Final 18/1?

Problem 1

$$\varepsilon_0 \operatorname{div}(\varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega)) = 0$$

$\cancel{\operatorname{div}\vec{E}}$

a) $\operatorname{rot}\vec{E}(\vec{r},\omega) = i\omega\mu_0\vec{H}(\vec{r},\omega)$

$\operatorname{rot}\vec{H}(\vec{r},\omega) = -i\omega\varepsilon_0\varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega)$

$\operatorname{div}\vec{H}(\vec{r},\omega) = 0$

b) $\cancel{\varepsilon_0}\operatorname{div}(\varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega)) = 0$

$$\underbrace{\operatorname{grad}\varepsilon(\vec{r},\omega)\cdot\vec{E}(\vec{r},\omega)}_{0} + \varepsilon(\vec{r},\omega)\operatorname{div}\vec{E}(\vec{r},\omega) = 0$$

$$\operatorname{div}\vec{E}(\vec{r},\omega) = 0$$

$$\operatorname{rot}\operatorname{rot}\vec{E}(\vec{r},\omega) = i\omega\mu_0(-i\omega\varepsilon_0\varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega))$$

$$-\Delta\vec{E}(\vec{r},\omega) = \frac{\omega^2}{c^2}\varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega)$$

$$\Delta\vec{E}(\vec{r},\omega) + \frac{\omega^2}{c^2}\varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega) = 0$$

c) $-\omega^2\vec{p}(\vec{r},\omega) \cancel{+} - i\omega g\vec{p}(\vec{r},\omega) + \omega_0^2\vec{p}(\vec{r},\omega) = \varepsilon_0 f\vec{E}(\vec{r},\omega)$

$$\vec{p}(\vec{r},\omega) = \varepsilon_0\cdot\frac{f}{\omega_0^2-\omega^2-ig\omega}\vec{E}(\vec{r},\omega)$$

$$\cancel{\varepsilon(\vec{r}}\ \varepsilon(\omega) = 1+\chi(\omega) = 1+\frac{f}{\omega_0^2-\omega^2-ig\omega}$$

$$\varepsilon'(\omega) = 1+\frac{f(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+g^2\omega^2}$$

$$\varepsilon''(\omega) = \frac{fg\omega}{(\omega_0^2-\omega^2)^2+g^2\omega^2}$$

blem 2.

a) k: wave vector (m^{-1})

ω: angular frequency ($rad\cdot s^{-1}$)

$k = \frac{2\pi}{\lambda}$

b) $\operatorname{div} \vec{E}(\vec{r},t) = 0$.

$ik \cdot E_0 \exp[i(k\cdot r - \omega t)] = 0$

$\therefore k \perp E_0$

$\Delta \vec{E}(\vec{r},\omega) + \frac{\omega^2}{c^2}\varepsilon(\omega)\vec{E}(\vec{r},\omega) = 0$

$\left[-k^2 + \frac{\omega^2}{c^2}\varepsilon(\omega)\right]\vec{E}(\vec{r},\omega) = 0$

DR: $-k^2 + \frac{\omega^2}{c^2}\varepsilon(\omega) = 0$.

c) $E(\vec{r},t) = E_0 e^{-i\omega t}\cdot \begin{pmatrix} e^{iax} \\ 0 \\ e^{-bz} \end{pmatrix}$ ~~plan~~

plane of constant amplitude: $b\cdot z = const. \rightsquigarrow z = \frac{c}{b}$

plane of constant phase: $a\cdot x = const. \rightsquigarrow x = \frac{c}{a}$.

normal vector of ~~first plane~~ constant amplitude: $\hat{e}_z$ $\quad \hat{e}_z \cdot \hat{e}_x = 0$. $\therefore$ Plane 1 $\perp$ plane 2

normal vector of constant phase: $\hat{e}_x$

d) ~~rot~~ $\vec{H}(\vec{r},\omega) = -\frac{i}{\omega\mu_0}\nabla\times\vec{E}(\vec{r},\omega) = \frac{1}{\omega\mu_0} k \times \vec{E}(\vec{r},\omega)$

$\langle S(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}[E\times H^*] = \frac{1}{2}\mathrm{Re}\left[E\times \frac{k^*}{\omega\mu_0}\times \vec{E}^*(\vec{r},\omega)\right]$

Because lossless, $k^* = k$.

$\therefore \langle S(\vec{r},t)\rangle = \frac{\vec{k}}{2\omega\mu_0}|E_0|^2$

blem 3.

a). 1° $\alpha^2+\beta^2 \ll k^2$, $N_F = \frac{a}{\lambda} \cdot \frac{a}{z_B} \lesssim 10$

2° $\alpha^2+\beta^2 \ll k^2$, $N_F = \frac{a^2}{\lambda z_B} \lesssim 0.1$

3° $N_F = \frac{a^2}{\lambda z_B} \leq 0.1$?

b) $u_{FR}(x, y, z=z_B) \sim U_0\left(\frac{kx}{z_B}, \frac{ky}{z_B}\right)$

c) According to the shift theorem,

$f(x-d) \longrightarrow F(\alpha) \cdot \exp(-id\alpha)$.

Single aperture: $u_{FR}(x,z) \sim \text{sinc}\left(\frac{kx}{z_B} \cdot \frac{b}{2}\right)$.

Periodic aperture: $u_{FR}(x,z) \sim \text{sinc}\left(\frac{kx}{z_B} \cdot \frac{b}{2}\right)\left[1 + e^{-i\frac{kx}{z_B}d} + e^{-i\frac{kx}{z_B}2d} + \cdots + e^{-i\frac{kx}{z_B}Nd}\right]$

$$u_{FR}(x,z) \sim \text{sinc}\left(\frac{kx}{z_B} \cdot \frac{b}{2}\right) \cdot \frac{\sin \frac{Nkx}{2z_B} d}{\sin \frac{kxd}{2z_B}} \cdot e^{i(1-N)\frac{kxd}{2z_B}}$$

N: defines the ~~position~~ order of maximum intensity

b: defines the global width

d: defines the distance between maximum intensity

d). $T(\alpha) = \frac{1}{2\pi}\int_{-\frac{b}{2}}^{+\frac{b}{2}} \left(\frac{1}{2} + \frac{x}{b}\right) \exp(-i\alpha x)\, dx$

$= \frac{1}{2\pi}\left[\frac{1}{2}\delta(\alpha) + \frac{1}{b}\left(-\frac{1}{i\alpha}\right)\left[x e^{-i\alpha x}\Big|_{-\frac{b}{2}}^{\frac{b}{2}} - \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{-i\alpha x} dx\right]\right]$

$= \frac{1}{2\pi}\left[\frac{1}{2}\delta(\alpha) + \frac{1}{b}\left(\frac{i}{\alpha}\right)\left(b\cos\left(\frac{\alpha b}{2}\right) - b\,\text{sinc}\left(\frac{\alpha b}{2}\right)\right)\right]$

$= \frac{1}{4\pi}\delta(\alpha) + \frac{i}{2\pi\alpha}\cos\left(\frac{\alpha b}{2}\right) - \frac{i}{2\pi\alpha}\text{sinc}\left(\frac{\alpha b}{2}\right)$

$$u_{FR} = \frac{1}{4\pi}\delta\left(\frac{kx}{z_B}\right) + \frac{i}{2\pi\alpha}\left[\cos\left(\frac{kxb}{2z_B}\right) - \text{sinc}\left(\frac{kxb}{2z_B}\right)\right]$$

Problem 4

a) $v_{ph} = \frac{w_0}{k_0}$ $\quad v_{gr} = \left(\frac{\partial k}{\partial w}\right)^{-1}$

v_{ph}: the velocity of phase fronts

v_{gr}: velocity of the centre of the pulse.

b) $n(w) = A + \frac{B}{\left(\frac{2\pi c}{w}\right)^2} = A + \frac{B w^2}{4\pi^2 c^2}$

$$\frac{1}{v_{gr}} = \frac{\partial k}{\partial w} = \frac{\partial\left[\frac{w}{c} n(w)\right]}{\partial w} = \frac{1}{c}\left[n(w_0) + \frac{\partial n(w)}{\partial w}\Big|_{w_0} \cdot w_0\right].$$

$$v_{gr} = \frac{c}{n(w_0) + \frac{\partial n(w)}{\partial w}\Big|_{w_0} \cdot w_0} = \frac{c}{n_g(w)} \quad w_0 = \frac{2\pi c}{\lambda_0}$$

$$n_g(w) = \left(A + \frac{B w_0^2}{4\pi^2 c^2}\right) + w_0 \cdot \left(\frac{B}{4\pi^2 c^2} \cdot 2 w_0\right).$$

$$= A + \frac{3 B w_0^2}{4\pi^2 c^2} \curvearrowright v_{gr} = \frac{c}{A + \frac{3 B w_0^2}{4\pi^2 c^2}} = \frac{c}{A + \frac{3B}{\lambda_0^2}}$$

$$t = \frac{L}{v_{gr}} = \frac{L}{c}\left(A + \frac{3 B w_0^2}{4\pi^2 c^2}\right)$$

$$= \frac{L}{c} \cdot \left(A + \frac{3B}{\lambda_0^2}\right).$$

c). $k(w) = k(w_0) + \frac{1}{v_g}(w - w_0) + \frac{D}{2}(w - w_0)^2$

$$H_{Fp}(\alpha, \beta, w; z) = \exp(i k_0 z) \cdot \exp\left(-i \frac{\alpha^2 + \beta^2}{2 k_0} z\right) \cdot \exp\left(\frac{i}{v_g}(w - w_0) z\right)$$

$$\times \exp\left(i \frac{D}{2}(w - w_0)^2 z\right)$$

$$\bar{H}_{Fp}(\alpha, \beta, \bar{w}; z) = \cancel{\exp(i k_0 z - w_0 t)}\, \exp\left(-i \frac{\alpha^2 + \beta^2}{2 k_0} z\right) \cdot \exp\left(\frac{i}{v_g} \bar{w} z\right) \exp\left(i \frac{D}{2} \bar{w} z\right)$$

$$\hat{H}_{Fp} = \bar{H}_{Fp} \cdot \exp(i(k_0 z - w_0 t)),$$

$$\tilde{H}_{Fp}(\alpha, \beta, \bar{w}; z) = \exp\left(-i \frac{\alpha^2 + \beta^2}{2 k_0} z\right) \exp\left(i \frac{D}{2} \bar{w} z\right).$$

$$v(x, y, t; z) = \iiint_{-\infty}^{\infty} V_0(\alpha, \beta, \bar{w}) \cdot \hat{H}_{Fp} \cdot \exp\left(-i \bar{w}\left(t - \frac{z}{v_g}\right)\right) d\bar{w}\, d\alpha\, d\beta$$

blem 5

a) $R_2 = R(d) = d + \frac{z_0^2}{d} = 2d.$

$z_0 = d.$

$z_0 = \frac{\pi W_1^2}{\lambda} \rightsquigarrow W_1 = \sqrt{\frac{\lambda}{\pi} d}$

$W(z) = W_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2}$

$w(d) = W_0 \sqrt{1 + \left(\frac{d}{d}\right)^2} = \sqrt{2} W_0 = \sqrt{\frac{2\lambda d}{\pi}}.$

$W_1 = \sqrt{\frac{\lambda d}{\pi}}, \quad W_2 = \sqrt{\frac{2\lambda d}{\pi}}$

b) $g_1 g_2 = \left(1 + \frac{d}{R_1}\right)\left(1 + \frac{d}{R_2}\right) = 0$. unstable.

c). set $q_1 = -iz_0$

$q_3 = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} q_1 = \begin{pmatrix} 0 & f \\ -\frac{1}{f} & 1 \end{pmatrix} q_1$

$= \frac{f}{\frac{iz_0}{f} + 1} = \frac{f^2}{f + iz_0}$

$\cancel{\frac{1}{q_3} =}$

$\frac{1}{q_3} = \frac{1}{f} + i\frac{z_0}{f^2} \qquad \mathrm{Im}\left(\frac{1}{q_3}\right) = \frac{z_0}{f^2} = \frac{\lambda}{\pi W_3^2}$

$W_3 = f\sqrt{\frac{\lambda}{\pi z_0}}$

$W_3 = f\frac{\lambda}{\pi}\frac{1}{W_1}$

Problem 6

a) 1° Take Fourier Transform of initial field.

$u_0 \rightarrow U_0(\alpha, \beta)$.

2° multiply Fresnel Transfer function.

$U_- = H_F \cdot U_0$

3° take convolution with Transfer function of Lens.

$U_+ = T_L \otimes U_-$

4° multiply Transferfunction again

$U(2f) = H_F \cdot U_+$

5°. Inverse Fourie Transform

$u(x, y, 2f) = FT^{-1}[U(2f)]$

b) $p(x, y) = \begin{cases} 1 & x, y = 0 \\ 0 & \text{elsewhere.} \end{cases} = p(\alpha, \beta) = \delta(\alpha, \beta)$

$u(-x, -y, 4f) \sim \iint_{-\infty}^{\infty} p(\frac{f}{k}\alpha, \frac{f}{k}\beta) U_0(\alpha, \beta) \exp[i(\alpha x + \beta y)] d\alpha d\beta$

$\sim U_0(0, 0)$. or Airy pattern ? I don't know its mean?

c) A Mask can filter horizental ~~fre~~ spatial frequency. i.e. α is filtered out.

d) impossible. The edge of lens will limit the resolution of this set-up.

e). $\alpha^2 + \beta^2 < (\frac{D}{2})^2 \sim (\frac{f}{k}\alpha)^2 + (\frac{f}{k}\beta)^2 < (\frac{D}{2})^2$

$\cancel{r^2 = x^2 + y^2 < \frac{D^2\lambda^2}{f^2\lambda^2}}$ $\alpha^2 + \beta^2 < \frac{D^2\lambda^2}{f^2\lambda^2}$

$\cancel{\Delta r_{min}} = \rho_{max} = \frac{\pi D}{\lambda f}$.

$\Delta r_{min} = \frac{2\pi}{\rho_{max}} = \frac{2\lambda f}{D} = 10 (\mu m)$

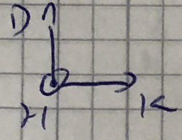
Problem 7

a). $D^{(i)} = D \cdot \exp[i(k_i \vec{r} - wt)]$ $\quad k_i = \frac{w}{c}\varepsilon_i(w)$

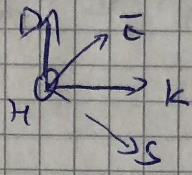
$E^{(i)} = \frac{D^{(i)}}{\varepsilon_0 \varepsilon^{(i)}}$. E and D are not parallel

b) $S = E_r \times H_r \Rightarrow S \perp E,\ S \perp H.$

$K \times H = -i w \mu_0 D \Rightarrow D \perp K,\ D \perp H$

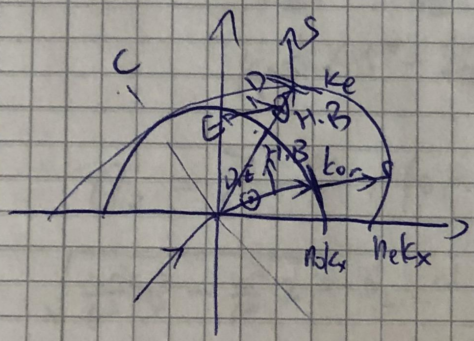


so if $E \nparallel D \Rightarrow$



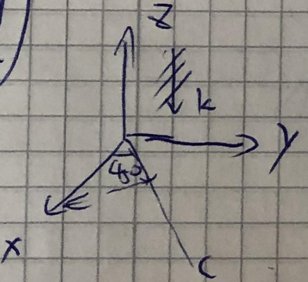
$S \nparallel K$

c)



$k_{or} = k n_o$

d)



TM: $E_{\parallel} = E\cos 45° = \frac{\sqrt{2}}{2}E$ $\quad k_e = n_e k = \frac{w}{c}n_e = \frac{2\pi n_e}{\lambda}$

TE: $E_{\perp} = E\sin 45° = \frac{\sqrt{2}}{2}E.$ $\quad k_o = n_o k = \frac{w}{c}n_o = \frac{2\pi n_o}{\lambda}$

after optical

TM: $E_{\parallel} = \frac{\sqrt{2}}{2}E \cdot \exp(ik_e L)$

TE: $E_{\perp} = \frac{\sqrt{2}}{2}E \cdot \exp(ik_o L)$

$\Delta\varphi = (k_e - k_o)L = \frac{2\pi L}{\lambda}(n_e - n_o) = \pi.$

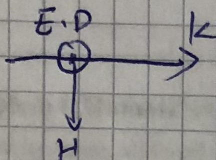
$L = \cancel{\frac{\lambda}{2(n_e - n_o)}}\ \frac{\lambda}{2(n_e - n_o)}$

oblem 8

a) $n_0 \sin\theta_0 = n_1 \sin\theta_1$

$$\theta_1 = 30^\circ$$

b)



$\bar{E}_0 = \bar{E}_1$

$H_{z_1} = H_{z_2} \Rightarrow H_1 \cdot \cos 45^\circ = H_2 \cdot \cos 30^\circ$

$\sqrt{2} H_1 = \sqrt{3} H_2$

$k_{z_1} = k_{z_2}$

$k_1 \cdot \sin 45^\circ = k_2 \cdot \sin 30^\circ$

c) $\theta_2 = 75^\circ$, for TIR, $Re(k_{0x}) < 0$

$$\sqrt{\frac{w^2}{c^2} n_2^2 - \frac{w^2}{c^2} \cdot 2 \cdot \sin\theta_2} < 0$$

$$n_2^2 < 2\sin 75^\circ$$

$$n_{2max} = 1.39$$