2020/21

problem 1

ⓐ in the time domain

$$\nabla \cdot H(r,t) = 0 \qquad \nabla \cdot D(r,t) = \rho(r,t)$$

$$\nabla \times H(r,t) = j(r,t) + \frac{\partial D(r,t)}{\partial t}, \qquad \nabla \times E(r,t) = -\frac{\partial B(r,t)}{\partial t}$$

in the Freq domain

$$\nabla \cdot \bar{H}(r,\omega) = 0, \qquad \nabla \cdot \bar{D}(r,\omega) = \bar{\rho}(r,\omega)$$

$$\nabla \times \bar{H}(r,\omega) = \bar{J}(r,\omega) - i\omega \bar{D}(r,\omega), \qquad \nabla \times \bar{E}(r,\omega) = i\omega \bar{B}(r,\omega)$$

ⓑ

$$\bar{D}(r,\omega) = \epsilon_0 \bar{E}(r,\omega) + \bar{P}(r,\omega)$$

but $\bar{P}(r,\omega) = \epsilon_0 \chi(\omega) \bar{E}(r,\omega)$

$$\Rightarrow \bar{D}(r,\omega) = \epsilon_0 \bar{E}(r,\omega) + \epsilon_0 \chi(\omega) \bar{E}(r,\omega)$$

$$= \epsilon_0 (1 + \chi(\omega)) \bar{E}(r,\omega)$$

$$= \epsilon_0 \epsilon(\omega) \bar{E}(r,\omega)$$

ⓒ $\bar{\rho}(r,\omega) = 0, \quad \bar{j}(r,\omega) = \sigma(\omega) \bar{E}(r,\omega)$

$$\nabla \times \nabla \times \bar{H}(r,\omega) = \nabla \times \sigma(\omega) \bar{E}(r,\omega) - i\omega \nabla \times (\epsilon_0 \epsilon(\omega) \bar{E}(r,\omega))$$

$$\nabla(\underbrace{\nabla \cdot \bar{H}(r,\omega)}_{0}) - \Delta \bar{H}(r,\omega) = \nabla \times \sigma(\omega) \bar{E}(r,\omega)) - i\omega \epsilon_0 \epsilon(\omega) (i\omega \mu_0 \bar{H}(r,\omega))$$

$$\Delta \bar{H}(r,\omega) + \frac{\omega^2}{c^2} \epsilon(\omega) H = -i\sigma(\omega) \omega \mu_0 \bar{H}(r,\omega)$$

(d)

$$\nabla \cdot (\epsilon_0 \epsilon(\omega) \mathcal{E}(r,t) = \rho$$

$$\Rightarrow \frac{\partial}{\partial t} \nabla \cdot \epsilon_0 \epsilon(\omega) \mathcal{E}(r,t) = \frac{\partial \rho}{\partial t}$$

$$\Rightarrow \nabla \cdot \epsilon_0 \epsilon(\omega) \frac{\partial \mathcal{E}(r,t)}{\partial t} = \frac{\partial \rho}{\partial t}$$

but $\frac{\partial \mathcal{E}}{\partial t}(r,t) = \frac{\nabla \times H(r,t) - j(r,t)}{\epsilon_0 \epsilon \omega}$

$$\Rightarrow \nabla \cdot \epsilon_0 \epsilon(\omega) \frac{\partial \mathcal{E}(r,t}{\partial t} = \nabla \cdot \left[\epsilon_0 \epsilon(\omega) \frac{\nabla \times H(r,t) - j(r,t)}{\epsilon_0 \cancel{\epsilon(\omega)}}\right]$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = \underbrace{\nabla \cdot (\cancel{\nabla} \times H(r))}_{0} - \nabla \cdot j(r,t)$$

$$\Rightarrow \frac{\partial \rho(r,t)}{\partial t} + \nabla \cdot j(r,t) = 0$$

a change in the charge density is done through a flux of current

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more correct solution

$\nabla \cdot \epsilon_0 \epsilon(\omega) \bar{\mathcal{E}}(r,\omega) = \bar{\rho}(r,\omega) \Rightarrow \epsilon_0 \epsilon(\omega)\ \nabla \cdot -i\omega \bar{\mathcal{E}}(r,\omega) = -i\omega\bar{\rho}(r,\omega)$

but $-i\omega \mathcal{E}(r,\omega) = \frac{\nabla \times \bar{H}(r,\omega) - \bar{j}(r,\omega)}{\epsilon_0 \epsilon(\omega)}$

Then $\epsilon_0 \cancel{\epsilon}(\omega) \frac{\underbrace{\nabla \cdot \nabla \times \bar{H}(r,\omega)}_{0} - \nabla \cdot \bar{j}(r,\omega)}{\epsilon_0 \cancel{\epsilon}(\omega)} = -i\omega \bar{\rho}(r,\omega)$

by doing $FT^{-1} \Rightarrow \frac{\partial}{\partial t} \rho(r,t) + \nabla \cdot j(r,t) = 0$

Problem 2

1)
$$H = H_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} - \omega t\right], \quad A = (x+y)\frac{k}{\sqrt{2}}$$

$$\Rightarrow \nabla \times H = \varepsilon_0 \varepsilon(\omega) \frac{\partial E}{\partial t}$$

$$\Rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_0 \sin A - \omega t \end{vmatrix} = +\frac{kH_0}{\sqrt{2}}\cos(A-\omega t)\hat{x} + \frac{kH_0}{\sqrt{2}}\cos A - \omega t\,\hat{y}$$

$$\Rightarrow E_x = \frac{-kH_0}{\omega\sqrt{2}\,\varepsilon\varepsilon_0}\sin(A-\omega t) + C$$

$$E_y = \frac{\mp kH_0}{\omega\sqrt{2}\,\varepsilon_0\varepsilon(\omega)}\sin(A-\omega t) + C$$

$$E_z = C$$

we can set $C = 0$

$$E(x,y,z,t) = \frac{H_0 k}{\varepsilon_0\varepsilon(\omega)\omega\sqrt{2}}\sin(A-\omega t)(-\hat{x} - \hat{y})$$

b)
$$\langle S(r,t)\rangle = \frac{1}{2}\mathrm{Re}[E \times H^*] = \frac{1}{2}\mathrm{Re}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_x & E_y & 0 \\ 0 & 0 & H_z \end{vmatrix}$$

c)
$$\Rightarrow E_y H_z \hat{x} - E_x H_z \hat{y} = \frac{-kH_0^2 \sin^2(A-\omega t)}{\omega\sqrt{2}\,\varepsilon_0\varepsilon(\omega)}\hat{x} \mp \frac{kH_0^2\sin^2(A-\omega t)}{\omega\sqrt{2}\,\varepsilon_0\varepsilon(\omega)}\hat{y}$$

$$\langle S(r,t)\rangle = \frac{-1}{2}\frac{kH_0\sin^2(A-\omega t)}{\omega\sqrt{2}\,\varepsilon_0\varepsilon(\omega)}(\hat{x}+\hat{y})$$

problem 3

ⓐ homogeneous, isotropic, dispersive

ⓑ $P(r,t) = \epsilon_0 \int_{-\infty}^{t} R(r, t-t')\, \mathcal{E}(r,t')\, dt'$

ⓒ $\chi(\omega) = \int_{-\infty}^{\infty} R(t)\, e^{i\omega t}\, dt$

$$= \frac{f}{2\Omega i} \int_{0}^{\infty} e^{-(\gamma - i\Omega - i\omega)t} - e^{-(\gamma + i\Omega - i\omega)t}\, dt$$

$$= \frac{f}{2\Omega i} \left[\frac{1}{(\gamma - i\Omega - i\omega)} - \frac{1}{(\gamma + i\Omega - i\omega)} \right]$$

$$= \frac{f}{2\Omega i} \left[\frac{\cancel{\gamma} + i\Omega - \cancel{i\omega} - \cancel{\gamma} + i\Omega + \cancel{i\omega}}{\gamma^2 + i\gamma\cancel{\Omega} - i\gamma\omega - i\cancel{\Omega}\gamma + \Omega^2 - \cancel{\Omega\omega} - i\gamma\omega + \cancel{\omega\Omega} - \omega^2} \right]$$

$$= \frac{f}{\cancel{2\Omega i}} \left[\frac{\cancel{2\Omega i}}{\gamma^2 + \Omega^2 - \omega^2 - 2i\gamma\omega} \right], \quad \Omega^2 = \omega_0^2 - \gamma^2$$

$$= f \left[\frac{1}{(\omega_0^2 - \omega^2) - 2i\gamma\omega} \right]$$

we see that there is a certain resonance freq at ω_0 and the damping will cause the excitation to decay $\Rightarrow$ dielectrics

(d) since $\mathcal{E}(r,t) = \mathcal{E}(r)\, e^{-i\omega_{cw} t}$

$$\bar{\mathcal{E}}(r,\omega) = \frac{\mathcal{E}(r)}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega_{cw} t}\, e^{i\omega t}\, dt = \frac{\mathcal{E}(r)}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega_{cw})t}\, dt$$

$$= \frac{\cancel{2\pi}\,\mathcal{E}(r)}{\cancel{2\pi}}\, \delta(\omega-\omega_{cw})$$

Then

$$\bar{P}(r,\omega) = \frac{\cancel{2\pi}\,\epsilon_0\, \mathcal{E}(r) f}{\cancel{2\pi}} \frac{\delta(\omega-\omega_{cw})}{(\omega_0^2-\omega^2)-2i\gamma\omega}$$

$$\Rightarrow \bar{P}(r,t) = \frac{\epsilon_0\, \mathcal{E}(r) f}{\cancel{(2\pi)^2}} \int_{-\infty}^{\infty} \frac{\delta(\omega-\omega_{cw})}{(\omega_0^2-\omega^2)-2i\gamma\omega}\, e^{-i\omega t}\, d\omega$$

$$= \frac{\epsilon_0\, \mathcal{E}(r) f}{\cancel{4\pi^2}} \left[\frac{e^{-i\omega_{cw} t}}{(\omega_0^2-\omega_{cw}^2)-2i\gamma\omega_{cw}} \right]$$

The complex suseptibity causes the polarization to Resonate when $\omega_0^2 = \omega_{cw}^2$, also damping the polarization which represents the dielectrics

when $\gamma = 0$

$$P(r,t) = \frac{\epsilon_0\, \mathcal{E}(r) f}{\cancel{4\pi^2}} \left[\frac{e^{-i\omega_{cw} t}}{(\omega_0^2-\omega_{cw}^2)} \right]$$ this will cause

$P \to \infty$ as $\omega_0 \to \omega_{cw}$ or in other words, ionize the dipoles

(e) $P = \epsilon_0 \int_{-\infty}^{\infty} R(t-t')\, \mathcal{E}(r)\, \delta(t'-t_0)\, dt = \epsilon_0\, \mathcal{E}(r)\, R(t-t_0)$

The delta excitation cause the response to be given by the respone Function itself. (Green function response)

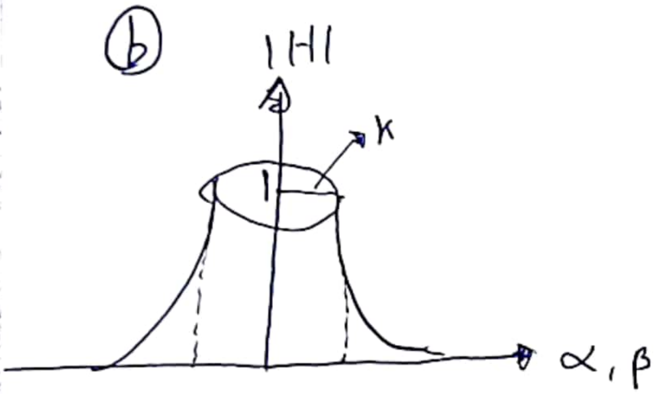
Problem 4

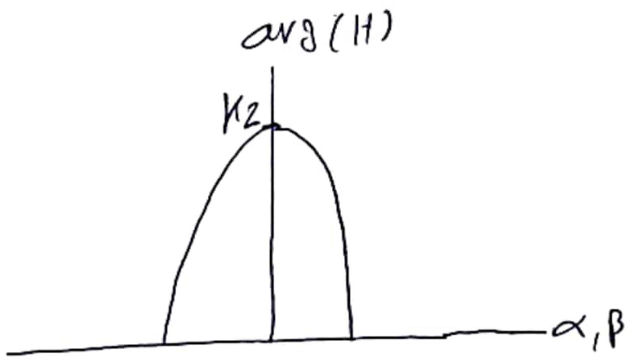
(a) $H(\alpha,\beta;z) = e^{i\gamma z}$ where $\gamma = \sqrt{K_0^2 - \alpha^2 - \beta^2}$

homogenous Reigons occur where $K_0 > \alpha^2 + \beta^2$ such that γ remains real => propagating phase, transport energy

while for $K_0 < \alpha^2 + \beta^2$ this makes γ complex => $e^{-\gamma z}$ resulting in evanescent waves causing exponential decaying in the direction of propagation, do not transport energy

(b)





(c)

$$U(x,y,z) = \iint_{-\infty}^{\infty} d\alpha\, d\beta \times U_0(\alpha,\beta) \times e^{i\gamma(\alpha,\beta)z} \times e^{i(\alpha x + \beta y)}$$

Problem 5

(a) $V_0(x,y,z=0) = A_0 e^{-\frac{x^2+y^2}{w_0^2}} e^{i\phi(x,y)} \Rightarrow A_0 e^{-\frac{x^2+y^2}{w_0^2}}$

$V(x,y) \xrightarrow{ft} V_0(\alpha,\beta) \xrightarrow[\text{using transfare function}]{\text{Propagte}} V_0(\alpha,\beta;z) \xrightarrow{FT^{-1}} V_0(x,y,z)$

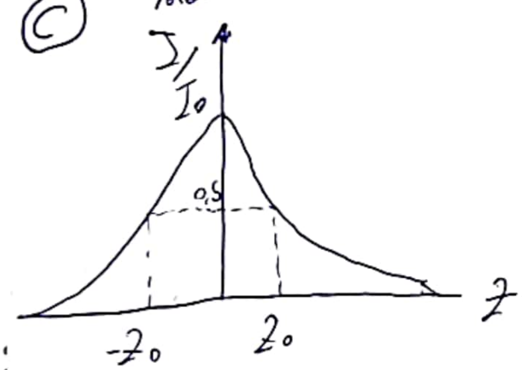
(b)

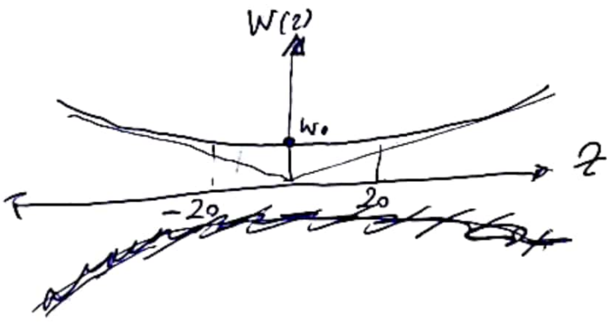
$$V(x,y,z) = A_0 \frac{1}{1+iz/z_0} e^{-\frac{x^2+y^2}{w_0^2(1+iz/z_0)}}$$

$$= A_0 \frac{1-iz/z_0}{1+(z/z_0)^2} e^{-\frac{x^2+y^2}{w_0^2(1+(z/z_0)^2)}} e^{\frac{i(x^2+y^2)z/z_0}{\underbrace{w_0^2}_{\frac{2}{k}z_0}(1+(z/z_0)^2)}}$$

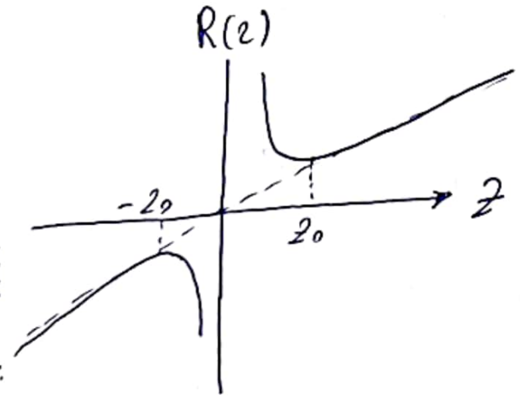
$$= A_0 \frac{1}{\sqrt{1+\left(\frac{z}{z_0}\right)^2}} e^{-\frac{x^2+y^2}{\underbrace{w_0^2(1+(z/z_0)^2)}_{W_z^2\ \text{beam waist}}}} e^{i\frac{k}{2}\frac{(x^2+y^2)}{\underbrace{z(1+(z_0/z)^2)}_{R(z)\ \text{Radius of curvature}}}} e^{i\phi(z)}$$

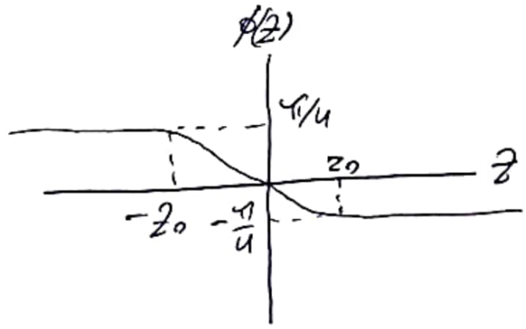
(c)





(d)





(d)

The paraxial approximation works for $w_0 \geq 10\frac{\lambda}{n}$

Then for $\lambda = 1\mu m$, $n_{air} = 1$

* $w_0 = 1\mu m \Rightarrow 1\mu m \overset{?}{\geq} 10\mu m$ is not valied
* $w_0 = 10\mu m \Rightarrow 10\mu m \geq 10\mu m$ is almost valied
* $w_0 = 1mm \Rightarrow 1mm \geq 10\mu m$ is valied
* the divergence is measured using Rayleigh length

$$\cancel{z_0 = \frac{\lambda}{\pi w_0^2} = \frac{10\mu m}{\pi\, 1\mu m^2} = \frac{10}{\pi\,\mu m} = 3,18\times10^{6} m}$$

$$\cancel{z_0 = \frac{10\mu m}{\pi \times 100\mu m^2} = \frac{1}{10\pi\,\mu m} = 3,183\times10^{3} m}$$

$$\cancel{z_0 = \frac{10\mu m}{\pi\, 1mm^2} = 3,18\ m}$$

* $z_0 = \frac{\pi w_0^2}{\lambda} = \frac{\pi(1\mu m)^2}{10\mu m} = 3,14\times10^{-7} m \Rightarrow z_B = 6,28\times10^{-7} m$
* $z_0 = \frac{\pi w_0^2}{\lambda} = \frac{\pi(10\mu m)^2}{10\mu m} = 3,14\times10^{-5} m \Rightarrow z_B = 6,28\times10^{-5} m$
* $z_0 = \frac{\pi w_0^2}{\lambda} = \frac{\pi(1mm)^2}{10\mu m} = 0,314\ m \Rightarrow z_B = 0,628\ m$

where the beam remains collimated for these distances

Problem 6

(a) $U(x, z=0) = A\cos(\alpha_0 x)$, $\alpha_0 < k_0$

$$U(\alpha) = \frac{A}{2\pi}\int_{-\infty}^{\infty} \cos(\alpha_0 x) e^{-i\alpha x} dx$$

$$= \frac{A}{2\pi}\int_{-\infty}^{\infty} \left[\frac{e^{i\alpha_0 x} + e^{-i\alpha_0 x}}{2}\right] e^{-i\alpha x} dx$$

$$= \frac{A}{4\pi}\int_{-\infty}^{\infty} e^{i(\alpha_0-\alpha)x} + e^{-i(\alpha_0+\alpha)x} dx$$

$$= \frac{A}{2}\left(\delta(\alpha-\alpha_0) + \delta(\alpha+\alpha_0)\right)$$

Then we propagte it with $H = e^{i\beta z} = e^{i\sqrt{k_0^2-\alpha^2}\, z}$

$$U(\alpha, z) = \frac{A}{2}\left[\delta(\alpha-\alpha_0) + \delta(\alpha+\alpha_0)\right] e^{i\sqrt{k_0^2-\alpha^2}\, z}$$

Then

$$U(x, z) = \frac{A}{2}\int_{-\infty}^{\infty} \delta(\alpha-\alpha_0) e^{i\sqrt{k_0^2-\alpha^2}\, z + i\alpha x} + \delta(\alpha+\alpha_0) e^{i\sqrt{k_0^2-\alpha^2}\, z + i\alpha x} d\alpha$$

$$= \frac{A}{2}\left[e^{i\sqrt{k_0^2-\alpha_0^2}\, z + i\alpha_0 x} + e^{i\sqrt{k_0^2-\alpha_0^2}\, z - i\alpha_0 x}\right]$$

$U(x, z=d) = A\, e^{i\sqrt{k_0^2-\alpha_0^2}\, d} \cos\alpha_0 x \Rightarrow$ it is a circular pattern

whose $I = A^2 \cos^2\alpha_0 x$ which have constructive and deconstructive interference pattern $\Rightarrow$ it is ~~free~~ diffraction free Beam that doesnot change ~~the~~ its amplitude distribution during propagation

(b)

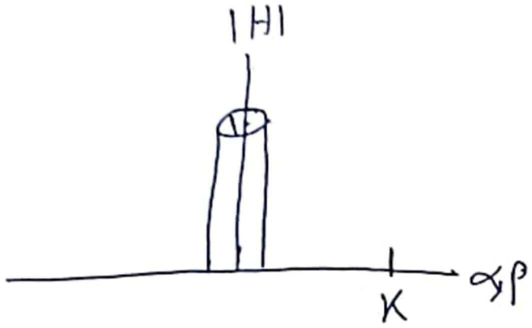
$$H_F(\alpha,\beta,;z) = e^{i\sqrt{K_0^2-\alpha^2-\beta^2}\,z}$$

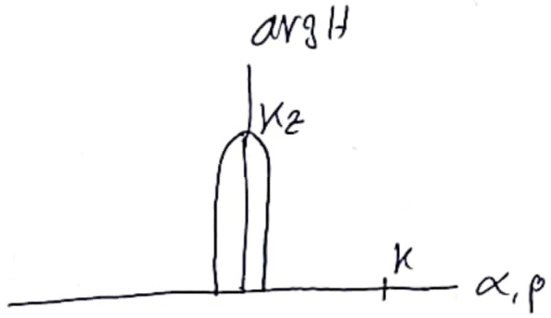
under the condition that $K_0^2 \gg \alpha^2+\beta^2$

$$\Rightarrow K_0\left(1-\frac{\alpha^2+\beta^2}{2K_0^2}\right) \Rightarrow K_0 - \frac{\alpha^2+\beta^2}{2K_0}$$

then $$H_F(\alpha,\beta,;z) = e^{iK_0 z}\, e^{-i\frac{\alpha^2+\beta^2}{2K_0}z}$$

(c)





(d)

narrow freq spectrum $\longrightarrow$ $|\Delta x|, |\Delta y| > 10\frac{\lambda}{n} \gg \frac{\lambda}{n}$

this arise from $\cancel{\alpha \ll \frac{2\pi}{K_0} \Rightarrow x \ll \frac{2\pi}{\alpha}}$

$$K_0^2 \gg \alpha^2+\beta^2$$

Then is $\frac{2\pi n}{\lambda} \gg \frac{2\pi}{a}$? $\Rightarrow$ $\frac{1}{\lambda} \gg \frac{1}{a}$

$\Rightarrow \lambda \ll a$

* $1\mu m \not\ll 100nm$
* $1\mu m \not\ll 1\mu m$
* $1\mu m \ll 10\mu m$ ✓ $\Rightarrow$ This one will be resolved