Task 1: Pulse compression (3+3+2+2 points)

A transform-limited Gaussian pulse given by

$$U(t) = A_0 \exp\left(-\frac{t^2}{\tau_0^2}\right) ,$$

can be compressed by transmitting it first through a quadratic phase modulator (QPM) and then through a chirp filter.

a) Using the QPM the pulse $U_1(t)$ is multiplied by a quadratic phase factor $\exp(i\zeta t^2)$ resulting in the chirping of the pulse. Consequently, the chirped pulse is defined by the chirp parameter C_1:

$$U_1(t) = A_{10} \exp\left(-(1 - iC_1)\frac{t^2}{\tau_1^2}\right) .$$

Find the chirp parameter C_1, the amplitude A_{10}, the pulse duration τ_1 and the spectral width ω_1 of the pulse.

b) In order to make the chirped pulse transform-limited it is sent through a chirp filter with the envelope transfer function:

$$H_e(\omega) = \exp\left(\frac{-ib\omega^2}{4}\right) .$$

The resulting pulse is given as:

$$U_2(t) = A_{20} \exp\left(-(1 - iC_2)\frac{t^2}{\tau_2^2}\right) .$$

Find the value of b for the pulse $U_2(t)$ to become transform-limited. *Hints: The condition for the pulse to be transform-limited is $C_2 = 0$. Also remember that in the Fourier domain $U_2(\omega) = H_e(\omega)U_1(\omega)$.*

c) Obtain the pulse duration τ_2, the spectral width ω_2, and the amplitude A_{20} of the pulse after the system.

d) Compare the QPM and the chirp filter. Shortly explain how they influence the pulse.

Solution Task 1:

a) The original (transform-limited) pulse is:

$$U(t) = A_0 \exp\left(-\frac{t^2}{\tau_0^2}\right) ,$$

which after the QPM becomes:

$$U_1(t) = U(t) \cdot \exp(i\zeta t^2) = A_0 \exp\left(-\frac{t^2}{\tau_0^2} + i\zeta t^2\right) = A_0 \exp\left(-(1 - i\zeta\tau_0^2)\frac{t^2}{\tau_0^2}\right) \stackrel{!}{=} A_{10} \exp\left(-(1 - iC_1)\frac{t^2}{\tau_1^2}\right)$$

By comparing the last equations, we find:

- $C_1 = \zeta\tau_0^2$, the chirp parameter. The initial transform-limited pulse is now chirped pulse. We will remove it in part (b) with a chirp filter.
- $\tau_1 = \tau_0$, the QPM does not alter the temporal width of the pulse.
- $A_{10} = A_0$, the pulse amplitude remains the same.

To find the spectral width ω_1 we go to Fourier domain:

$$\begin{aligned} U_1(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} A_{10} \exp(-(1 - iC_1)t^2/\tau_1^2) \exp(i\omega t)\, \mathrm{d}t \\ &= \frac{A_{10}\tau_1}{2\sqrt{\pi}} \frac{1}{\sqrt{1 - iC_1}} \exp\left(-\frac{\omega^2\tau_1^2}{4(1 - iC_1)}\right) \end{aligned}$$

To find $1/e$ of the maximum field envelope, we need to use the exponentially decaying part (real part in exponent). Thus ω:

$$\omega = \pm\sqrt{\frac{4(1 + C_1^2)}{\tau_1^2}} = \pm\frac{2}{\tau_1}\sqrt{1 + C_1^2} \quad \Longrightarrow \quad \omega_1 = \frac{2}{\tau_1}\sqrt{1 + C_1^2} = \omega_0\sqrt{1 + C_1^2}.$$

b) The chirp filter must be evaluated in the Fourier domain.

$$
\begin{aligned}
U_2(\omega) &= H_e(\omega)U_1(\omega) \\
&= \frac{A_{10}\tau_1}{2\sqrt{\pi}} \frac{1}{\sqrt{1-iC_1}} \exp\left(-\frac{\omega^2}{4} \frac{\tau_1^2 + ib(1-iC_1)}{1-iC_1}\right)
\end{aligned}
$$

If we calculate $U_2(t)$ by Fourier back transforming:

$$
\begin{aligned}
U_2(t) &= \int_{-\infty}^{\infty} \frac{A_{10}\tau_1}{2\sqrt{\pi}} \frac{1}{\sqrt{1-iC_1}} \exp\left(-\frac{\omega^2}{4} \frac{\tau_1^2 + ib(1-iC_1)}{1-iC_1}\right) \exp(-i\omega t)\, \mathrm{d}\omega \\
&= \frac{A_{10}\tau_1}{\sqrt{\tau_1^2 + ib(1-iC_1)}} \exp\left(-t^2 \frac{1-iC_1}{\tau_1^2 + ib(1-iC_1)}\right) \\
&\stackrel{!}{=} A_{20} \exp\left(-t^2 \frac{1-iC_2}{\tau_2^2}\right)
\end{aligned}
$$

Now we need to find the chirp parameter C_2 from $\exp\left(-t^2 \frac{1-iC_1}{\tau_1^2+ib(1-iC_1)}\right)$. We want to have a chirp parameter of zero so $C_2 = (b + C_1\tau_1^2 + bC_1^2)/\tau_1^2 \stackrel{!}{=} 0$.

$$
b(1+C_1^2) = -C_1\tau_1^2 \implies b = -C_1\tau_1^2/(1+C_1^2)
$$

c) In this task, in order to obtain the amplitude A_{20}, the temporal width τ_2 and the spectral width ω_2 of the pulse, we use our previous findings, e.g., the pulse amplitude can be found from b:

$$
A_{20} = \frac{A_{10}\tau_1}{\sqrt{\tau_1^2 + ib(1-iC_1)}} = A_{10}\sqrt{1+iC_1}\ .
$$

Pulse width can also be found from b:

$$
\begin{aligned}
\tau_2^2 &= \frac{(\tau_1^2 + bC_1)^2 + b^2}{\tau_1 2^2} = \frac{\tau_1^2}{1+C_1^2}, \\
\tau_2 &= \frac{\tau_1}{\sqrt{1+C_1^2}} = \frac{\tau_1}{\sqrt{1+(\zeta\tau_1^2)^2}}\ ,
\end{aligned}
$$

while the pulse spectral width ω_2 is not changed when it is transmitted through the chirp filter, thus:

$$
\omega_2 = \omega_1 = \omega_0\sqrt{1+C_1^2},
$$

Since $C_0 > 0$, after the QPM and the chirp filter the initial pulse is spectrally broadened and temporally compressed.

d) The comparison of the QPM and the chirp filters can be done via examining their influence on the pulse:

- Chirp: Both the chirp filter and the QPM introduces chirp the pulse.
- Temporal width: The QPM preserves the temporal width of the pulse while the chirp filter alters it.
- Spectral width: The QPM changes the spectral width of the pulse while the Chirp filter preserves it.