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Series 4
Fundamentals of Modern Optics

to be returned on **17.11.2022**, at the beginning of the lecture

Task 1: Traveling and standing waves (2+2+2 points)

Consider a monochromatic field of frequency ω with the complex amplitude vector

$$\mathbf{E}(\mathbf{r}) = \exp(\mathrm{i}\beta x)\left[A_1 \exp(\mathrm{i}\kappa z) + A_2 \cos(\kappa z)\right]\mathbf{e}_y,$$

where β, κ, A_1, and A_2 are real-valued numbers.

a) Derive an expression for the complex amplitude vector of the magnetic field $\mathbf{H}(\mathbf{r})$.
Hint: All calculations become easier if you first try to reformulate the given electric field as a sum of plane waves.

b) Calculate the time-averaged Poynting vector.

c) Use the Helmholtz equation in vacuum to find the relation between the parameters β, κ and λ_0 (the vacuum wavelength).

Task 2: Poynting vector for a surface guided wave (2*+2+2+2+2+2 points)

We have learned in the previous exercise series about the separation of EM modes in TE and TM if we have invariance of the structure in one direction. Let us investigate one such structure with invariance in y direction: a metal-to-air interface at $x = 0$ with $\varepsilon(x < 0) = \varepsilon_\mathrm{M} < 0$ and $\varepsilon(x > 0) = 1$. It turns out that this geometry supports a TM surface guided mode (propagating in the z-direction and confined to the interface along the x-direction from both sides) with frequency ω with the following magnetic field:

$$\mathbf{H} = \begin{bmatrix} 0 \\ H_y \\ 0 \end{bmatrix}; \quad H_y = H_0 e^{ik_z z}\begin{cases} \exp\{-\sqrt{k_\mathrm{z}^2 - \frac{\omega^2}{c^2}}\, x\} & , x > 0 \\ \exp\{+\sqrt{k_\mathrm{z}^2 - \frac{\omega^2}{c^2}\varepsilon_\mathrm{M}}\, x\} & , x < 0 \end{cases}$$

with $k_\mathrm{z} > 0$ and $k_z^2 > \frac{\omega^2}{c^2}$.

a*) Show that if we want to have such a guided/confined wave with the above mentioned characteristics, H_y would indeed have the form given above. You can start with the general form of the H-field of a wave in y-invariant space: $H_0 \exp\left[i\left(k_x x + k_z z\right)\right]$. (This proof is not needed to proceed with the next tasks.)

b) Using the given expression for $\mathbf{H}$ and Maxwell's equations, find the expression for the electric field.

c) Use the continuity of E_z at the interface to show that

$$-\varepsilon_\mathrm{M}\sqrt{k_\mathrm{z}^2 - \frac{\omega^2}{c^2}} = \sqrt{k_\mathrm{z}^2 - \frac{\omega^2}{c^2}\varepsilon_\mathrm{M}}$$

From this relation it can be seen that having $\varepsilon_\mathrm{M} < 0$ is essential for having a confined wave in the x-direction. But is it enough for having it propagating in the z-direction? Find k_z from this equation and find the condition on ε_M for having a real valued k_z.

d) Make drawings of the electric and magnetic fields and how they behave close to the metal-air interface.

e) Calculate the time-averaged Poynting vector as a function of distance x from the metal-air interface. Make a schematic drawing of it.

f) Calculate the net flow of energy per unit length (per unit length of the y-direction) by integrating the time-averaged Poynting vector from e) over $x \in [-\infty, \infty]$. In which direction does the net energy flow?

Hints: The eigenmodes of each of these half spaces are still plane waves of the form $H_\mathrm{y} \propto e^{i(k_\mathrm{x}x + k_\mathrm{z}z)}$. The wave numbers satisfy the dispersion relations of their corresponding media. The key to have a mode that coexists in both media is that both waves in metal and air should have the same k_z (satisfying the boundary condition for tangential magnetic field across the interface).

Task 3: Normal modes in low loss materials (2+3+3 points)

Assume an isotropic and homogeneous medium with the complex-valued relative permittivity $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$. A plane wave is propagating in this medium with the frequency-domain electric-field expression of $\mathbf{E}(\mathbf{r},\omega) = \mathbf{E}_0(\omega)\exp(\mathrm{i}\mathbf{k}(\omega)\cdot\mathbf{r})$, where the wave vector $\mathbf{k} = \mathbf{k}' + \mathrm{i}\mathbf{k}''$ is complex-valued. The electric field satisfies the Helmholtz equation in the frequency domain

$$\left[\Delta + \frac{\omega^2}{c^2}\varepsilon(\omega)\right]\mathbf{E}(\mathbf{r},\omega) = 0.$$

a) Derive the set of equations which connect $\mathbf{k}'$ and $\mathbf{k}''$ with ε' and ε''.

Now assume $\mathbf{k}' \parallel \mathbf{k}'' \parallel \hat{k}$, which gives $\mathbf{k}' + i\mathbf{k}'' = \hat{k}\frac{\omega}{c}(n + i\kappa)$. We would like to use the *perturbation theory* to find n and κ as functions of ε' and ε'', in other words $n(\varepsilon',\varepsilon'')$ and $\kappa(\varepsilon',\varepsilon'')$. Perturbative approaches are applied to many different physics problems, and the idea is as follows: Assume that you have a mathematical problem that is hard to solve analytically (although our particular problem has an analytical solution given in the lecture note, which is derived by solving a second order polynomials, but let's assume we do know how to do that and we are stuck at the two equations derived in part (a)). However, it turns out that if you set a certain parameter in the problem equal to zero, then it becomes much simpler to find an exact solution. In our case that parameter is ε''. By putting $\varepsilon'' = 0$, you can find the *zeroth-order* approximation solutions, let us call them $n_0(\varepsilon',\varepsilon'')$ and $\kappa_0(\varepsilon',\varepsilon'')$. If ε'' is very small (in this problem it means $|\varepsilon'| \gg \varepsilon''$), this could already be a good enough approximation. But say you want to improve the accuracy of your answer. You can calculate what is called a corrected *first-order* answer. To find this, you take the original problem with $\varepsilon'' \neq 0$, and if $\kappa_0 = 0$, then substitute n with n_0 and then try to find κ. The κ you find in this way is your corrected first-order answer, call it κ_1. Now go back to the original problem again and this time substitute κ with κ_1 and then try to find n. The n you find is your corrected first-order answer n_1. For the case of $n_0 = 0$, follow the same mentioned steps just with n and κ exchanged in the instructions. You can do this repeatedly to find the higher order corrections to $n(\varepsilon',\varepsilon'')$ and $\kappa(\varepsilon',\varepsilon'')$.

Find $n_1(\varepsilon',\varepsilon'')$ and $\kappa_1(\varepsilon',\varepsilon'')$ for the two practical situations below:

b) The case of a low loss dielectric with $\varepsilon'(\omega) > 0$ and $\varepsilon''(\omega) > 0$ and $\varepsilon'(\omega) \gg \varepsilon''(\omega)$.

c) The case of a low loss metal with $\varepsilon'(\omega) < 0$ and $\varepsilon''(\omega) > 0$ and $|\varepsilon'(\omega)| \gg \varepsilon''(\omega)$.