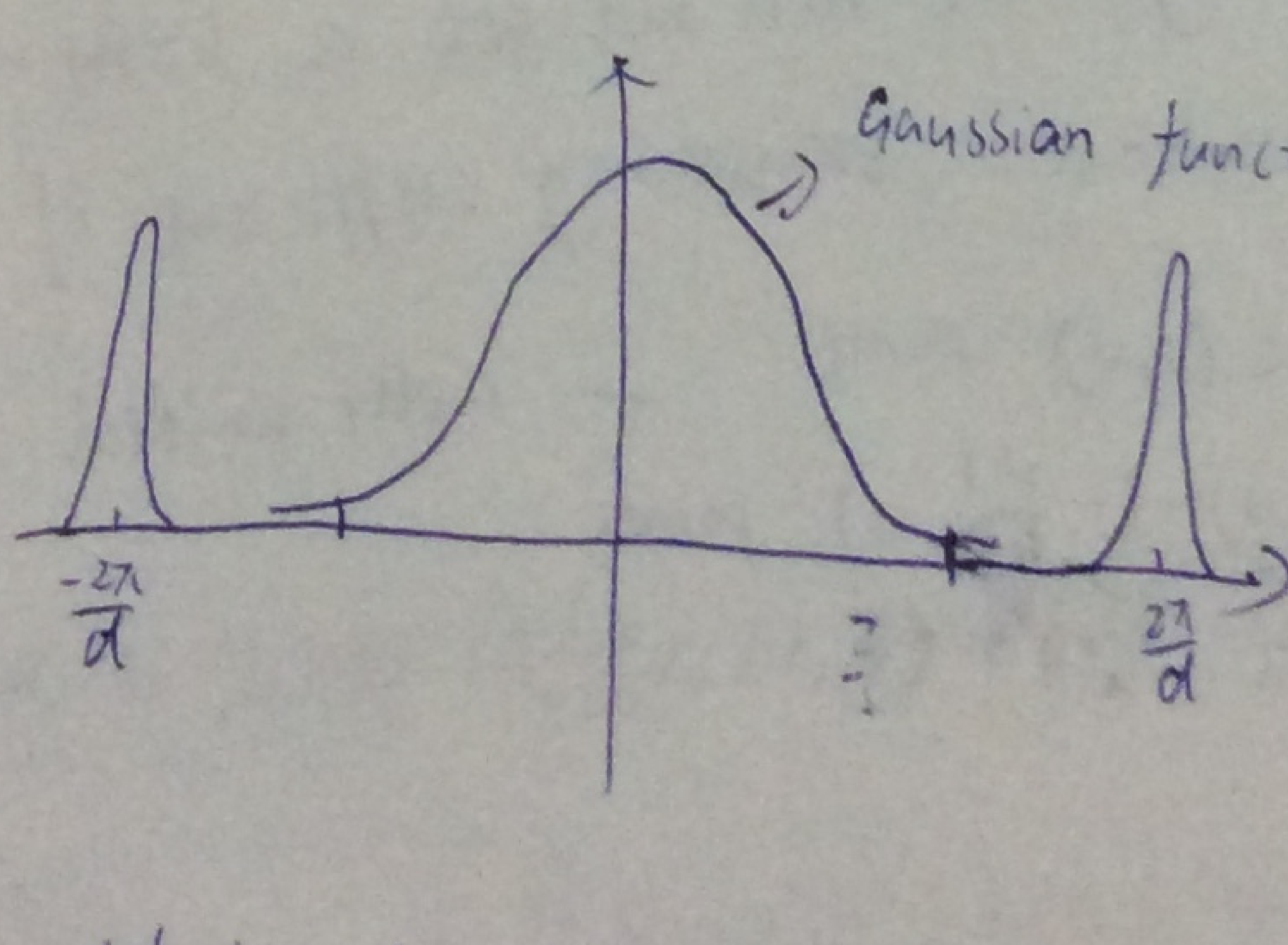
$\{\Delta_i^y\} = (\frac{k}{5}, -\frac{k}{2}, \frac{k}{2}, \frac{k}{2})$

So $FT[g(x)]$ is a combination of three δ-function with center at $(0,0)$ $(\frac{2\pi}{d},0)$ $(-\frac{2\pi}{d},0)$

$FT[f(x,y)]$ is a combination of Gaussion function with widths $\frac{2}{w}$ or $\frac{2}{l}$, and some of the functions are modulated by frequency $\frac{k}{2}$ or $\frac{k}{5}$ because of factor $e^{-i(\Delta_i^x\alpha + \Delta_i^y\beta)}$

Since $k > l > w \gg d \Rightarrow \frac{2\pi}{d} \gg \frac{2}{w}, \frac{2}{l}$. ✓

$\delta(\alpha - \frac{2\pi}{d})$ and $\delta(\alpha + \frac{2\pi}{d})$ are outside the main part of Gaussion function



✓ 2

$\Rightarrow$ So we can block the two δ-function without unpleasent effect on recover image.

The center δ-function $\delta(\alpha)\delta(\beta)$ can be blocked by some point overlap, but the image will be distorted a little bit.

the aperture function should be:

$$H\left(\frac{x\alpha}{k},\frac{x\beta}{k}\right)=H(x,y)=\begin{cases}1 & \text{otherwise}\\ 0 & \text{at points } (x,y)=(0,0),\ \left(\frac{2\pi f}{kd},0\right),\ \left(-\frac{2\pi f}{kd},0\right)\end{cases}$$

b) we can use the 4f-system

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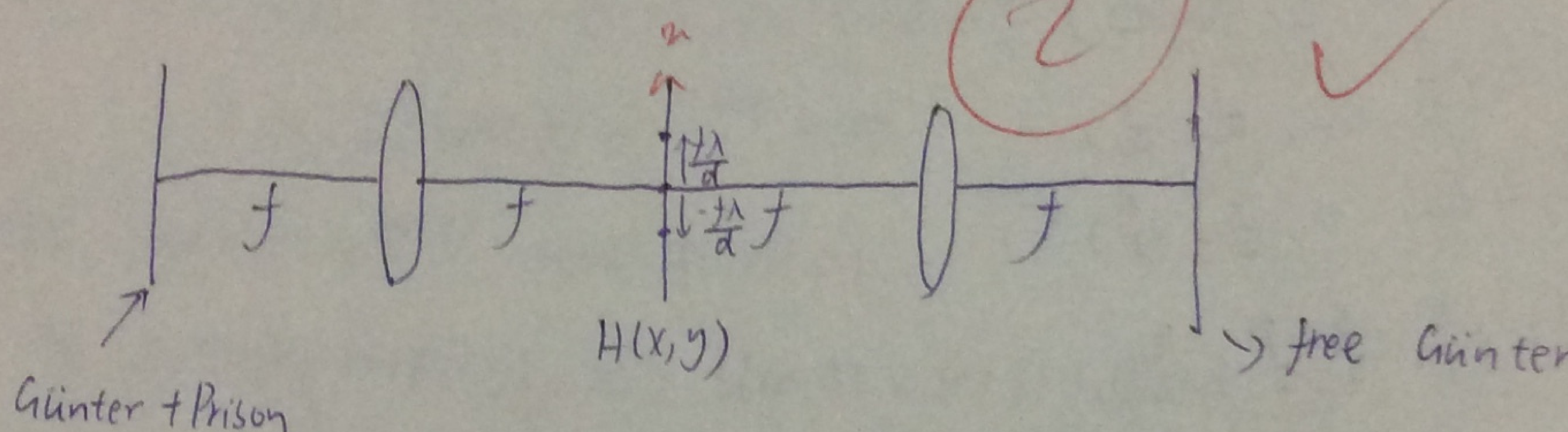

c) as discussion in part (a), the central δ-function $\delta(\alpha)\delta(\beta)$ is blocked necessarily, which will leads to blocking of center of Gaussian function. That means we will lost some information.

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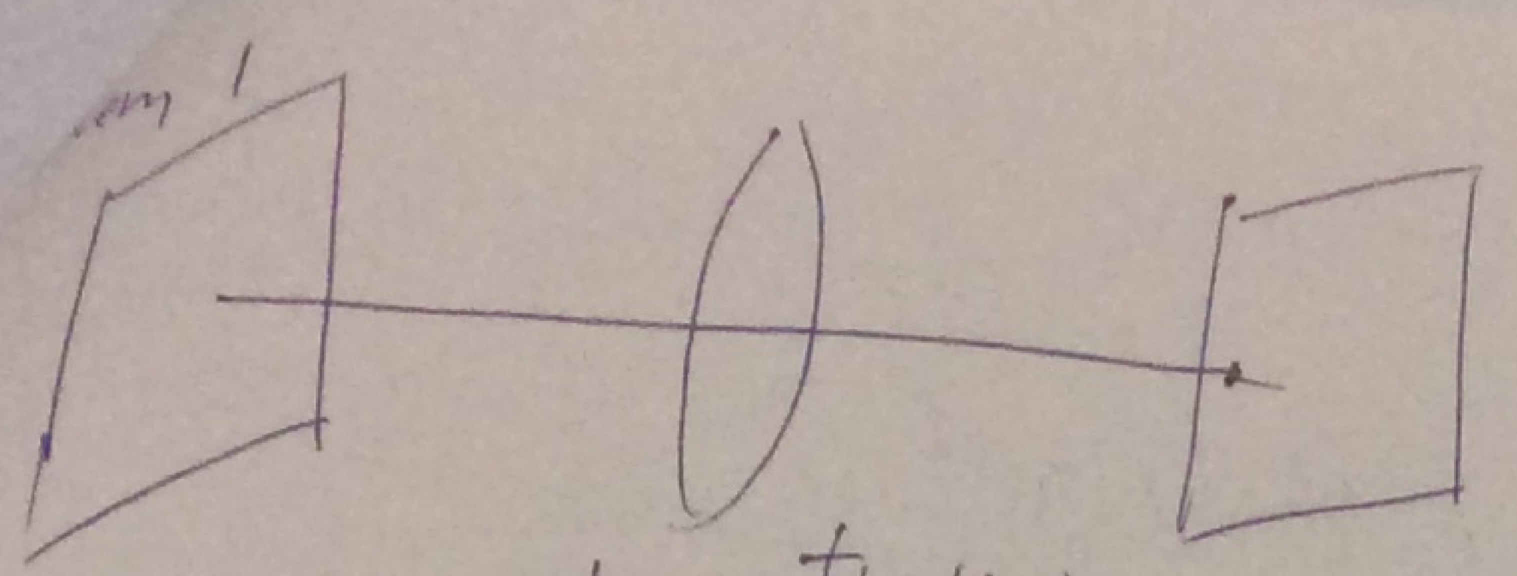
In order to improve Free Günter image, this central overlap should be as small as possible. In reality, we will not get a δ-function with infinite peak, so we can set a non-zero transparency coefficient of central overlap

$$H=\begin{cases}K & \text{at } x,y=(0,0) \text{ point}\\ 0 & \text{at } \left(\frac{2\pi f}{kd},0\right)\ \left(-\frac{2\pi f}{kd},0\right) \text{ points}\\ 1 & \text{otherwise}\end{cases}$$

— with unideal δ-function.



$u_0(x,y)$ $t_{lens}(x,y)$ $u(x,y,2f)$

$$t_{lens} = e^{-\frac{ik}{2f}(x^2+y^2)}$$

$$u_0(x,y) \xrightarrow{FT} U_0(\alpha,\beta)\, e^{ikf} e^{\frac{if}{2k}(\alpha^2+\beta^2)} * T_{lens}(\alpha,\beta)$$

$$u(x,y,2f) \xleftarrow{FT^{-1}}$$

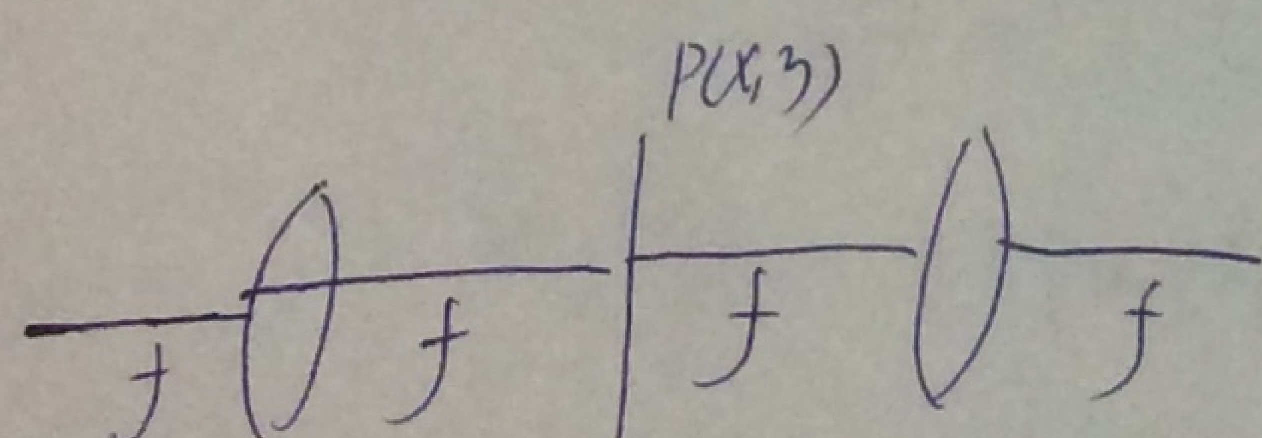
$$T_{lens} = \frac{-i\lambda f}{(2\pi)^2} e^{\frac{if}{2k}(\alpha^2+\beta^2)}$$

$$U(\alpha,\beta,2f) = -\frac{i\lambda f}{(2\pi)^2} e^{2ikf} \iint_{-\infty}^{+\infty} U_0(\alpha',\beta')\, e^{-\frac{if}{2k}(\alpha'^2+\beta'^2)} e^{\frac{if}{2k}[(\alpha-\alpha')^2+(\beta-\beta')^2]} e^{-\frac{if}{2k}(\alpha^2+\beta^2)}$$

$$= \frac{-i\lambda f}{(2\pi)^2} e^{i2kf} \iint_{-\infty}^{+\infty} U_0(\alpha',\beta')\, e^{-\frac{if}{k}(\alpha\alpha'+\beta\beta')} d\alpha' d\beta'$$

$$= \frac{-i\lambda f}{(2\pi)^2} e^{i2kf} u_0\left(\frac{-f}{k}\alpha, \frac{-f}{k}\beta\right)$$

$$u(x,y,2f) = -i\frac{(2\pi)^2}{\lambda f} e^{i2kf} U_0\left(\frac{k}{f}x, \frac{k}{f}y\right)$$



$u_-(x,y,2f)$ $u_+(x,y,2f) = u_-(x,y,2f) \times P(x,y)$

$$u(x,y,4f) = \iint_{-\infty}^{\infty} u_+(x',y',2f)\, e^{-i\frac{k}{f}(xx'+yy')} dx'dy' \propto \iint_{-\infty}^{\infty} U_0\left(\frac{kx'}{f}, \frac{ky'}{f}\right) P(x',y')\, e^{-i\frac{k}{f}(xx'+yy')} dx'dy'$$

$$\left.\begin{array}{l} x \to -x \\ y \to -y \\ \frac{kx'}{f} \to \alpha \\ \frac{ky'}{f} \to \beta \end{array}\right\} \Rightarrow u(-x,-y,4f) \propto \iint U_0(\alpha,\beta)\, P\left(\frac{f\alpha}{k}, \frac{f\beta}{k}\right) e^{+i(\alpha x+\beta y)} d\alpha d\beta$$

$$= \iint u_0(x',y')\, h(x-x', y-y')\, dx'dy'$$

$$h(x,y) = \iint P\left(\frac{\alpha f}{k}, \frac{f\beta}{k}\right) e^{i(\alpha x+\beta y)} d\alpha d\beta = \hat{P}\left[\frac{-kx}{f}, \frac{-ky}{f}\right]$$

Problem 2

$$\tilde{u}_0(x,y) = e^{i\varphi(x,y)} \approx 1 + i\varphi(x,y) \qquad \varphi(x,y) \ll 1, \pi$$

$$u(-x,-y,4f) \propto FT^{-1}\left[U_0(\alpha,\beta)\, H\left(\tfrac{f\alpha}{k}, \tfrac{f\beta}{k}\right)\right](x,y)$$

$$U_0(\alpha,\beta) \propto \delta(\alpha)\delta(\beta) + i\hat{\varphi}(\alpha,\beta)$$

$$\cancel{\iint_{-\infty}^{\infty} U_0(\alpha,\beta)\, H\left(\tfrac{f\alpha}{k}, \tfrac{f\beta}{k}\right) e^{-i(\alpha x+\beta y)}\, d\alpha\, d\beta = \begin{cases} \delta(\alpha,\beta), & \alpha,\beta = 0 \\ -\hat{\varphi}(\alpha,\beta), & \text{otherwise} \end{cases}}$$

$$U_0(\alpha,\beta)\, H\left(\tfrac{f\alpha}{k}, \tfrac{f\beta}{k}\right) = \begin{cases} \delta(\alpha)\delta(\beta) & \alpha,\beta = 0 \\ -\hat{\varphi}(\alpha,\beta) & \text{otherwise} \end{cases}$$

$$FT^{-1}\left\{U_0(\alpha,\beta)\, H\left(\tfrac{f\alpha}{k}, \tfrac{f\beta}{k}\right)\right\} = 1 - \varphi(x,y)$$

$$\Rightarrow I = |u(4f)|^2 = 1 + \underbrace{\varphi^2}_{\approx 0} - 2\varphi \approx 1 - 2\varphi$$

$$I_{before} = |U_0|^2 = |1+i\varphi| \cdot |1-i\varphi| = 1 - \varphi^2$$