Name: _____ Date of birth: _____ Student ID No.: _____

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Midterm Exam

Fundamentals of Modern Optics

to be written on December 16, 8:15 am – 9:45 am

Problem 1: Maxwell's Equations — 4 + 2 + 3 = 9 points

a) Write down Maxwell's equations (MwE) in the time domain and in the frequency domain in a linear, non-magnetizable, homogeneous, isotropic, and dispersive medium without external charges but with currents.

b) Write down the relation between $\mathbf{D}$ and $\mathbf{E}$ in the time and frequency domain in this medium. Name additional functions used in the relations!

c) Derive the Helmholtz equation for the magnetic field $\mathbf{H}(\mathbf{r},\omega)$ from MwE in this medium. Assume that $\mathbf{j}(\mathbf{r},\omega) = \sigma(\omega)\,\mathbf{E}(\mathbf{r},\omega)$.

Problem 2: Poynting Vector and Normal Mode — 2 + 3 + 1 + 3 = 9 points

Consider a monochromatic plane wave of frequency ω, propagating in a homogeneous isotropic lossy dielectric medium of relative permittivity $\varepsilon(\omega) = \varepsilon'(\omega) + \varepsilon''(\omega)$ (where $\varepsilon', \varepsilon'' > 0$ and $\varepsilon' >> \varepsilon''$). Its electric field has the form $\mathbf{E}_r(\mathbf{r},t) = E_0\,\mathbf{e}_x\,e^{-k''z}\cos(k'z - \omega t + \varphi)$, where the subscript r is used for the real valued fields.

a) Express k' and k'' (approximately) with respect to ω, ε', and ε''.

b) Find the real valued magnetic field $\mathbf{H}_r(\mathbf{r},t)$.

c) Write down the formula for the instantaneous Poynting vector $\mathbf{S}_r(\mathbf{r},t)$.

d) Find the time averaged Poynting vector using the formula $\langle \mathbf{S}_r(\mathbf{r},t)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\mathbf{S}_r(\mathbf{r},t)dt$. You also can directly use the formula for time averaged Poynting vector, which uses the complex amplitudes. Your answer should be as simplified as possible.

Hint: You may, in all the steps of your calculations, use the complex representation as a mean to simplify your calculations. However, the final answers have to be real-valued physical quantities.

Problem 3: Beam propagation and Talbot effect — 2 + 3 + 2 + 2 = 9 points

Given is a monochromatic field, of vacuum wavelength λ, directly behind a one dimensional amplitude mask

$$u_0(x,z_0) = A\left[1 + \cos\left(\frac{2\pi}{G}x\right)\right].$$

The field is propagating through vacuum.

a) Calculate the spatial frequency spectrum $U_0(\alpha,z_0)$.

b) Calculate the field $u(x,z)$ for all $z > z_0$ without approximation.

c) The field will reproduce itself periodically except for a constant phase factor $e^{i\Phi}$ after a certain propagation length z_T (Talbot effect). Calculate the shortest repetition length z_T as a function of G and the wavelength λ.

d) Specify the condition for the applicability of the Fresnel approximation for the given field distribution. What will z_T approximately be under the Fresnel approximation?

Problem 4: Pulse propagation — 3 + 3 + 2 = 8 points

A Gaussian pulse travels through a $L = 200$ meters long medium whose dispersive refractive index is defined as:

$$n(\omega) = B + C\omega^2$$

where $B = 2$ and $C = 10^{-32}\mathrm{s}^2$. Before entering the medium, the pulse is transform limited (has a flat phase) and has a temporal width of $T_0 = 4$ ps and is centered around the carrier frequency $\omega_0 = 2\times10^{15}$Hz.

a) What are the phase velocity of the ω_0-frequency-component and the group velocity of the pulse? Give also the numbers, and not just the general formulas. You may leave your answers in terms of the velocity of light c_0.

b) Calculate the pulse width after propagating the length $z = L$ through this medium. If you cannot remember the exact formula, try to make a simple approximation for the asymptotic behavior in long propagation distances.

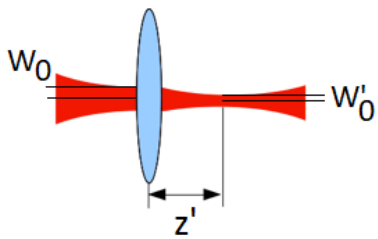
c) An identical pulse was simultaneously launched in a different medium whose $n(\omega)$ is the same as before with a small difference that $C = 0$ now. Calculate the difference between the time it takes for the two pulses to reach $z = L$.

Problem 5: Gaussian beams — 3 + 4 = 7 points

Consider a monochromatic Gaussian beam of wavelength λ and Rayleigh length $z_0 = \pi W_0^2/\lambda$, where W_0 is the waist radius.

a) Use the q-parameter formulation to find the expression for the beam width $W(z)$ and the radius of phase curvature $R(z)$ after propagating distance z away from the beam waist in free space. You can use the relation $\frac{1}{q(z)} = \frac{1}{R(z)} + \mathrm{i}\frac{\lambda}{\pi W^2(z)}$.

b) A positive (focusing) thin lens of focal length f is placed right at the beam waist of this Gaussian beam, as shown in the figure. Find, without any approximations, the distance z' of the waist of the transformed beam and also its beam waist W_0'. What is z' approximately in the limit of $z_0 >> f$? The ABCD matrix of a thin lens is $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$.



Problem 6: Fraunhofer diffraction — 3 + 1 + 2 = 6 points

a) Calculate the intensity of the diffracted monochromatic (of wavelength λ) field pattern $I(x, z_B) = |u(x, z_B)|^2$ in paraxial Fraunhofer approximation for two slits illuminated with a normally incident plane wave (prefactors are not important, the functional dependencies are important). The width of each slit is $2a$ and separated by a distance d ($d >> 2a$):

$$u_0(x, z = 0) = \begin{cases} 1, & \text{for } |x \pm d/2| \leq a \\ 0, & \text{elsewhere.} \end{cases}$$

b) What conditions should the parameters of the initial field satisfy, for the paraxial Fraunhofer approximation to be valid?

c) Try to roughly sketch a figure of the intensity, and explain how parameters a and d influence the main features of the intensity distribution.

Hint: The Fouriertransform of a single slit of width $2a$ is $\propto \mathrm{sinc}(\alpha a)$.