Institute of Applied Physics
Friedrich-Schiller-Universität Jena

Optical System Design Fundamentals

Lecture 5: Optical Transfer Function, Point Spread Function and Performance Criteria

2024 / 06 / 04

Vladan Blahnik

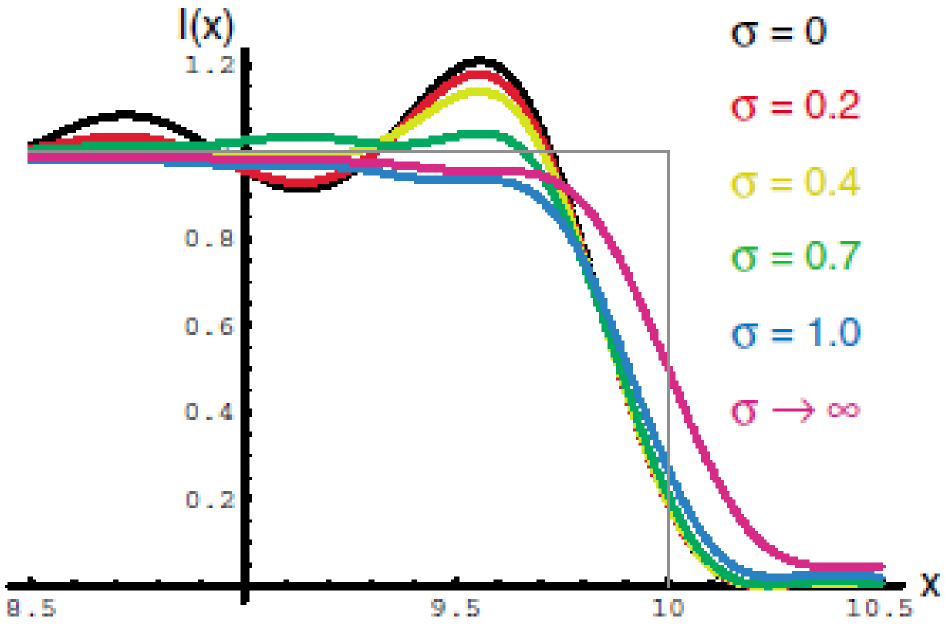
Preliminary Schedule - OSDF 2024

Institute of Applied Physics
Friedrich-Schiller-Universität Jena

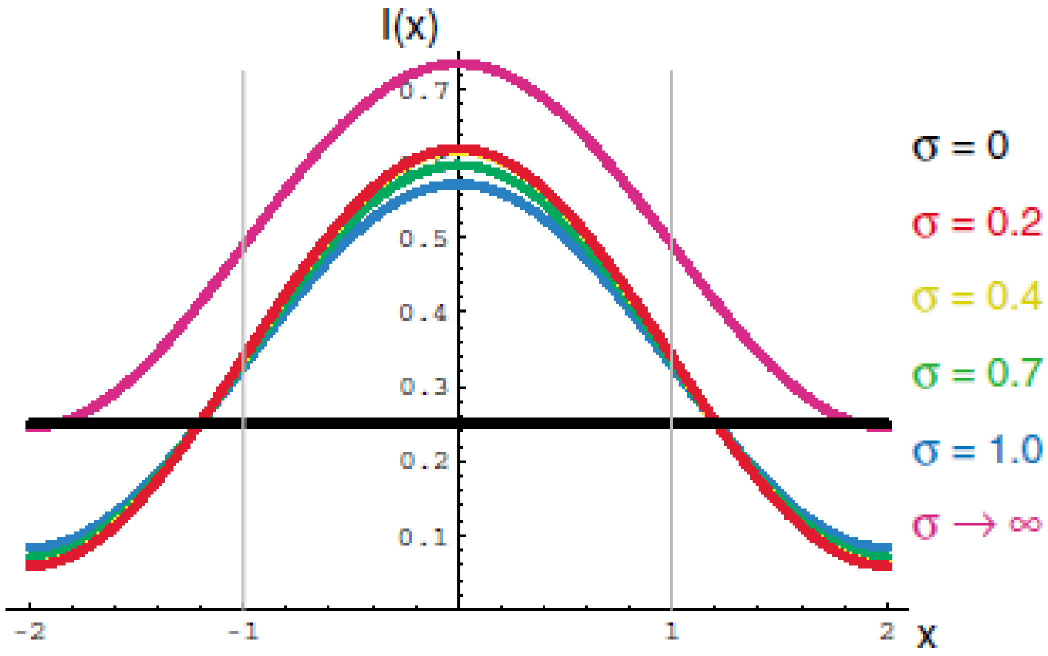
1	07.05.2024	Optical Imaging Fundamentals	Fermat principle, modeling light propagation with ray and wave model; diffraction; Snell´s law, total internal reflection, light propagation in optical system, raytracing; ray and wave propagation in optical systems; pinhole camera; central perspective projection; étendue, f-number and numerical aperture, focal length; ideal projection laws; fish,eye imaging; anamorphic projection; perspective distortion, Panini projection; massive ray-tracing based imaging model	(S) (optic
2	14.05.2024	The ideal image	Paraxial approximation, principal planes, entrance and exit pupil, paraxial imaging equations; Scheimpflug condition; paraxial matrix formulation; Delano diagram; Aplanatic imaging: Abbe sine condition, numerical aperture, sine scaled pupil for microscope imaging	(S)
3	21.05.2024	First order layout, system structures and properties	application of imaging equations; examples: layout of visual imaging system; layout of microscope lens and illumination system; telecentricity, "4f-setups"; tele and reversed tele (retrofocus) system; depth of field, bokeh; equivalence to multi-perspective 3D image acquisition; focusing methods of optical systems; afocal systems; connecting optical (sub-)systems: eye to afocal system; illumination system and its matching to projection optics); zoom systems; first order aberrations (focus and magnification error) and their compensation: lens breathing, aperture ramping; scaling laws of optical imaging	S
4	28.05.2024	Imaging model including diffraction	Gabor analytical signal; coherence function; degree of coherence; Rayleigh-Sommerfeld diffraction; optical system transfer as complex, linear operator; Fraunhofer-, Fresnel-approximation and high-NA field transfer wave-optical imaging equations for non-coherent, coherent and partially coherent imaging; isoplanatic patches, Hopkins Transfer Function for partially coherent imaging, optical transfer functions for non-coherent and coherent imaging; ideal wave-optical imaging equations; Fourier-Optics; spatial filtering; Abbe theory of image formation in microscope, phase contrast microscopy	S
5	04.06.2024	Wavefront deformation, Optical Transfer Function, Point Spread Function and performance criteria	wave aberrations, Zernike polynomials, measurement of system quality; point spread function (PSF), optical transfer function (OTF) and modulation transfer function (MTF); optical system performance criteria based on PSF and MTF: Strehl definition, Rayleigh and Marechal criteria, 2-point resolution, MTF-based criteria, coupling efficiency / insertion loss at photonics interface	S
6	11.06.2024	Aberrations: Classification, diagrams and identification in real images	measurement of wave front deformation: Shack-Hartmann sensor, Fizeau interferometer Symmetry considerations as basis for aberration theory; Seidel polynomial, chromatic aberrations; surface contributions and Pegel diagram, stop-shift-equation; Longitudinal and transverse aberrations, spot diagram; computing aberrations with Hamilton Eikonal Function; examples: defocus, plane parallel plate longitudinal and lateral chromatic aberration, Abbe diagram, dispersion fit; achromat and apochromat aberration in real images of (extended) objects	no
7	18.06.2024	Optimization process and correction principles	pre- and post-computer era correction methodology, merit function, system specification, initial setups, opto-mechanical constraints, damped-least-square optimization; symmetry principles, lens bending, aplanatic surface insertation, lens splitting, aspheres and freeforms	S
8	25.06.2024	Aberration correction and optical system structure	Petzval theorem, field flattening, achromat and apochromat, sensitivity analysis, diffractive elements; system structure of optical systems, e.g. photographic lenses, microscopic objectives, lithographic systems, eyepieces, scan systems, telescopes, endoscopes	(S)
9	02.07.2024	Tolerancing and straylight analysis	optical technology constraints, sensitivity analysis, statistical fundamentals, compensators, Monte-Carlo analysis, Design-To-Cost; false light; ghost analysis in optical systems; AR coating; opto-mechanical straylight prevention	S

Imaging with partially coherent light

Edge spread function

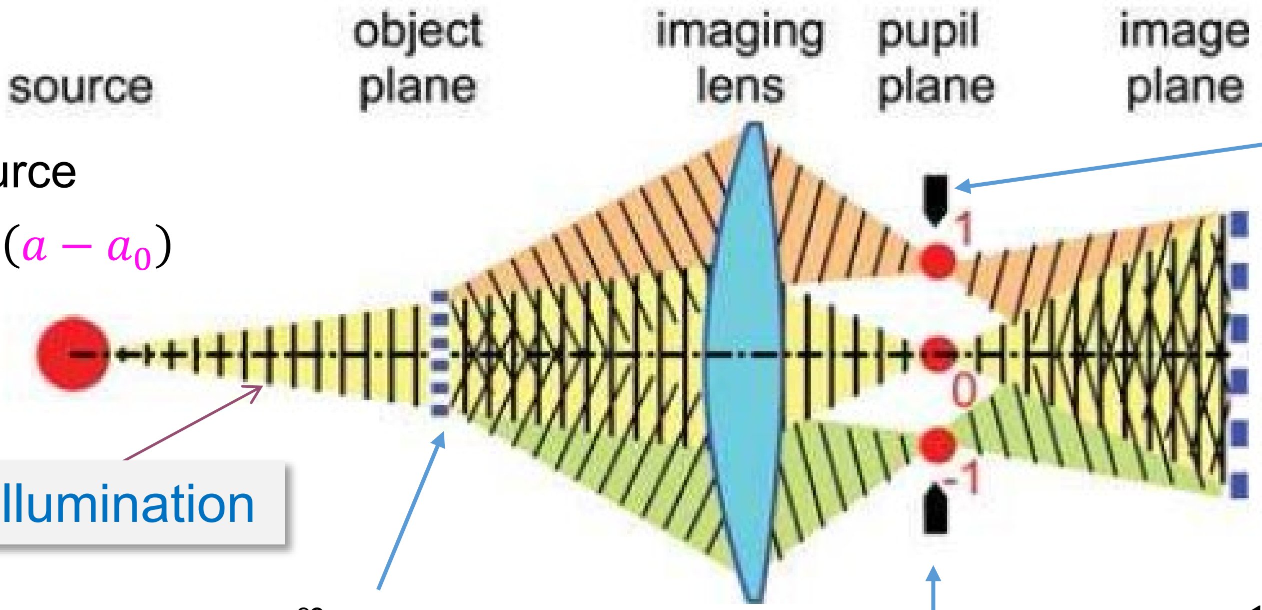


Periodic object $2\frac{\lambda}{NA} \geq p \geq \frac{\lambda}{NA}$



Near Resolution limit:
coherent case not resolved

Image formation in microscope (Abbe)



point source

$$Q(a) = \delta(a - a_0)$$

pupil filter $L\left(\frac{j}{w\,p}\right)$

Coherent illumination

periodic object $t_{ob}(\xi) = \sum_{j=-\infty}^{\infty} T_j \, exp\left(i2\pi\,\xi\frac{j}{p}\right)$

image intensity distribution:

$$I(x) = \frac{1}{p^2}\left|\sum_{j=-\infty}^{\infty} L\left(\frac{j}{w\,p}\right) T_j \; exp\left(-i2\pi\, w\frac{j}{p}\cdot x\right)\right|^2$$

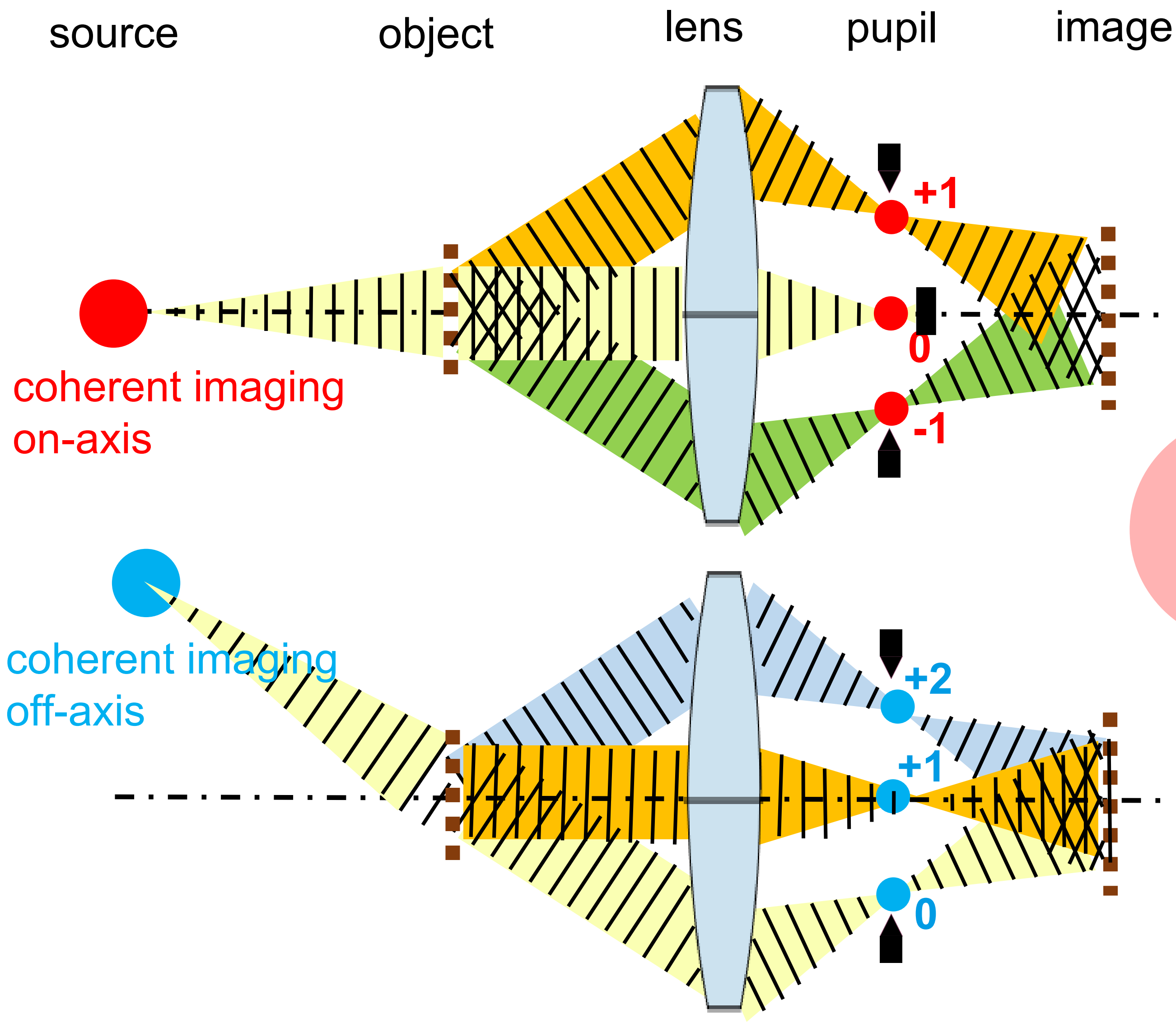
amplitude spectrum in pupil $T_{ob}(\alpha) = FT(t_{ob})(\alpha) = \frac{1}{p}\sum_{j=-\infty}^{\infty} T_j \,\delta\left(w\alpha - \frac{j}{p}\right)$

Partially coherent imaging equations

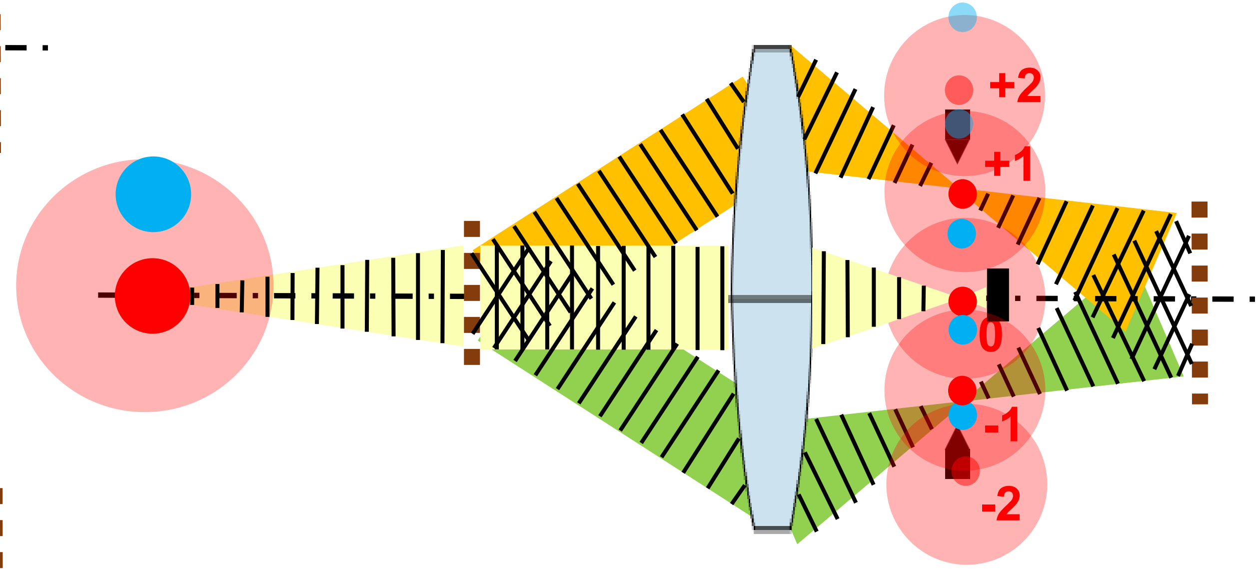
(assuming spatially incoherent light source & spatially invariant (isoplanatic) optical system transfer)

Order of integrations	Imaging equation	remark
1. source object 2. pupil (a, ξ, α)	$I(x) = \iint \iint d\alpha_1\, d\alpha_2\, exp(-i2\pi\, w(\alpha_1 - \alpha_2) \cdot x) \times HTF(\alpha_1, \alpha_2) FT(A_{ob})(\alpha_1) FT(A^*_{ob})(\alpha_2)$ $C(\alpha_1, \alpha_2) = \iint da\; Q(a)\, L(\alpha_1 + \sigma\, a)\, L^*(\alpha_2 + \sigma\, a)$ $FT(A_{ob})(\alpha) = \iint d\xi\; A_{ob}(\xi)\, exp(i2\pi w\, \alpha \cdot \xi)$	Hopkins transfer function (HTF) enables Fourier analysis and synthesis of spatial frequency transfer in lens pupil; Analytical transfer function for ideal cases, e.g., no aberrations and circular or annular source & pupil Analytical calculation of object spectrum e.g., for periodic (or aliased) structures → Fourier series of intensity distribution
1. object 2. pupil 3. source (ξ, α, a)	$I(x) = \iint da\; Q(a) \left\| \iint d\alpha\; L(\alpha) FT(A_{ob})(\alpha - \sigma\, a) exp(-i2\pi\, w\alpha \cdot x) \right\|^2$ $FT(A_{ob})(\alpha - \sigma\, a) = \iint d\xi\; A_{ob}(\xi)\, exp(i2\pi w(\alpha - \sigma\, a) \cdot \xi)$	suitable if object spectrum analytical, e.g., periodic (aliased) objects & complex lens pupil functions (Litho-simulators Solid-C, Prolith)
1. pupil 2. object 3. source (α, ξ, a)	$I(x) = \iint da\; Q(a) \left\| \iint d\xi\; A_{ob}(\xi)\, K(x - \xi) exp(-i2\pi\sigma w\, a \cdot \xi) \right\|^2$ $K(x - \xi) = \iint d\alpha\; L(\alpha)\, exp\big(-i2\pi w\, \alpha \cdot (x - \xi)\big) = FT(L)(x - \xi)$	aPSF analytical or efficiently stored (non-isoplanatic case Karhunen-Loewe)

From oblique illumination to extended light source



The transition from oblique illumination to **extended light sources** is just the **superposition of all oblique directions** contained in the extended distribution:



Partially coherent imaging with periodic structures

Partially coherent imaging with periodic object

$$I(x) = \iint da\, Q(a) \left| \sum_{j=-\infty}^{\infty} L\left(\frac{\sigma}{w}\, a + \frac{j}{w\, p}\right) T_j\ exp\left(-i2\pi\, w \left(\sigma\, a + \frac{j}{p}\right) \cdot x\right) \right|^2 =$$

$$= \iint da\, Q(a) \left(\sum_{j=-\infty}^{\infty} L\left(\frac{\sigma}{w}\, a + \frac{j}{w\, p}\right) T_j\ exp\left(-i2\pi\, w \left(\sigma\, a + \frac{j}{p}\right) \cdot x\right) \right) \left(\sum_{k=-\infty}^{\infty} L^*\left(\frac{\sigma}{w}\, a + \frac{j}{w\, p}\right) T_k^*\, exp\left(+i2\pi\, w \left(\sigma\, a + \frac{k}{p}\right) \cdot x\right) \right) =$$

$$= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} HTF\left(\frac{j}{p}, \frac{k}{p}\right) T_j T_k^* exp\left(i2\pi x \frac{j-k}{p}\right).$$

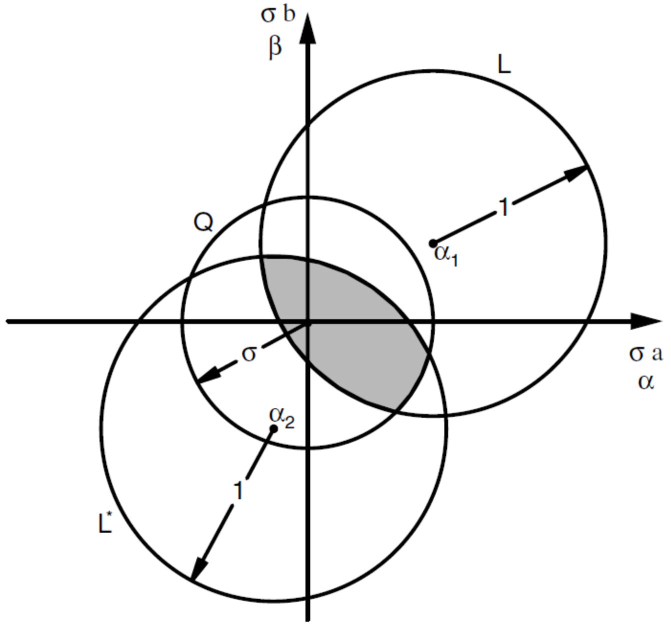
interchanging order of summation / integration

$$\boxed{HTF(\alpha_1, \alpha_2) = \iint da\, Q(a)\, L(\alpha_1 + \sigma\, a)\, L^*(\alpha_2 + \sigma\, a)\, .}$$

Hopkins Transfer Function or **Transmission Cross Coefficients**

Partially coherent transfer function

Hopkins, H. H. (1953). *On the diffraction theory of optical images*, Proc. Roy. Soc.(London) A **217**, S. 408-432.



Harold H. Hopkins (1918-1994)

Optics professor Imperial College London & University of Reading
- First TV zoom lens (1948 for BBC)
- First endoscope with fibre glass (1950s; Fibrescope)
- Endoscope with rod lens system (1960s with Storz)
- Laser Disk at Philips 1970s

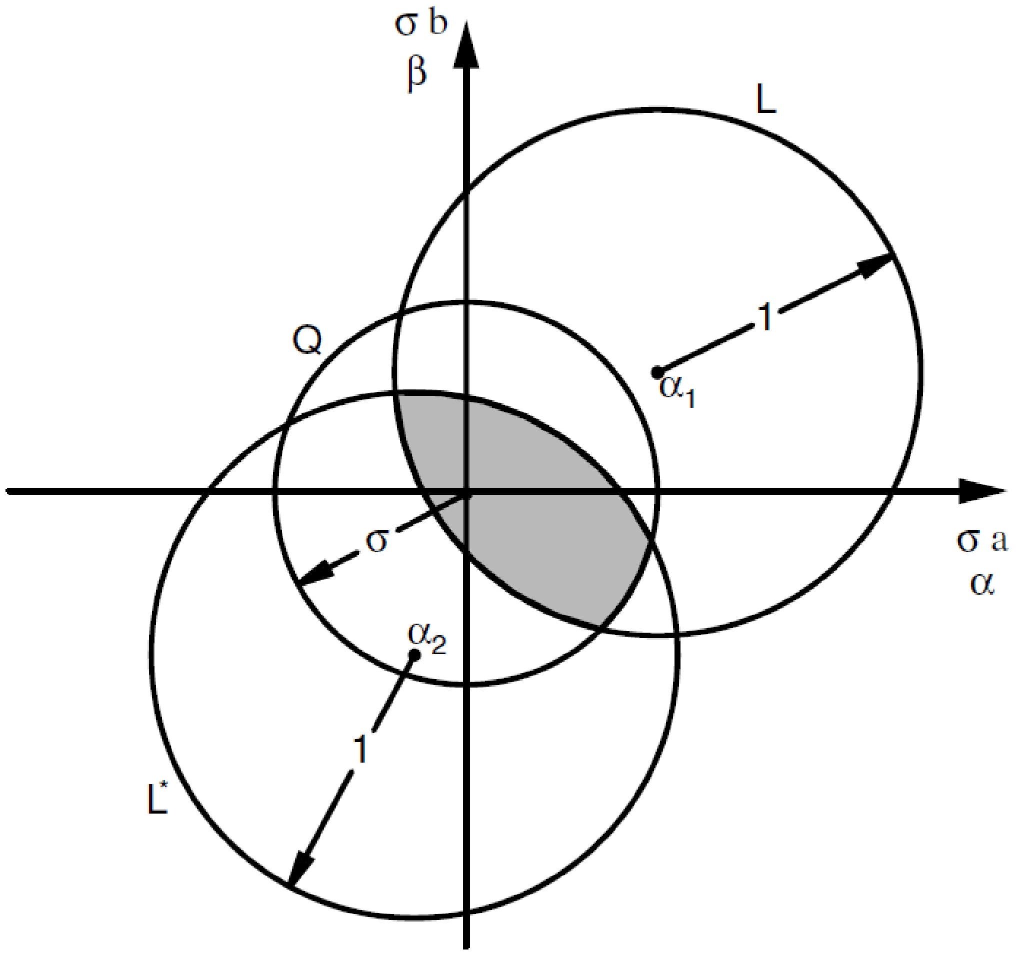
Intensity distribution

$$I(x) = \iint \iint d\alpha_1\, d\alpha_2\, exp(-i2\pi\, w(\alpha_1 - \alpha_2) \cdot x) \times HTF(\alpha_1, \alpha_2) FT(A_{ob})(\alpha_1) FT(A_{ob}^*)(\alpha_2)$$

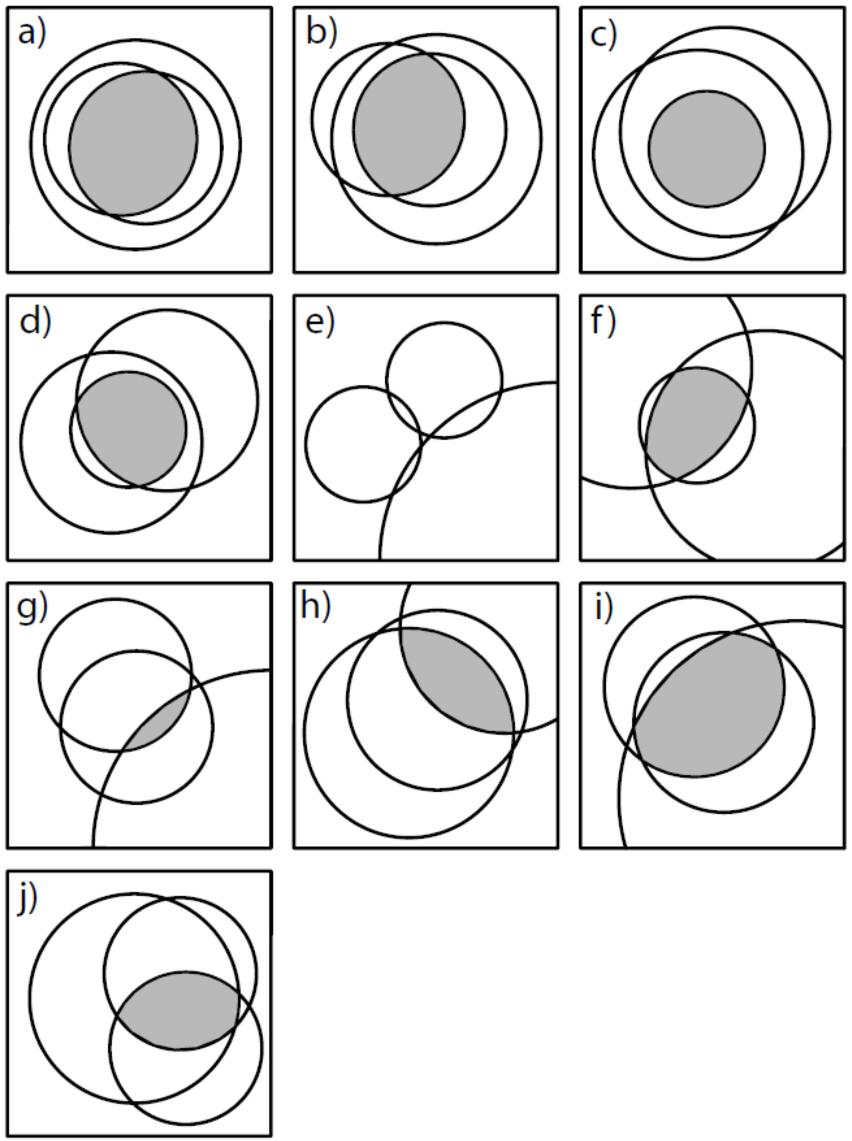
Hopkins Transfer Function

$$HTF(\alpha_1, \alpha_2) = \iint da\; Q(a)\, L(\alpha_1 + \sigma\, a\,) L^*(\alpha_2 + \sigma\, a\,)$$

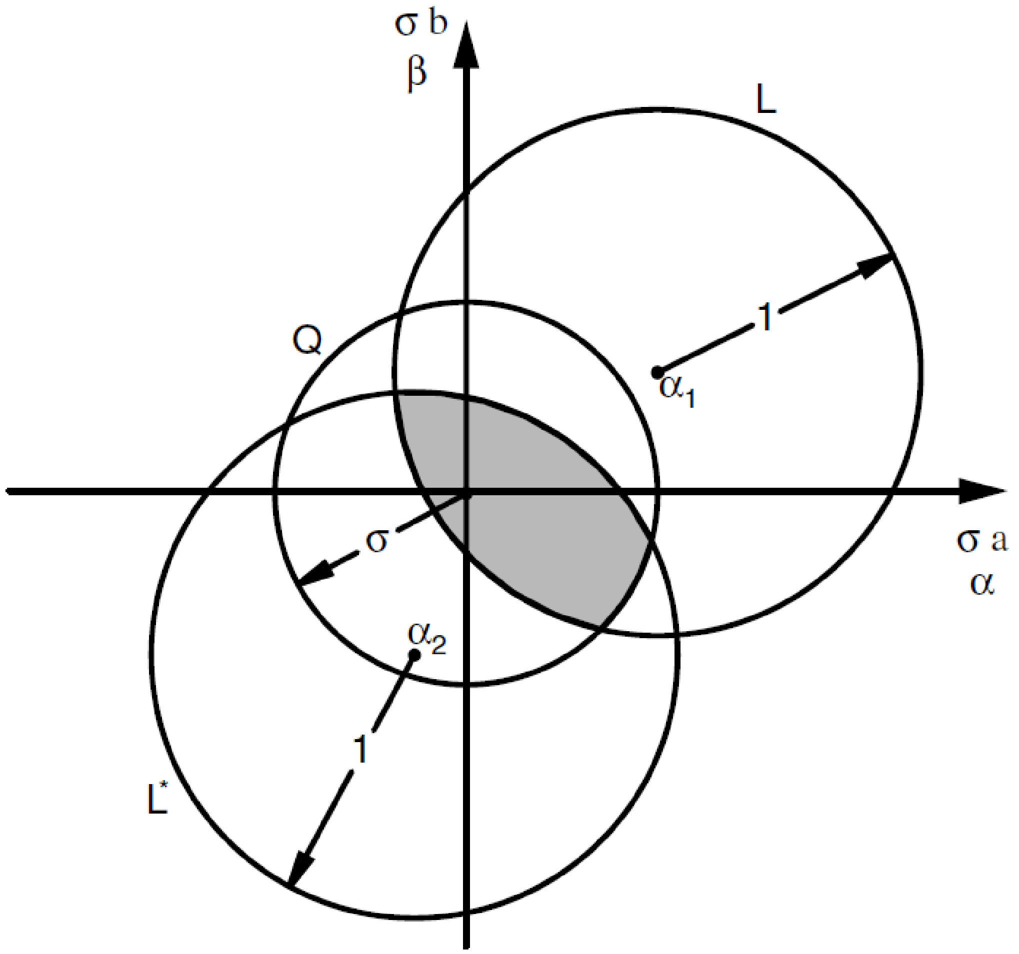
Calculation of Hopkins Transfer Function

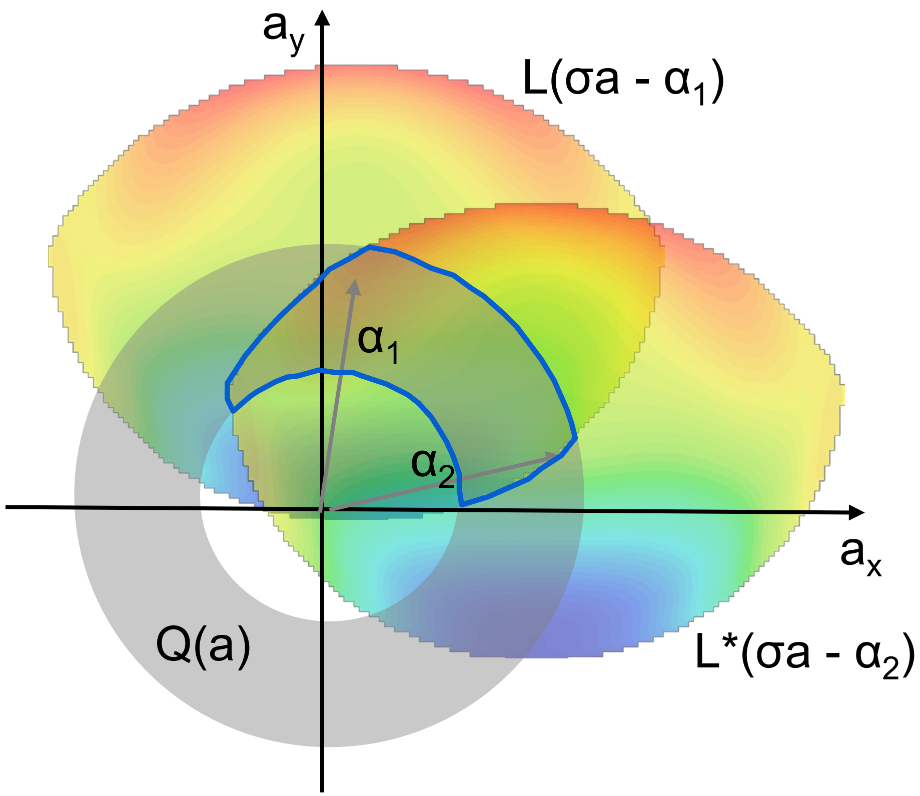


Analytical solution possible for ideal homogeneous source and pupil (1D-case: Kintner (1977), 2D-case: Blahnik (2002))



Calculation of Hopkins Transfer Function

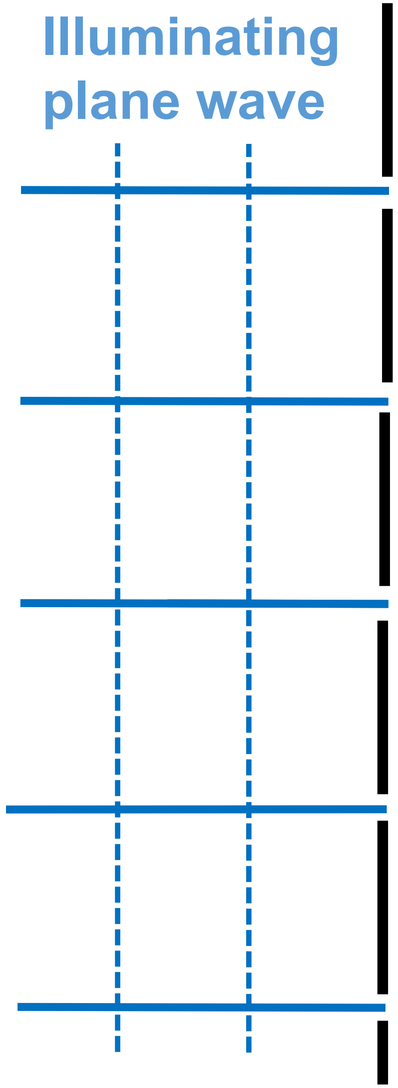




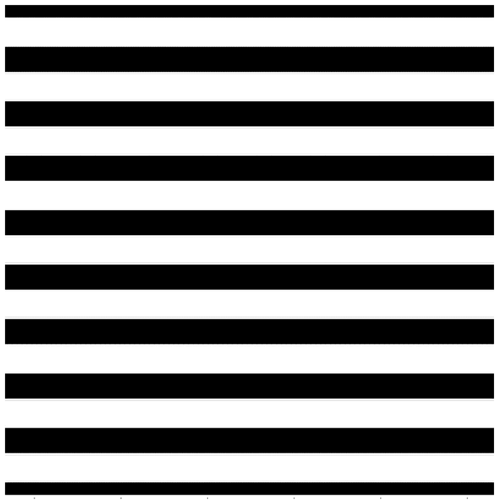
In general complex valued (lens aberrations) and complex shape of pupil and source

Requires numerical 2D integration (system data can be stored for computation of different objects)

Linear Grating: Illumination with Plane Wave (normal incidence angle)



Linear grating



Linear Grating: First diffraction order

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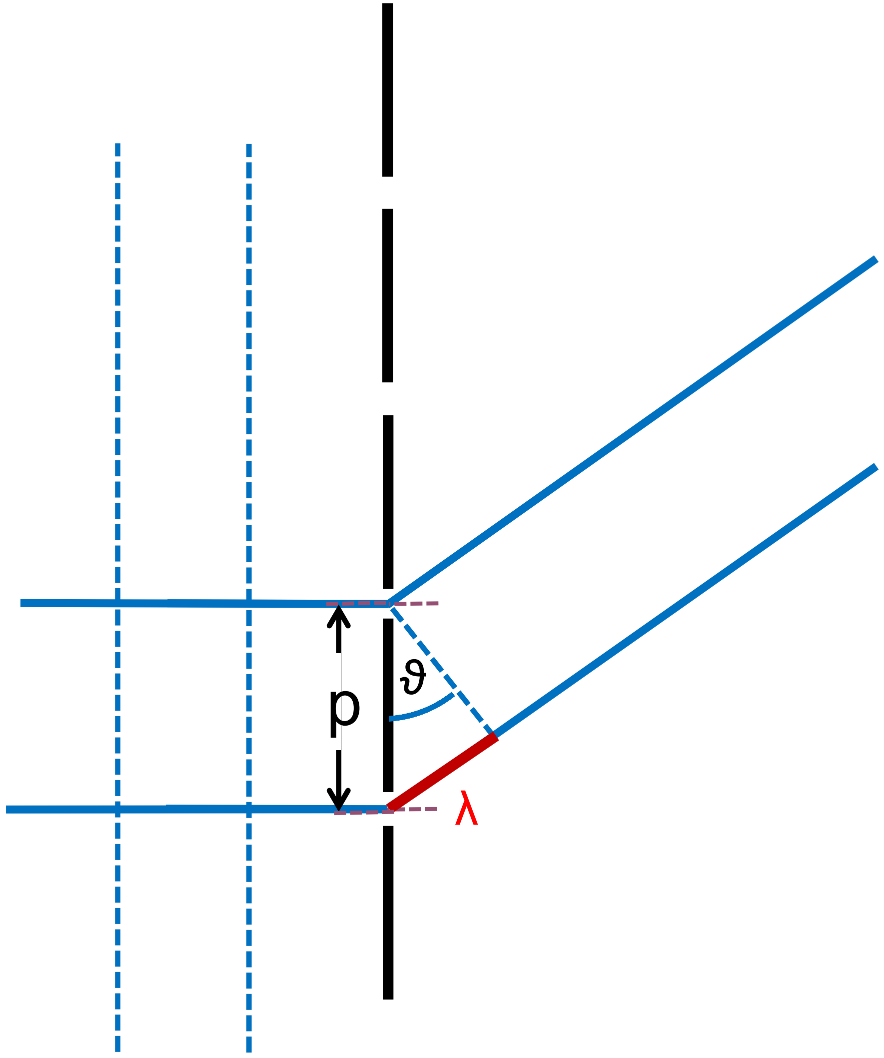
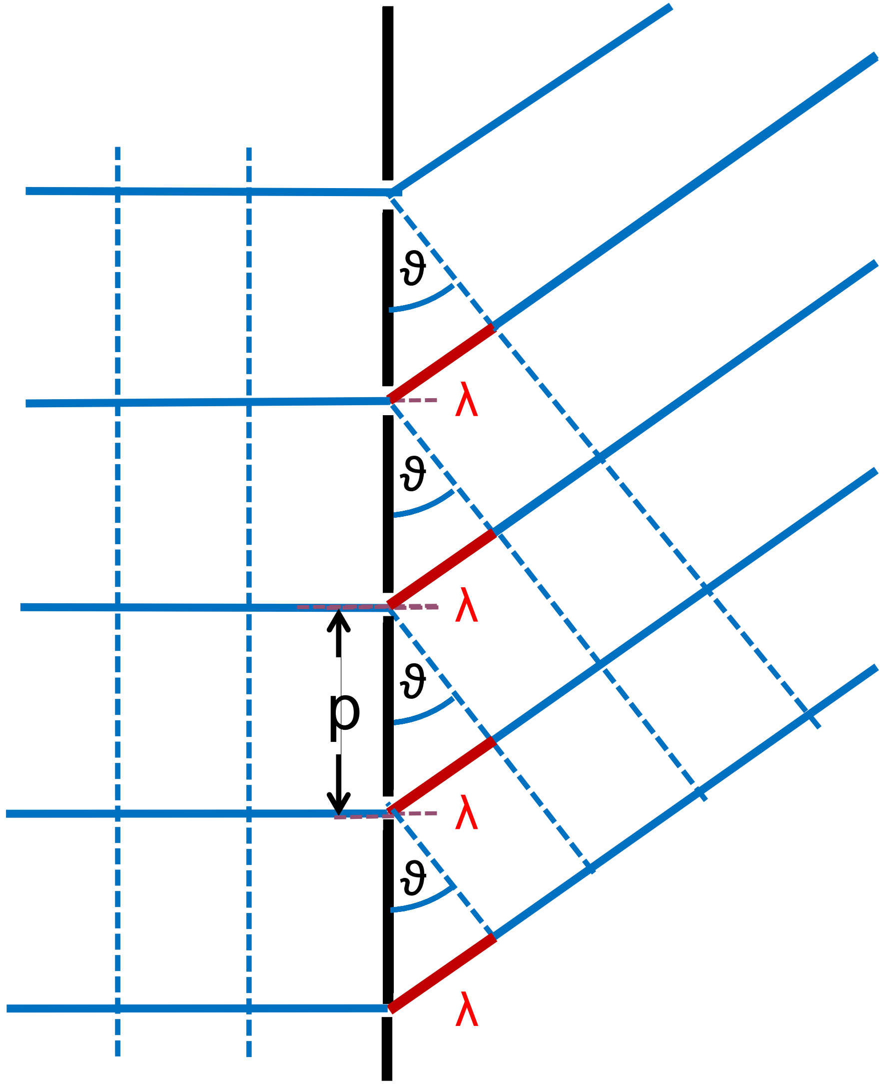


Oscillation in same phase also in direction where optical path difference differs by exactly one wavelength:
1st diffraction order

$$\sin\vartheta = \frac{\lambda}{p}$$

Linear Grating: First diffraction order



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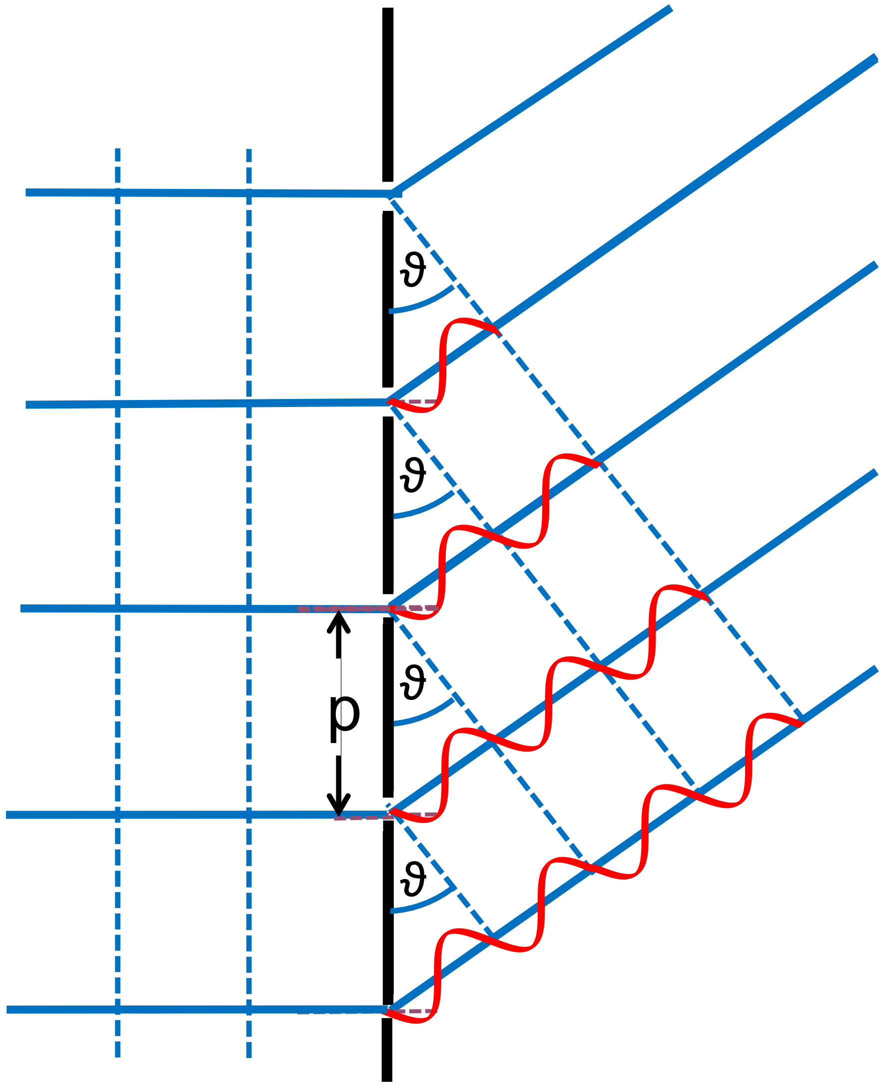
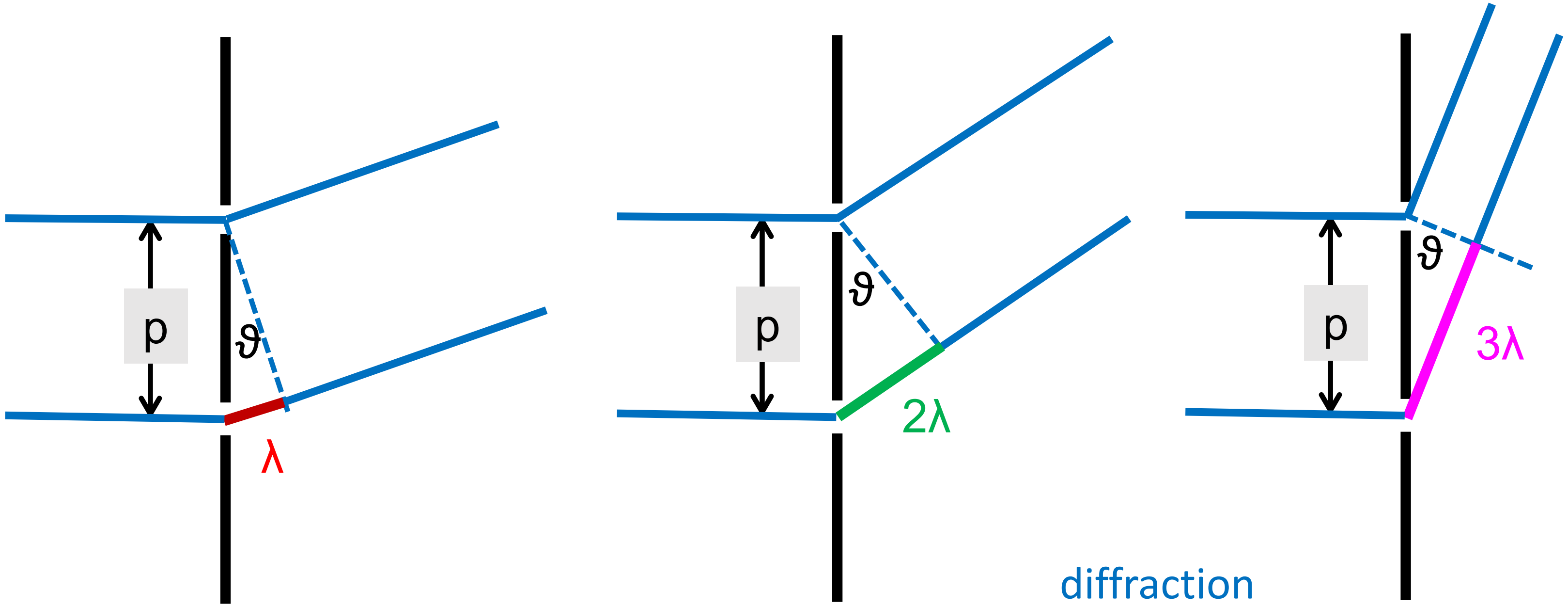
Linear Grating: First diffraction order



Oscillation in same phase also in direction where optical path difference differs by exactly one wavelength:
1st diffraction order

$$\sin\vartheta = \frac{\lambda}{p}$$

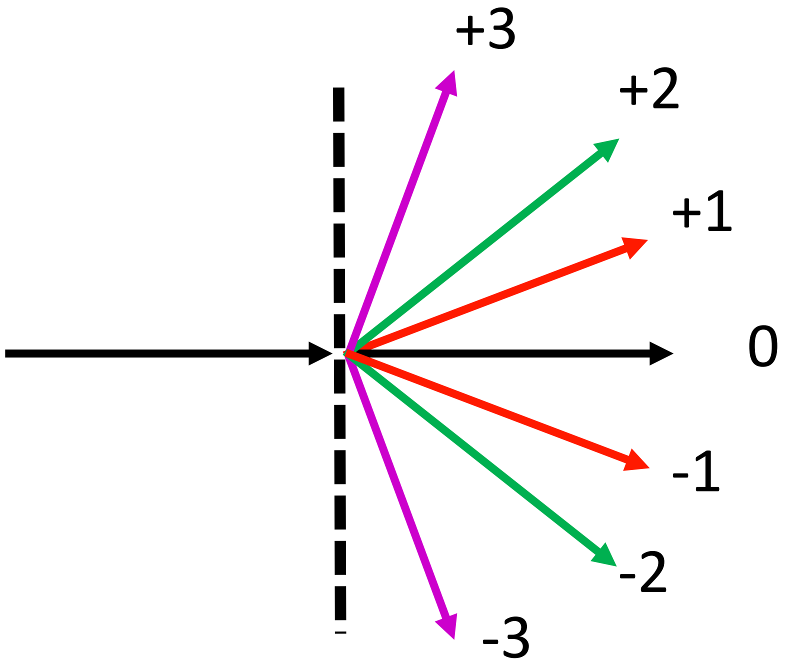
Diffraction orders



Multiples of $\frac{\lambda}{p}$ are also diffraction orders
(as long as <1, as the sine function)

diffraction order

$$\sin\vartheta = m\frac{\lambda}{p}$$



Diffraction at linear grating: Binary vs Cosine Grating

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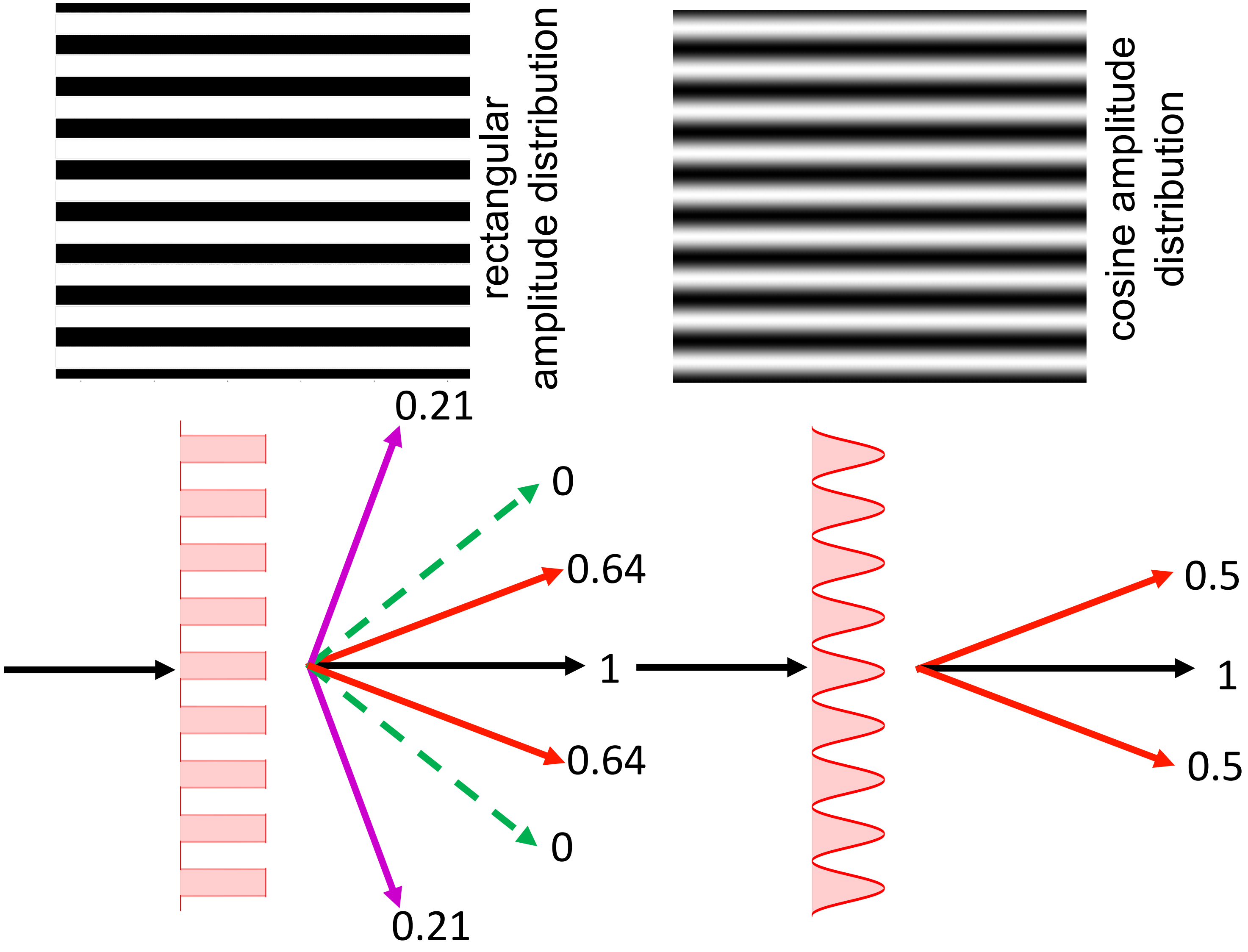

Direction ϑ of diffraction orders determined by: grating period p and wavelength λ:

$$\sin\vartheta = m\frac{\lambda}{p}$$

Relative amplitude of diffraction orders: amplitude and phase distribution of grating. Fourier Transform over period:

$$T_j = \frac{1}{p}\int_{-p/2}^{p/2} t_{ob}(\xi) exp\left(-i2\pi\xi\frac{j}{p}\right) d\xi$$

Coherent image formation with periodic object

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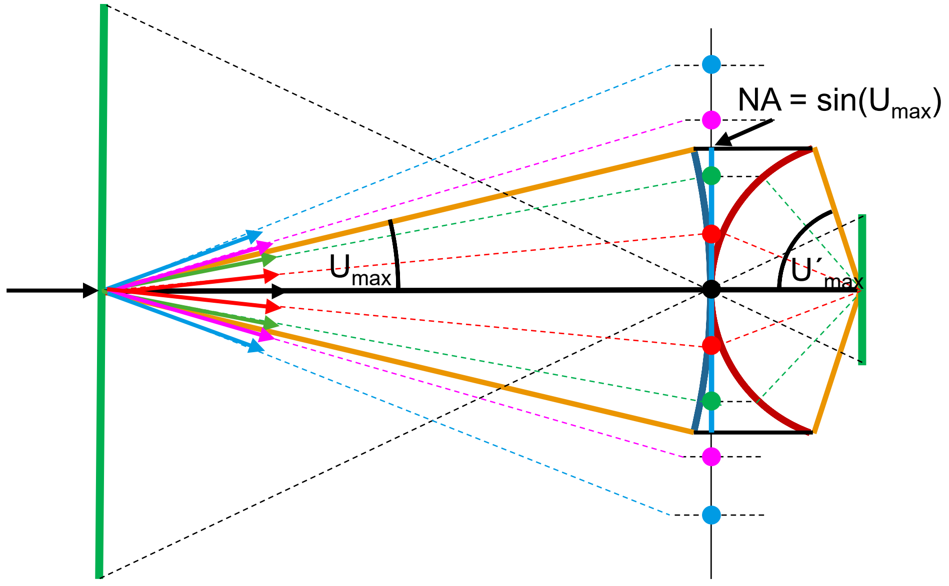
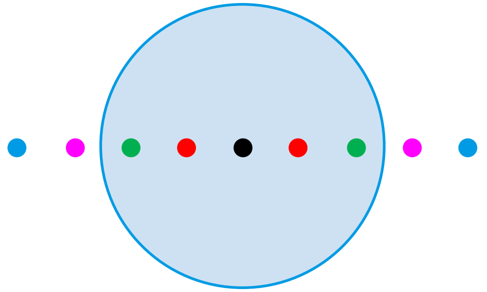
- Aplanatic system fulfills "Abbe sine-condition $\sin U = m \sin U'$, therefore we chose sine-scaled variables $\alpha = \sin U$

- As the object angles of a diffraction grating also scale according to the sine of the object angle, $\sin U = m\frac{\lambda}{p}$, the diffraction orders are <u>equidistant in the pupil</u>





Abbe's sine condition: Aplanatic image condition corresponds to equal OPLs of diffracted plane waves

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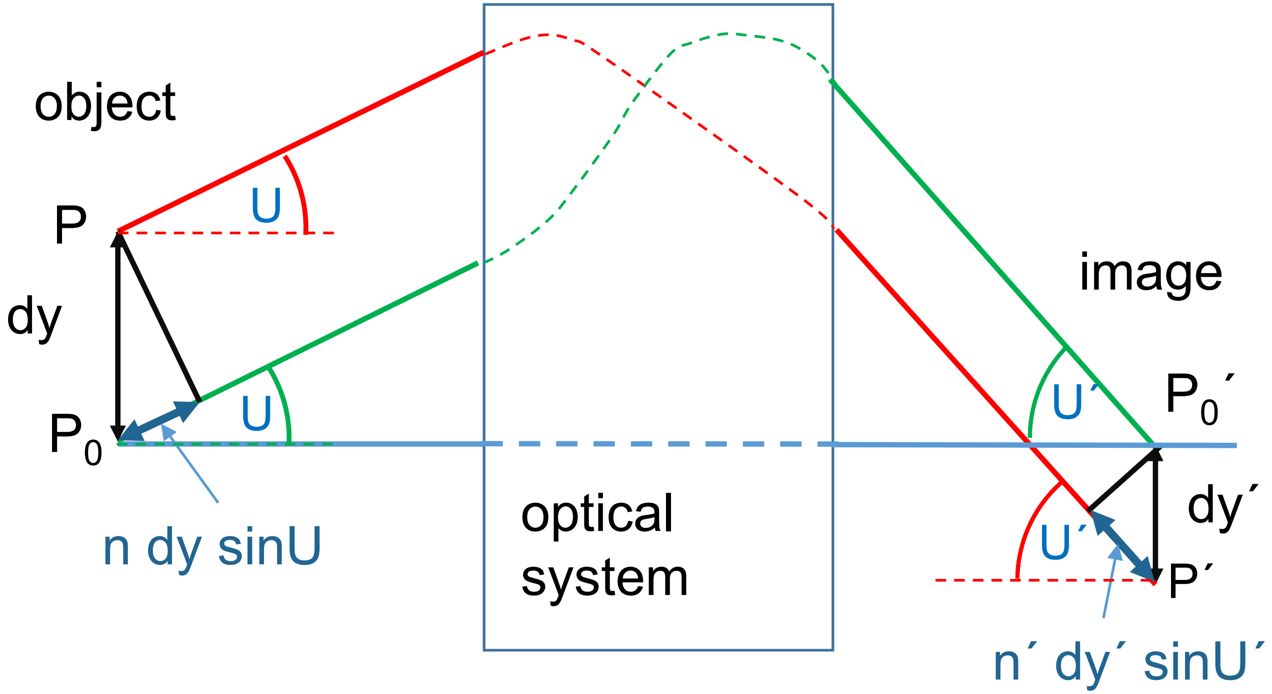


Abbe's sine condition:

$$m \frac{n´}{n} \sin U' = \sin U$$

Consequently, for $\mathrm{NA} = n \ \sin U_{max}$:

$$m\, NA´ = NA$$

In accordance with local étendue conservation law.

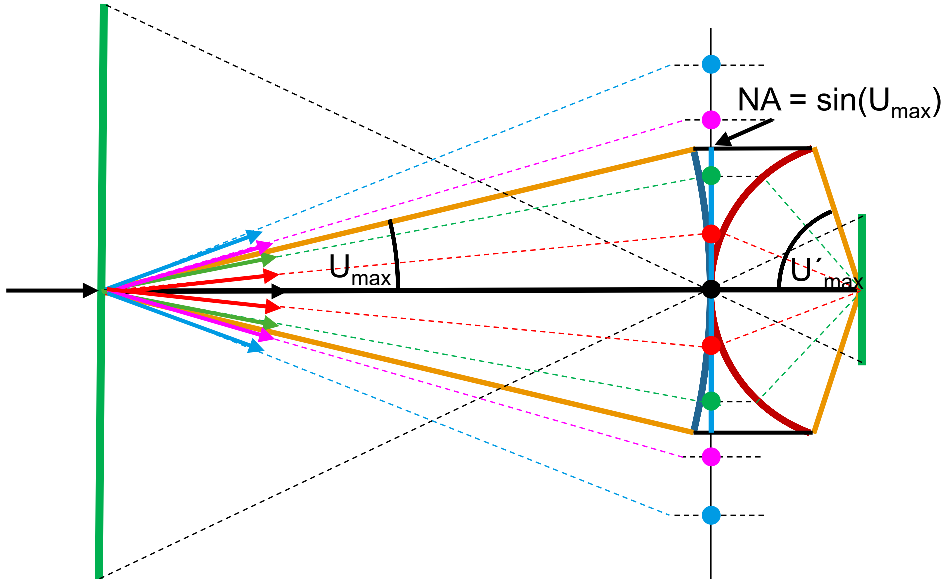
Assume that the image on axis, between P_0 and P_0´ is stigmatic (or free of spherical aberration) even for large aperture angles U and U´ (denoted with large letter U to distinguish paraxial u). Then, according to Fermat´s principle, the optical path length is equal for any optical path between P_0 and P_0´ (green path and e.g., on-axis path).

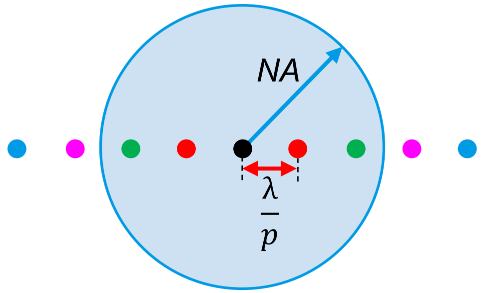
To achieve a stigmatic image between object and image plane, oriented normal to the optical axis, the optical path length along a "directly adjacent" (infinitesimal distances dy and dy´ respectively) off-axis optical path (drawn red) must also be equal to the on-axis path length. Consequently, as the change of the aperture angles U, U´ is negligible over the distance dy, dy´:

$$\mathrm{n´\ dy´} \sin U' = n\, dy \sin U$$

Coherent image formation with periodic object

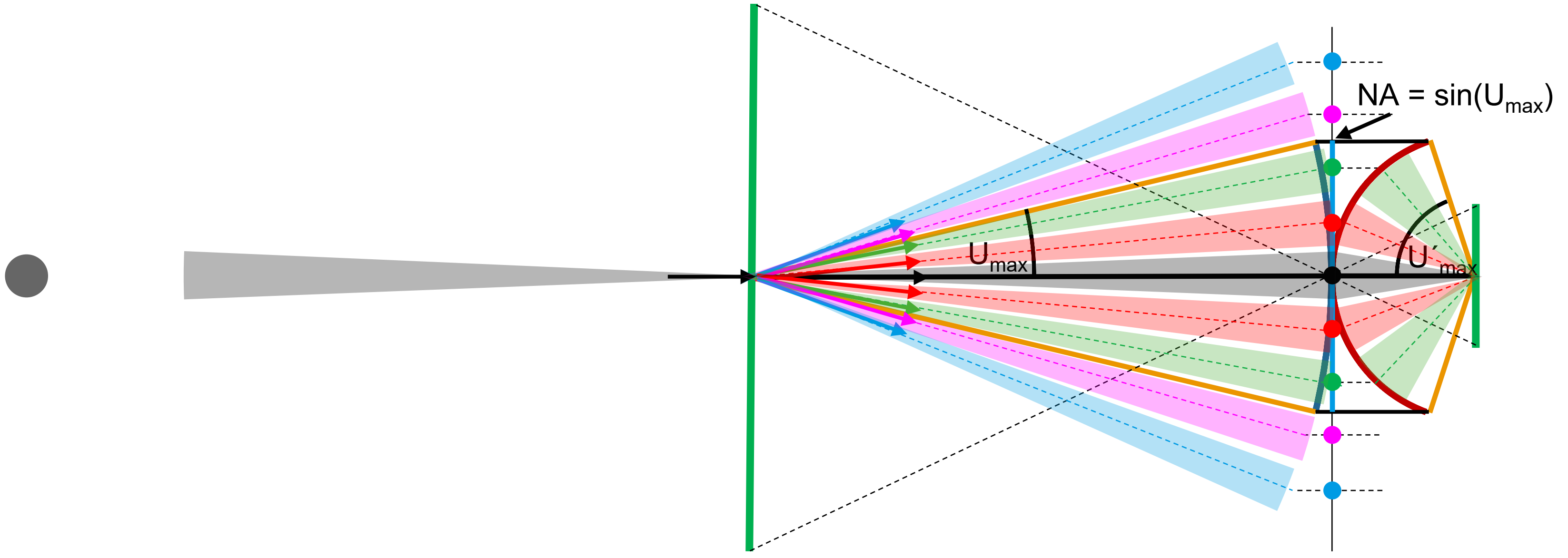
- Aplanatic system fulfills "Abbe sine-condition $\sin U = m \sin U'$, therefore we chose sine-scaled variables $\alpha = \sin U$
- As the object angles of a diffraction grating also scale according to the sine of the object angle, $\sin U = m \frac{\lambda}{p}$, the diffraction orders are <u>equidistant in the pupil</u>



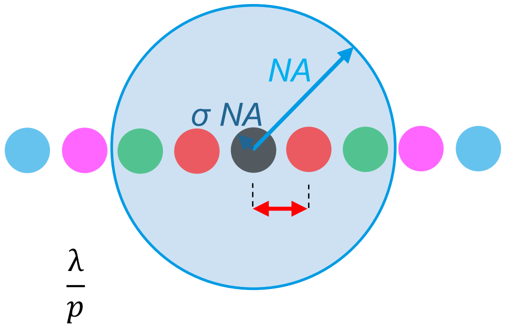


Partially coherent image formation with periodic object

effective light source
(in illumination pupil
conjugate to projection
lens pupil)

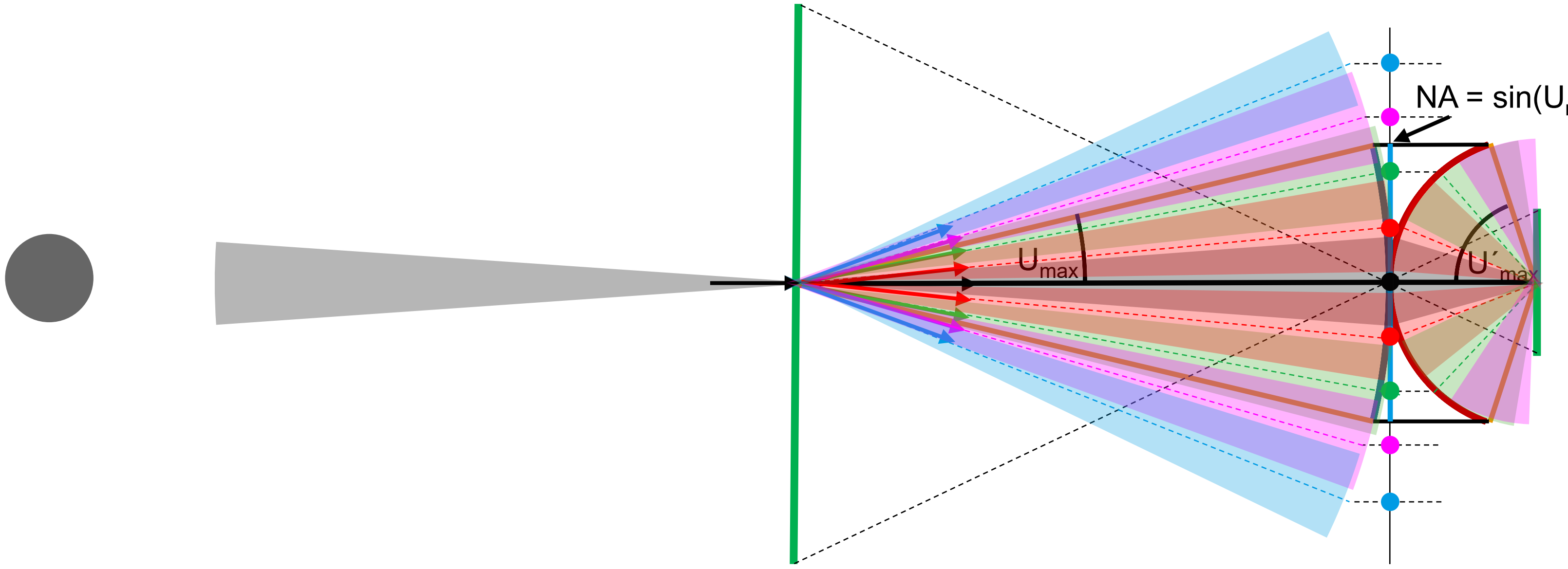


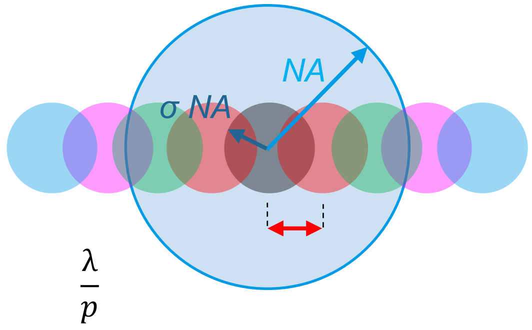
Every source point within the effective source distribution is an off-axis light source giving rise to a shift of the diffraction orders. Altogether in the lens pupil the effective source distribution appears at the position of each (coherent) diffraction order, again with the relative weight given by the diffraction order efficiency.



Partially coherent image formation with periodic object

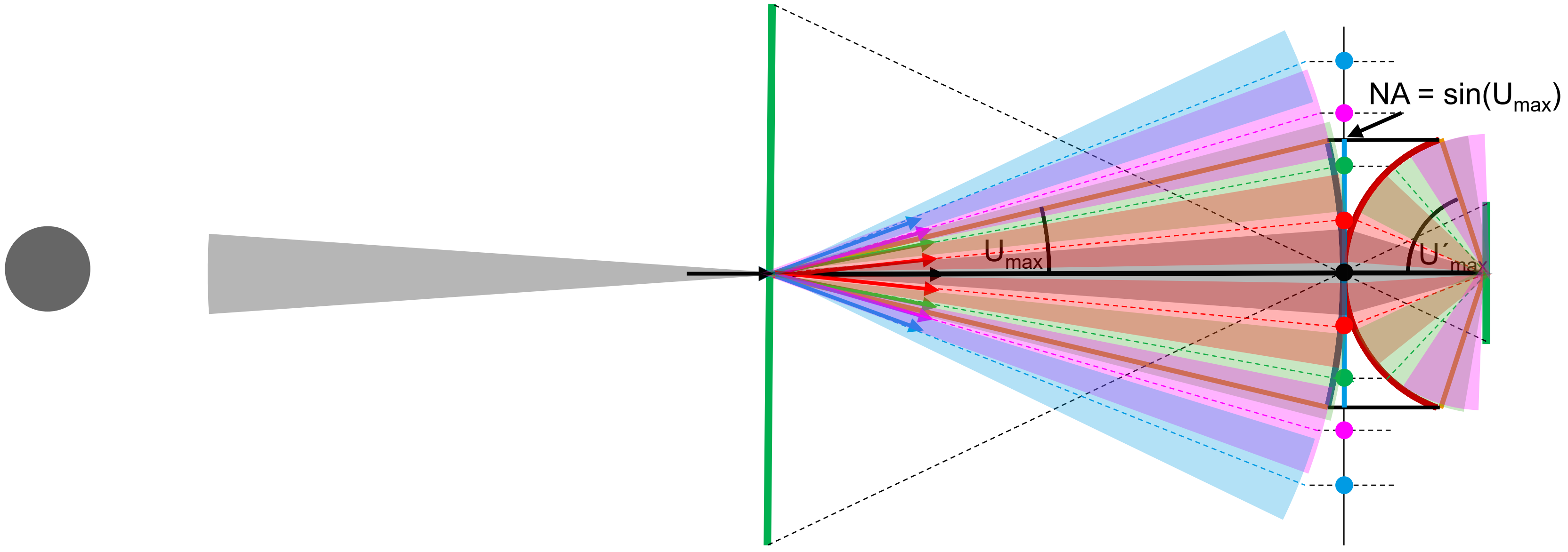
Larger light source

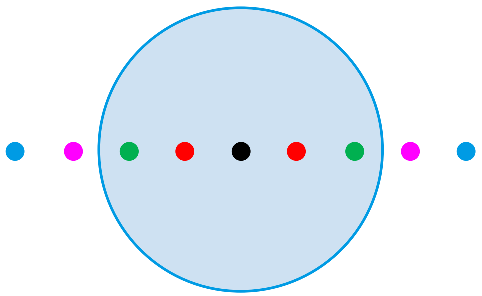




Partially coherent image formation with periodic object

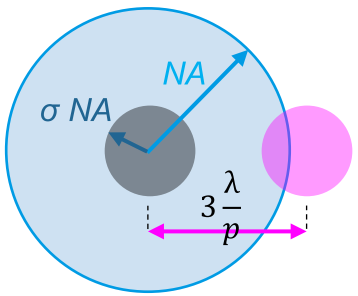
Larger light source



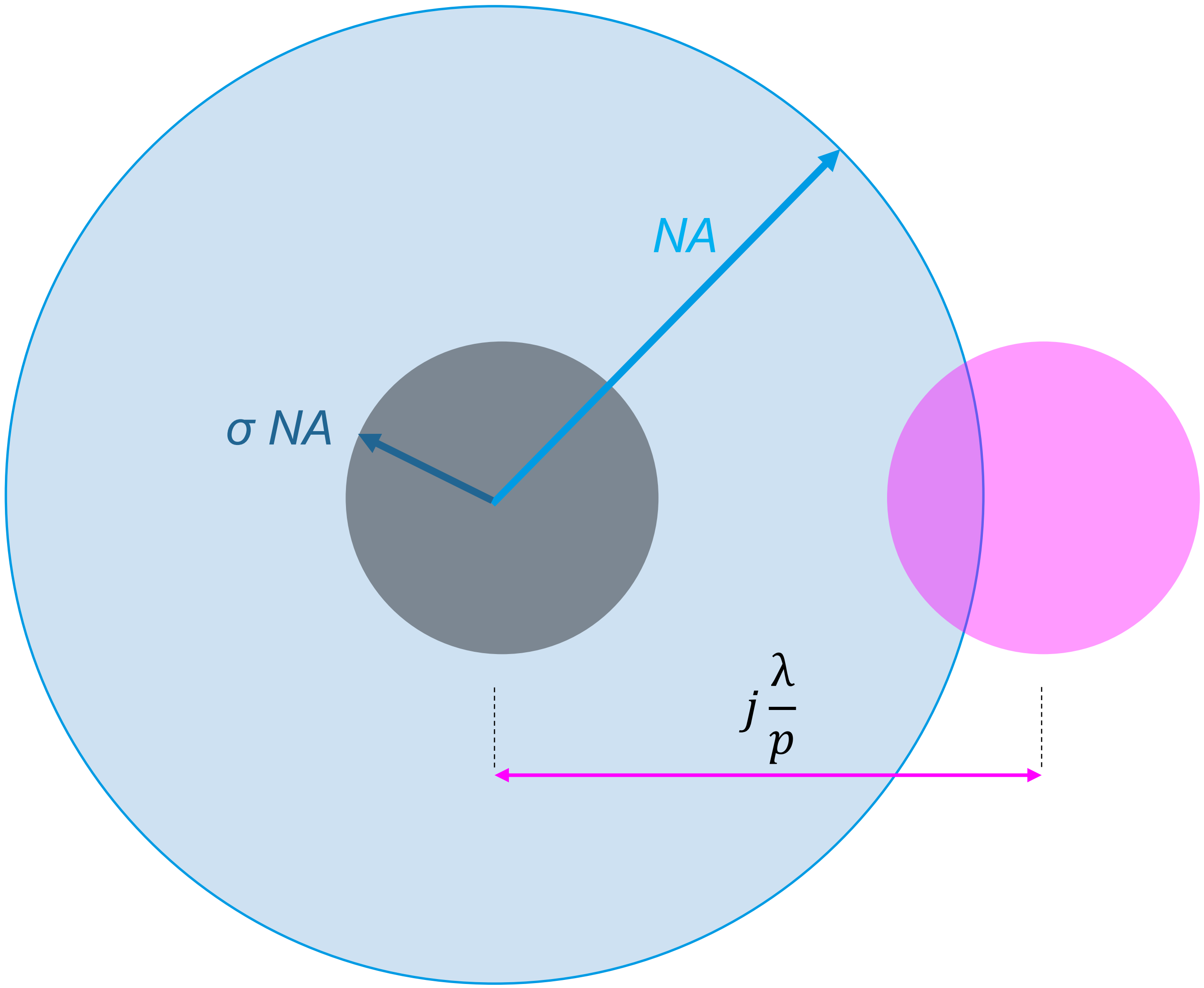


3rd diffraction order now also contributes to image formation

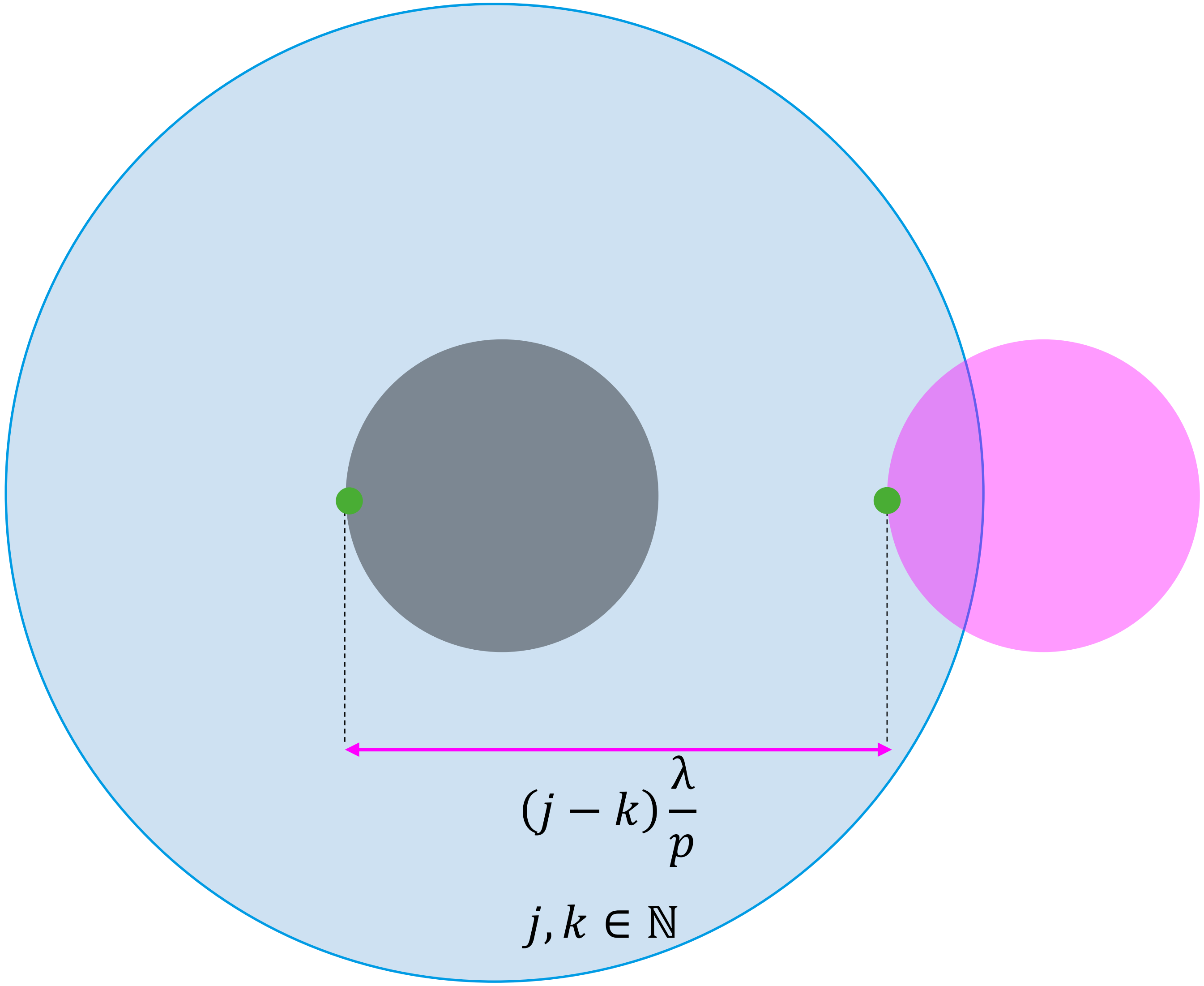
$$3\frac{\lambda}{p} - \sigma\, NA < NA$$



Partially coherent image formation with periodic object



Partially coherent image formation with periodic object



Interfering points between the source distribution of the diffraction orders have a distance

$$(j-k)\frac{\lambda}{p}$$

They contribute a harmonic order

$$I_{j,k}\, cos\left(2\pi x \frac{j-k}{p}\right).$$

In case of a pure amplitude grating the coefficient is given by

$$I_{j,k} = T_j\, T_k$$

with the diffraction efficiencies

$$T_j = \frac{1}{p}\int_{-\frac{p}{2}}^{\frac{p}{2}} t_{ob}(\xi) exp\left(-i2\pi\, \xi\, \frac{j}{p}\right) d\xi.$$

Partially coherent image formation with periodic object

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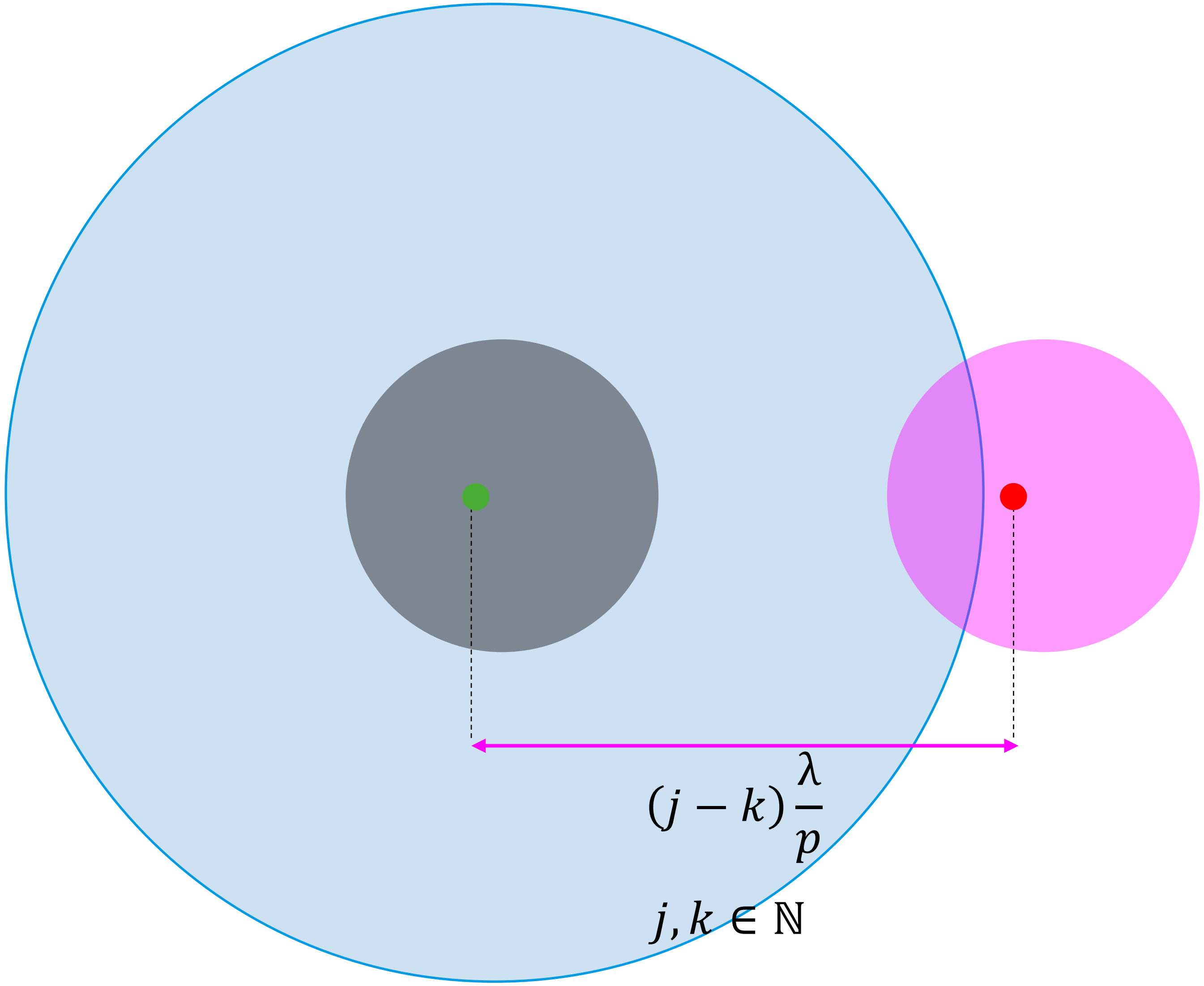
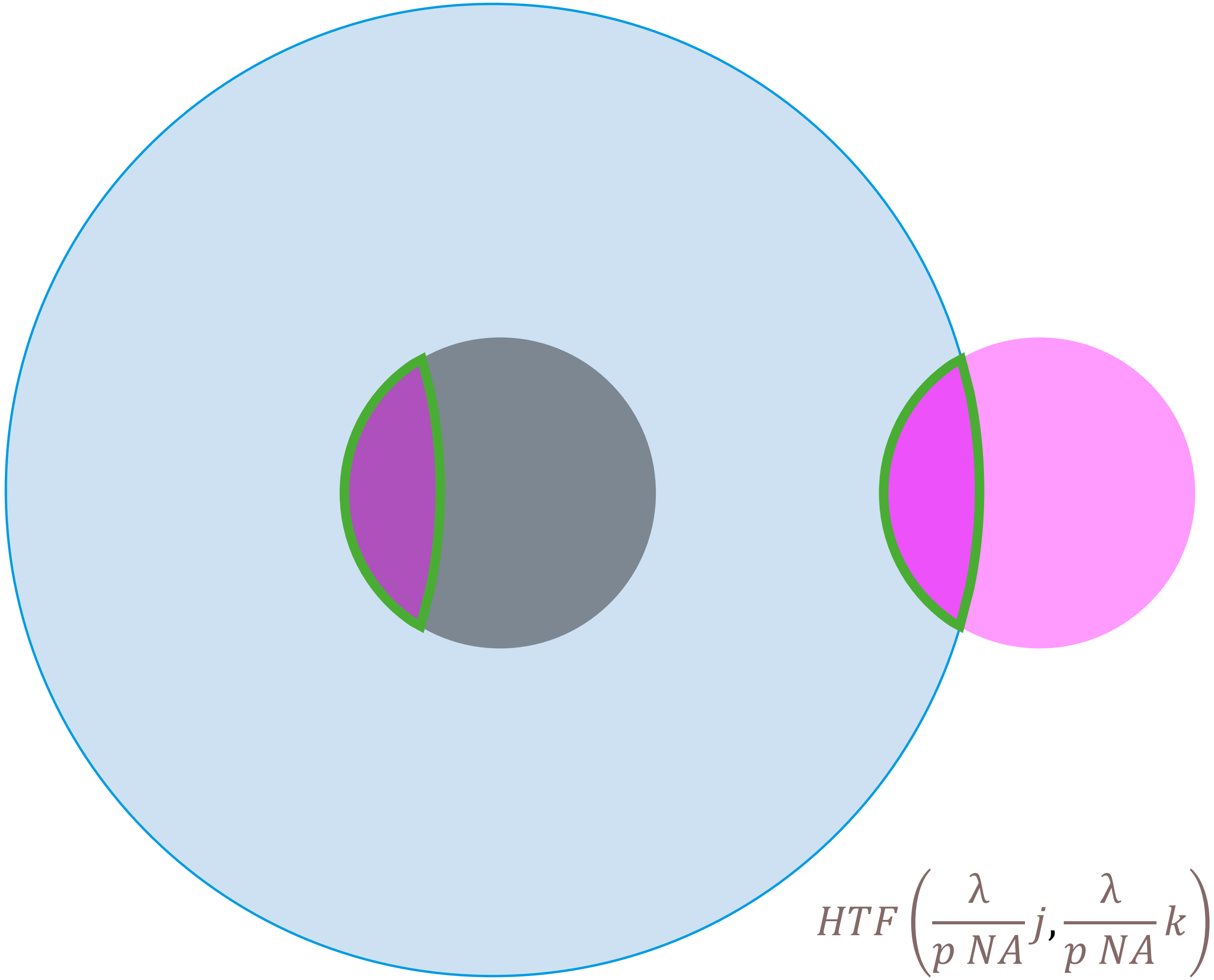


If one of the interfering points is outside of the lens pupil, this pair if does not contribute a harmonic component in the image distribution.

Partially coherent image formation with periodic object



For a pair of diffraction orders only the area indicated in the figure contributes to harmonic components.

The mutual interference area is given by the relatively shifted common area of the source pattern at the diffraction order and the pupil

$$Q(a)\, L\left(\frac{\lambda}{p\, NA} j + \sigma\, a\right)$$

respectively for the interference partner

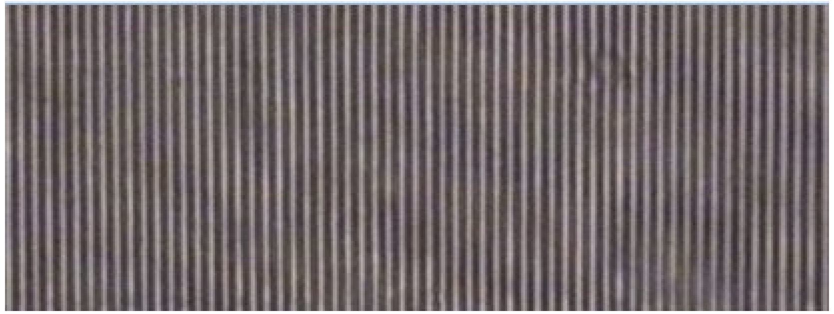
$$Q(a)\, L^{*}\left(\frac{\lambda}{p\, NA} k + \sigma\, a\right),$$

that is:

$$HTF\left(\frac{\lambda}{p\, NA} j, \frac{\lambda}{p\, NA} k\right) = \iint da\, Q(a)\, L\left(\frac{\lambda}{p\, NA} j + \sigma\, a\right) L^{*}\left(\frac{\lambda}{p\, NA} \mathrm{k} + \sigma\, a\right).$$

Resolution, Spatial Frequencies and wavelength dependence

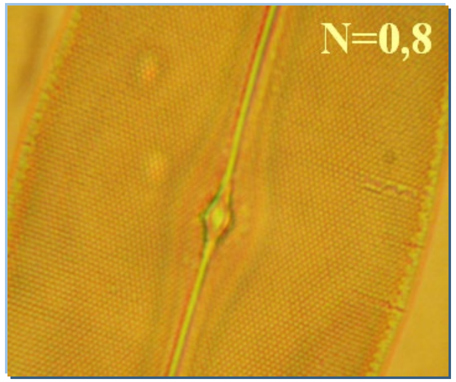
Grating object

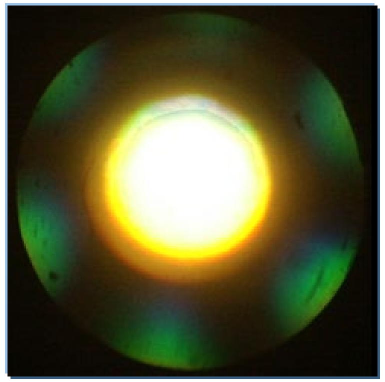


pupil

Note that with $\sin\vartheta = m\frac{\lambda}{p}$ the diffraction order position depend on wavelength.
→ it may happen, that red is not transferred to image, whereas blue is

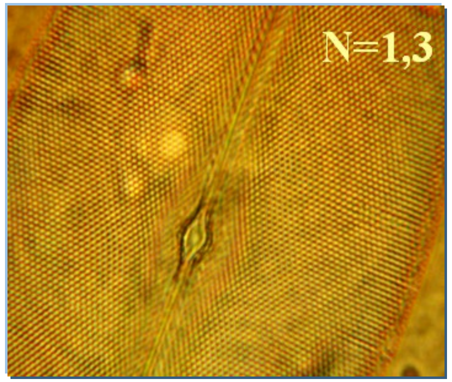
Imaging with **NA = 0.8**





Periodic objects only partly transferred in low order

Imaging with **NA = 1.3**



Increased NA enables visibility of extremely fine periodic structures

Ref: L. Wenke

Image Synthesis with Hopkins Transfer Function

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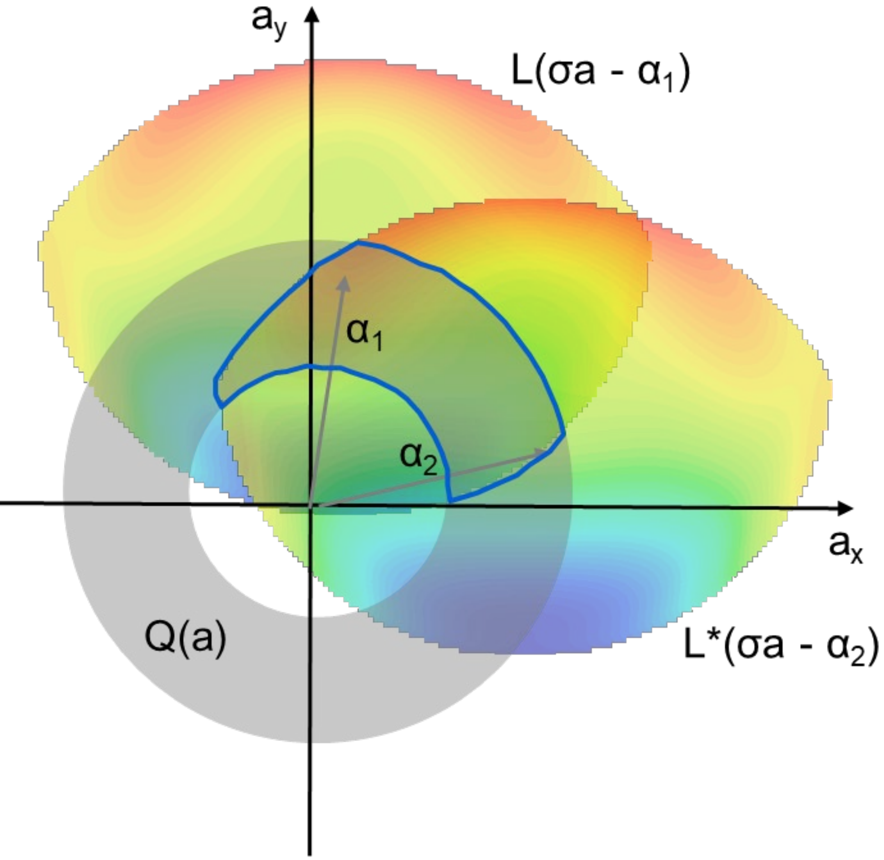
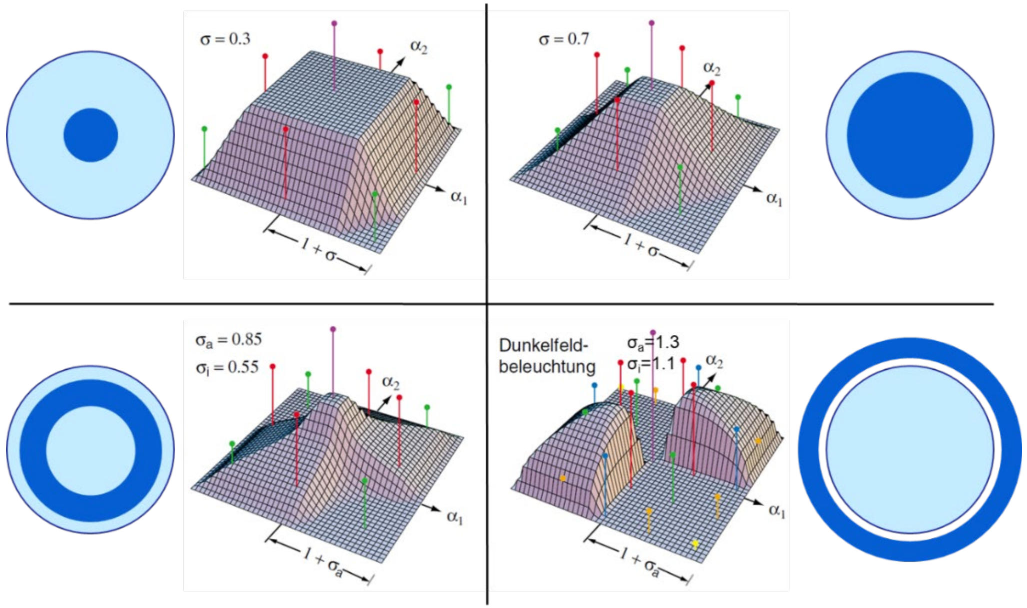
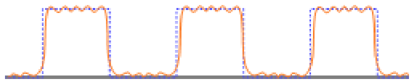




$$I(x) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} I_{j,k} \cos\left(2\pi x \frac{j-k}{p} - \Phi_{jk}\right)$$



$$R_{j,k} = I^r_{j,k} HTF^r(j,k) - I^i_{j,k} HTF^i(j,k),$$

$$I_{j,k} = \sqrt{R^2_{j,k} + Q^2_{j,k}} \qquad \Phi_{jk} = \arctan\left(\frac{Q_{j,k}}{R_{j,k}}\right)$$

HTF in general complex.

For ideal pupil real function.

Applications:

- Visibility optimization depending on object in microscopy
- Optical proximity correction and Source mask optimization in optical lithography

Zernike Phase Contrast Microscopy

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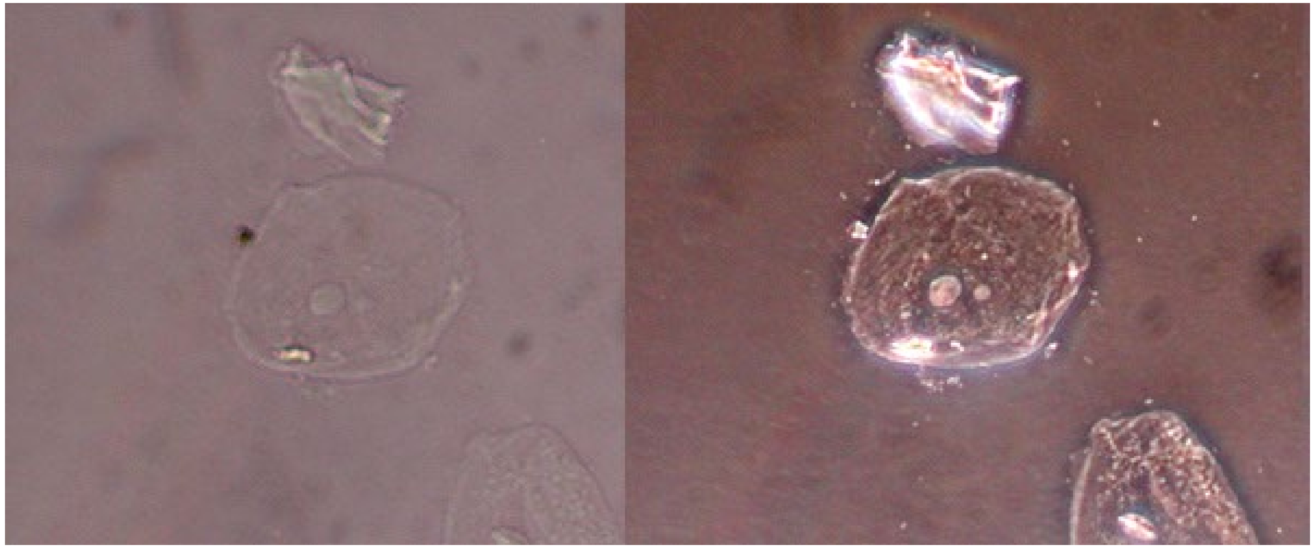
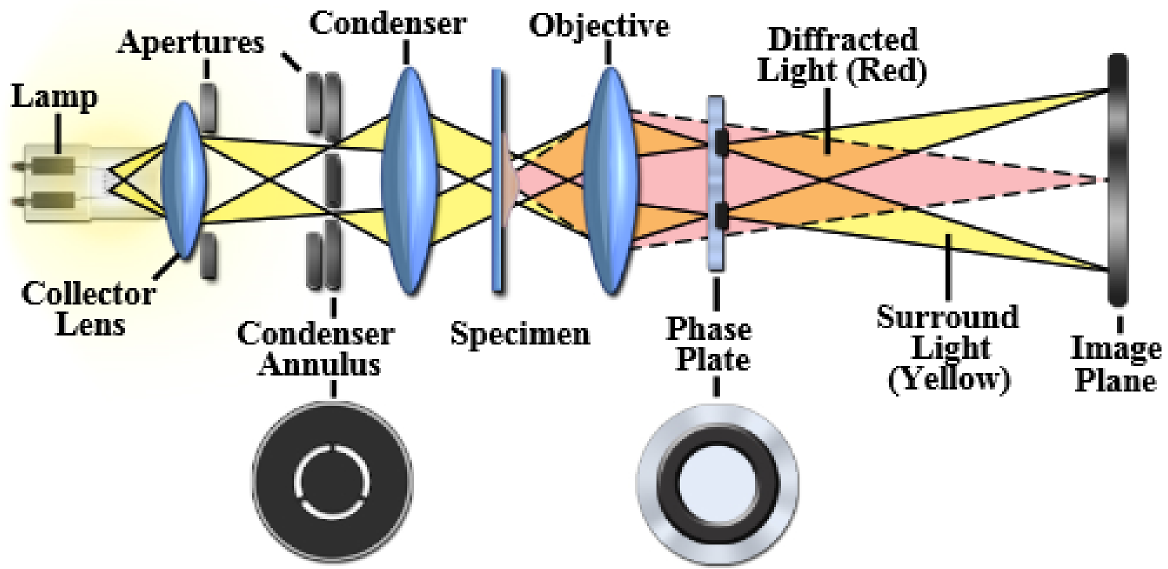
Deliberately introduced “aberration” in lens pupil makes phase objects visible!

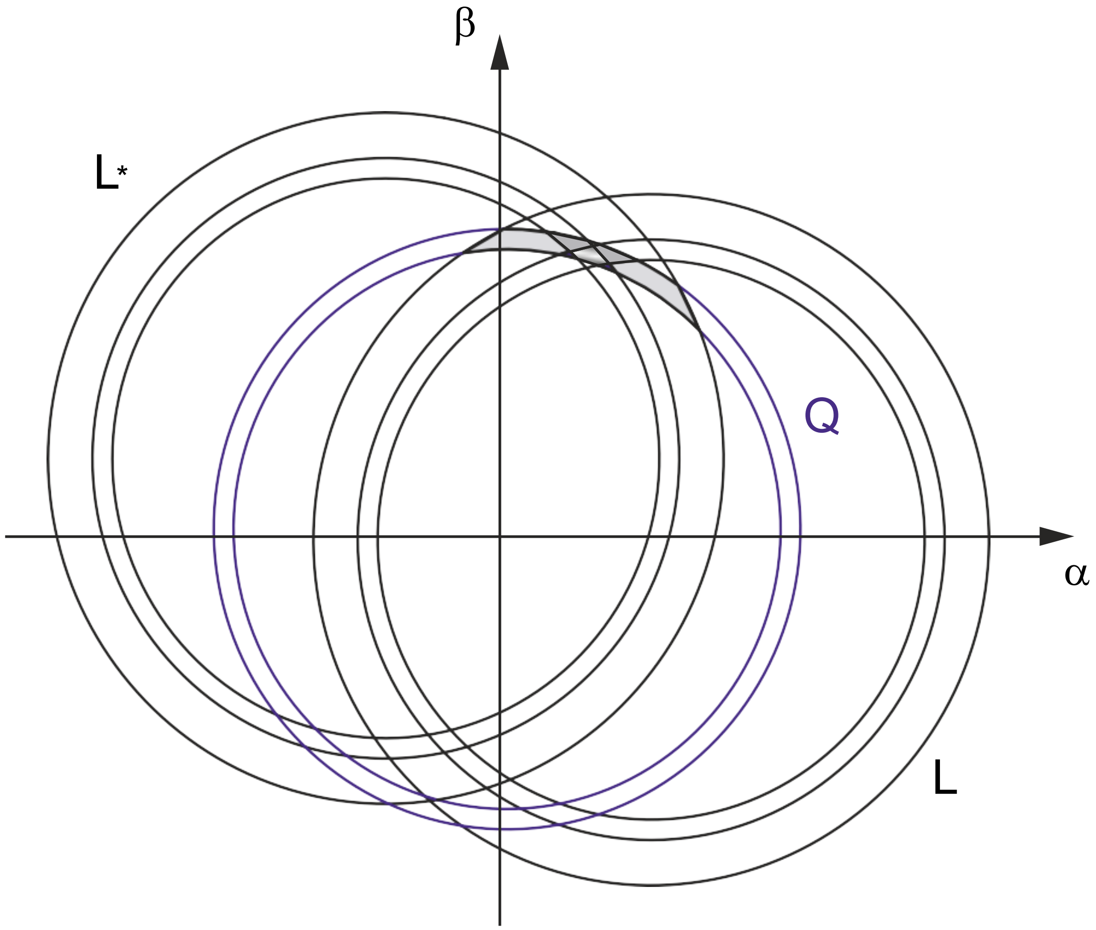
Frits Zernike, inventor of phase-contrast method for microscopy (Nobel prize physics 1953)



Cell with traditional bright-field microscopy (left) and phase-contrast microscopy (right)



Phase contrast microscopy

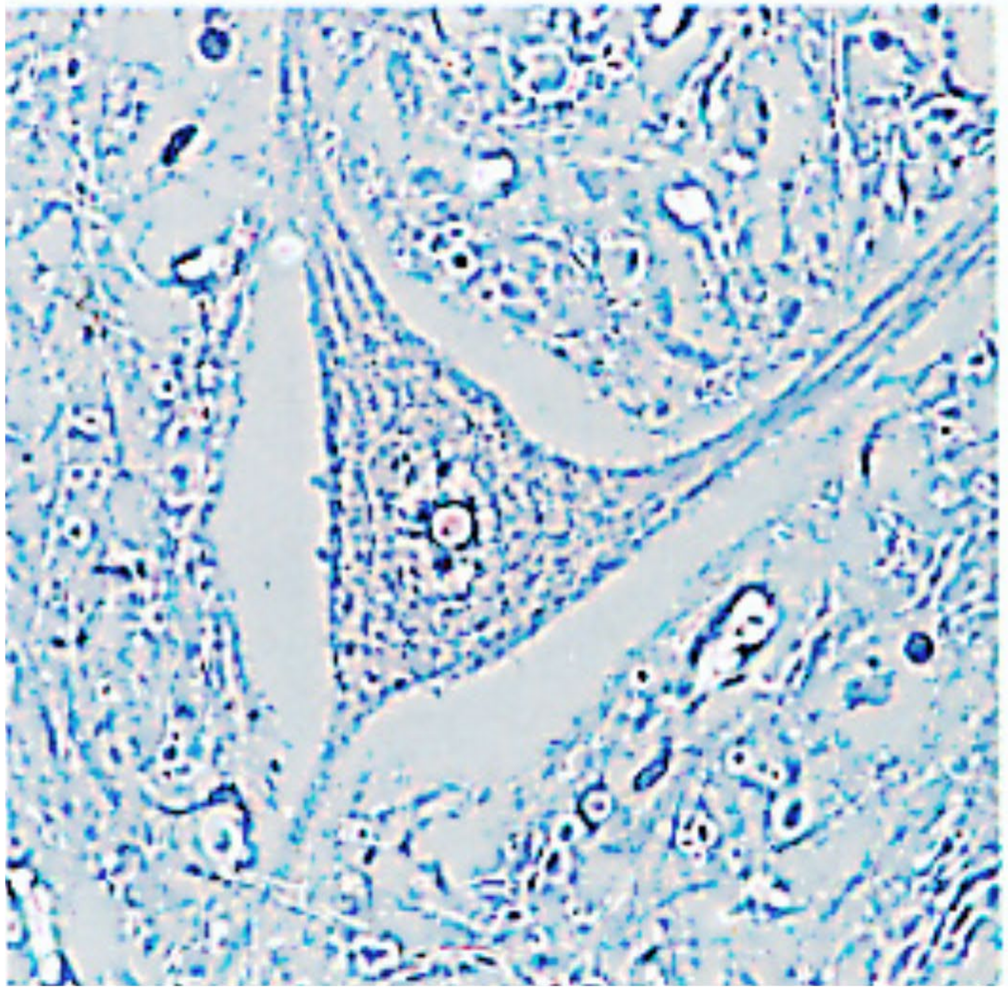


Hopkins Transfer Function:

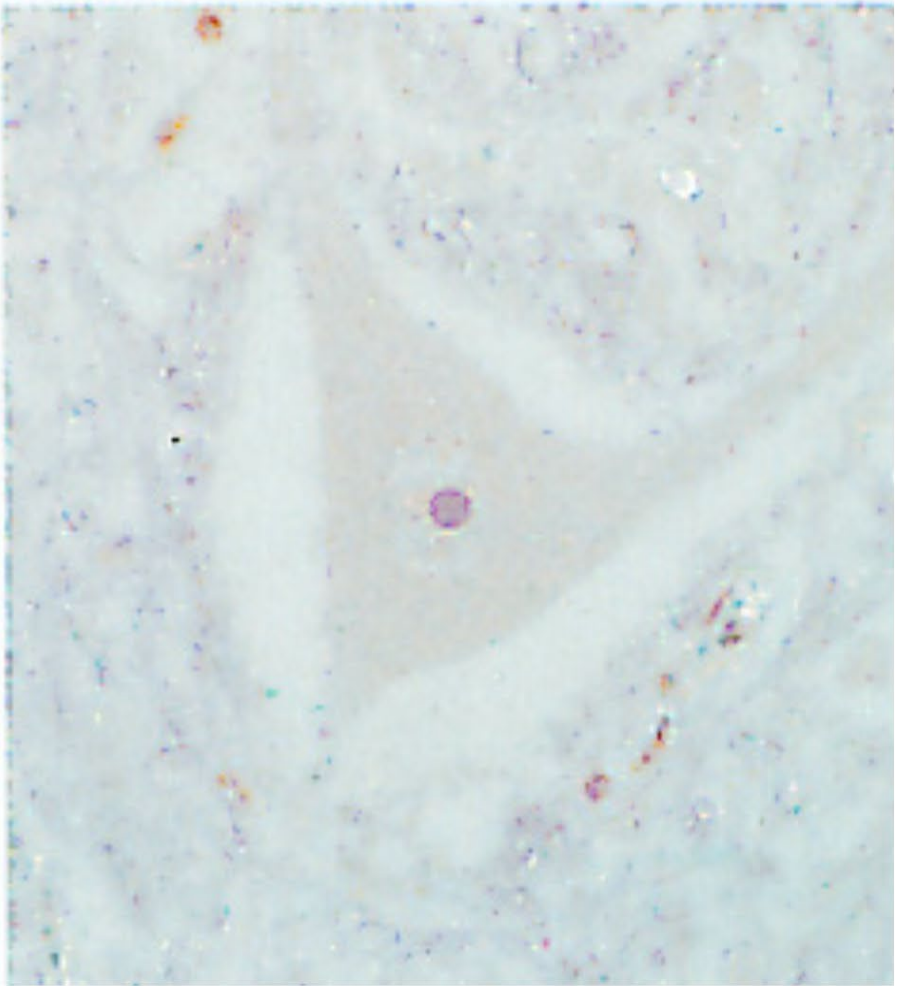
- Annular illumination
- Matched size phase ring in projection lens pupil

Zernike Phase Contrast

Phase contrast

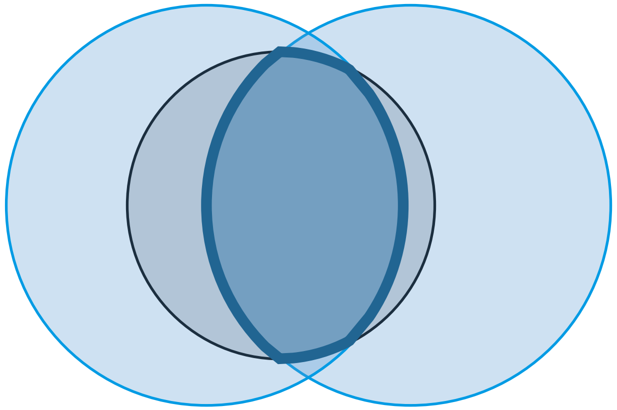


Conventional bright field



HTF: Non-coherent and coherent limit

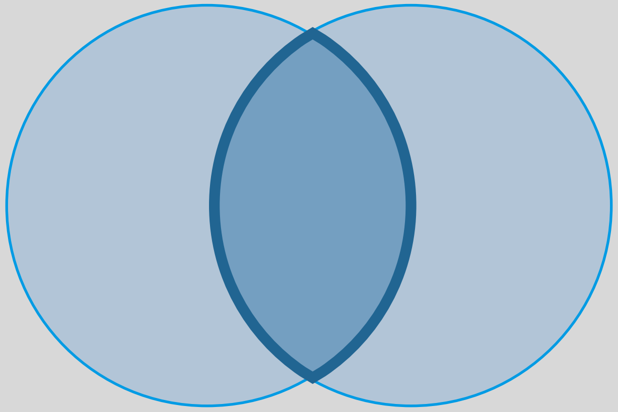
Partially coherent HTF



$$HTF(\alpha_1, \alpha_2) = \iint da\, Q(a)\, L(\alpha_1 + \sigma\, a\,) L^*(\alpha_2 + \sigma\, a\,)$$

Non-coherent limit

"$Q(a) \rightarrow \infty$"



"$Q(a) \rightarrow \infty$"

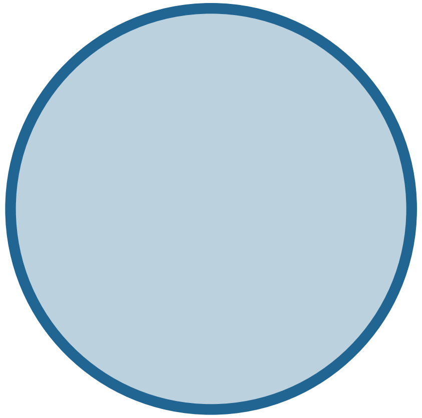
$$HTF(\alpha_1 - \alpha_2) = \iint da\;\, L(\alpha_1 - \alpha_2 + \sigma\, a\,) L^*(\sigma\, a\,)$$

$$OTF(\alpha) = \iint d\alpha'\, L(\alpha + \alpha') L^*(\alpha')$$

Transfer function for <u>intensity</u> spectrum

coherent limit

"$Q(a) \rightarrow \delta(a)$"



$$HTF(\alpha_1, \alpha_2) = \iint da\, \delta(a)\, L(\alpha_1 + \sigma\, a\,) L^*(\alpha_2 + \sigma\, a\,)$$

$$= L(\alpha_1\,) L^*(\alpha_2\,)$$

Transfer function for <u>amplitude</u> spectrum

Coherent approximation

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In the case of coherent imaging, the coherence function separates in the object coordinates:

$$\Gamma_{ob}(\xi_1,\xi_2) = A_{ob}(\xi_1)A^*_{ob}(\xi_2)$$

and it results with the integration order α, ξ from $I(x) = \iint d\xi_1 d\xi_2\, \Gamma_{ob}(\xi_1,\xi_2) A_{ob}(\xi_1)A^*_{ob}(\xi_2)K(x,\xi_1)K^*(x,\xi_2)$ directly the coherent imaging equation $I_{im\,,coh}(x) = \left|\iint d\xi\, A_{ob}(\xi)\, K(x,\xi)\right|^2$.

If the integration is first carried out via the object coordinate ξ and then the pupil integration, you get $I_{im\,,coh}(x) = \left|\iint d\alpha\, exp(-i2\pi\, w\alpha\cdot x/m) \iint d\xi\, A_{ob}(\xi)\, L(\xi,\alpha) exp(i2\pi\, w\alpha\cdot\xi)\right|^2$

In the spatial-stationary case, one can factor out L(α) from the ξ-integrals:

$$I_{im\,,coh}(x) = \left|\iint d\alpha\, L(\alpha) exp(-i2\pi\, w\alpha\cdot x/m) L(\alpha) \iint d\xi\, A_{ob}(\xi)\, exp(i2\pi\, w\alpha\cdot\xi)\right|^2$$

or with the notation „FT“ for Fourier transform compactly written as

$$I_{im\,,coh}(x) = \left|FT_\alpha\{L(\alpha) FT_\xi(A_{ob})(w\alpha)\}(x)\right|^2$$

The pupil function L can be interpreted as a **coherent transfer function** in the pupil variables because of the product relationship in the amplitude. Note: **coherent transfer function L acts on amplitude, non-coherent OTF on intensity**!

Coherent image formation for periodic object

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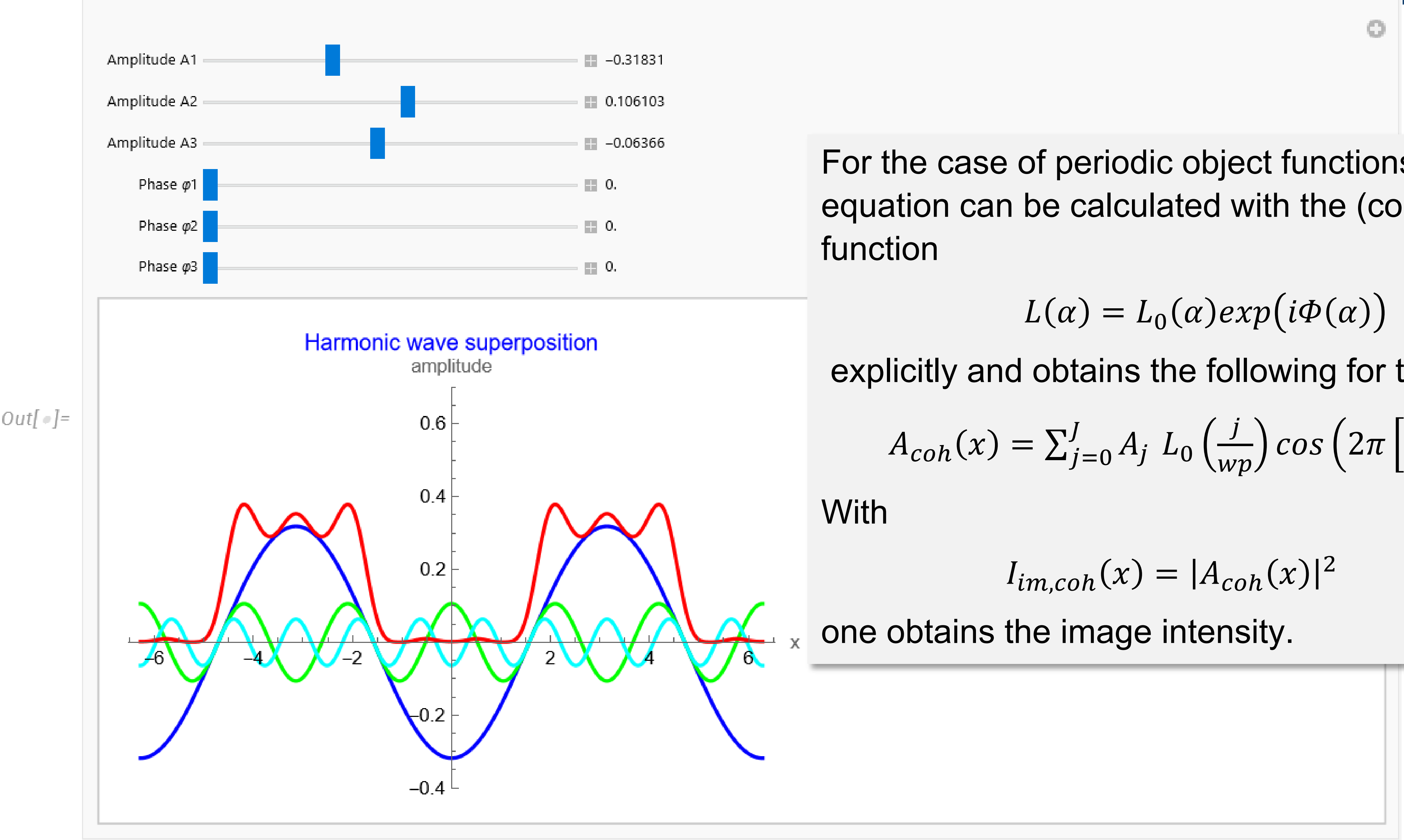

For the case of periodic object functions, the imaging equation can be calculated with the (coherent) transfer function

$$L(\alpha) = L_0(\alpha)exp\left(i\Phi(\alpha)\right)$$

explicitly and obtains the following for the amplitude

$$A_{coh}(x) = \sum_{j=0}^{J} A_j\ L_0\left(\frac{j}{wp}\right) cos\left(2\pi\left[\frac{j}{p}x + \Phi\left(\frac{j}{wp}\right)\right]\right).$$

With

$$I_{im,coh}(x) = |A_{coh}(x)|^2$$

one obtains the image intensity.

Resolution of Fourier Components

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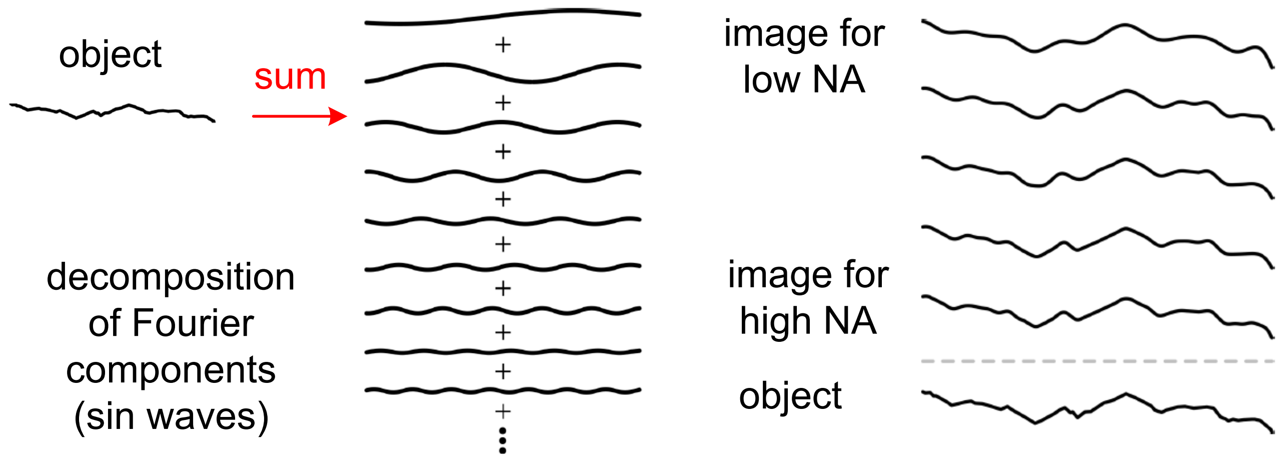
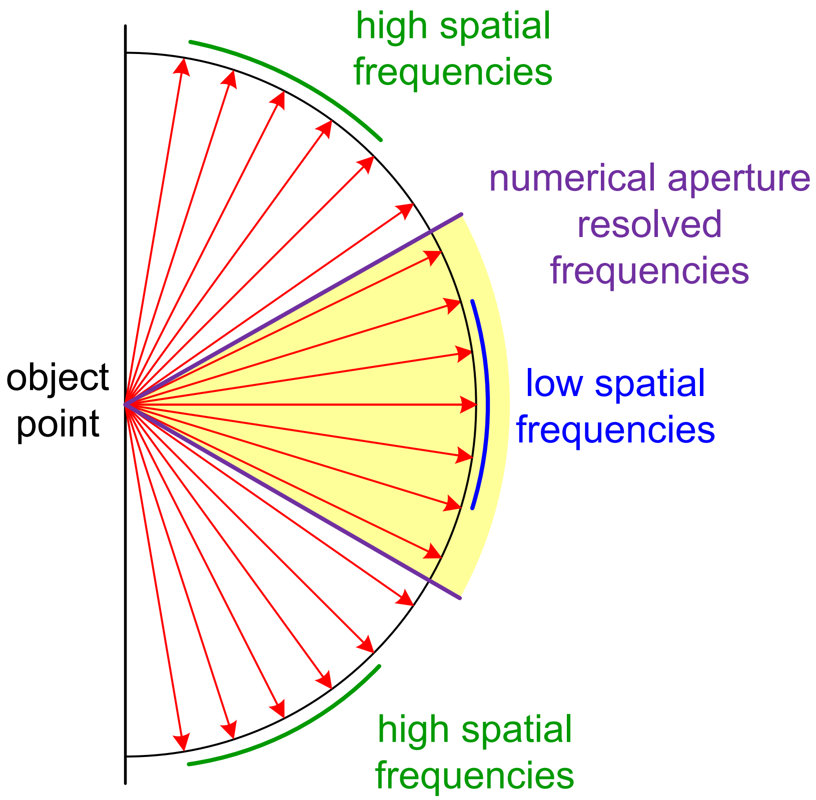
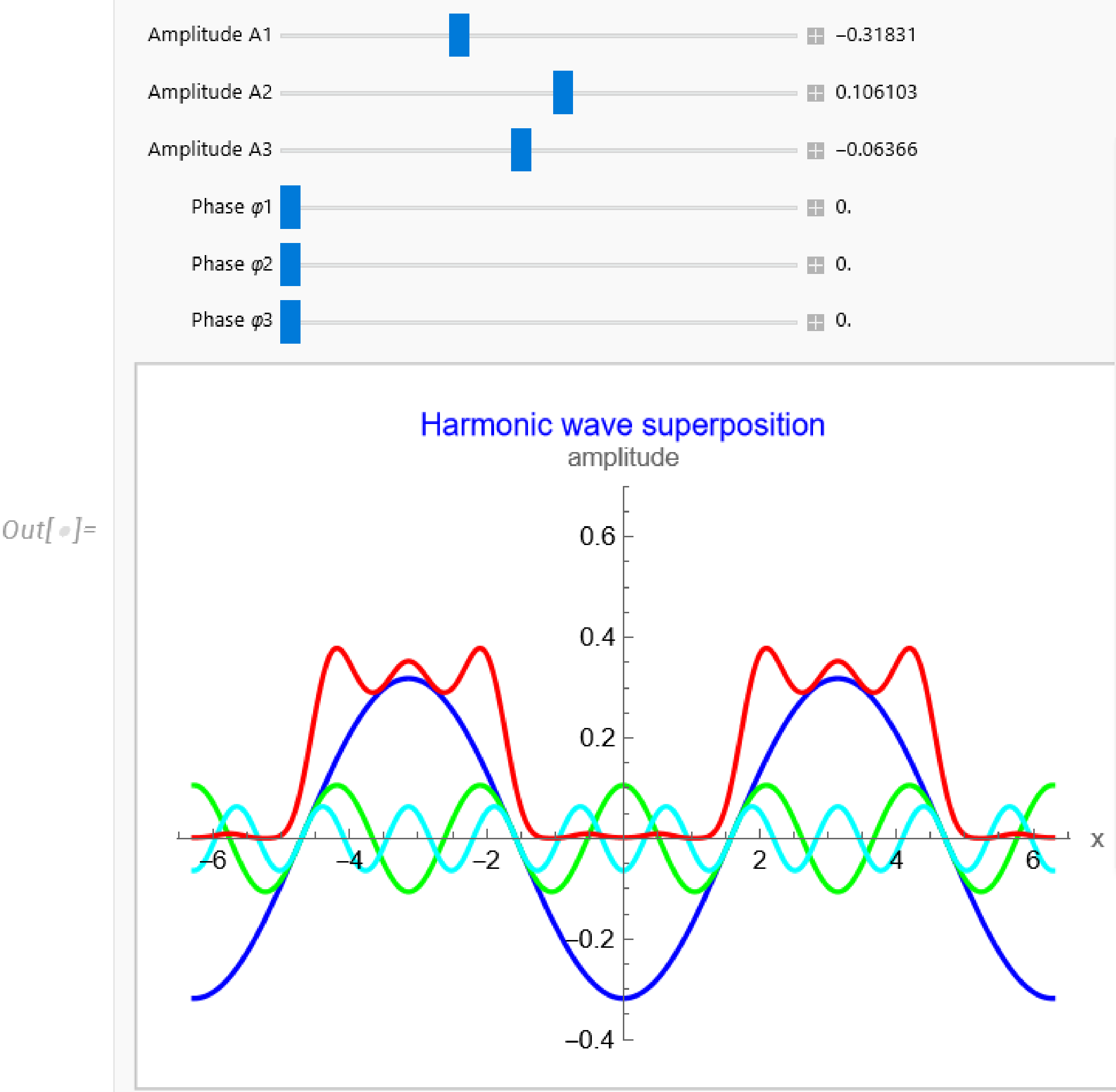




Ref: D.Aronstein / J. Bentley

Coherent image formation for periodic object



For the case of periodic object functions, the imaging equation can be calculated with the (coherent) transfer function

$$L(\alpha) = L_0(\alpha) exp\big(i\Phi(\alpha)\big)$$

explicitly and obtains the following for the amplitude

$$A_{coh}(x) = \sum_{j=0}^{J} A_j \; L_0\left(\frac{j}{wp}\right) cos\left(2\pi\left[\frac{j}{p}x + \Phi\left(\frac{j}{wp}\right)\right]\right).$$

With

$$I_{im\,,coh}(x) = |A_{coh}(x)|^2$$

one obtains the image intensity.

Phase transfer function introduces relative shifts of harmonic orders

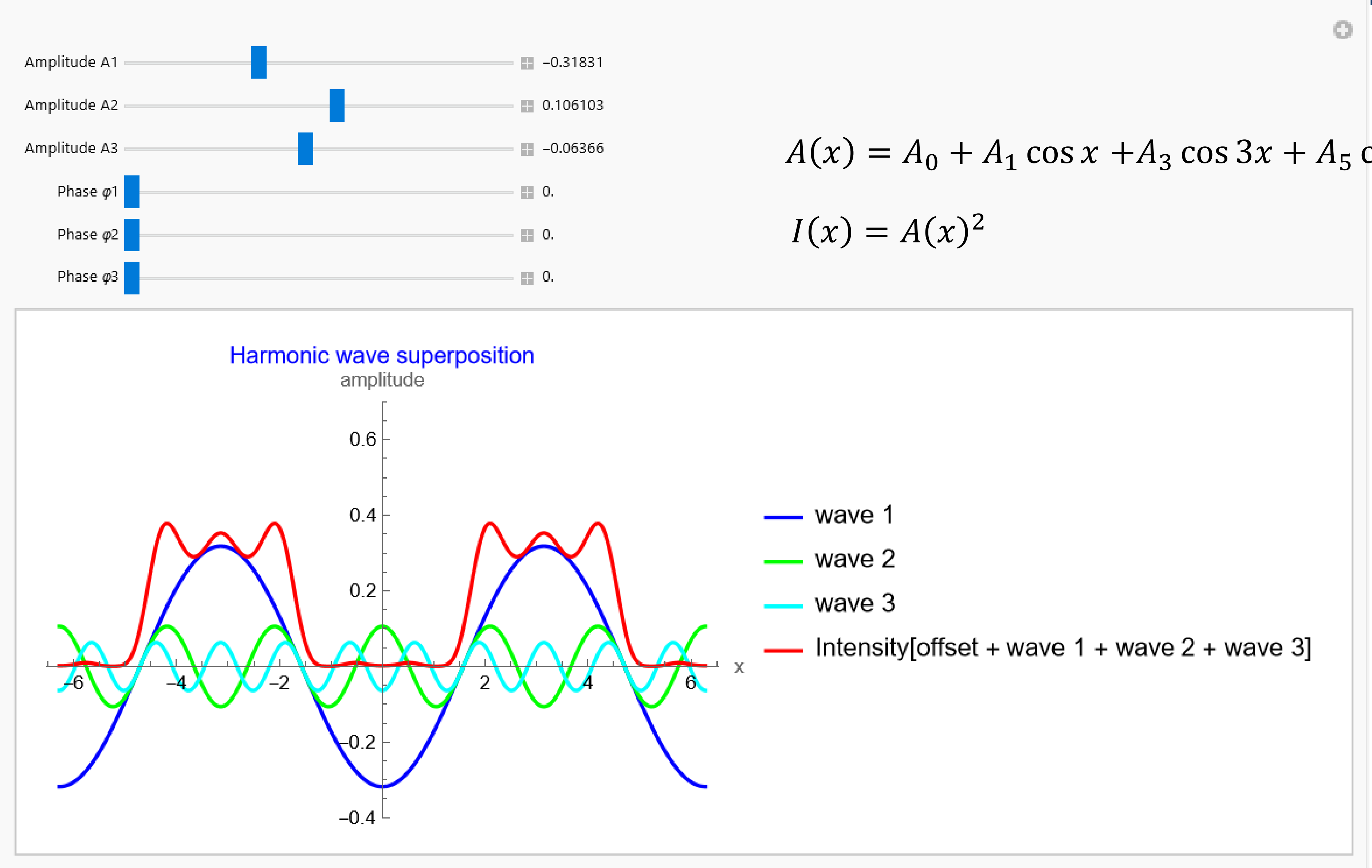
Completely analog in non-coherent case

Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

$$A(x) = A_0 + A_1 \cos x + A_3 \cos 3x + A_5 \cos 5x$$

$$I(x) = A(x)^2$$

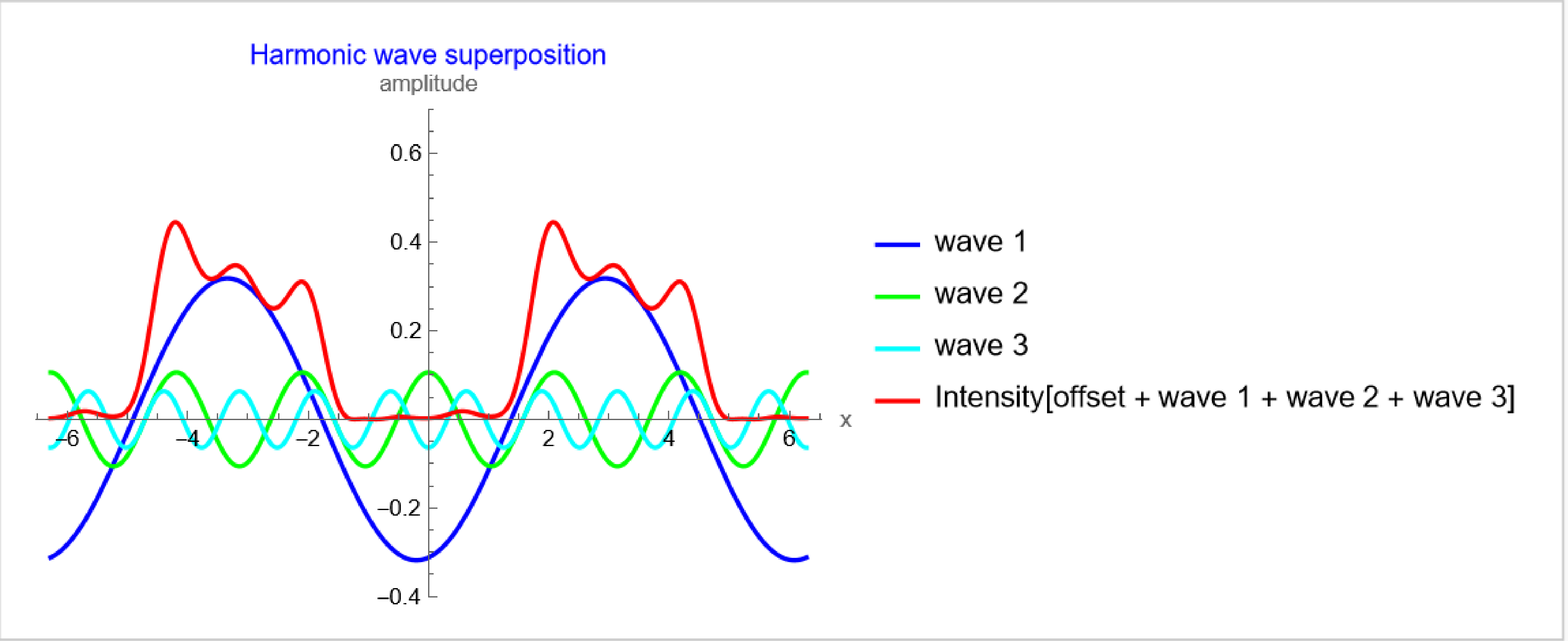
Out[ ]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

Amplitude A1	−0.31831
Amplitude A2	0.106103
Amplitude A3	−0.06366
Phase φ1	0.2
Phase φ2	0.
Phase φ3	0

Out[ ]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

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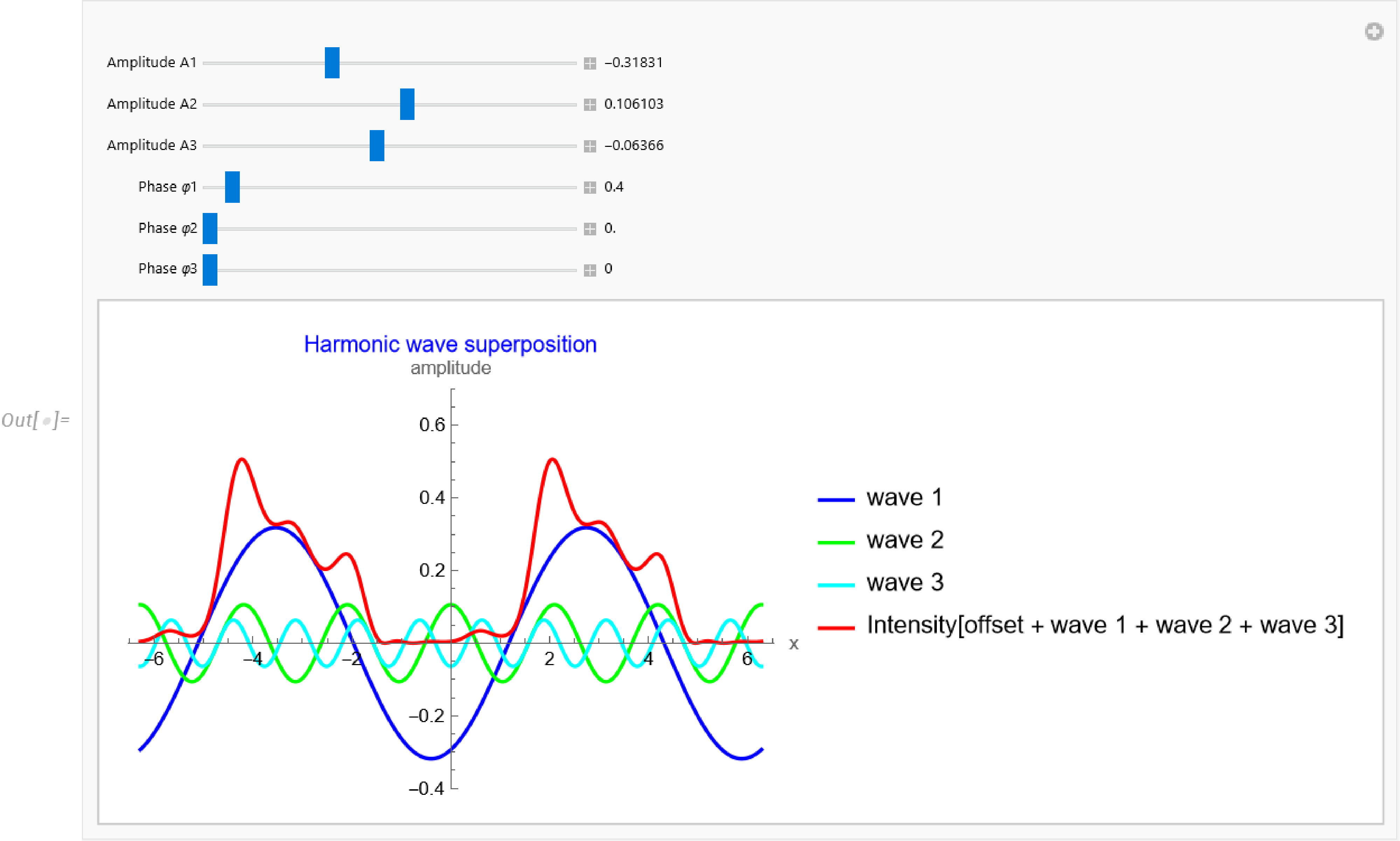
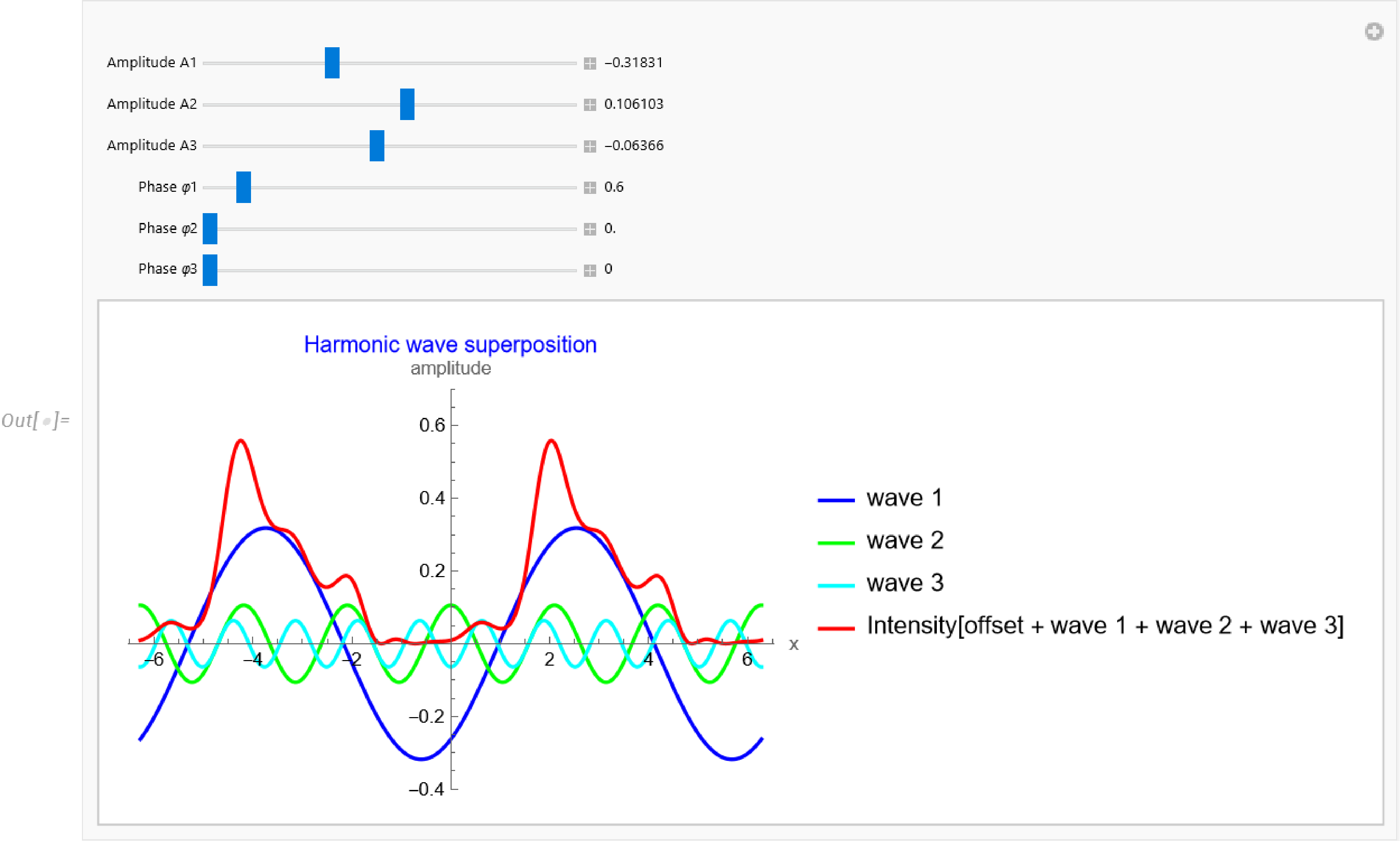


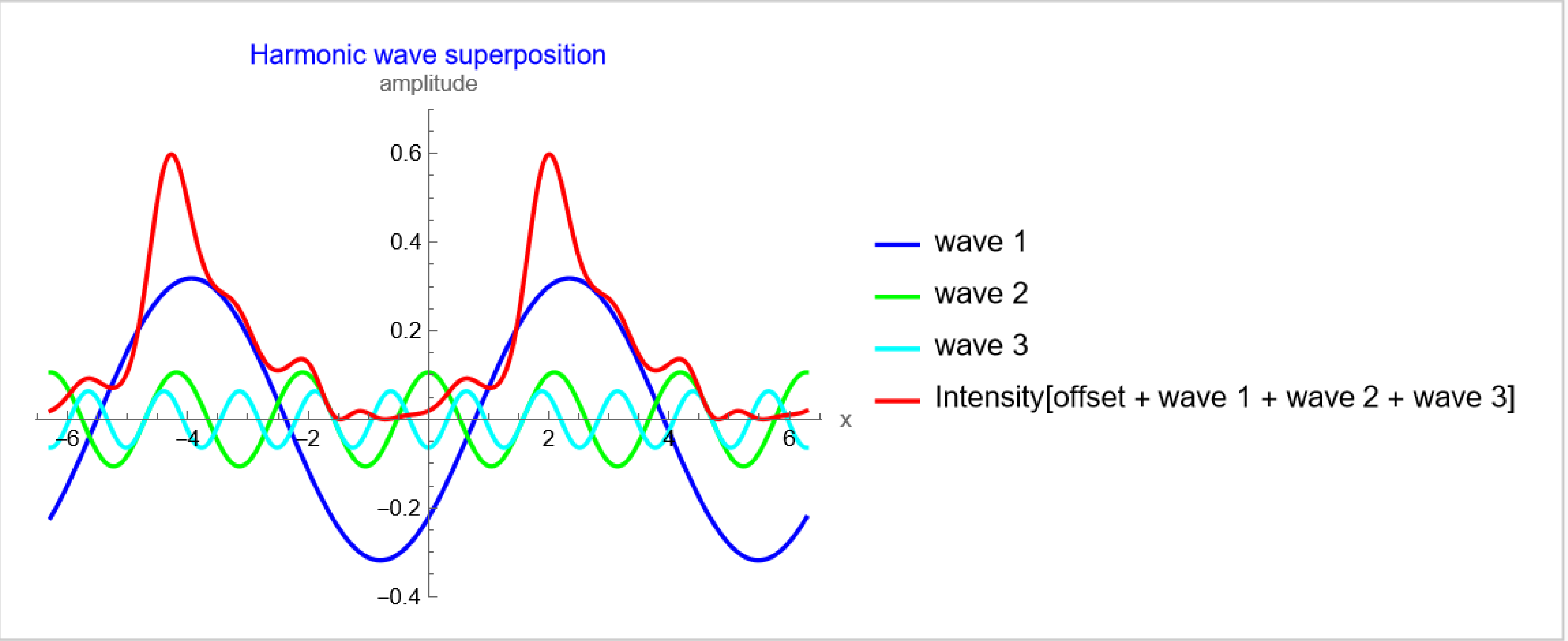
Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

Amplitude A1 −0.31831
Amplitude A2 0.106103
Amplitude A3 −0.06366
Phase φ1 0.8
Phase φ2 0.
Phase φ3 0

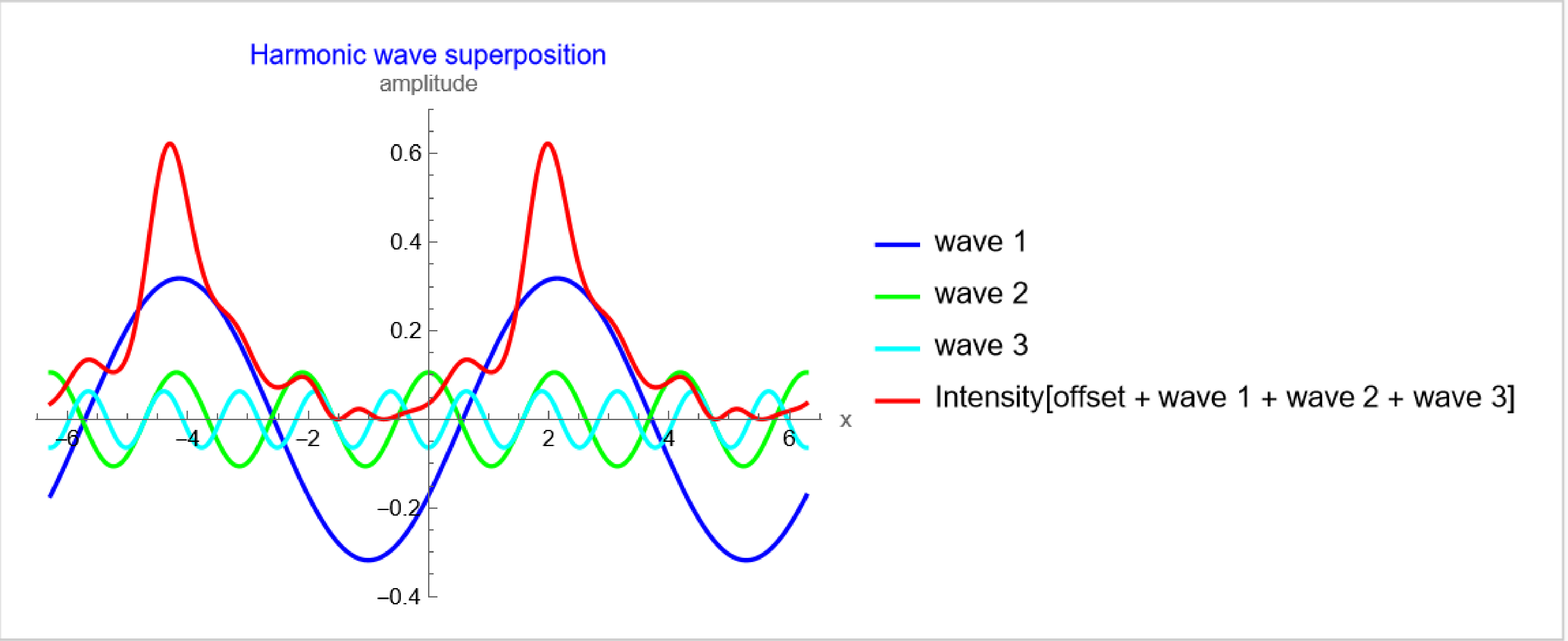
Out[ ]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

Amplitude A1 −0.31831
Amplitude A2 0.106103
Amplitude A3 −0.06366
Phase φ1 1
Phase φ2 0.
Phase φ3 0

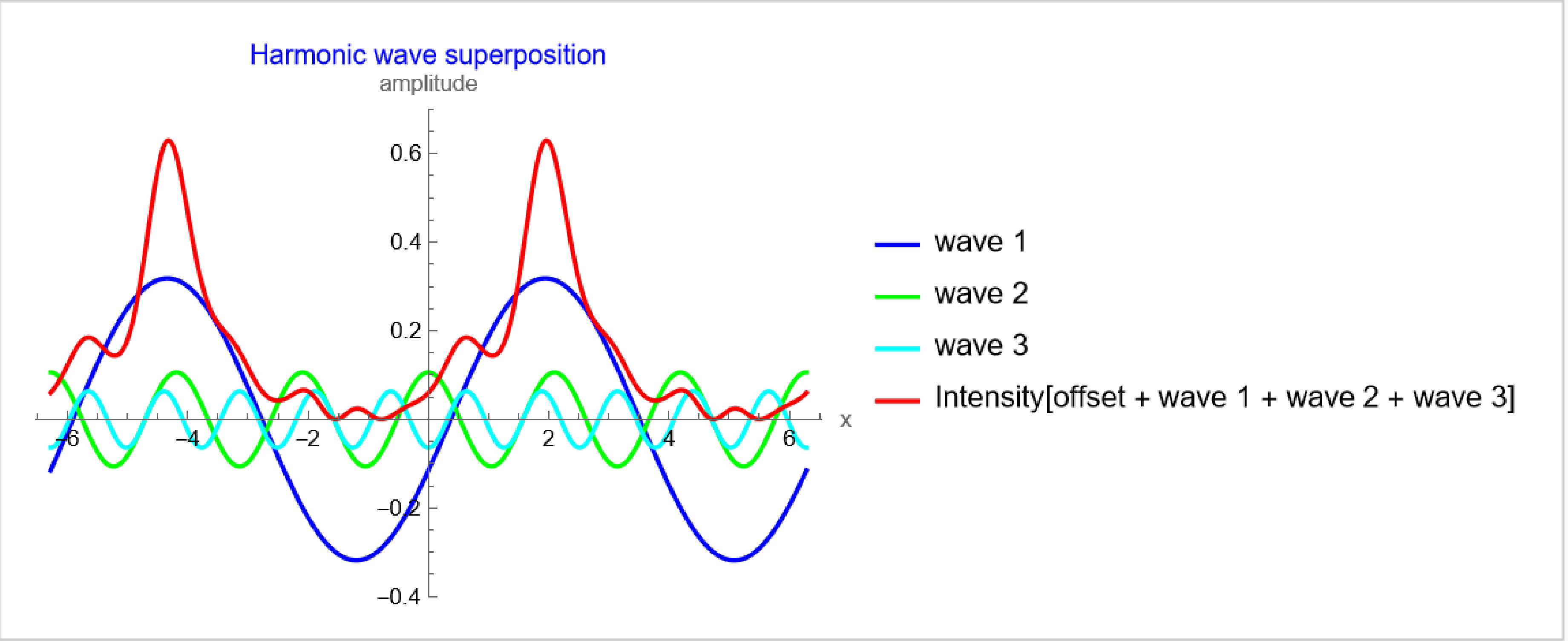
Out[ ]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

Amplitude A1 −0.31831
Amplitude A2 0.106103
Amplitude A3 −0.06366
Phase φ1 1.2
Phase φ2 0.
Phase φ3 0

Out[ ]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

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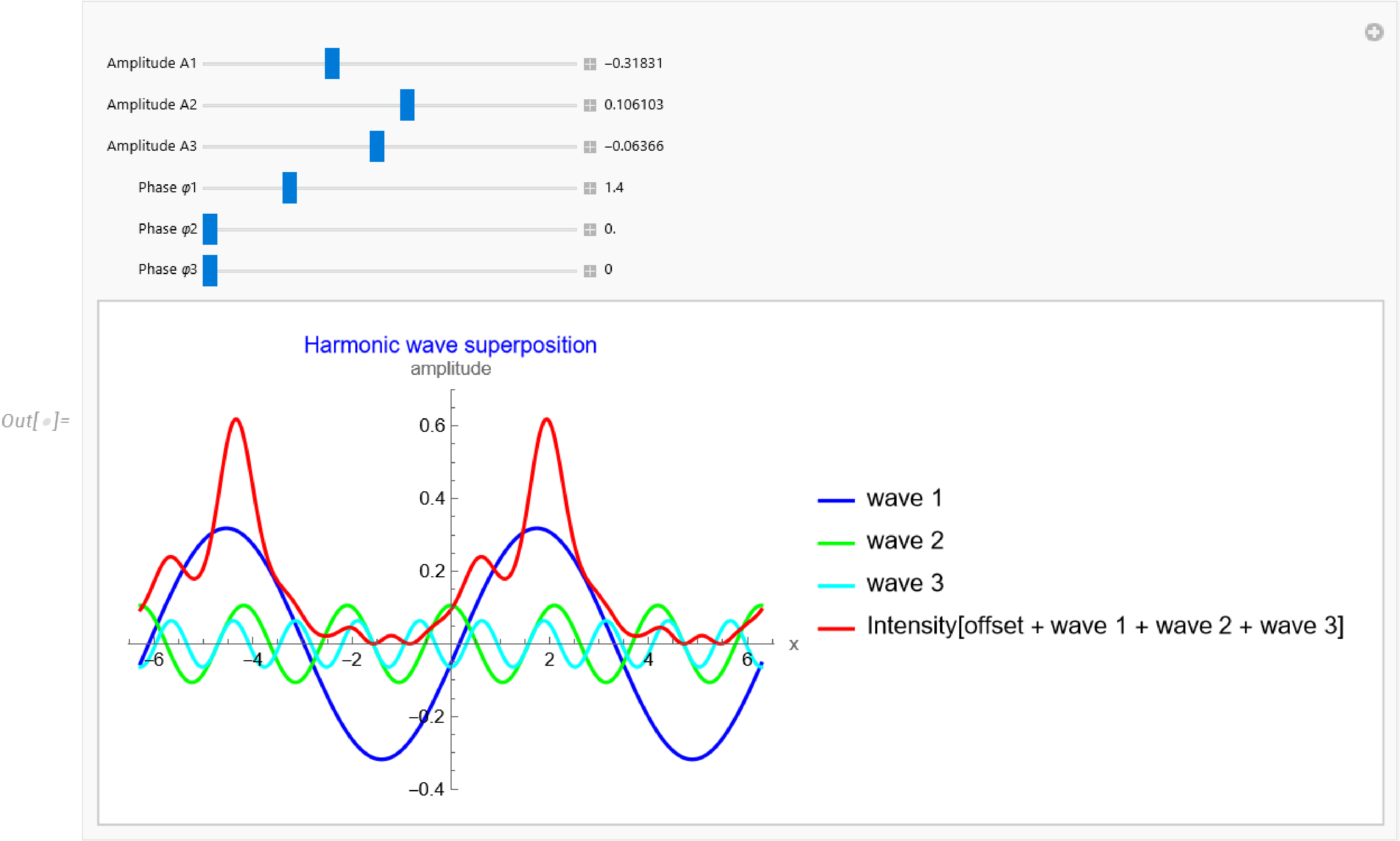
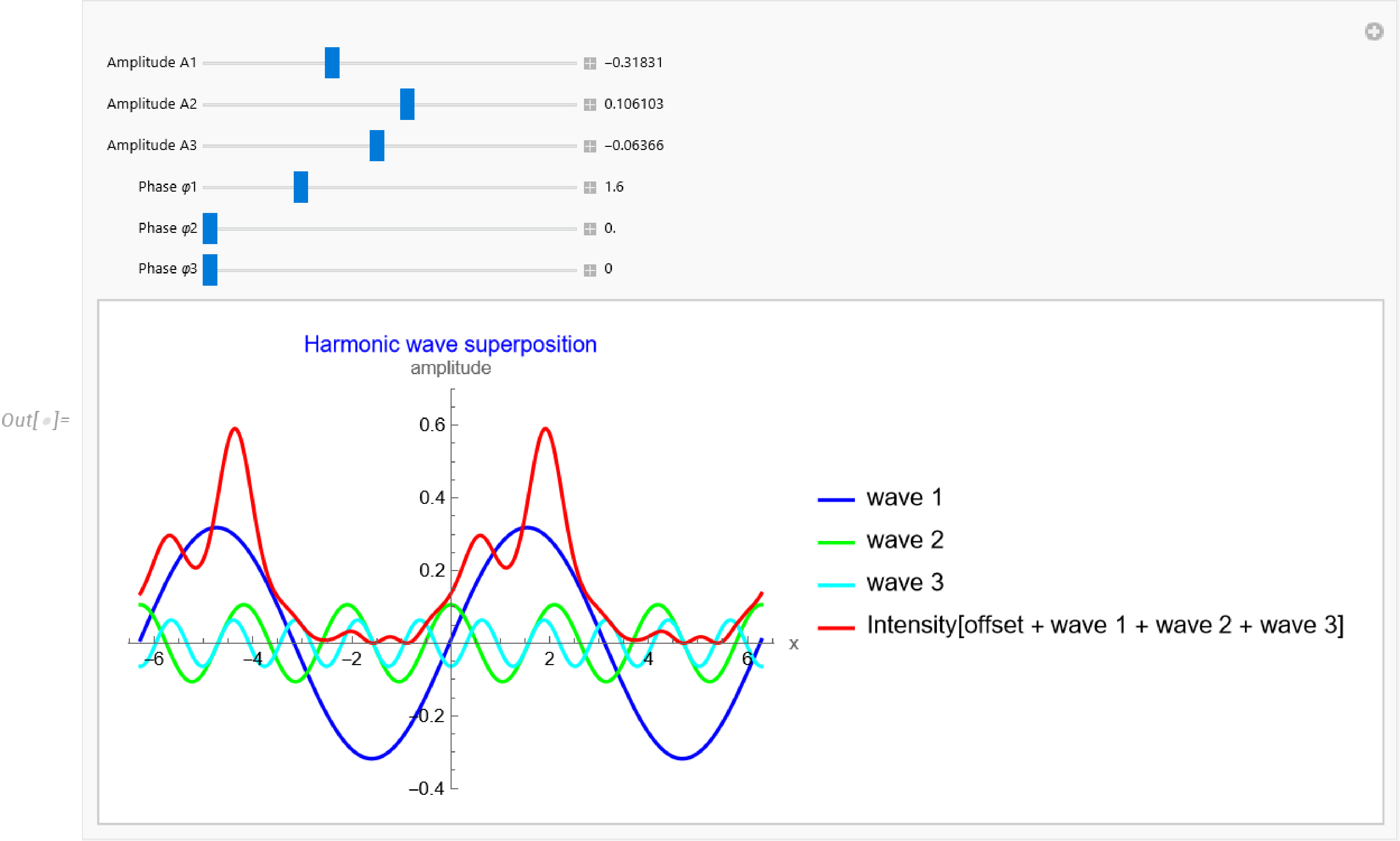


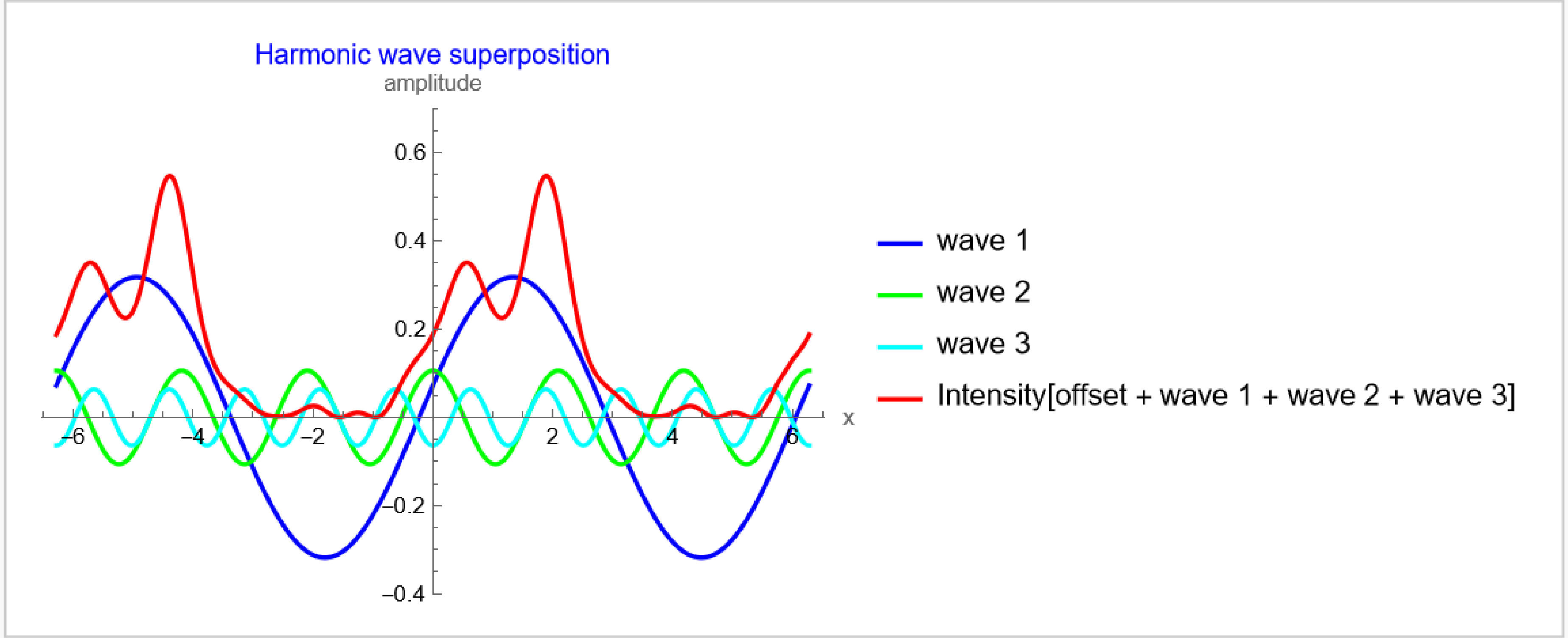
Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

Amplitude A1 −0.31831
Amplitude A2 0.106103
Amplitude A3 −0.06366
Phase φ1 1.8
Phase φ2 0.
Phase φ3 0

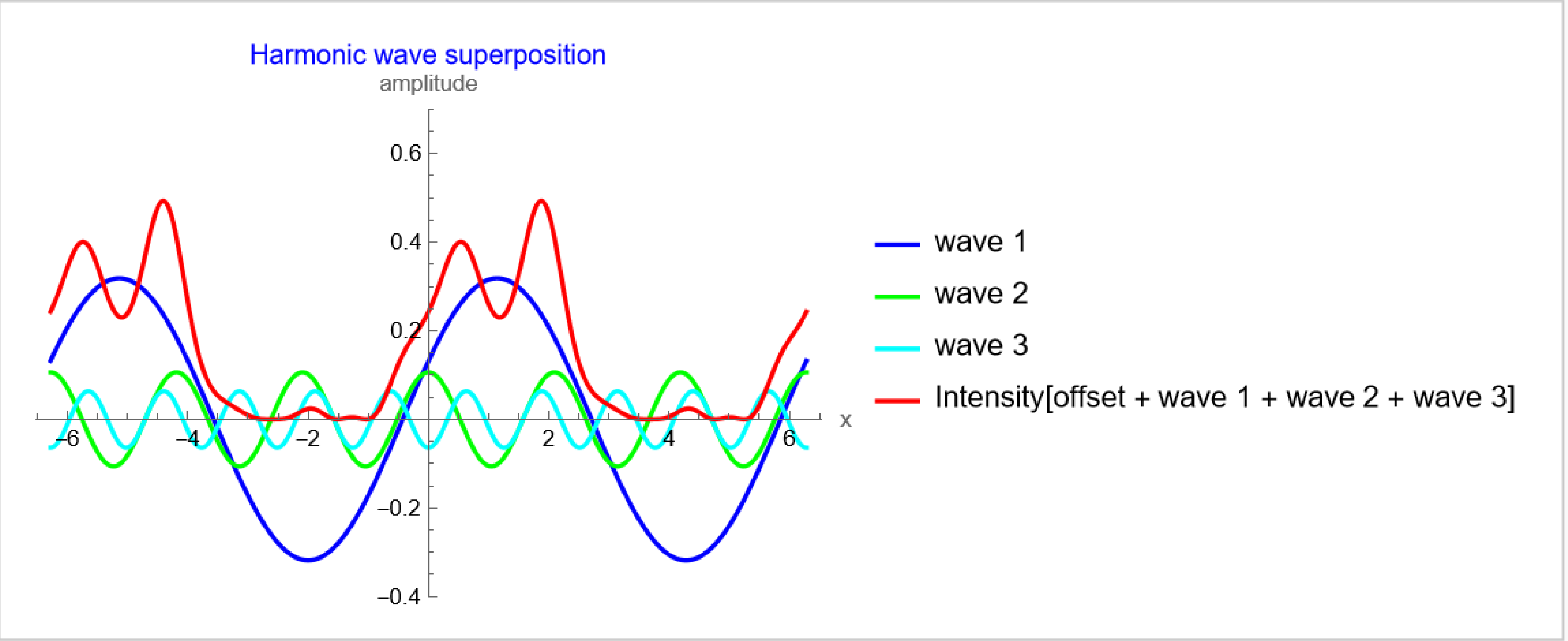
Out[ ]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

Amplitude A1 −0.31831
Amplitude A2 0.106103
Amplitude A3 −0.06366
Phase φ1 2
Phase φ2 0.
Phase φ3 0

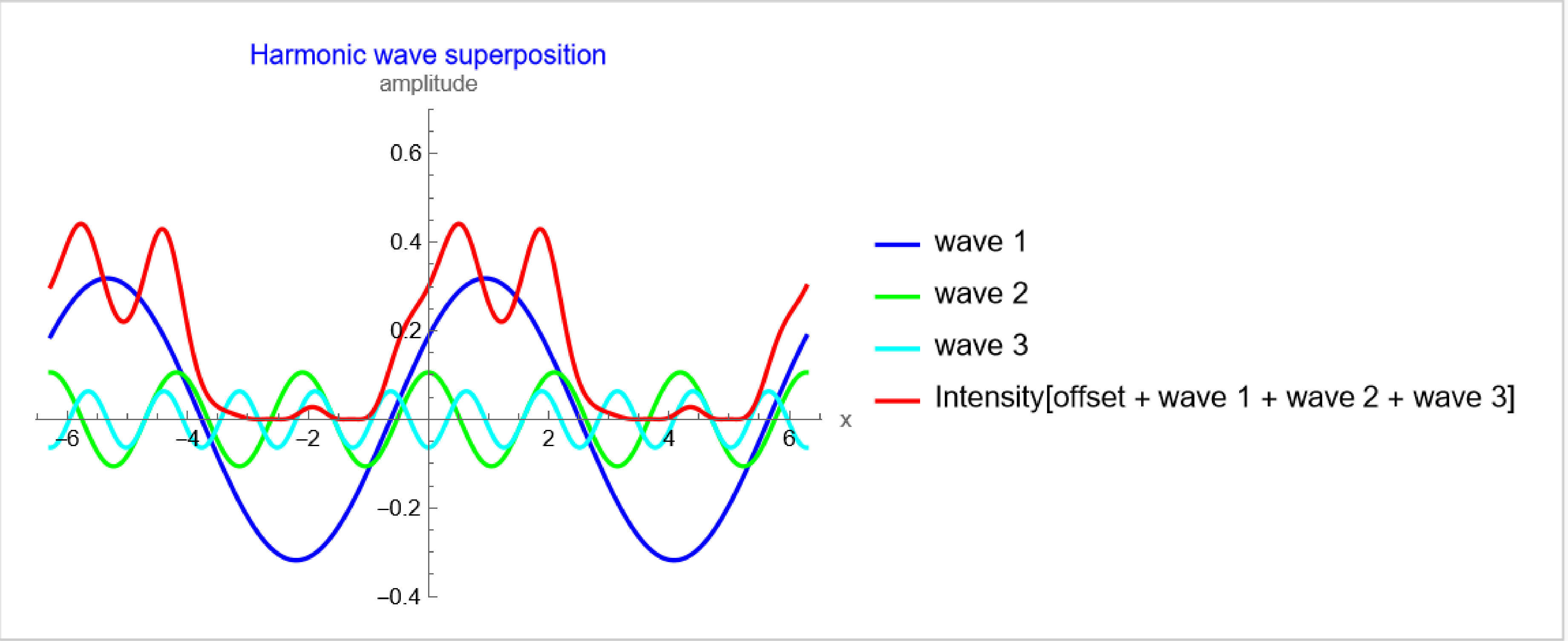
Out[ ]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

Amplitude A1	−0.31831
Amplitude A2	0.106103
Amplitude A3	−0.06366
Phase φ1	2.2
Phase φ2	0.
Phase φ3	0

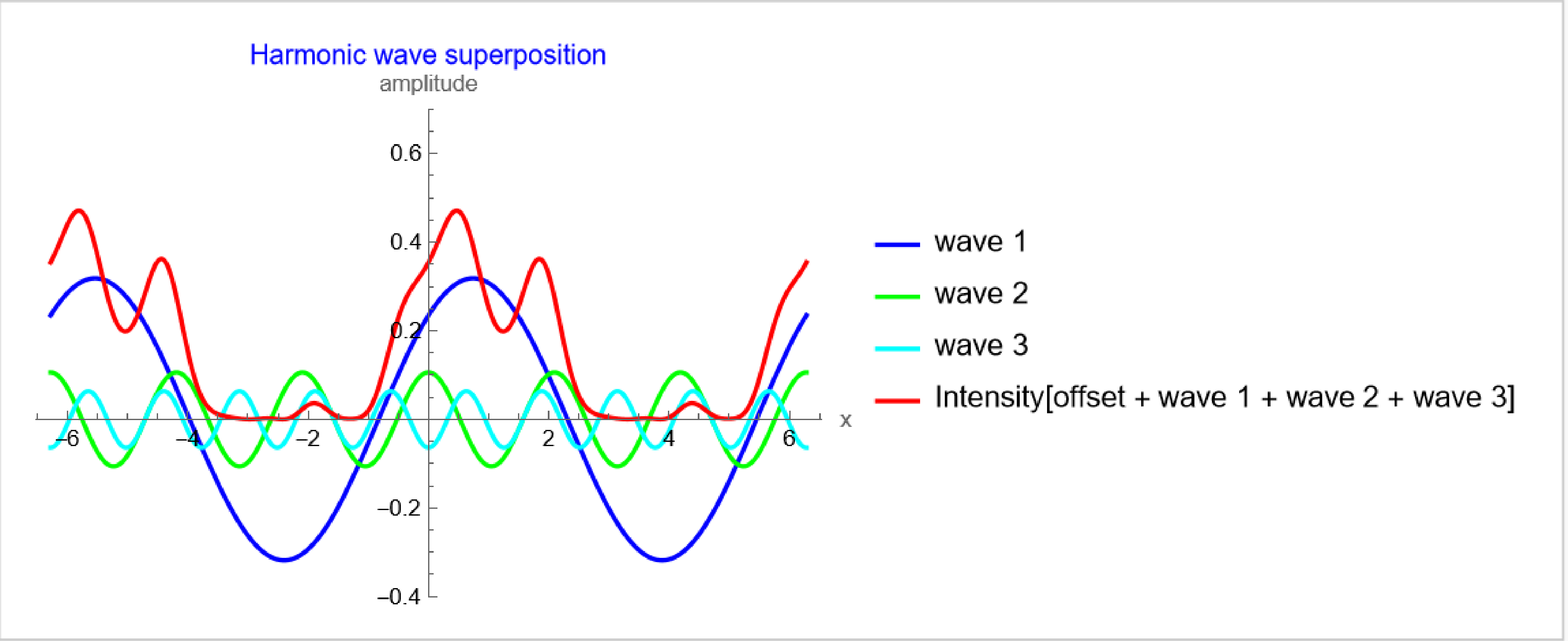
Out[ ]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

Amplitude A1 −0.31831
Amplitude A2 0.106103
Amplitude A3 −0.06366
Phase φ1 2.4
Phase φ2 0.
Phase φ3 0

Out[ ]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

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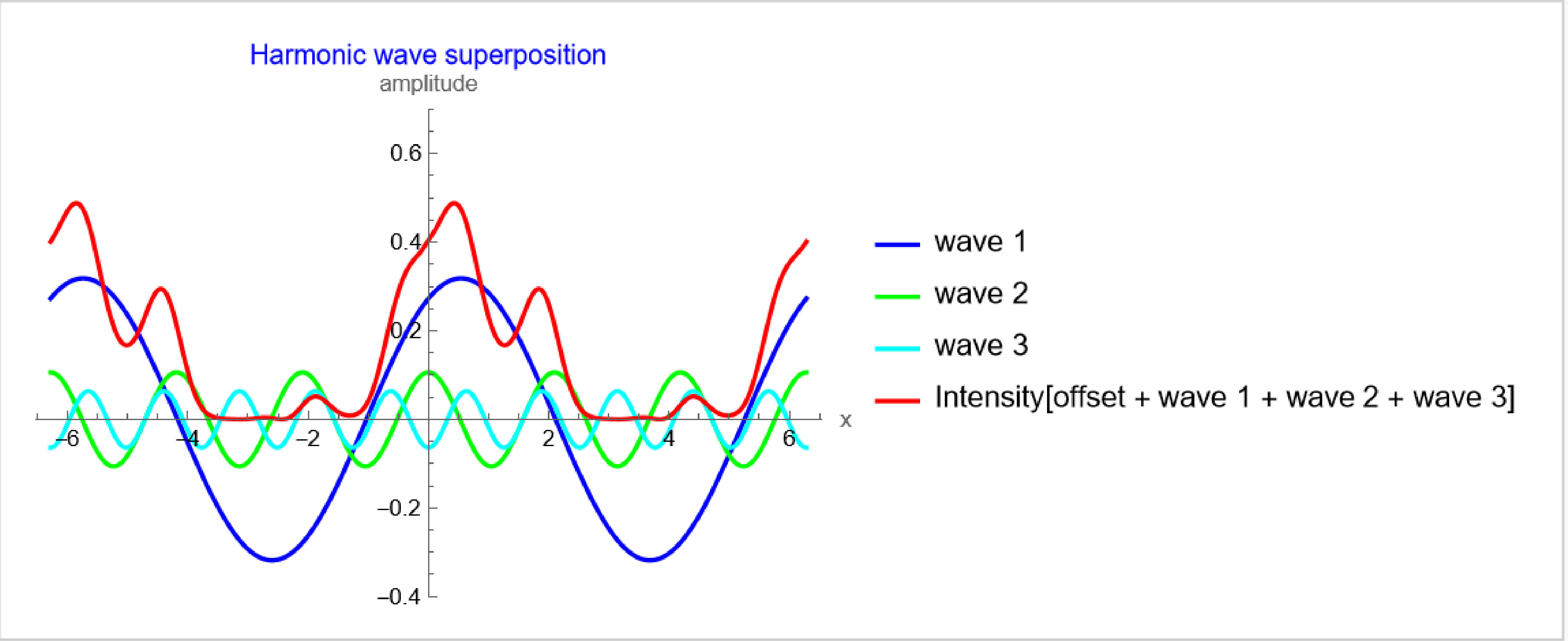
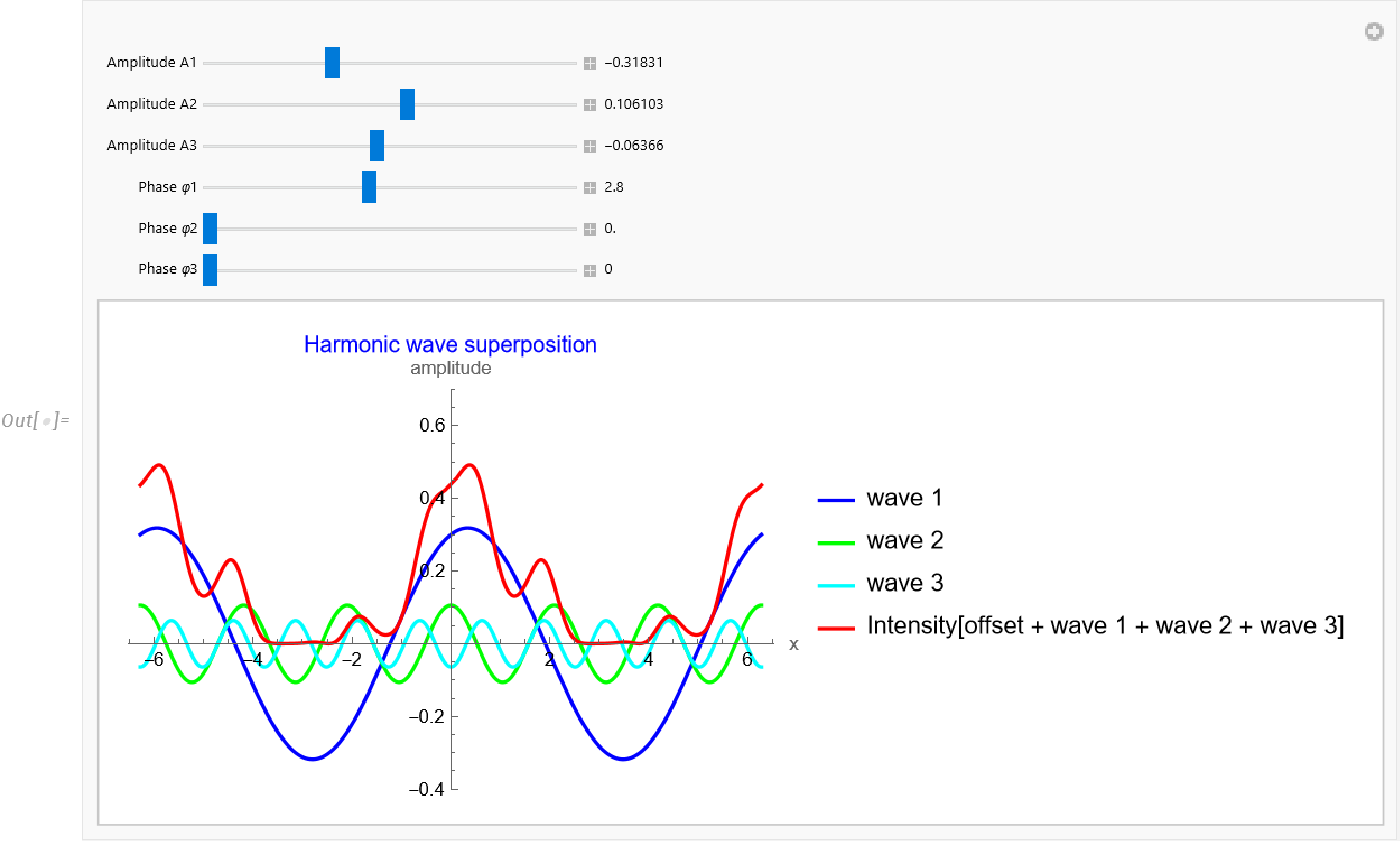
Amplitude A1 −0.31831
Amplitude A2 0.106103
Amplitude A3 −0.06366
Phase φ1 2.6
Phase φ2 0.
Phase φ3 0

Out[ ]=



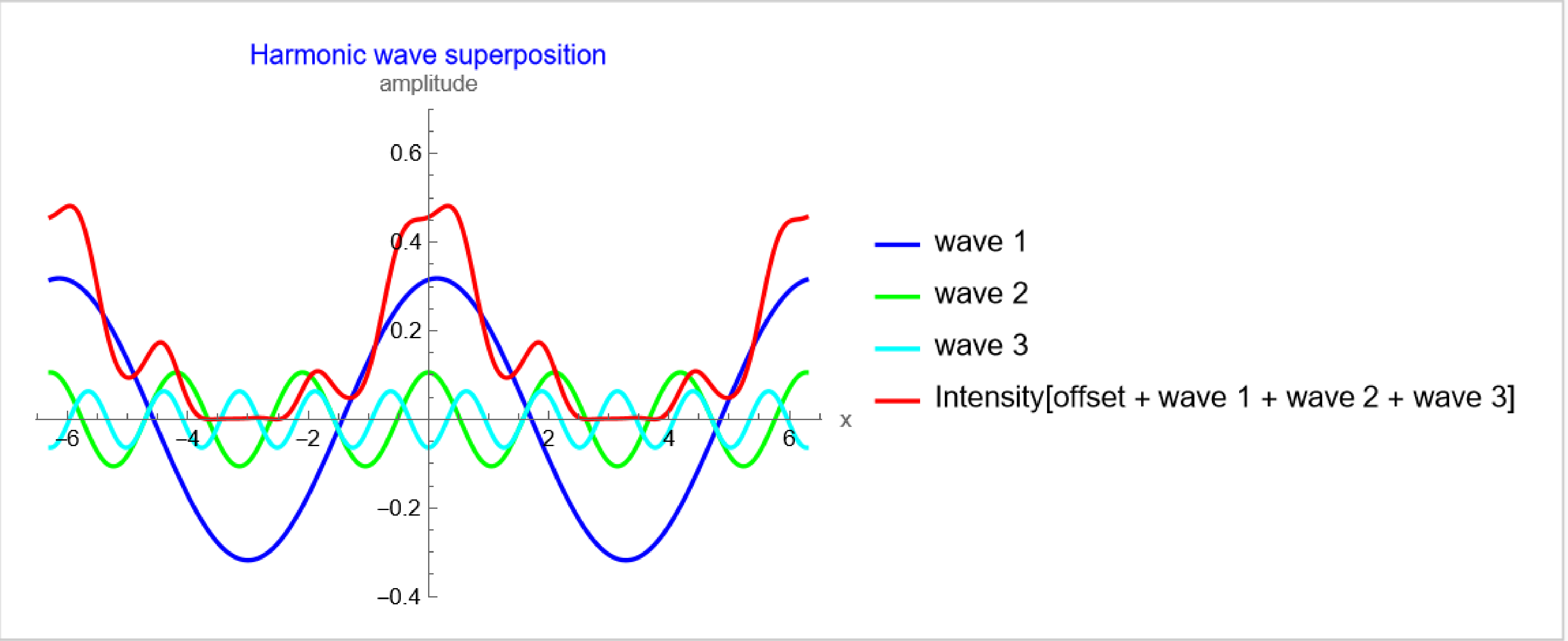
Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)



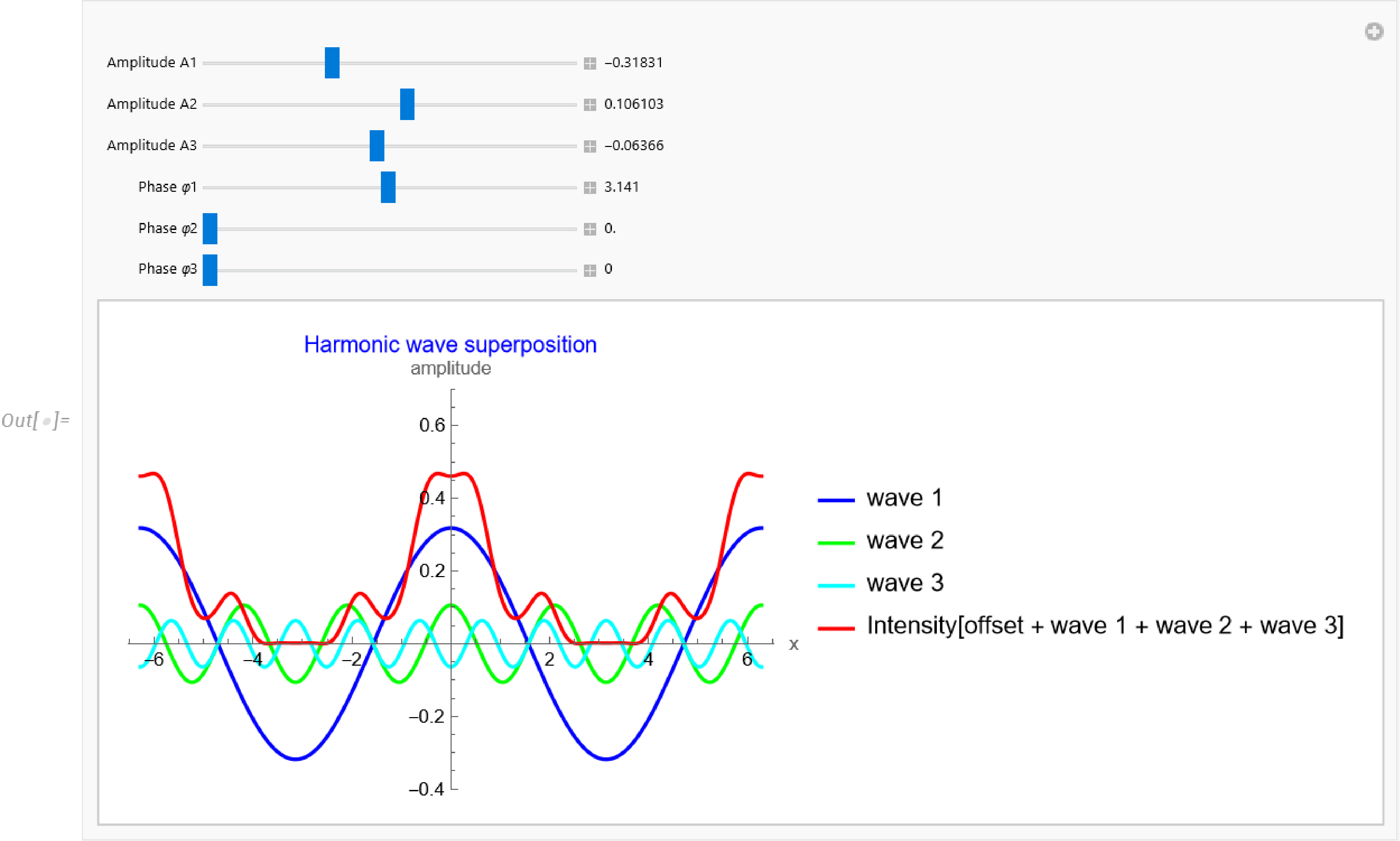
Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

Amplitude A1 −0.31831
Amplitude A2 0.106103
Amplitude A3 −0.06366
Phase φ1 3
Phase φ2 0.
Phase φ3 0

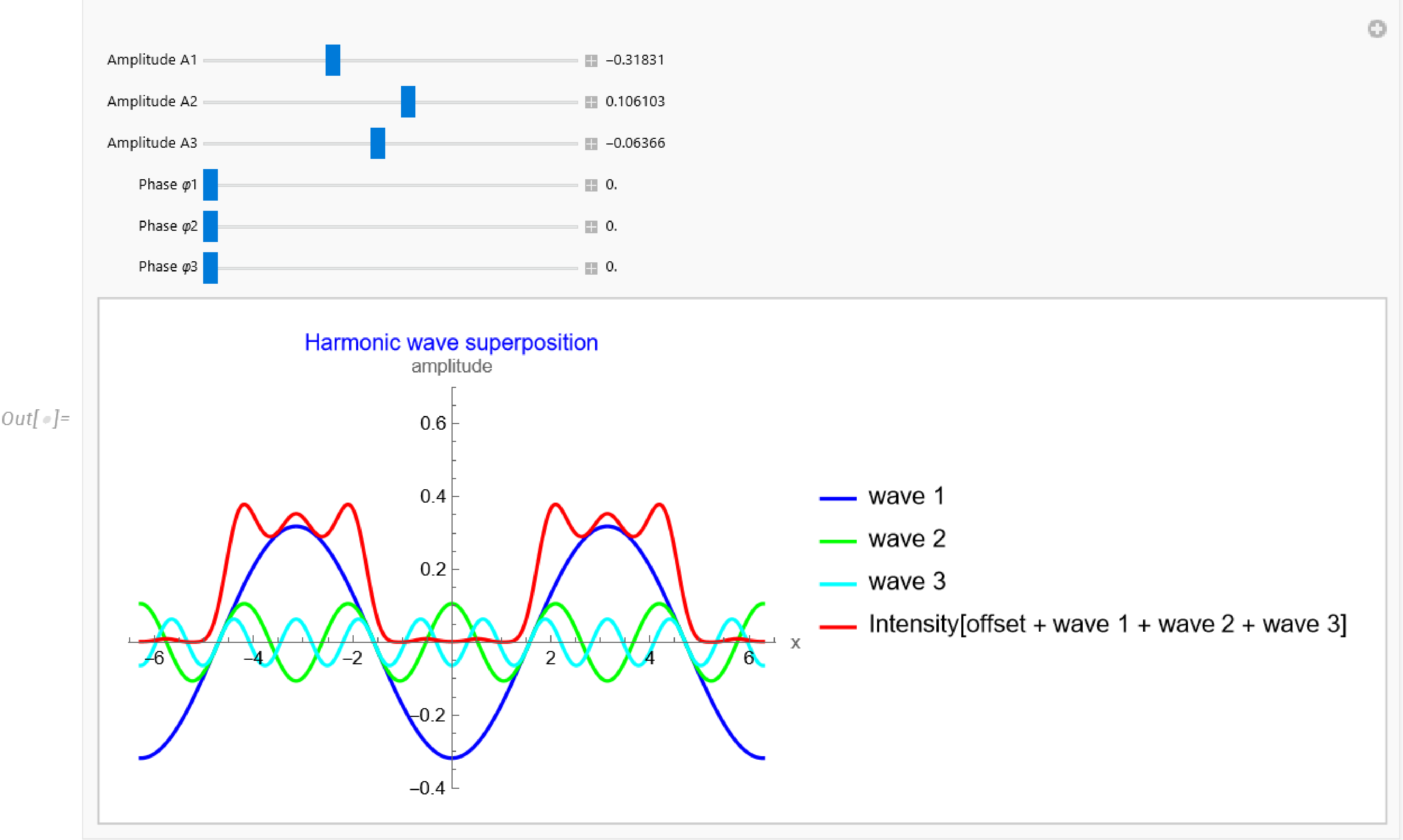
Out[•]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)



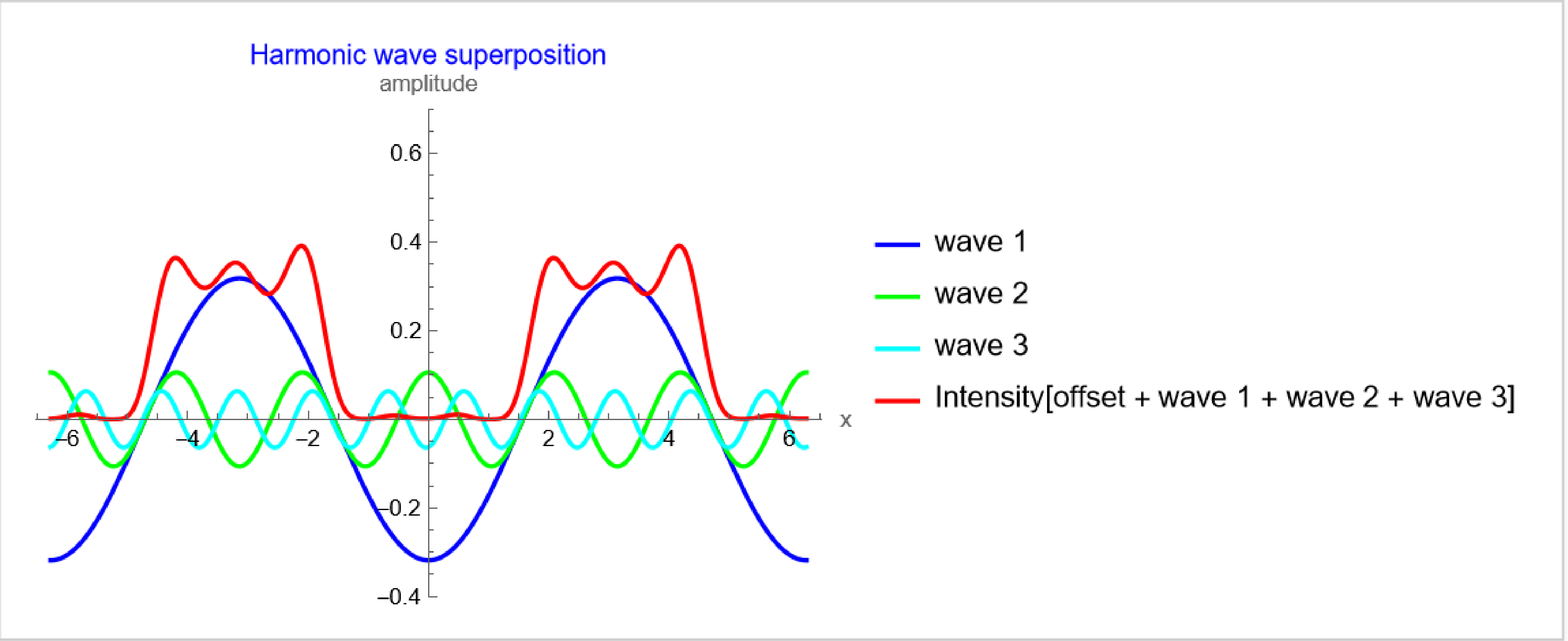
Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

Amplitude A1 −0.31831
Amplitude A2 0.106103
Amplitude A3 −0.06366
Phase φ1 0
Phase φ2 0.
Phase φ3 0.2

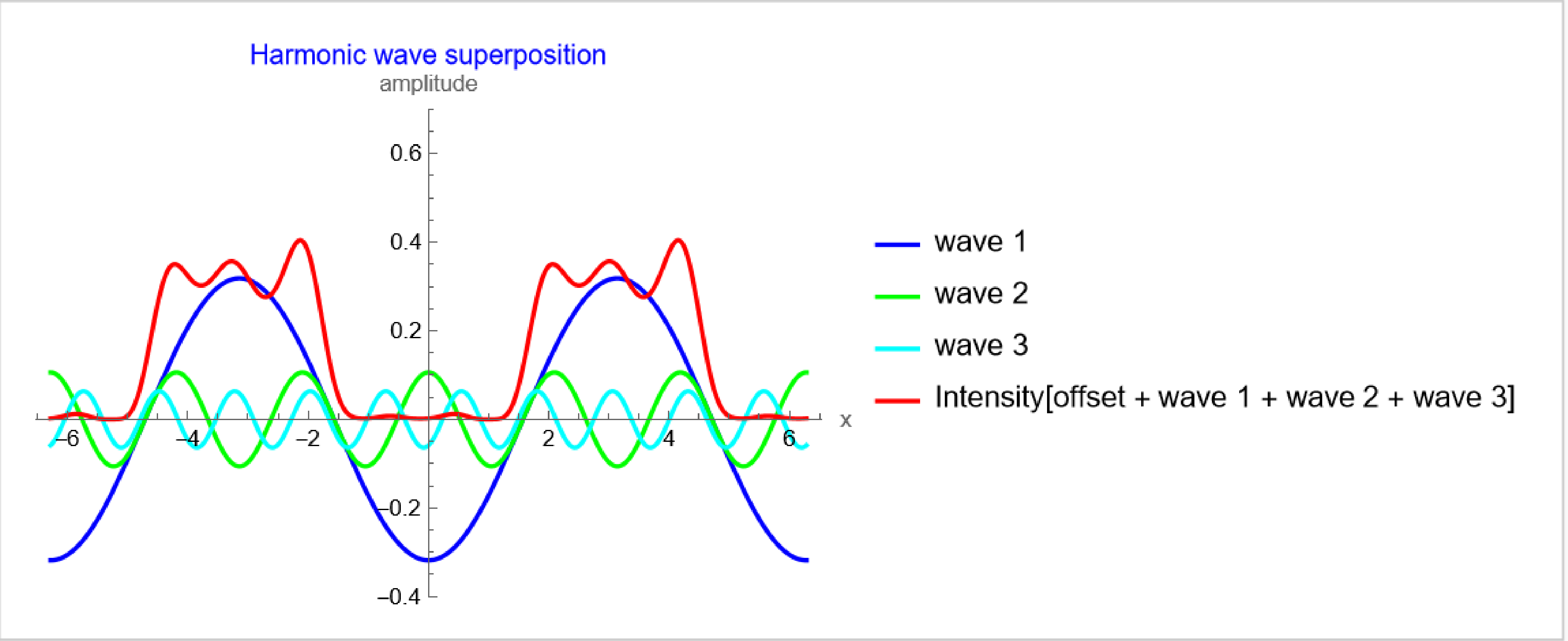
Out[ ]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

Amplitude A1 −0.31831
Amplitude A2 0.106103
Amplitude A3 −0.06366
Phase φ1 0
Phase φ2 0.
Phase φ3 0.4

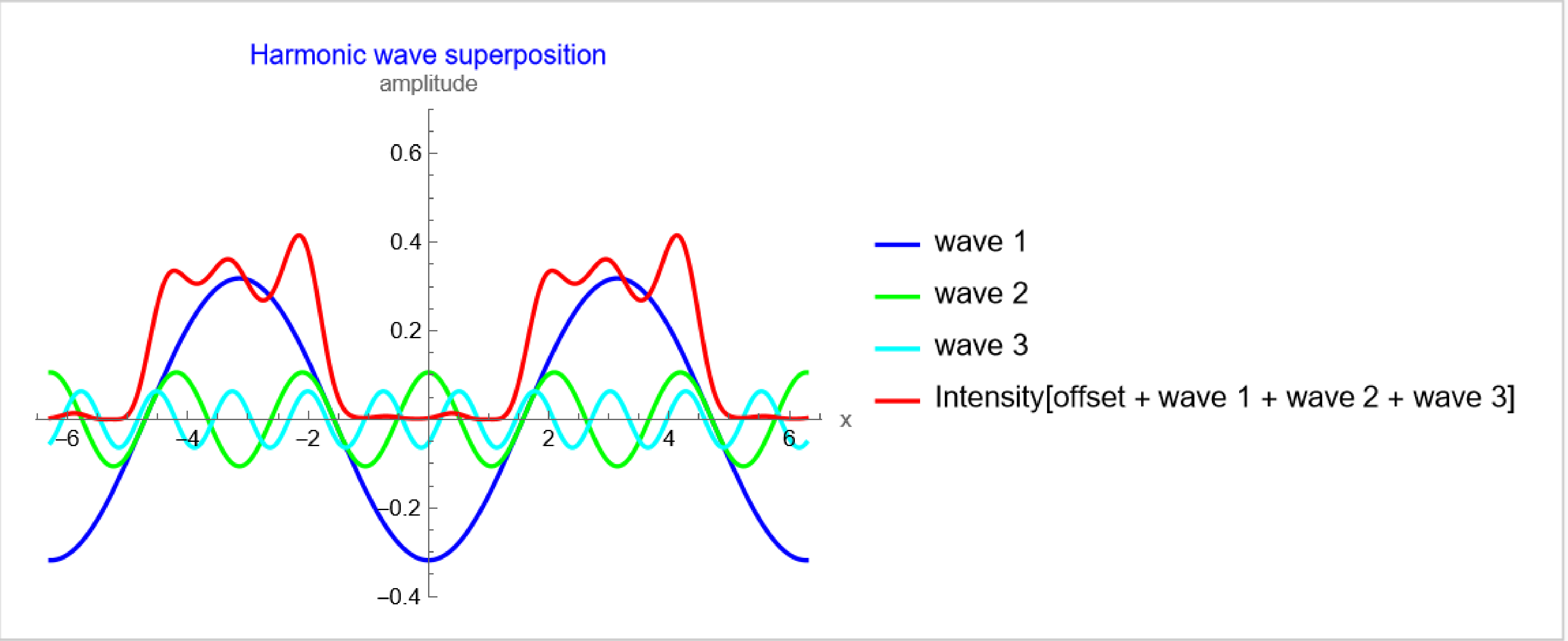
Out[ ]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

Amplitude A1		−0.31831
Amplitude A2		0.106103
Amplitude A3		−0.06366
Phase φ1		0
Phase φ2		0.
Phase φ3		0.6

Out[∘]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

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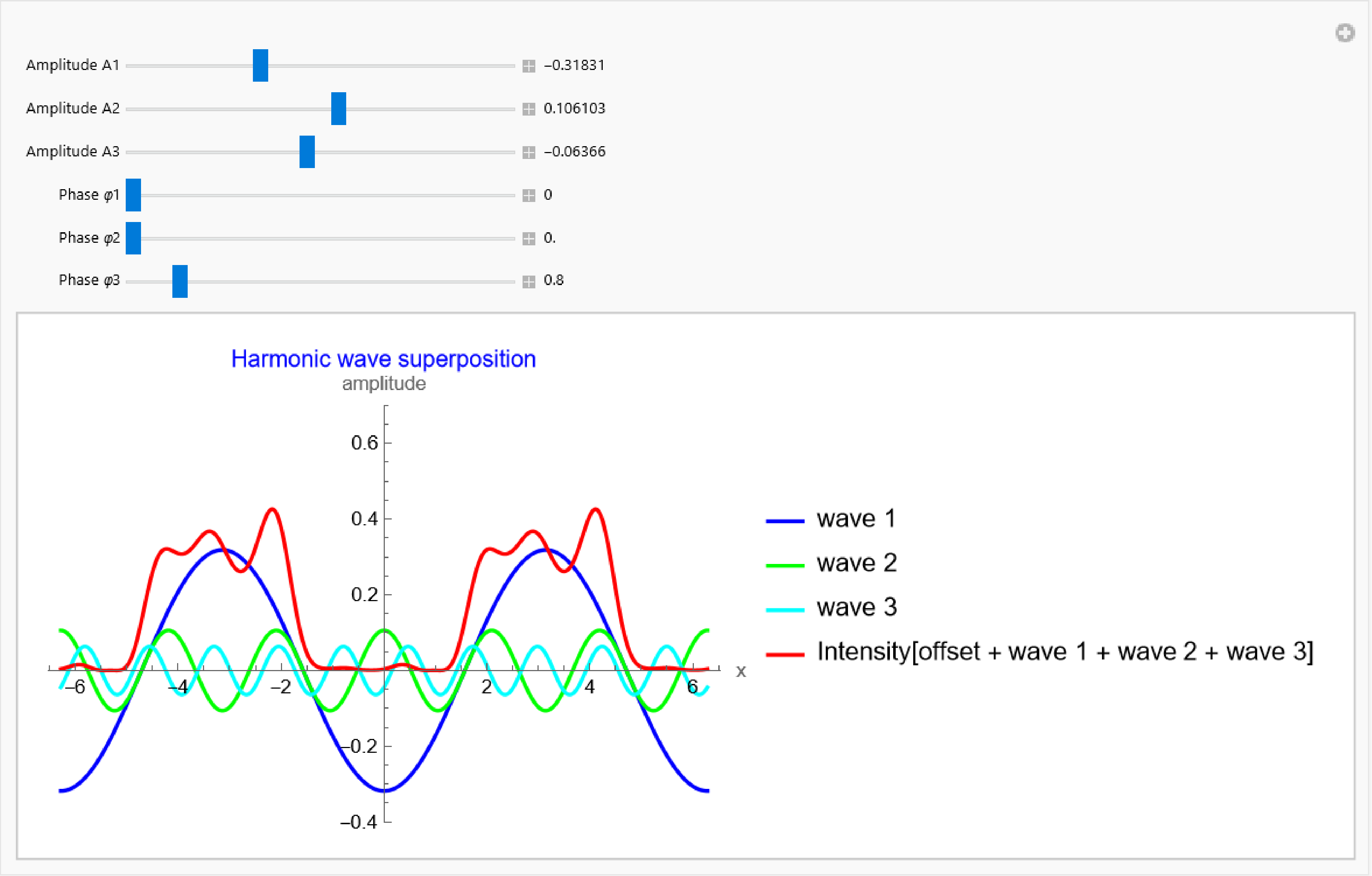
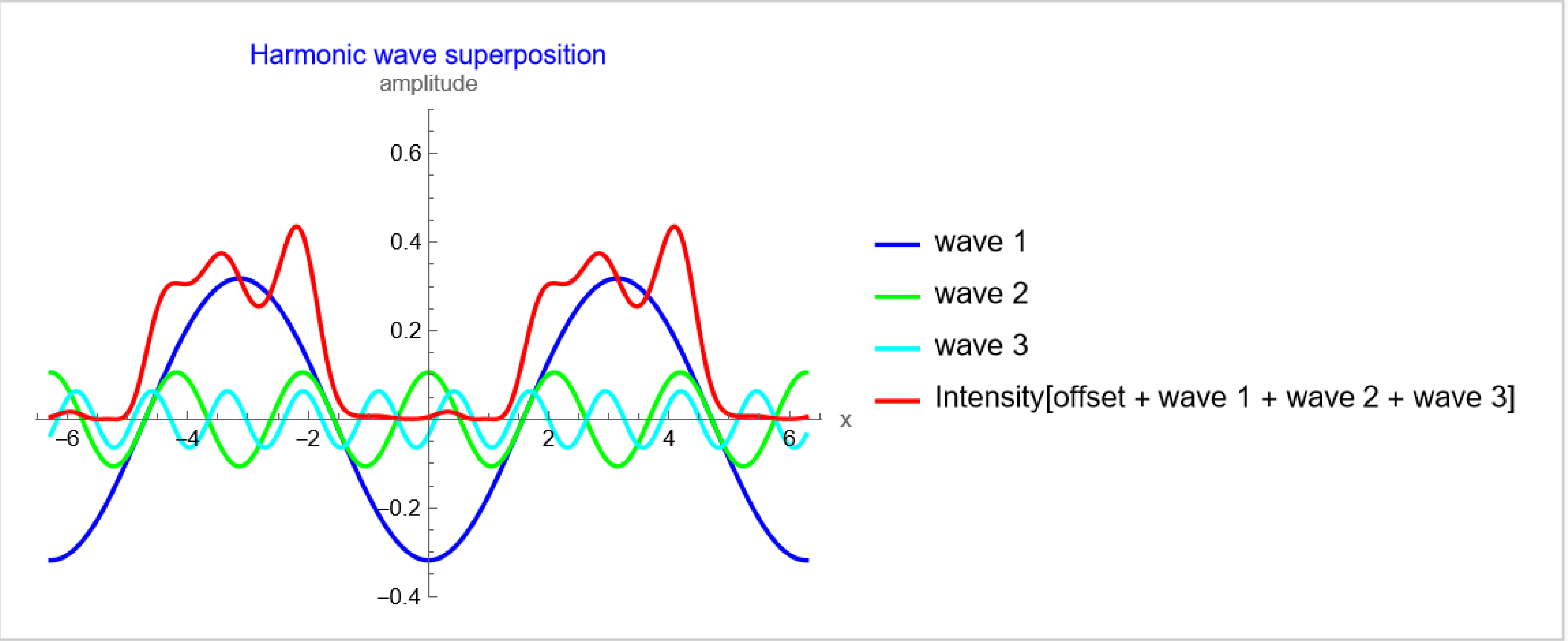


Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

Amplitude A1 −0.31831
Amplitude A2 0.106103
Amplitude A3 −0.06366
Phase φ1 0
Phase φ2 0.
Phase φ3 1

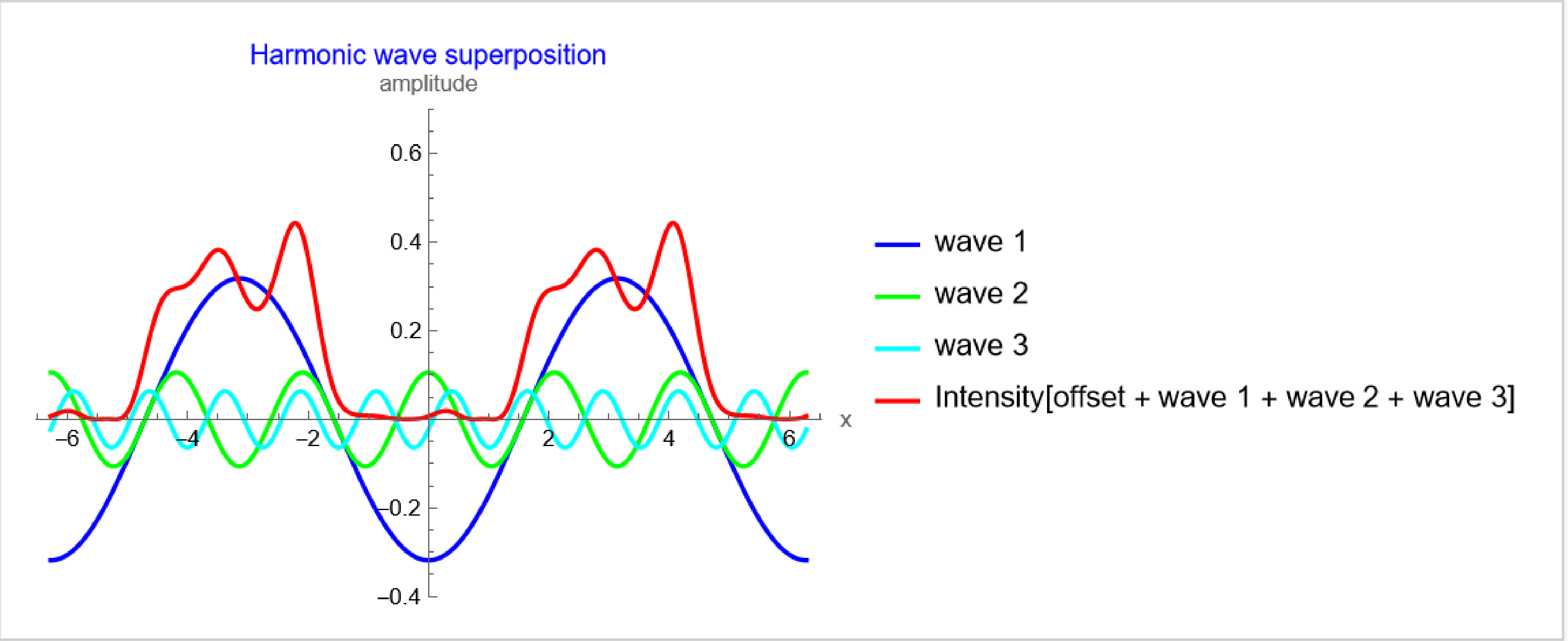
Out[∘]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

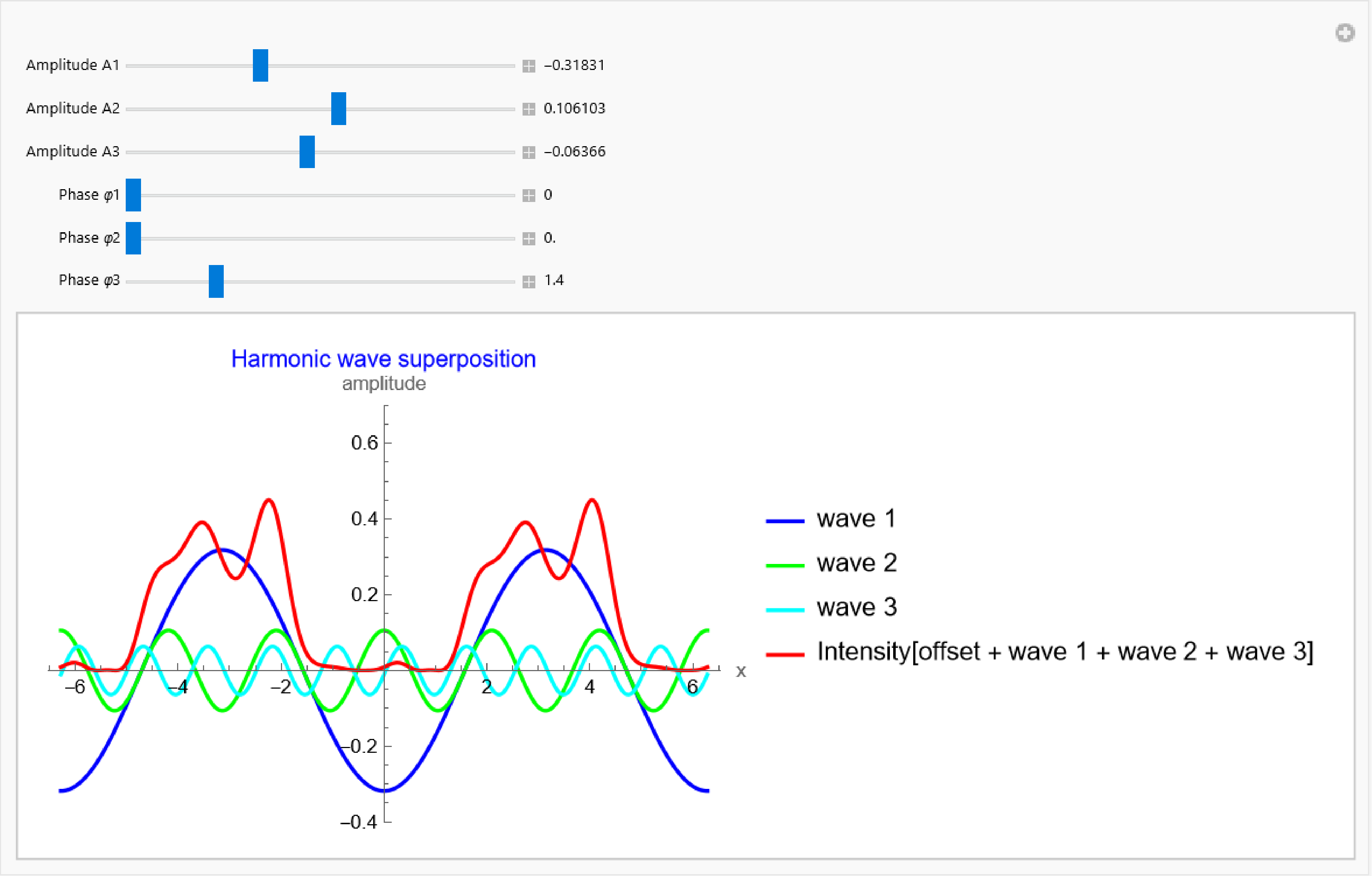
Amplitude A1 −0.31831
Amplitude A2 0.106103
Amplitude A3 −0.06366
Phase φ1 0
Phase φ2 0.
Phase φ3 1.2

Out[ ]=



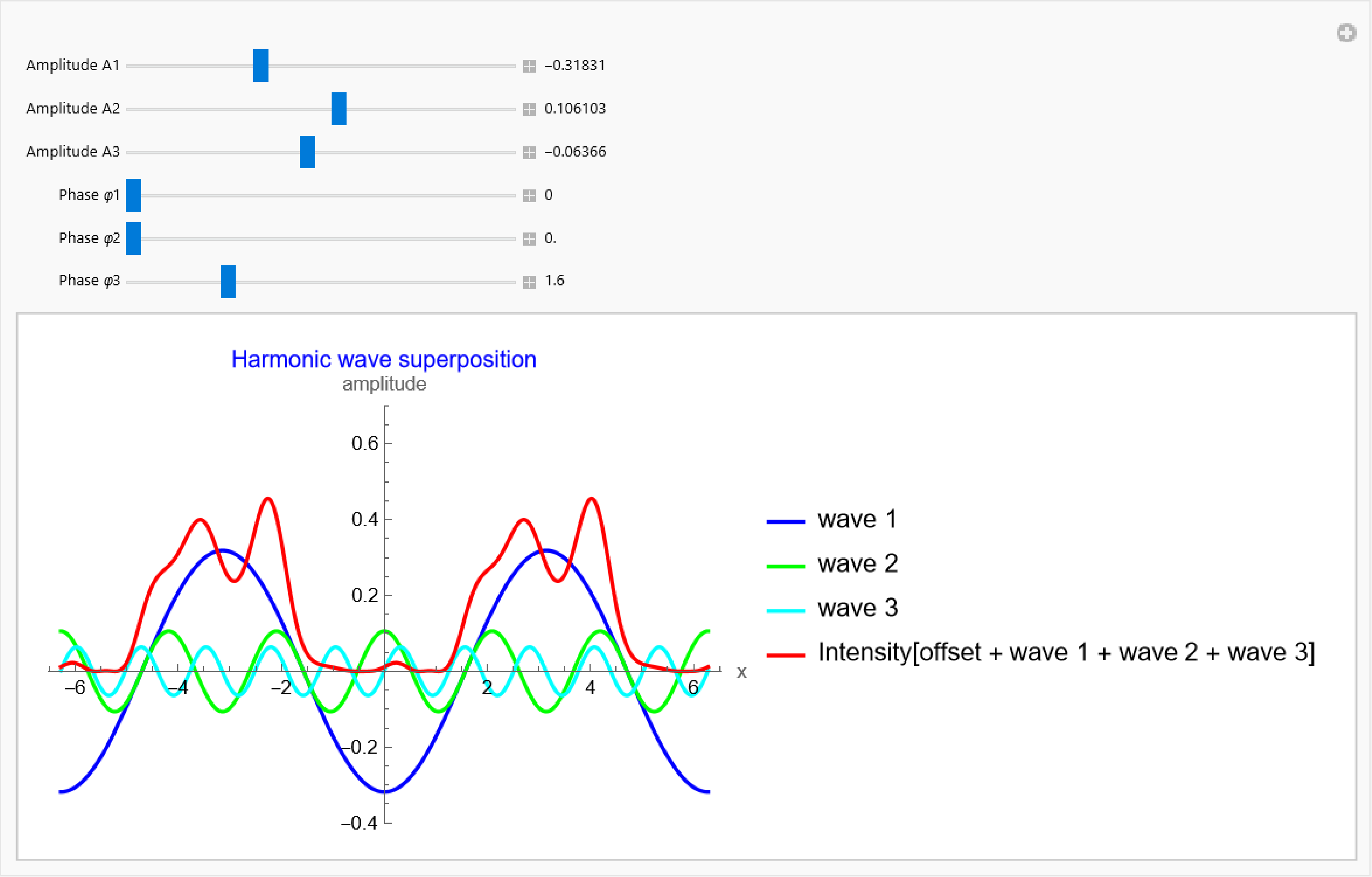
Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

Out[ ]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

Out[•]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

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Amplitude A1 −0.31831

Amplitude A2 0.106103

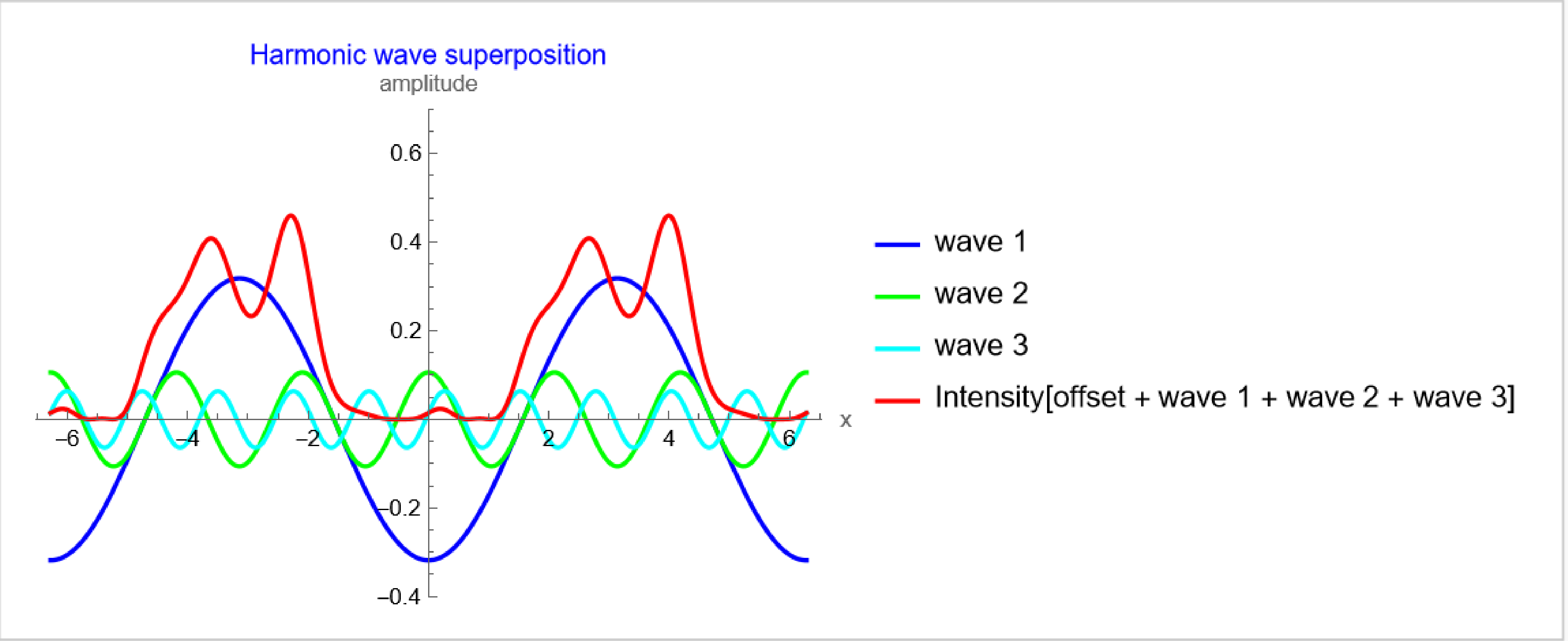
Amplitude A3 −0.06366

Phase φ1 0

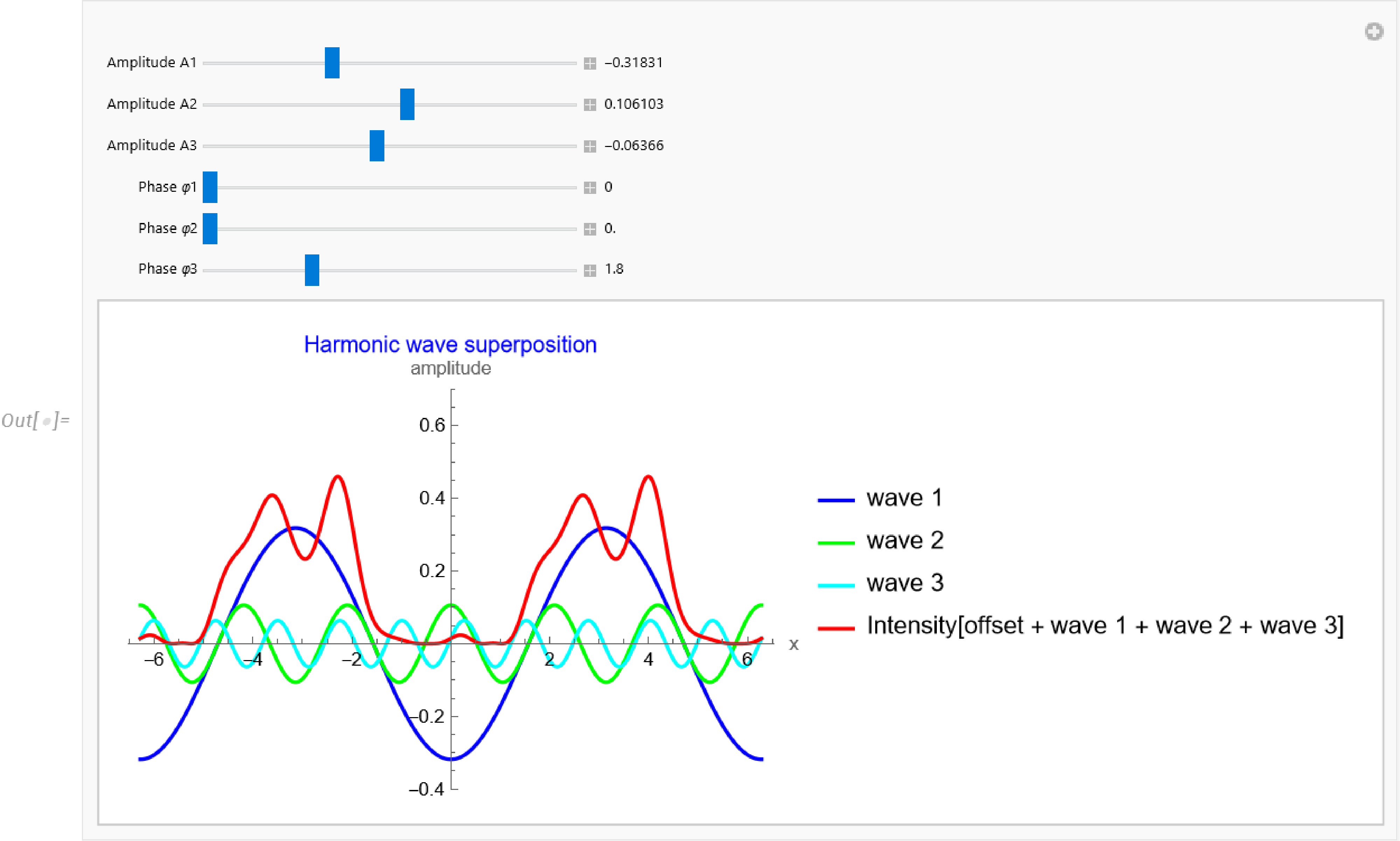
Phase φ2 0.

Phase φ3 1.8

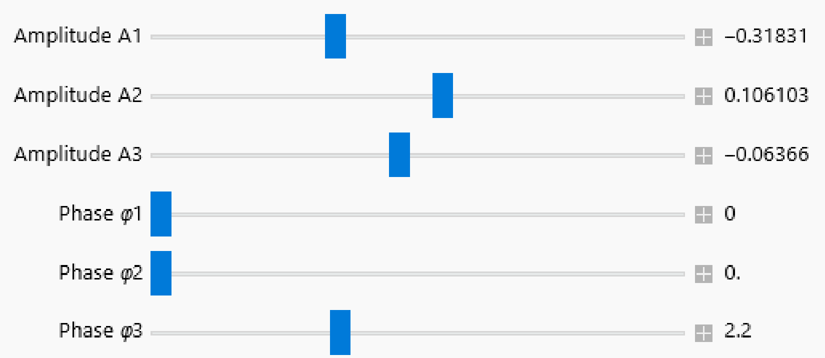
Out[ ]=



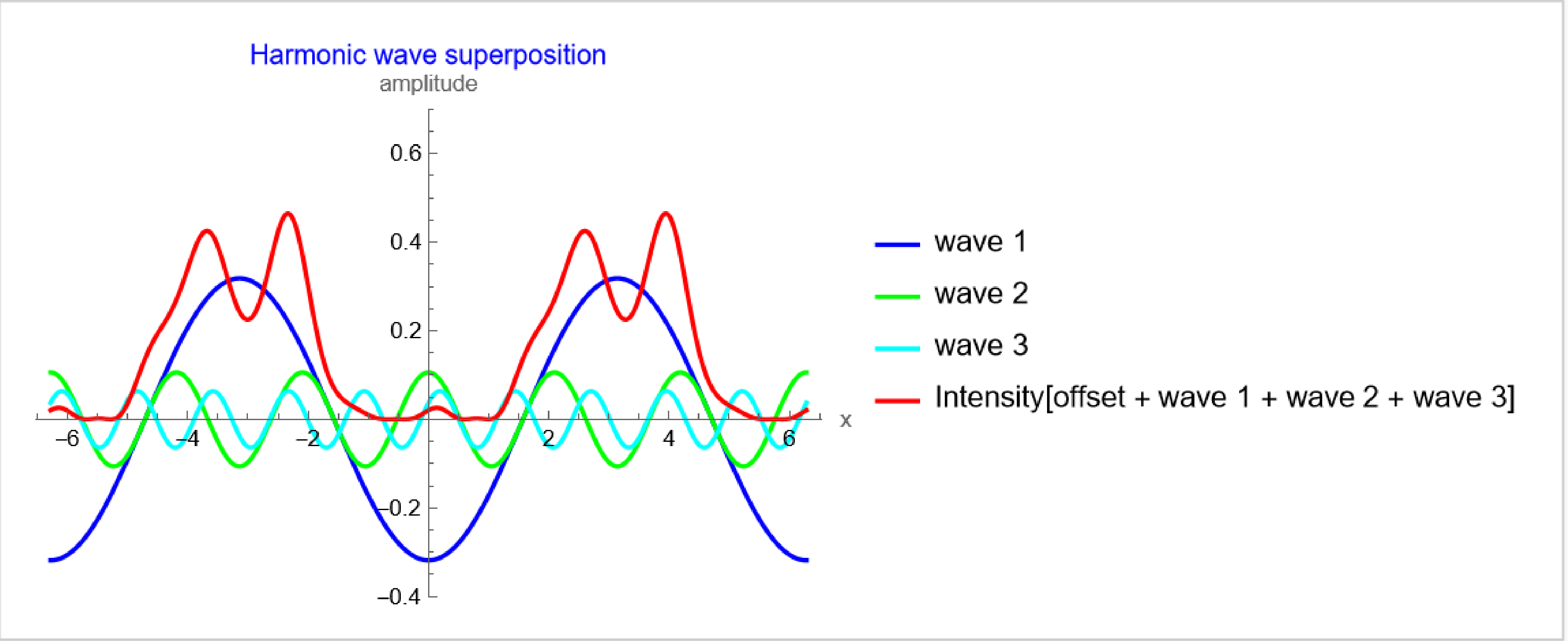
Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)



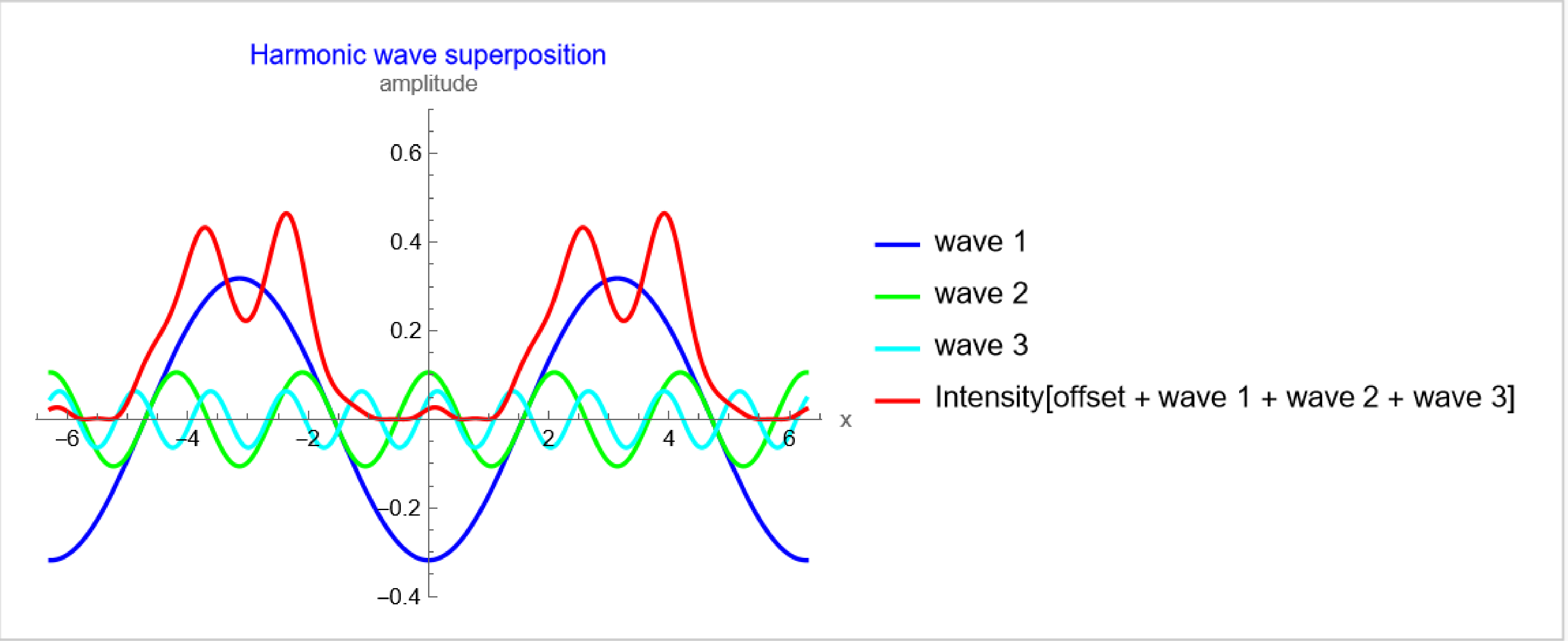
Out[ ]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

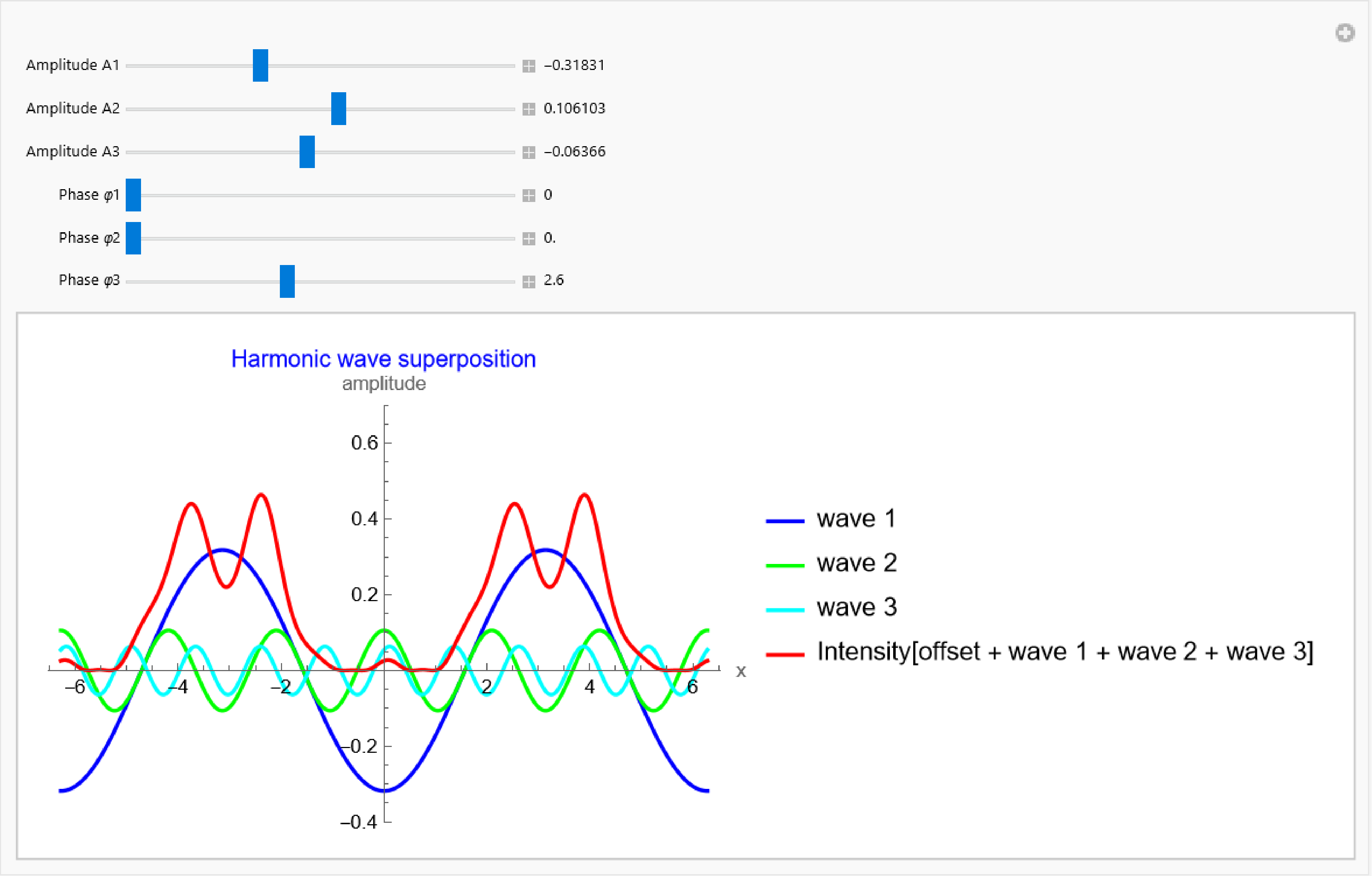
Amplitude A1 −0.31831
Amplitude A2 0.106103
Amplitude A3 −0.06366
Phase φ1 0
Phase φ2 0.
Phase φ3 2.4

Out[∘]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

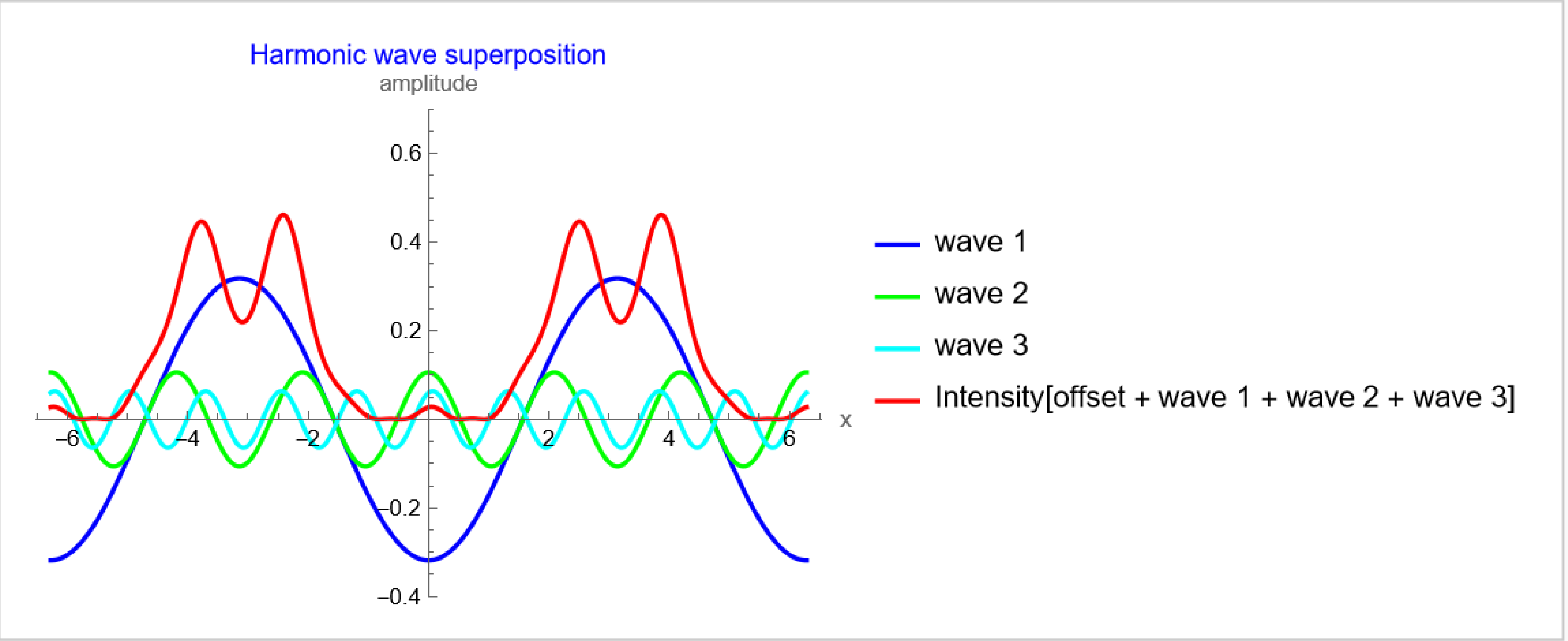
Out[ ]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

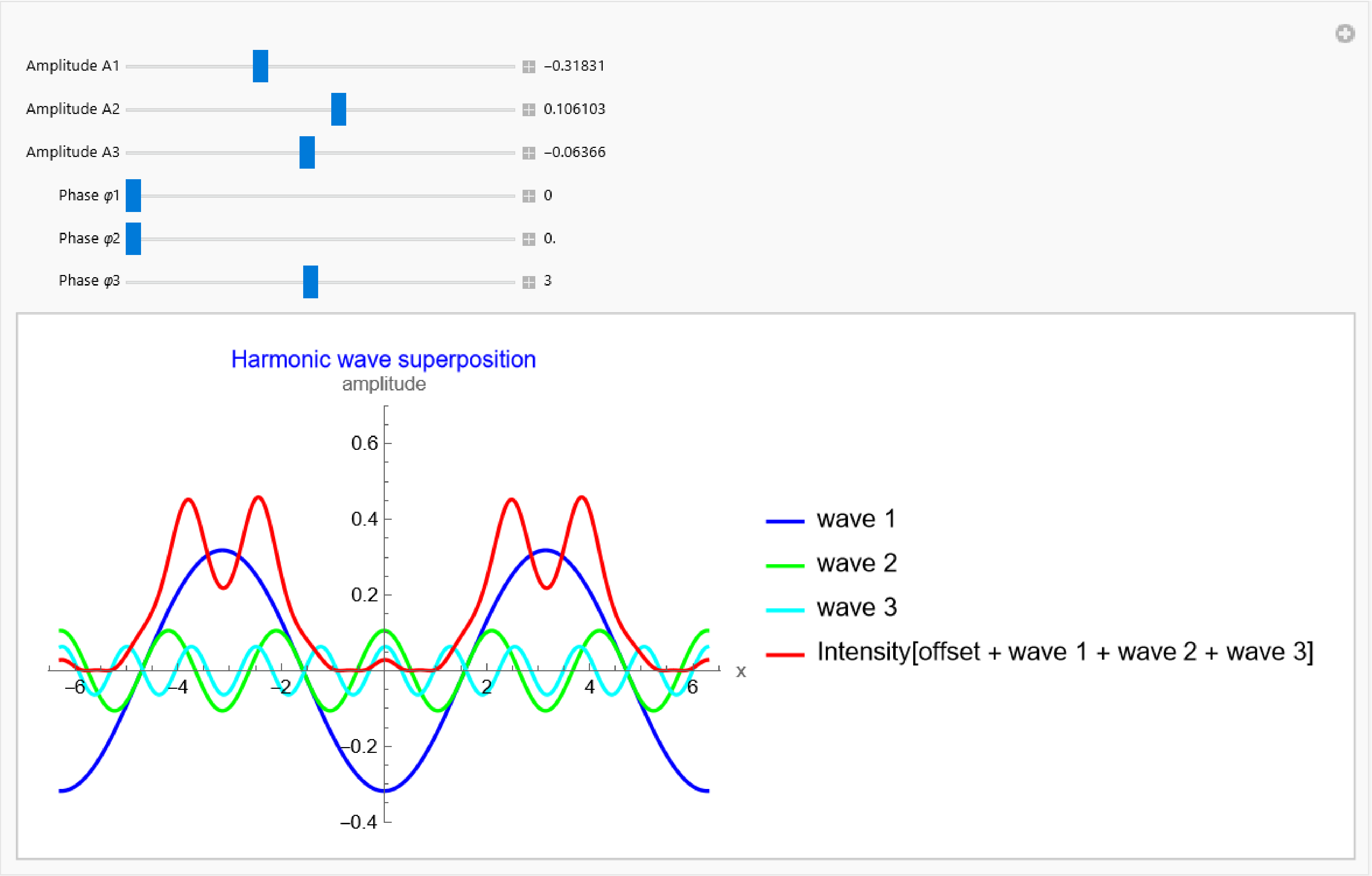
Amplitude A1 −0.31831
Amplitude A2 0.106103
Amplitude A3 −0.06366
Phase φ1 0
Phase φ2 0.
Phase φ3 2.8

Out[ ]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

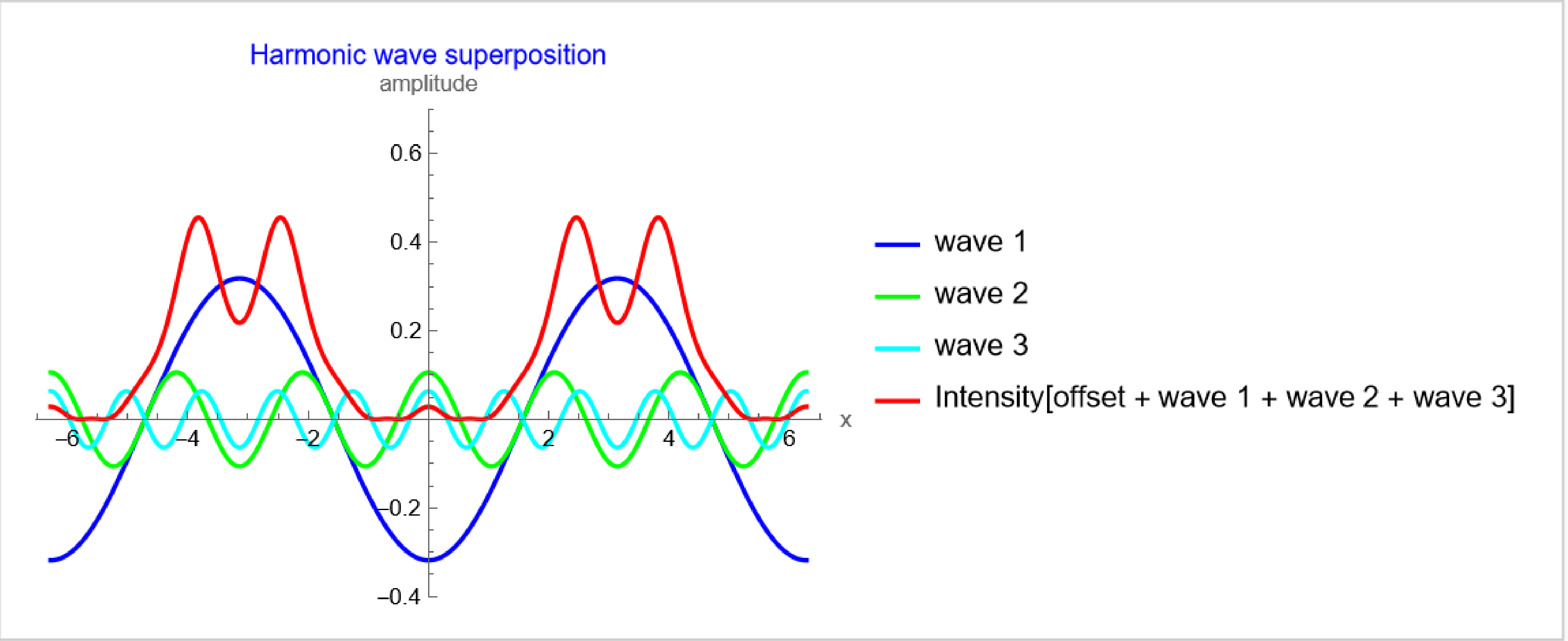
Out[•]=



Periodic pattern: Effect of relative phase shifts (Phase Transfer Function)

Amplitude A1 −0.31831
Amplitude A2 0.106103
Amplitude A3 −0.06366
Phase φ1 0
Phase φ2 0.
Phase φ3 3.14159

Out[ ]=



Quasi-coherent approximation of partially coherent imaging

Hopkins´ Transfer Function (HTF) couples the variables α_1, α_2 like a correlation function additionally coupled with the source distribution

$$\mathrm{HTF}(\alpha_1,\alpha_2) = \iint da\, Q(a)\, L(\alpha_1 + \sigma\, a\,)L^*(\alpha_2 + \sigma\, a\,)$$

Sometimes it is also called **bilinear** transfer function.
With following approximation the variables α_1, α_2 can be decoupled (with $HTF(\alpha_1,\alpha_2) = HTF^*(\alpha_2,\alpha_1)$)

$$HTF_{qc}(\alpha_1,\alpha_2) \approx HTF(\alpha_1,0)HTF(0,\alpha_2) = HTF(\alpha_1,0)\; HTF^*(\alpha_2,0)$$

Now **object to image transfer of the coherence function is as in the coherence case** the, only that $L(\alpha) \rightarrow HTF(\alpha,0)$ or $HTF_{coh}(\alpha_1,\alpha_2) = L(\alpha_1)L^*(\alpha_2\,)$ and the image intensity is

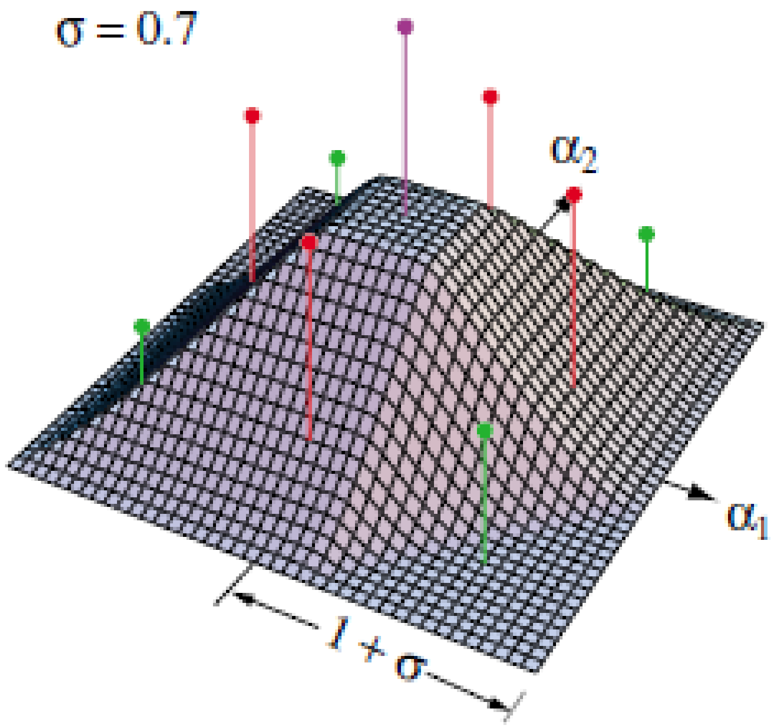
$$I(x) = \left| \iint d\alpha\; exp(-i2\pi w\alpha \cdot x/m) HTF_{qc}(\alpha) \iint d\xi\, A_{ob}(\xi) exp(i2\pi w\alpha \cdot \xi) \right|^2$$

$$= \left| FT_\alpha \{ \underset{\uparrow}{HTF_{qc}}(\alpha) FT_\xi \{A_{ob}\}(w\alpha)\}(x) \right|^2$$

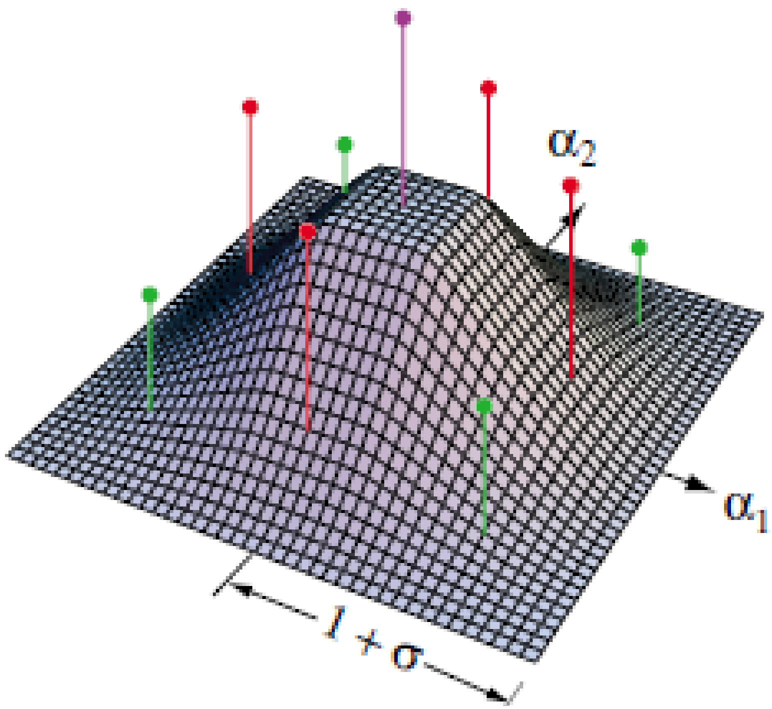
Quasi-coherent (linearized) partially coherent transfer function

with $HTF_{qc}(\alpha) = \iint da\; Q(a)\, L(\alpha + \sigma\, a\,)L^*(\sigma\, a\,)$

Quasi-coherent approximation of Hopkins Transfer Function

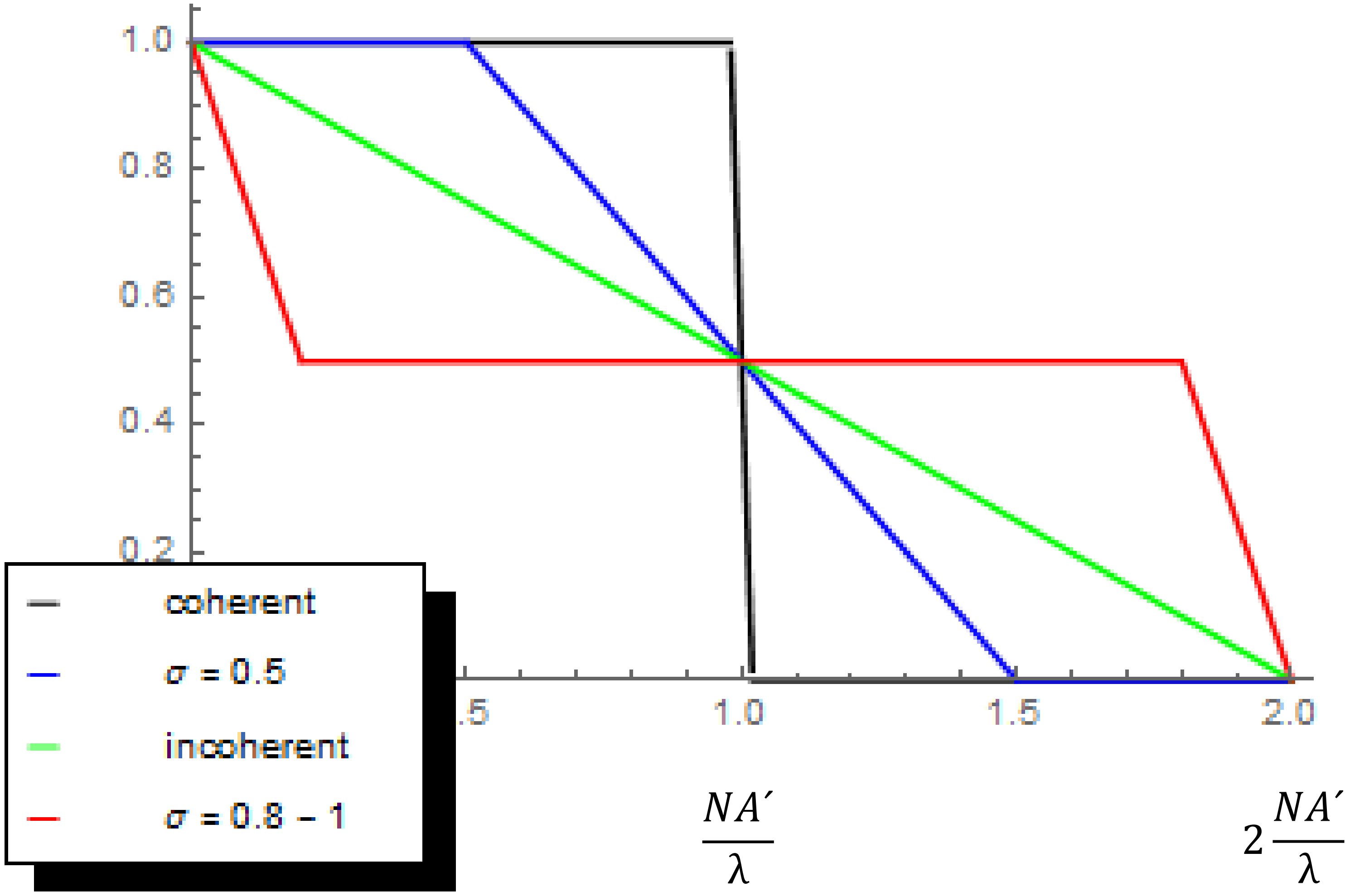


$HTF(\alpha_1, \alpha_2)$



$HTF_{qc}(\alpha_1, \alpha_2) \approx HTF(\alpha_1, 0)\ \mathrm{HTF}(0, \alpha_2)$

Linearized Transfer Function



Max. resolution in terms of spatial frequency:

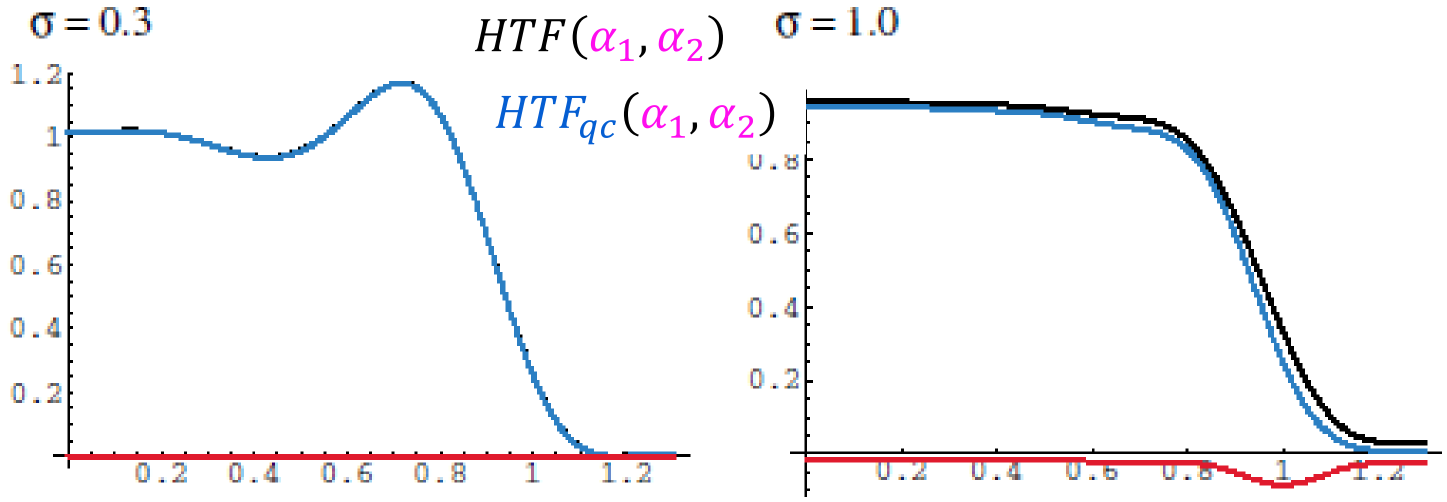
Coherent $\leq \frac{NA}{\lambda}$,

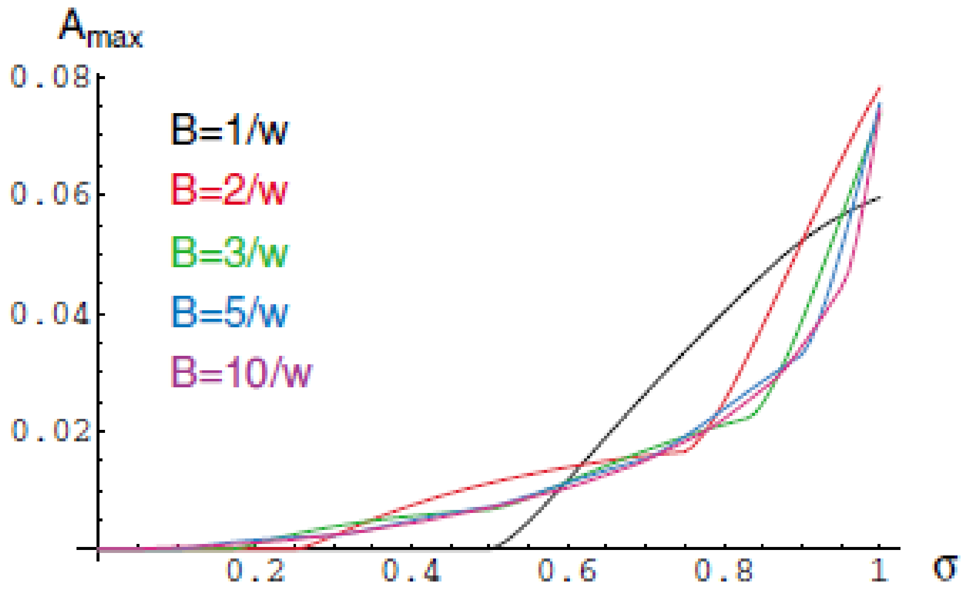
Partially coherent $\leq (1+\sigma)\frac{NA}{\lambda}$ *

Non-coherent $\leq 2\frac{NA}{\lambda}$

* If σ>1, $\leq 2\frac{NA}{\lambda}$

On approximation error
Periodic structure with different pitch vs σ





Example of quality of approximation: dependence on coherence parameter σ and structure size of periodic 1D-grating

Imaging equations for non-coherent light (1/2)
General spatially non-stationary ("non-isoplanatic") case

For **non-coherent imaging** all points in the object plane are uncorrelated, that is:

$$\Gamma_{ob}(\xi_1, \xi_2) = I_{ob}(\xi_1)\delta(\xi_1 - \xi_2)$$

and we obtain from the partially coherent equation $I(x) = \iint d\xi_1 d\xi_2\, \Gamma_{ob}(\xi_1, \xi_2) A_{ob}(\xi_1) A^*_{ob}(\xi_2) K(x, \xi_1) K^*(x, \xi_2)$ the following imaging equation ($I_{ob}(\xi) = |A_{ob}(\xi)|^2$)

$$I_{im,ncoh}(x) = \iint d\xi\, I_{ob}(\xi)\, |K(x, \xi)|^2 . \qquad [1]$$

The absolute square of the amplitude point spread function $K(x, \xi)$, the intensity distribution of an object point in image space, is called **point spread function** (**PSF**):

$$PSF(x, \xi) \coloneqq |K(x, \xi)|^2$$

Changing the order of integration first over object coordinates ξ then pupil coordinates α performed together with the substitution of variables $\alpha = \alpha_1 - \alpha_2$ and renaming $\alpha' = \alpha_2$ gives

$$I_{im,ncoh}(x) = \iint d\alpha\, exp(-i2\pi\, w\alpha \cdot x/m) \iint d\xi\, I_{ob}(\xi)\, exp(i2\pi\, w\alpha \cdot \xi) \iint d\alpha'\, L(\xi, \alpha + \alpha') L^*(\xi, \alpha') \qquad [2]$$

This is a 6-fold integral, as pupil and object variables can not be further separated.

Imaging equations for non-coherent light (2/2) Spatially stationary ("isoplanatic") case

For (locally) spatially stationary (isoplanatic) imaging, if the pupil function is independent (invariant) on the object height ξ: $L(\xi,\alpha) = L(\alpha)$

For most optical system this condition rarely holds over the complete object field, but approximately holds locally for the size of object structures of interest

Eq. [2] yields: $I_{im,ncoh}(x) = \iint d\alpha\, exp(-i2\pi\, w\alpha \cdot x/m) \iint d\xi\, I_{ob}(\xi)\, exp(i2\pi\, w\alpha \cdot \xi) \iint d\alpha'\, L(\alpha+\alpha')L^*(\alpha')$ [2,iso]

The convolution integral over pupil coordinates α (**autocorrelation**) is called **Optical Transfer Function** (**OTF**):

$$OTF(\alpha) = \iint d\alpha'\, L(\alpha+\alpha')L^*(\alpha') = MTF(\alpha) exp\big(i2\pi\, PTF(\alpha)\big)$$

The "abs"-value of the OTF is called **modulation transfer function** (**MTF**), its "arg" **phase transfer function** (**PTF**). By comparing the imaging equations, we see that for the spatially stationary case **PSF and OTF are a Fourier-Transform pair**: $PSF(x/m-\xi) = \iint d\alpha\, exp\big(-i2\pi\, w\alpha \cdot (x/m-\xi)\big) OTF(\alpha)$

Writing the Fourier-Transforms in compact form and $FT_\xi\,\{I_{ob}\}$ for the intensity spectrum of the object distribution in the lens pupil we can write the equation [2,iso] more compactly:

$$I_{im,ncoh}(x) = FT_\alpha^{-1}\{OTF(\alpha) FT_\xi(I_{ob})(w\alpha)\}(x) \qquad [2,iso]$$

For spatially stationary imaging eq. [1] becomes a convolution integral of the object distribution and the PSF:

$$I_{im,ncoh}(x) = \iint d\xi\, I_{ob}(\xi)\, PSF(x/m-\xi) \qquad [1,iso]$$

Summary: Imaging equations and optical system transfer functions PSF and OTF

$$I_{im,ncoh}(x) = \iint d\xi \; I_{ob}(\xi) \; PSF(x/m - \xi)$$ [1,iso]

where $PSF(x, \xi) := |K(x/m - \xi)|^2$ and $K(x/m - \xi) = \iint d\alpha \; L(\alpha) \; exp\big(-i2\pi w \; \alpha \cdot (x - \xi)\big)$

$$I_{im,ncoh}(x) = \iint d\alpha \; exp(-i2\pi \; w\alpha \cdot x/m) \; OTF(\alpha) FT_{\xi}(I_{ob})$$ [2,iso]

where $FT_{\xi}(I_{ob}) = \iint d\xi \; I_{ob}(\xi) \; exp(i2\pi \; w\alpha \cdot \xi)$ and $OTF(\alpha) = MTF(\alpha) exp\big(i2\pi \; PTF(\alpha)\big) = \iint d\alpha' \; L(\alpha + \alpha') L^*(\alpha')$

- These imaging equations are based on **spatial stationarity**, that is: within the size of the object region the variations of the system transfer function (either PSF or OTF) must be sufficiently small, such that the changes of the intensity distribution is small (condition $L(\xi, \alpha) = L(\alpha)$ or equivalently $K(x, \xi) = K(x/m - \xi)$)
- **PSF and OTF are equivalent** (Fourier-Transform pair) and completely describe the optical system
- The transfer functions PSF or OTF are **completely separated from the object intensity distribution** or its spectrum (due to the assumption of spatial stationarity)
- Although the **MTF** is frequently used for system quality evaluation, it **does <u>not</u> contain the complete information** of the system (missing: relative phase shifts of spatial frequency components)

Incoherent Imaging: mathematical model

Institute of Applied Physics
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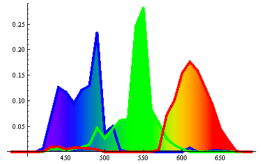
spectral response



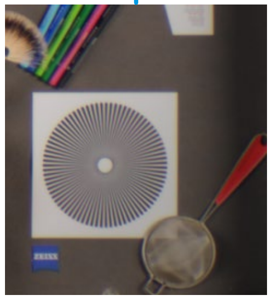
spectral light source distribution

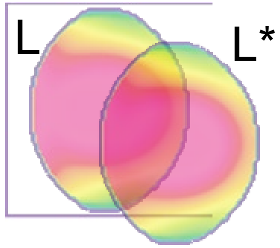
Optical Transfer Function

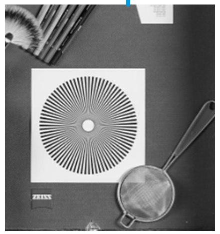
„Spatial frequency distribution“ of object

image intensity

$$I_{\{R,G,B\}}(x) = \int_{\text{wavelength}} d\lambda \; s_{\{R,G,B\}}(\lambda) q(\lambda) \iint_{pupil} d\alpha \; \exp(i2\pi w\alpha \cdot x) \; OTF(\alpha,\lambda) \; FT_{\xi}\{I_{ob}\}(w\alpha,\lambda)$$







monochromatic image distribution

“resolution parameter”

$$w = \frac{NA_{im}}{\lambda} = \frac{1}{2\,K\,\lambda}$$

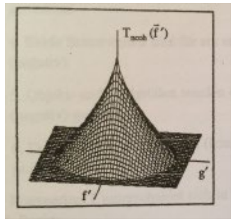
monochromatic OTF

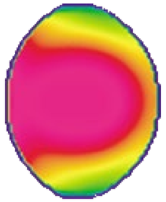
Autocorrelation of lens pupil function

monochromatic MTF (Modulation)

monochromatic PTF (Phase)

$$OTF(\alpha,\lambda) = \iint_{pupil} d\bar{\alpha} \; L(\bar{\alpha}+\alpha,\lambda) \; L^{*}(\bar{\alpha},\lambda) = MTF(\alpha,\lambda)\exp\big(i2\pi PTF(\alpha,\lambda)\big)$$





$$L(\alpha,\lambda) = L_0(\alpha,\lambda)\exp\big(i2\pi W(\alpha)\big)$$

lens pupil shape

wavefront deviation

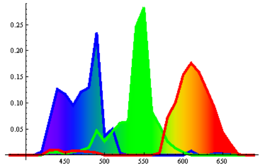
$\xi \in R^2$ object coordinates (ob),
$\alpha \in R^2$ entrance pupil coordinates (EP)
$\alpha' \in R^2$ exit pupil coordinates (AP)
$x \in R^2$ image coordinates (im)

Incoherent Imaging: mathematical model

image intensity

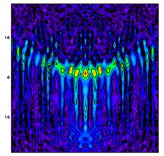
object

spectral response

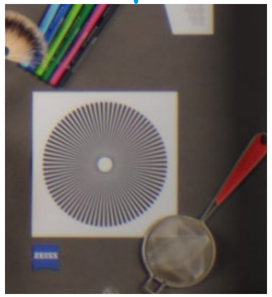


spectral light source distribution

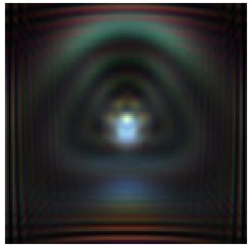
monochromatic PSF



$$\underbrace{I_{\{R,G,B\}}(x)}\; = \iint_{\text{object surface}} d\xi d\eta\; \underbrace{I_{ob}(\xi)} \int_{\text{wavelength}} \underbrace{d\lambda\; s_{\{R,G,B\}}(\lambda) q(\lambda)\;\; PSF(x - m\xi, \lambda)}$$



photorealistic PSF



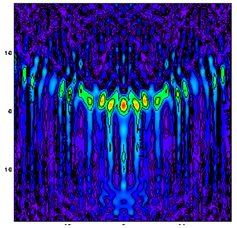
“resolution parameter”

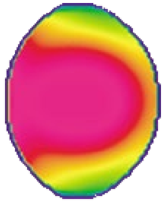
$$w = \frac{NA_{im}}{\lambda} = \frac{1}{2\,K\,\lambda}$$

monochromatic PSF

lens pupil function

$$\underbrace{PSF(x - m\xi, \lambda)} = \left| \iint_{\text{lens pupil}} d\alpha d\beta\;\; \underbrace{L(\alpha, \lambda)} \exp(-i2\pi w[\alpha \cdot (x - m\xi)]) \right|^2$$





$$L(\alpha, \lambda) = \underbrace{L_0(\alpha, \lambda)}_{\text{lens pupil shape}} \exp\big(i2\pi \underbrace{W(\alpha)}_{\text{wavefront deviation}}\big)$$

$\xi \in R^2$ object coordinates (ob),
$\alpha \in R^2$ entrance pupil coordinates (EP)
$\alpha' \in R^2$ exit pupil coordinates (AP)
$x \in R^2$ image coordinates (im)

Incoherent Imaging: mathematical model

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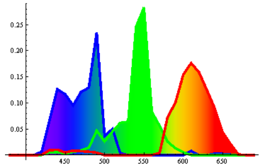
depth map
$z(\xi,\eta)$

spectral response



spectral light source distribution

monochromatic PSF

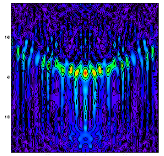
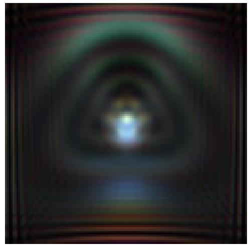


image intensity

$$I_{\{R,G,B\}}(x) = \iint_{\text{object surface}} d\xi d\eta \; I_{ob}(\xi) \int_{\text{wavelength}} d\lambda \; s_{\{R,G,B\}}(\lambda) q(\lambda) \; PSF(x - m\xi, \lambda)$$

Object ("texture")

photorealistic PSF



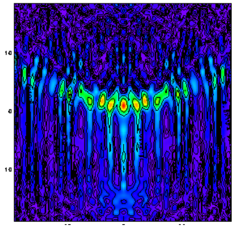
"resolution parameter"

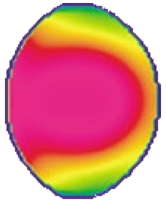
$$w = \frac{NA_{im}}{\lambda} = \frac{1}{2\,K\,\lambda}$$

monochromatic PSF

lens pupil function

$$PSF(x - m\xi, \lambda) = \left| \iint_{\text{lens pupil}} d\alpha d\beta \; L(\alpha,\lambda) \exp(-i2\pi w[\alpha \cdot (x - m\xi)]) \right|^2$$





$$L(\alpha,\lambda) = \underbrace{L_0(\alpha,\lambda)}_{\text{lens pupil shape}} \exp\big(\underbrace{i2\pi W(\alpha)}_{\text{wavefront deviation}}\big)$$

$\xi \in R^2$ object coordinates (ob),
$\alpha \in R^2$ entrance pupil coordinates (EP)
$\alpha' \in R^2$ exit pupil coordinates (AP)
$x \in R^2$ image coordinates (im)

Monochromatic point spread function (PSF) over field

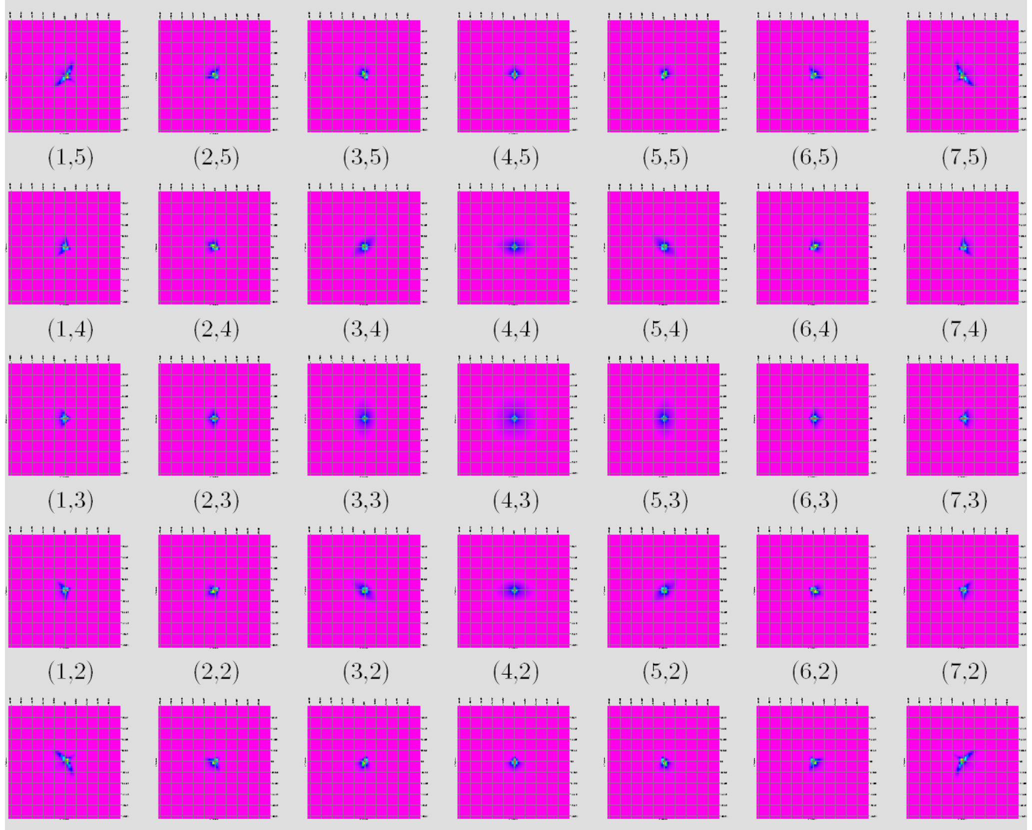
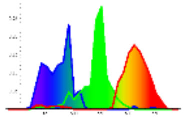
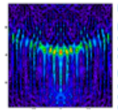
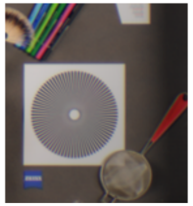
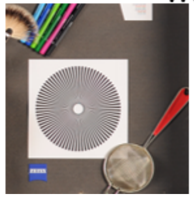


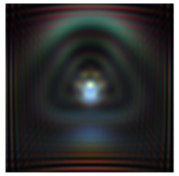
image intensity — object — spectral response — spectral light source distribution — space-variant, monochromatic PSF

$$I_{\{R,G,B\}}(x) = \iint_{\text{object surface}} d\xi d\eta \; I_{ob}(\xi) \int_{\text{wavelength}} d\lambda \; s_{\{R,G,B\}}(\lambda) q(\lambda) \; PSF(x,\xi,\lambda)$$

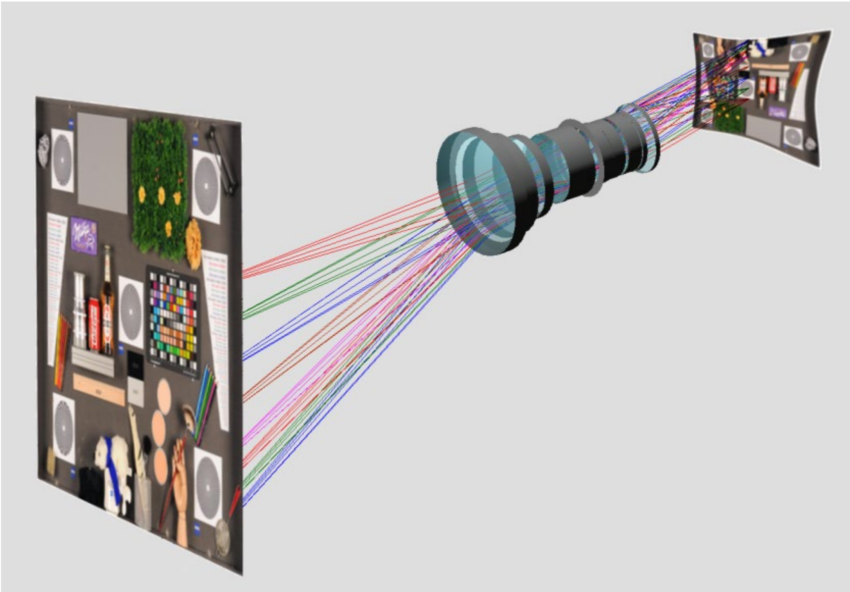



photorealistic PSF



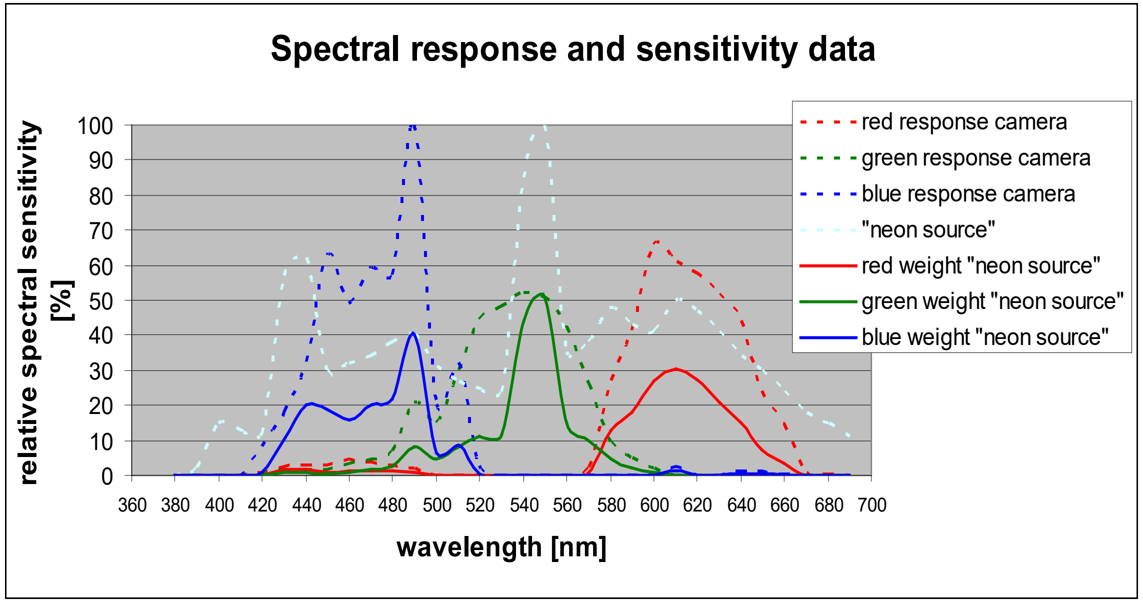
variable over field

The PSF of camera lenses often changes significantly over the field of view.
In smaller areas, ca. 1/10 - 1/100 of the field of view.
This "isoplanasy condition" holds only locally.



Spectral weights

Spectral weight = spectral source distribution
***spectral lens transmission**
***spectral sensor response**

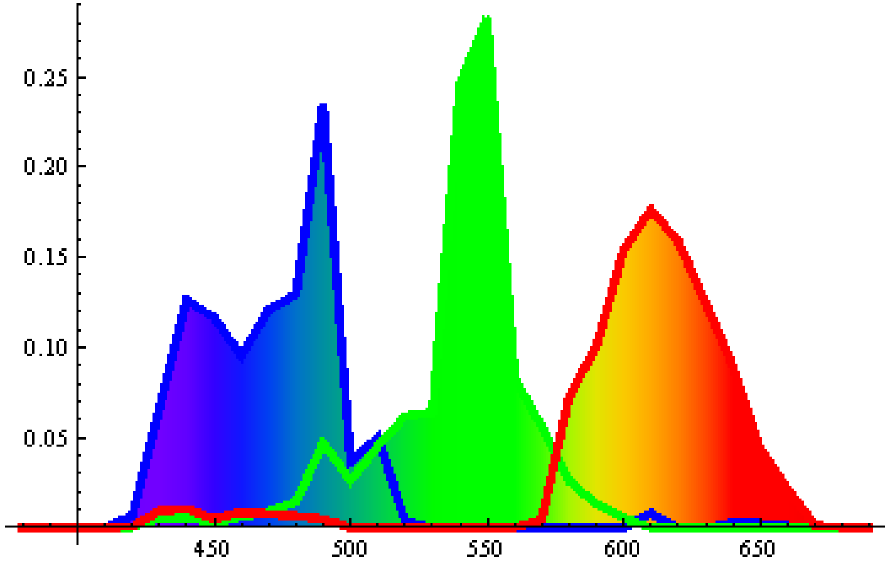


Spectral weight contributions

$$\int d\lambda\, g_{\mathrm{Red}}(\lambda) = 1$$

$$\int d\lambda\, g_{\mathrm{Green}}(\lambda) = 1$$

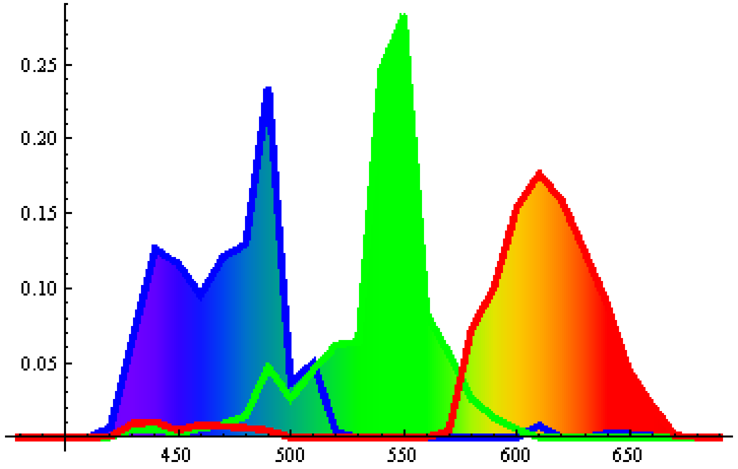
$$\int d\lambda\, g_{\mathrm{Blue}}(\lambda) = 1$$



Normalized („white-balanced“) spectral weight

Polychromatic point spread functions of color channels

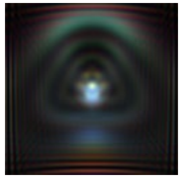
After integration of the monochromatic PSFs over wavelengths with the weights given by the spectral response of the image sensor and the light source distribution one obtains the PSFs of the RGB-channels repectively



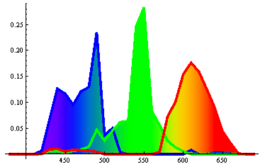
$$s_{\{R,G,B\}}(\lambda)q(\lambda)t_{lens}(\lambda)$$

sensor illumination lens spectral transmission

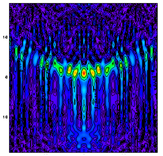
photorealistic PSF



spectral response

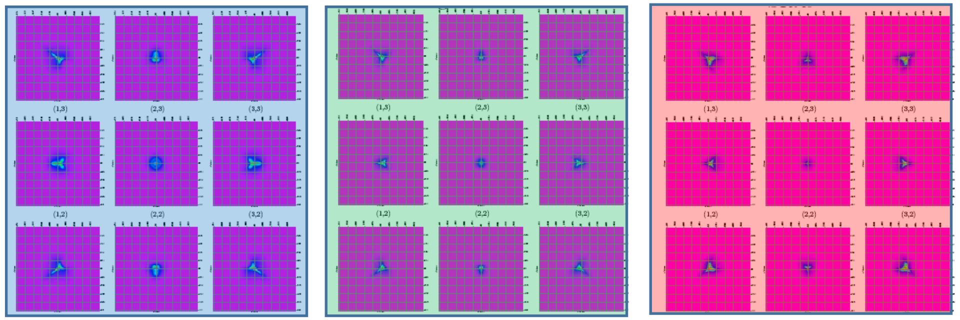


spectral light source distribution



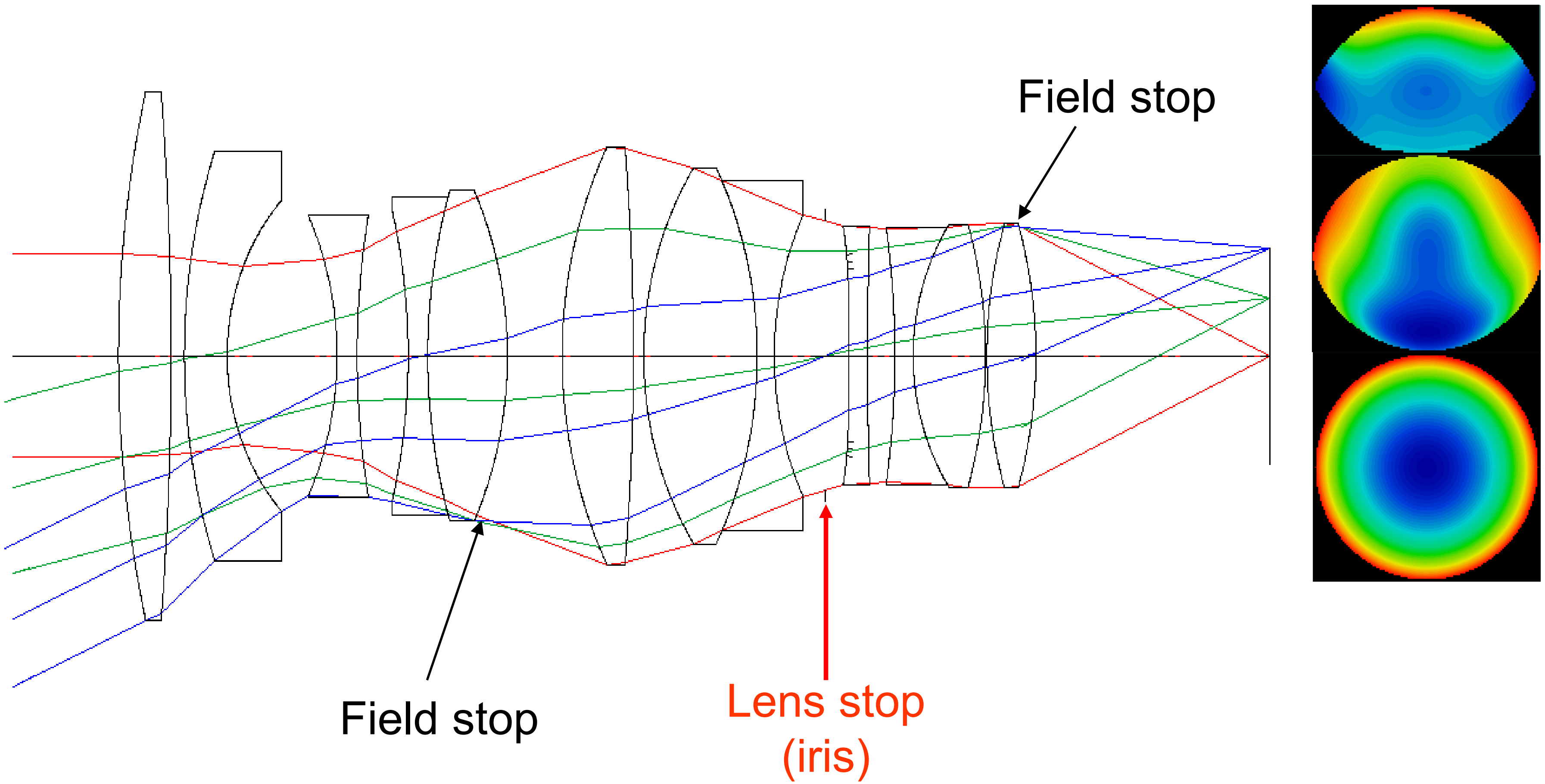
space-variant, monochromatic PSF

$$PSF_{\{R,G,B\}}(x,\xi) = \int_{\text{wavelength}} d\lambda\ s_{\{R,G,B\}}(\lambda)q(\lambda)\ PSF(x,\xi,\lambda)$$



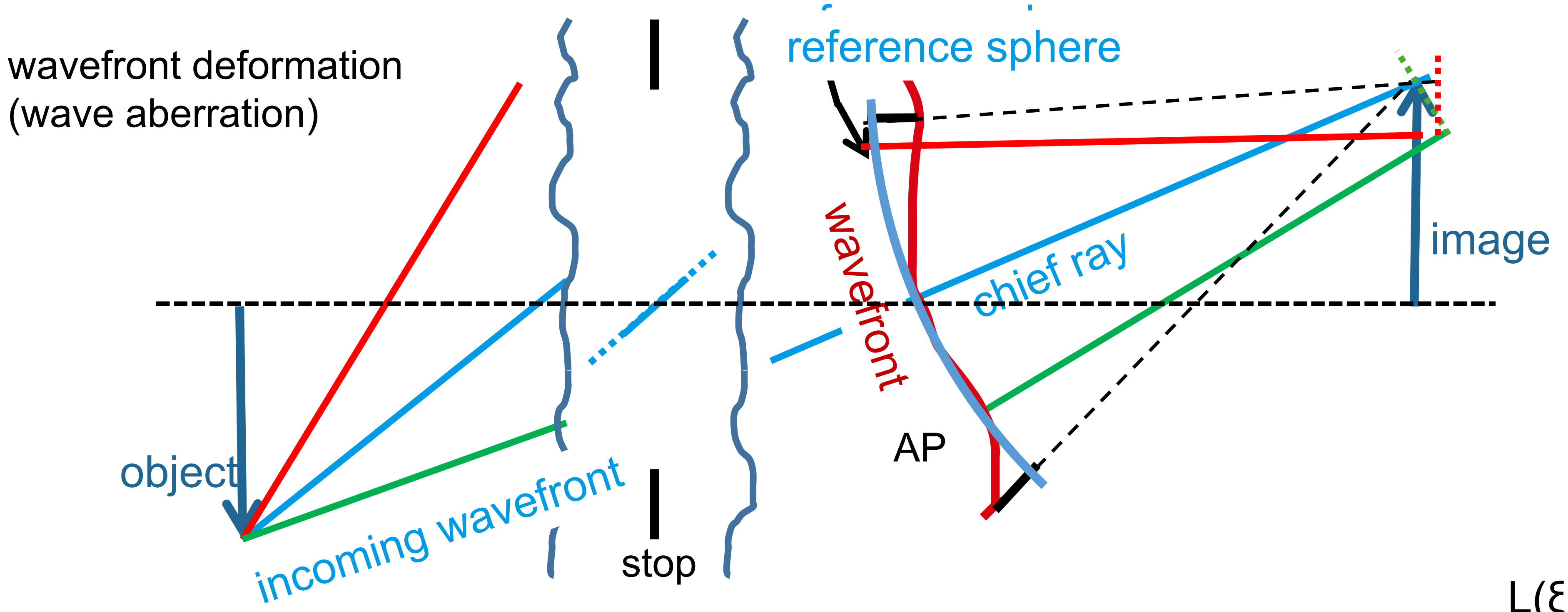
e.g. 15 x 15 PSFs calculated over field and afterwards interpolated for each grid point

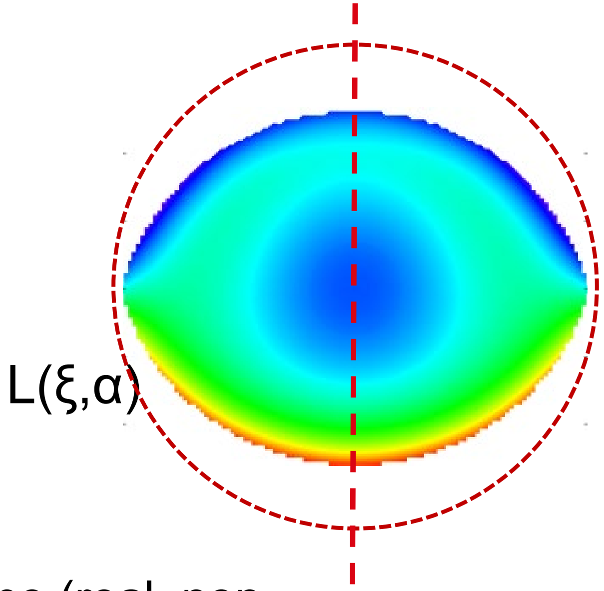
Lens pupil
(shape and wavefront aberration)



Real objects are complicated. The lens as well (pupil shape, aberrations,…)
How can the image be calculated?

Computation of wavefront deformation





Entrance and Exit Pupil are non-paraxial here meaning the (virtual) intersection of the (real, non-paraxial) chief ray in object and image space respectively.
In paraxial approximation EP and AP are system invariants; however for real rays the exit (entrance) pupil is changing with object height.

Computation of wavefront deformation

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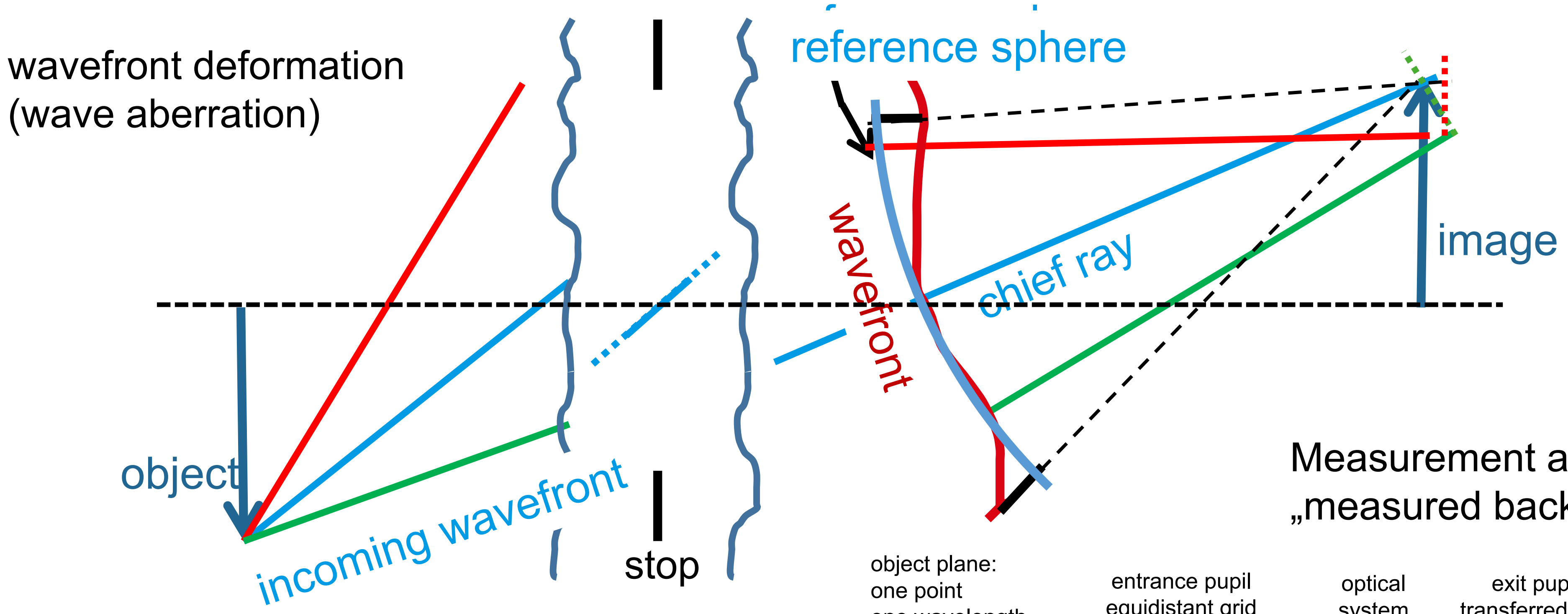
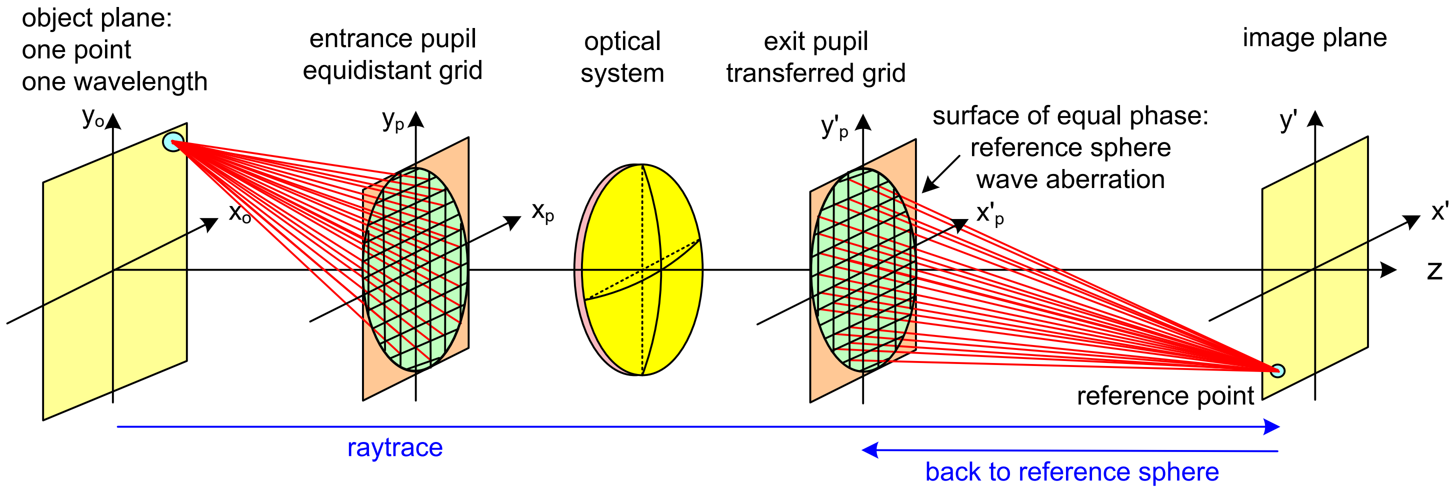
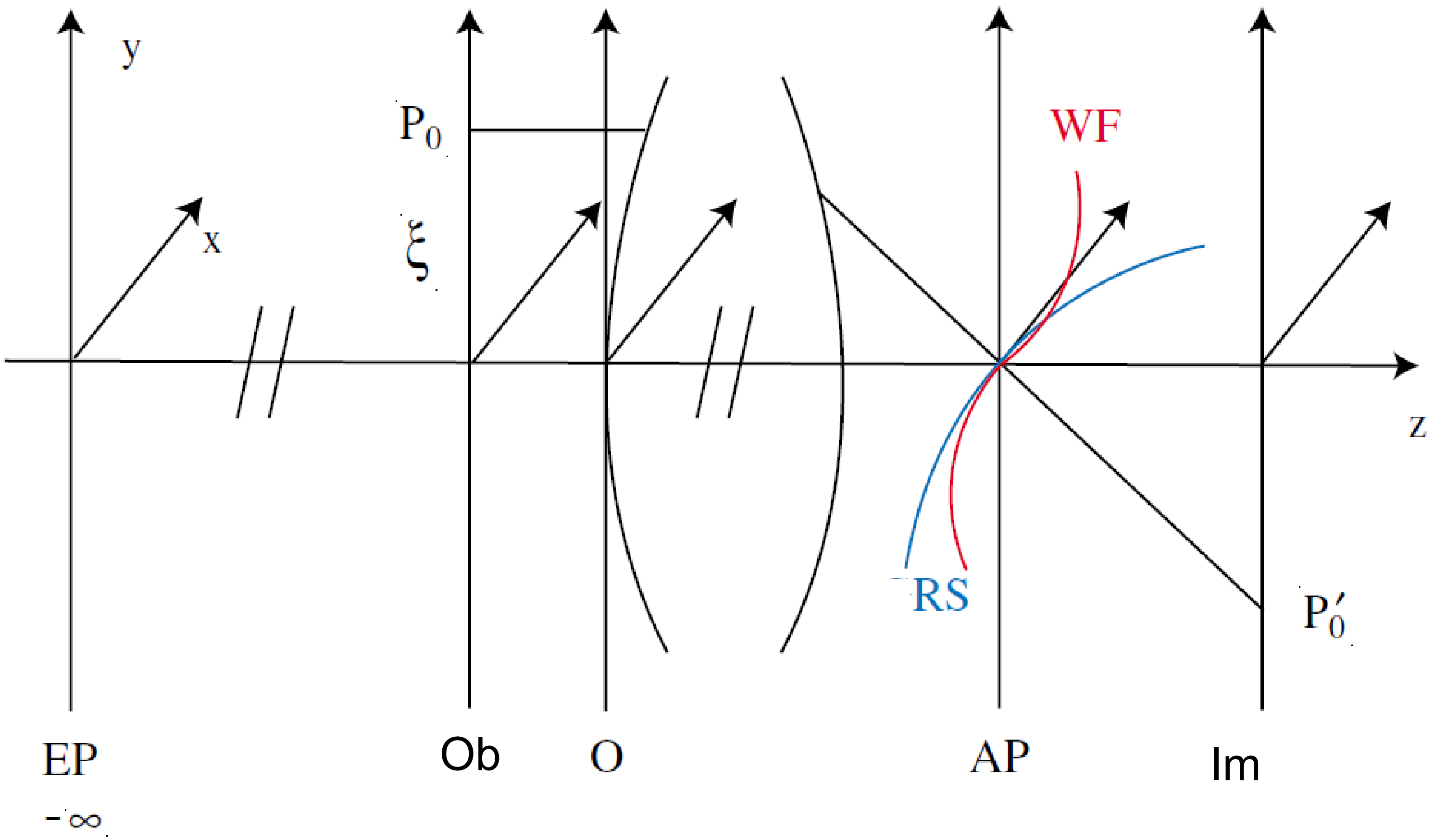


Measurement along reference is, as if „measured backwards from image point“:



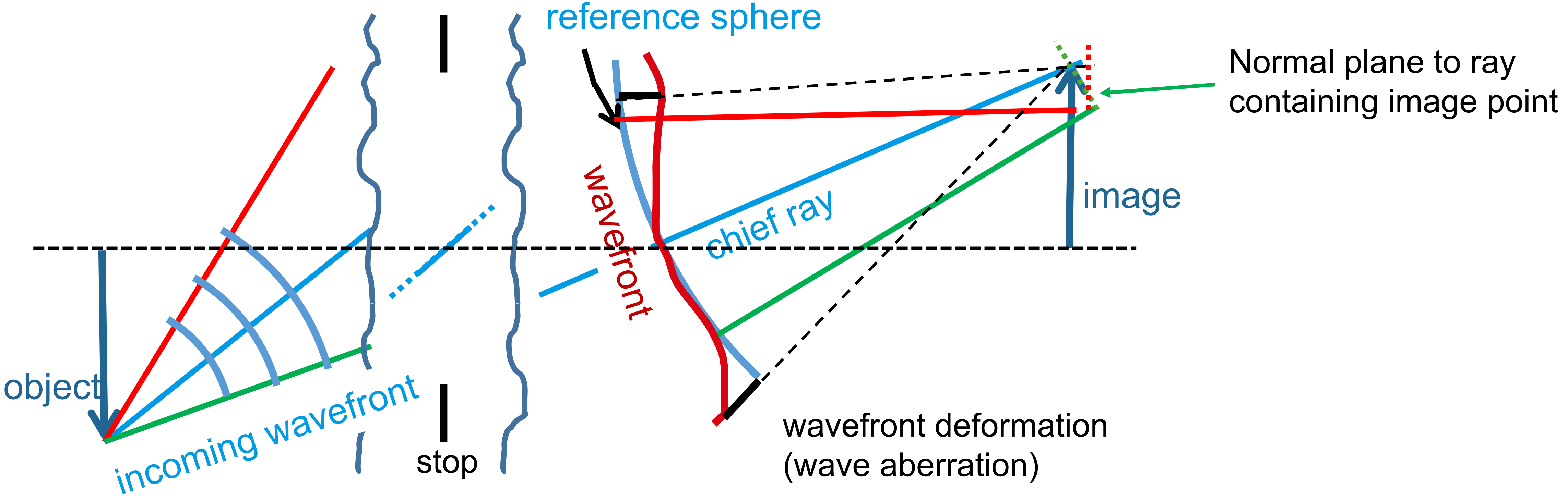
Wavefront deformation for an object-side telecentric optical system



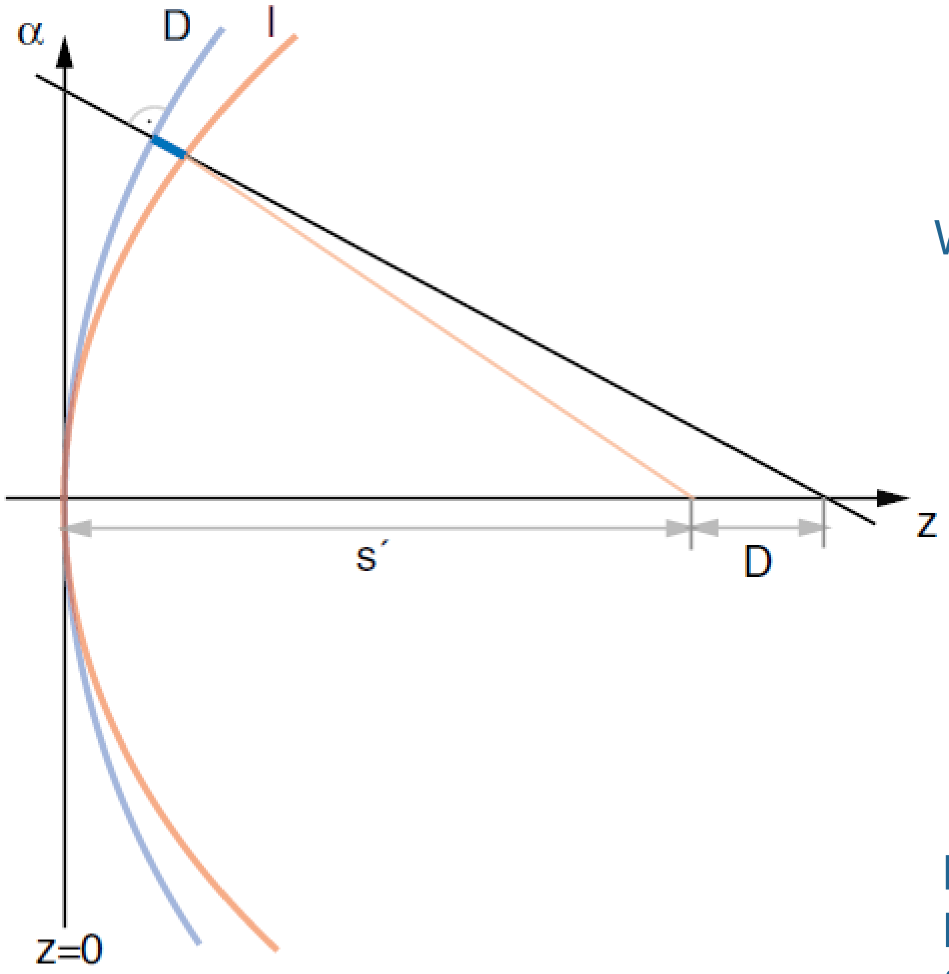
Construction of the wave aberration for microscope imaging according to Gerlich and Schomäcker.

The entrance pupil (EP) of a microscope objective lies at (minus) infinity.
Accordingly, the chief ray determines the exit pupil (AP) as the intersection with the optical axis. The distance to the intersection $P_0{}'$ with the given image plane gives the radius of the reference sphere (RS), with respect to which wavefront deformation is measured as the distance to the wavefront (WF).

Alternative computation method of wavefront deformation



On dependence of wavefront deformation on distance to image



$$\overline{\alpha}^2 + (z - s')^2 = s'^2,$$ „Ideal“

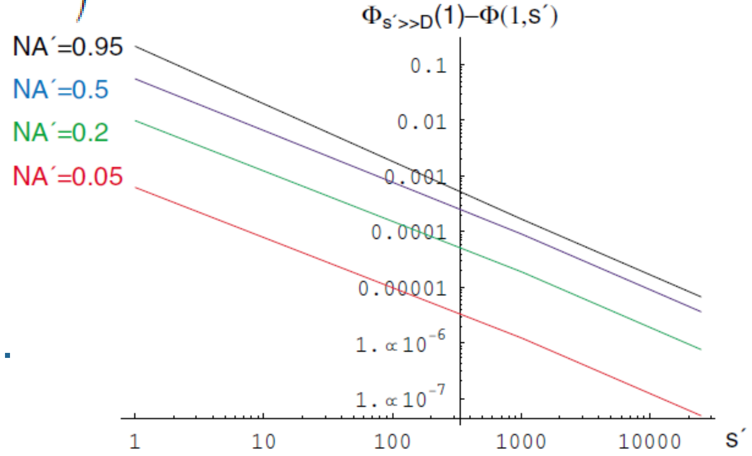
$$\overline{\alpha}^2 + \left(z - (s' + D)\right)^2 = (s' + D)^2$$ „Defocus“

Wavefront deformation depends on distance to „ideal image point“

$$\Phi(\alpha, s', D) = \frac{D + s'\sqrt{1 + \dfrac{s' - D}{s' + D}\,\dfrac{(NA'/n')^2\alpha^2}{1 - (NA'/n')^2\alpha^2}}}{\sqrt{1 + \left(\dfrac{s'}{s' + D}\right)^2 \dfrac{(NA'/n')^2\alpha^2}{1 - (NA'/n')^2\alpha^2}}} - (s' + D)$$

$$\Phi_{s' >> D}(\alpha, D) = D\left(1 - \sqrt{1 - (NA'/n')^2\alpha^2}\right)$$

Difference usually negligible.
However a location of RS at only few orders of λ should be avoided.



Calculation of wavefront deformation for defocus

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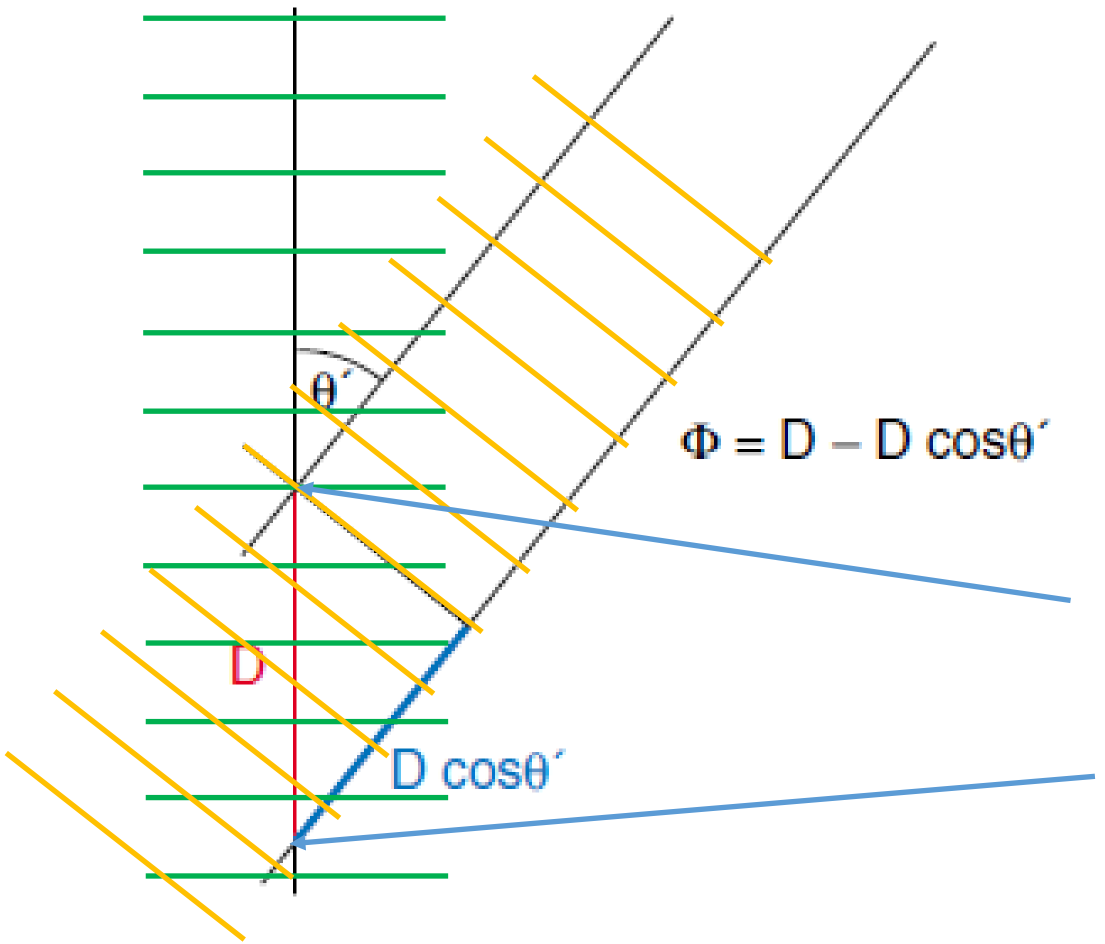
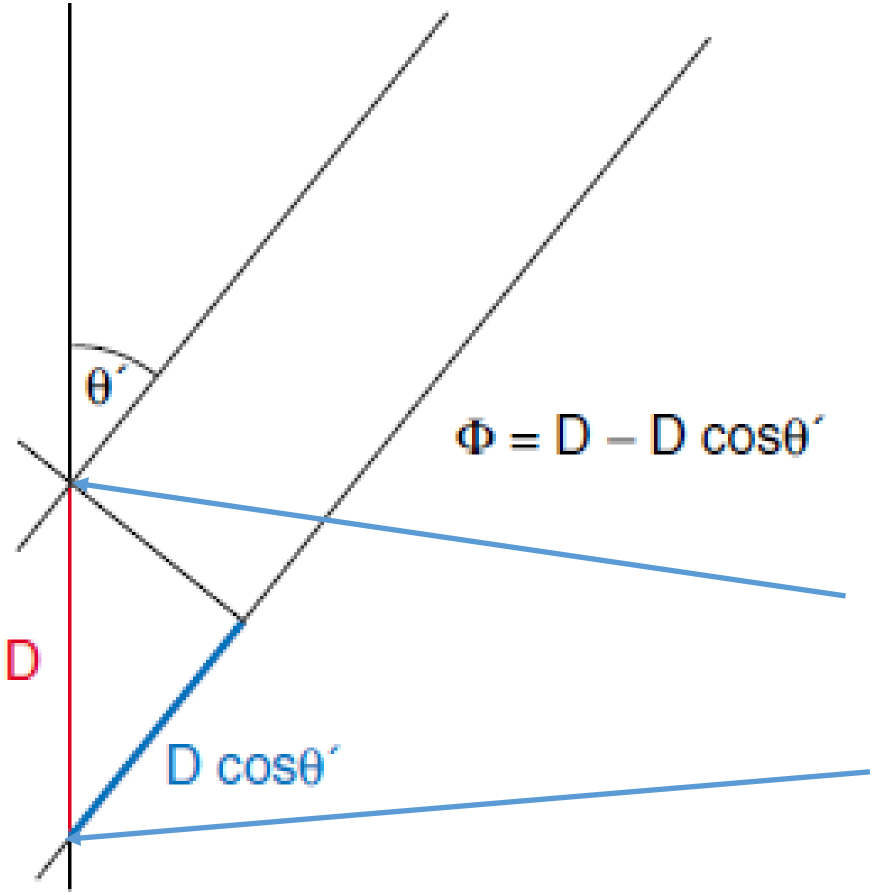


Defocus wavefront deformation („Eikonal method"):

Both plane waves are in phase here…

So how far are they out of phase here?
(→ wavefront deformation W(θ´))

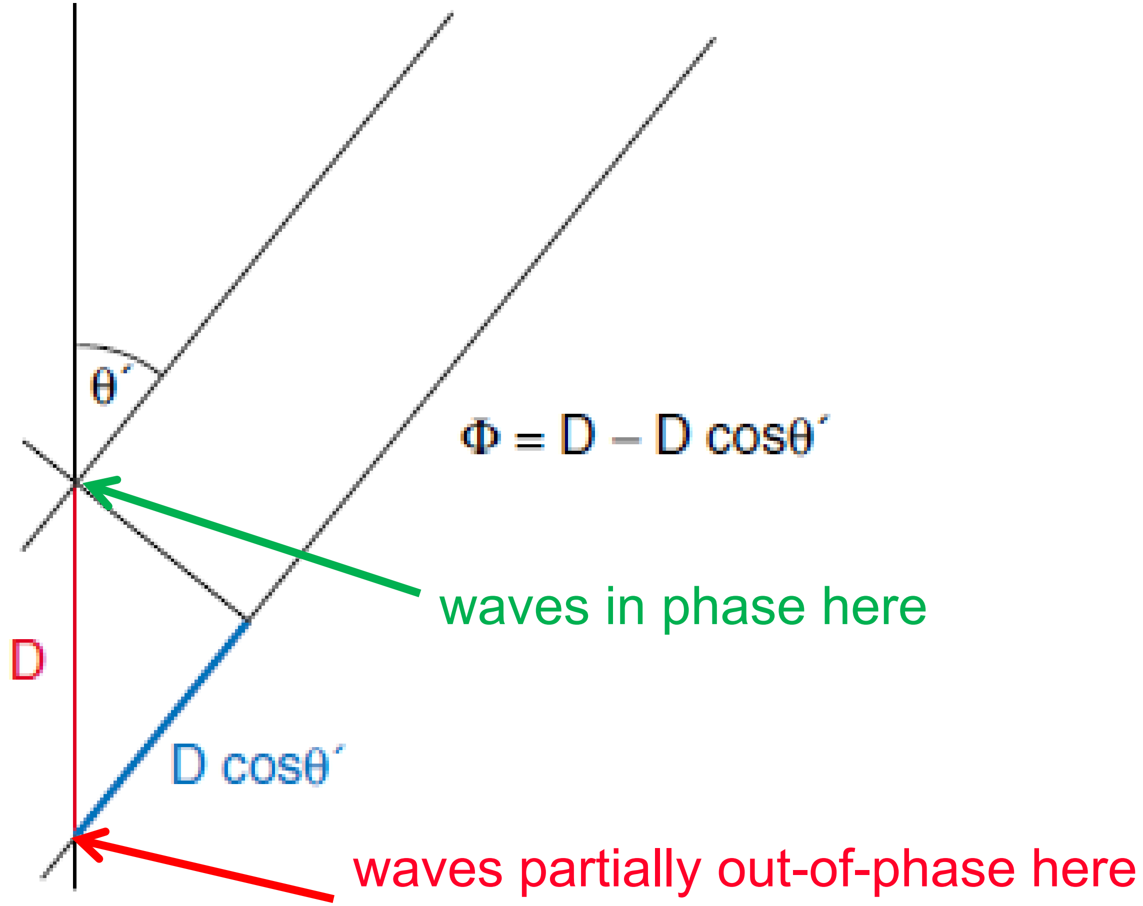
Calculation of wavefront deformation for defocus



Both plane waves are in phase here…

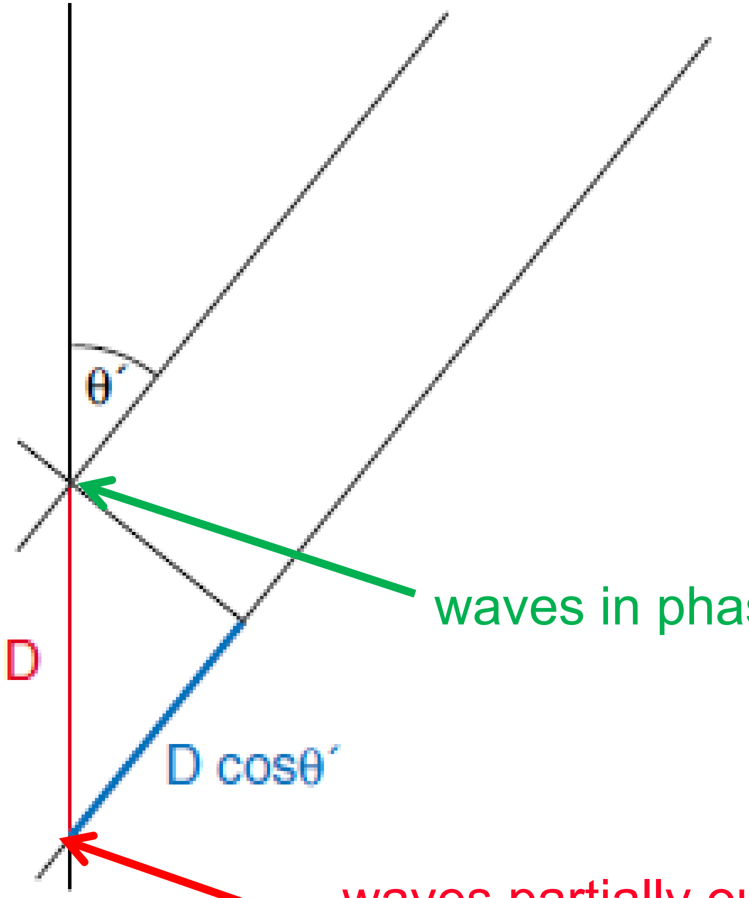
So how far are they out of phase here?
(→ wavefront deformation W(θ´))

Calculation of wavefront deformation for defocus



Optical path difference $\boldsymbol{\Phi} = \mathrm{D} - \mathrm{D}\cos\boldsymbol{\vartheta}'$

Calculation of wavefront deformation for defocus



waves in phase here

waves partially out-of-phase here
Optical path difference $\boldsymbol{\Phi} = \mathrm{D} - \mathrm{D}\cos\boldsymbol{\vartheta}'$

Optical path difference $\mathrm{OPD} = \mathrm{D} - \mathrm{D}\cos\boldsymbol{\vartheta}'$

Expressed in normalized pupil coordinates NA´$\boldsymbol{\alpha}$ pupil (NA´=sin $\boldsymbol{\vartheta}'$; 0≤ $|\alpha| \leq 1$):

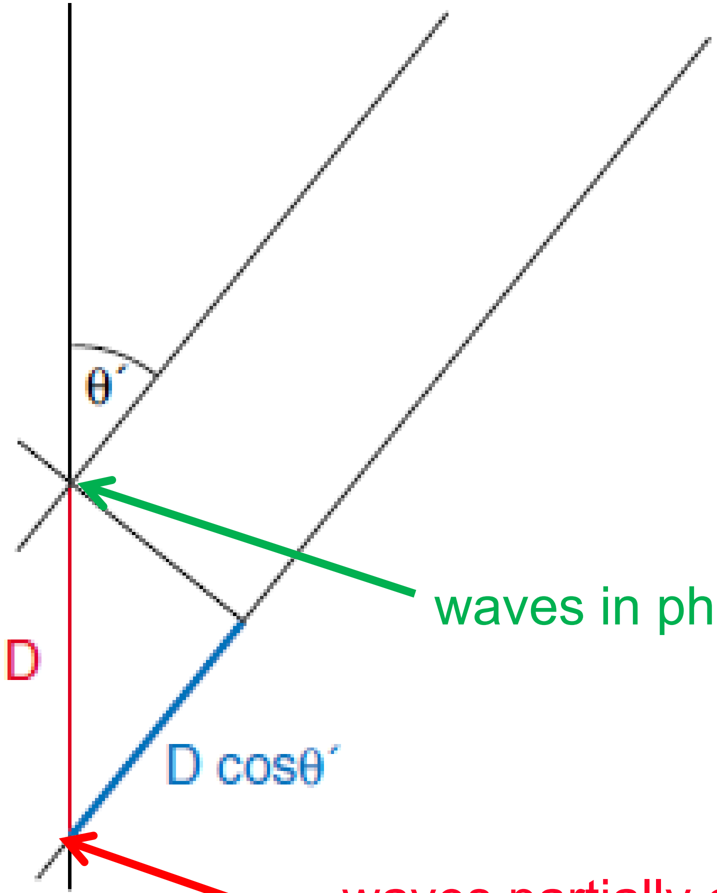
$$OPD = D\left(1 - \sqrt{1 - sin^2\boldsymbol{\vartheta}'}\right)$$
$$= D\left(1 - \sqrt{1 - NA'^2\alpha^2}\right)$$

With a **series-expansion** of the square-root we find an approximate expression for not-too-large apertures **in quadratic order**

$$OPD = D\left(1 - \left(1 - \frac{NA'^2\alpha^2}{2} - \frac{NA'^4\alpha^4}{8} - \cdots\right)\right)$$
$$\approx D\,\frac{NA'^2\alpha^2}{2}$$

typical Rayleigh approximation (not valid for high-NA)

Calculation of wavefront deformation for defocus



waves in phase here

waves partially out-of-phase here
Optical path difference $\boldsymbol{\Phi} = \mathrm{D} - \mathrm{D}\cos\boldsymbol{\vartheta}'$

The phase delay between the waves is

$$\varphi = \frac{2\pi}{\lambda} OPD$$

That is

$$\varphi \approx \frac{\pi}{\lambda} NA'^2 \alpha^2$$

The waves are out of phase cos(φ)=0

$$\rightarrow \frac{\pi}{\lambda} NA'^2 \alpha^2 = \frac{\pi}{2}$$

Image field depth of focus scales with

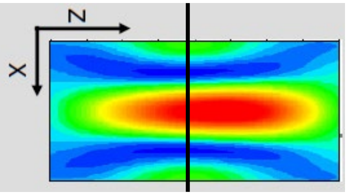
$$\mathrm{DOF} = \frac{\lambda}{NA'^2}$$

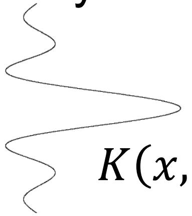
Point spread function from pupil function

Amplitude point spread function, aPSF), which is related to the pupil function L of the system as follows:

$$K(x,\xi,z) = \iint d\alpha\, L(\alpha,\xi,z) exp\left(-i2\pi w\, \alpha \cdot (x/m - \xi)\right)$$

Complex function





$K(x,\xi_0,z_0)$

Point Spread Function (PSF)

$$PSF(x,\xi,z) = |K(x,\xi,z)|^2$$

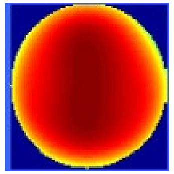
Intensity of „point object“ („star“)

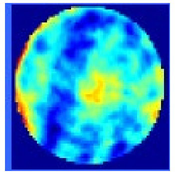
Real function

The pupil function L describes the limitation of the aperture(s) and the transmission distribution in the pupil, contained in the function $L_0(\alpha,\xi)$, as well as the aberrations $W(\alpha,\xi)$ of the optical system:

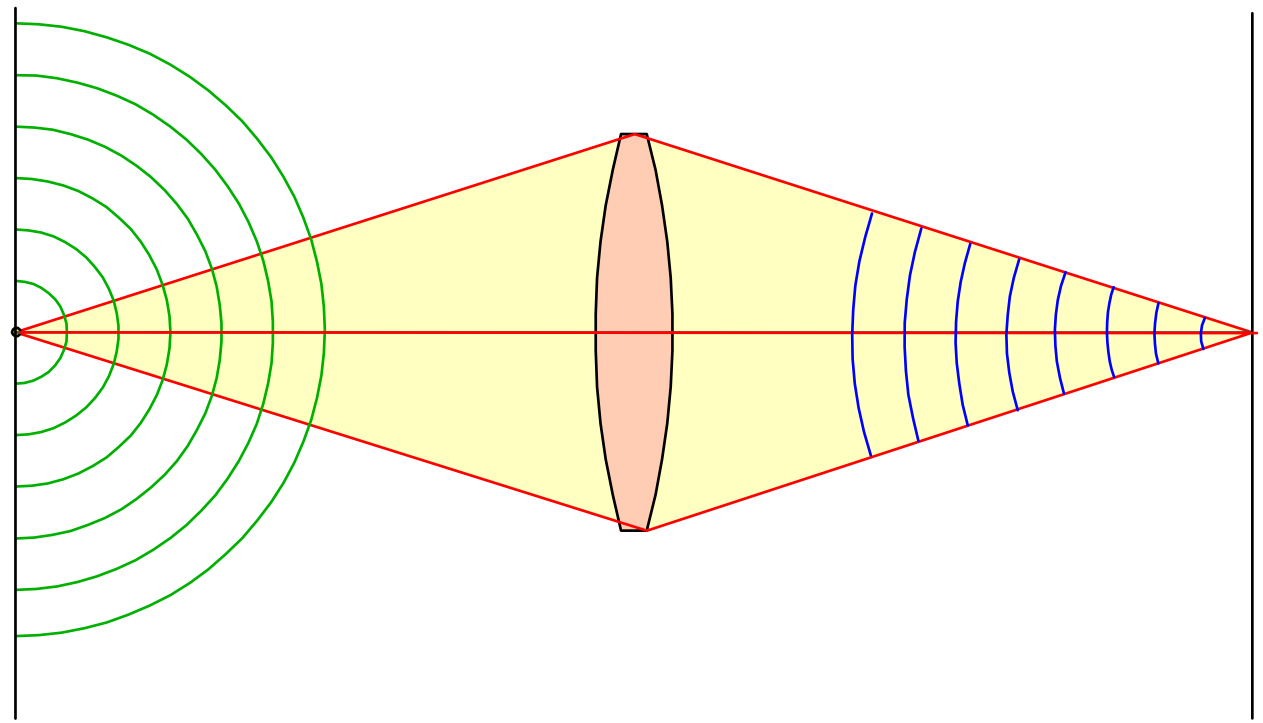
$$L(\alpha,\xi,z) = L_0(\alpha,\xi) exp\left(i2\pi W(\alpha,\xi)\right) exp\left(-i\frac{2\pi}{\lambda} z\sqrt{1-(NA/n)^2\alpha^2}\right).$$

Complex function





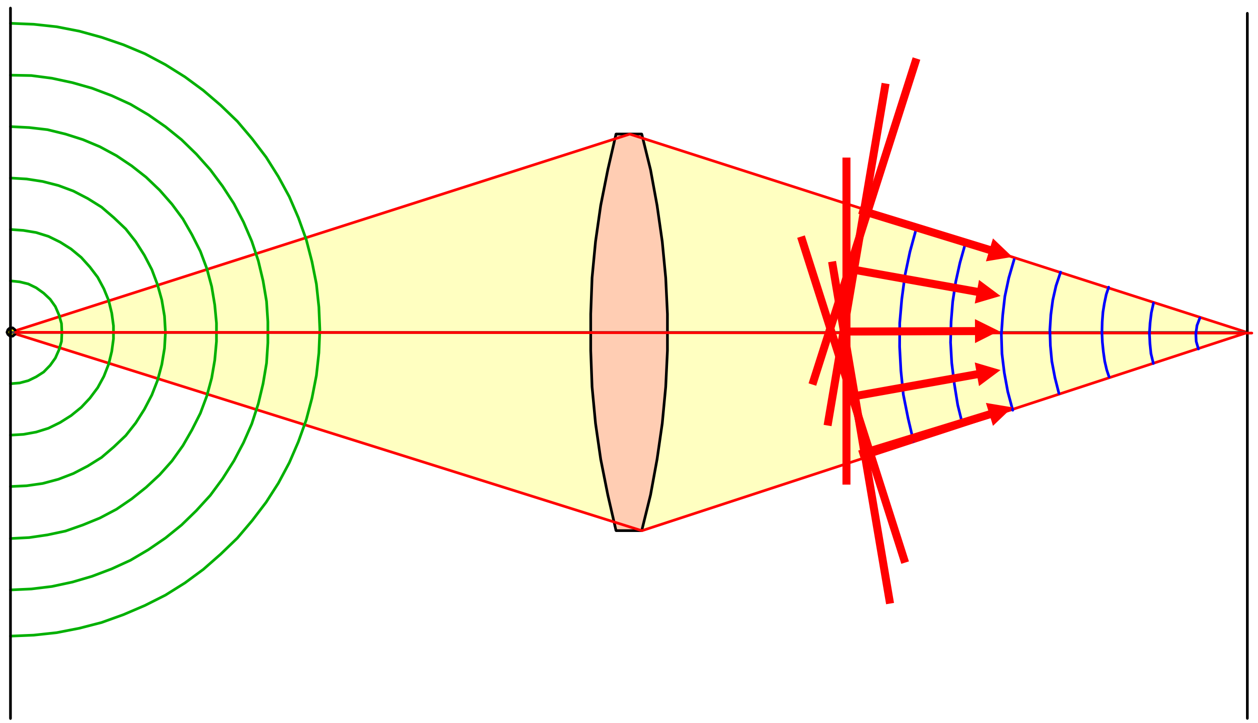
Point spread function (PSF)



Point spread function (PSF)

Fourier integral

$$\frac{e^{ikr}}{r} = \int \frac{1}{m} e^{ikmz}\, e^{ik(px+qy)} dpdq$$



Point spread function (PSF): ideal amplitude PSF of circular pupil

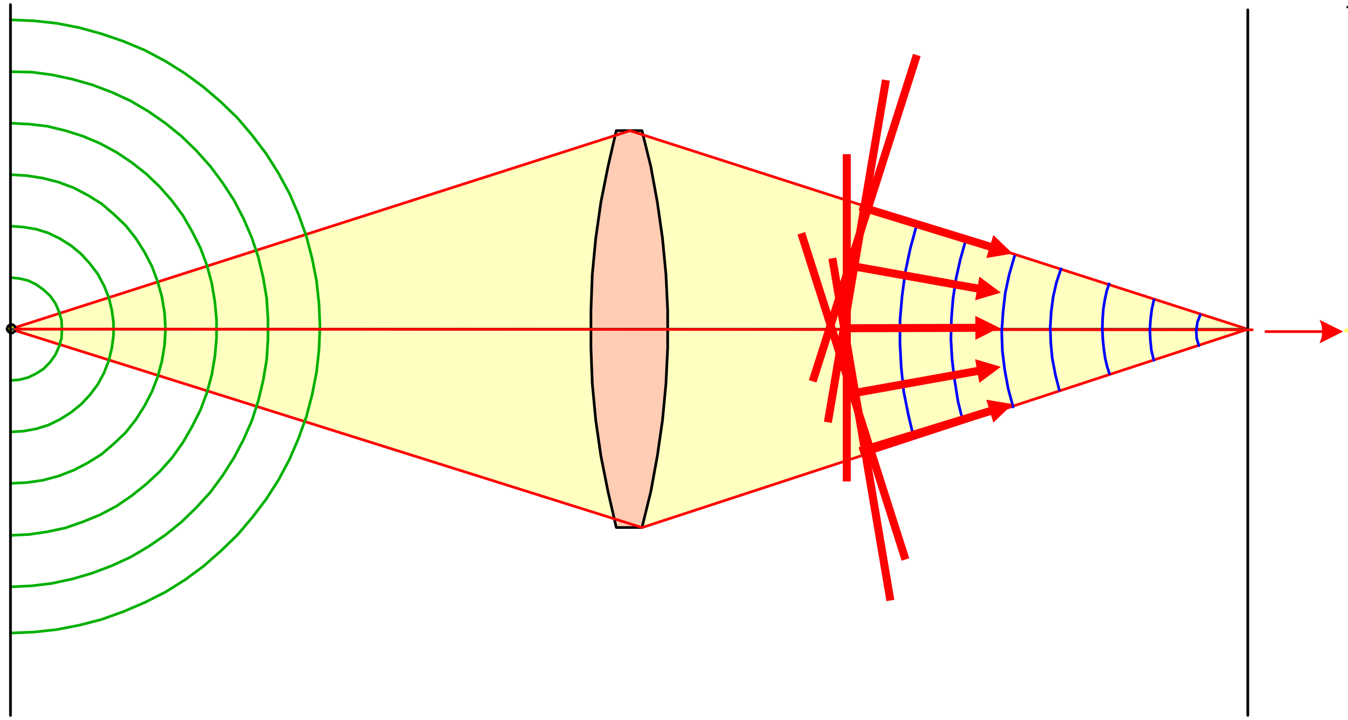
Ideal amplitude PSF:
circular pupil

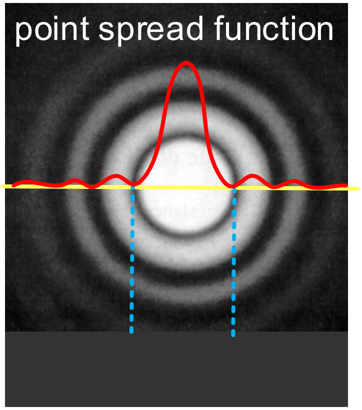
$$K(r) = aPSF(r) = \int_0^{2\pi}\int_0^{1} exp\left(i2\pi\frac{NA}{\lambda}\rho r\right)\rho d\rho d\varphi$$

$$= 2\pi\int_0^1 J_0\left(2\pi\frac{NA}{\lambda}\rho r\right)\rho d\rho = \frac{2J_1\left(2\pi\frac{NA}{\lambda}\rho r\right)}{2\pi\frac{NA}{\lambda}\rho r} = Besinc\left(2\pi\frac{NA}{\lambda}r\right)$$

radial coordinates $r = \sqrt{x^2+y^2}$, $\rho = \sqrt{\alpha^2+\beta^2}$

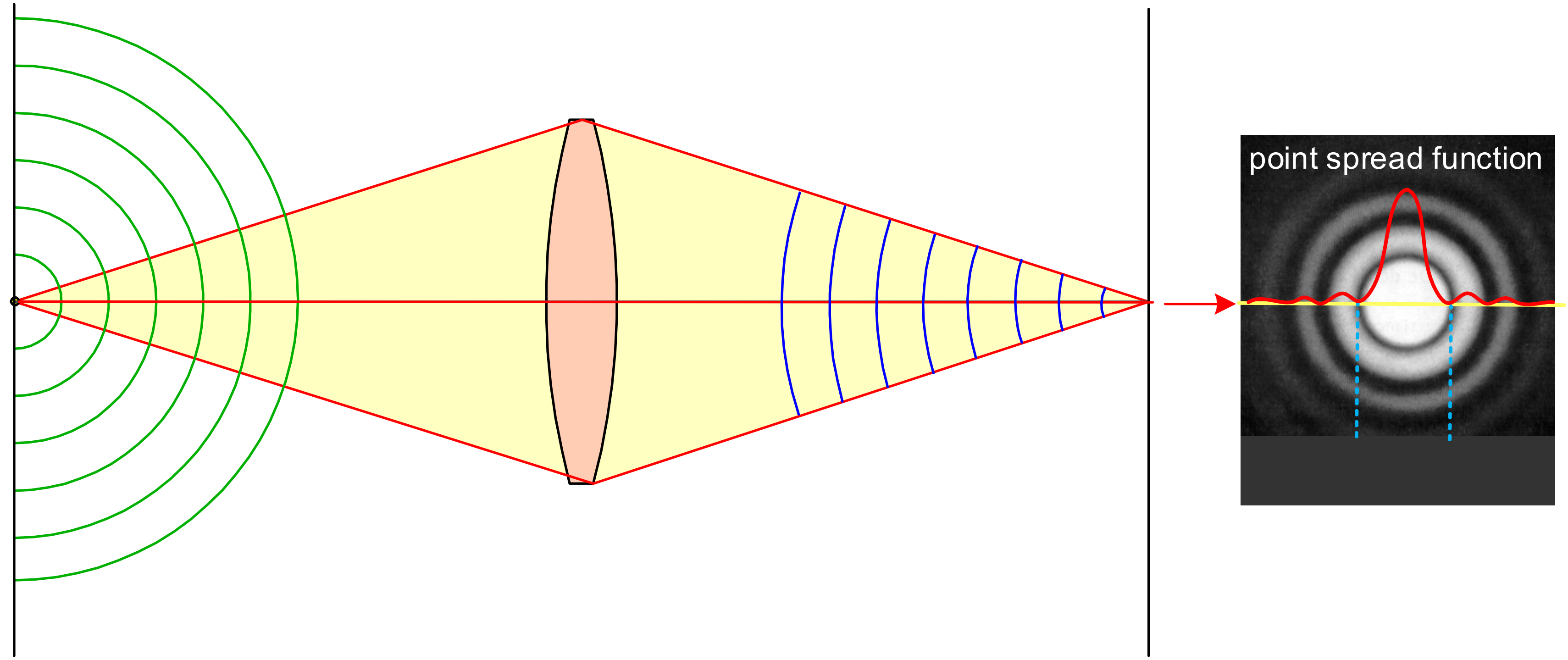
J_0, J_1 Bessel function of 0^{th} and 1^{st} order





Airy radius = first zero of Besinc function

Point spread function (PSF)



Point spread function (PSF)

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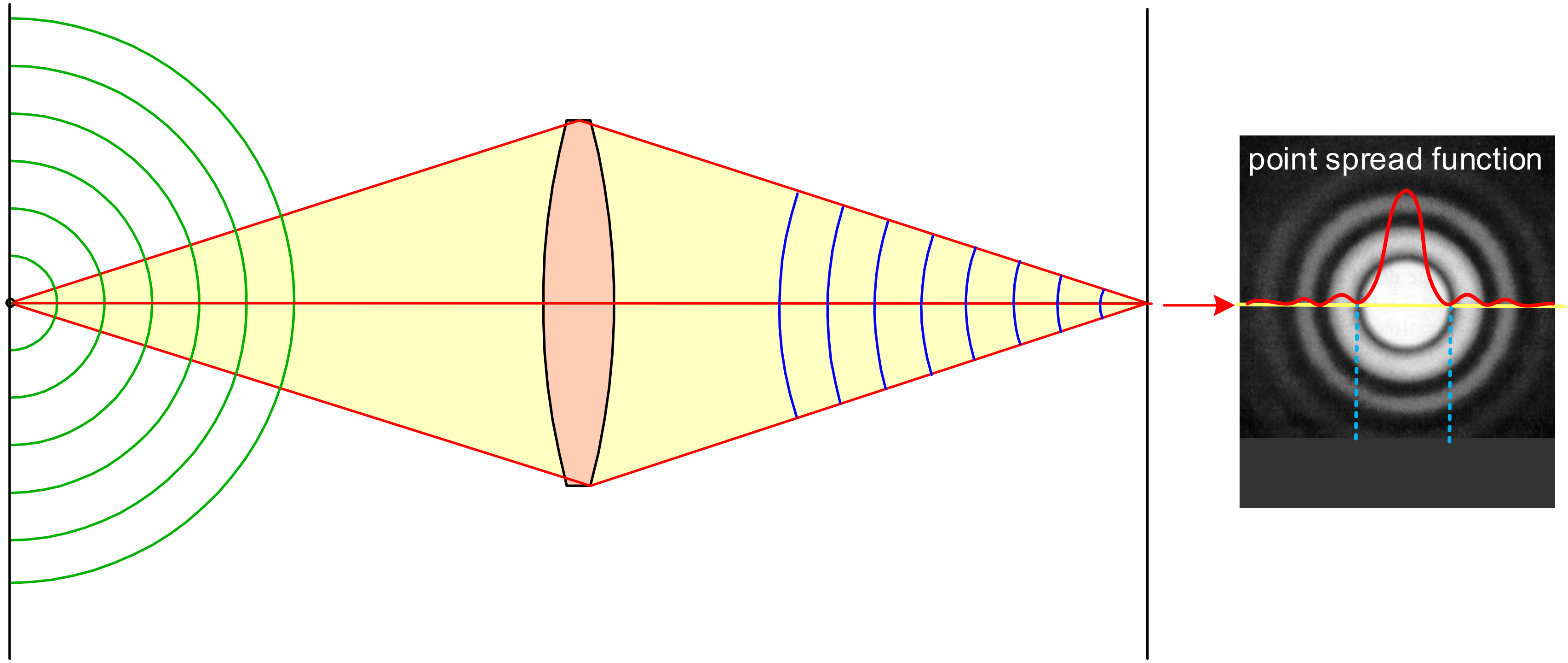
- Self luminous points: emission of spherical waves
- Optical system: only a limited solid angle is propagated, the truncation of the spherical wave results in a finite angle light cone
- In the image space: uncomplete constructive interference of partial waves, the image point is spread
- The optical systems works as a low pass filter



Interpretation of Point Spread Function

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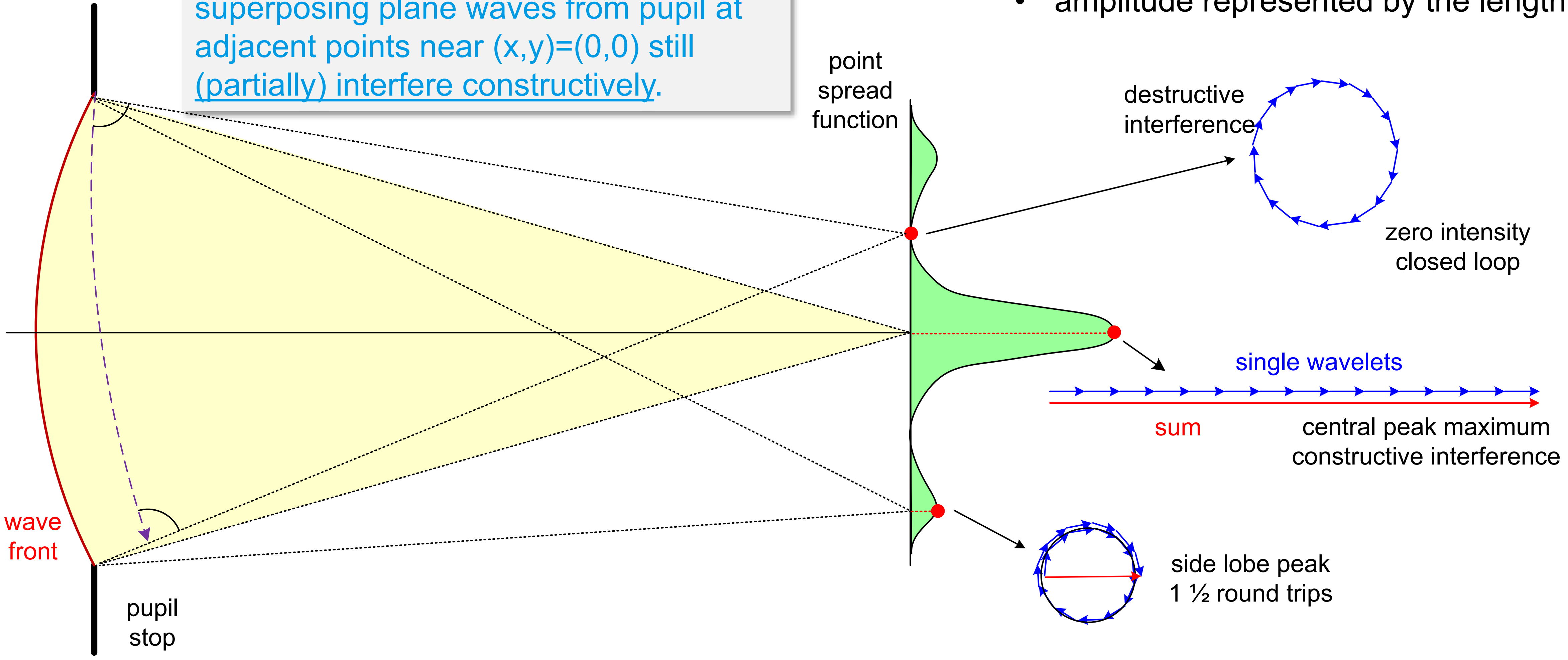
Ideal PSF: Finite size of intensity spot (not as geometrical spot), as superposing plane waves from pupil at adjacent points near (x,y)=(0,0) still <u>(partially) interfere constructively</u>.

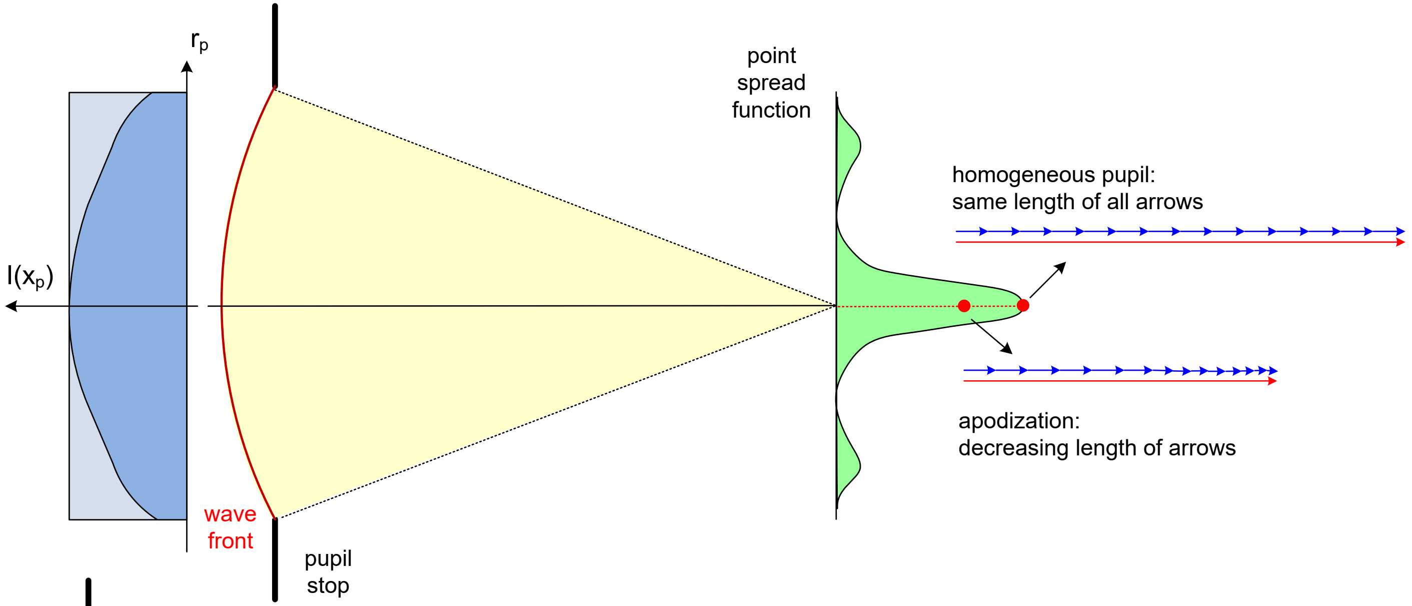
$A\ exp(i\varphi)$ diagram

- phase represented by the direction
- amplitude represented by the length

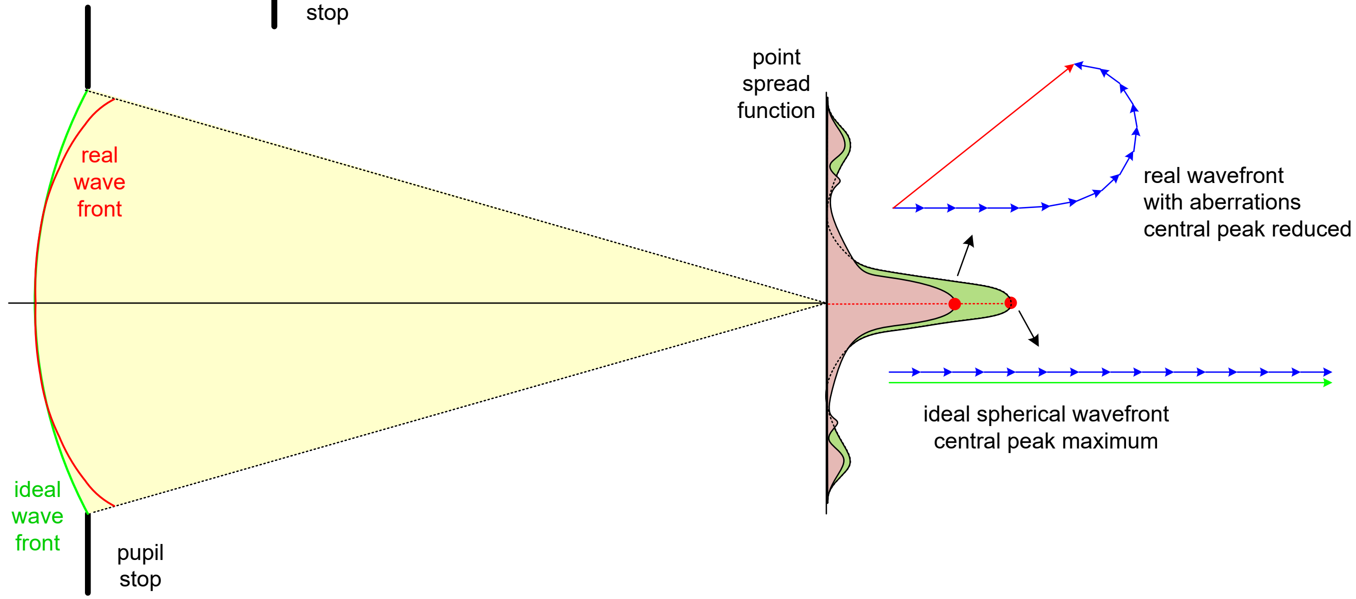


Interpretation of Point Spread Function: Apodization and Aberration

Transmission variation (Apodization):
variable lengths
of arrows



Wave front deformation:
variable orientation
of arrows



Ideal amplitude and intensity PSFs of ideal pupil

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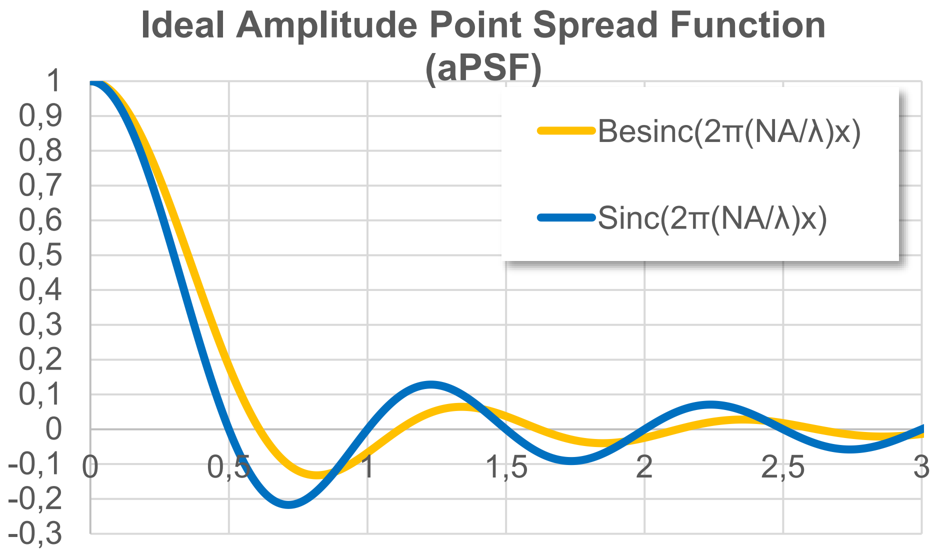
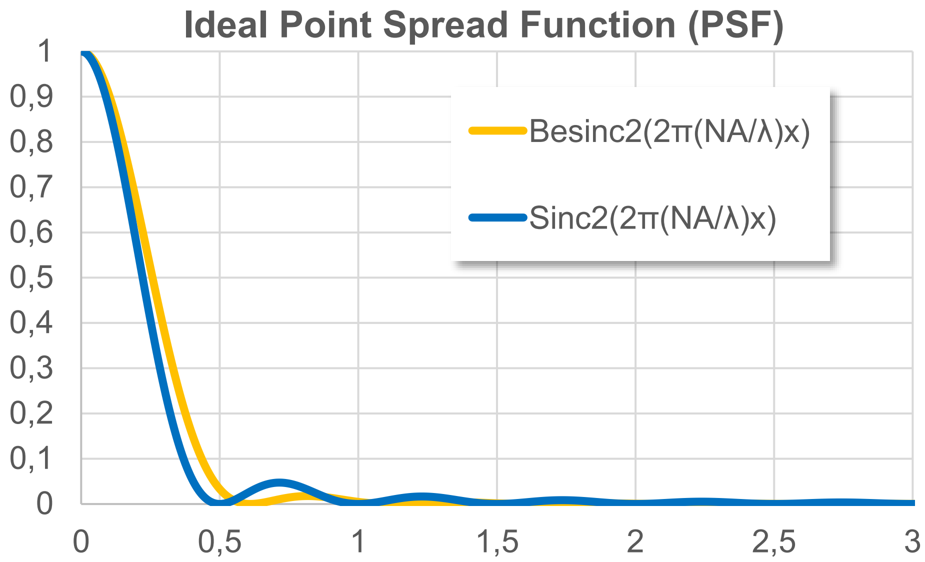


Example data:

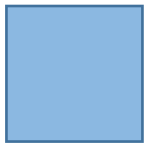
$$\frac{NA}{\lambda} = 0.1\mu m^{-1}$$

axis: $x[\mu m]$



$$aPSF(x) = Sinc\left(2\pi\frac{NA}{\lambda}x\right)$$

$$PSF(x) = |aPSF(x)|^2 = Sinc^2\left(2\pi\frac{NA}{\lambda}x\right)$$



$$aPSF(x) = Besinc\left(2\pi\frac{NA}{\lambda}x\right)$$

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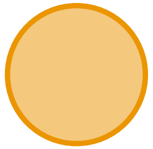
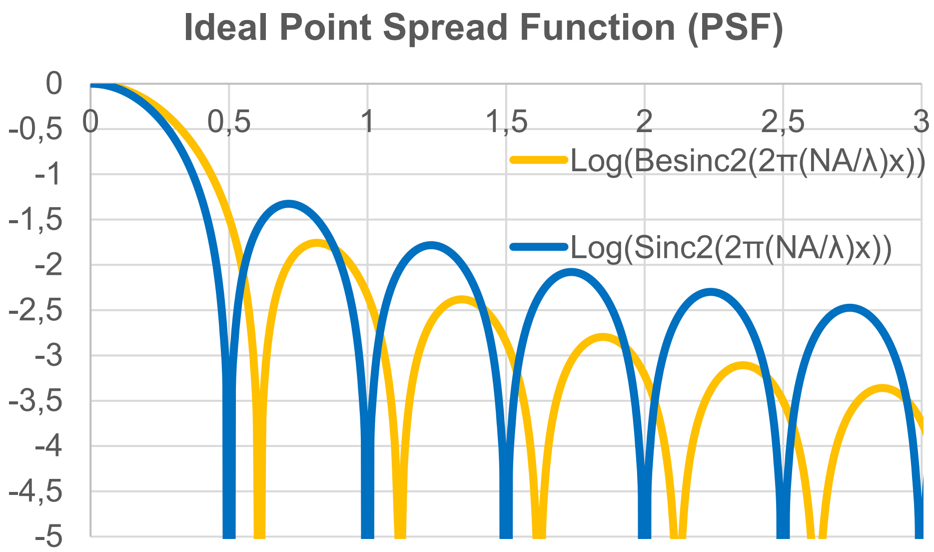
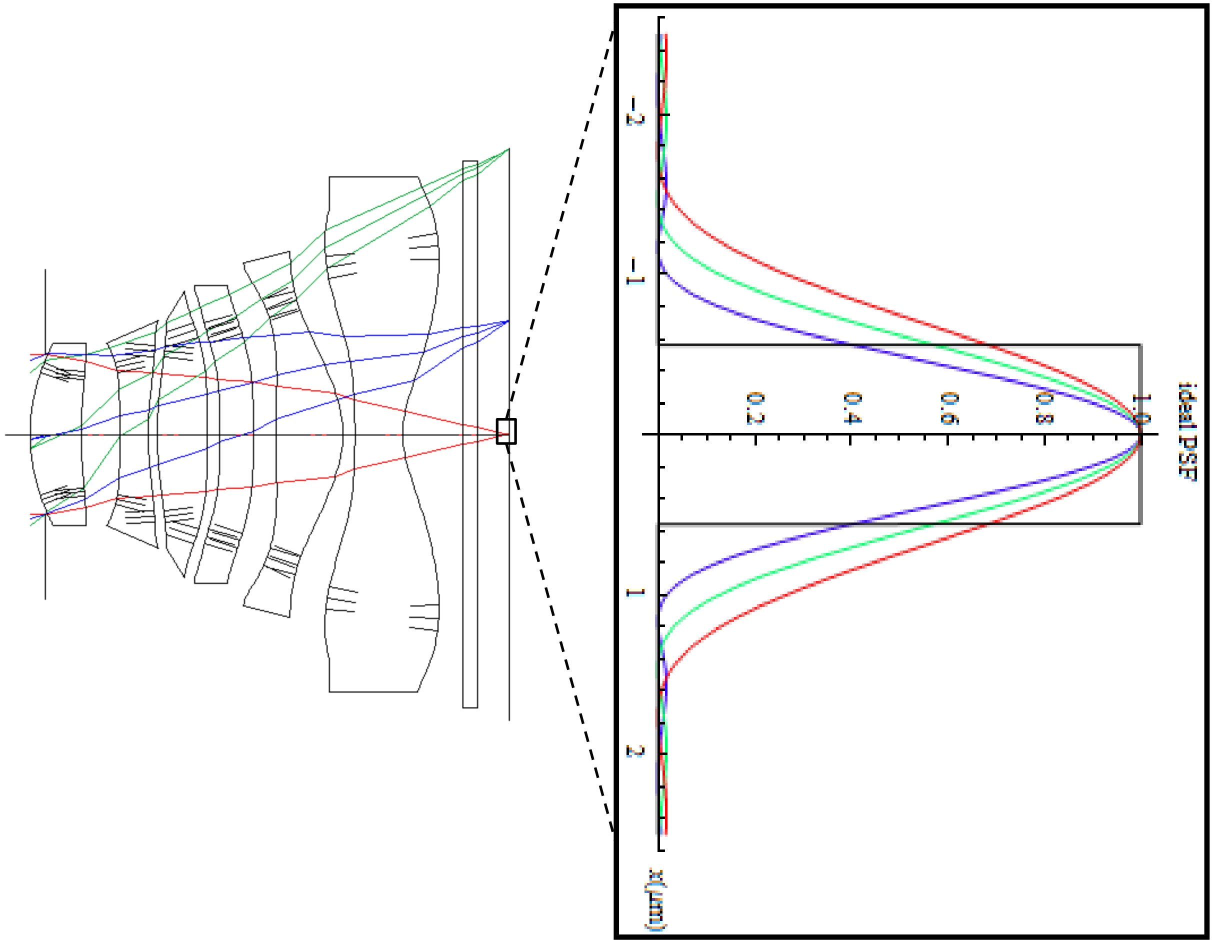


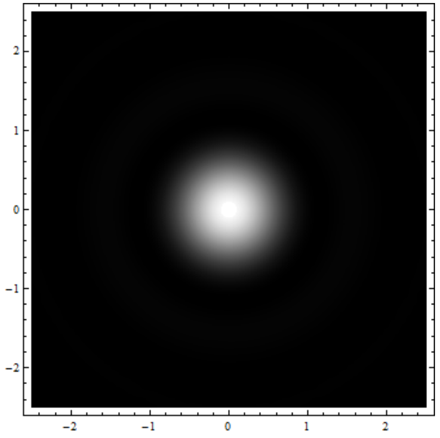


Ideal point spread function (PSF)

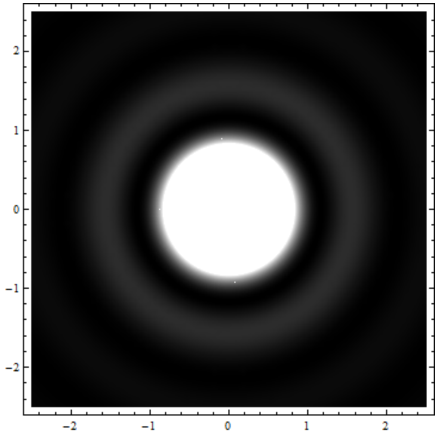
object point far away



Airy-spot = PSF of ideal lens



range
0 - 1 (max)



range
0 - 0.1

„over-exposed“

Ideal PSF (point spread function)

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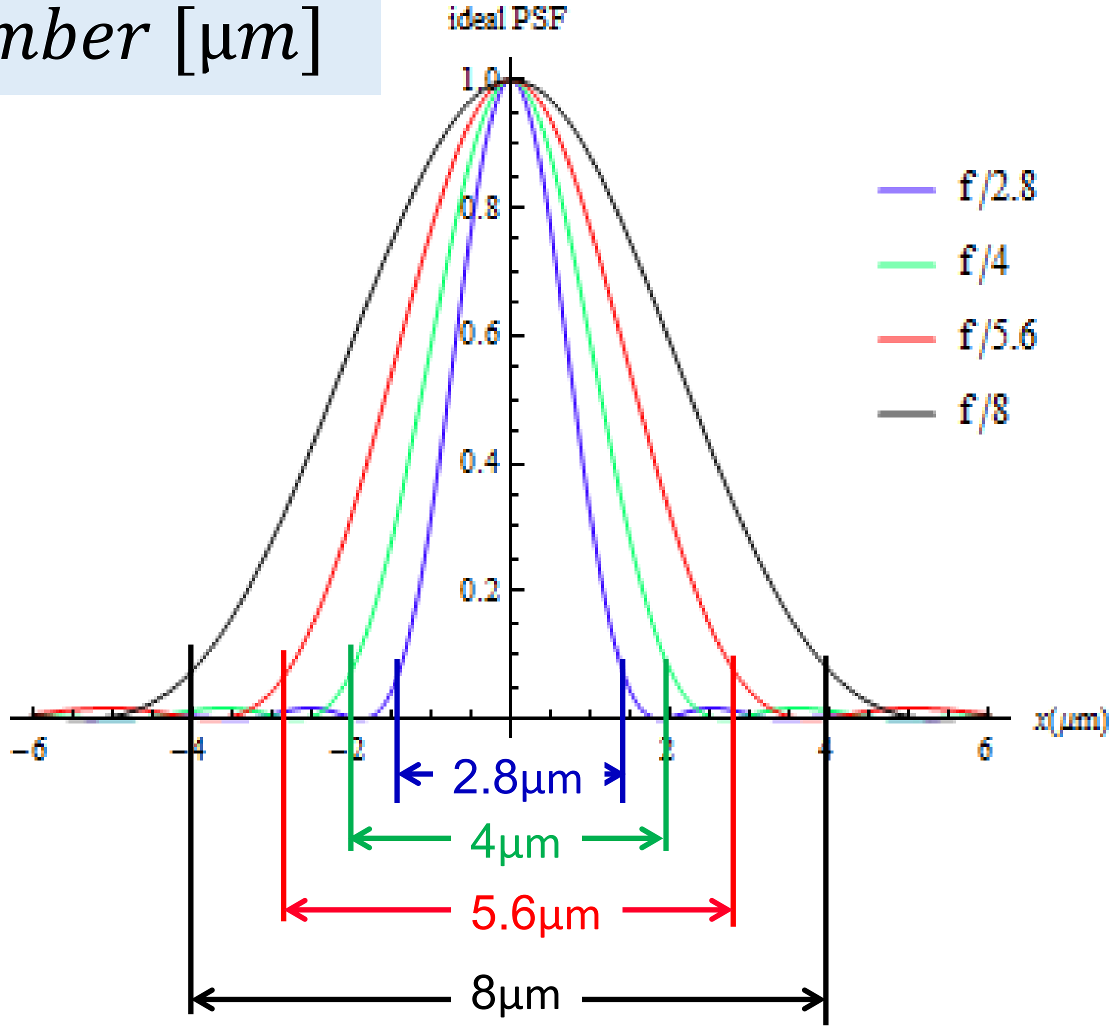
$$\varnothing_{Airy} \approx f - number\ [\mu m]$$



Rule of thumb:

„The diameter of the ideal PSF corresponds to the f-number in micrometers."

Effect of point spread function (PSF) on image details

Like a brushstroke in painting, the PSF determines the details rendered in the image.

$$I(x,y) = \iint I_{ob}(\xi,\eta) PSF(x-\xi, y-\eta)\, d\xi d\eta$$

$$= I_{ob} * PSF$$

convolution operation

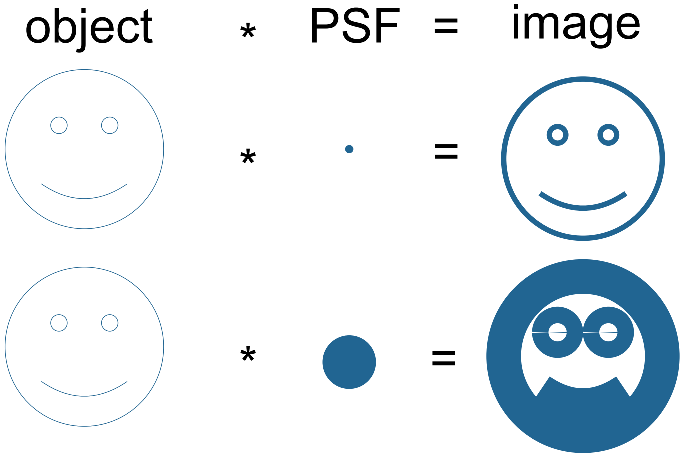


Image of an edge (edge spread function)

PSF connects object and image: $$I(x) = \int_{object} d\xi \; I_{ob}(\xi)\, PSF(x-\xi)$$ (convolution integral)

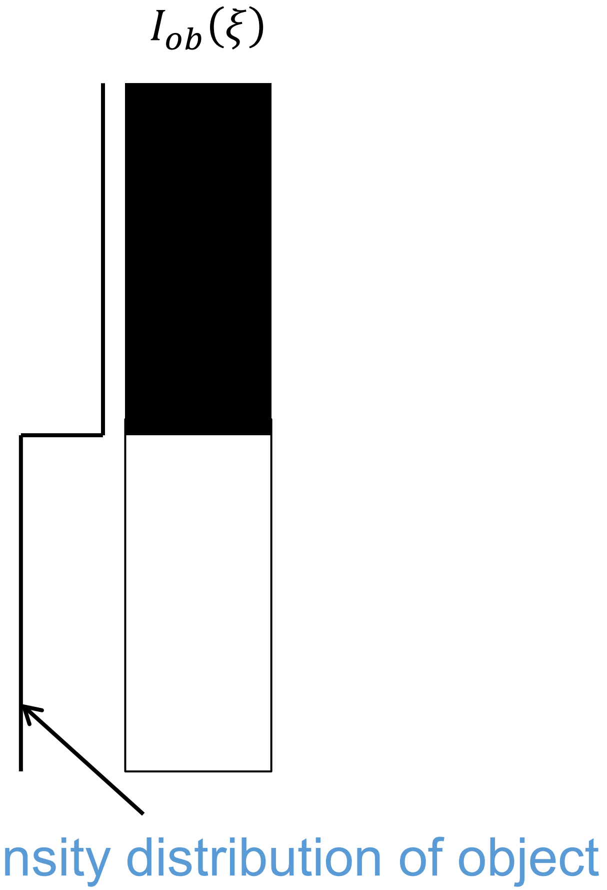


Image of an edge (edge spread function)

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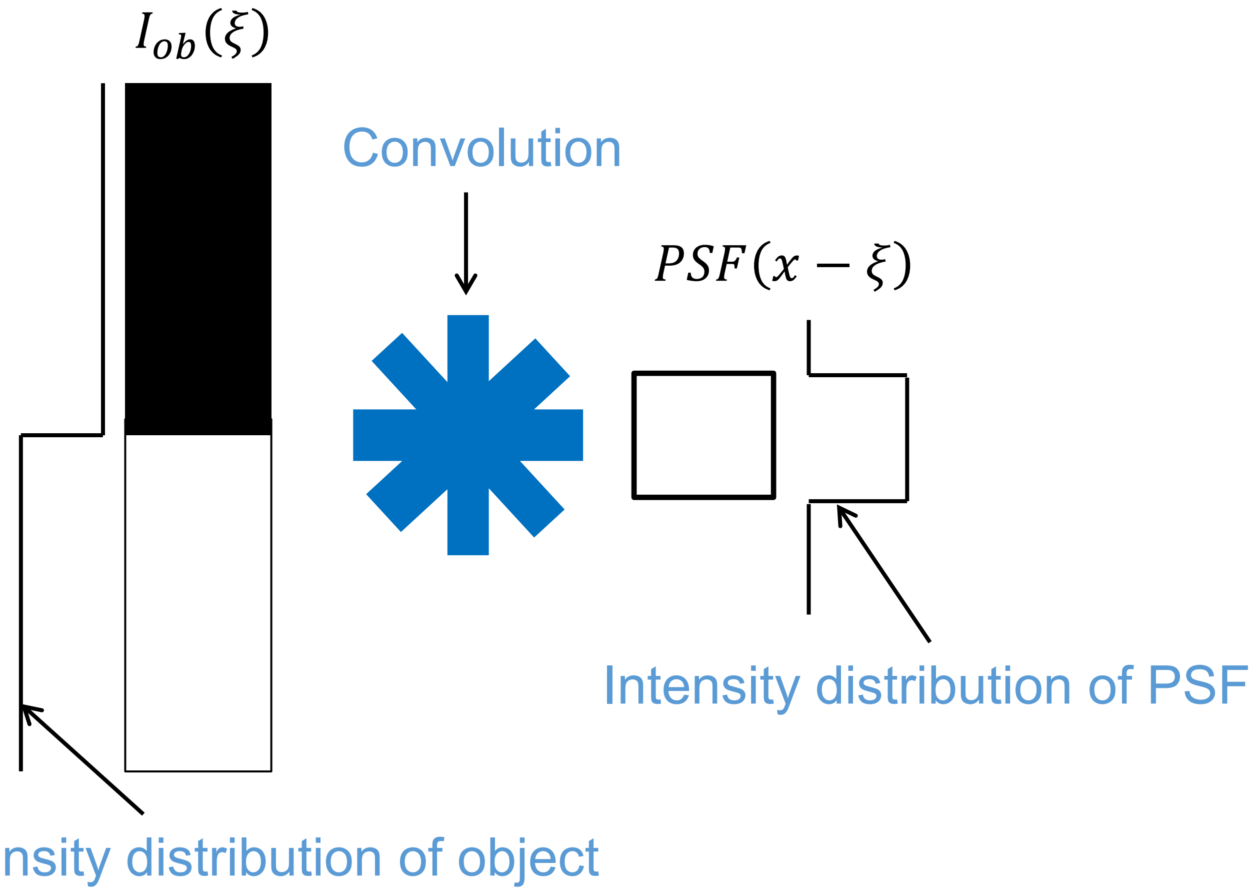

Image of an edge (edge spread function)

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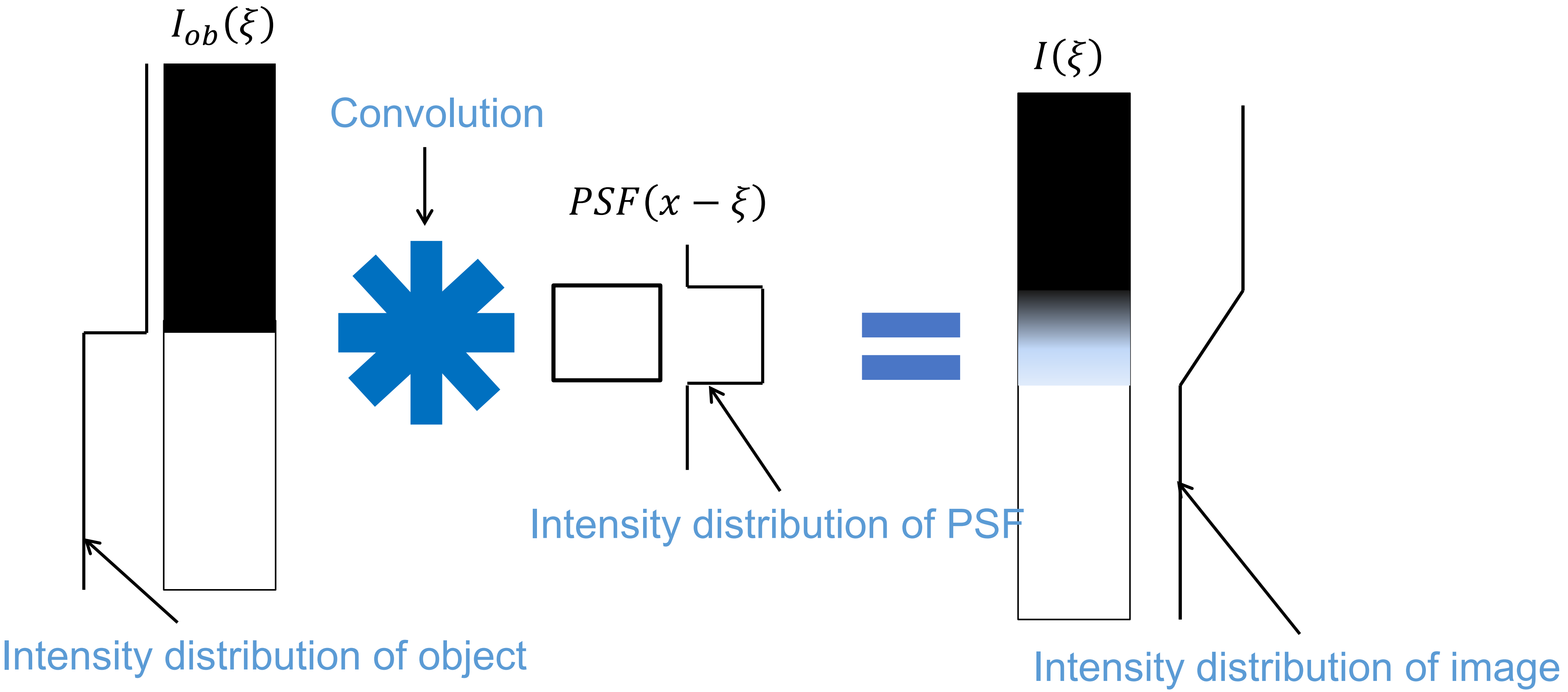


Image of an edge (edge spread function)

PSF connects object and image: $I(x) = \int_{object} d\xi \; I_{ob}(\xi)\, PSF(x-\xi)$ (convolution integral)

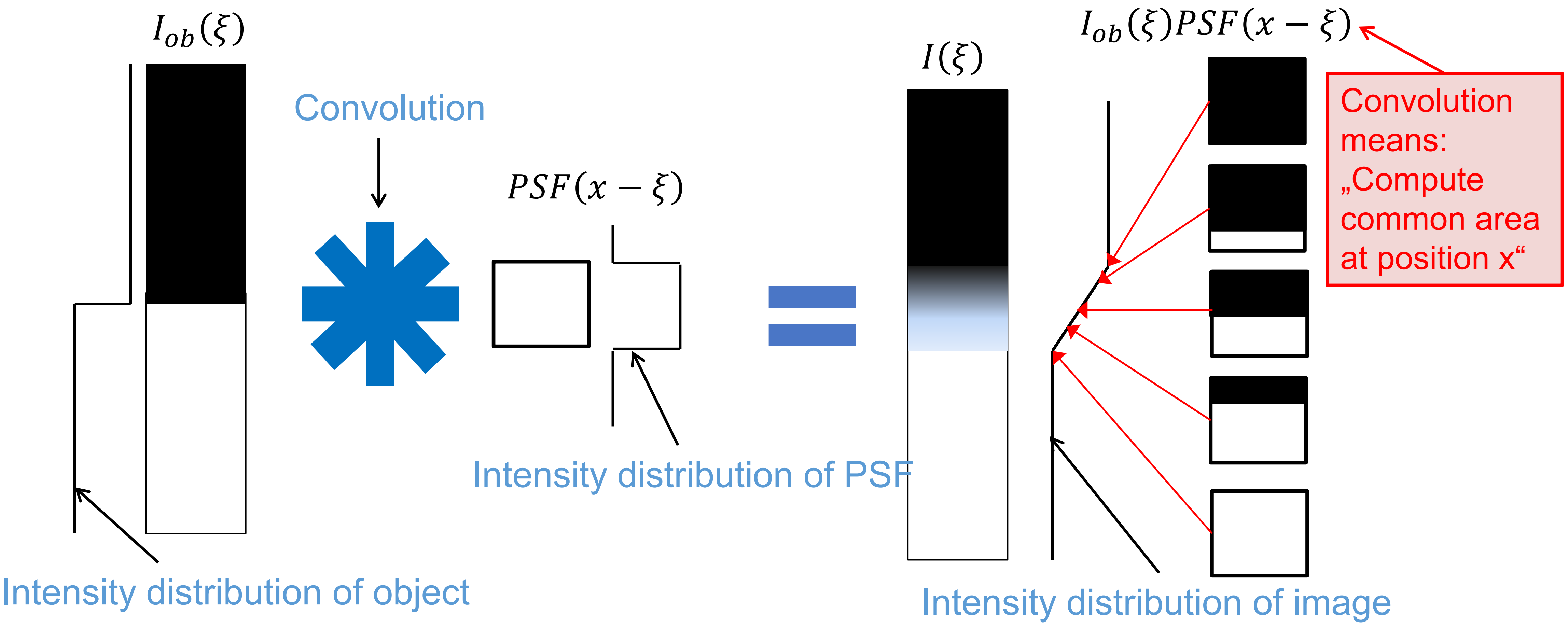
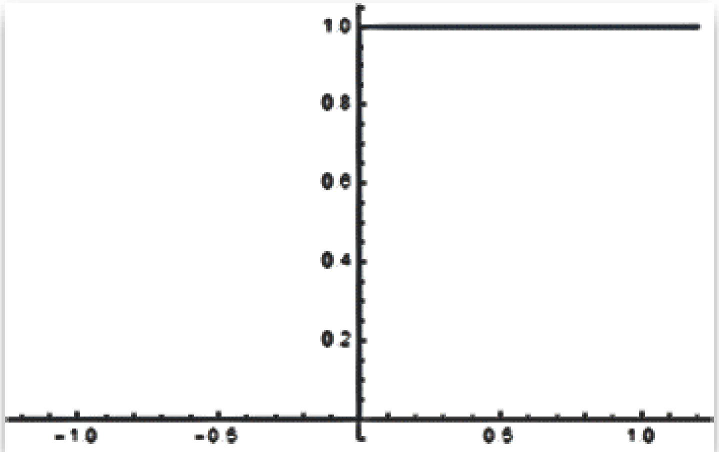


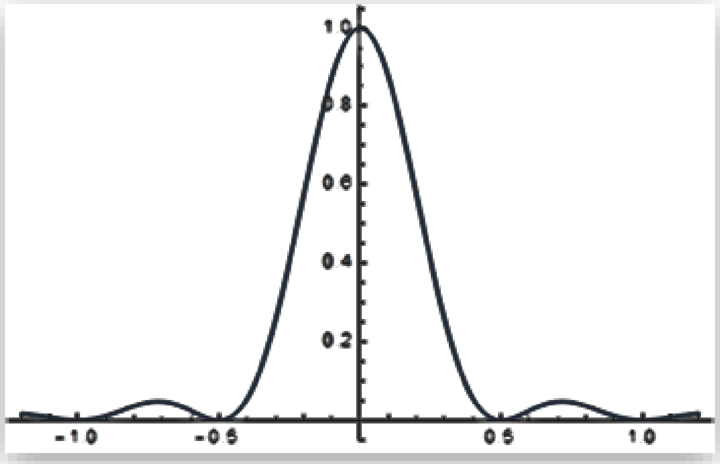
Image of an edge (edge spread function) with ideal PSF (1D)

object | convolution | PSF | = | image

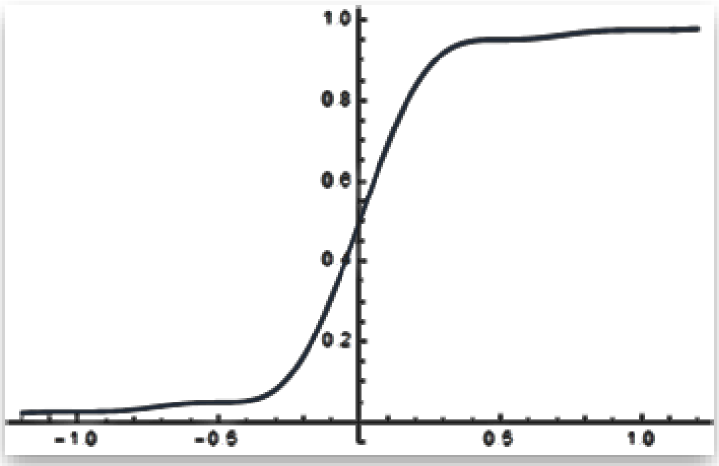
$I_{ob}(\xi)$ | $PSF(x-\xi)$ | $I(x)$



$*$



$=$

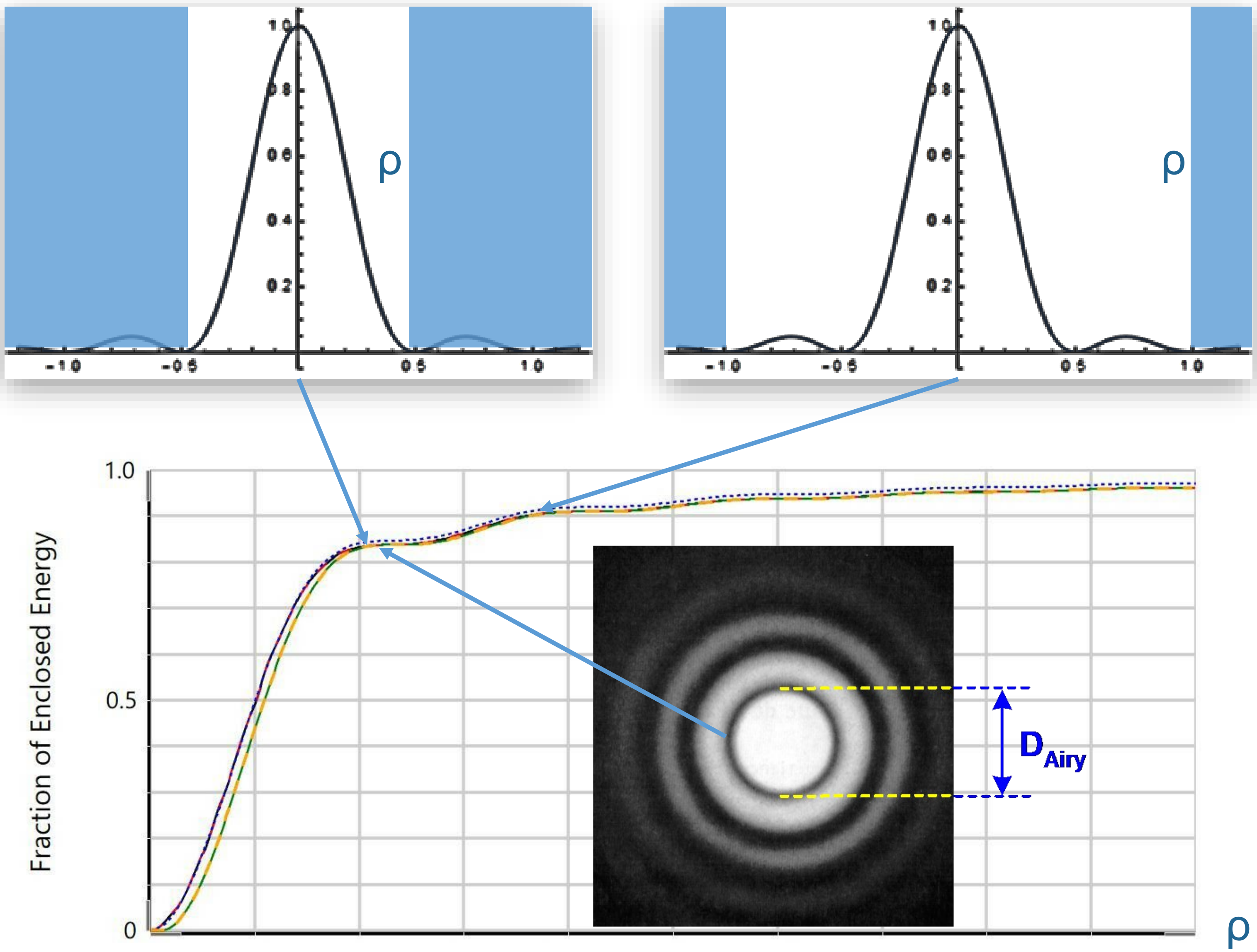


$$I_{ob}(\xi) = \begin{cases} 1, & x \geq 0; \\ 0, & x < 0. \end{cases}$$

$$PSF(x-\xi) = \left[\frac{\sin\left(2\pi w(x-\xi)\right)}{2\pi w(x-\xi)}\right]^2$$

$$I(x) = \frac{1}{2} + \frac{1}{\pi}\left[Si(4\pi w x) + \frac{\cos(4\pi w x) - 1}{4\pi w x}\right]$$

Encircled energy



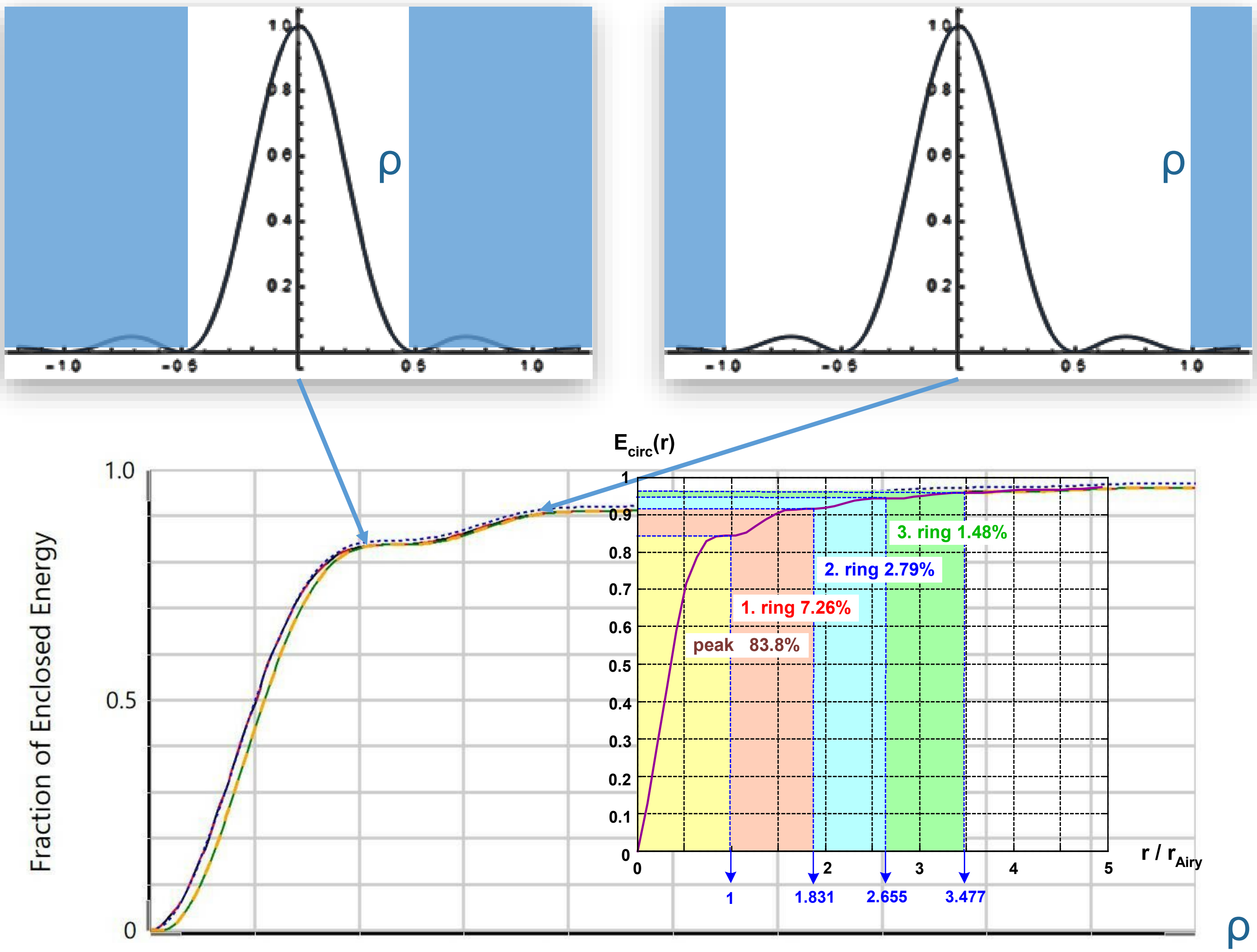
Integral analysis of PSF:

$$EE\{PSF\}(\rho) = \int_0^{\rho} PSF(\rho',\varphi)\,\rho'\mathrm{d}\rho'$$

Integral criterium: „How much energy (irradiance) falls in area with radius ρ?“

Alternative criteria, e.g., „energy in box“ = integration over quadratic area

Encircled energy



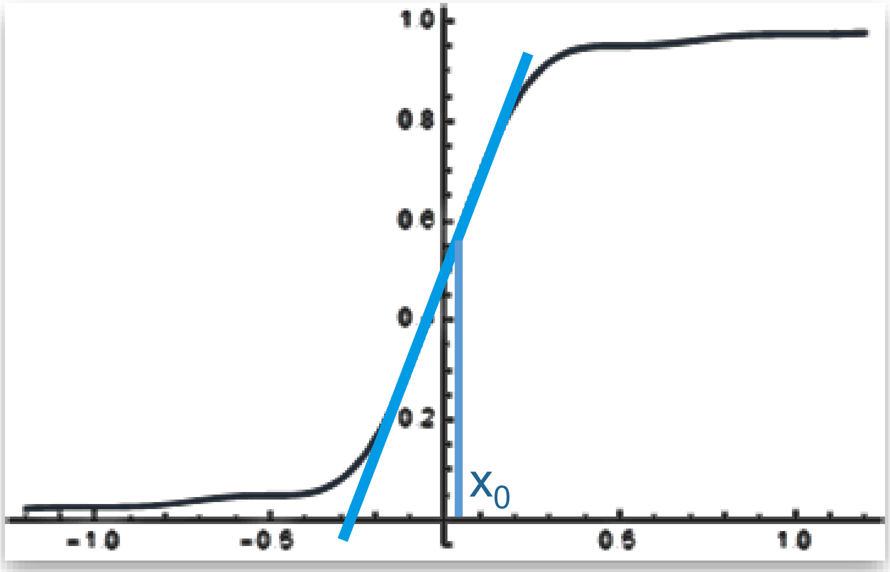
Integral analysis of PSF:

$$EE\{PSF\}(\rho) = \int_0^{\rho} PSF(\rho', \varphi)\, \rho' \mathrm{d}\rho'$$

Integral criterium: „How much energy (irradiance) falls in area with radius ρ?“

Alternative criteria, e.g., „energy in box“ = integration over quadratic area

Image slope or normalized image log-slope



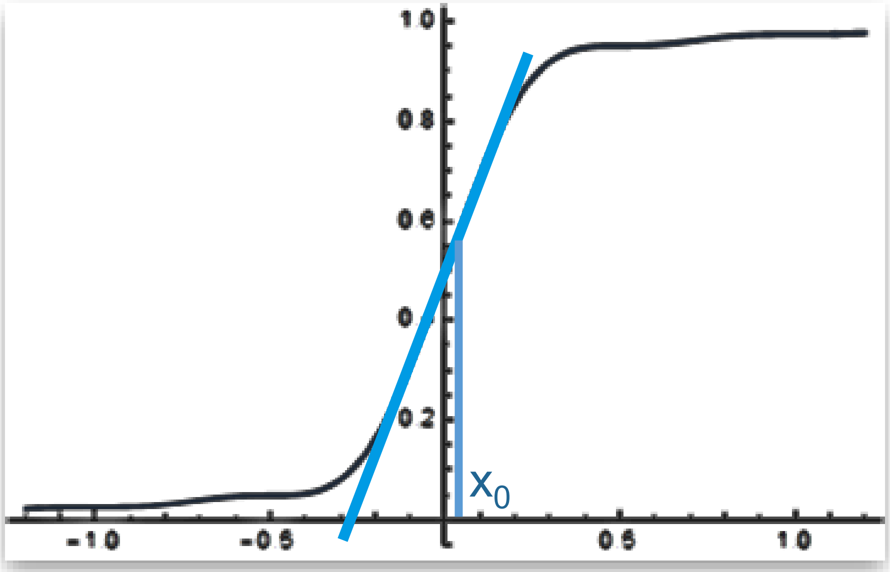
Normalized image log-slope (NILS)

$$NILS = w\,\frac{1}{I}\frac{\partial I}{\partial x}\bigg|_{x=x_0} = w\,\frac{\partial ln(I)}{\partial x}\bigg|_{x=x_0}$$

Slope of aerial image.
Relative logarithmic scale for image formation in resist.

Optical lithography or metrology applications: measure of sensitivity for line width deviation.

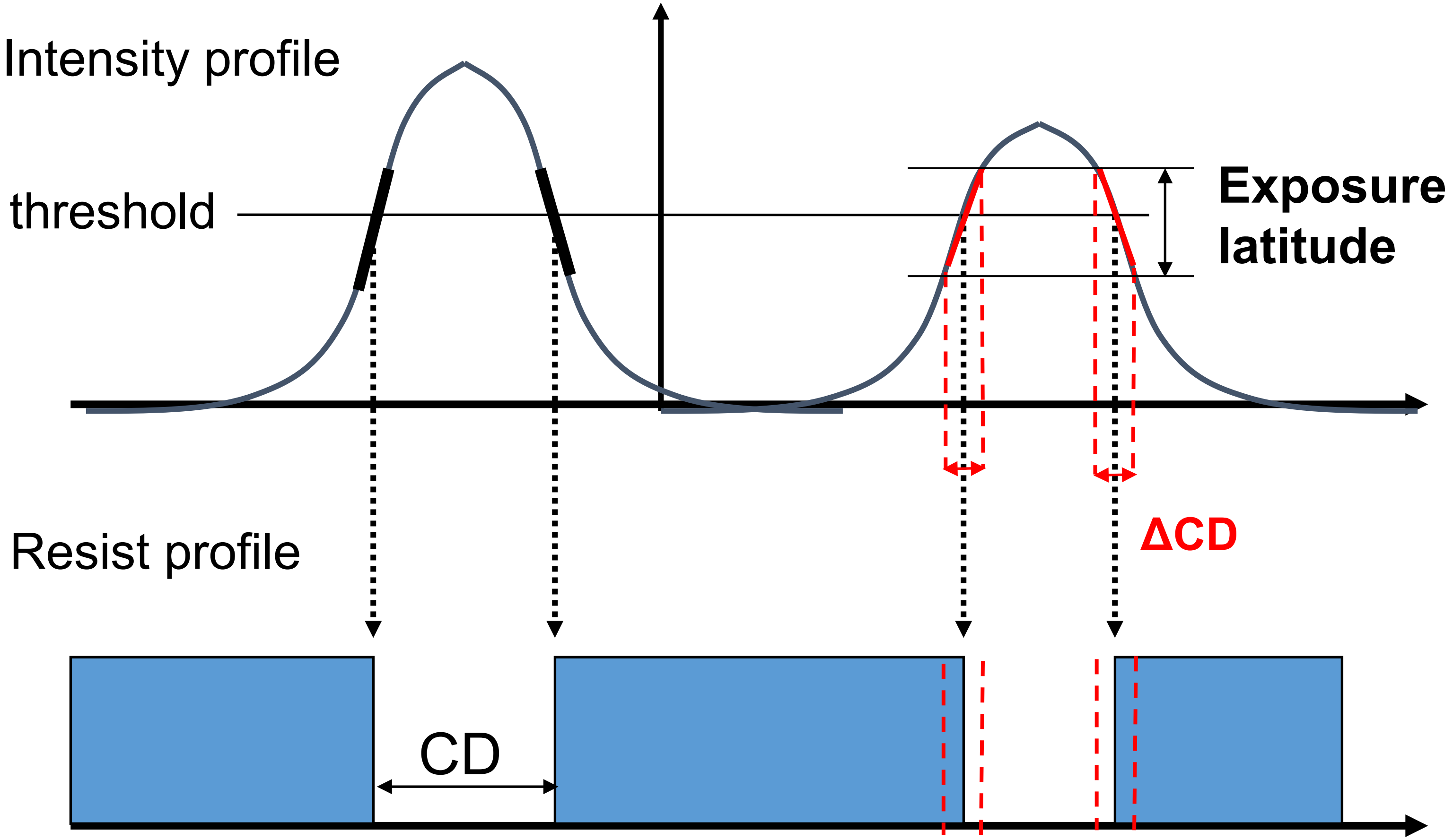
Image slope or normalized image log-slope



Normalized image log-slope (NILS)

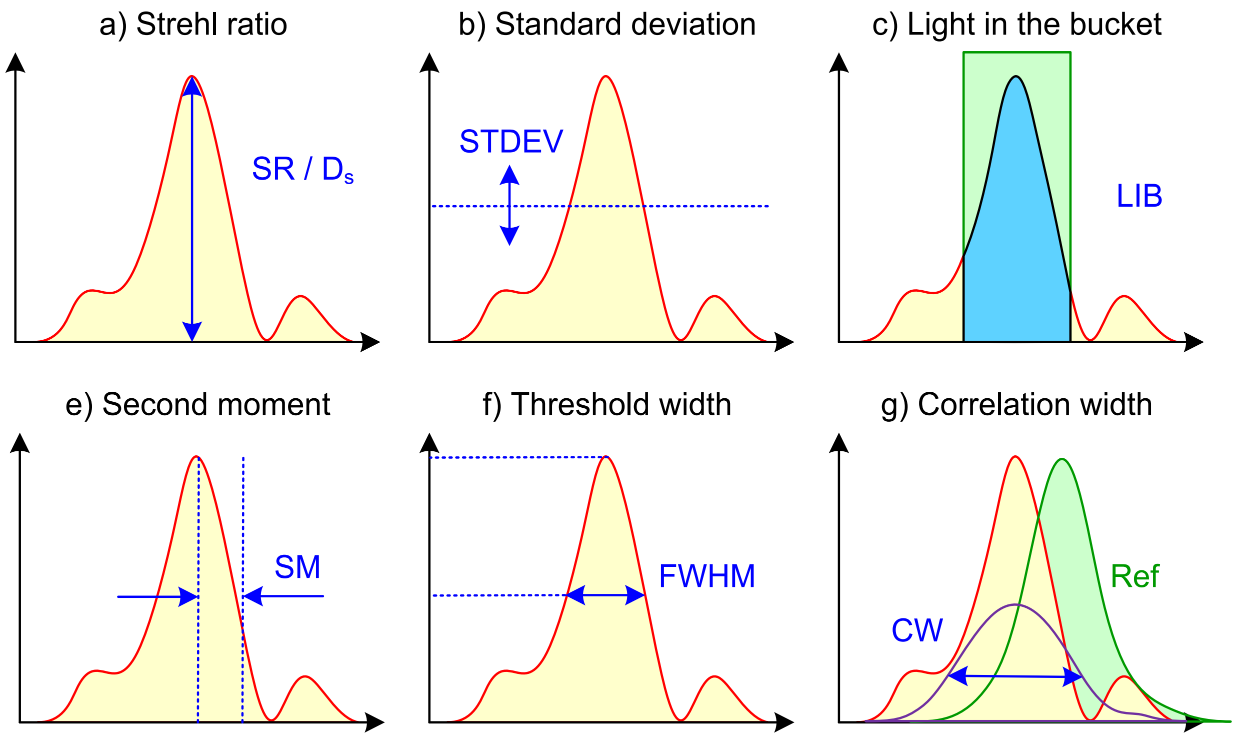
$$NILS = w\,\frac{1}{I}\frac{\partial I}{\partial x}\bigg|_{x=x_0} = w\,\frac{\partial ln(I)}{\partial x}\bigg|_{x=x_0}$$

Slope of aerial image.
Relative logarithmic scale for image formation in resist.



Exposure latitude is another parameter to determine linewidths (or „CD“ = critical dimension).
A large slope of the intensity distribution near the threshold helps to reduce undesired variations of linewidth.

Quality Criteria for Point Spread Function



Criteria for measuring the degradation of the point spread function:

1. Strehl ratio
2. width/threshold diameter
3. second moment of PSF
4. area equivalent width
5. correlation with perfect PSF
6. power in the bucket

Usefulness of Criterium needs to be carefully considered **individually for the particular application**!
E.g.: threshold criterium unsuited for high dynamic range evaluation!
Strehl ratio just a measure relative to diffraction limit, but not relative to a target size (e.g. pixel pitch)!
…

Examples of typical PSF´s of different camera lenses in best focus

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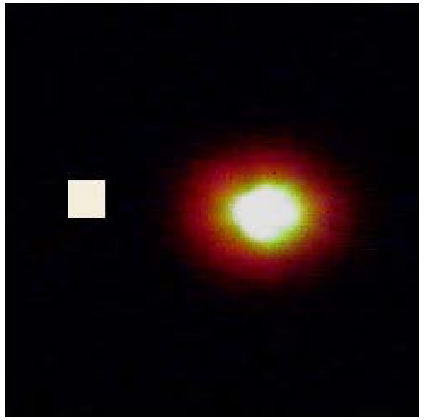
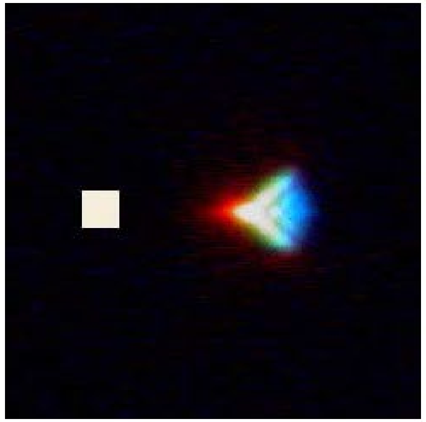


1



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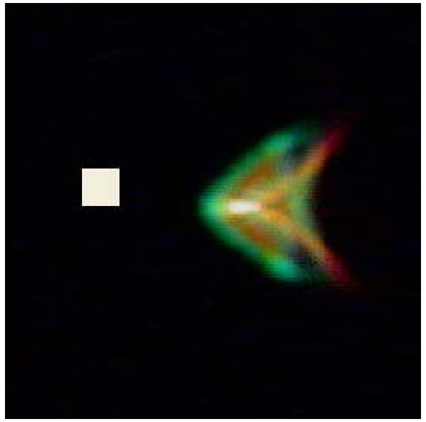

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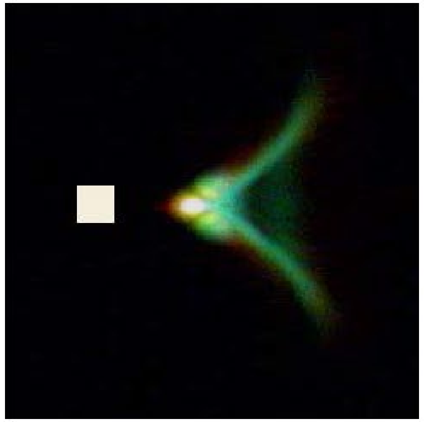

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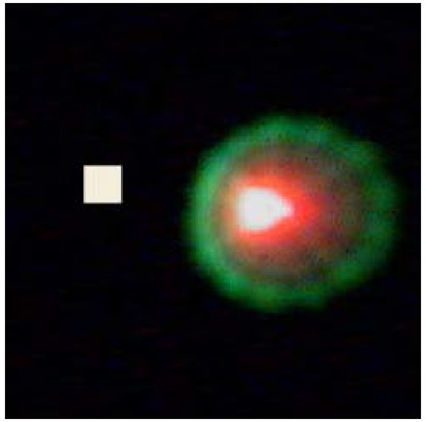
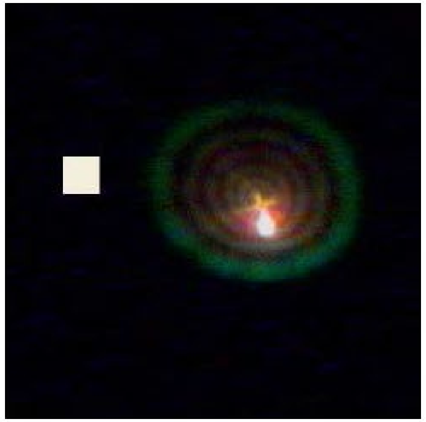
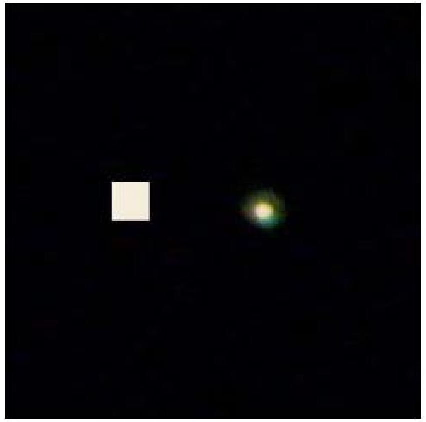


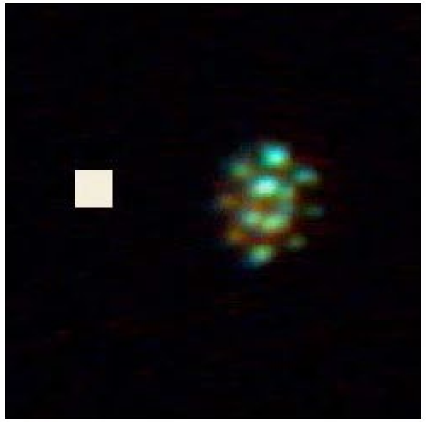
5



6

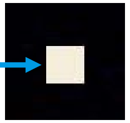


7



8

„sensor pixel“



1 – 6: Typical high-quality camera lenses PSFs in comparison to the size of a pixel (8.5µm) of a 12MP camera for 36x24mm^2-format

7 extremely good lens

8 same lens as on 7, but camera has a anti-aliasing filter

Star test

Foto: Dr. Benjamin Voelker

Star test

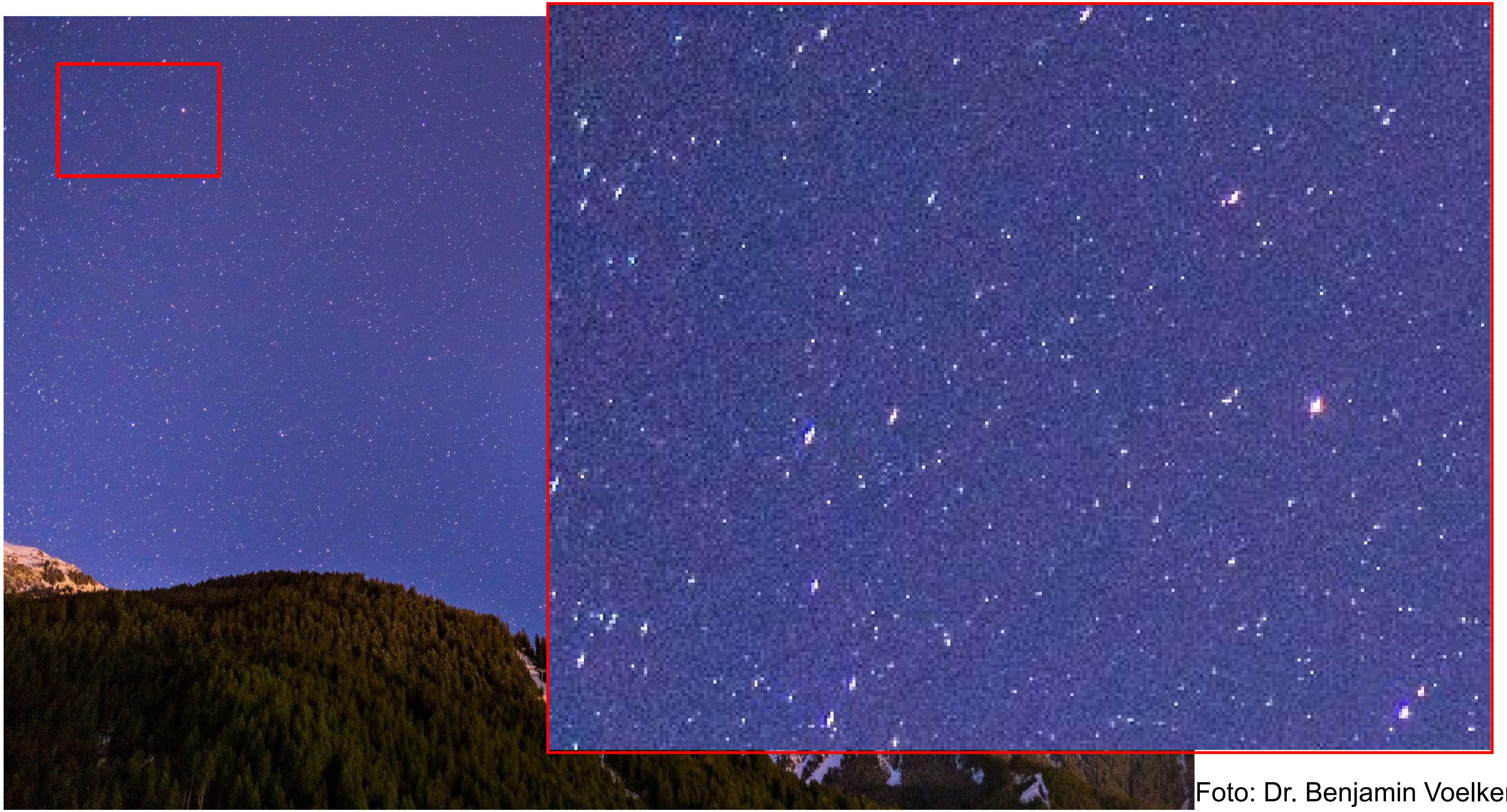


Foto: Dr. Benjamin Voelker

Star test

Foto: Dr. Benjamin Voelker

Star test

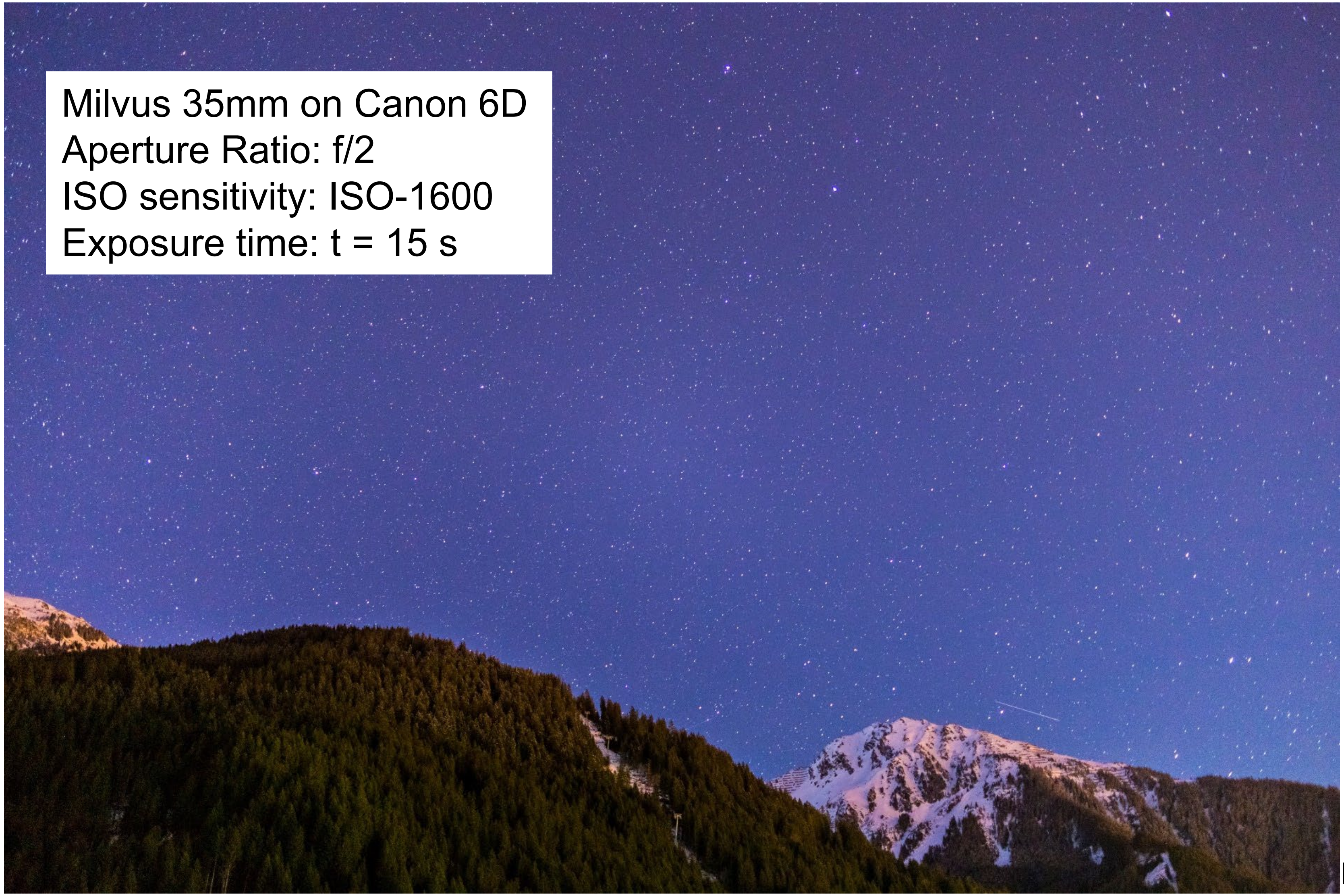
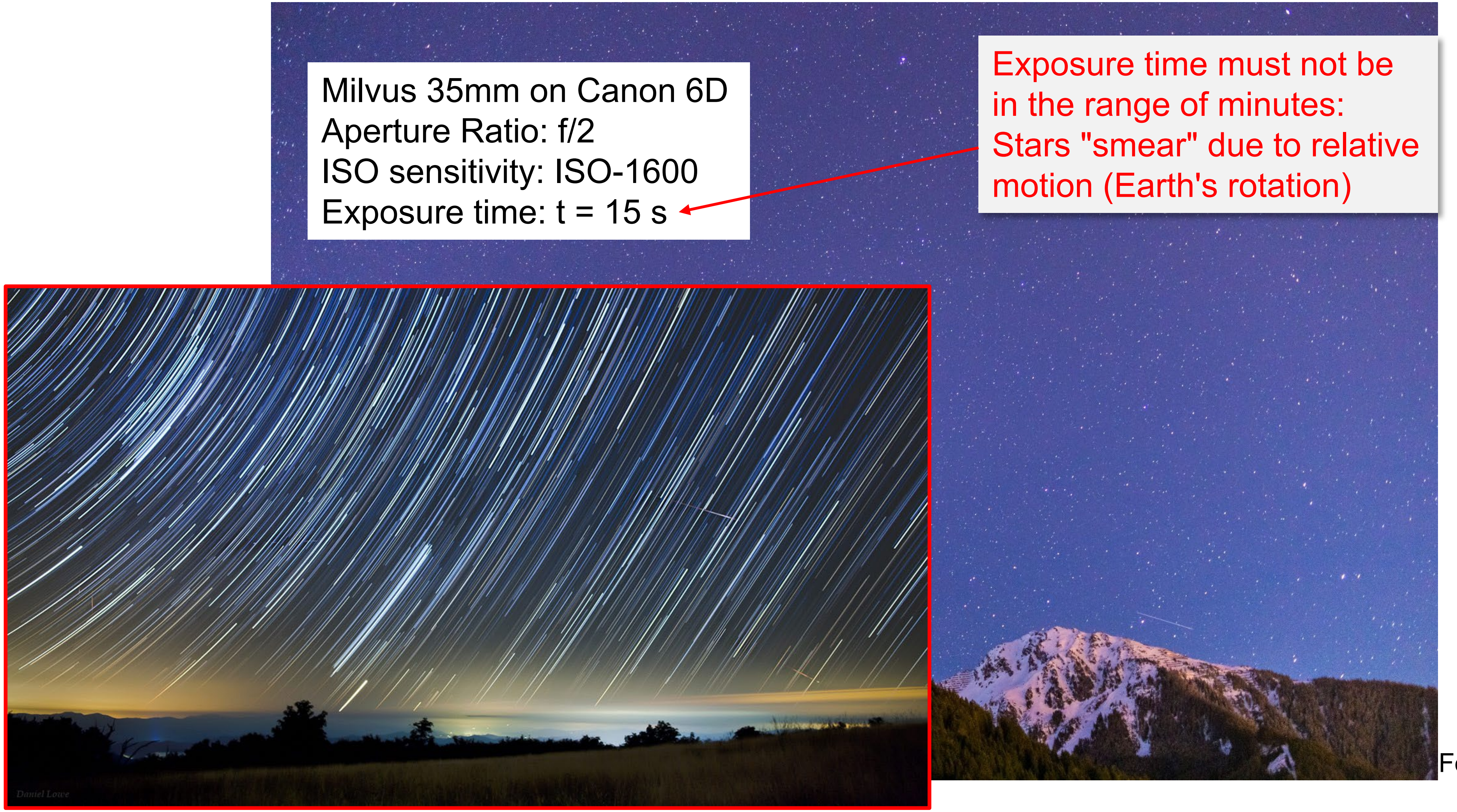


Foto: Dr. Benjamin Voelker

Star test

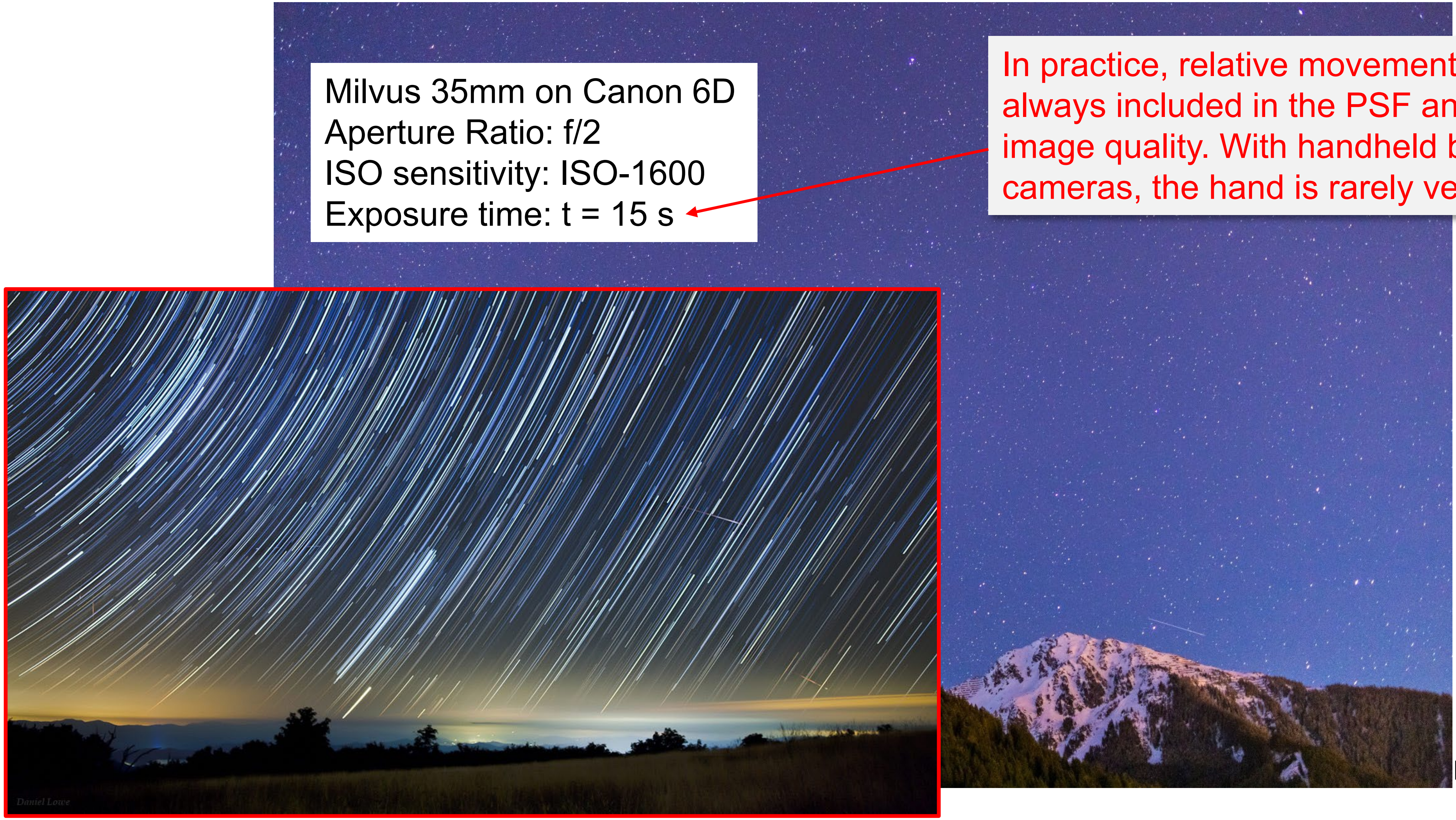


Milvus 35mm on Canon 6D
Aperture Ratio: f/2
ISO sensitivity: ISO-1600
Exposure time: t = 15 s

Exposure time must not be in the range of minutes: Stars "smear" due to relative motion (Earth's rotation)

Foto: Dr. Benjamin Voelker

Star test



Milvus 35mm on Canon 6D
Aperture Ratio: f/2
ISO sensitivity: ISO-1600
Exposure time: t = 15 s

In practice, relative movements are almost always included in the PSF and reduce the image quality. With handheld binoculars and cameras, the hand is rarely very steady...

Foto: Dr. Benjamin Voelker

Star test shows:

- Image of "almost point-shaped" light sources (→ "PSF"´s) of different light intensity over the entire field of view
- PSF generally varies across the field of view; usually the quality of the optics in the center of the image is much better than in the corners
- Image defects are more visible in bright light sources (because the intensity outside the nominal image location is usually very small and only becomes visible in stronger exposure)

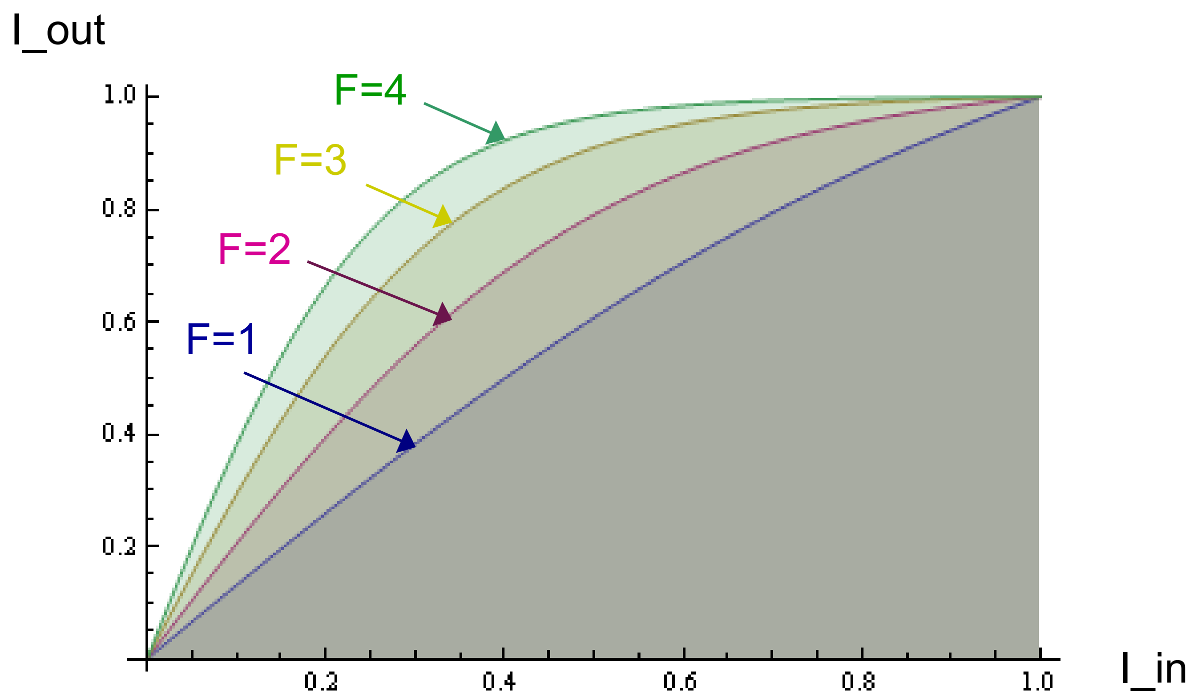
Foto: Dr. Benjamin Voelker

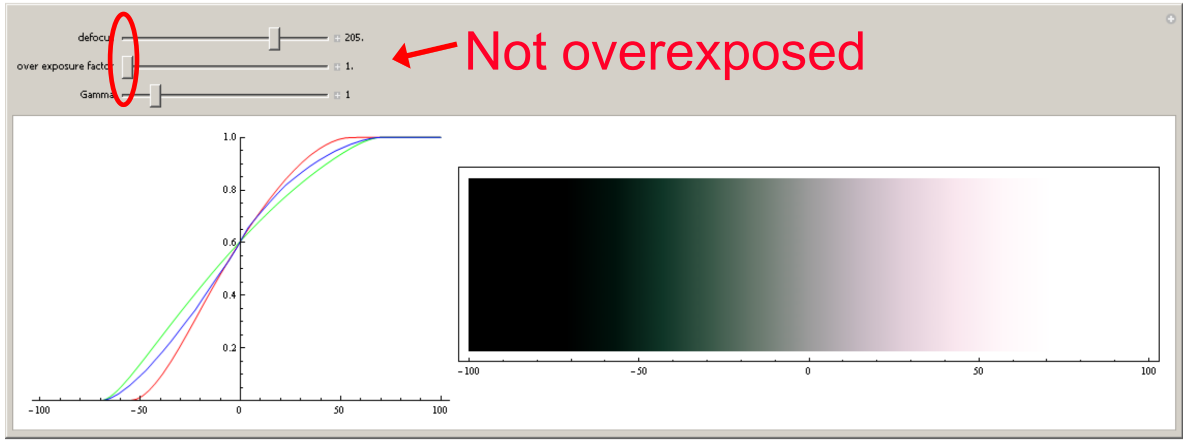
Overexposure

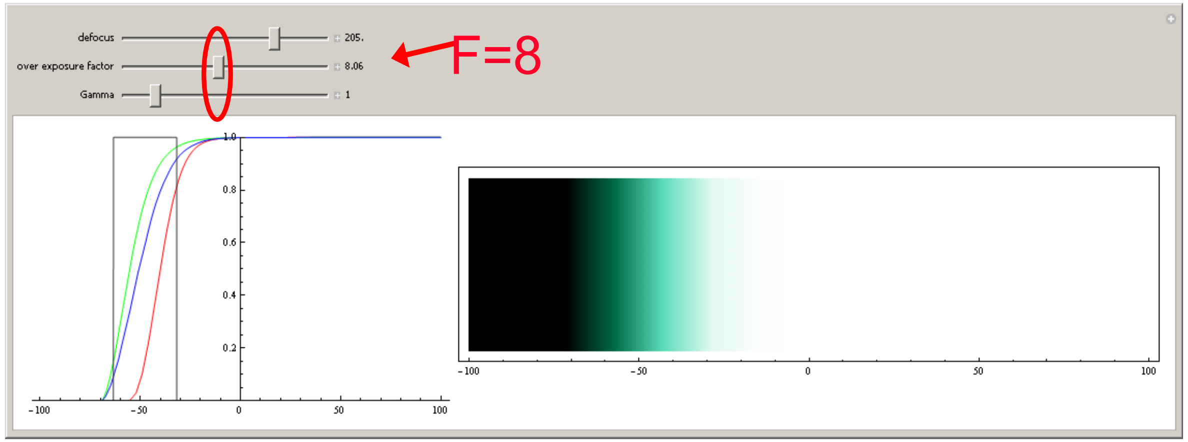
$$I \rightarrow \frac{\tanh\left(F \cdot \frac{I}{I_{max}}\right)}{\tanh(F)} \cdot I_{max}$$

Simplified model of (over)exposure is determined by the opto-electronic conversion function (OECF).

I_{max} = maximum intensity level, e. g. for 8bit = 255
I = actual intensity value
F = over exposure factor

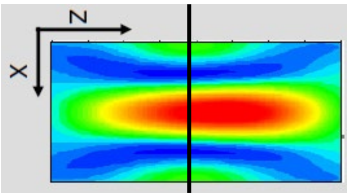


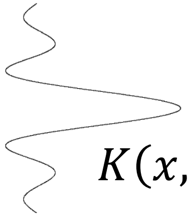




Amplitude point spread function, aPSF), which is related to the pupil function L of the system as follows:

$$K(x,\xi,z) = \iint d\alpha\, L(\alpha,\xi,z) exp\big(-i2\pi w\,\alpha \cdot (x/m - \xi)\big)$$





$K(x,\xi_0,z_0)$

Point Spread Function (PSF)

Complex function

$$PSF(x,\xi,z) = \left| \iint d\alpha L_0(\alpha,\xi) exp\big(i2\pi W(\alpha,\xi)\big) exp\left(-i\frac{2\pi}{\lambda} z\sqrt{1-(NA/n)^2\alpha^2}\right) exp\big(-i2\pi w\,\alpha \cdot (x/m - \xi)\big)\right|^2$$

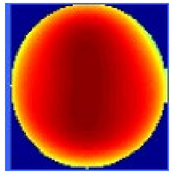
Real function

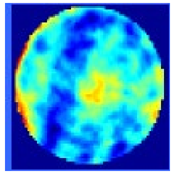
Intensity of „point object“ („star“)

The pupil function L describes the limitation of the aperture(s) and the transmission distribution in the pupil, contained in the function $L_0(\alpha,\xi)$, as well as the aberrations $W(\alpha,\xi)$ of the optical system:

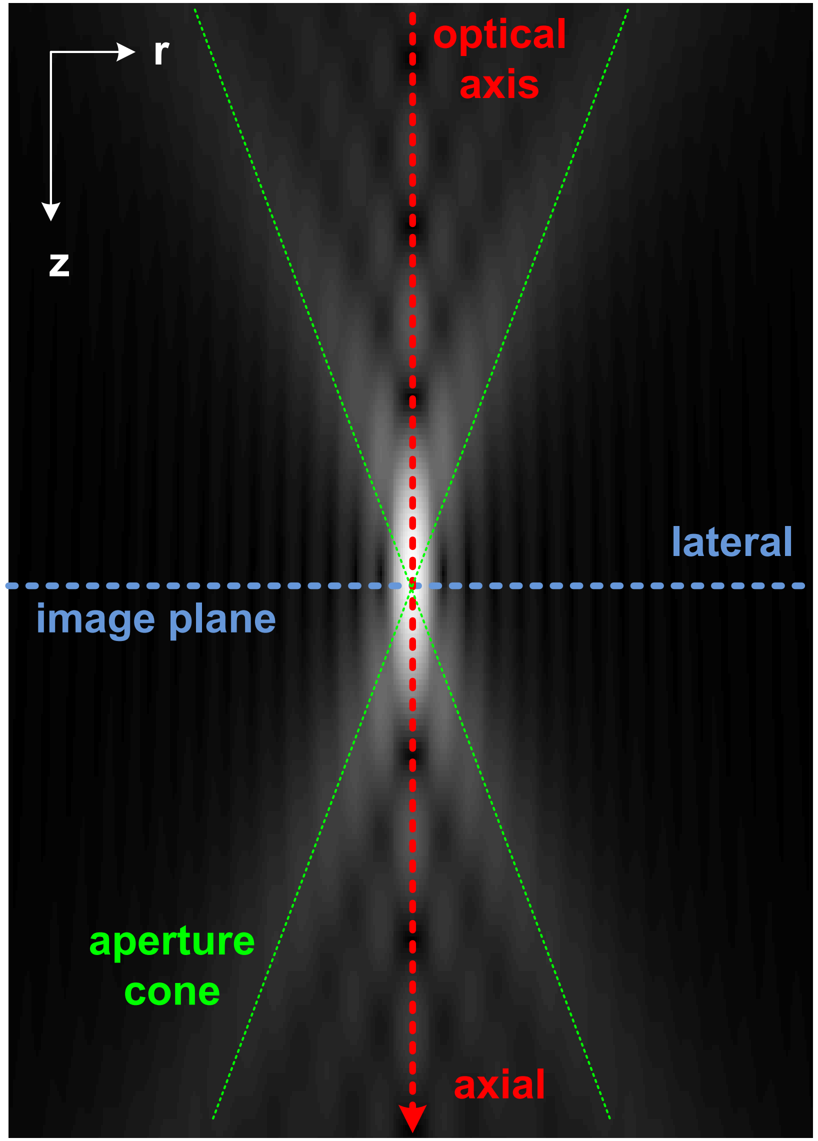
$$L(\alpha,\xi,z) = L_0(\alpha,\xi) exp\big(i2\pi W(\alpha,\xi)\big) exp\left(-i\frac{2\pi}{\lambda} z\sqrt{1-(NA/n)^2\alpha^2}\right).$$

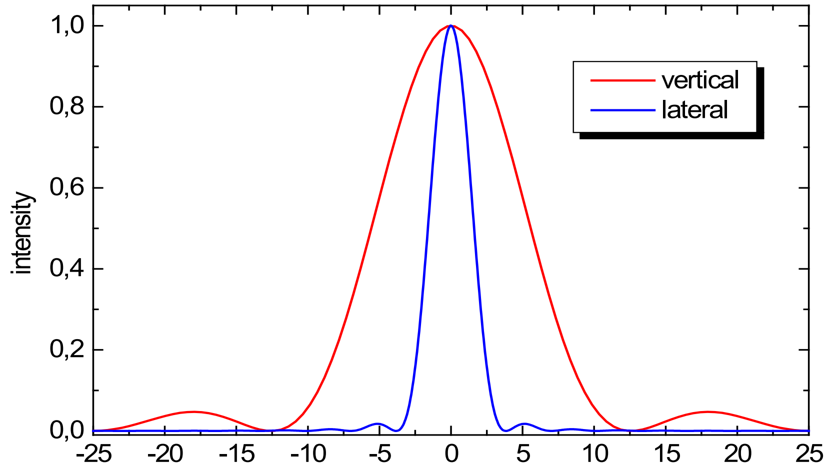
Complex function





Ideal Point Spread Function PSF(x,y,z)





$$PSF(x,\xi,z) = \left| \iint d\alpha L_0(\alpha,\xi) exp\big(i2\pi W(\alpha,\xi)\big) exp\left(-i\frac{2\pi}{\lambda} z\sqrt{1-(NA/n)^2\alpha^2}\right) exp\big(-i2\pi w\, \alpha\cdot(x/m-\xi)\big)\right|^2$$

$$PSF_{ideal}(x,0) = Besinc\left(2\pi\frac{NA}{\lambda}x\right)$$

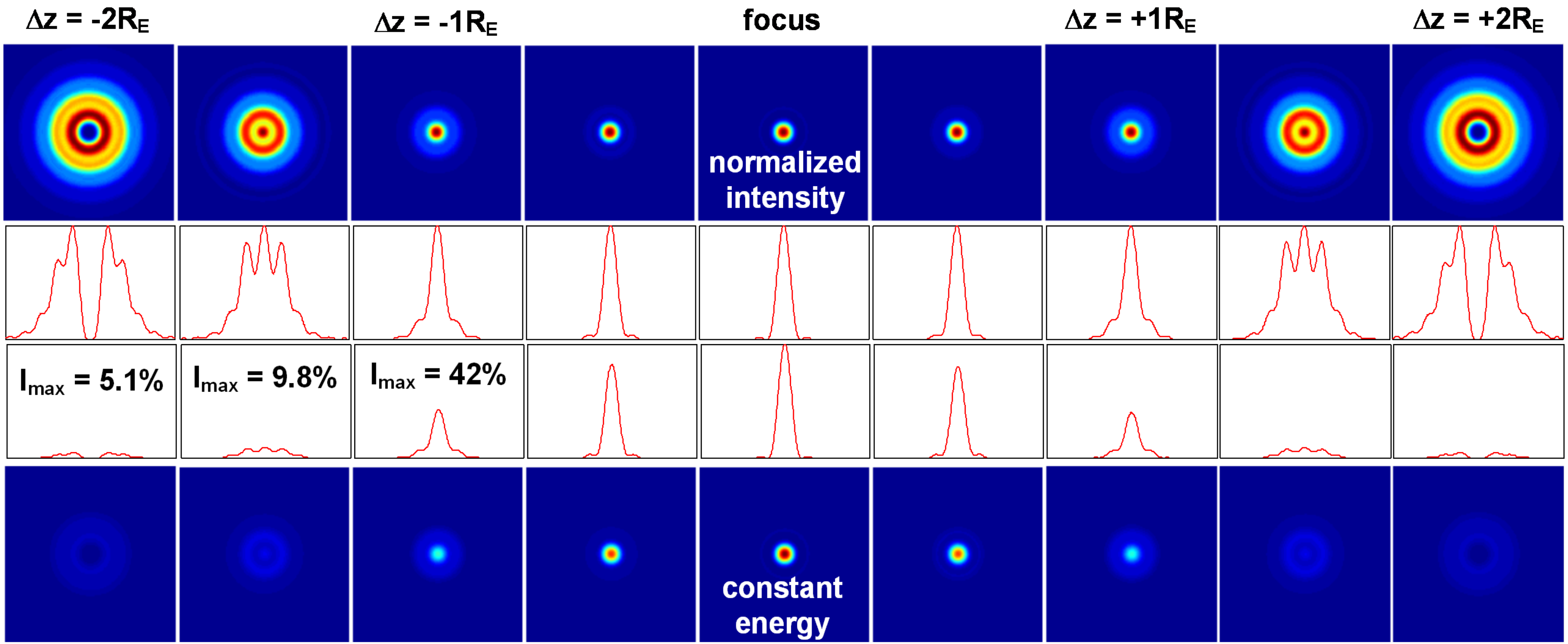
$$D_{Airy} = \frac{1.22\cdot\lambda}{NA}$$

Small/moderate NA:

$$PSF_{ideal}(0,z) \approx Sinc\left(2\pi\frac{NA^2}{n\lambda}z\right)$$

$$R_E = \frac{n\cdot\lambda}{NA^2}$$

PSF(x,z)



Δz = -2R_E
Δz = -1R_E
focus
Δz = +1R_E
Δz = +2R_E
normalized intensity
I_max = 5.1%
I_max = 9.8%
I_max = 42%
constant energy

PSF versus focus (standard lens)

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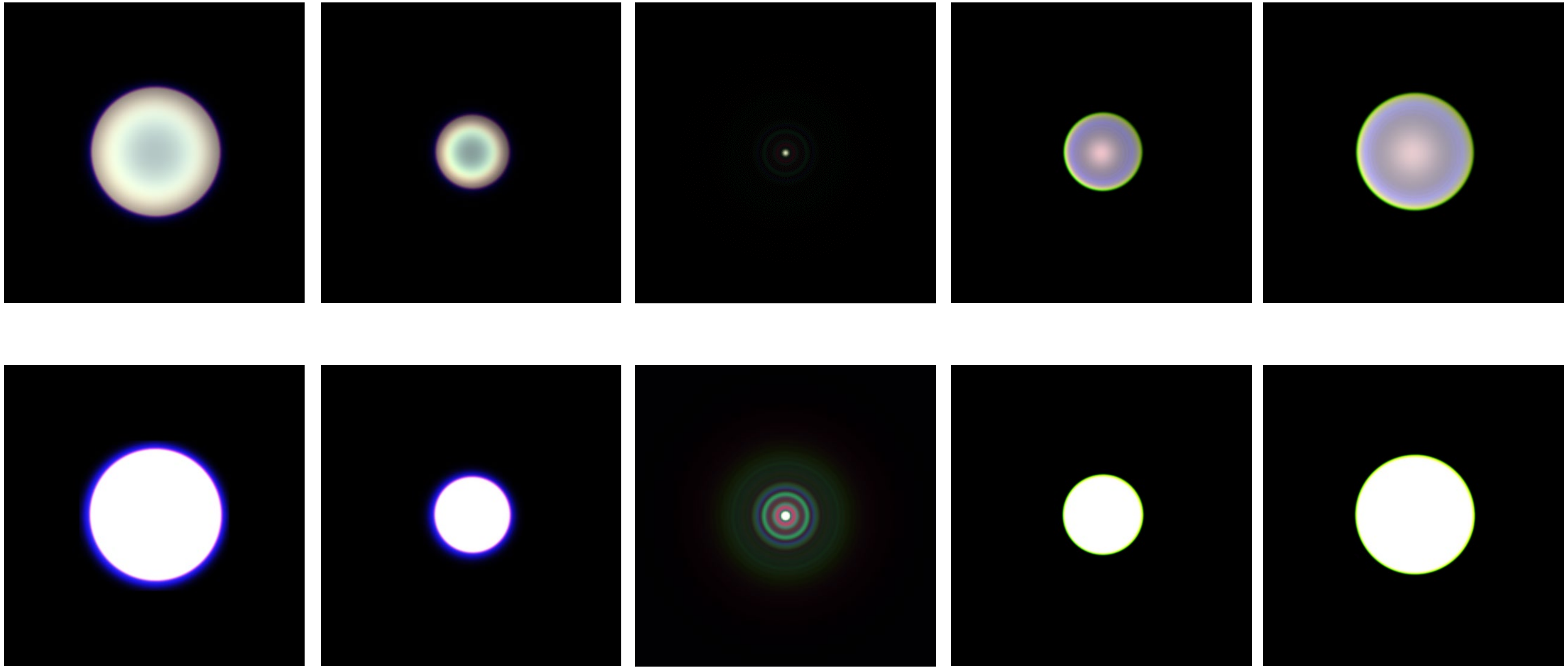

1.33m 1.53m 2.03m 2.53m 3.03m

Top row: normal exposure; 2nd row: overexposure 3 stops (factor 8)

PSF versus focus (excellent lens)

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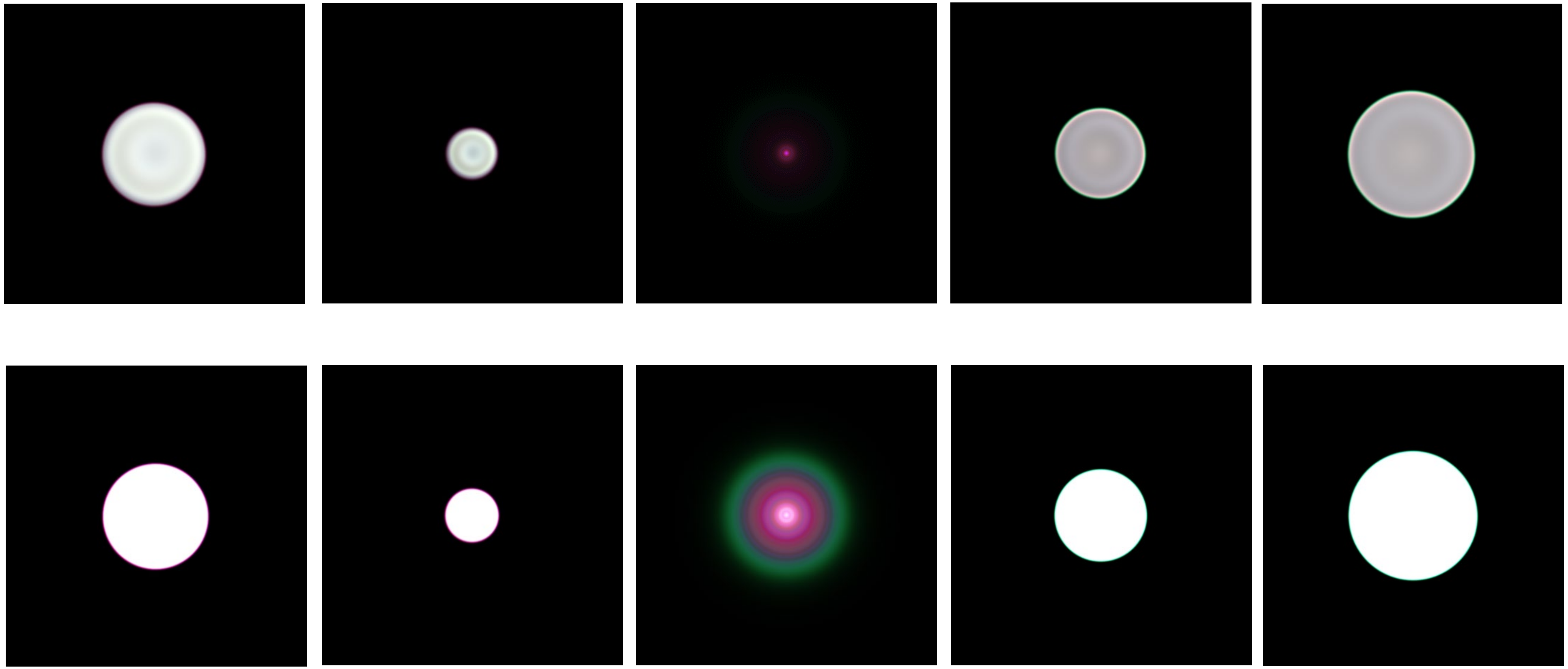

1.33m 1.53m 2.03m 2.53m 3.03m

Top row: normal exposure; 2nd row: overexposure 3 stops (factor 8)

Calculation of strongly defocused PSF

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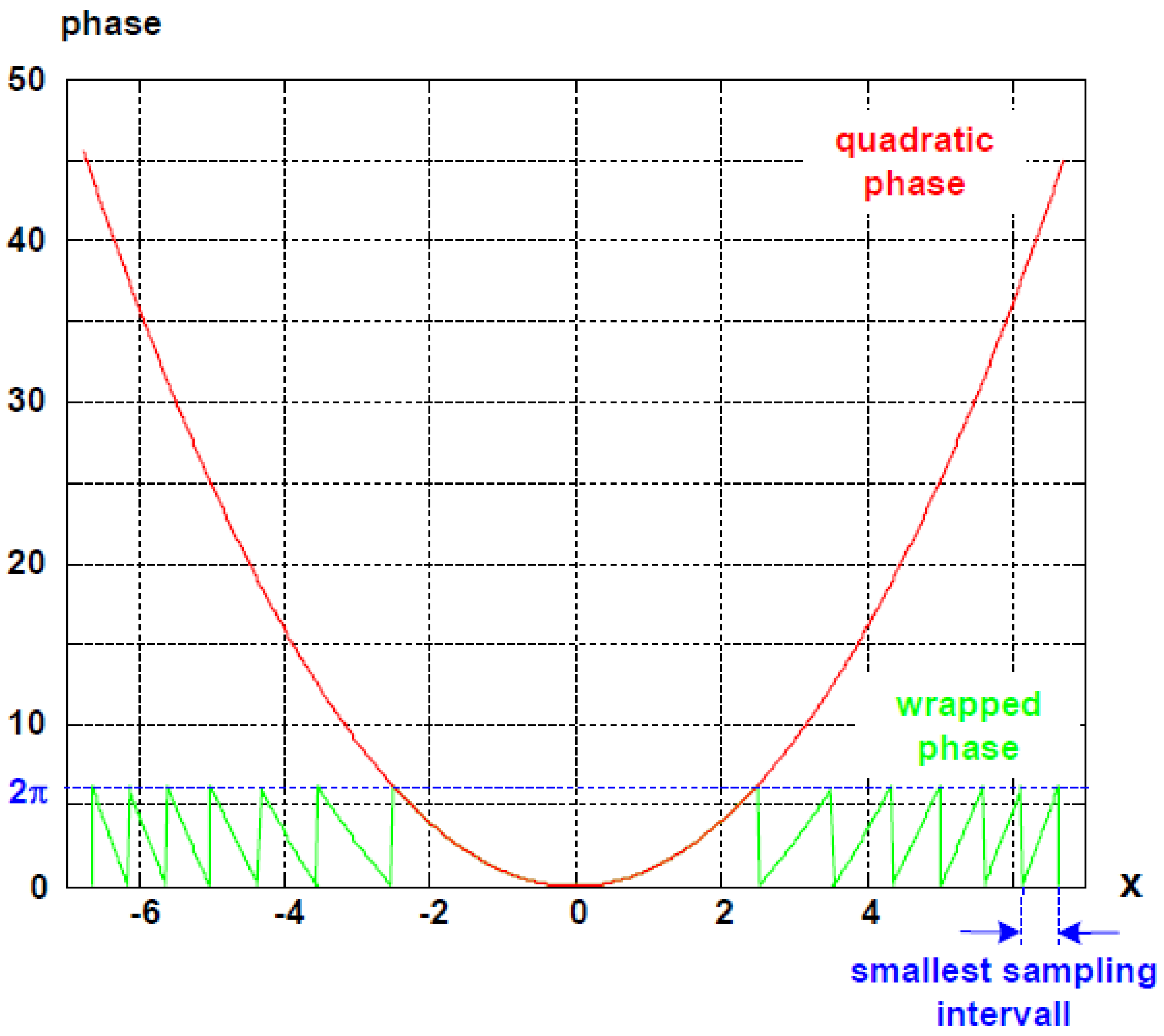

The wave-optical calculation out of focus **requires an increased sampling** for the calculation of the integral as defocusing increases.

The **wave-optical calculat**ion will be very **time-consuming**.

PSF data of a camera lens versus field and depth

„Geometrical PSF“ neglecting diffraction effects

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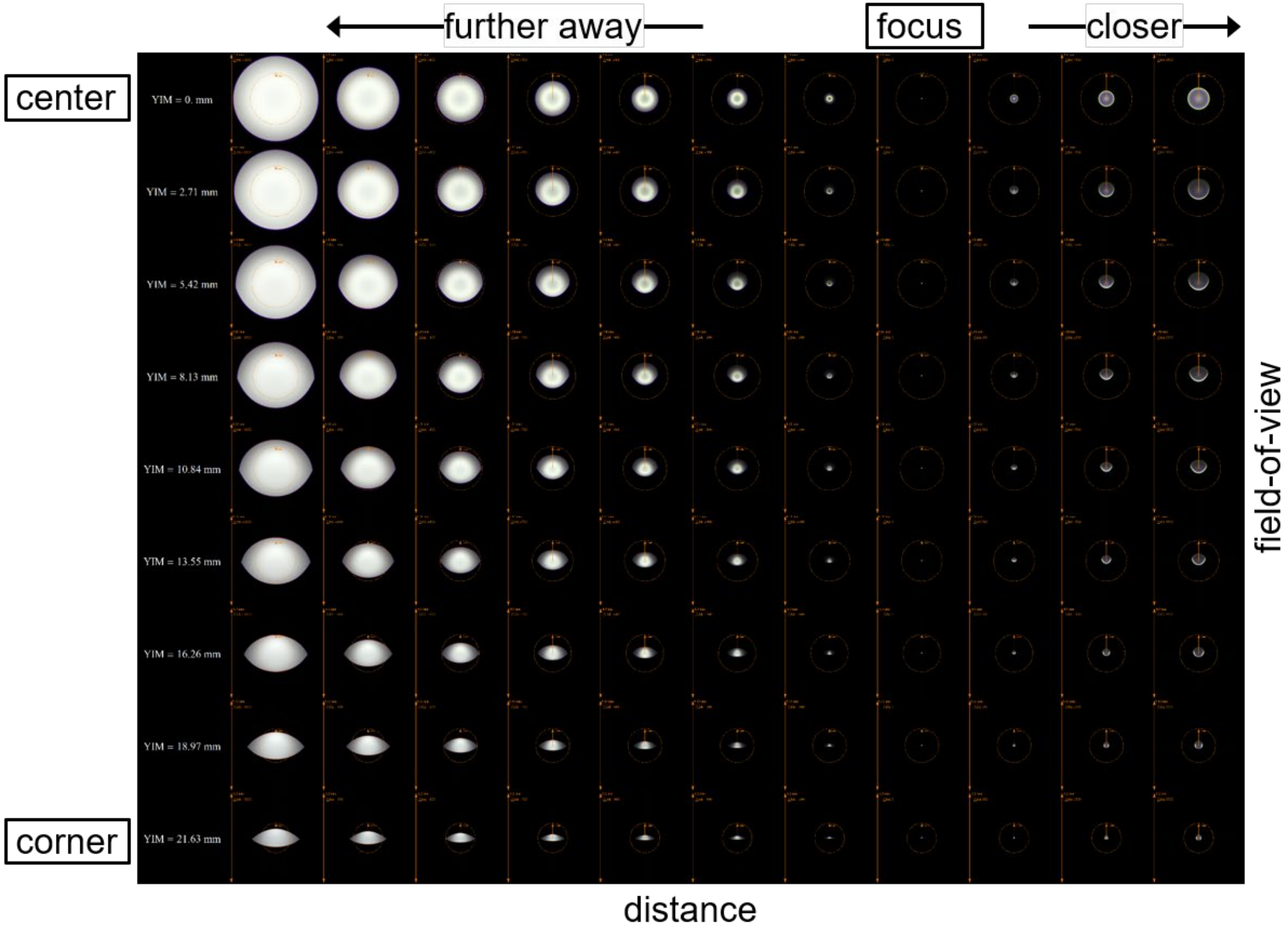
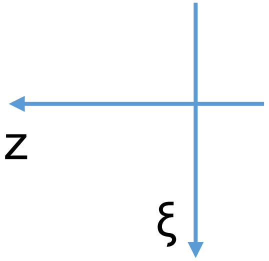


Photorealistic PSF lens data vs field & focus (simulated data)

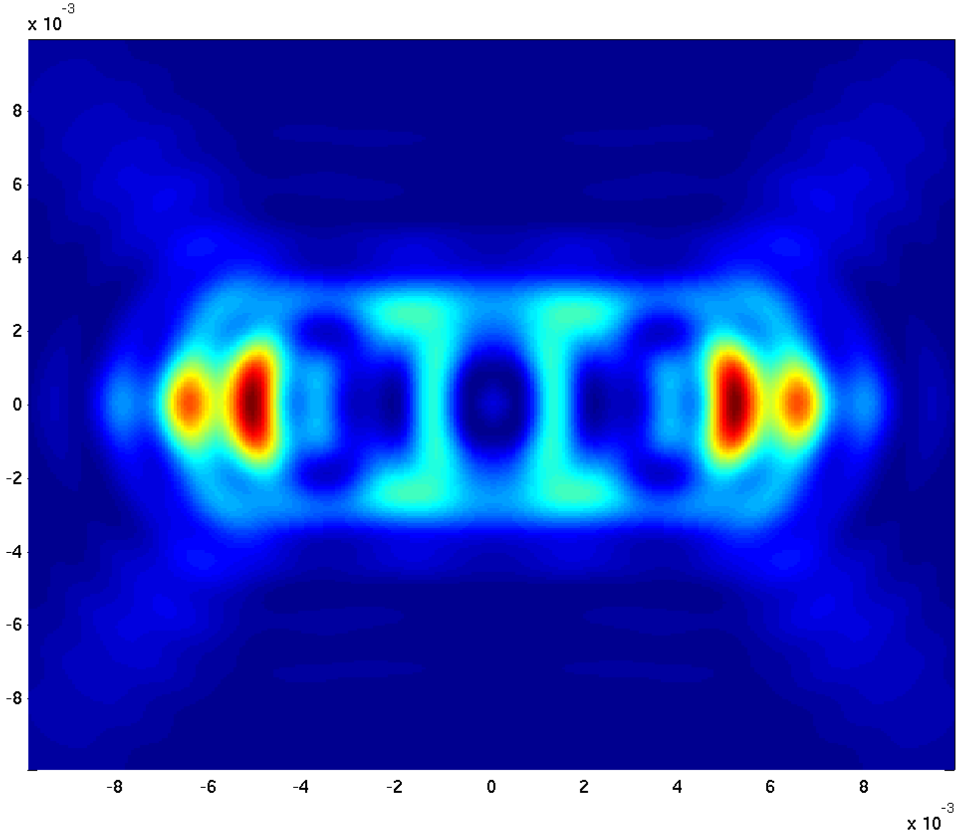
3D **PSF(**x,y,**ξ,η,z)** kernels depend on

1. Focusing distance s_F
2. f-number
3. Focal length (zoom)

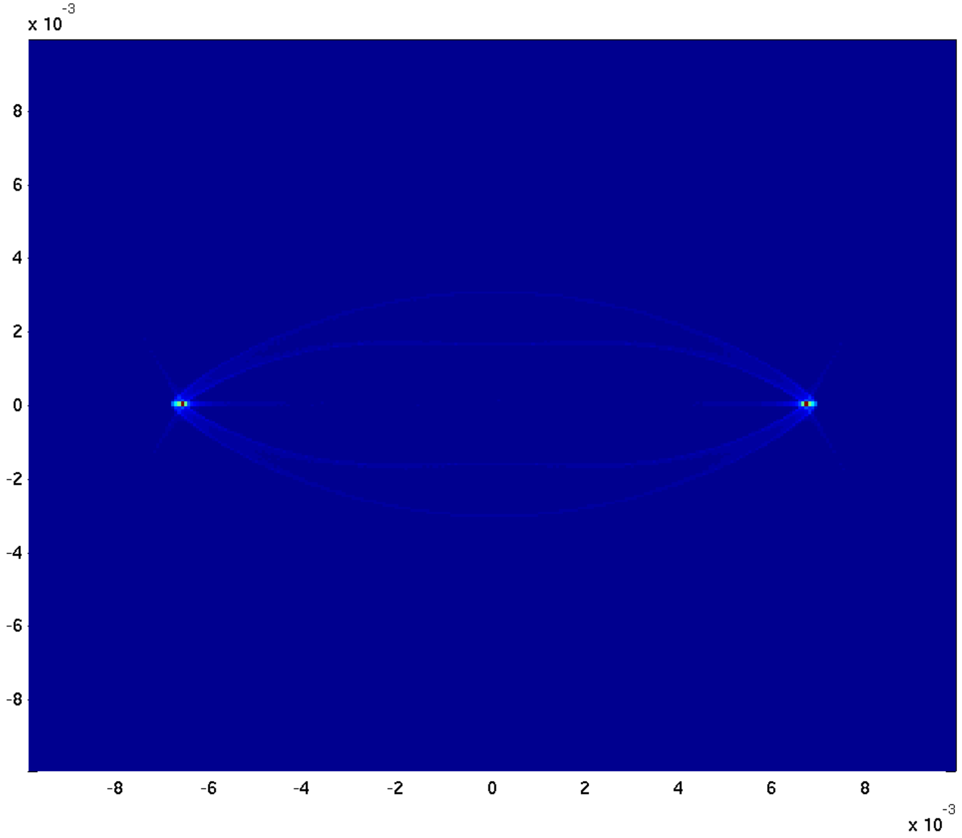


Wave-optical vs Geometrical-Optical PSF
Wavelength 656nm

Wave-Optical PSF



Geometric-Optical PSF



λ 656.2730 587.5620 546.0740 486.1330 435.8340 404.6560

Wave-optical vs Geometrical-Optical PSF
Wavelength 587nm

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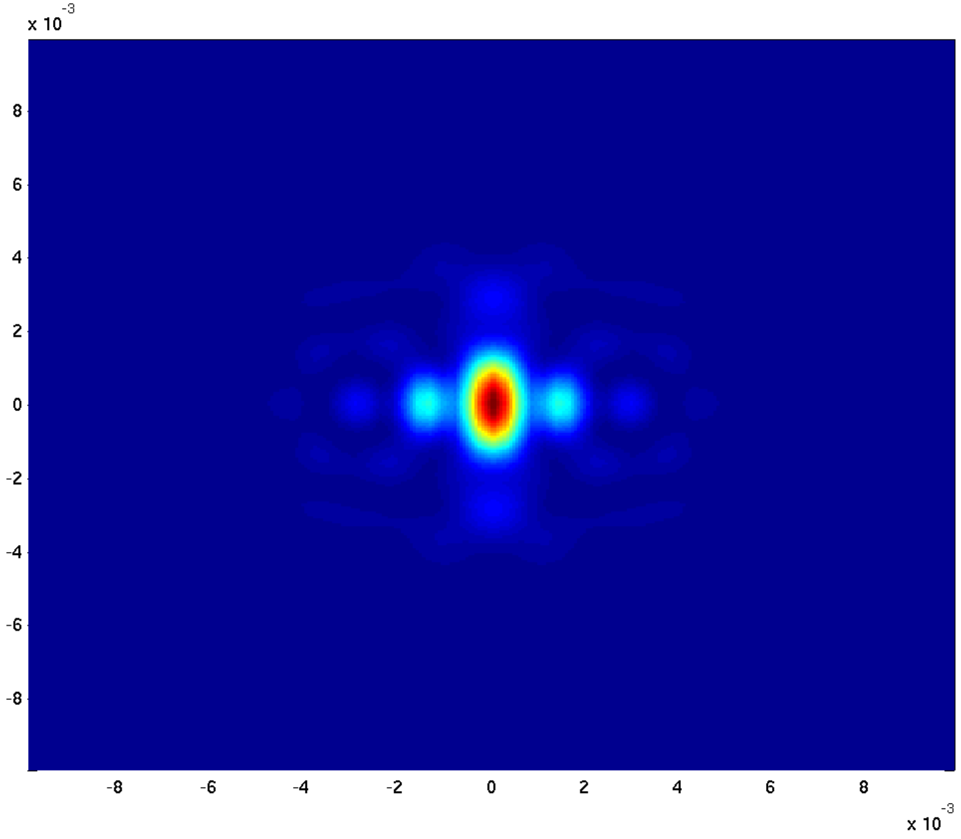
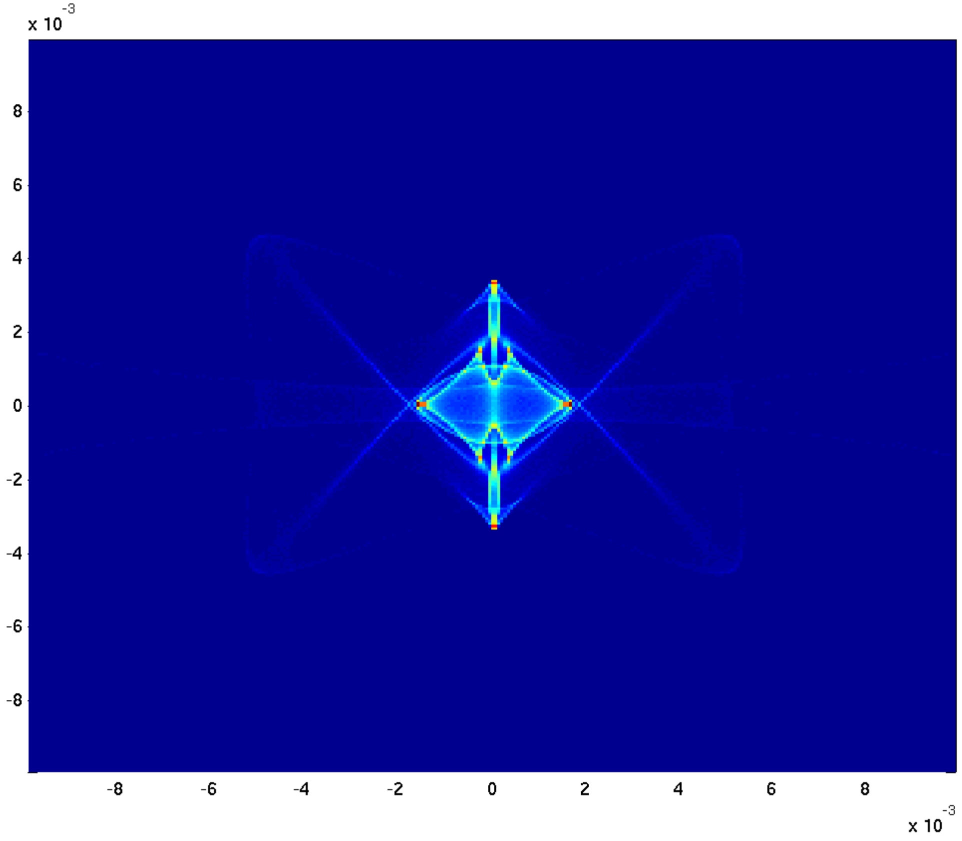
Wave-Optical PSF



Geometric-Optical PSF



λ 656.2730 587.5620 546.0740 486.1330 435.8340 404.6560

Wave-optical vs Geometrical-Optical PSF
Wavelength 546nm

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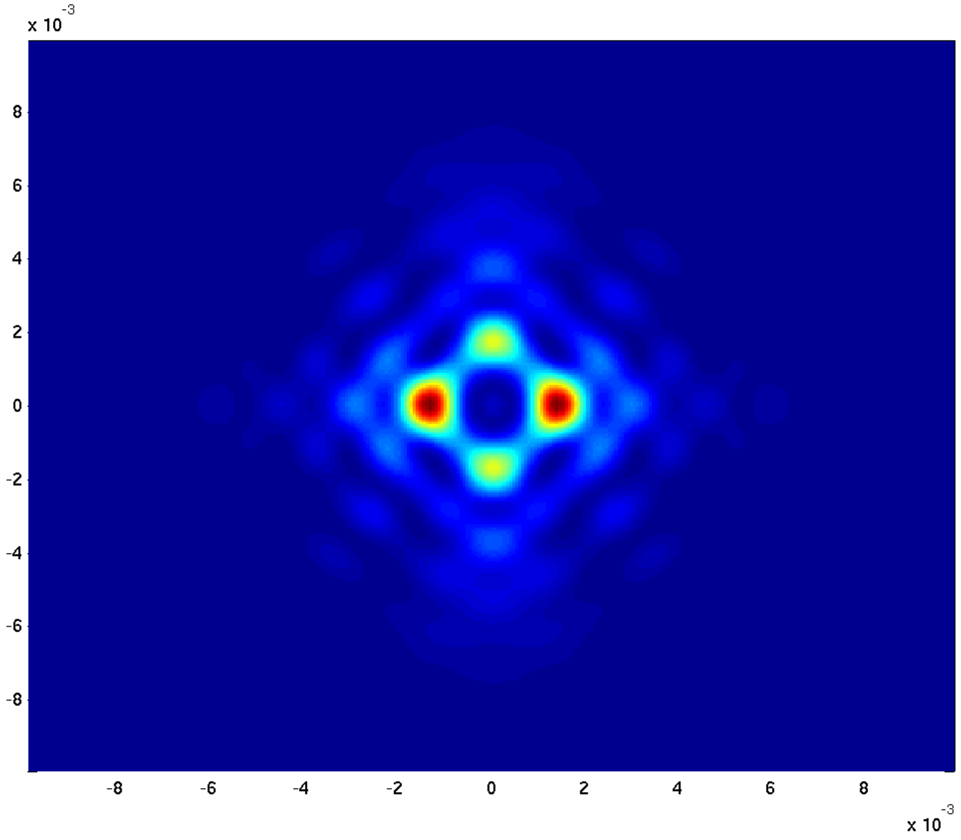
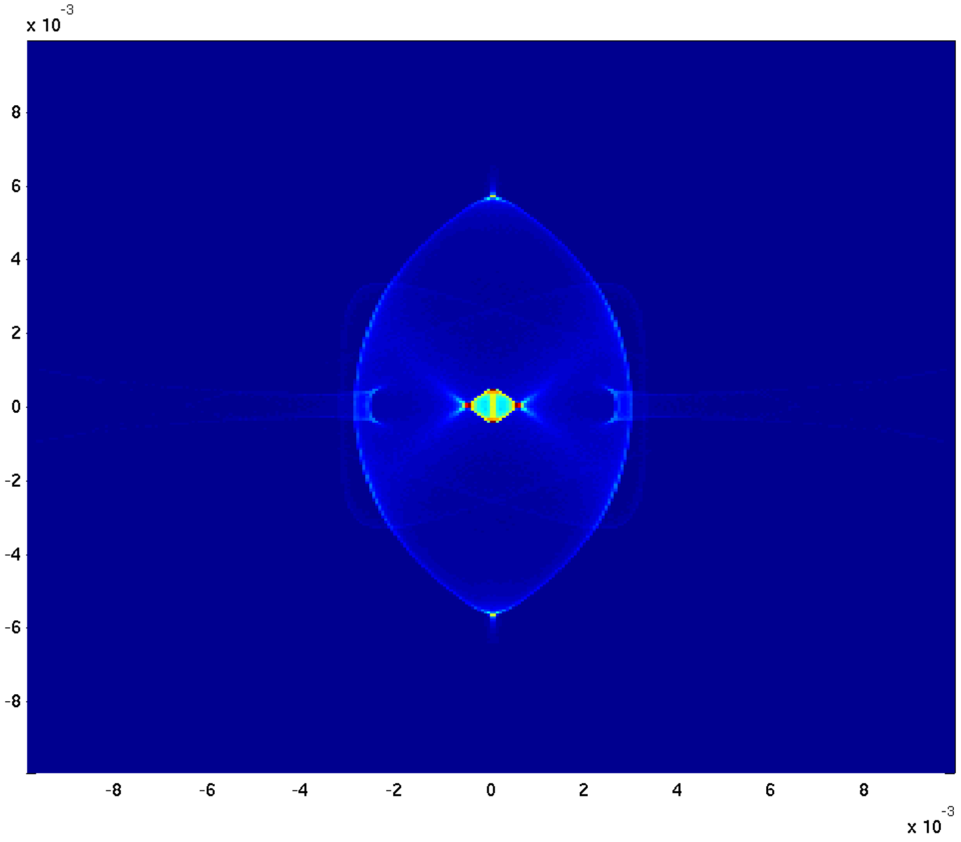
Wave-Optical PSF



Geometric-Optical PSF



λ 656.2730 587.5620 546.0740 486.1330 435.8340 404.6560

Wave-optical vs Geometrical-Optical PSF
Wavelength 486nm

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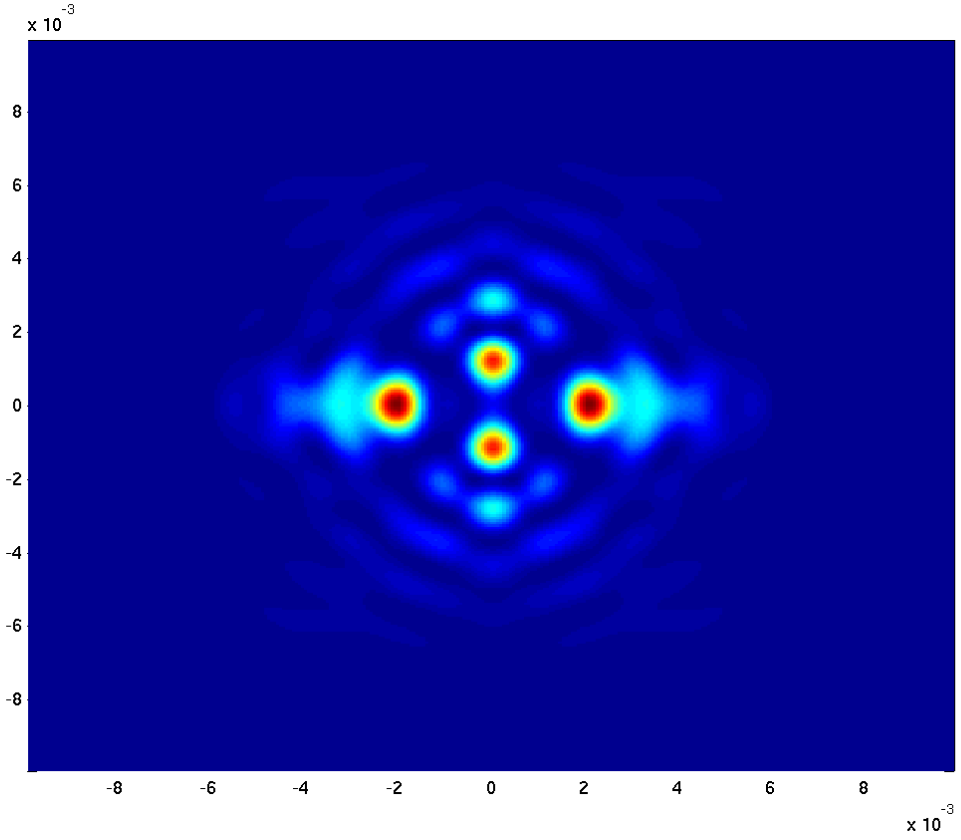
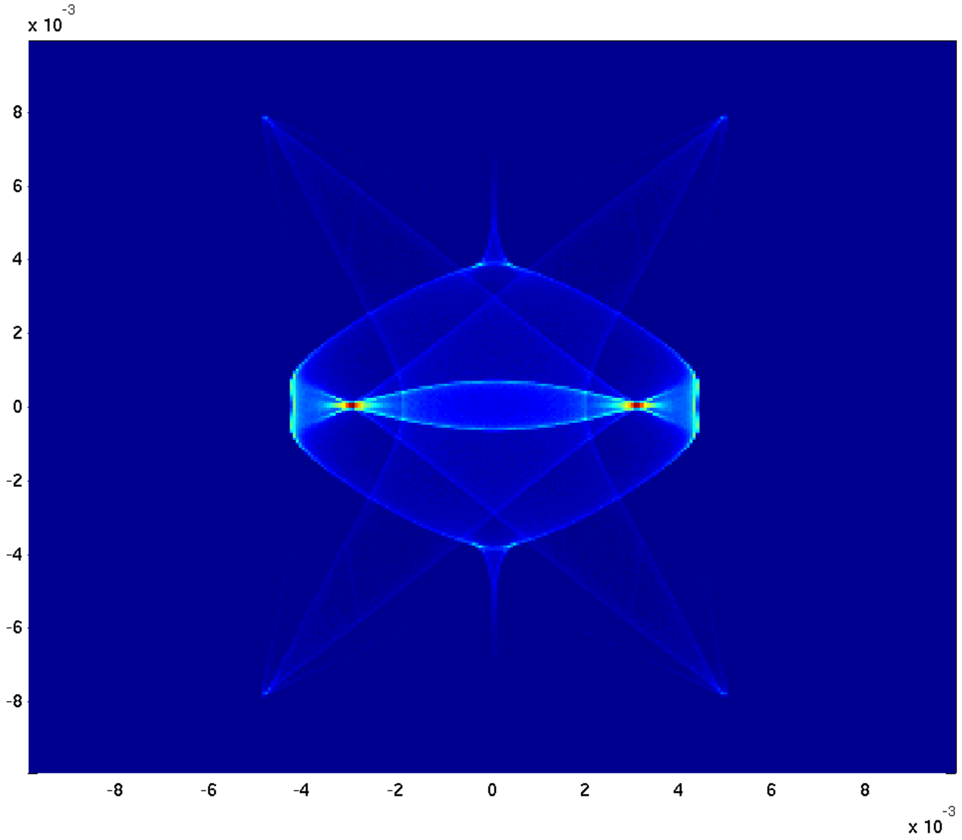
Wave-Optical PSF



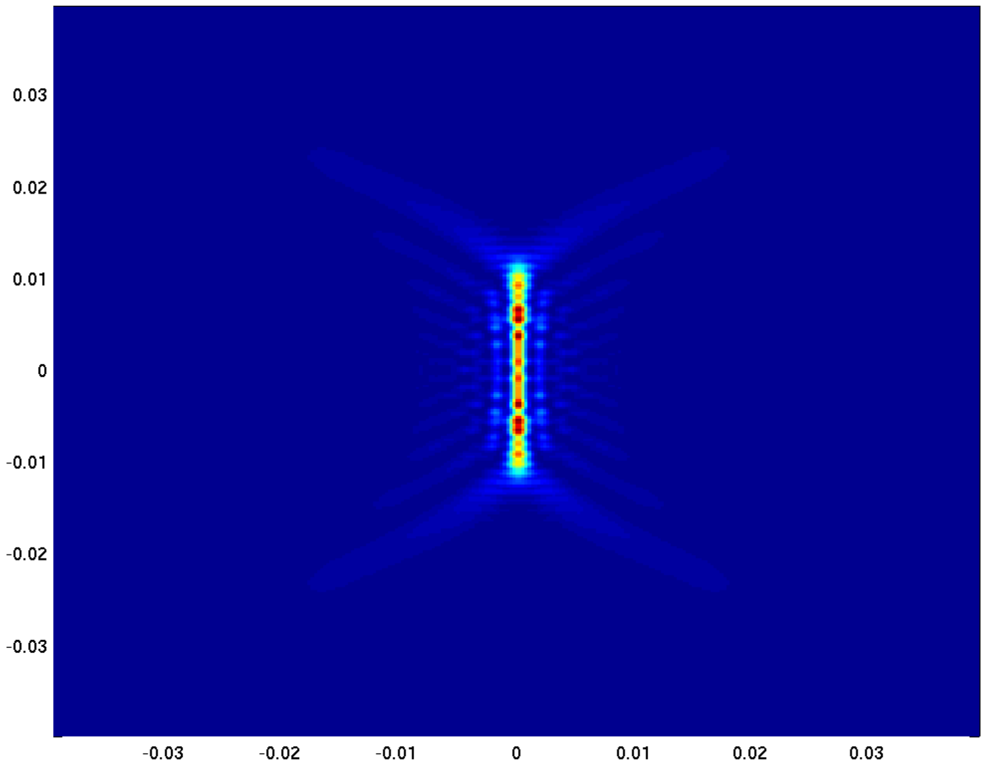
Geometric-Optical PSF



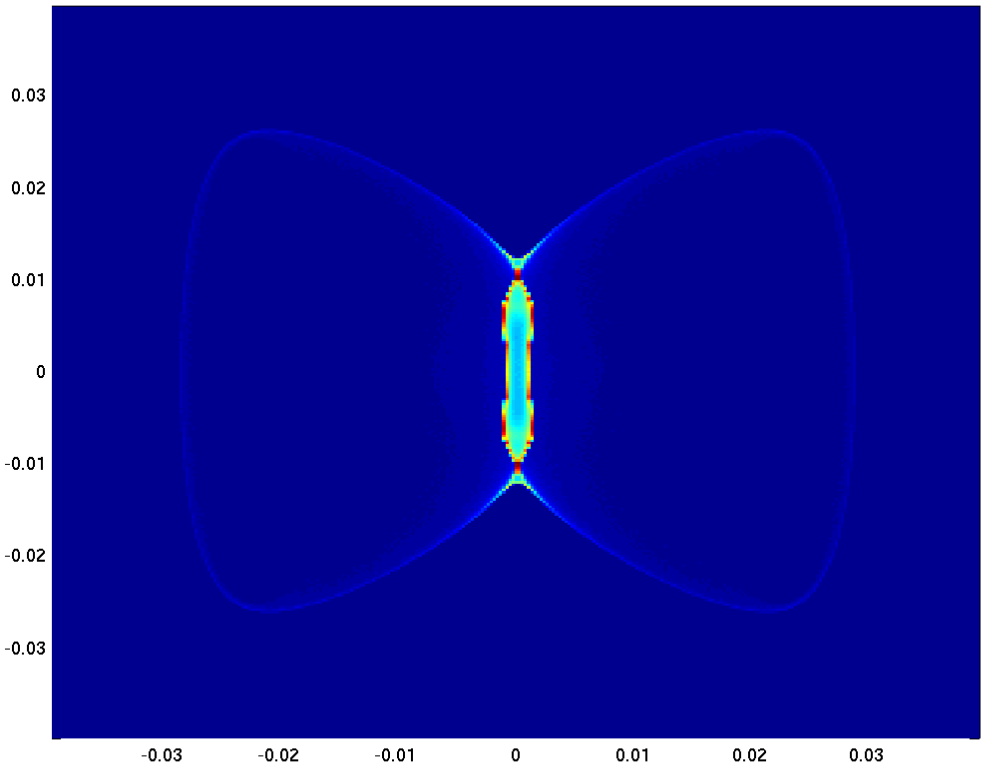
λ 656.2730 587.5620 546.0740 486.1330 435.8340 404.6560

Wave-optical vs Geometrical-Optical PSF Wavelength 435nm

Wave-Optical PSF



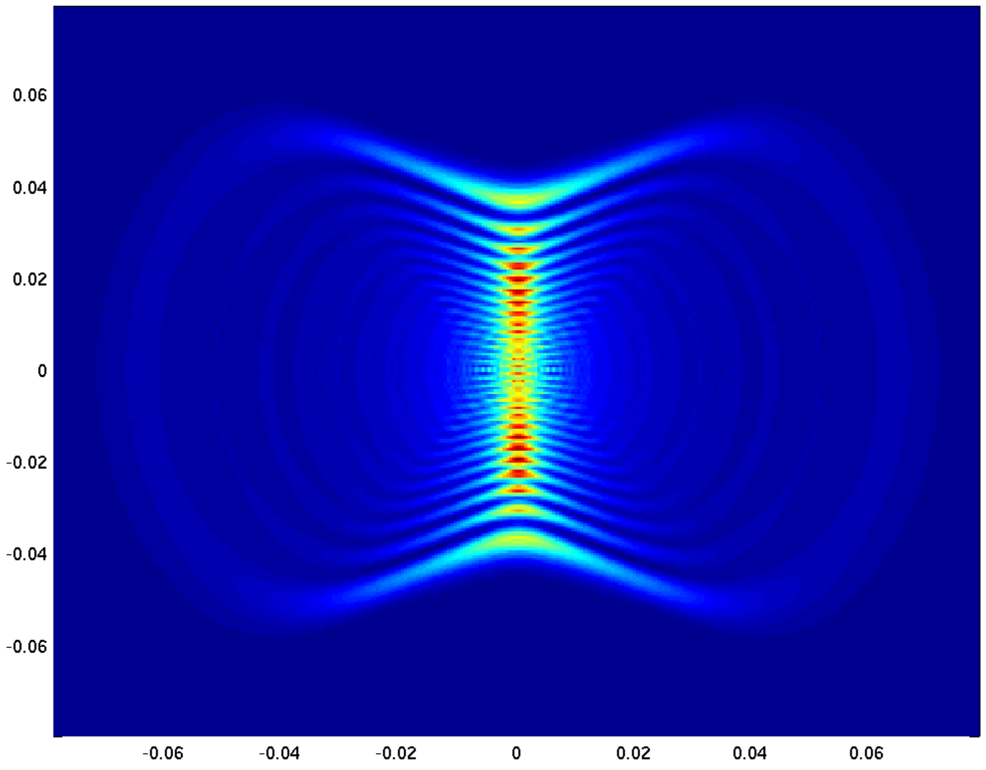
Geometric-Optical PSF



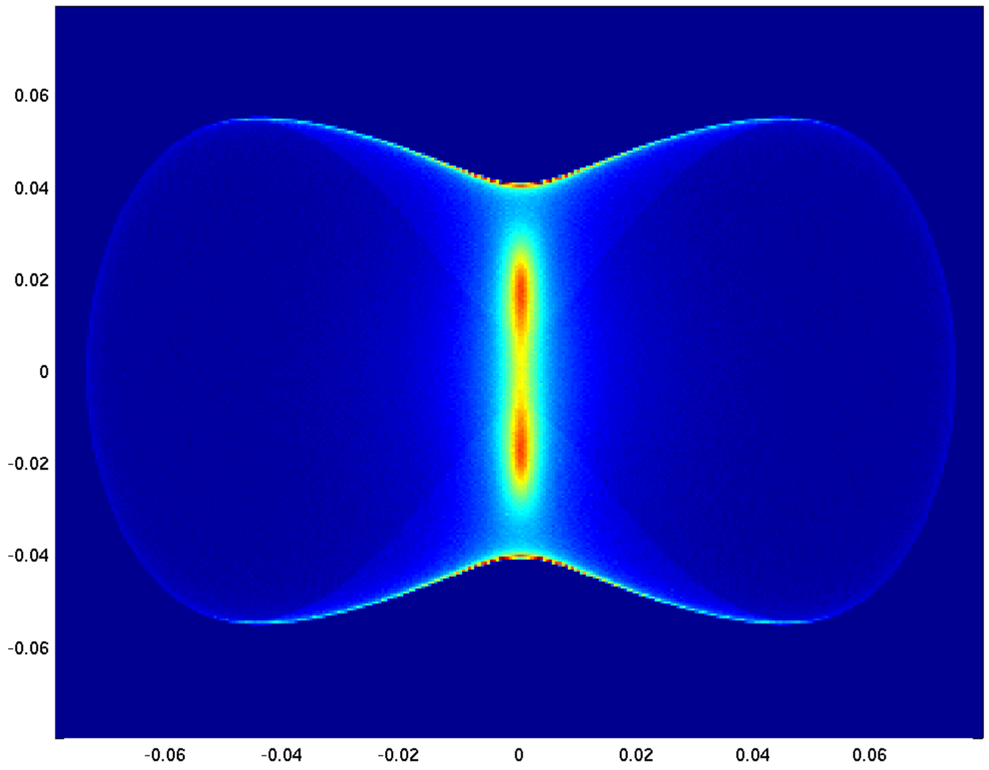
λ 656.2730 587.5620 546.0740 486.1330 435.8340 404.6560

Wave-optical vs Geometrical-Optical PSF
Wavelength 405nm

Wave-Optical PSF



Geometric-Optical PSF



λ 656.2730 587.5620 546.0740 486.1330 435.8340 404.6560

Wave-optical vs Geometrical-Optical PSF
Fine-sampled image distribution (best focus)

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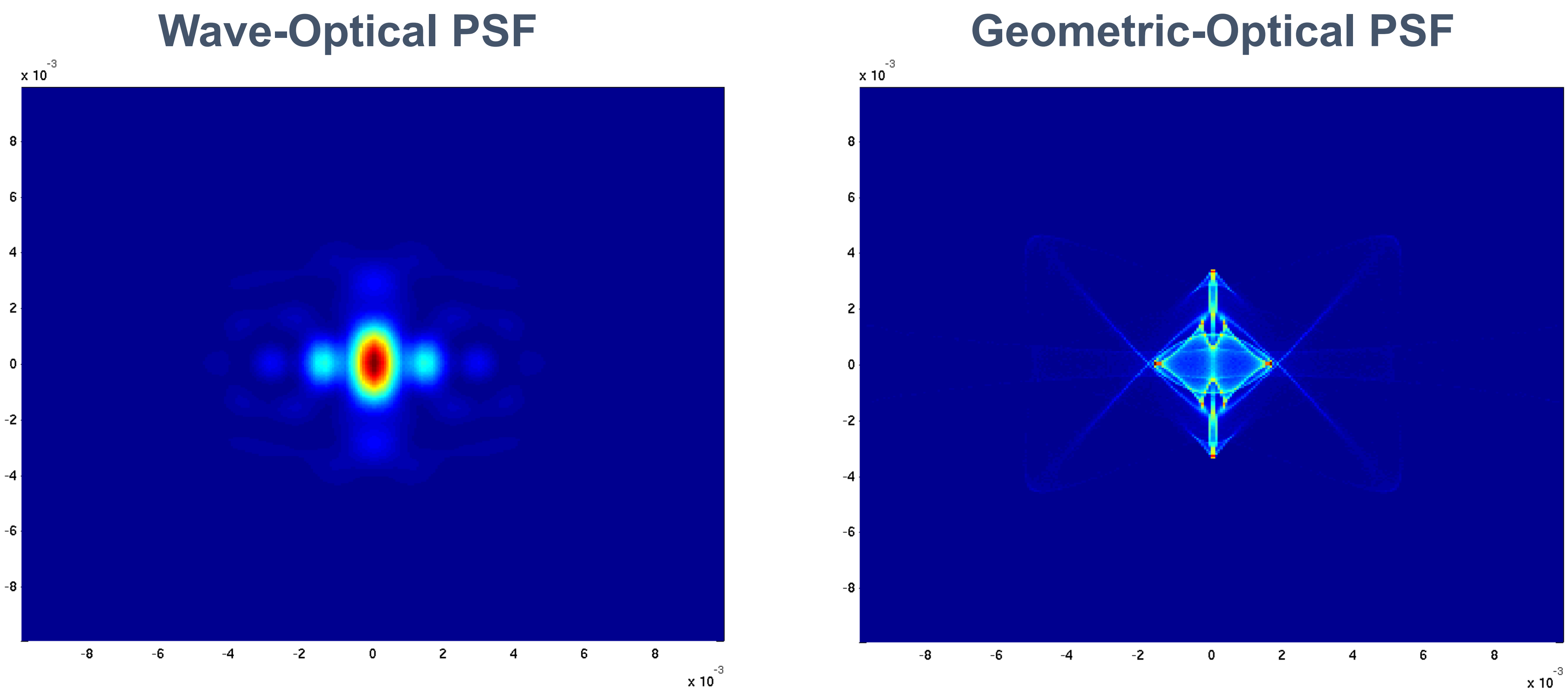
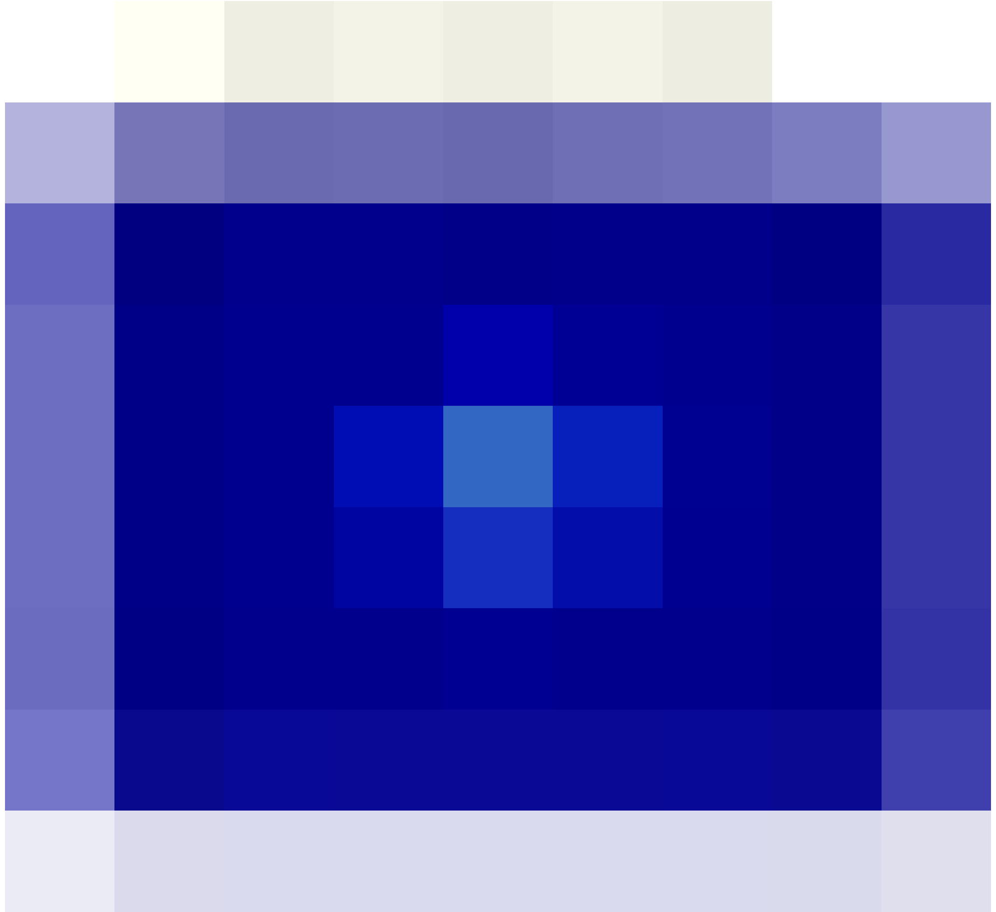
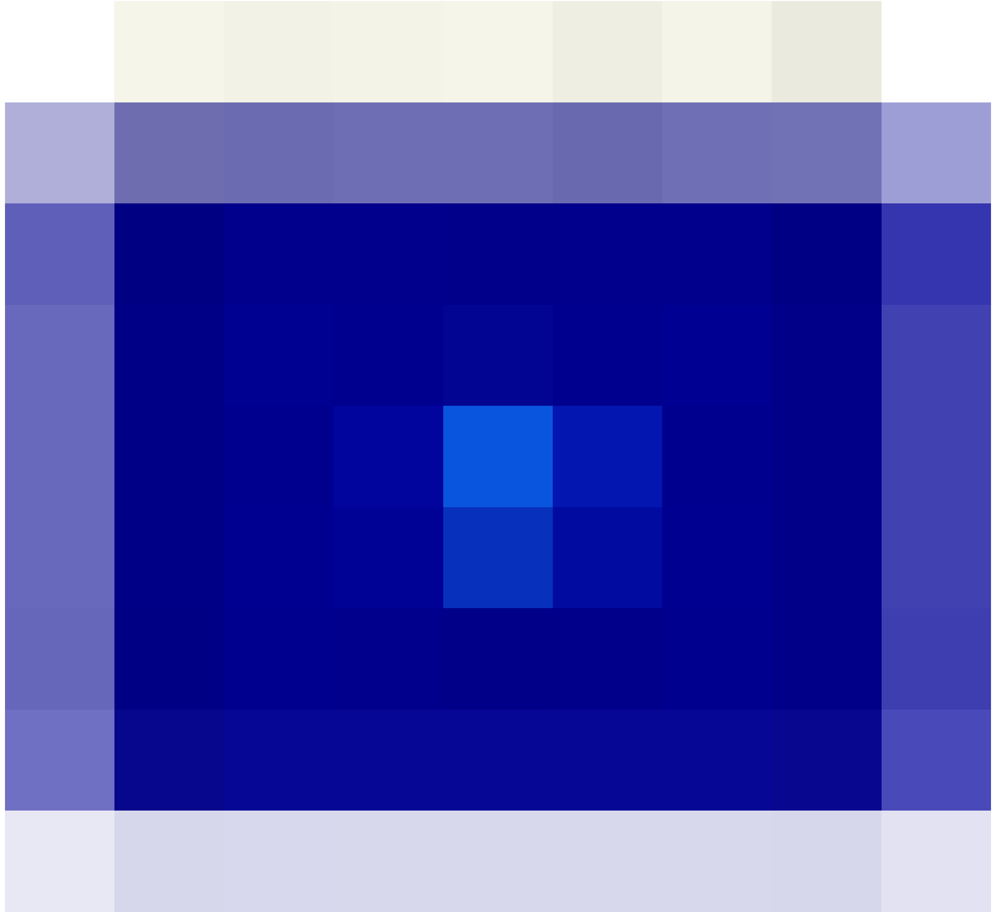


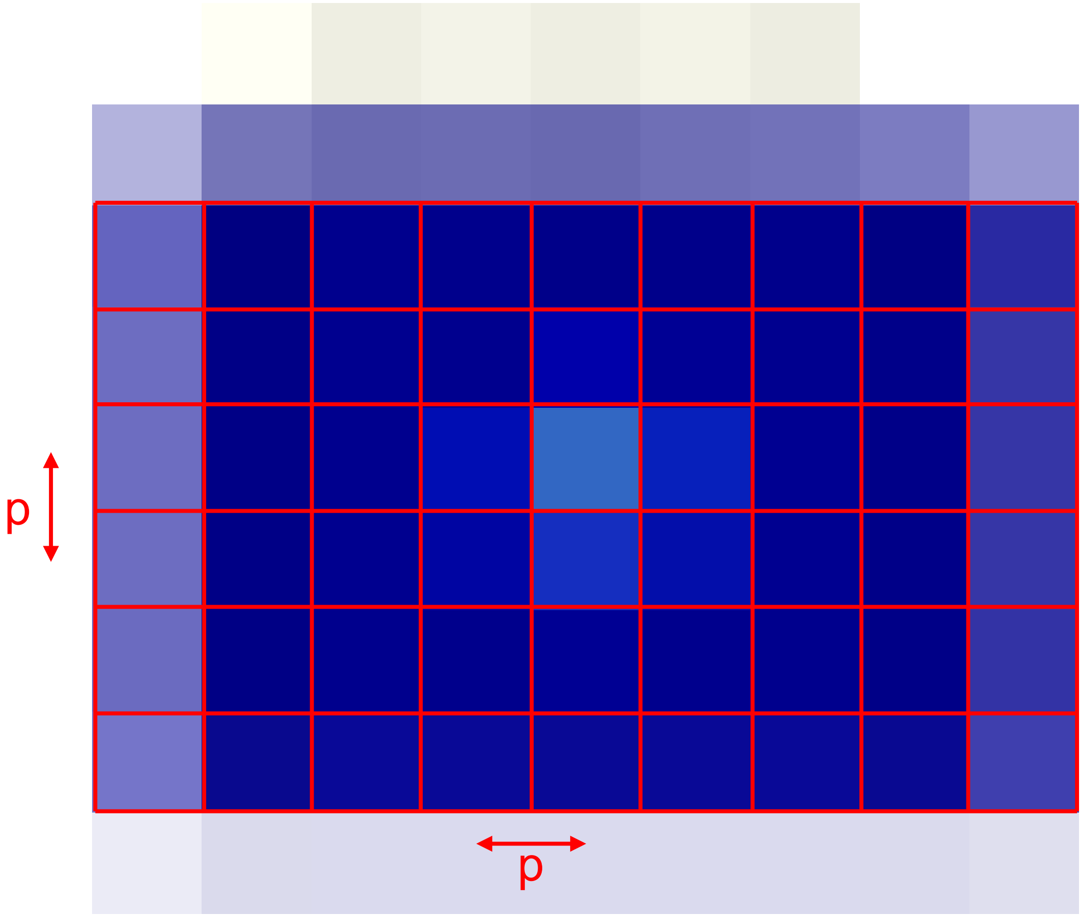
Wave-optical vs Geometrical-Optical PSF
Pixel sampling

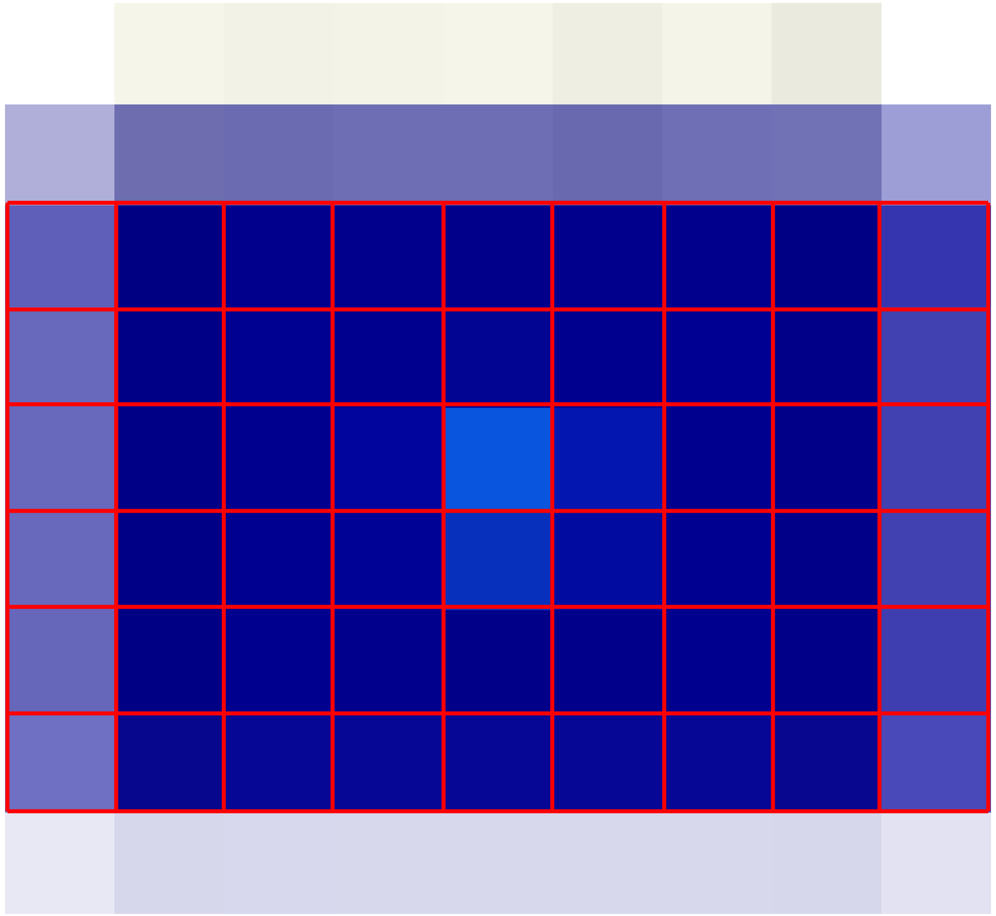




2µm pixel pitch

Wave-optical vs Geometrical-Optical PSF
Pixel sampling

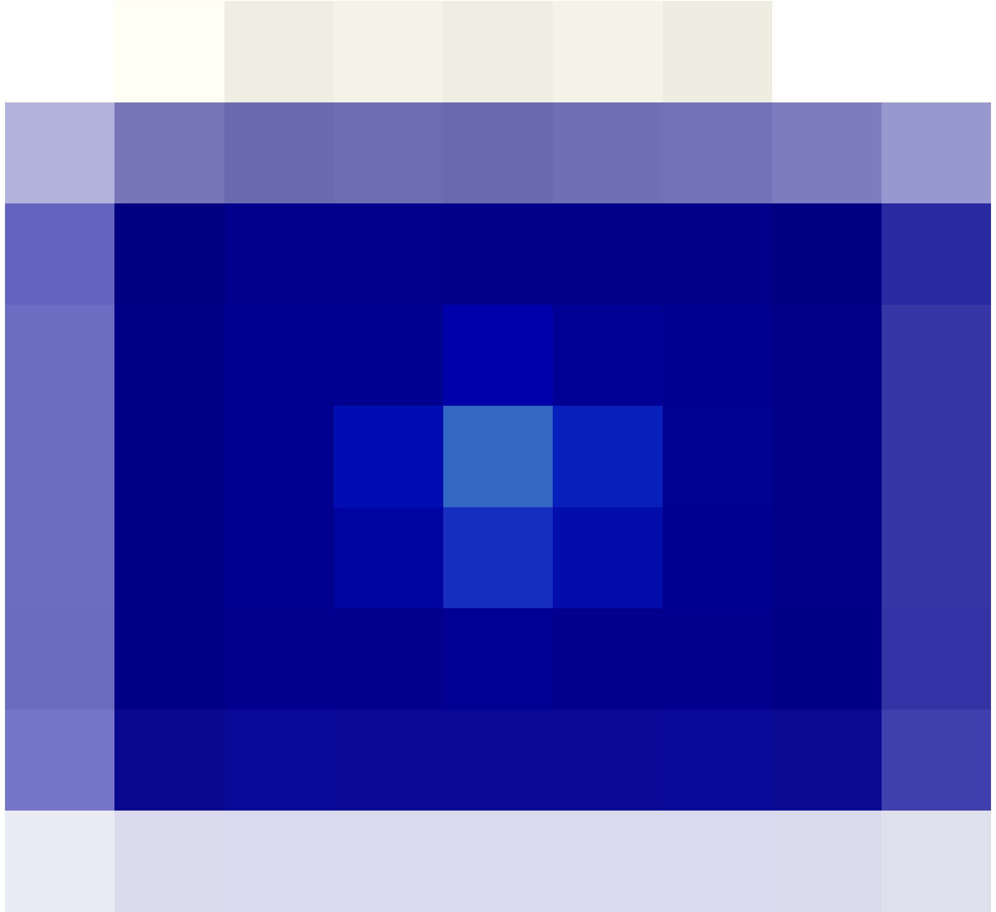


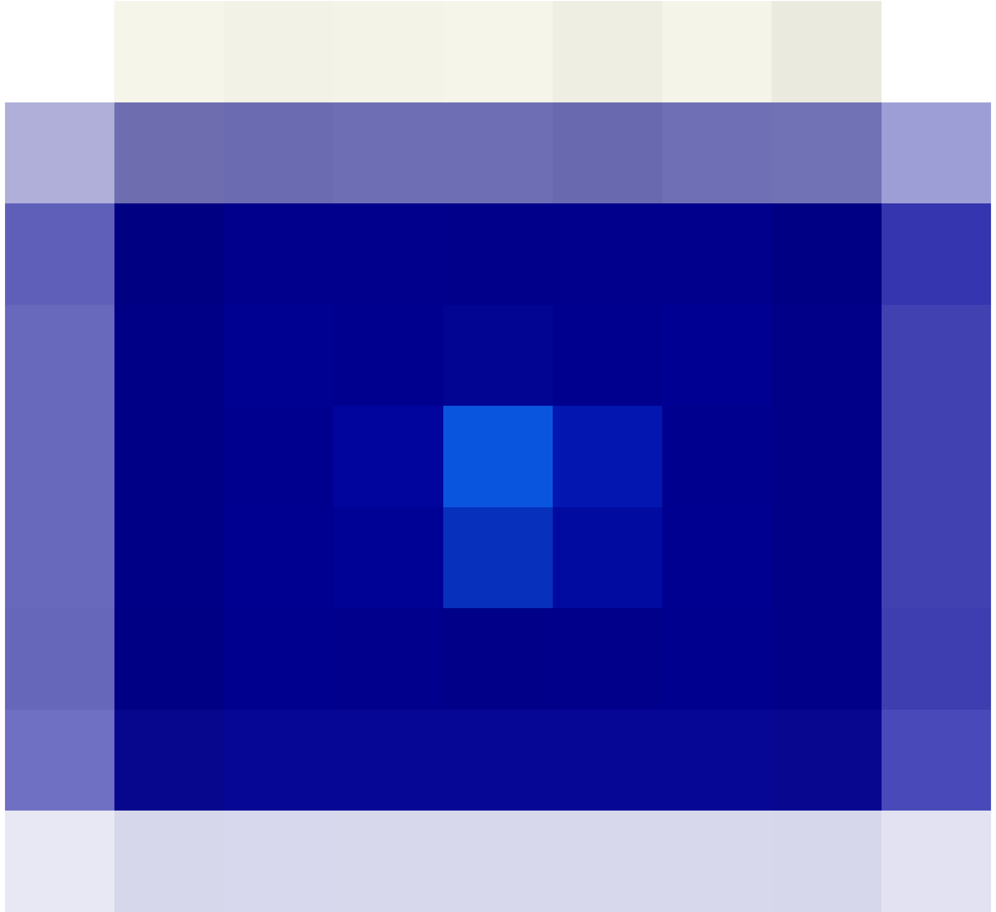


$$PSF(n,m) = \int_{-p/2}^{p/2} \int_{-p/2}^{p/2} PSF(n\,p - x, m\,p - y)\,dx\,dy$$

Pixel array averages intensity distribution in pixel area.

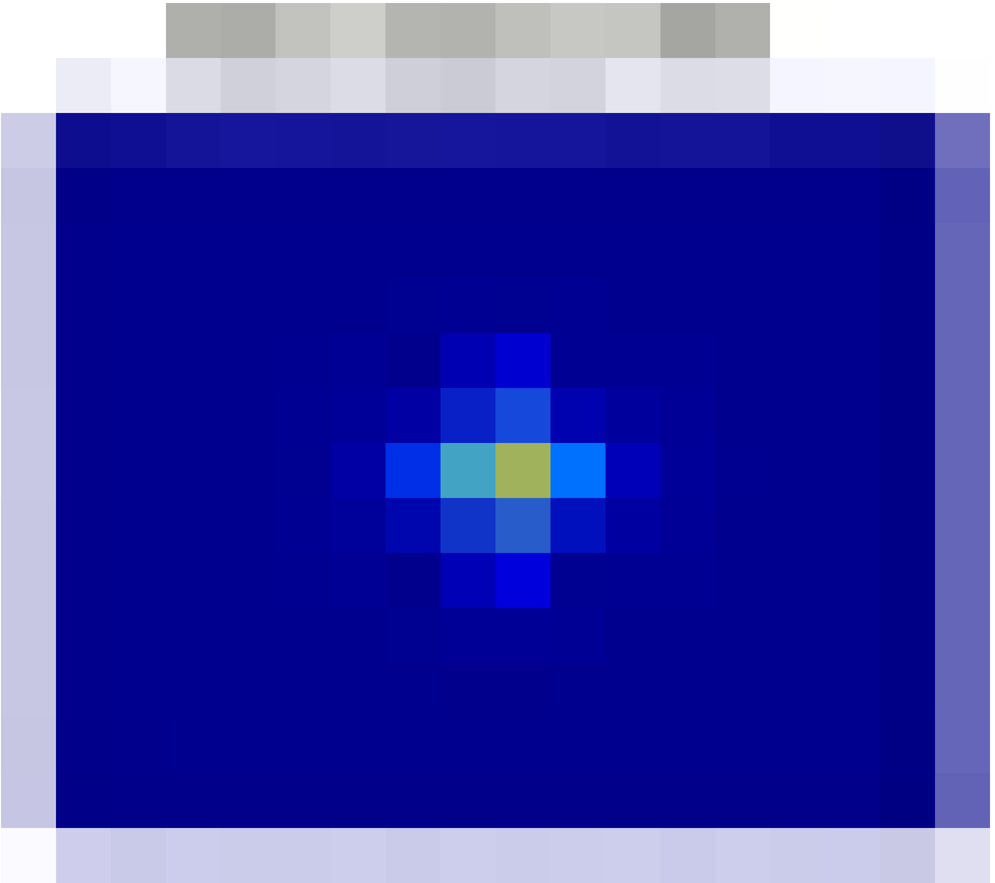
Wave-optical vs Geometrical-Optical PSF
Pixel sampling

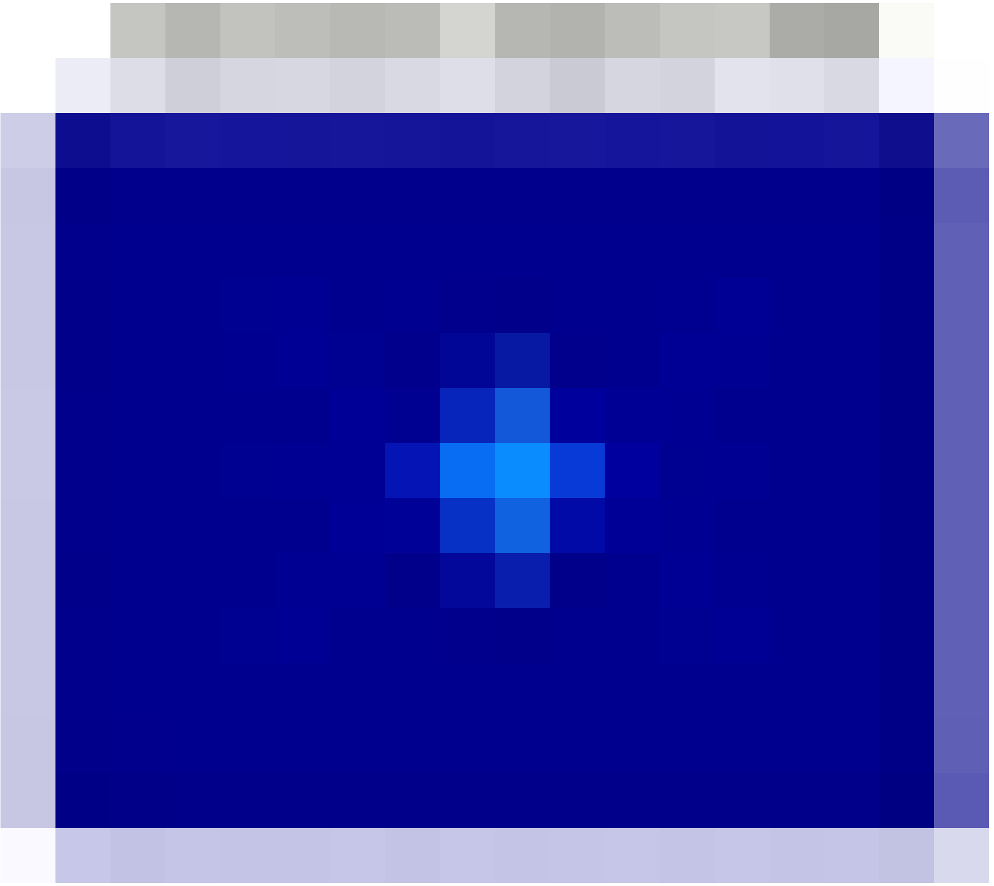




2µm pixel pitch

Wave-optical vs Geometrical-Optical PSF
Pixel sampling

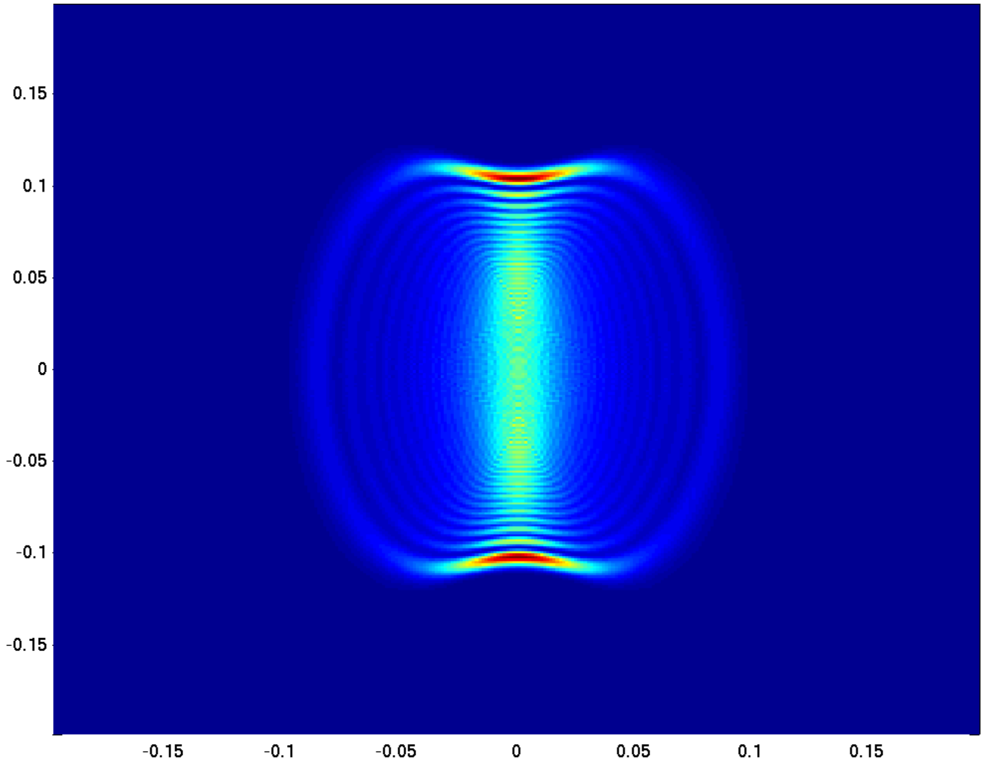




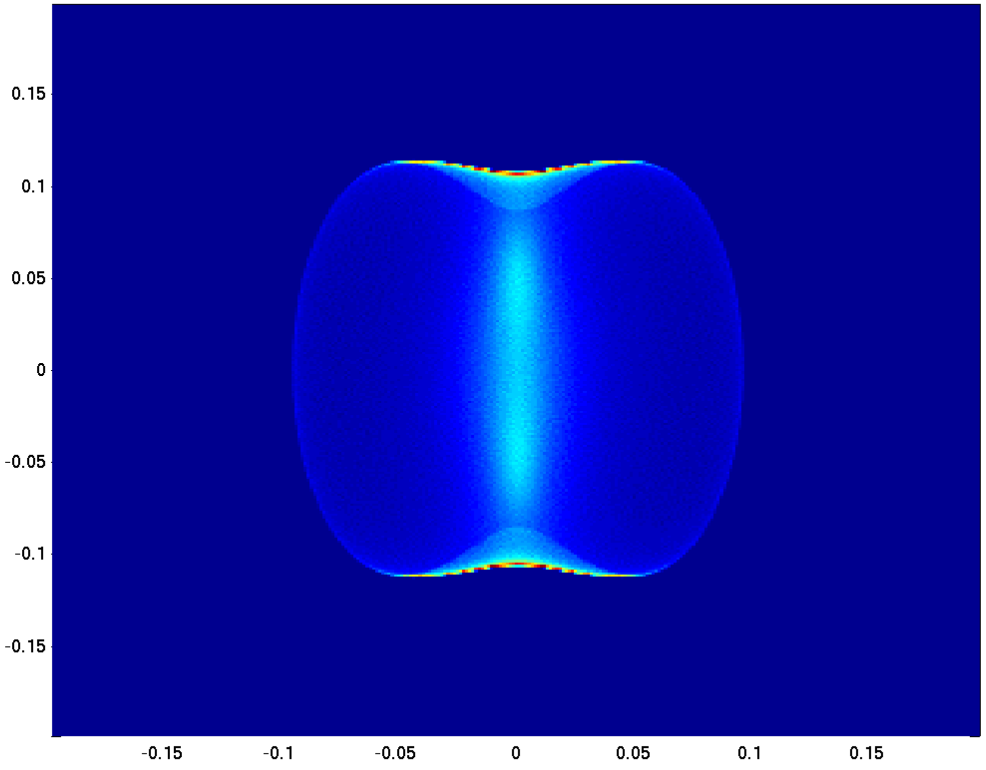
1µm pixel pitch

Wave-optical vs Geometrical-Optical PSF
Out-of-focus PSF

Wave-Optical PSF

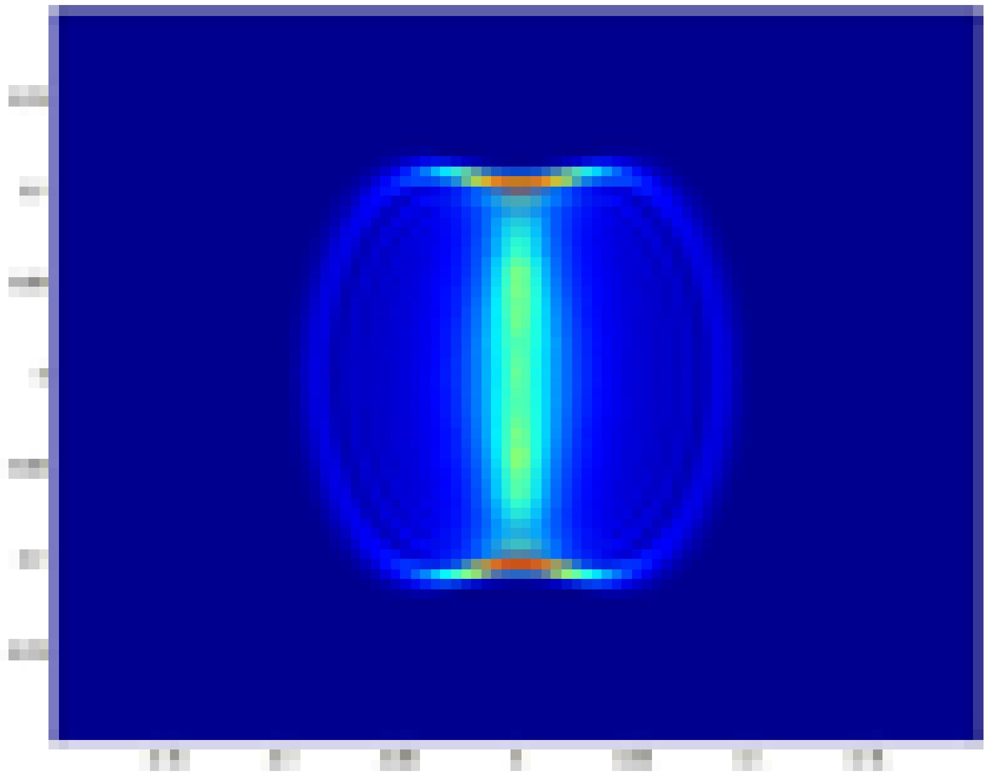


Geometric-Optical PSF

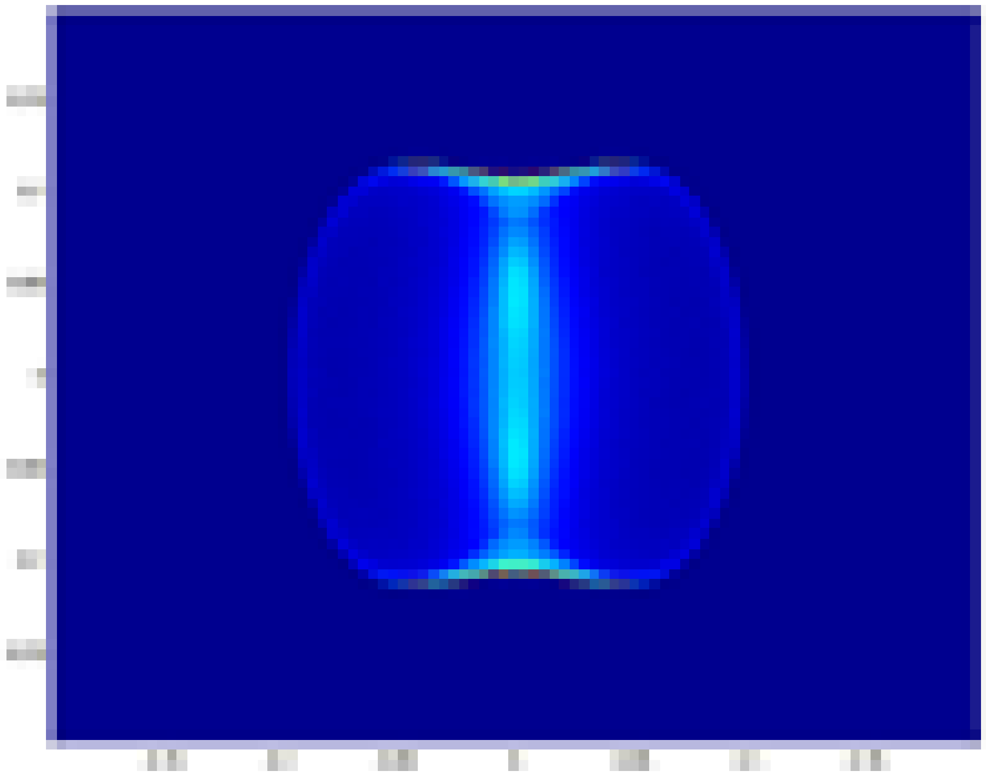


Wave-optical vs Geometrical-Optical PSF
Out-of-focus PSF, Pixel sampling

Wave-Optical PSF

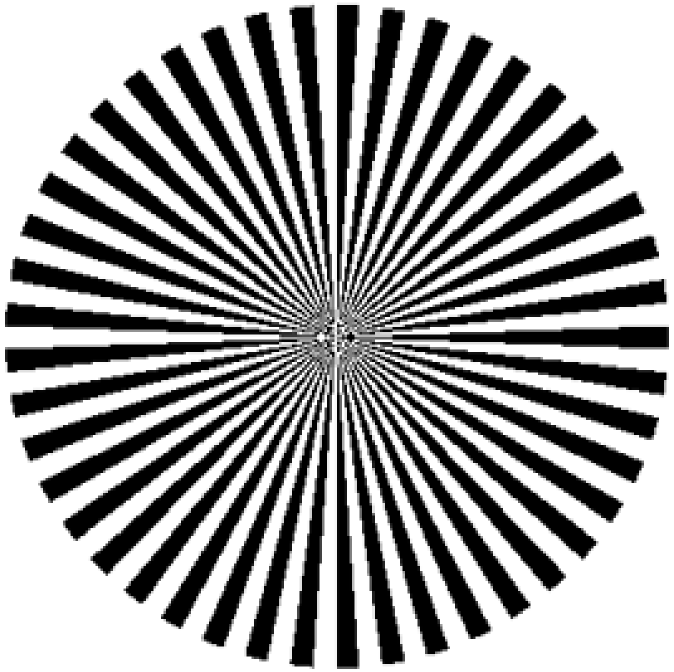


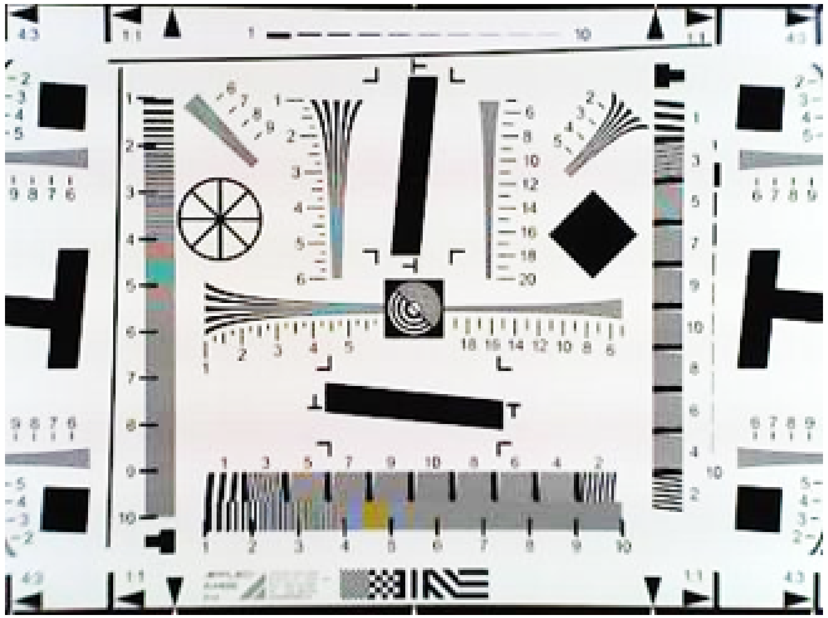
Geometric-Optical PSF

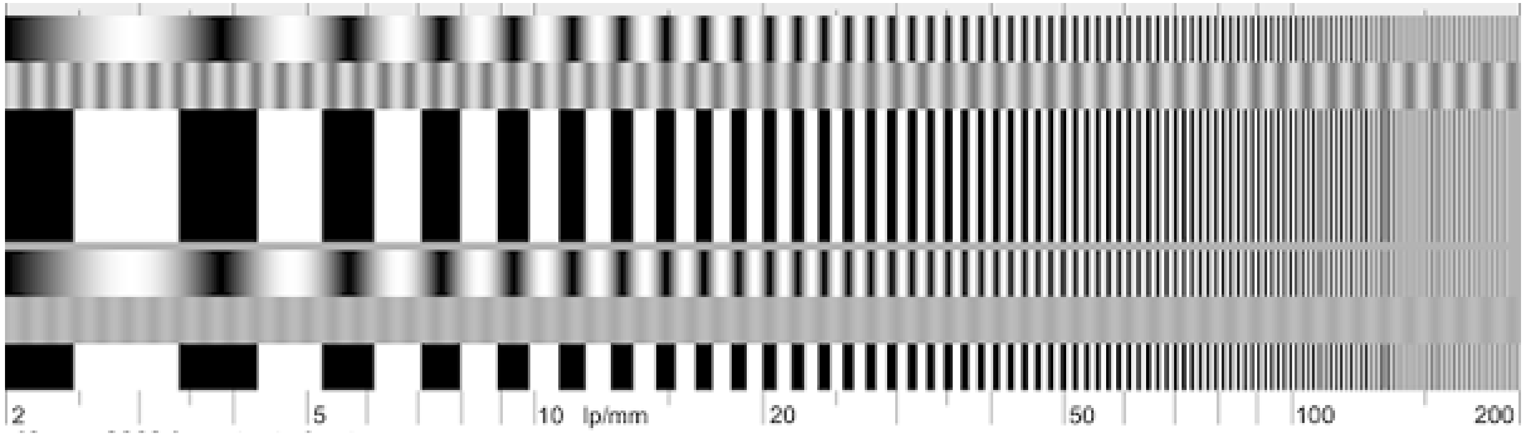


5µm pixel pitch

Test pattern for MTF evaluation.







Test pattern for MTF evaluation.

Non-coherent imaging of periodic object

For non-coherent imaging the Fourier transform of a 1D periodic intensity distribution is a "Dirac comb“ in the lens pupil: $FT_{\xi}\{I_{ob}\}(w\alpha) = \frac{1}{p}\sum_{j=-J}^{J} \tilde{I}_j\ \delta\left(w\alpha - \frac{j}{p}\right)$

and the Fourier coefficients $\tilde{I}_j$ are calculated with $\tilde{I}_j = \frac{1}{p}\int_{-p/2}^{p/2} d\xi\ I_{ob}(\xi)\, exp\left(-i2\pi\xi\frac{j}{p}\right)$. Normalizing those coefficients with $I_j = \frac{2\tilde{I}_j}{\tilde{I}_0}$ the non-coherent image intensity reads:

$$I(x) = \sum_{j=0}^{J} I_j\ MTF\left(\frac{j}{wp}\right) cos\left(2\pi\left[\frac{j}{p}x + \Phi\left(\frac{j}{wp}\right)\right]\right).$$

If only one diffraction order passes the pupil:

$$I(x) = 1 + I_1\ MTF\left(\frac{1}{wp}\right) cos\left(2\pi\left[\frac{x}{p} + \Phi\left(\frac{1}{wp}\right)\right]\right).$$

The maximum is located at $\frac{x_{max}}{p} + \Phi\left(\frac{1}{wp}\right) = 0$ and the minimum at $\frac{x_{min}}{p} + \Phi\left(\frac{1}{wp}\right) = 1$ with following values for the intensity $I\left(\frac{1}{wp}\right)_{max} = 1 + I_1\ MTF\left(\frac{1}{wp}\right)$, and $I\left(\frac{1}{wp}\right)_{min} = 1 - I_1\ MTF\left(\frac{1}{wp}\right)$ respectively. With that the **contrast** c, defined as follows is directly related with MTF:

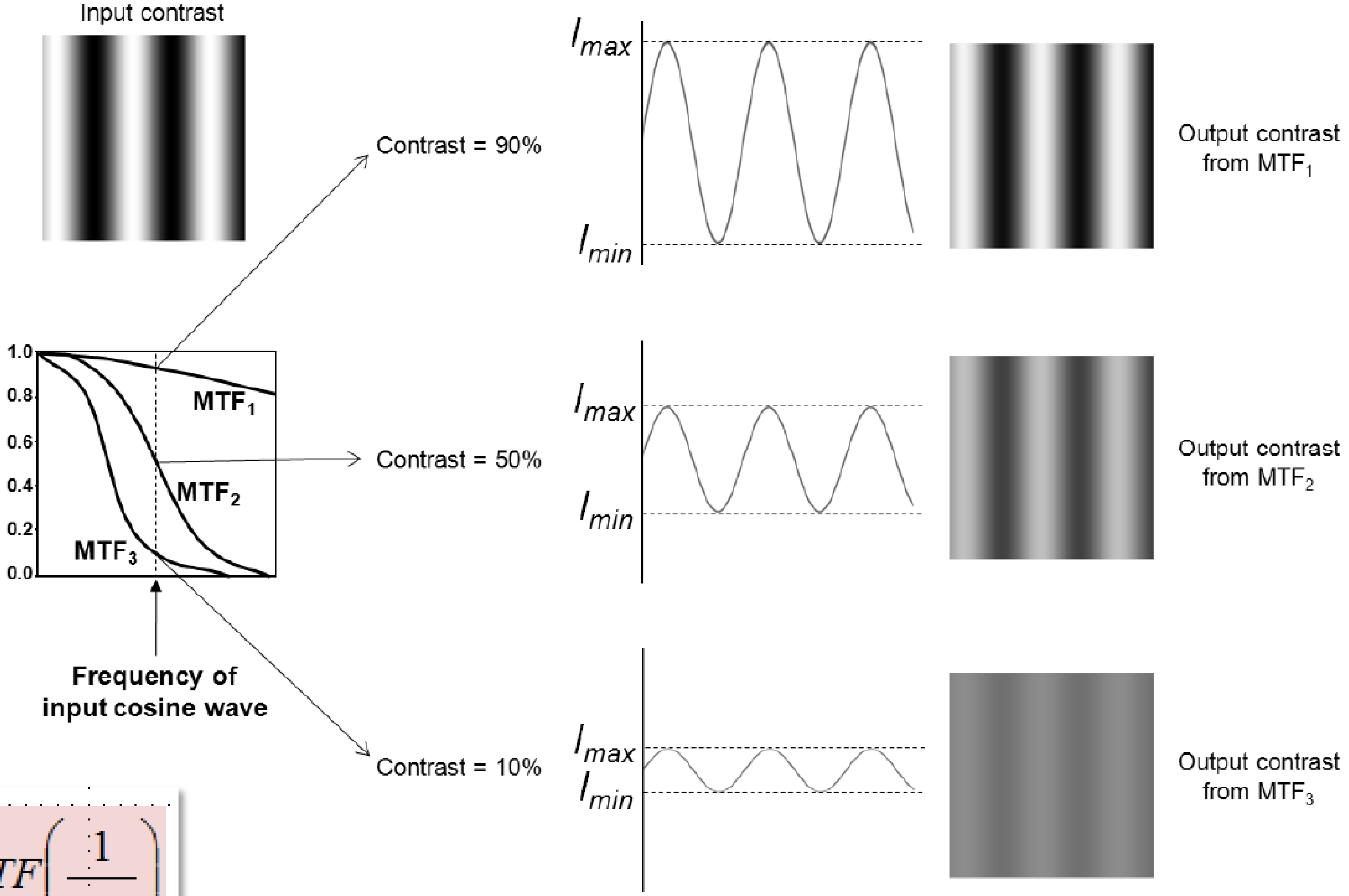
$$\boxed{c := \frac{I_{max}-I_{min}}{I_{max}+I_{min}} = MTF\left(\frac{1}{wp}\right)}$$

The MTF is therefore sometimes also called „contrast transfer function“.

It can be shown that for even aberrations only and an even periodic object <u>for a larger number of harmonics</u> the contrast is

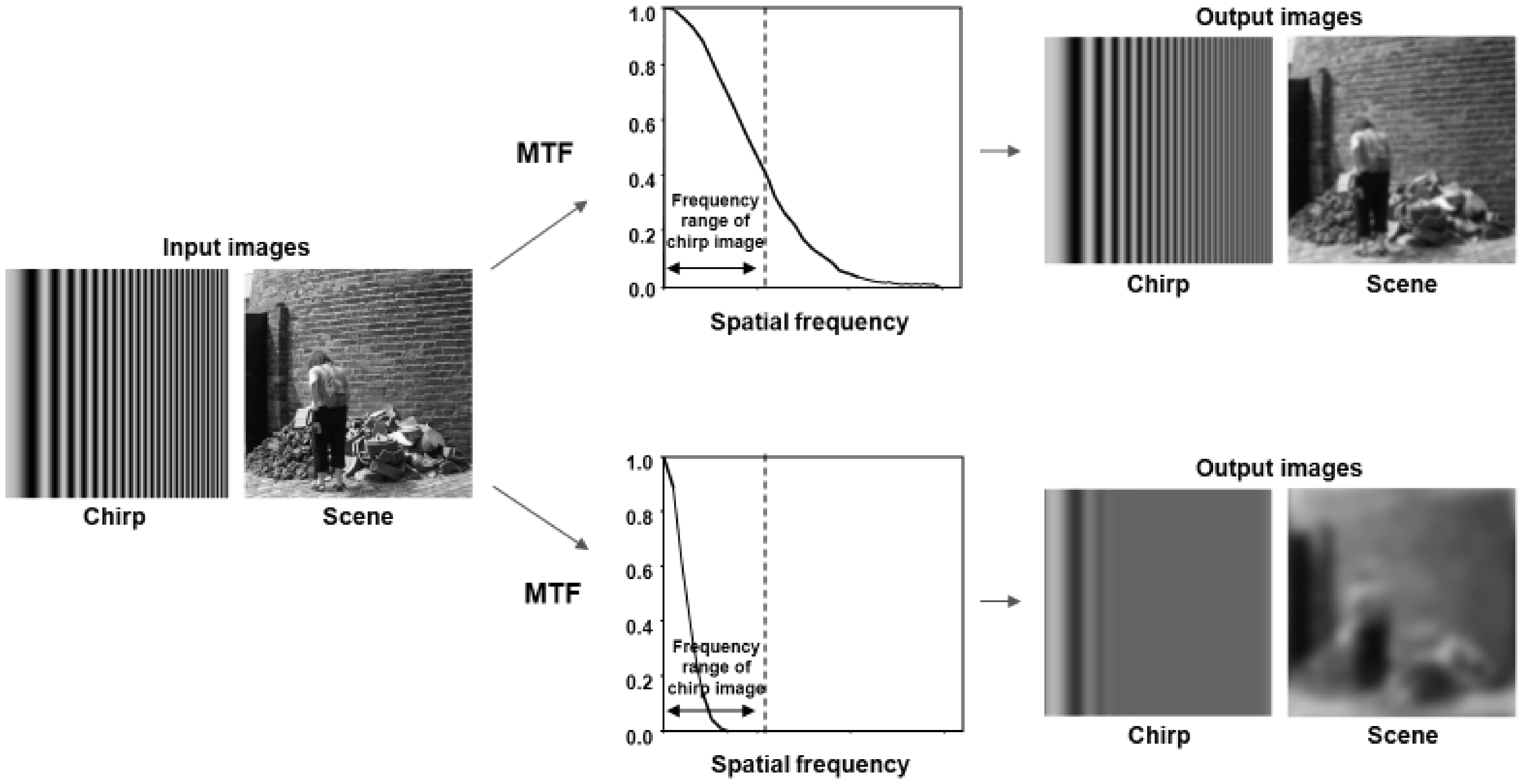
$$c := \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{\sum_{j=1}^{J} I_j MTF_j}{I_0 MTF_0}$$

Characterizing „blur“: MTF (modulation transfer function) and Contrast

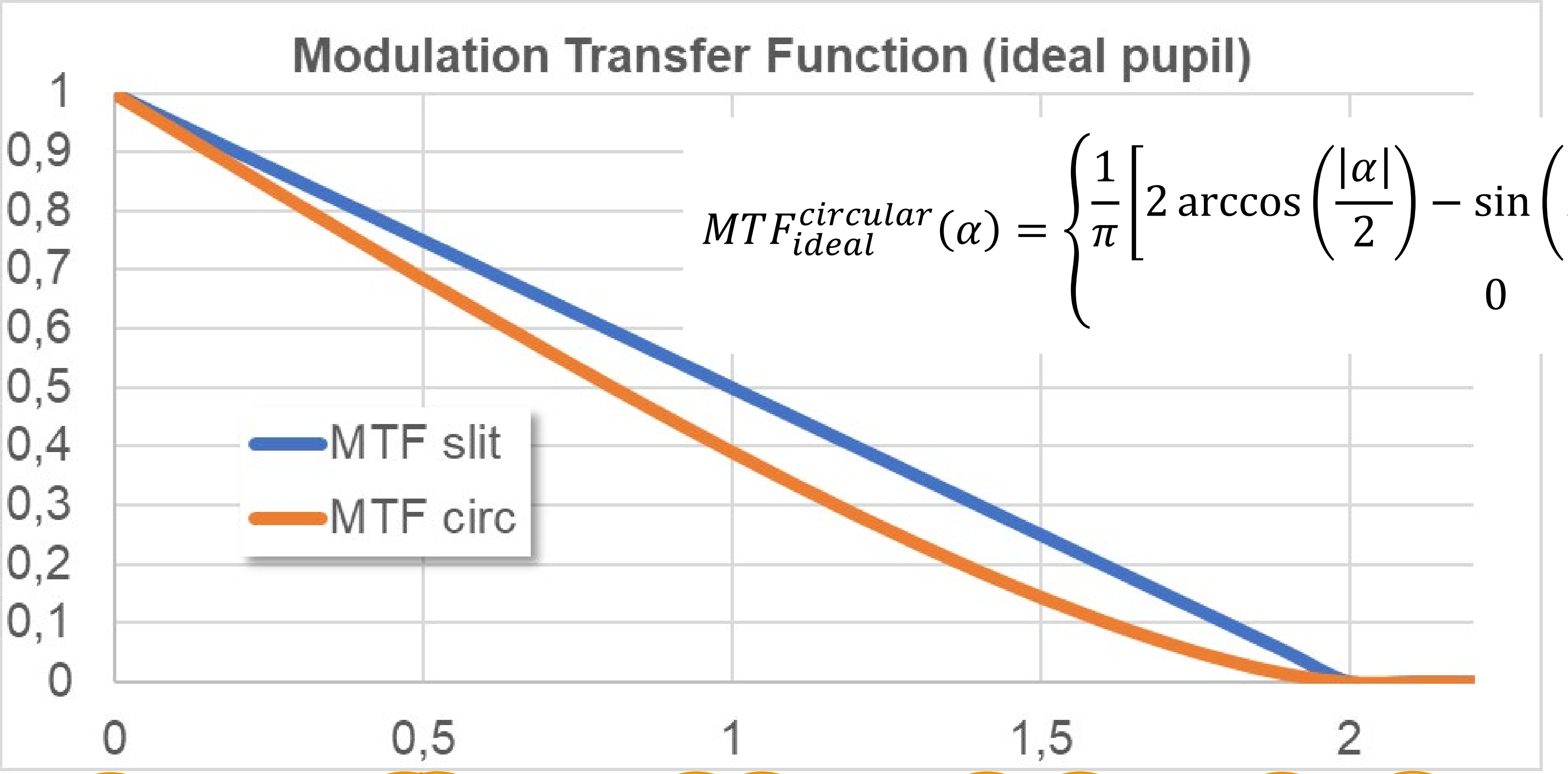


$$c := \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = MTF\left(\frac{1}{w\,p}\right)$$

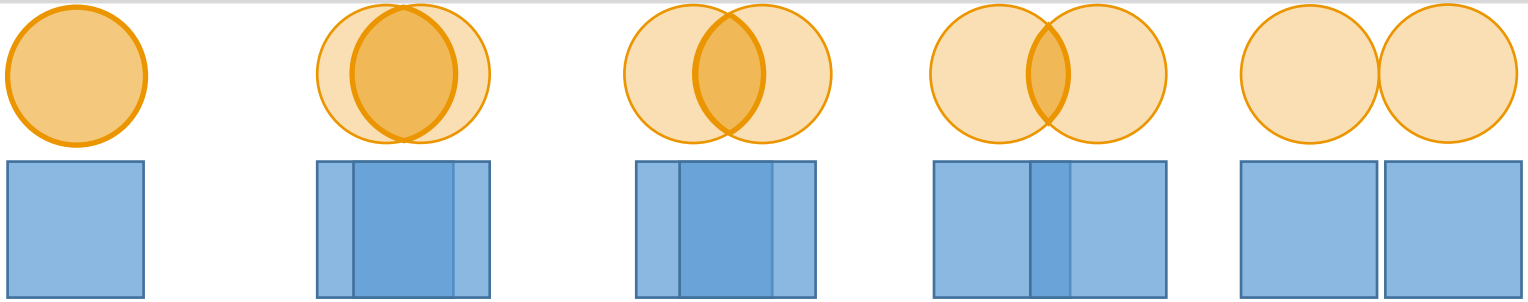
Imaging with different MTFs



MTF for circular pupil, free of aberration



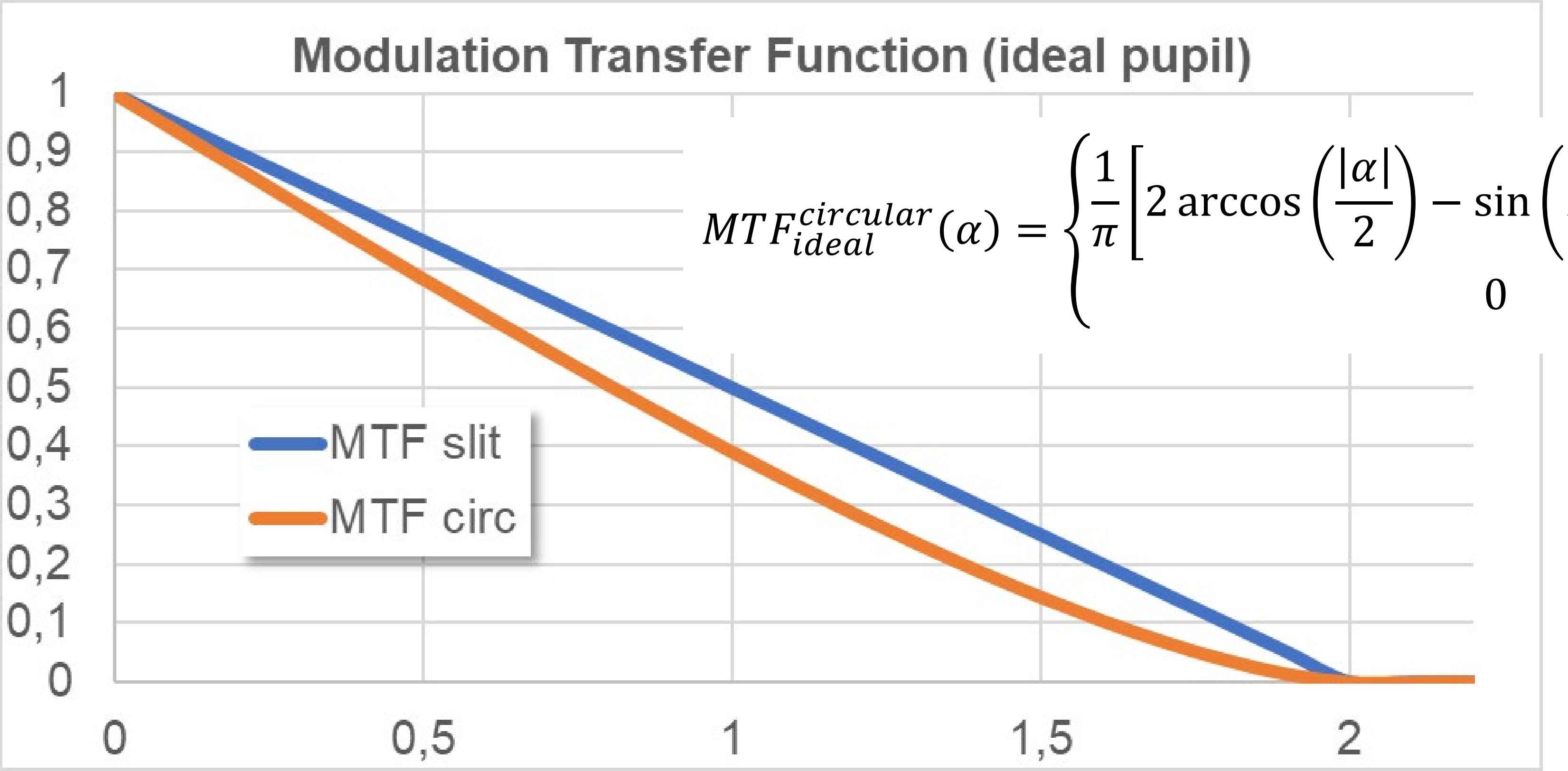
$$MTF_{ideal}^{circular}(\alpha) = \begin{cases} \dfrac{1}{\pi}\left[2\arccos\left(\dfrac{|\alpha|}{2}\right) - \sin\left(2\arccos\left(\dfrac{|\alpha|}{2}\right)\right)\right], & |\alpha| \leq 2\,NA'/\lambda\,; \\ 0 & |\alpha| > 2\,NA'/\lambda\,. \end{cases}$$



Limiting spatial frequency (λ=0.5µm):

$$\frac{2\,NA'}{\lambda} = \frac{1}{\lambda\,K} \approx \frac{2000}{K}\,\frac{lp}{mm}$$

MTF for circular pupil, free of aberration



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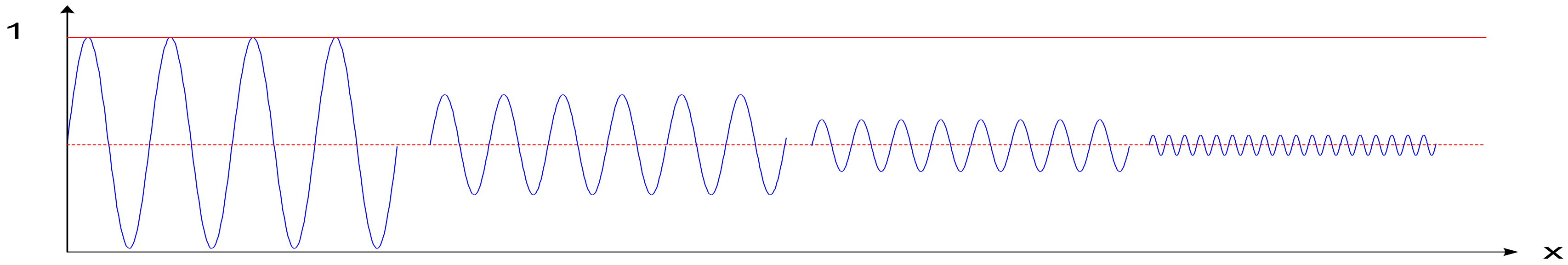
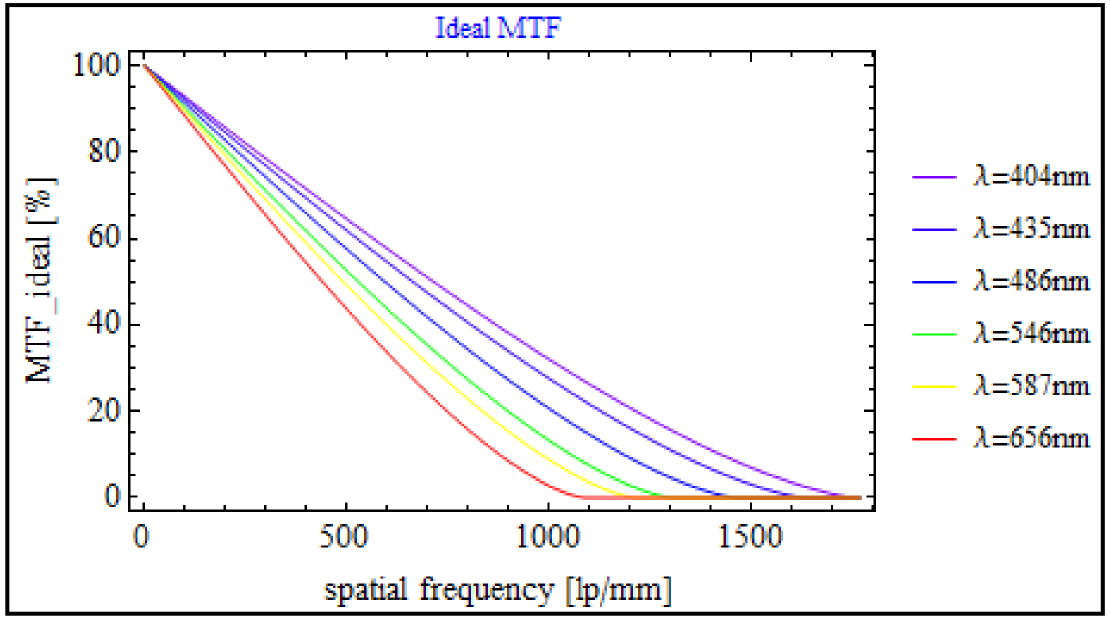
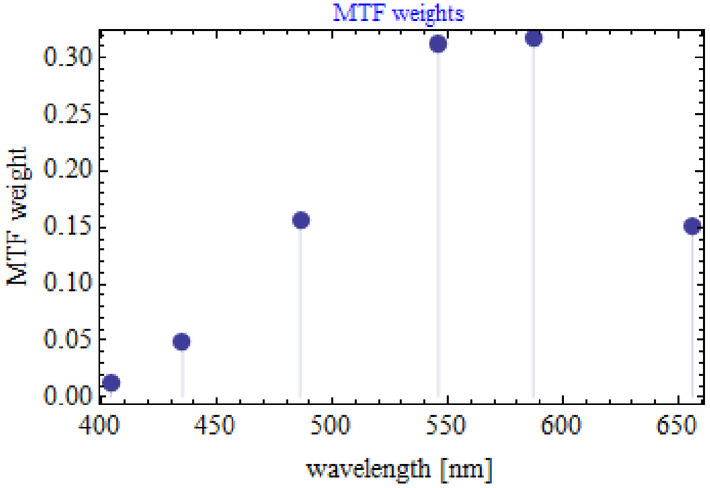
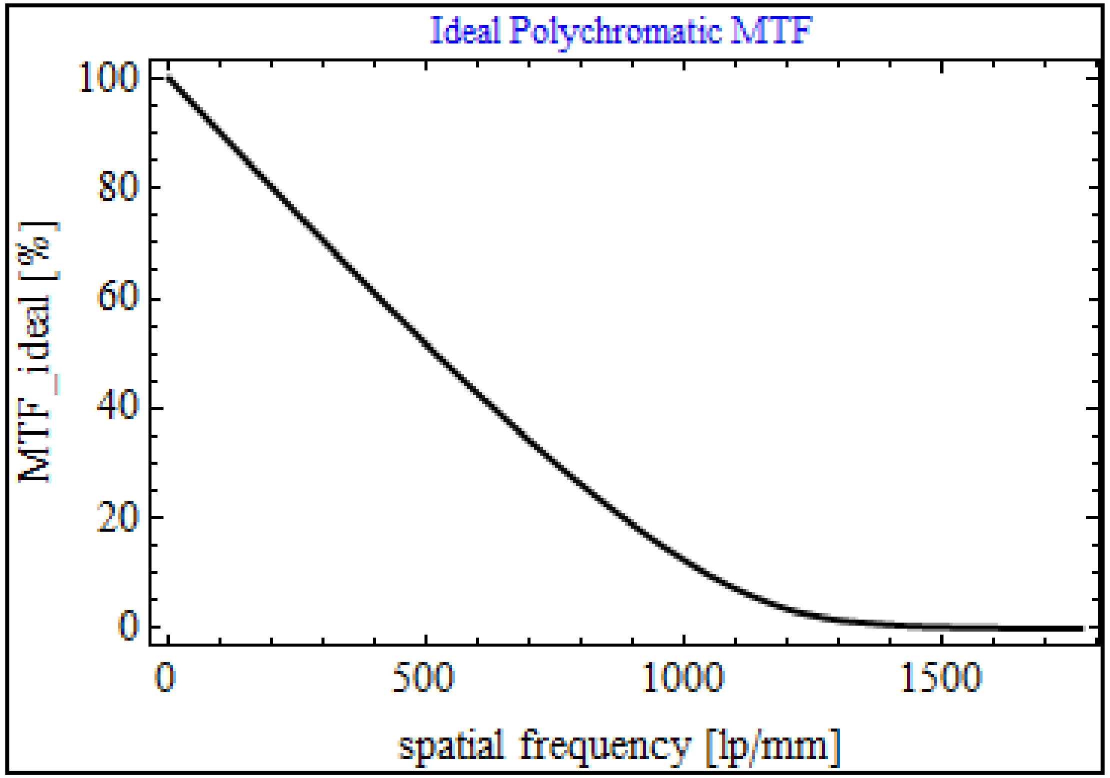


Ideal polychromatic MTF

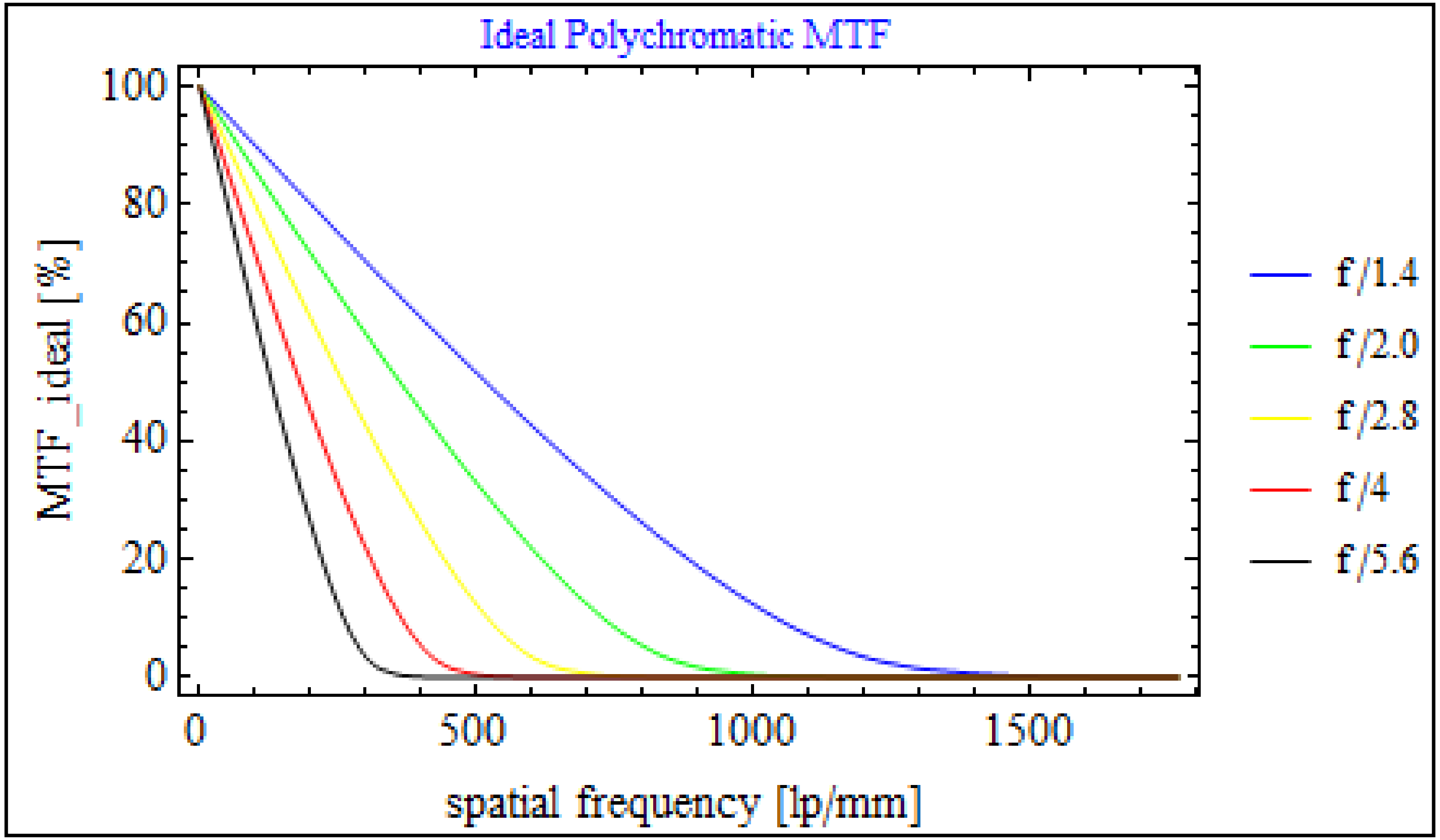


spectral weights





Ideal polychromatic MTF vs f-number



Pixel-Sampling (Pixel-Averaging)

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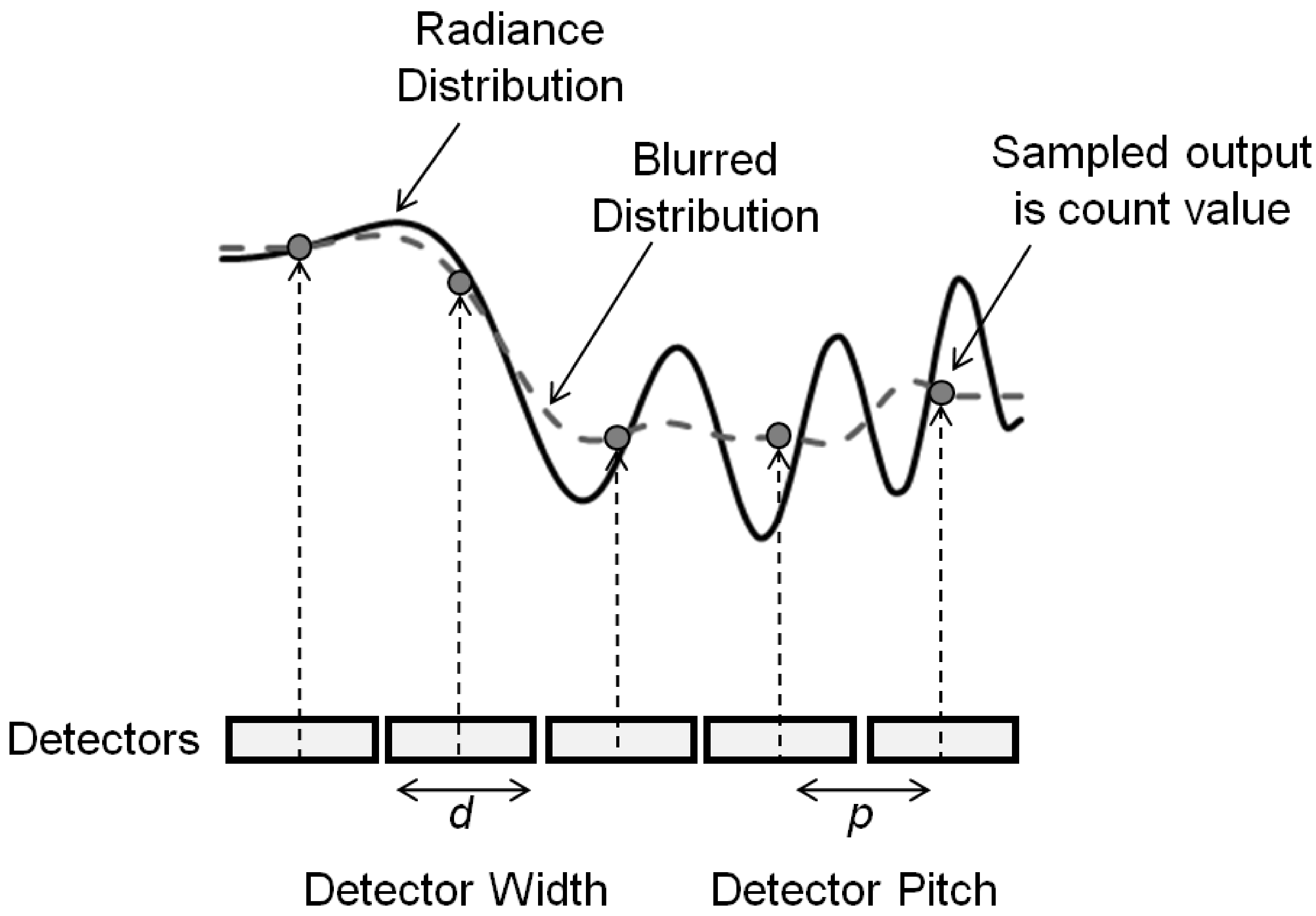
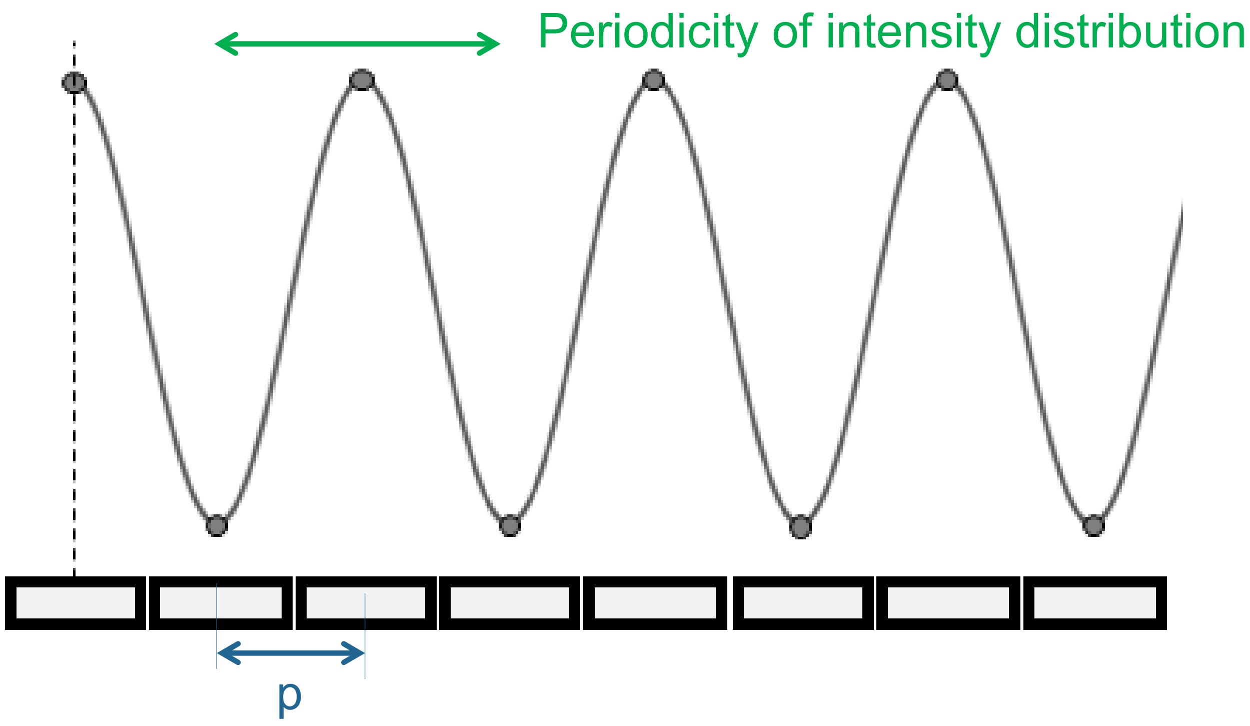


Nyquist-Frequency



Nyquist-frequency $$f_{Nyquist} = \frac{1}{2p}$$

„Sensor-Limit“
Higher spatial frequencies cause aliasing artefacts

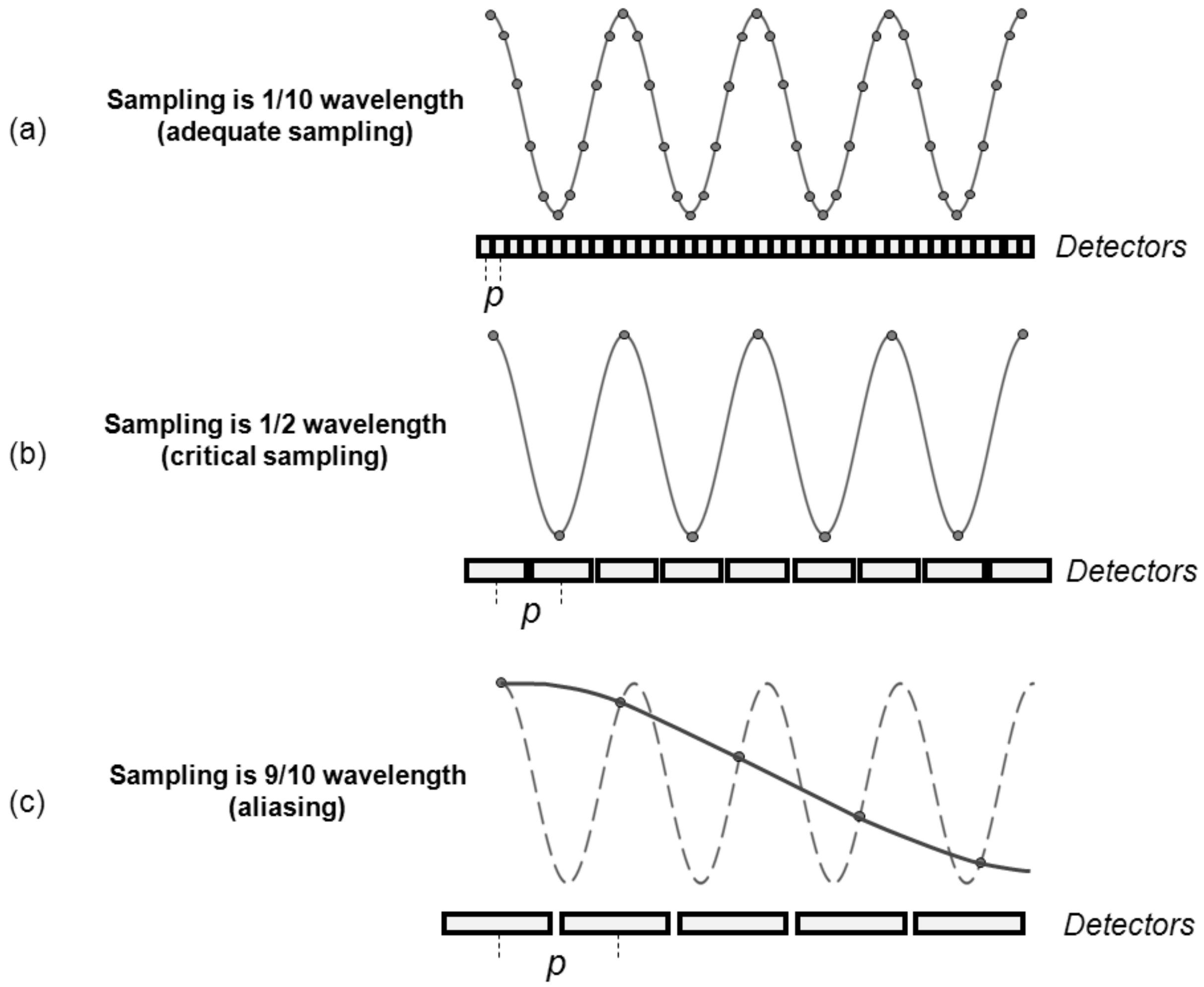
Fine structures and sensor-sampling

Sampling pitch = p

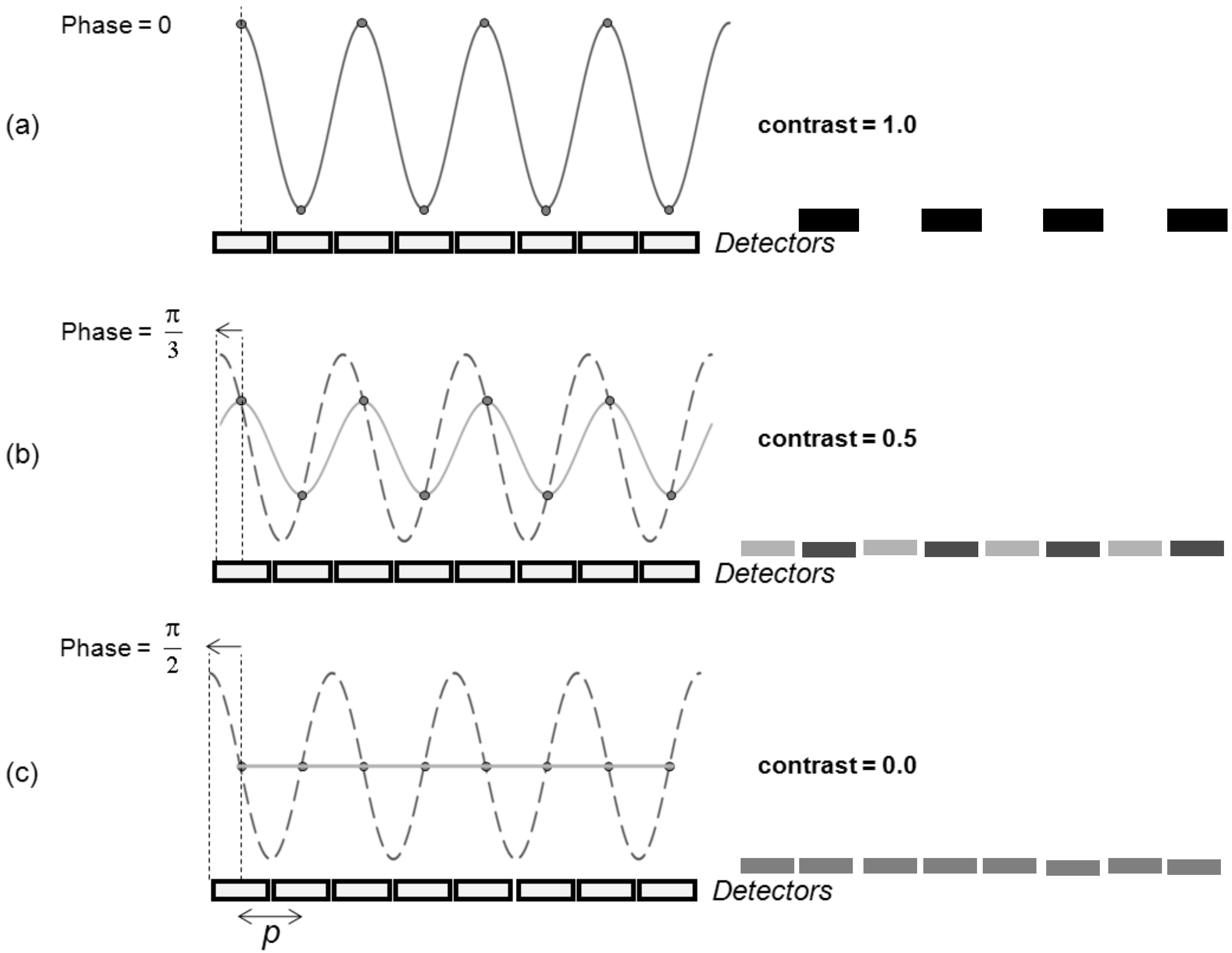
Sampling pitch = $4p$

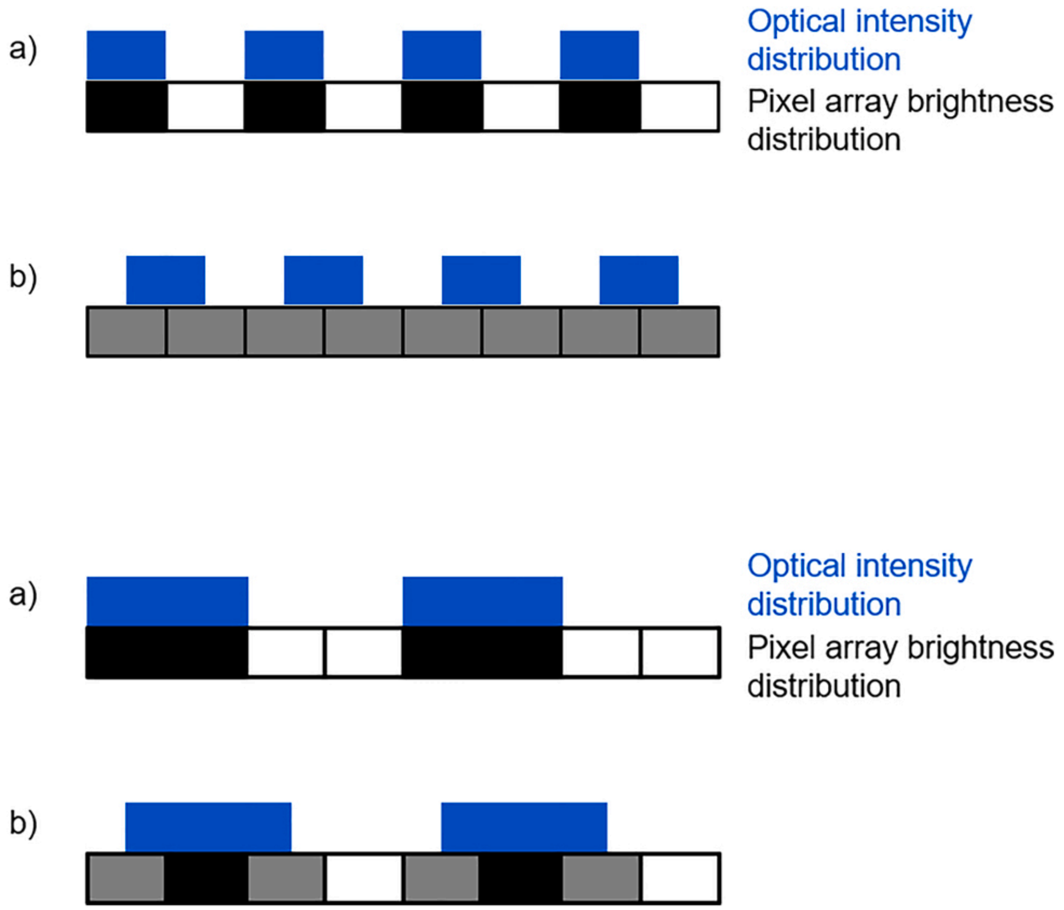
Sampling pitch = $6p$

Pixel-Sampling: critical and uncritical



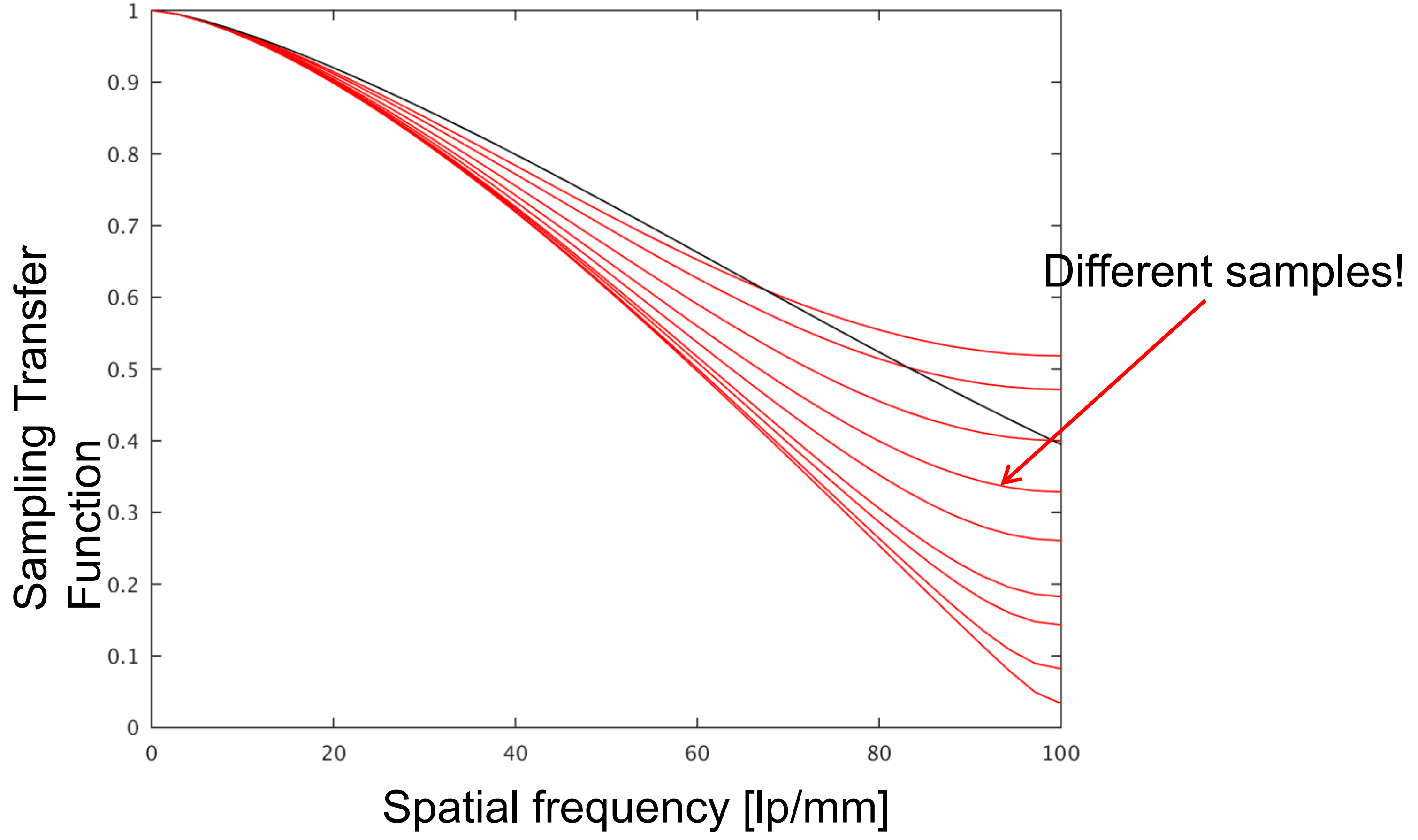
Imaging at Nyquist-Frequency





Effective Sampling-Transfer-Function(s):

Sensor-Sampling-Transfer-Function depends on the position of the image structure relative to the sensor!



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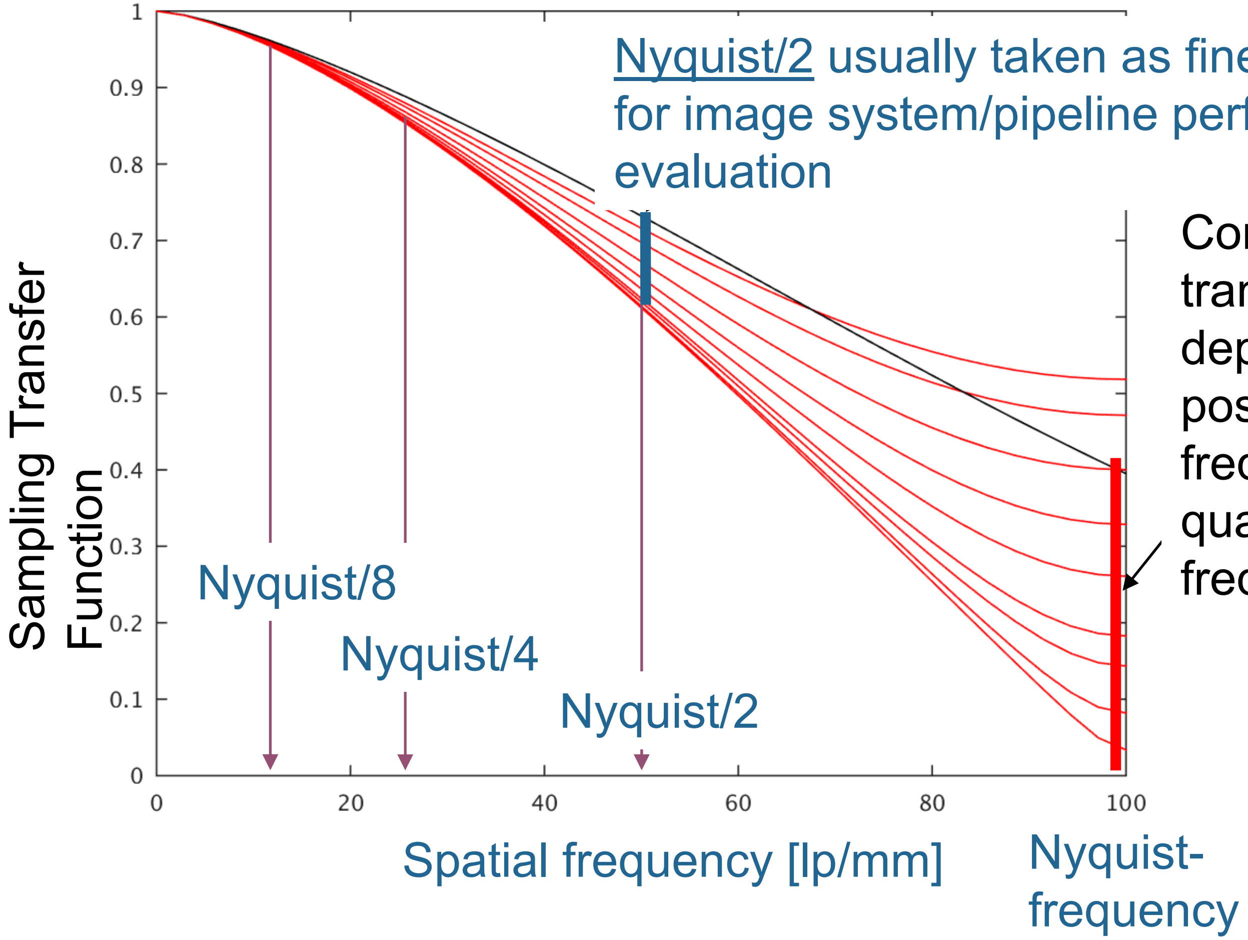

Contrast of pixel sampling transfer function is strongly dependant on structure positioning at Nyquist-frequency. Therefore a system qualification at Nyquist frequency is not useful.

Whittaker-Shannon Sampling Theorem

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In numerical integration, the variable range at (equidistant points) is discretized and their function values are added together with the sampling intervals (integration limits$[-\infty, +\infty]$):

$$\int f(x)\,dx = \Delta x \sum_n f(n\,\Delta x) = \Delta x \int f(x) \sum_n \delta(x - n\,\Delta x)\,dx$$

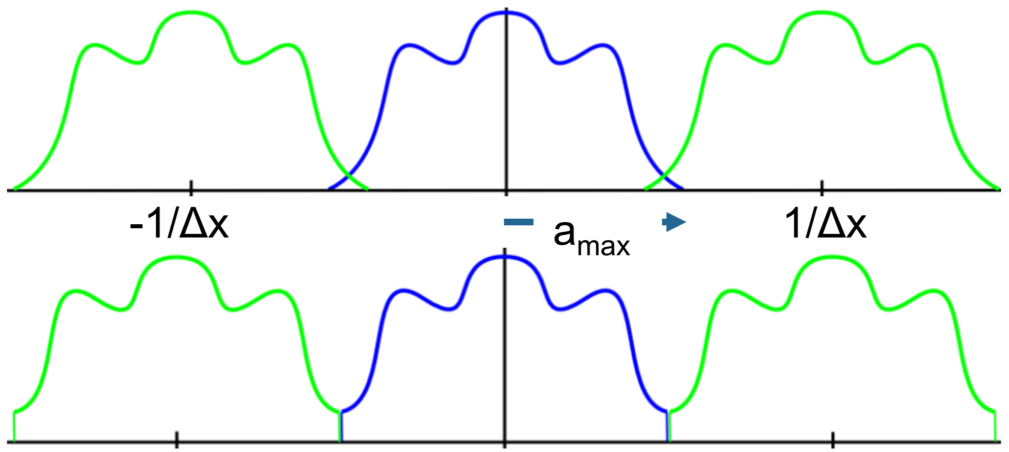
The Fourier transform of the discretized function, formally written in the form $f(x) = \Delta x\, f(x) \sum_n \delta(x - n\,\Delta x)$, is obtained with the convolution theorem to

$$FT_x\{f(x)\}(a) = \Delta x\, FT_x\left\{f(x) \sum_n \delta(x - n\,\Delta x)\right\}(a) = \Delta x \int FT_x\{f(x)\}(a')\, FT_x\left\{\sum_n \delta(x - n\,\Delta x)\right\}(a' - a)da'$$

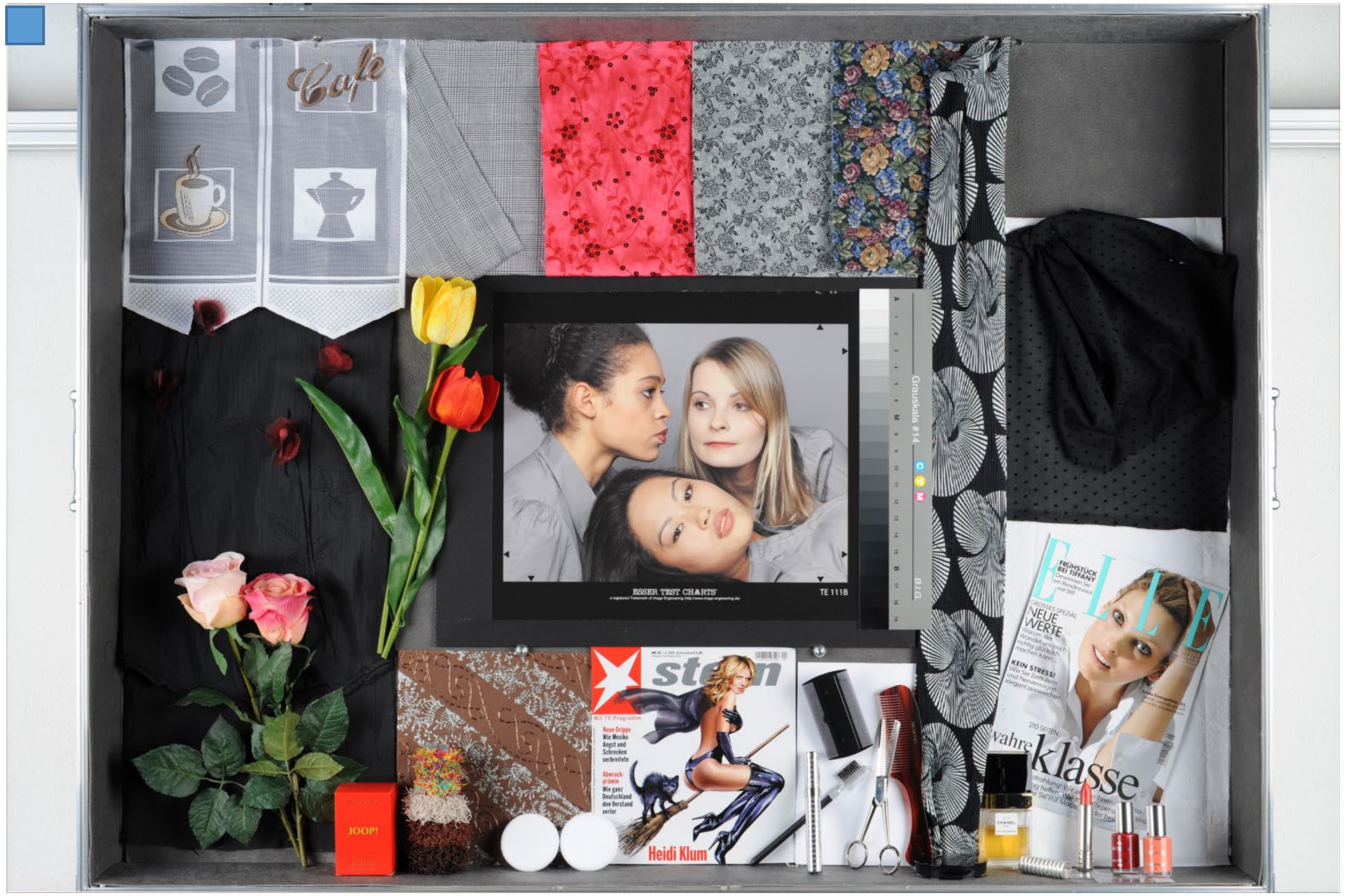
$$= \Delta x \int FT_x\{f(x)\}(a') \frac{1}{\Delta x} \sum_m \delta\left(a' - a + \frac{m}{\Delta x}\right) da' = \sum_m FT_x\{f(x)\}\left(a - \frac{m}{\Delta x}\right)$$

By the equidistant sampling of the Fourier integral, $FT_x\{f\}$ repeats at distances $1/\Delta x$.

Whittaker-Shannon sampling theorem: If $FT_x\{f\}(a)$ band-limited with a value a_{max}, the condition $\Delta x \leq \frac{1}{2a_{max}}$ assures that adjacent $FT_x\{f\}\left(a - \frac{m}{\Delta x}\right)$ do not overlap.

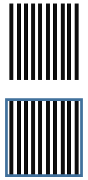


1mm x 1mm

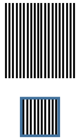


24mm

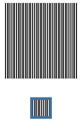
36mm



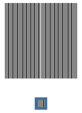
5 lp/mm



10 lp/mm

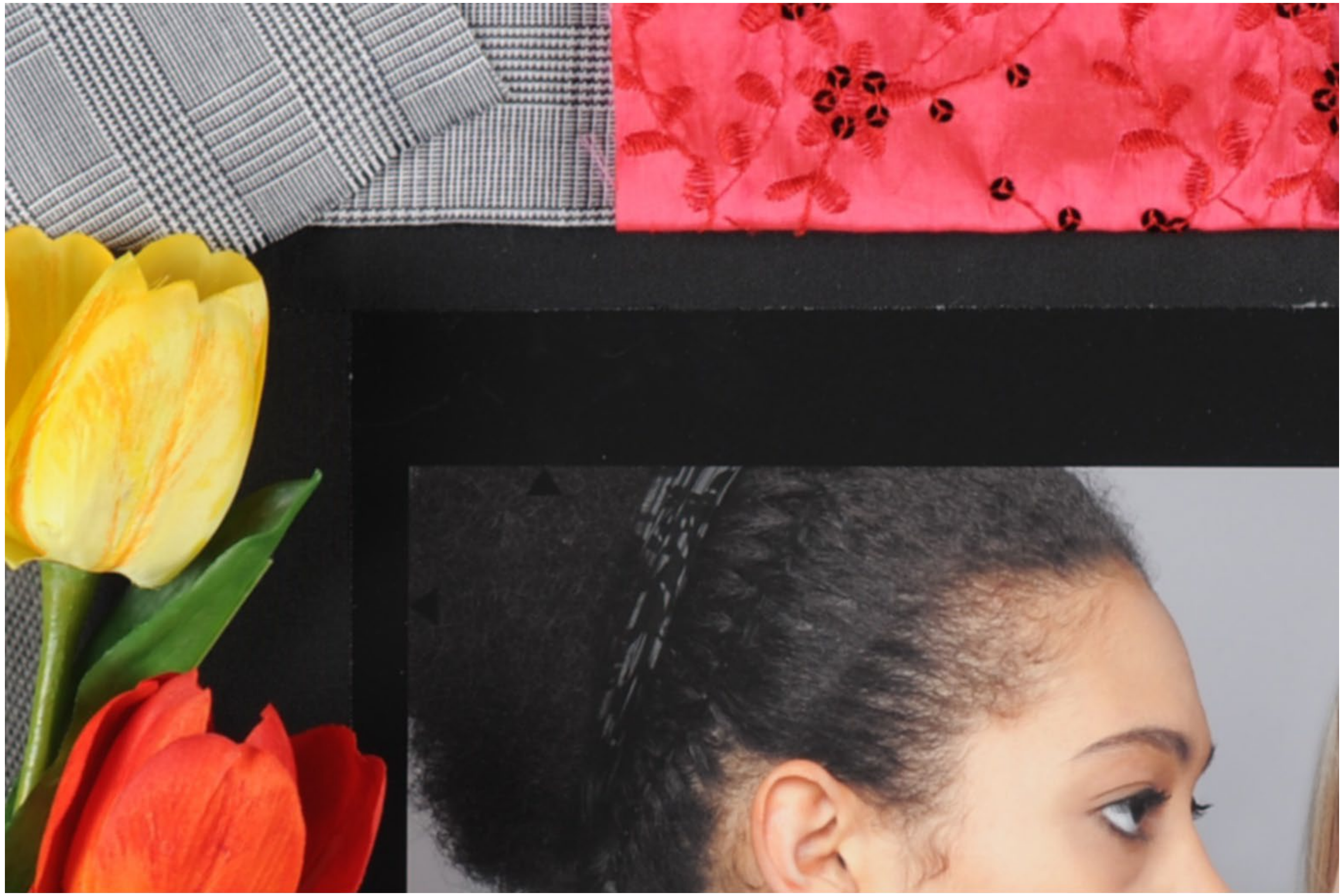


20 lp/mm



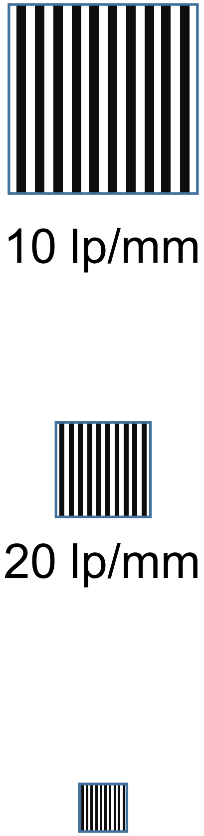
40 lp/mm

20% crop



4.8mm

7.2mm



10 lp/mm

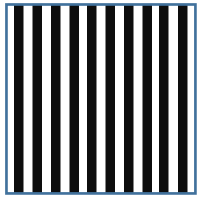
20 lp/mm

40 lp/mm

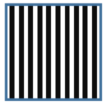
10% crop

2.4mm

3.6mm



20 lp/mm



40 lp/mm

Ideal polychromatic MTF vs f-number

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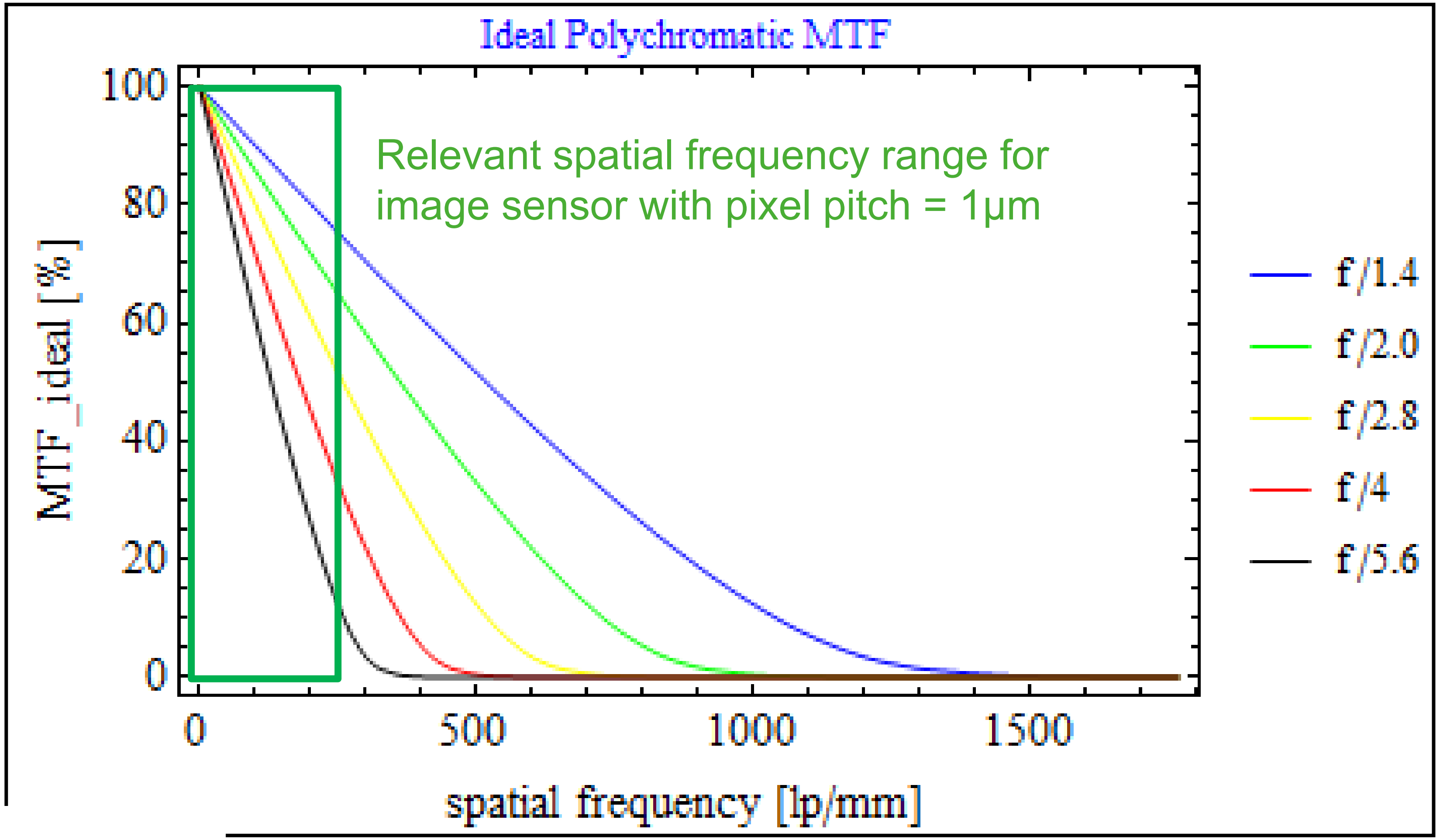
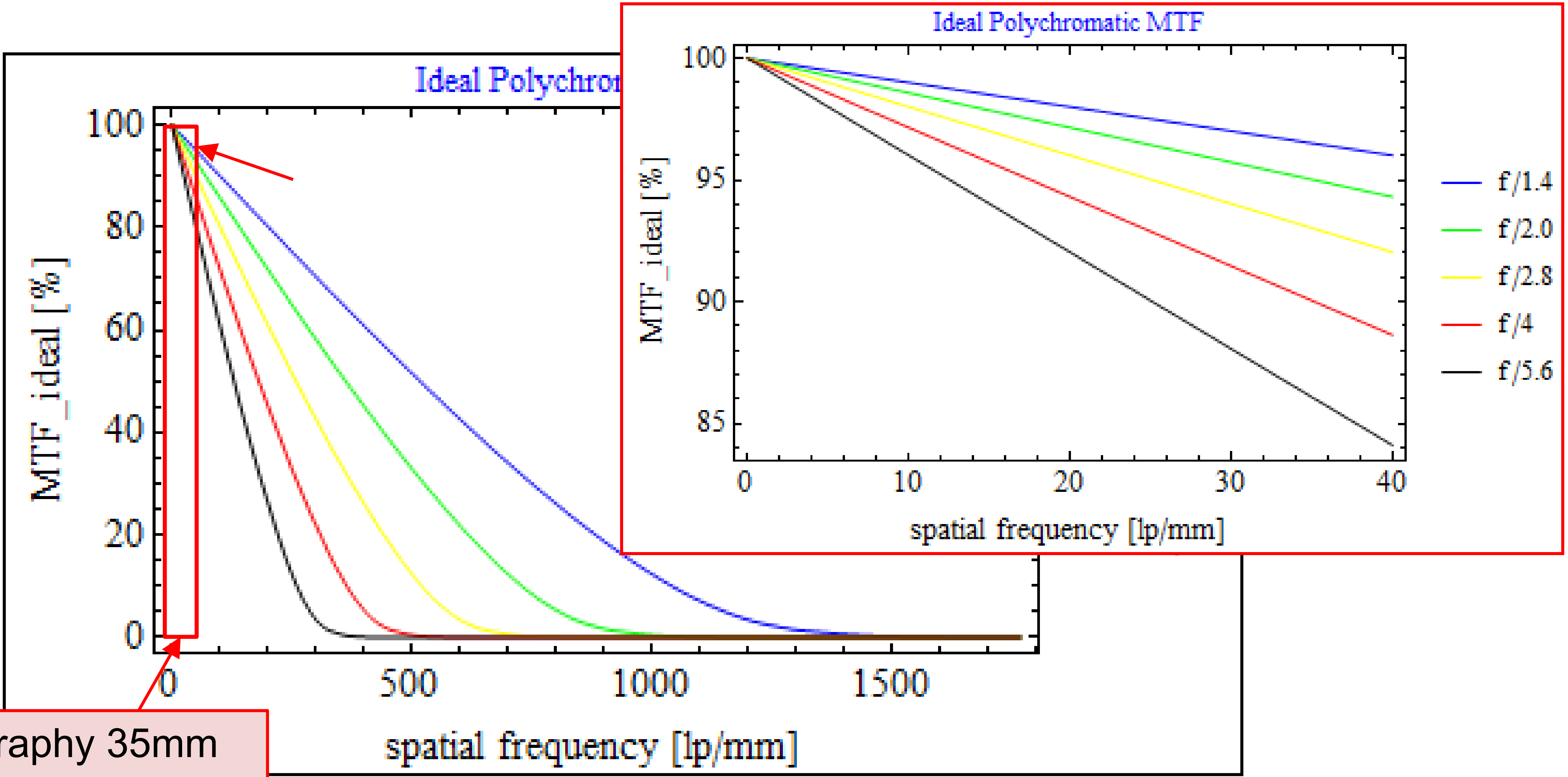


Pixel pitch p=**1µm**
Nyquist frequency=500lp/mm

Ideal polychromatic MTF vs f-number



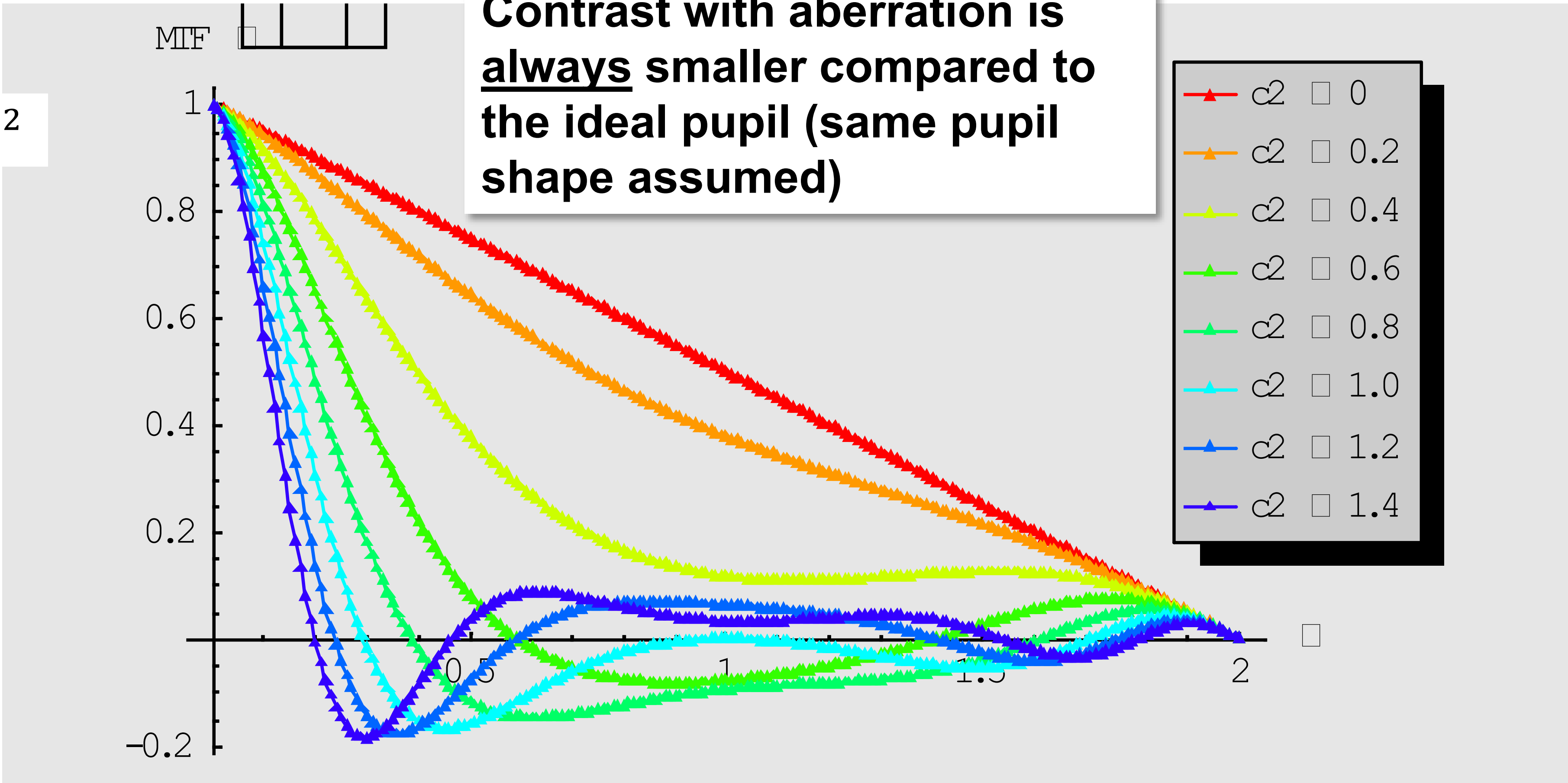
film & photography 35mm format relevant

Defocused MTF

$$W(\alpha) = c_2 \alpha^2$$

c_2 in waves

Contrast with aberration is <u>always</u> smaller compared to the ideal pupil (same pupil shape assumed)



MTF with aberration is always below ideal MTF curve.
Nonlinear relationship between MTF and wavefront aberration

MTF with pupil „apodization“ (Transmission varies over pupil)

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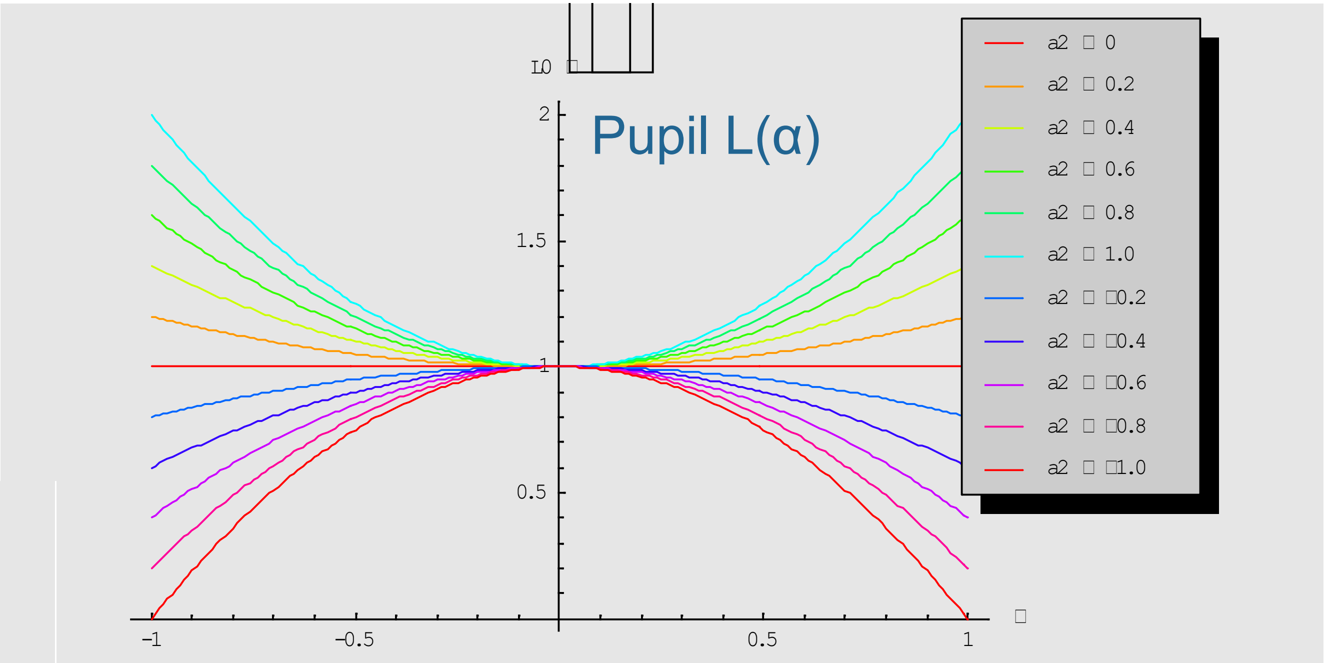
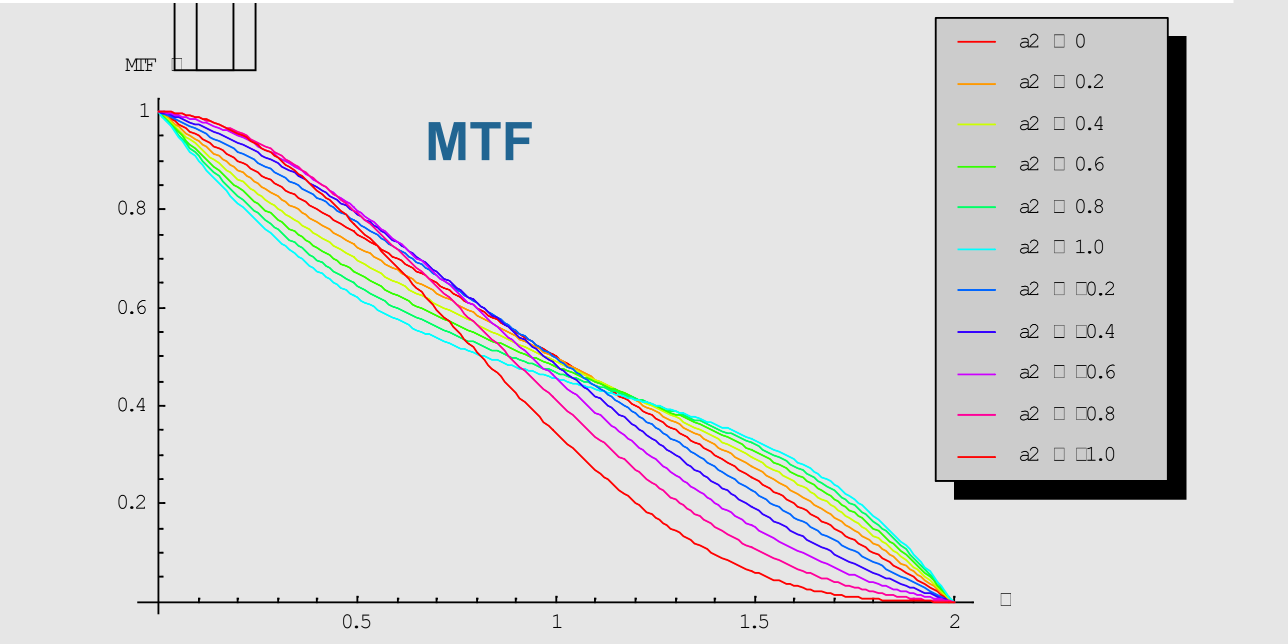
Pupil L(α) variation in transmission with α:





A transmission falling towards the edge of the pupil causes a contrast increase at low spatial frequencies and a contrast decrease at high spatial frequencies.

Properties of the Optical Transfer Function (OTF)

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For practical calculations it is convenient to shift the coordinates: $\alpha' = \bar{\alpha} - \frac{\alpha}{2}$, and have symmetrical arguments with respect to α': $OTF(\alpha) = \int d\alpha' \; L\left(\alpha' - \frac{\alpha}{2}\right) \; L^*\left(\alpha' + \frac{\alpha}{2}\right)$

and a symmetrical integration range: $\alpha' = (\alpha'_x, \alpha'_y) \in \left(\left[-1 + \frac{\alpha_x}{2}, 1 - \frac{\alpha_x}{2}\right], \left[-1 + \frac{\alpha_y}{2}, 1 - \frac{\alpha_y}{2}\right]\right)$.

Following properties of OTF can be derived from its definition:

1. $OTF(0)$ is a real and positive number
2. $OTF(\alpha) = OTF^*(-\alpha)$
3. $|OTF(\alpha)| \leq OTF(0)$
4. if the pupil function L is real, then OTF is also real
5. for a symmetrical pupil function, $L(\alpha) = L(-\alpha)$, the OTF is real

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Proofs:

1. with $OTF(0) = \int d\bar{\alpha} \; |L(\bar{\alpha})|^2$, since the integrand is real and positive.

2. $OTF^*(-\alpha) = \left(\int d\alpha' \; L(\alpha' - \alpha) \; L^*(\alpha')\right)^* = \int d\alpha' \; L(\alpha') \; L^*(\alpha' - \alpha)$

$$\overset{\bar{\alpha} := \alpha' - \alpha}{=} \int d\bar{\alpha} \; L(\bar{\alpha} + \alpha) \; L^*(\bar{\alpha}) = OTF(\alpha)$$

3. with Schwartz inequality $\left(\int d\alpha \; L(\alpha) \; M^*(\alpha)\right)^2 \leq \int d\alpha \; |L(\alpha)|^2 \cdot \int d\alpha \; |M(\alpha)|^2$ with replacement $M(\alpha) \rightarrow L(\alpha + \alpha')$.

4. directly with definition of OTF: integral of real function is real.

5. imaginary part is $OTF^{(i)}(\alpha) = \int d\bar{\alpha} \left[L^{(i)}\left(\bar{\alpha} - \frac{\alpha}{2}\right) L^{(r)}\left(\bar{\alpha} + \frac{\alpha}{2}\right) - L^{(r)}\left(\bar{\alpha} - \frac{\alpha}{2}\right) L^{(i)}\left(\bar{\alpha} + \frac{\alpha}{2}\right)\right]$. For a symmetrical pupil $L(\alpha) = L(-\alpha)$ this becomes

$$\frac{1}{2}\int d\bar{\alpha} \left[L^{(i)}\left(\bar{\alpha} - \frac{\alpha}{2}\right) L^{(r)}\left(\bar{\alpha} + \frac{\alpha}{2}\right) - L^{(r)}\left(\bar{\alpha} - \frac{\alpha}{2}\right) L^{(i)}\left(\bar{\alpha} + \frac{\alpha}{2}\right)\right] +$$
$$+\frac{1}{2}\int d\bar{\alpha} \left[L^{(i)}\left(-\bar{\alpha} - \frac{\alpha}{2}\right) L^{(r)}\left(-\bar{\alpha} - \frac{\alpha}{2}\right) - L^{(r)}\left(-\bar{\alpha} + \frac{\alpha}{2}\right) L^{(i)}\left(-\bar{\alpha} - \frac{\alpha}{2}\right)\right],$$

Substituting $\alpha' = -\bar{\alpha}$ in second integral makes obvious, that the summands just compensate, and imaginary part vanishes, $OTF^{(i)}(\alpha) = 0$.

Properties of the Optical Transfer Function (OTF)

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4. if the pupil function L is real, than OTF is also real
5. for a symmetrical pupil function $L(\alpha) = L(-\alpha)$ the OTF is real

Re 4: Consequently, apodization, i.e. a transmission variation across the pupil in any case, i.e. also for asymmetrical variations, does not cause any distortion or asymmetries of the imaged structures, but leads to altered weights of the transmitted harmonic orders. However, asymmetric apodization leads to an image offset when passing through the focus, i.e. a telecentric error.

Re 5: This means that a focus error or spherical aberration (symmetric pupil) does not cause an image shift and the effect is fully characterized in the MTF. This is not the case with coma, where the asymmetric pupil phase leads to both a degradation of the MTF and an image shift for any particular harmonic order.

Modulation Transfer Function (MTF)

This property is used to conveniently reference the transfer of spatial frequencies to an ideal lens by defining the modulation transfer function (MTF) as the normalised OTF with respect to its absolute value:

$$MTF(\alpha) = \frac{|OTF(\alpha)|}{OTF(0)}.$$

The MTF value for the local frequency value "zero" is always equal to one: $MTF(0) = 1.$

The phase value Φ (units of wavelength) of the OTF in

$$MTF(\alpha)\, exp\big(i2\pi\Phi(\alpha)\big) = \frac{OTF(\alpha)}{OTF(0)}$$

is referred to as a phase transfer function (PTF).

In the case of gratings, the phase determines shifts in the harmonic orders of the image (see below). The phase part is also relevant for image generation, although it is usually omitted. This is partly justified, since image shifts due to wavefront tilt are included in the distortion and are represented as such: they have no effect on the MTF. However, higher-order odd aberrations such as coma affect both MTF and distortion.

It follows directly from the definition that the MTF is symmetrical and the PTF is antisymmetrical:

$$MTF(\alpha) = MTF(-\alpha),$$

$$\Phi(\alpha) = -\Phi(-\alpha).$$

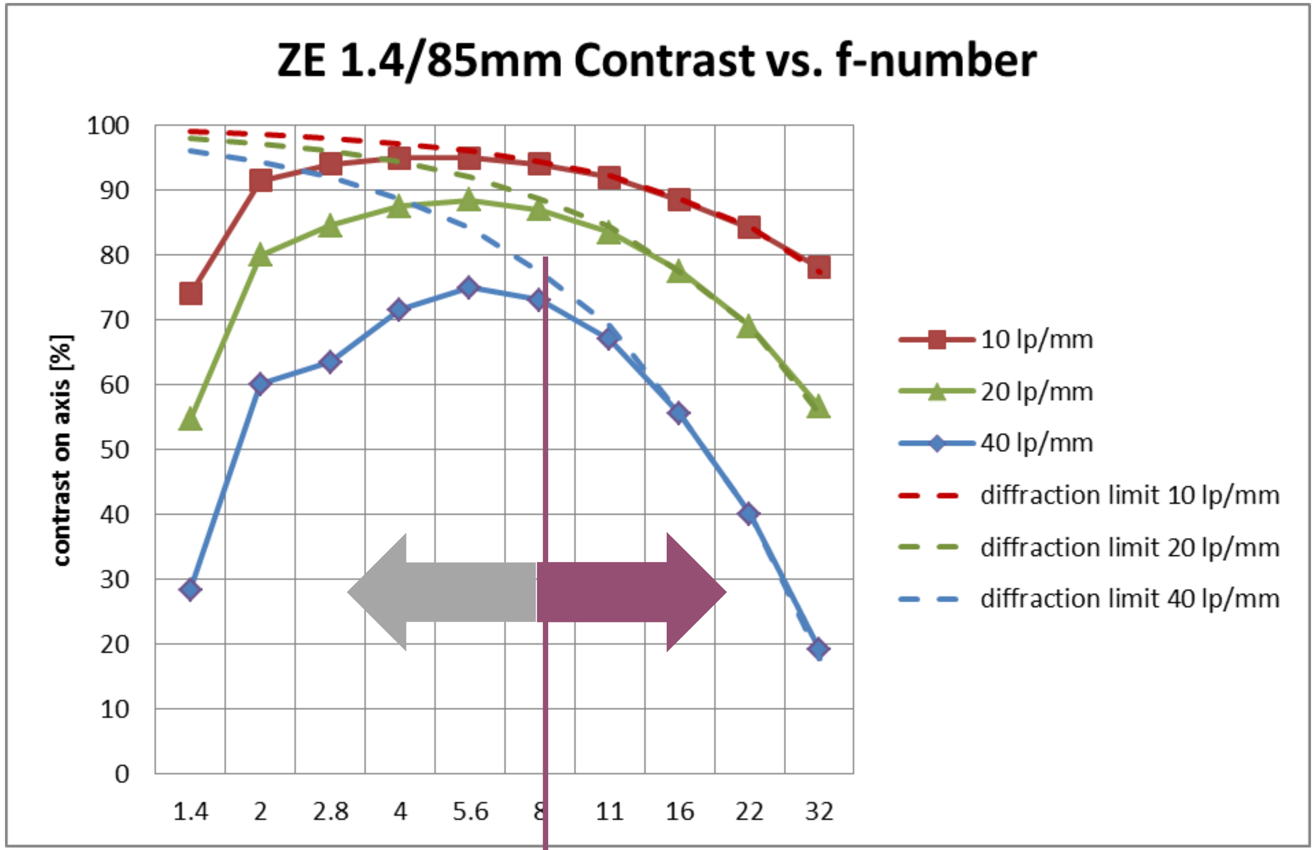
From the anti-symmetry of the phase function follows: $\Phi(0) = 0.$

Thus, it is sufficient to represent only one branch of the transfer function, in normalized coordinates$0 \leq \alpha \leq 2$ or in spatial frequency coordinates in units [lp/mm]:

$$0 \leq lp/mm \leq 2\frac{\lambda}{NA}.$$

Since we theoretically and in many measurements and the representation of performance data look at periodic structures and here at certain discrete structural periods, it is useful to explicitly write down the mapping equations for this case.

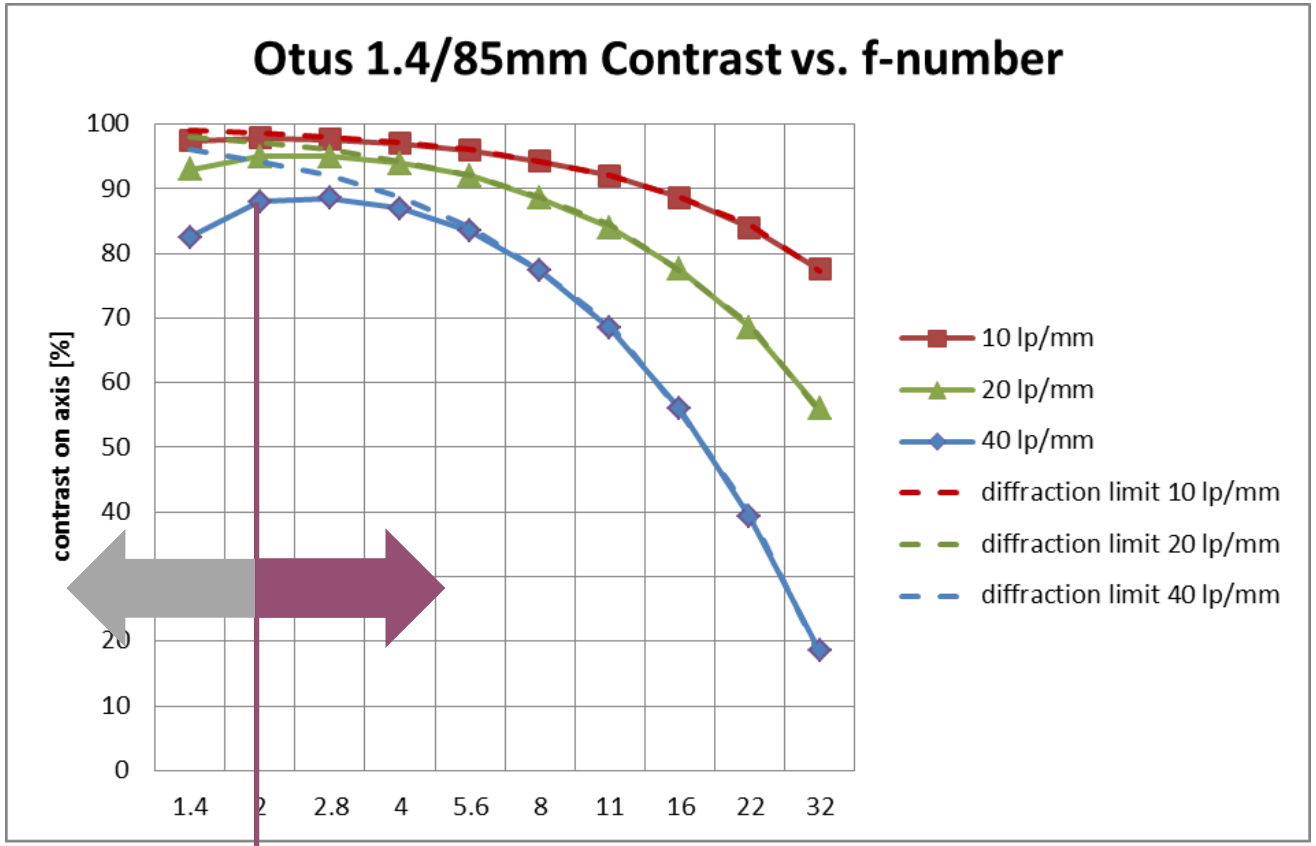
Standard camera lens: „diffraction-limited performance at lower aperture (higher f-number")



„aberration – limited"

optics "diffraction-limited"

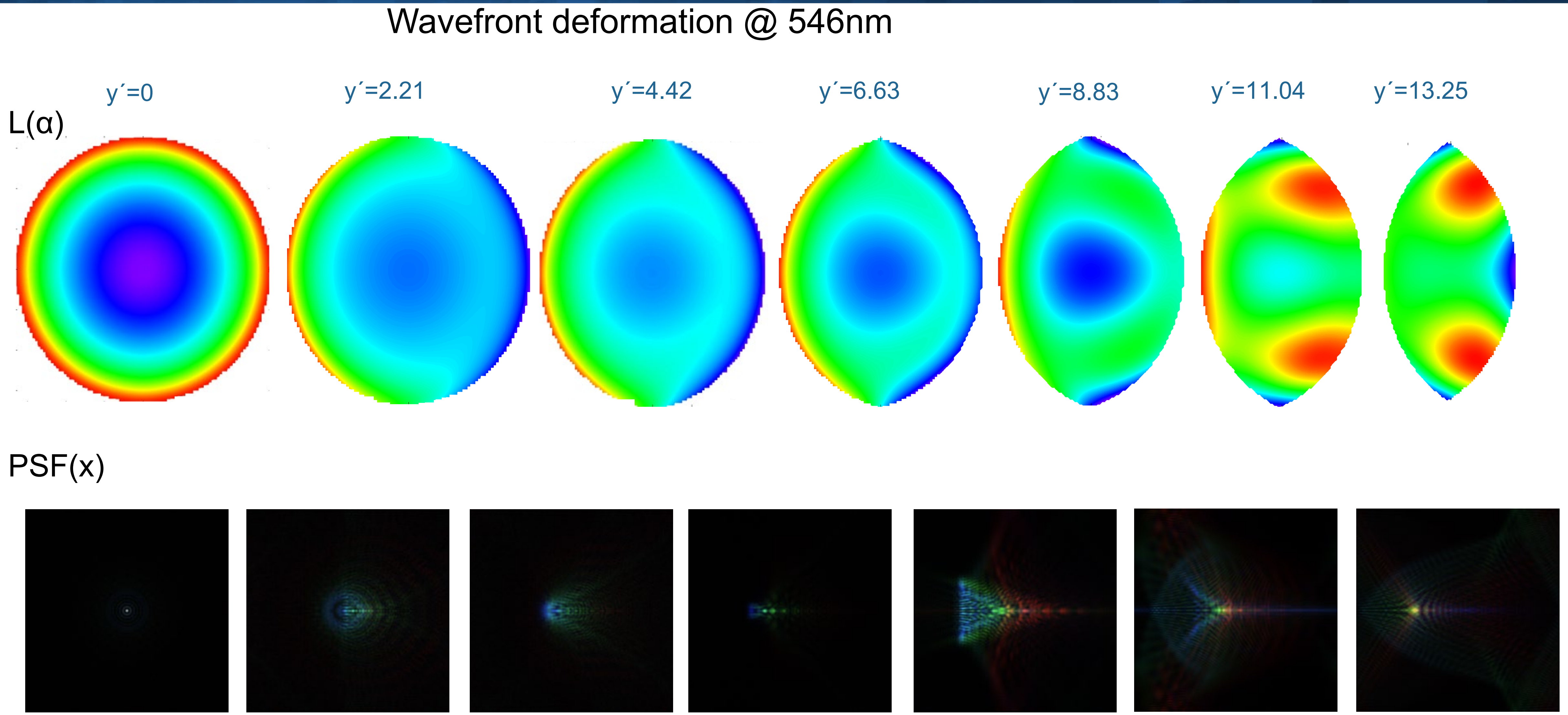
Extreme high-quality camera lens: „diffraction-limited performance at high aperture (low f-number“)



„aberration – limited“

optics
“diffraction-limited”

Pupil function and vs image field height of a lens



PSF vs defocus vs image field height

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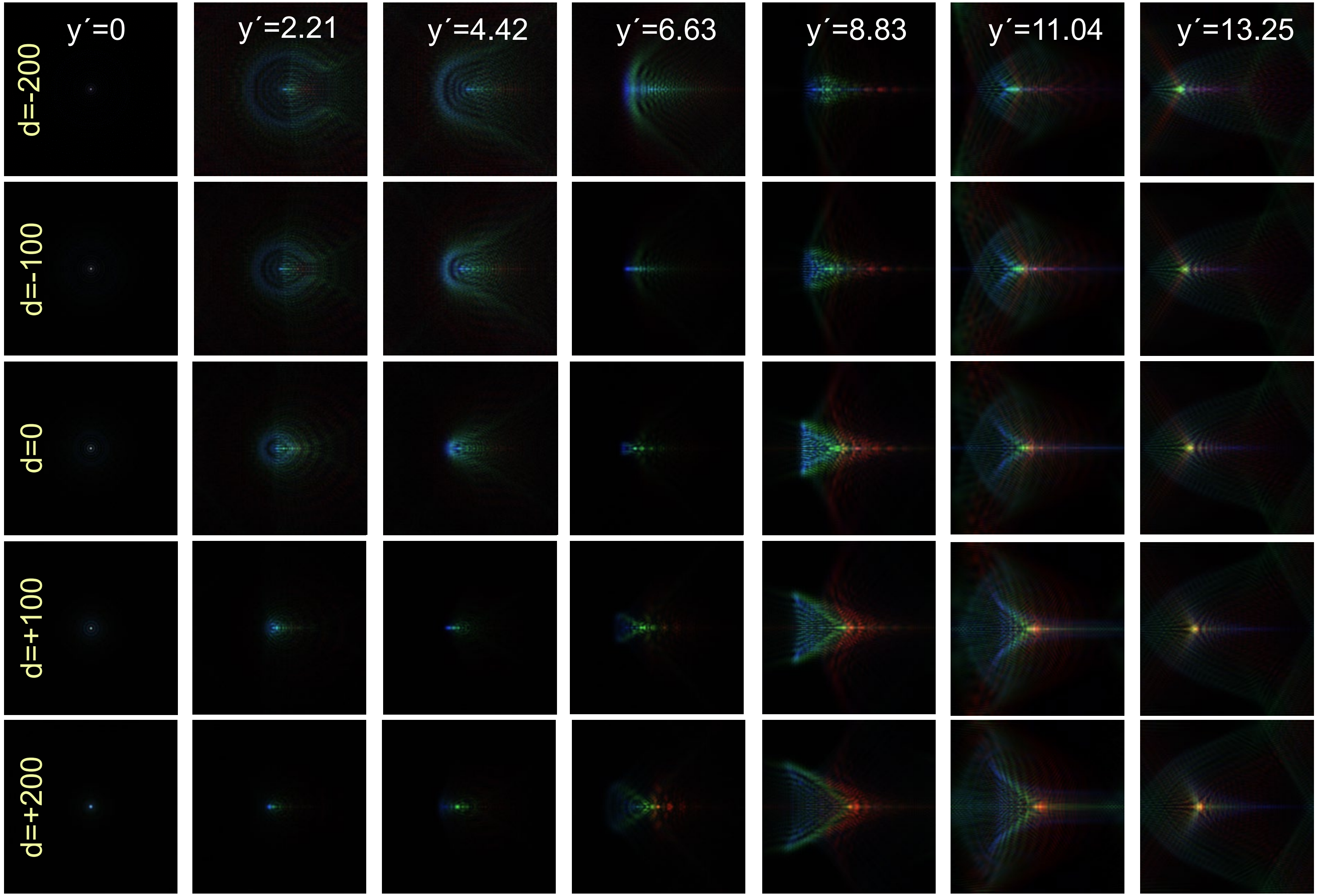
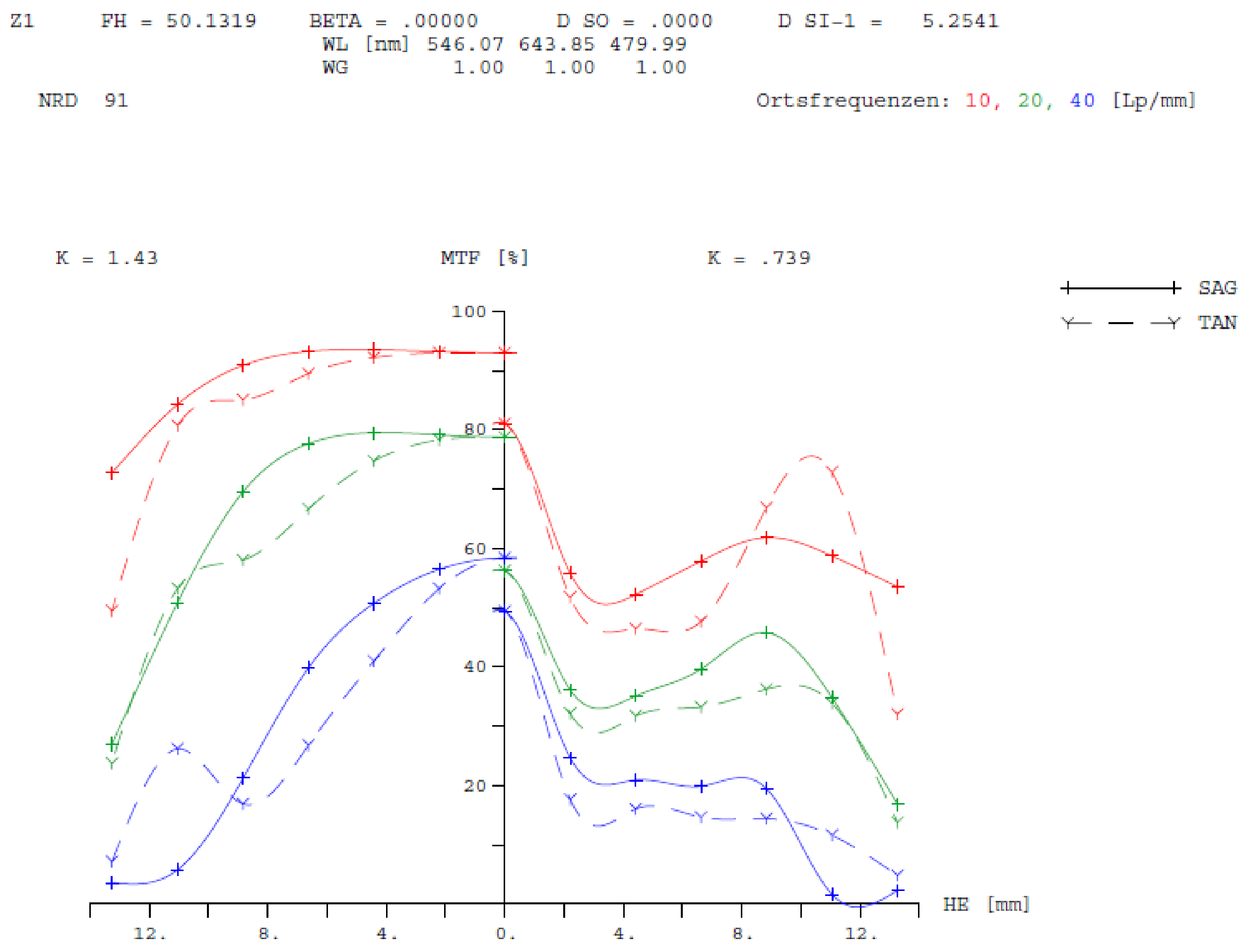
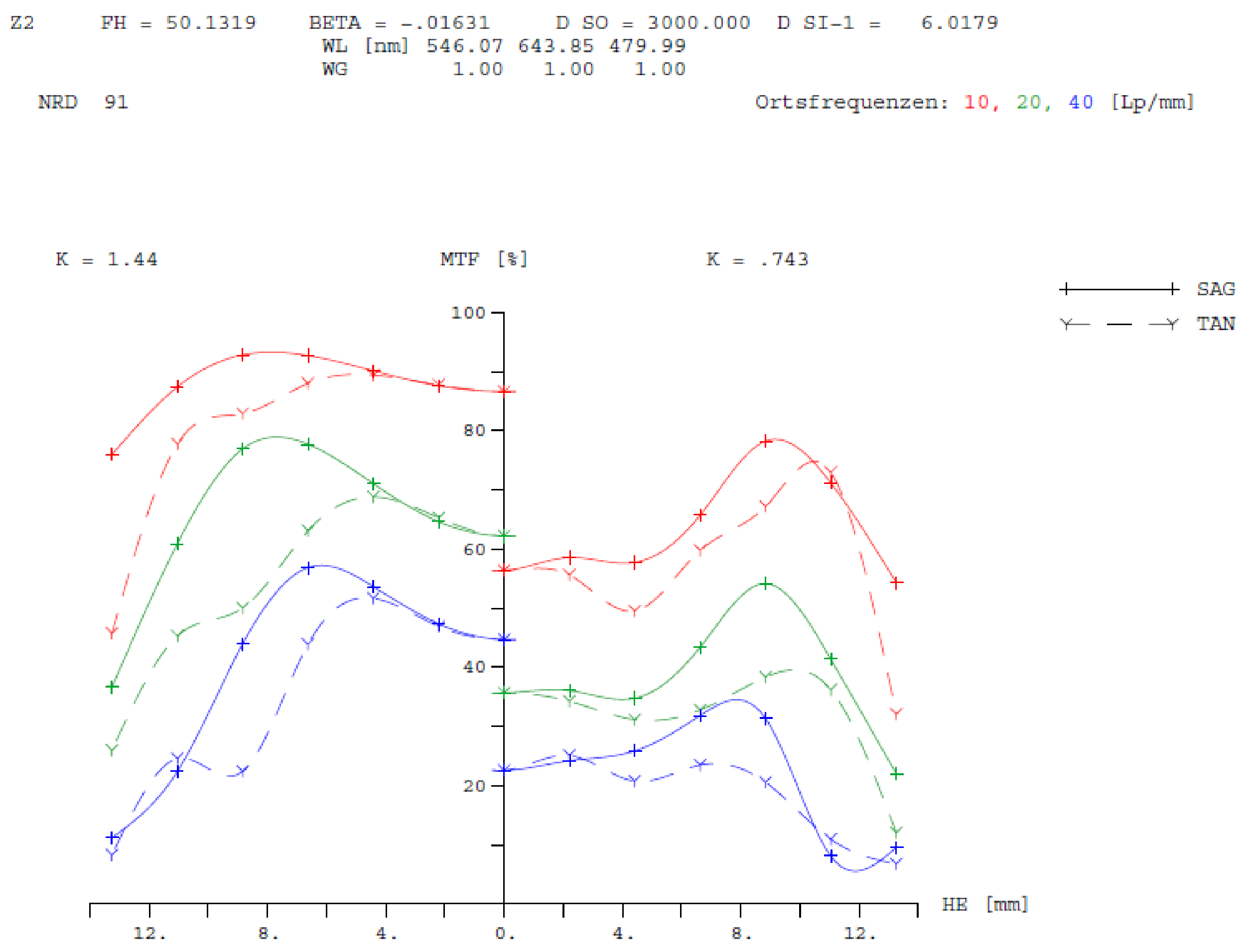
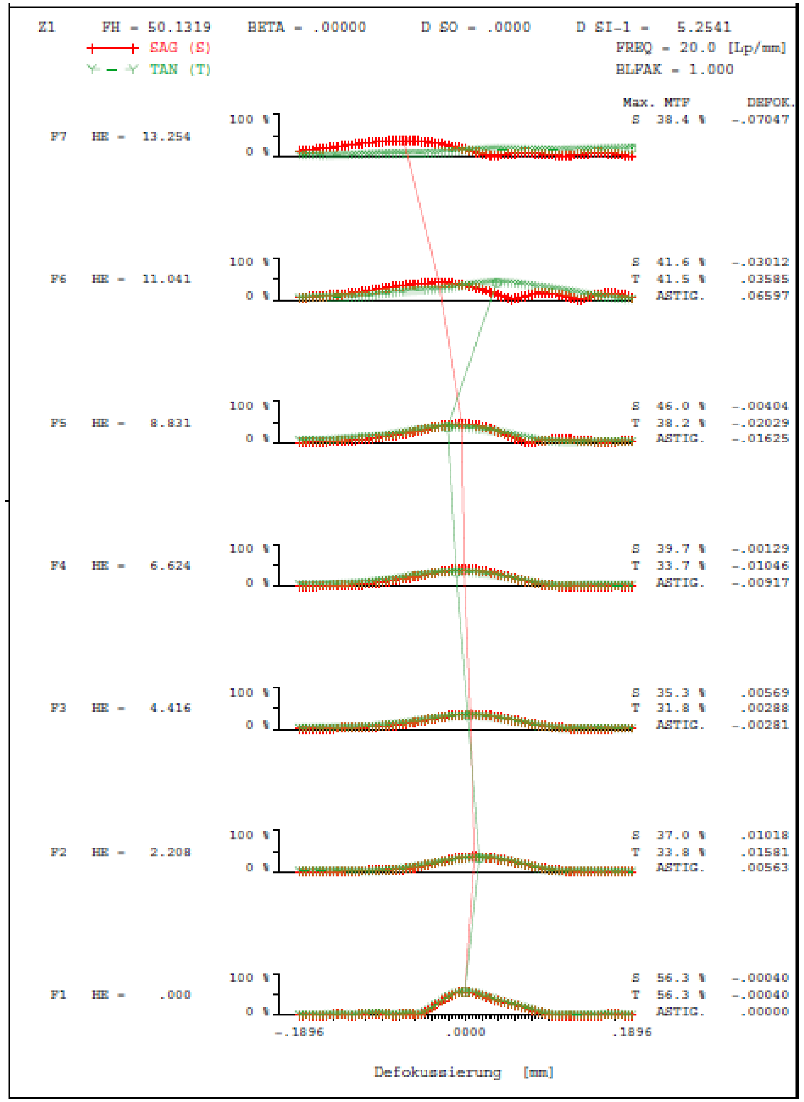


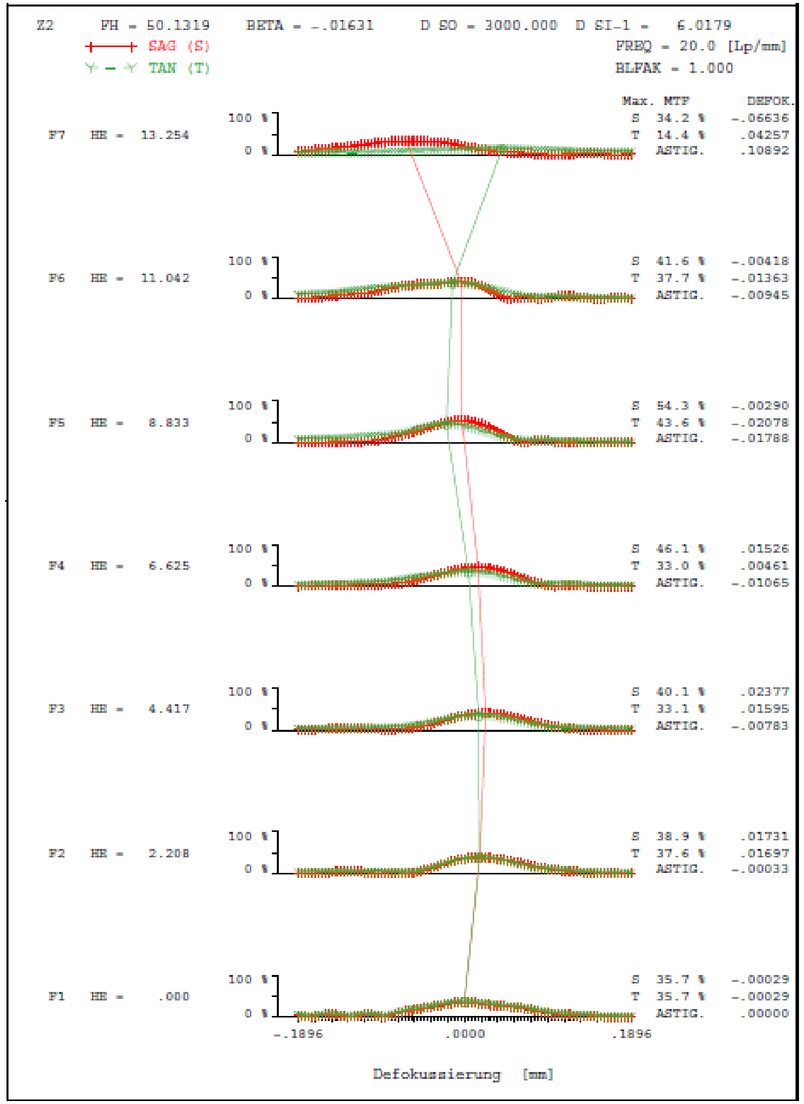
MTF analysis of a camera lens



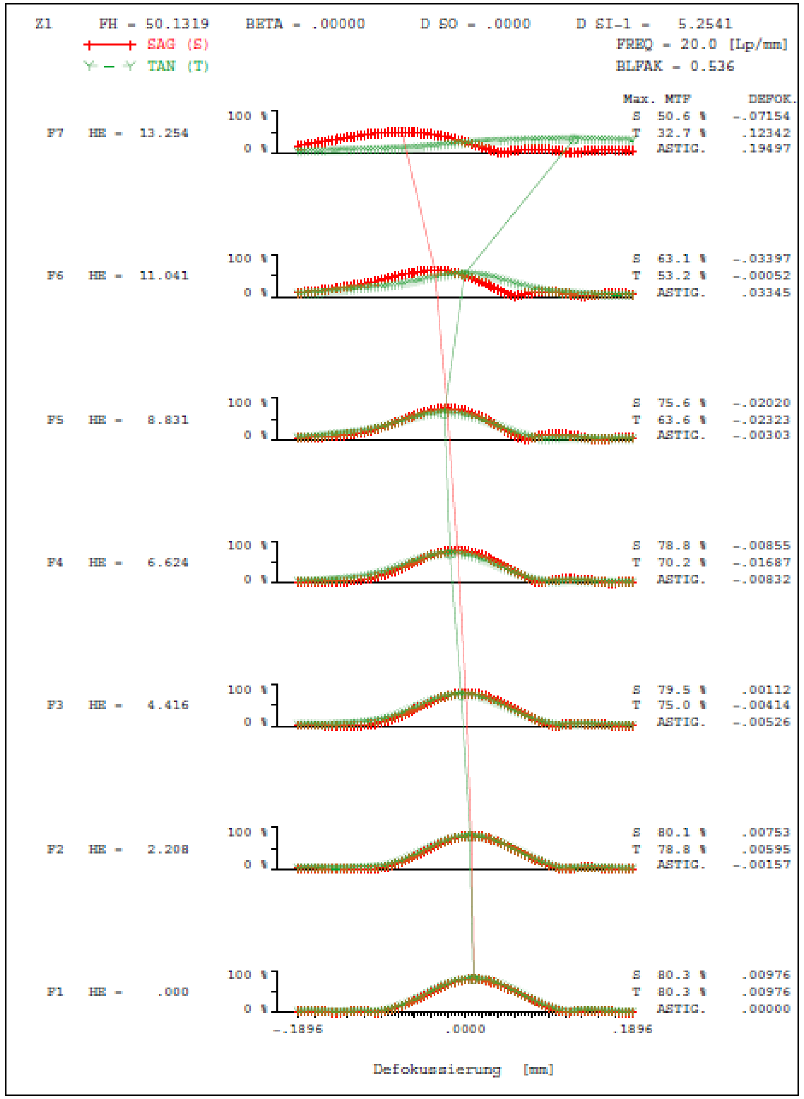


MTF vs defocus vs image field height





MTF vs defocus vs image field height: increased f-number (larger depth-of-field)



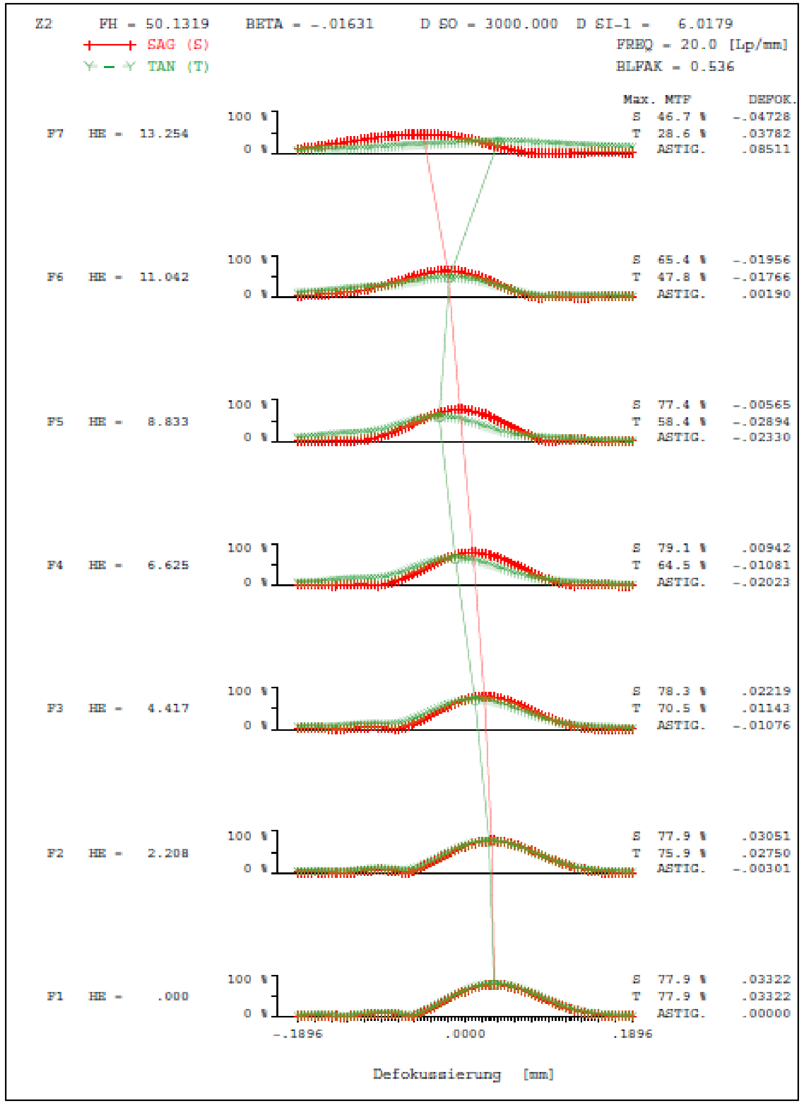


Image Simulation
Reference Image

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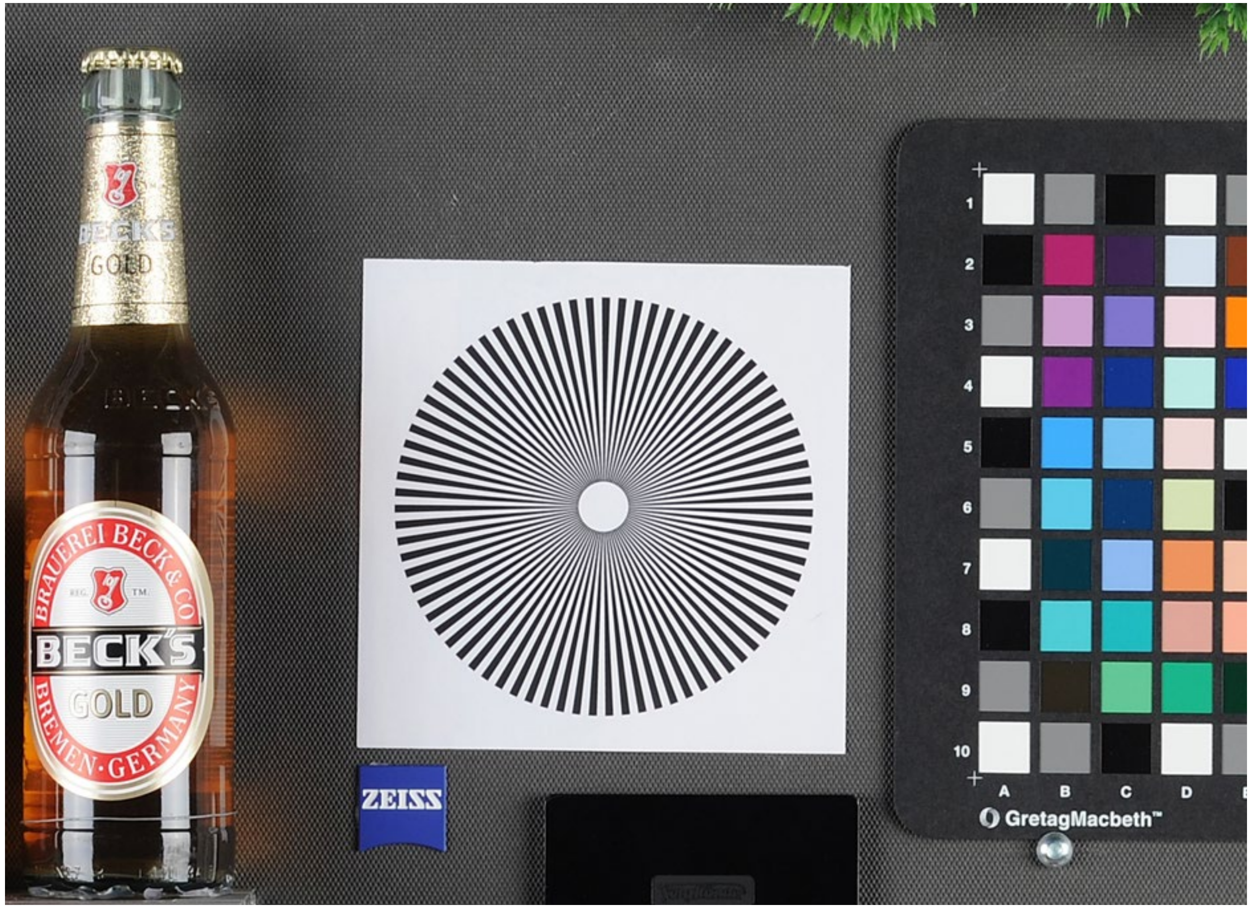
Image Simulation
Simulated Image of Lens

Image Simulation
Simulated Image of Lens



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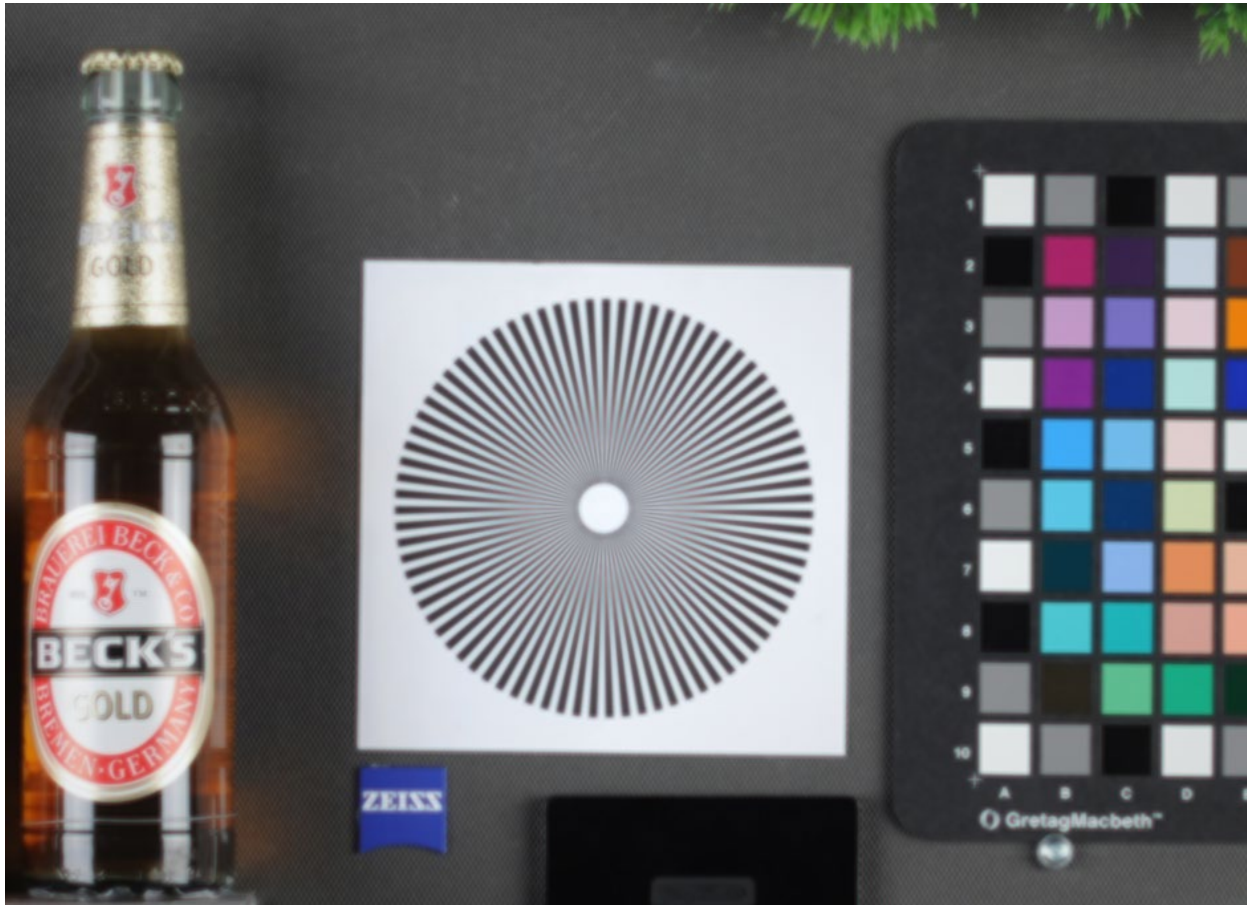
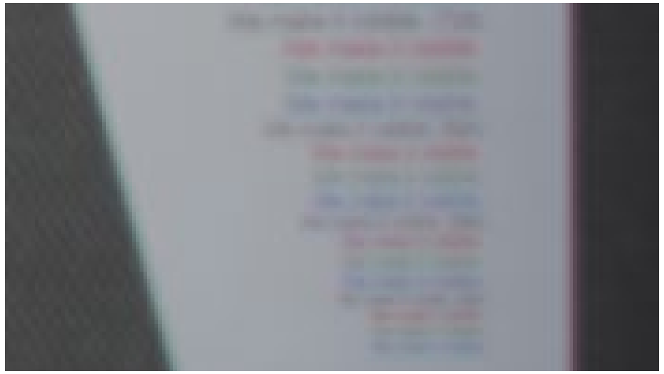


Lens Image

Image Simulation
Simulated Image of Lens

Reference



Lens Image

Literature Lecture 5

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Summary / Take-Aways

- Partially coherent imaging is a natural generalization of coherent off-axis imaging incoherently superposing the coherent images for all illumination directions defined by the extended source
- The overlap areas which contribute to image formation correspond to the overlap areas of Hopkins´ Transfer Function
- For periodic objects with aplanatic objects the diffraction orders appear equidistant in the lens pupil
- For spatially stationary imaging in all cases, whether coherent, non-coherent or partially coherent a transfer function can be defined, which depends solely on the system characteristic (pupil and eventually light source distribution) being separated from the object (or the object spectrum respectively)
- For coherent imaging the transfer function is the lens pupil function operating on the amplitude object spectrum, whereas for non-coherent imaging it is the OTF (autocorrelation of lens pupil) which operates in the intensity object spectrum; for partially coherent imaging the imaged Fourier coefficients are given by the HTF combined with correlations of the amplitude spectrum

Summary / Take-Aways

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- OTF and PSF are related by a Fourier transform and equivalent descriptions of the system
- As MTF=abs(OTF) represents only part of the system transfer function neglecting the phase part it is representative to characterize the transfer of single, isolated harmonic orders (spatial frequency);
- to characterize the full image formation also the phase transfer function (PTF) is relevant which characterizes the relative shifts of the harmonic components
- To determine the wavefront deformation ray tracing with optical path measurement is required; different methods (“back tracing”, “Eikonal”) have been presented
- The “Airy spot” scales with λ/NA and determines the principle limitation of resolution; the finite spot size is a consequence of (partially) constructive interference near the “ideal image point” (defined as all contributors being mutually in phase)
- There are different PSF based quality criteria like maxima-, threshold-, integral-(correlation)-criteria which usefulness strongly depends on the specific application
- “Geometric” (purely ray-tracing based) intensity distributions neglect diffraction, however might still be useful approximation, e.g. for out-of-focus distributions and/or after sampling with a sensor array

Summary / Take-Aways

- MTF evaluations for digital-optical system require specifications adapted to the pixel pitch of the sensor related with sampling theorems: Nyq/2 is a reasonable assumption for the highest transferred spatial frequency; Nyquist frequency is defined by Nyq=1/(2p)
- „diffraction-limited“ imaging means „close to the diffraction limit“; however „diffraction limited“ does not automatically mean a high resolution, as the resolution towards low apertures is poor