4. Diffraction Theory

4.1 Interaction with plane masks

i) propagation from light source to aperture

→ plane wave

ii) multiply field distribution of illuminating wave by transmission function $u_+(x,y,z_A) = t(x,y)u_-(x,y,z_A)$

iii) propagation of the modified field distribution behind the aperture through homogeneous space

$$u(x,y,z) = \iint_{-\infty}^{\infty} H(\alpha,\beta;z-z_A)\,U_+(\alpha,\beta;z_A)\exp[i(\alpha x+\beta y)]\,d\alpha\, d\beta$$

or

$$u(x,y,z) = \iint_{-\infty}^{\infty} h(x-x',y-y',z-z_A)\,u_+(x',y',z_A)\,dx'dy'$$

with $h = \frac{1}{(2\pi)^2} FT^{-1}[H]$

4.2 Propagation using different approximations

4.2.1 General case - small aperture

$$h(x,y,z) = -\frac{1}{2\pi}\frac{\partial}{\partial z}\left[\frac{1}{r}\exp(ikr)\right] \quad r=\sqrt{x^2+y^2+z^2}$$

$$u(x,y,z_A+z_B) = \iint_{-\infty}^{\infty} h(x-x',y-y',z_B)\,u_+(x',y',z_A)\,dx'dy'$$

4.2.2 Fresnel approximation

Thus, this approximation is valid only for a limited angular spectrum, which corresponds to a large size of the structures inside the aperture.

$$\begin{cases} H_F(\alpha,\beta;z_B) = \exp(ikz_B)\exp\left[-i\frac{z_B}{2k}(\alpha^2+\beta^2)\right] \\ h_F(x,y,z_B) = -\frac{i}{\lambda z_B}\exp(ikz_B)\exp\left[i\frac{k}{2z_B}(x^2+y^2)\right] \end{cases}$$

4.2.3 Paraxial Fraunhofer approximation

$$\alpha^2+\beta^2 \ll k^2$$

and the additional condition for the so-called Fresnel number N_F

$$N_F \lesssim 0.1 \text{ with } N_F = \frac{a}{\lambda}\frac{a}{z_B}$$

where a is the largest size of the aperture.

$u_+(x,y) = t(x,y)u_-(x,y)$, and $t(x,y)=0$ for $|x|,|y|>a$ (aperture)

$\longrightarrow u_+(x,y)=0$ for $|x|,|y|>a$

$$u_F(x,y,z_B) = -\frac{i}{\lambda z_B}\exp(ikz_B)\iint_{-a}^{a} u_+(x',y')\exp\left\{i\frac{k}{2z_B}\left[(x-x')^2+(y-y')^2\right]\right\}dx'dy'$$

$$u_F(x,y,z_B) = -\frac{i}{\lambda z_B}\exp(ikz_B)\iint_{-a}^{a} u_+(x',y')\exp\left\{i\frac{k}{2z_B}[x^2+y^2]\right\}\exp\left\{-i\left[\frac{kx}{z_B}x'+\frac{ky}{z_B}y'\right]\right\}\exp\left\{i\frac{k}{2z_B}[x'^2+y'^2]\right\}dx'dy'$$

$$\exp\left\{i\frac{k}{2z_B}(x'^2+y'^2)\right\} \approx 1$$

$$u_{FR}(x,y,z_B) = -\frac{i}{\lambda z_B}\exp(ikz_B)\exp\left[i\frac{k}{2z_B}(x^2+y^2)\right] \times \underline{\iint_{-\infty}^{\infty} u_+(x',y')\exp\left\{-i\left(\frac{kx}{z_B}x'+\frac{ky}{z_B}y'\right)\right\}dx'dy'} \text{ FT}$$

$$= -i\frac{(2\pi)^2}{\lambda z_B}\exp(ikz_B)\,U_+\left(k\frac{x}{z_B},k\frac{y}{z_B}\right)\exp\left[i\frac{k}{2z_B}(x^2+y^2)\right]$$

$$I_{FR}(x,y,z_B) \sim \frac{1}{(\lambda z_B)^2}\left|U_+\left(k\frac{x}{z_B},k\frac{y}{z_B};z_A\right)\right|^2$$

Interpretation

For any plane $z=z_B$ in the far field, only one spatial frequency ($\alpha = kx/z_B$; $\beta = ky/z_B$) with spectral amplitude $U_+\left(\frac{kx}{z_B},\frac{ky}{z_B}\right)$ contributes to the field distribution at each point x,y. This is in contrast to the previously considered cases, where all spatial frequencies contributed to the field at a given point.

check:

A) $\alpha^2+\beta^2 \ll k^2 \rightarrow$ smallest features $\Delta x, \Delta y \gg \lambda$

→ narrow angular spectrum

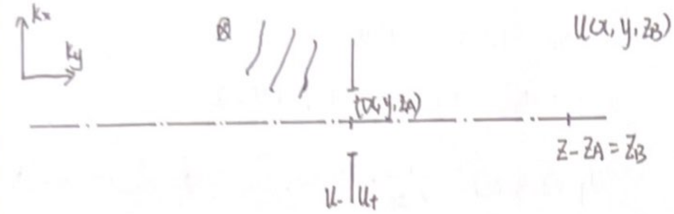
B) $N_F = \frac{a^2}{\lambda z_B} \ll 1 \rightarrow$ largest feature a detemines

$z_B \gg \frac{a^2}{\lambda}$

→ minimum propagation distance to the far field.

Example: $\Delta x, \Delta y = 10\lambda$, $a = 100\lambda$, $\lambda = 1\mu m \rightarrow z_B \gg 10^4\lambda \approx 1cm$

4.3 Paraxial Fraunhofer diffraction at plane masks



$$U_-(x,y,z_A)=A\exp\left[i(k_x x+k_y y+k_z z_A)\right] \quad \text{(is inclined with z axis)}$$

$$U_+(x,y,z_A)=U_-(x,y,z_A)\,t(x,y)=A\exp\left[i(k_x x+k_y y+k_z z_A)\right]t(x,y)$$

$$I(x,y,z_B)\sim|U(x,y,z_B)|^2\sim\frac{1}{(\lambda z_B)^2}\left|U_+\!\left(k\frac{x}{z_B},k\frac{y}{z_B}\right)\right|^2$$

$$\alpha=k\frac{x}{z_B},\quad \beta=k\frac{y}{z_B}$$

$U_+\left(k\frac{x}{z_B},k\frac{y}{z_B}\right)$ (Fourier transform)

$$=\frac{A}{(2\pi)^2}\exp(ik_z z_A)\iint_{-\infty}^{\infty}t(x',y')\exp\left[-i\left(k\frac{x}{z_B}-k_x\right)x'-i\left(k\frac{y}{z_B}-k_y\right)y'\right]dx'dy'$$

$$=A\exp(ik_z z_A)\,T\!\left(k\frac{x}{z_B}-k_x,\,k\frac{y}{z_B}-k_y\right)$$

Examples

A) Rectangular aperture illuminated by normal plane wave

$$t(x,y)=\begin{cases}1 & \text{for } |x|\le a,\ |y|\le b\\ 0 & \text{elsewhere}\end{cases}$$

$$I(x,y,z_B)\sim \mathrm{sinc}^2\!\left(ka\frac{x}{z_B}\right)\mathrm{sinc}^2\!\left(kb\frac{y}{z_B}\right)$$

Half-angular widths $\theta_x=\lambda/D_x \quad \theta_y=\lambda/D_y$

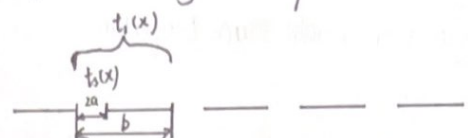
B) Circular aperture (pinhole) illuminated by normal plane wave

$$t(x,y)=\begin{cases}1, & \text{for } x^2+y^2\le a^2\\ 0 & \text{elsewhere}\end{cases}$$

$$I(x,y,z_B)\sim\left[\frac{J_1\left(\frac{ka}{z_B}\sqrt{x^2+y^2}\right)}{\frac{ka}{z_B}\sqrt{x^2+y^2}}\right]^2 \rightarrow \text{Bessel function}$$

The Airy pattern with the radius of the central disk subtending an angle $\theta\simeq 1.22\lambda/D$

C) One-dimensional periodic structure (grating) illuminated by normal plane wave



$$t(x)=\sum_{n=0}^{N-1}t_1(x-nb) \text{ with } t_1(x)=\begin{cases}t_s(x) & \text{for } |x|\le a\\ 0 & \text{elsewhere.}\end{cases}$$

$$T\!\left(k\frac{x}{z_B}\right)\sim T_s\!\left(k\frac{x}{z_B}\right)\frac{\sin\left(N\frac{k}{2}\frac{x}{z_B}b\right)}{\sin\left(\frac{k}{2}\frac{x}{z_B}b\right)}$$

For the particular case of a simple grating of slit apertures with $t_s(x)=1$ we have

$$T_1\!\left(k\frac{x}{z_B}\right)=\mathrm{sinc}\left(k\frac{x}{z_B}a\right)$$

$$I\sim\mathrm{sinc}^2\!\left(k\frac{x}{z_B}a\right)\frac{\sin^2\left(N\frac{k}{2}\frac{x}{z_B}b\right)}{\sin^2\left(\frac{k}{2}\frac{x}{z_B}b\right)}$$

Global width of the diffraction pattern

$\rightarrow$ first zero of slit function T_s

$$k\frac{x_s}{z_B}a=\pi \rightarrow x_s=\frac{\lambda z_B}{2a}$$

The width x_s of the entire far-field diffraction pattern is determined by the size a of the individual slit.

Positions of local maxima of the diffraction pattern

$$\frac{k}{2}\frac{x_p}{z_B}b=n\pi \rightarrow x_p=n\frac{\lambda z_B}{b}$$

These are the so-called diffraction orders, which are determined exclusively by the grating period.

Width of local maxima

$$N\frac{k}{2}\frac{x_N}{z_B}b=\pi \rightarrow x_N=\frac{\lambda z_B}{Nb}$$

The width of a maximum in the far-field diffraction pattern x_N is determined by $N*b$, which is the total size of the mask.

These observations are consistent with the general property of the Fourier-transform: small scales in position give rise to a broad angular spectrum and vice versa.

5. Fourier optics

5.1 Imaging of arbitrary optical fields with a thin lens

5.1.1 Transfer function of a thin lens

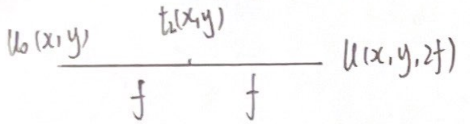
response function

$$t_L(x,y) = \exp\left[-i\frac{k}{2f}(x^2+y^2)\right]$$

Transfer function

$$T_L(\alpha,\beta) = -i\frac{\lambda f}{(2\pi)^2}\exp\left[i\frac{f}{2k}(\alpha^2+\beta^2)\right]$$

5.1.2 Optical imaging using the 2f-setup



A)

$$U_0(\alpha,\beta) = FT[u_0(x,y)]$$

B) Propagation from object to lens in the paraxial approximation

$$U_-(\alpha,\beta;f) = \underline{H_F(\alpha,\beta;f)}\,U_0(\alpha,\beta)$$

$$U_-(\alpha,\beta;f) = \underline{\exp(ikf)\exp\left[-\frac{i}{2k}(\alpha^2+\beta^2)f\right]}U_0(\alpha,\beta)$$

c) interaction with lens

$$U_+(x,y,f) = t_L(x,y)\,u_-(x,y,f)$$

$$U_+(\alpha,\beta;f) = T_L(\alpha,\beta) * U_-(\alpha,\beta;f)$$

$$= -i\frac{\lambda f}{(2\pi)^2}\exp(ikf)\iint_{-\infty}^{\infty}\exp\left\{i\frac{f}{2k}\left[(\alpha-\alpha')^2+(\beta-\beta')^2\right]\right\}$$

$$\cdot\exp\left[-\frac{i}{2k}(\alpha'^2+\beta'^2)f\right]U_0(\alpha',\beta')\,d\alpha'd\beta'$$

D) Propagation from lens to image plane

$$U(\alpha,\beta;2f) = H_F(\alpha,\beta;f)\,U_+(\alpha,\beta;f)$$

$$U(\alpha,\beta;2f) = -i\frac{\lambda f}{(2\pi)^2}\exp(2ikf)\iint_{-\infty}^{\infty}U_0(\alpha',\beta')\exp\left[-i\frac{f}{k}(\alpha\alpha'+\beta\beta')\right]d\alpha'd\beta'$$

$$= -i\frac{\lambda f}{(2\pi)^2}\exp(2ikf)\,u_0\left(-\frac{f}{k}\alpha,-\frac{f}{k}\beta\right)$$

E) Fourier back transform in the image plane

$$U(x,y,2f) = FT^{-1}\left[U(\alpha,\beta;2f)\right]$$

$$= -i\frac{\lambda f}{(2\pi)^2}\exp(2ikf)\iint_{-\infty}^{\infty}u_0\left(-\frac{f}{k}\alpha,-\frac{f}{k}\beta\right)\exp[i(\alpha x+\beta y)]\,d\alpha d\beta$$

$$x' = -\frac{f}{k}\alpha,\quad y' = -\frac{f}{k}\beta \rightarrow d\alpha = -\frac{2\pi}{\lambda f}dx',\; d\beta = -\frac{2\pi}{\lambda f}dy'$$

$$\rightarrow U(x,y,2f) = -i\frac{1}{\lambda f}\exp(2ikf)\iint_{-\infty}^{\infty}u_0(x',y')\exp\left[-i\frac{k}{f}(xx'+yy')\right]dx'dy'$$

$$U(x,y,2f) = -i\frac{(2\pi)^2}{\lambda f}\exp(2ikf)\,U_0\left(\frac{k}{f}x,\frac{k}{f}y\right)$$

5.2 Optical filtering and Image processing

5.2.1 4f setup

A) Field behind the transmission Mask

$$U_+(x,y,2f) = U(x,y,2f)\,p(x,y) \sim A\,U_0\left(\frac{k}{f}x,\frac{k}{f}y\right)p(x,y)$$

B) Second lens → Fourier back transform of field distribution

$$U(x,y,4f) = -i\frac{(2\pi)^2}{\lambda f}\exp(2ikf)\,U_+\left(\frac{k}{f}x,\frac{k}{f}y;2f\right)$$

$$\rightarrow U(x,y,4f) \sim \iint_{-\infty}^{\infty}u_+(x',y',2f)\exp\left[-i\frac{k}{f}(xx'+yy')\right]dx'dy'$$

$$\sim \iint_{-\infty}^{\infty}U_0\left(\frac{k}{f}x',\frac{k}{f}y'\right)p(x',y')\exp\left[-i\frac{k}{f}(x'x+y'y)\right]dx'dy'$$

$$\rightarrow \sim\iint_{-\infty}^{\infty}U_0(\alpha,\beta)\,p\left(\frac{f}{k}\alpha,\frac{f}{k}\beta\right)\exp[-i(\alpha x+\beta y)]\,d\alpha d\beta$$

$$\therefore U(-x,-y,4f) \sim \iint_{-\infty}^{\infty}p\left(\frac{f}{k}\alpha,\frac{f}{k}\beta\right)U_0(\alpha,\beta)\exp[i(\alpha x+\beta y)]\,d\alpha d\beta$$

$$H_A(\alpha,\beta,4f) \sim p\left(\frac{f}{k}\alpha,\frac{f}{k}\beta\right)$$

In position space

$$U(-x,-y,4f) = \iint_{-\infty}^{\infty}h_A(x-x',y-y')\,u_0(x',y')\,dx'dy'$$

$$U(-x,-y,4f) \propto \iint_{-\infty}^{\infty}P\left[\frac{k}{f}(x'-x),\frac{k}{f}(y'-y)\right]u_0(x',y')\,dx'dy'$$

5.2.2 Examples of aperture functions

1 The ideal image (infinite aperture)

$$u(-x,-y,4f) \sim u_0(x,y)$$

2 Finite circular aperture

$$p(x,y) = \begin{cases} 1 & \text{for } x^2+y^2 \le (D/2)^2 \\ 0 & \text{elsewhere} \end{cases}$$

$$H_A(\alpha,\beta;4f) \sim \begin{cases} 1 & \text{for } \left(\frac{f}{k}\alpha\right)^2+\left(\frac{f}{k}\beta\right)^2 \le (D/2)^2 \\ 0 & \text{elsewhere} \end{cases}$$

3. Optical resolution

$$\rho^2 = \alpha^2+\beta^2 \qquad \rho_{max}^2 = \frac{k^2}{f^2}\left(\frac{D}{2}\right)^2 \rightarrow \rho_{max} = \frac{2\pi n}{\lambda f}\frac{D}{2}$$

$$\Delta r_{min} = \frac{1.22\lambda f}{nD}$$

7. Principles of optics in crystals

7.1 Susceptibility and dielectric tensor

Assumptions:

monochromaticity $\rightarrow$ single ω

plane wave $\rightarrow$ single spatial frequency

no absorption $\rightarrow$ real valued $\varepsilon = \varepsilon'$

Anisotropic

$$P_i(\vec{r},\omega) = \varepsilon_0 \sum_{j=1}^{3} \underbrace{\chi_{ij}(\omega)}_{3\times 3 = 9 \text{ components}} E_j(\vec{r},\omega)$$

$\vec{P} \nparallel \vec{E}$: the polarization is not necessarily parallel to the electric field.

$$D_i(\vec{r},\omega) = \varepsilon_0 \sum_{j=1}^{3} \varepsilon_{ij}(\omega) E_j(\vec{r},\omega)$$

$$\vec{D}(\vec{r},\omega) = \varepsilon_0 \hat{\varepsilon}(\omega) \vec{E}(\vec{r},\omega)$$

$$\vec{D} \nparallel \vec{E}$$

Notation:

$\hat{\chi} = (\chi_{ij}) \rightarrow$ susceptibility tensor

$\hat{\varepsilon} = (\varepsilon_{ij}) \rightarrow$ dielectric tensor

$\hat{\sigma} = (\hat{\varepsilon})^{-1} = (\sigma_{ij}) \rightarrow$ inverse dielectric tensor

$$\sum_{j=1}^{3} \sigma_{ij}(\omega) D_j(\vec{r},\omega) = \varepsilon_0 E_i(\vec{r},\omega)$$

σ_{ij}, ε_{ij} are real in the transparent region

The tensors are symmetric ($\varepsilon_{ij} = \varepsilon_{ji}, \sigma_{ij} = \sigma_{ji}$)

$\rightarrow$ orthogonal transformation

Directions where $\vec{D} \parallel \vec{E}$

$$\varepsilon_0 E_i = \sum_{j=1}^{3} \sigma_{ij} D_j = \lambda D_i$$

$$\downarrow$$

$$\det[\sigma_{ij} - \lambda I_{ij}] = 0 \quad I_{ij} = \delta_{ij}$$

$$\rightarrow \sum_{j=1}^{3} \sigma_{ij} D_j^{(\alpha)} = \lambda^{(\alpha)} D_i^{(\alpha)}$$

$$\sum_{i=1}^{3} D_i^{(\beta)} D_i^{(\alpha)} = 0 \quad \text{for } \lambda^{(\alpha)} \neq \lambda^{(\beta)}$$

The eigenvectors are orthogonal

$$\varepsilon_{ij} = \varepsilon_i \delta_{ij}, \quad \sigma_{ij} = \sigma_i \delta_{ij} = \frac{1}{\varepsilon_i} \delta_{ij}$$

(diagonalized)

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{bmatrix}$$

in the principal coordinate system

7.2 Optical classification of crystals

A isotropic

cubic crystals

$\varepsilon_1(\omega) = \varepsilon_2(\omega) = \varepsilon_3(\omega) \rightarrow D_i = \varepsilon_0 \varepsilon(\omega) E_i$

B uniaxial

trigonal, tetragonal, hexagonal

$\varepsilon_1(\omega) = \varepsilon_2(\omega) \neq \varepsilon_3(\omega)$

C biaxial

$\varepsilon_1(\omega) \neq \varepsilon_2(\omega) \neq \varepsilon_3(\omega)$

7.3 Index ellipsoid

$$\hat{\sigma} = [\hat{\varepsilon}]^{-1}$$

$$\sum_{i,j=1}^{3} \sigma_{ij} x_i x_j = 1$$

The index ellipsoid defines a surface of constant electric energy density in three-dimensional field space:

$$\sum_{i,j=1}^{3} \sigma_{ij} D_i D_j = \varepsilon_0 \sum_{i=1}^{3} E_i D_i = 2W_{el}$$

$$\downarrow$$

$$\frac{x_1^2}{\varepsilon_1} + \frac{x_2^2}{\varepsilon_2} + \frac{x_3^2}{\varepsilon_3} = 1$$

$$n = \sqrt{\varepsilon_i}$$

7.4 Normal modes in anisotropic media

Before - isotropic media

$$\vec{E}(\vec{r},t) = \vec{E}\exp\{i[\vec{k}\cdot\vec{r} - \omega t]\}$$

Dispersion relation

$$\vec{k}^2(\omega) = k^2(\omega) = \frac{\omega^2}{c^2}\varepsilon(\omega)$$

$\varepsilon(\omega) > 0$ $\qquad \vec{k}\cdot\vec{E} = \vec{k}\cdot\vec{D} = 0$

7.4.1 Normal modes propagating in principal directions

light propagates in z - direction

$$\vec{D}^{(a)} = \{D_x \exp[i(\vec{k}_a\cdot\vec{r} - \omega t)]\}\vec{e}_x \rightarrow \vec{k}_a^2 = \frac{\omega^2}{c^2}\varepsilon_x$$

$$\vec{D}^{(b)} = \{D_y \exp[i(\vec{k}_b\cdot\vec{r} - \omega t)]\}\vec{e}_y \rightarrow \vec{k}_b^2 = \frac{\omega^2}{c^2}\varepsilon_y$$

For light propagation in a principal direction, we find two perpendicular linearly polarized normal modes with $\vec{E} \parallel \vec{D}$

7.4.2 Normal modes for arbitrary propagation direction

$$k_a = \frac{\omega}{c} n_a \qquad k_b = \frac{\omega}{c} n_b$$

$$E_i^{(a)} = \frac{D_i^{(a)}}{\varepsilon_0 \varepsilon_i}, \quad E_i^{(b)} = \frac{D_i^{(b)}}{\varepsilon_0 \varepsilon_i}$$

$\vec{D}^{(a,b)} \nparallel \vec{E}^{(a,b)}$, and $\vec{E}^{(a,b)}$ are not perpendicular to $\vec{k}$

$$\langle \vec{S} \rangle = \frac{1}{2} R(\vec{E} \times \vec{H}^*)$$

$$\langle \vec{S} \rangle \perp \vec{E}$$

Mathematical derivation

In anisotropic case

$$k = k(\omega, u)$$

the polarizations of the normal modes are not elliptic

$$\vec{k} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = k \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{with } u_1^2 + u_2^2 + u_3^2 = 1$$

Start from Maxwell's equations for the plane wave Ansatz:

$$\vec{k}\cdot\vec{D} = 0 \qquad \vec{k}\times\vec{E} = \omega\mu_0\vec{H}$$

$$\vec{k}\cdot\vec{H} = 0 \qquad \vec{k}\times\vec{H} = -\omega\vec{D}$$

$$-[\vec{k}\times(\vec{k}\times\vec{E})] = \frac{\omega^2}{c^2}\frac{1}{\varepsilon_0}\vec{D} \rightarrow -\vec{k}(\vec{k}\cdot\vec{E}) + \vec{k}^2\vec{E} = \frac{\omega^2}{c^2}\frac{1}{\varepsilon_0}\vec{D}$$

In the principal coordinate system

$$D_i = \varepsilon_0 \varepsilon_i E_i$$

$$\downarrow$$

$$\left(\frac{\omega^2}{c^2}\varepsilon_i - k^2\right)E_i = -k_i \sum_j k_j E_j$$

$$\because \operatorname{div}\vec{E} = \sum_j k_j E_j \neq 0$$

$\longrightarrow$ dispersion relation:

$$\sum_i \frac{k_i^2}{\left(k^2 - \frac{\omega^2}{c^2}\varepsilon_i\right)} = 1$$

$$\rightarrow \sum_i \frac{u_i^2}{[n^2 - \varepsilon_i]} = \frac{1}{n^2}$$

$$E_i = \frac{k_t}{\left(\frac{\omega^2}{c^2}\varepsilon_i - k^2\right)} \underbrace{\sum_j k_j E_j}_{= const}$$

How are the normal modes polarized?

- The ratio between the field components is real
  $\rightarrow$ phase difference 0 $\rightarrow$ linear polarization

7.4.3 Normal surfaces

there are two geometrical constructions:

A index ellipsoid

fix propagation direction $\rightsquigarrow$ index ellipse

$\rightsquigarrow$ semi-major / minor axes give n_a, n_b

optical axis $\rightarrow$ index ellipse is a circle

(for uniaxial cystals the optical axis coincides with one principal axis)

B normal surfaces

- fix propagation direction $\rightsquigarrow$ intersection with surfaces

  $\rightsquigarrow$ distances from origin give n_a, n_b

- optical axis connects points with $n_a = n_b$

In anisotropic media and for a given propagation direction we find two normal modes, which are linearly polarized monochromatic plane waves with two different phase ~~velocity~~ velocities c/n_a, c/n_b and two orthogonal polarization directions $\vec{D}^{(a)}$, $\vec{D}^{(b)}$

7.4.4 Special case: uniaxial crystals

7.4.4

$\varepsilon_{or} > \varepsilon_e \rightarrow$ negative uniaxial

$\varepsilon_{or} < \varepsilon_e \rightarrow$ positive uniaxial

A) ordinary wave

n_a independent of propagation direction

The ordinary wave $\vec{D}^{(or)}$ is polarized perpendicular to the z-axis and the k-vector and it does not interact with ε_e

B) extraodinary wave

n_b depends on propagation direction

$\vec{D}^{(e)}$ is polarized perpendicular to the k-vector and $\vec{D}^{(or)}$

$\downarrow$

$$\frac{u_1^2}{[n^2-\varepsilon_{or}]} + \frac{u_2^2}{[n^2-\varepsilon_{or}]} + \frac{u_3^2}{[n^2-\varepsilon_e]} = \frac{1}{n^2}$$

Normal surfaces

A

$$n_a^2 = \varepsilon_{or} \rightarrow k_a^2 = \frac{\omega^2}{c^2} n_a^2 = k_0^2 \varepsilon_{or}$$

$$k_a^2 = k_1^2 + k_2^2 + k_3^2 = k_0^2 \varepsilon_{or}$$

$\vec{D} \perp \vec{K}$, $\vec{D} \parallel \vec{E}$

B

$$\frac{(u_1^2+u_2^2)}{\varepsilon_e} + \frac{u_3^2}{\varepsilon_{or}} = \frac{1}{n_b^2},$$

$$k_b^2 = \frac{\omega^2}{c^2} n_b^2 (u_1, u_2, u_3)$$

$$n_b^2(\theta) = \frac{\varepsilon_e \varepsilon_{or}}{\varepsilon_{or} \sin^2\theta + \varepsilon_e \cos^2\theta}$$

$$\frac{1}{\varepsilon_e} \frac{(k_1^2+k_2^2)}{k_0^2} + \frac{1}{\varepsilon_{or}} \frac{k_3^2}{k_0^2} = 1$$

$\vec{D} \perp \vec{K}$, $\vec{D} \nparallel \vec{E}$

$\because D_2 = \varepsilon_0 \varepsilon_{or} E_2,$

$D_3 = \varepsilon_0 \varepsilon_e E_3$

8. Optical fields in isotropic, dispersive and piecewise homogeneous media

Translational invariance in y-direction

$$\vec{E}_{\perp} = \begin{bmatrix} 0 \\ E_y \\ 0 \end{bmatrix}, \quad \vec{E}_{\parallel} = \begin{bmatrix} E_x \\ 0 \\ E_z \end{bmatrix}$$

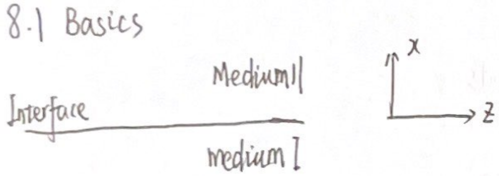
perpendicular : $\perp \rightarrow S \rightarrow TE$

parallel : $\parallel \rightarrow P \rightarrow TM$

$$\vec{E}_{TE} = \begin{bmatrix} 0 \\ E_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ E \\ 0 \end{bmatrix}, \quad \vec{H}_{TE} = \begin{bmatrix} H_x \\ 0 \\ H_z \end{bmatrix}$$

$$\vec{E}_{TM} = \begin{bmatrix} E_x \\ 0 \\ E_z \end{bmatrix}, \quad \vec{H}_{TM} = \begin{bmatrix} 0 \\ H_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ H \\ 0 \end{bmatrix}$$

8.1 Basics



Transition conditions

A) Continuity of fields

TE: $E = E_y$ and H_z continuous

TM: E_z and $H = H_y$ continuous

B) Continuity of wave vectors

homogeneous in z-direction $\rightarrow$ phase $e^{ik_z z}$

$\downarrow$

k_z continuous

Therefore, the electric field on both sides of the planar interface can be written as:

$$\vec{E}(x,z,t) = \vec{E}_{TE}(x)\exp[i(k_z z - \omega t)] + \vec{E}_{TM}(x)\exp[i(k_z z - \omega t)]$$

8.2 Fields in a layer system

8.2.1 Fields in one homogeneous layer

thickness d

dielectric function $\varepsilon_f(\omega)$

Generalized transverse fields F & G

$E, H \rightarrow F$ generalized field 1

$i\omega\mu_0 H_z, -i\omega\varepsilon_0 E_z \rightarrow G$ generalized field 2

$$\begin{cases} \left[\frac{\partial^2}{\partial x^2} + k_{fx}^2(k_z,\omega)\right] F(x) = 0 \\ G(x) = \alpha_f \frac{\partial}{\partial x} F(x) \quad \text{with } \alpha_{fTE} = 1 \end{cases}$$

$$\alpha_{fTM} = \frac{1}{\varepsilon_f}$$

$\xrightarrow[\text{equation}]{\text{harmonic oscillator}}$ $F(x) = C_1\exp(ik_{fx}x) + C_2\exp(-ik_{fx}x)$

$$G(x) = \alpha_f \frac{\partial}{\partial x} F(x) = i\alpha_f k_{fx}[C_1\exp(ik_{fx}x) - C_2\exp(-ik_{fx}x)]$$

initial conditions

$$F(0) = C_1 + C_2$$

$$G(0) = i\alpha_f k_{fx}[C_1 - C_2]$$

$$\rightarrow \quad C_1 = \frac{1}{2}\left[F(0) - \frac{i}{\alpha_f k_{fx}} G(0)\right]$$

$$C_2 = \frac{1}{2}\left[F(0) + \frac{i}{\alpha_f k_{fx}} G(0)\right]$$

$$F(x) = \cos(k_{fx}x)F(0) + \frac{1}{\alpha_f k_{fx}}\sin(k_{fx}x)G(0)$$

$$G(x) = -\alpha_f k_{fx}\sin(k_{fx}x)F(0) + \cos(k_{fx}x)G(0)$$

$$\left(k_{fx}^2(k_z,\omega) = \frac{\omega^2}{c^2}\varepsilon_f(\omega) - k_z^2\right)$$

8.2.2 Fields in a system of layers

$$\begin{Bmatrix} F(x) \\ G(x) \end{Bmatrix} = \hat{m}(x) \begin{Bmatrix} F(0) \\ G(0) \end{Bmatrix}$$

$$\hat{m}(x) = \begin{Bmatrix} \cos(k_{fx} x) & \frac{1}{k_{fx}\alpha_f} \sin(k_{fx} x) \\ -k_{fx}\alpha_f \sin(k_{fx} x) & \cos(k_{fx} x) \end{Bmatrix}$$

$$\|\hat{m}(x)\| = 1$$

To compute the fields at the end of the layer we set $x = d$

A) Two layers

$$\begin{bmatrix} F \\ G \end{bmatrix}_{d_1+d_2} = \hat{m}_2(d_2) \begin{bmatrix} F \\ G \end{bmatrix}_{d_1} = \hat{m}_2(d_2)\hat{m}_1(d_1) \begin{bmatrix} F \\ G \end{bmatrix}_0$$

B) N layers

......

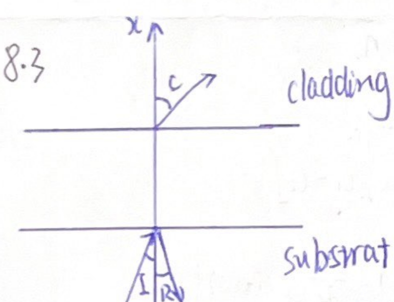
Summary: Incoming fields $F(0)$ and $G(0)$ given

k_z, α_f^i, ε_i, d_i given → matrix elements

multiplication of matrices

Outgoing fields $F(D)$ and $G(D)$

8.3



$$\vec{k}_I = \begin{bmatrix} k_{sx} \\ 0 \\ k_z \end{bmatrix}, \quad \vec{k}_R = \begin{bmatrix} -k_{sx} \\ 0 \\ k_z \end{bmatrix}, \quad \vec{k}_T = \begin{bmatrix} k_{cx} \\ 0 \\ k_z \end{bmatrix}$$

$$k_{sx} = \sqrt{\frac{w^2}{c^2}\varepsilon_s - k_z^2} = \sqrt{k_s^2(w) - k_z^2}$$

$$k_{cx} = \sqrt{\frac{w^2}{c^2}\varepsilon_c - k_z^2} = \sqrt{k_c^2(w) - k_z^2}$$

A) Field in substrate

$$F_s(x,z) = \exp(ik_z z)\left[F_I \exp(ik_{sx} x) + F_R \exp(-ik_{sx} x)\right]$$

$$G_s(x,z) = i\alpha_s k_{sx} \exp(ik_z z)\left[F_I \exp(ik_{sx} x) - F_R \exp(-ik_{sx} x)\right]$$

B) Field in layer system

$$F_f(x,z) = \exp(ik_z z) F(x)$$

$$G_f(x,z) = \exp(ik_z z) G(x)$$

$$\begin{bmatrix} F \\ G \end{bmatrix}_x = \hat{M}(x) \begin{bmatrix} F \\ G \end{bmatrix}_0$$

C) Field in cladding

$$F_c(x,z) = \exp(ik_z z) F_T \exp[ik_{cx}(x-D)]$$

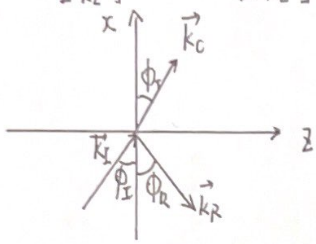
$$G_c(x,z) = i\alpha_c k_{cx} \exp(ik_z z) F_T \exp[ik_{cx}(x-D)]$$

R_{TE} $\quad T_{TE}$ $\quad R_{TM}$ $\quad T_{TM}$

$$\rho_{TE,TM} = |R_{TE,TM}|^2, \quad \tau_{TE,TM} = \frac{\mathrm{Re}(k_{cx})}{k_{sx}} |T_{TE,TM}|$$

Single interface

$$\vec{k_I} = \begin{bmatrix} k_{sx} \\ 0 \\ k_z \end{bmatrix}, \quad \vec{k_R} = \begin{bmatrix} -k_{sx} \\ 0 \\ k_z \end{bmatrix}, \quad \vec{k_T} = \begin{bmatrix} k_{cx} \\ 0 \\ k_z \end{bmatrix}$$



$$k_z = \frac{\omega}{c}\sqrt{\varepsilon_s}\sin\varphi_I = \frac{\omega}{c} n_s \sin\varphi_I$$

$$\rightarrow k_{ix} = \sqrt{\frac{\omega^2}{c^2}\varepsilon_i - k_z^2} = \sqrt{\frac{\omega^2}{c^2}\varepsilon_i - \frac{\omega^2}{c^2}\varepsilon_s \sin^2\varphi_I}$$

(discontinuous component) $\quad = \frac{\omega}{c}\sqrt{n_i^2 - n_s^2 \sin^2\varphi_I}$

$$\curvearrowright \; k_{sx} = \frac{\omega}{c} n_s \cos\varphi_I, \quad k_{cx} = \frac{\omega}{c}\sqrt{n_c^2 - n_s^2 \sin^2\varphi_I} = \frac{\omega}{c} n_c \cos\varphi_T \qquad \hat{M} = \hat{m}(d=0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

k_{cx} is real for $n_c > n_s \sin\varphi_I$, but imaginary for $n_c < n_s \sin\varphi_I$ (total internal reflection)

$$R_{TE} = \frac{k_{sx} - k_{cx}}{k_{sx} + k_{cx}} \qquad T_{TE} = \frac{2k_{sx}}{k_{sx} + k_{cx}} \quad \curvearrowright \rho_{TE} + \tau_{TE} = 1$$

$$\boxed{\cancel{R_{TM} = \frac{2k_{sx}\sqrt{\varepsilon_c \varepsilon_s}}{k_{sx}\varepsilon_c + k_{cx}\varepsilon_s}}}$$

$$R_{TM} = \frac{k_{sx}\varepsilon_c - k_{cx}\varepsilon_s}{k_{sx}\varepsilon_c + k_{cx}\varepsilon_s} \qquad T_{TM} = \frac{2k_{sx}\sqrt{\varepsilon_c \varepsilon_s}}{k_{sx}\varepsilon_c + k_{cx}\varepsilon_s} \quad \sim \rho_{TM} + \tau_{TM} = 1$$

Series + quiz

$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

$\nabla\cdot(\nabla\times\vec{A}) = 0$

$\nabla\times\nabla\times\vec{a} = \nabla(\nabla\cdot\vec{a}) - \Delta\vec{a}$

$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$

$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\exp(i\omega t)\,dt$

$\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$

$f(t) = \int_{-\infty}^{\infty} \tilde{f}(\omega)\exp(-i\omega t)\,d\omega$

$\nabla\times\vec{A}\times\vec{B} = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A})$

$\vec{A}\cdot\vec{B} = A_i B_i$

$F\{f\otimes g\} = 2\pi F\{f\} F\{g\}$

$(\vec{A}\times\vec{B})_i = \epsilon_{ijk} A_j B_k$

$F\{f(t)\}\otimes F\{g(t)\} = 2\pi F\{f(t)g(t)\}$

$$\begin{aligned}\vec{A}\cdot\vec{B} &= (A_i\vec{e_i})\cdot(B_j\vec{e_j})\\ &= (A_i B_j)(\vec{e_i}\cdot\vec{e_j})\\ &= (A_i B_j)\delta_{ij}\\ &= A_i B_i\end{aligned}$$

$\text{div}\,\vec{A} = \lim_{V\to 0}\frac{\oint_S \vec{A}\cdot d\vec{S}}{V}$

$\int_{-\infty}^{\infty} e^{-at^2\pm bt}\,dt = \sqrt{\frac{\pi}{a}}\cdot e^{\frac{b^2}{4a}}$

$\int_{-\infty}^{+\infty} e^{-at^2\pm ibt}\,dt = \sqrt{\frac{\pi}{a}}\cdot e^{-\frac{b^2}{4a}}$

$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega_0)x}\,dx = \delta(\omega-\omega_0)$

$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\,dx = \delta(\alpha)$

$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots$

Fourier series

$f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{in\omega_0 x} \quad (n\in\mathbb{Z})$