$\psi_n = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac14} \frac{1}{\sqrt{2^n n!}} e^{-\frac{m\omega_0}{2\hbar}x^2} H_n\left(x\sqrt{\frac{m\omega_0}{\hbar}}\right) \quad H_n(\xi) = (-1)^n e^{\xi^2}\frac{d^n e^{-\xi^2}}{d\xi^n}$

$n=1 \quad H_1(\xi) = -1\, e^{\xi^2}(-2\xi)\, e^{-\xi^2} = 2\xi \Rightarrow \psi_1 = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac14}\frac{1}{\sqrt2}\, e^{-\frac{m\omega_0}{2\hbar}x^2}\cdot 2x\sqrt{\frac{m\omega_0}{\hbar}} = \sqrt2\sqrt{\frac{m\omega_0}{\hbar}}\left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac14} x e^{-\frac{m\omega_0}{2\hbar}x^2} = A x e^{-\frac{m\omega_0}{2\hbar}x^2}$

Structure of matter: Homework to exercise 8 $\langle \hat p\rangle = \int_x \psi^* \hat p \psi\, dx = -i\hbar A^2\int_x x e^{-\frac{m\omega_0}{2\hbar}x^2}\frac{d}{dx}\left(x e^{-\frac{m\omega_0}{2\hbar}x^2}\right) dx$

MATRIX ELEMENTS/UNCERTAINTY RELATION

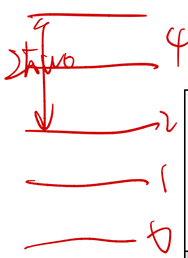
Due on november 28th 2023 at noon! $f_{nk} = -\frac{2}{m}\frac{|\langle n|\hat p|k\rangle|^2}{E_n - E_k}$

Please indicate your name on the solution sheets and send it to your seminar leader! $\sum_\ell f_{\ell n} = \frac{2m}{\hbar}\sum_\ell \omega_{\ell n}|x_{n\ell}|^2 = 1$ $\quad n=0 \Rightarrow \sum_\ell f_{\ell 0} = \frac{2m}{\hbar}\sum_\ell \omega_{0}|x_{\ell 0}|^2 = 1 = f_{10}$

$n=1 \quad f_{21}+f_{01} = 1 \Rightarrow f_{21} = 2$

$n=2 \quad f_{32}+f_{12} \Rightarrow f_{32} = 3$

$n=3 \quad f_{43}+f_{23} \Rightarrow f_{43} = 4$



1. Multiple-choice test: Please tick all box(es) with correct answer(s)!
   (correctly ticked box: +1/2 point; wrongly ticked box: -1/2 point)

Consider an electric dipole transition between the two eigenstates m = 2 and n = 4 of a harmonic oscillator. The oscillator strength f_{nm} of this transition is f_{42}	2	
	4	
	-24	
	42i	
Consider the first excited state ($E_1 = \frac{3}{2}\hbar\omega_0$) of a one-dimensional harmonic oscillator. In this state, the expectation value of the momentum will be $n=1$	$+\sqrt{3\hbar\omega_0 m}$	
	$-\sqrt{3\hbar\omega_0 m}$	
	0	✓

2. True or wrong? Make your decision (tick the appropriate box):

   (2 points): 1 point per correct decision, 0 points per wrong or no decision

Assertion	true	wrong
The oscillator strength is dimensionless.	✓	
The oscillator strength cannot be negative.		✓

3. Assume a quantum state described by a wavefunction $\Psi(\mathbf{r},t) = e^{-i\frac{E_n}{\hbar}t}\psi(\mathbf{r})$ with $\psi(\mathbf{r})$ - real. Show that in such a state, the expectation value of the momentum must be zero. (2 points)

4. From Heisenbergs uncertainty relation, find an estimation for the minimum energy of the ground state of a harmonic oscillator! (4 Points)

5. Assume an electron confined in a rectangular potential box of length L = 1nm with impermeable walls (one-dimensional case). Obtain the transition matrix element x_{jk} and the corresponding oscillator strength for k = 1 and j = 2. (6 points)

6. By explicitly calculating the variances of coordinate x and momentum p_x, show that in any stationary state of the particle confined in a 1D box with impermeable walls, Heisenbergs uncertainty relation $\langle(\Delta x)^2\rangle\langle(\Delta p_x)^2\rangle \geq \frac{\hbar^2}{4}$ for the coordinate and the momentum p_x is fulfilled! (10points)

(3) $\langle p\rangle = \int_V \varphi^*(r)\,(-i\hbar\nabla)\varphi(r)\,dv$ $p(r)$—real $\Rightarrow \langle p\rangle = -i\hbar\int_V \varphi\nabla\varphi\,dv$

$\nabla\varphi^2 = 2\varphi\nabla\varphi \Rightarrow \langle p\rangle = -\frac{i\hbar}{2}\int_V \nabla\varphi^2\,dv = -\frac{i\hbar}{2}\varphi^2\Big|_{-\infty}^{\infty} = 0$

(4) $\langle(\Delta x)^2\rangle\langle(\Delta p_x)^2\rangle \ge \frac{\hbar^2}{4}$ $\sigma^2 = \langle(\Delta j)^2\rangle = \langle j^2\rangle - \langle j\rangle^2$ $\langle p^2\rangle = p^2$ $\langle x^2\rangle = x^2$

$\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 \Rightarrow \langle x^2\rangle \ge \langle(\Delta x)^2\rangle$ $\langle(\Delta p_x)^2\rangle = \langle p_x^2\rangle - \langle p_x\rangle^2 \Rightarrow \langle p_x^2\rangle \ge \langle(\Delta p_x)^2\rangle$

$E = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2 \Rightarrow \langle E\rangle = \frac{\langle p_x^2\rangle}{2m} + \frac{1}{2}m\omega^2\langle x^2\rangle \ge \frac{\langle\Delta p_x^2\rangle}{2m} + \frac{1}{2}m\omega^2\langle\Delta x^2\rangle$

$(a-b)^2 = a^2+b^2-2ab \ge 0 \Rightarrow a^2+b^2 \ge 2ab$

$\Rightarrow \frac{\langle\Delta p_x^2\rangle}{2m} + \frac{m\omega^2}{2}\langle\Delta x^2\rangle \ge 2\sqrt{\frac{\langle\Delta p_x^2\rangle}{2m}\cdot\frac{1}{2}m\omega^2\langle\Delta x^2\rangle} = \sqrt{\omega^2\langle\Delta p_x^2\rangle\langle\Delta x^2\rangle} \ge \sqrt{\omega^2\cdot\frac{\hbar^2}{4}} = \frac{\omega\hbar}{2}$

(5) $x_{nm} = \int_x x\,\varphi_n\varphi_m\,dx$ $\varphi_n = \sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x \Rightarrow x_{nm} = \frac{2}{L}\int_0^L x\sin\frac{n\pi}{L}x\,\sin\frac{m\pi}{L}x\,dx$

$x_{jk} = x_{2,1} = \frac{2}{L}\int_0^L x\sin\frac{2\pi x}{L}\,x\,\sin\frac{\pi}{L}x\,dx$ $\frac{\pi}{L}x = t \Rightarrow dx = \frac{L}{\pi}dt$

$\Rightarrow x_{2,1} = \frac{2L}{\pi^2}\int_0^\pi t\sin 2t\,\sin t\,dt$ $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \Rightarrow \sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

$\Rightarrow x_{2,1} = \frac{L}{\pi^2}\int_0^\pi t(\cos t - \cos 3t)\,dt$

$= \frac{L}{\pi^2}\int_0^\pi t\cos t\,dt - \frac{L}{\pi^2}\int_0^\pi t\cos 3t\,dt = \frac{L}{\pi^2}\int_0^\pi t\,d\sin t - \frac{L}{3\pi^2}\int_0^\pi t\,d\sin 3t$

$= \frac{L}{\pi^2}\left(t\sin t\Big|_0^\pi - \int_0^\pi \sin t\,dt\right) - \frac{L}{3\pi^2}\left(t\sin 3t\Big|_0^\pi - \int_0^\pi \sin 3t\,dt\right)$

$= \frac{L}{\pi^2}\cos t\Big|_0^\pi - \frac{L}{9\pi^2}\cos 3t\Big|_0^\pi = -\frac{2L}{\pi^2} + \frac{2L}{9\pi^2} = -\frac{16L}{9\pi^2}$

$f_{nm} = \frac{2m}{\hbar}\omega_{nm}|x_{nm}|^2 = \frac{2m}{\hbar}\frac{E_n - E_m}{\hbar}|x_{nm}|^2 \Rightarrow f_{2,1} = \frac{2m}{\hbar}\frac{E_2 - E_1}{\hbar}|x_{21}|^2$; $E_n = \frac{h^2n^2}{8mL^2}$ $E_2 = \frac{4h^2}{8mL^2}$ $E_1 = \frac{h^2}{8mL^2}$

$\Rightarrow f_{2,1} = \frac{2m}{\hbar^2}\frac{3h^2}{8mL^2}\cdot\frac{16^2L^2}{81\pi^4} = \frac{8\pi^2\cdot 3}{8}\cdot\frac{16^2}{81\pi^4} = \frac{16^2}{27\pi^2} = 0.96$

(6) $[\hat F,\hat G] = i\hat M \Rightarrow var(F)\,var(G) \ge \left(\frac{1}{2i}[\hat F,\hat G]\right)^2 = \frac{\langle\hat M\rangle^2}{4}$ $\sigma \equiv \langle(\Delta j)^2\rangle = \langle j^2\rangle - \langle j\rangle^2$

thus $[\hat x,\hat p_x] = i\hbar \Rightarrow var(x)\,var(p_x) \ge \frac{\hbar^2}{4}$ $var(x) = \langle x^2\rangle - \langle x\rangle^2 = \langle\Delta x\rangle^2$ $var(p_x) = \langle p_x^2\rangle - \langle p_x\rangle^2 = \langle\Delta p_x\rangle^2$

$\Rightarrow \langle(\Delta x)^2\rangle\langle(\Delta p_x)^2\rangle \ge \frac{\hbar^2}{4}$

$\varphi_n = \sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x$ $\langle F\rangle = \int_V \varphi^*\hat F\varphi\,dv \Rightarrow \frac{2}{L}\int_0^L \sin\frac{n\pi}{L}x\cdot\hat F\sin\frac{n\pi}{L}x\,dx$ $p_x = -i\hbar\frac{d}{dx}$

$\langle p_x\rangle = \frac{2}{L}\int_0^L \sin\frac{n\pi}{L}x\left(-i\hbar\frac{d}{dx}\sin\frac{n\pi}{L}x\right)dx = \frac{-2i\hbar}{L}\int_0^L \sin\frac{n\pi}{L}x\cdot\frac{d}{dx}\sin\frac{n\pi}{L}x\,dx = \frac{-i\hbar}{L}\int_0^L \frac{d}{dx}\sin^2\frac{n\pi}{L}x\,dx = \frac{-i\hbar}{L}\sin^2\frac{n\pi}{L}x\Big|_0^L = 0$

$\langle p_x^2\rangle = \frac{2}{L}\int_0^L \sin\frac{n\pi}{L}x\left(-\hbar^2\frac{d^2}{dx^2}\sin\frac{n\pi}{L}x\right)dx = \frac{2\hbar^2n^2\pi^2}{L^3}\int_0^L \sin^2\frac{n\pi}{L}x\,dx = \frac{\hbar^2n^2\pi^2}{L^3}\int_0^L\left(1 - \cos\frac{2n\pi}{L}x\right)dx$

$= \frac{\hbar^2n^2\pi^2}{L^3}\left(\int_0^L dx - \int_0^L \cos\frac{2n\pi}{L}x\,dx\right) = \frac{\hbar^2n^2\pi^2}{L^3}\left(L - \frac{L}{2n\pi}\sin\frac{2n\pi}{L}x\Big|_0^L\right) = \frac{\hbar^2n^2\pi^2}{L^2}$

$\langle x\rangle = \frac{2}{L}\int_0^L x\sin^2\frac{n\pi}{L}x\,dx = \frac{1}{L}\int_0^L x\left(1 - \cos\frac{2n\pi}{L}x\right)dx = \frac{1}{L}\left(\int_0^L x\,dx - \int_0^L x\cos\frac{2n\pi}{L}x\,dx\right)$

$= \frac{1}{L}\left(\frac{1}{2}L^2 - \frac{L}{2n\pi}\int_0^L x\,d\sin\frac{2n\pi}{L}x\right) = \frac{L}{2} - \frac{1}{2n\pi}\left(x\sin\frac{2n\pi}{L}x\Big|_0^L - \int_0^L \sin\frac{2n\pi}{L}x\,dx\right) = \frac{L}{2} - \frac{L}{4n^2\pi^2}\cos\frac{2n\pi}{L}x\Big|_0^L = \frac{L}{2}$

$\langle x^2\rangle = \frac{2}{L}\int_0^L x^2\sin^2\frac{n\pi}{L}x\,dx = \frac{1}{L}\int_0^L\left(x^2 - x^2\cos\frac{2n\pi}{L}x\right)dx = \frac{1}{L}\left(\int_0^L x^2\,dx - \int_0^L x^2\cos\frac{2n\pi}{L}x\,dx\right) = \frac{1}{L}\left(\frac{1}{3}L^3 - \frac{L}{2n\pi}\int_0^L x^2\,d\sin\frac{2n\pi}{L}x\right)$

$$= \frac{1}{3}L^2 - \frac{1}{2n\pi}\left(x^2 \sin\frac{2n\pi}{L}x\Big|_0^L - 2\int_0^L x \sin\frac{2n\pi}{L}x\,dx\right) = \frac{1}{3}L^2 + \frac{1}{n\pi}\int_0^L x \sin\frac{2n\pi}{L}x\,dx = \frac{1}{3}L^2 - \frac{L}{2n^2\pi^2}\int x\,d\cos\frac{2n\pi}{L}x$$

$$= \frac{1}{3}L^2 - \frac{L}{2n^2\pi^2}\left(x\cos\frac{2n\pi}{L}x\Big|_0^L - \int_0^L \cos\frac{2n\pi}{L}x\,dx\right) = \frac{1}{3}L^2 - \frac{L^2}{2n^2\pi^2} + \frac{L}{4n^3\pi^3}\sin\frac{2n\pi}{L}x\Big|_0^L = \left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)L^2$$

$$Var(x) = \langle \Delta x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 = \left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)L^2 - \frac{1}{2}L$$

$$Var(p_x) = \langle \Delta p_x^2 \rangle = \langle p_x^2 \rangle - \langle p_x \rangle^2 = \frac{\hbar^2 n^2 \pi^2}{L^2} = \frac{h^2 n^2}{4L^2}$$

$$\langle \Delta x^2 \rangle \langle \Delta p_x^2 \rangle = \left(\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)L^2 - \frac{1}{2}L\right)\frac{\hbar^2 n^2 \pi^2}{L^2} = \frac{\hbar^2 n^2 \pi^2}{3} - \frac{\hbar^2}{2} - \frac{\hbar^2 n^2 \pi^2}{2L} = \left(\frac{n^2\pi^2}{3} - \frac{n^2\pi^2}{2L} - \frac{1}{2}\right)\hbar^2 = \frac{4n^2\pi^2 L - 6n^2\pi^2 - 6L}{12L}\hbar^2$$

$$= \frac{4n^2\pi^2 L - 6n^2\pi^2 - 6L}{3L} \cdot \frac{\hbar^2}{4} > \frac{\hbar^2}{4}$$