Fundamentals of Modern Optics
Series 09
23.12.2016

to be returned on 06.01.2017, at the beginning of the lecture

Problem 1 – Gratings **3+3 points**

A periodic, one-dimensional grating with N elements is given by a period b and a grating function $\tilde{f}(x)$ which is just defined for $0 \leq x < b$, so that the transmission function of the whole grating is given by

$$t(x) = \sum_{l=0}^{N-1} \tilde{f}(x - lb).$$

a) Prove that the spatial spectrum is given by

$$T(\alpha) = \tilde{F}(\alpha) \frac{\sin(N\alpha b/2)}{\sin(\alpha b/2)} e^{i(1-N)\alpha b/2},$$

where $\tilde{F}(\alpha)$ is the Fourier transform of $\tilde{f}(x)$.

Hint: You can make use of the Fourier shifting theorem!

b) Derive the spatial spectrum for an infinite grating:

$$t(x) = \sum_{l=-\infty}^{+\infty} \tilde{f}(x - lb),$$

as a function of $\tilde{F}(\alpha)$.
Hint: You can make use of the fact that any periodic function has a Fourier series expansion!

Problem 2 – Fraunhofer diffraction **2+3+2 points**

Calculate the diffraction pattern in Fraunhofer approximation for:

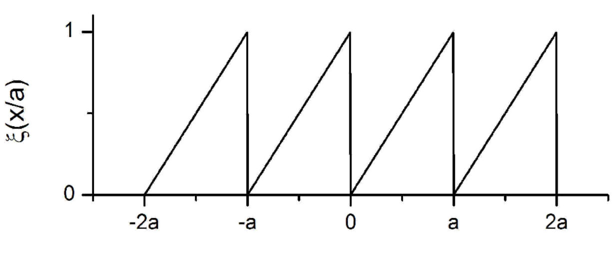
a) A blazed phase grating

$$u_0(x) = \exp\left\{ i\varphi_0\, \xi\left(\frac{x}{a}\right) \right\},$$

with N illuminated periods, where

$$\xi\left(\frac{x}{a}\right) = \frac{x}{a} - \left\lfloor \frac{x}{a} \right\rfloor$$

is the saw tooth function below ($\lfloor x \rfloor$ is the largest integer number smaller than x).



b) A pinhole with radius a.
(*Hint:* Use polar coordinates for $\mathbf{k}$ and $\mathbf{r}$ to solve the Fourier transform. A useful integral is

$$\int_0^a \int_0^{2\pi} e^{iky\cos\varphi} y \, dy d\varphi = 2\pi a^2 \frac{J_1(ka)}{ka},$$

where J_1 is a Bessel function.)

c) A sequence of N pinholes placed along the x-axis with distances of $b > 2a$.

Problem 3 – Phase Contrast Microscopy — 4 points

In biology one often has to deal with samples which are mostly transparent. Those samples introduce a phase modulation only and it is hard to see them in the microscope by normal means. By measuring the light intensity the phase information gets lost. We will now look at a solution to this problem which was introduced by Zernike: the phase contrast microscopy.
Suppose you shine with a plane wave onto a transparent sample which introduces a phase modulation $u_0(x,y) = e^{i\varphi(x,y)}$. Now you pass this light into a $4f$-setup with the following filter function in the Fourier plane:

$$H(x,y) = \begin{cases} 1 & , x \approx 0 \text{ and } y \approx 0 \\ e^{i\frac{\pi}{2}} = i & , \text{else} \end{cases} .$$

This means that a part of the spectrum at the 2f plane is retarded by $\pi/2$. Assume small phase variations $|\varphi(x,y)| \ll 1$ and show that such a setup introduces an intensity modulation which depends on the local phase $\varphi(x,y)$ of the sample.

Problem 4 – Prison Break — 2+1+2 points

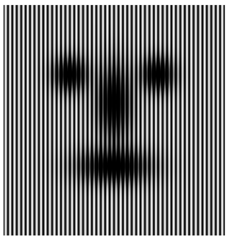


Figure 1: Hello! I am Günther! Please help me to get out of jail! I am innocent. I swear!

Your German friend Günther Gaußig was sent to prison for admitting that it is possible to overcome the resolution limit. He was locked by sheriff Peter Periodic using a grating in x-direction which has the following form:

$$\text{PRISON} = g(x) = \frac{1}{2}\left[1 + \cos\left(\frac{2\pi}{d}x\right)\right]$$

Luckily, Günther is a very "Gaussian" person in real space, where he usually lives

$$\text{GÜNTHER} = f(x,y) = m(x,y) + n(x,y) + e_l(x,y) + e_r(x,y),$$

with

$$\text{Nose} = n(x,y) = \exp\left[-\frac{x^2}{w^2} - \frac{\left(y - \frac{R}{5}\right)^2}{l^2}\right], \text{ Mouth} = m(x,y) = \exp\left[-\frac{x^2}{l^2} - \frac{\left(y + \frac{R}{2}\right)^2}{w^2}\right]$$

$$\text{LeftEye} = e_l(x,y) = \exp\left[-\frac{\left(x - \frac{R}{2}\right)^2}{w^2} - \frac{\left(y - \frac{R}{2}\right)^2}{w^2}\right], \text{ RightEye} = e_r(x,y) = \exp\left[-\frac{\left(x + \frac{R}{2}\right)^2}{w^2} - \frac{\left(y - \frac{R}{2}\right)^2}{w^2}\right]$$

and $R > l > w \gg d$. As an Abbe School of Photonics student you want to free your friend by optical means. You find Günther superimposed by the prison in the spatial domain as a starting point, i.e. the function $u_0(x,y) = f(x,y) + g(x)$ (see Fig. 1.).

a) Develop a plan to free Günther, i.e. to remove the prison, by Fourier optical filtering.

b) Construct and sketch an optical setup which is able to perform that task.

c) Why does Günther necessarily get battered (at least a little bit) by the escape out of prison? How can you minimize that effect?