Task 1 : 1.5

a) Solution: 1

$$F\{f(t)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\,dt$$

$$=\frac{1}{2\pi}\int_{0}^{\infty} A e^{-\gamma t}\cos(\omega_0 t)e^{i\omega t}\,dt$$

$$=\frac{A}{2\pi}\boxed{\int_0^{\infty} e^{(-\gamma+i\omega)t}\cos(\omega_0 t)\,dt}$$

$$=\frac{A}{2\pi}\cdot\frac{1}{\omega_0}\int_0^{\infty} e^{(-\gamma+i\omega)t}\,d\sin(\omega_0 t)$$

$$=\frac{A}{2\pi\omega_0}\left(0-(-\gamma+i\omega)\int_0^{\infty}\sin(\omega_0 t)e^{(-\gamma+i\omega)t}\,dt\right)$$

$$=-\frac{A(-\gamma+i\omega)}{2\pi\omega_0}\cdot\left(-\frac{1}{\omega_0}\int_0^{\infty} e^{(-\gamma+i\omega)t}\,d\cos(\omega_0 t)\right)$$

$$=\frac{A(-\gamma+i\omega)}{2\pi\omega_0^2}\left(-1-(-\gamma+i\omega)\boxed{\int_0^{\infty} e^{(-\gamma+i\omega)t}\cos(\omega_0 t)\,dt}\right)$$ -0.5

$$\therefore \int_0^{\infty} e^{(-\gamma+i\omega)t}\cos(\omega_0 t)\,dt=\frac{A(-\gamma+i\omega)}{-2\pi\omega_0^2+\cancel{A}(-\gamma+i\omega)^2}$$

$2\pi\cdot(-i\gamma-\omega)^2$

-0.5

b) Solution: 0.5

$$F\{f(t)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty} A e^{-\frac{t^2}{2t_0^2}} e^{i\omega t}\,dt$$

if you use $\frac{t}{i\sqrt{2}t_0}$, you will get $+\frac{t^2}{2t_0^2}$

$$I=\frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{t}{\cancel{i}\sqrt{2}t_0}-\frac{it_0}{\sqrt{2}}\omega\right)^2-\frac{t_0^2}{2}\omega^2}\,dt$$

$$I=\frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{t}{i\sqrt{2}t_0}-\frac{t_0}{\sqrt{2}}\omega_0\right)^2}\cdot e^{-\frac{t_0^2}{2}\omega^2}\,dt$$

$$I=\frac{Ae^{-\frac{t_0}{2}\omega^2}}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{t}{i\sqrt{2}t_0}-\frac{t_0}{\sqrt{2}}\omega\right)^2}\,dt$$

$dz=\frac{dt}{i\sqrt{2}t_0}\Rightarrow dt=i\sqrt{2}t_0\,dz$

suppose $z=\frac{t}{i\sqrt{2}t_0}-\frac{t_0}{\sqrt{2}}\omega$

$$I=\frac{Ae^{-\frac{t_0}{2}\omega^2}}{2\pi}\int_{-\infty}^{\infty} e^{-z^2}\cdot\frac{1}{i\sqrt{2}t_0}\,dz$$

$$I=-\frac{iAe^{-\frac{t_0}{2}\omega^2}}{2\sqrt{2}\pi t_0}\cdot 2\int_0^{\infty} e^{-z^2}\,dz$$

$$I=-\frac{iAe^{-\frac{t_0}{2}\omega^2}}{2\sqrt{2}\pi t_0}\cdot 2\int_0^{\infty} z^0\cdot e^{-z^2}\,dz$$

$\because$ Gamma function $\Gamma(x)=2\int_0^{\infty} z^{2x-1}e^{-z^2}\,dz$

when $x=\frac{1}{2}$ $\quad\Gamma(\frac{1}{2})=\sqrt{\pi}$

$$\therefore I=-\frac{iAe^{-\frac{t_0}{2}\omega^2}}{2\sqrt{2\pi}\,t_0}$$ ×

Task 2 2.5

(a) $f(t-t_0)$

Solution:

$\because$ $\tilde{f}(\omega)$ is the frequency representation of $f(t)$

$$\therefore \tilde{f}(\omega)=\int_{-\infty}^{\infty}\frac{1}{2\pi}f(t)e^{i\omega t}\,dt$$

Suppose $F(\omega)$ is the frequency representation of $f(t-t_0)$

$$\therefore F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t-t_0)e^{i\omega t}\,dt$$ 2

$$=\frac{e^{i\omega t_0}}{2\pi}\int_{-\infty}^{\infty} f(t-t_0)e^{i\omega(t-t_0)}\,d(t-t_0)$$

$$=e^{i\omega t_0}\tilde{f}(\omega)$$

b) $\frac{d}{dt}f(t)$

Solution:

$\because$ $\tilde{f}(\omega)$ is the frequency representation of $f(t)$

$$\therefore \cancel{\tilde{f}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)}\qquad \tilde{f}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt$$ 0.5

$$f(t)=\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-i\omega t}\,d\omega$$

$$\therefore \frac{d}{dt}f(t)=\int_{-\infty}^{\infty}\frac{d}{dt}\tilde{f}(\omega)e^{-i\omega t}\,d\omega$$ No!

$$\frac{d}{dt}f(t)=-i\omega\int_{-\infty}^{\infty}\tilde{f}(\omega)e^{-i\omega t}\,d\omega=F^{-1}\{-i\omega\tilde{f}(\omega)\}$$

$$\therefore F\left\{\frac{d}{dt}f(t)\right\}=-i\omega\tilde{f}(\omega)$$

You can not take variable of integration out of integral

Task 3 5

a)

Solution:

$\forall \varepsilon > 0$

$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt$$

$$= \int_{-\infty}^{0-\varepsilon} \delta(t) f(t)\, dt + \int_{0-\varepsilon}^{0+\varepsilon} \delta(t) f(t)\, dt + \int_{0+\varepsilon}^{+\infty} \delta(t) f(t)\, dt$$

$$= \int_{0-\varepsilon}^{0+\varepsilon} \delta(t) f(t)\, dt = \lim_{\varepsilon \to 0} \int_{0-\varepsilon}^{0+\varepsilon} \delta(t) f(t)\, dt$$

$$\therefore \int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0) \int_{0-\varepsilon}^{0+\varepsilon} \delta(t)\, dt$$

$$= \cancel{f(0)}\ f(0)$$

1

b)

Solution:

$\forall \varepsilon > 0$

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = \int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t-t_0) f(t)\, dt$$

$$= \lim_{\varepsilon \to 0} \int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t-t_0) f(t)\, dt$$

$$= f(t_0) \int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t-t_0)\, dt = f(t_0)$$

1

c)

Solution:

$$\because \int_{-\infty}^{\infty} \delta(t)\, dt = 1 \quad \int_{-\infty}^{\infty} \delta(at)\, dat = 1$$

$$\therefore \int_{-\infty}^{\infty} \delta(at)\, dat = \int_{-\infty}^{\infty} \delta(t)\, dt$$

$$|a| \int_{-\infty}^{\infty} \delta(at)\, dt = \int_{-\infty}^{\infty} \delta(t)\, dt$$

$$\therefore |a|\, \delta(at) = \delta(t)$$

$$\delta(at) = \frac{\delta(t)}{|a|}$$

1

$$\therefore \int_{-\infty}^{\infty} \delta(at) f(t)\, dt = \frac{1}{|a|} \int_{-\infty}^{\infty} \delta(t) f(t)\, dt = \frac{f(0)}{|a|}$$

e) Solution:

$$F\{\delta(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(t) e^{i\omega t}\, dt = \frac{1}{2\pi}$$

1

d)

Solution: suppose when $t = t_i$, $g(t_i) = 0$

$$\because \int_{-\infty}^{\infty} \delta(g(t))\, dg(t) = \int_{-\infty}^{\infty} \delta(t)\, dt$$

$\forall \varepsilon > 0$

$$\therefore \int_{g(t_i-\varepsilon)}^{g(t_i+\varepsilon)} \delta(g(t))\, dg(t) = \int_{t_i-\varepsilon}^{t_i+\varepsilon} \delta(t-t_i)\, dt$$

$$\int_{t_i-\varepsilon}^{t_i+\varepsilon} g'(t)\, \delta(g(t))\, dt = \int_{t_i-\varepsilon}^{t_i+\varepsilon} \delta(t-t_i)\, dt$$

$$\therefore \delta(g(t)) = \boxed{\cancel{g'(t)}}\ \sum_i \frac{\delta(t-t_i)}{|g'(t_i)|}$$

1

$$\therefore \int_{-\infty}^{\infty} \delta(g(t)) f(t)\, dt = \sum_i \frac{f(t_i)}{|g'(t_i)|}$$

5

Task 4:

a)

Proof:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}[f\otimes g](t)e^{i\omega t}dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\tau)g(t-\tau)d\tau e^{i\omega t}dt \checkmark$$

4

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\tau)g(t-\tau)d\tau e^{i\omega(t-\tau)}e^{i\omega\tau}dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\tau)e^{i\omega\tau}d\tau\int_{-\infty}^{\infty}g(t-\tau)e^{i\omega(t-\tau)}d(t-\tau) \checkmark$$

$$=2\pi F\{f\}F\{g\} \checkmark$$

b)

Solution:

$$F\{\Pi(t)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Pi(t)e^{i\omega t}dt$$

$$=\frac{1}{2\pi}\int_{-\frac{t_0}{2}}^{\frac{t_0}{2}}e^{i\omega t}dt \checkmark$$

1

$$=\frac{e^{i\omega\frac{t_0}{2}}-e^{-i\omega\frac{t_0}{2}}}{2\pi i\omega}=\frac{t_0\sin(\omega\frac{t_0}{2})}{2\pi\omega\frac{t_0}{2}}=\frac{t_0}{2\pi}Sa\left(\frac{\omega t_0}{2}\right) \checkmark$$

$$F\{\cos(\omega_0 t)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(\omega_0 t)e^{i\omega t}dt$$

$$=\frac{1}{4\pi}\int_{-\infty}^{\infty}\left(e^{i(\omega+\omega_0)t}+e^{i(\omega-\omega_0)t}\right)dt$$

$$\because \delta(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega t}dt$$

$$\therefore F\{\cos(\omega_0 t)\}=\frac{1}{2}[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)] \checkmark$$

According to the convolution theorem

$$F\{f\}\otimes F\{g\}=2\pi F\{f\cdot g\}$$

$$F\{\Pi(t)\cdot\cos(\omega_0 t)\}=\frac{1}{2\pi}F\{f\}\otimes F\{g\}=\frac{t_0}{(2\pi)^2}Sa\left(\frac{\omega t_0}{2}\right)\otimes\frac{1}{2}[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$$

$$=\cancel{\frac{t_0^2}{16\pi^2}}\left[Sa\left(\frac{t_0(\omega+\omega_0)}{2}\right)+Sa\left(\frac{t_0(\omega-\omega_0)}{2}\right)\right]$$

$$\frac{t_0}{8\pi^2}$$