Task 1: Vector analysis (a=1, b=1, c=1, d=1, + 1* pts.)

Prove the following vector identities where **a**, **b** and **c** are vector-valued functions of the coordinate vector **r** and α is a scalar function (In general, quantities set in bold font represent vectors, while non-bold quantities refer to scalars.)

a) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

b) $\nabla\times(\alpha\mathbf{a}) = \alpha\nabla\times\mathbf{a} - \mathbf{a}\times\nabla\alpha$

c) $\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$

d) $\nabla\cdot(\nabla\times\mathbf{a}) = 0$

Hint: You can prove each relation by brute-force, simply writing each side using all the Cartesian vector components and crossing out similar terms. While that is perfectly acceptable here, we encourage you to use the so called *Einstein summation notation*, that makes use of the *Levi-Civita symbol*, ε_{ijk}. Look it up in any reference on mathematical physics and try to make use of the identities of the Levi-Civita symbol (which you do not need to prove). The bonus point goes to anyone who proves at least one of the relations in (a-c) in this way.

Einstein summation notation:

According to the notation, when an index variable appears twice in a single term and is not otherwise defined, it implies summation of that term over all the value of the index

So where the indices can range over the set $\{1,2,3\}$, $y=\sum_{i=1}^{3} c_i x^i = c_1x^1 + c_2x^2 + c_3x^3$

and can be simplified by the notation to $y = c_i x^i$ $\quad (x^1,x^2,x^3)$ is equivalent to (x,y,z)

upper indices represent components of vector

lower indices represent components of covectors

$$v = v^i e_i = [e_1, e_2 \cdots e_n]\begin{bmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \\ v^n \end{bmatrix}$$ where v is the vector and v^i are its components

$$w = w_i e^i = [w_1\ w_2 \cdots w_n]\begin{bmatrix} e^1 \\ e^2 \\ e^3 \\ \vdots \\ e^n \end{bmatrix}$$ Where w is the covector and w_i are its components

The basis vector elements e_i are each column vectors, and the covector basis elements e^i are each raw covectors

Inner product (also vector dot product)

$\vec{u}\cdot\vec{v} = u_j v^j$ $\qquad \nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} = \partial_i = \frac{\partial}{\partial x_i}$

vector cross product

$\vec{u}\times\vec{v} = \varepsilon^i_{\ jk} u^j v^k \vec{e_i}$ $\qquad \varepsilon^i_{\ jk} = \delta^{il}\varepsilon_{ljk}$

ε_{ijk} is the Levi-Civita symbol $\qquad \varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1), \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (2,1,3), \text{ or } (1,3,2) \\ 0 & \text{if } i=j \text{ or } j=k, \text{ or } k=i \end{cases}$

$\varepsilon_{ijk} A_i B_j C_k = \varepsilon_{jki} B_j C_k A_i$

$\varepsilon_{ijk}\varepsilon^{pqk} = \delta_i^p\delta_j^q - \delta_i^q\delta_j^p$

δ^{il} is the generalized Kronecker delta $\quad \delta_{ij} = \begin{cases} 0 & \text{if } i\neq j \\ 1 & \text{if } i=j \end{cases}$ $\qquad \varepsilon_{jmn}\varepsilon^{imn} = 2\delta_j^i \qquad \varepsilon_{ijk}\varepsilon^{ijk} = 6$

$\vec{e}^i\cdot\vec{e}_j = \delta^i_j \qquad a_i\delta_{ij} = a_j$

Based on this definition of ε, $\varepsilon^i_{\ jk} = \varepsilon_{ijk}$

(a) $\vec{a}\times(\vec{b}\times\vec{c}) = \varepsilon_{ijk} a_j (\vec{b}\times\vec{c})_k \hat{e_i} = \varepsilon_{ijk} a_k \varepsilon_{kmn} b_m c_n \hat{e_i} = \varepsilon_{ijk}\varepsilon_{kmn} a_j b_m c_n = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) a_j b_m c_n \hat{e_i}$

$= \delta_{im}\delta_{jn} a_j b_m c_n \hat{e_i} - \delta_{in}\delta_{jm} a_j b_m c_n \hat{e_i} = a_j b_i c_j \vec{e_i} - b_j c_i a_j \vec{e_i} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

(b) $\nabla\times(\alpha\vec{a}) = \varepsilon_{ijk}\nabla_j(\alpha\vec{a})_k\hat{e_i} = \varepsilon_{ijk}\nabla_j(\alpha a_k)\hat{e_i} = \varepsilon_{ijk}(a_k\nabla_j\alpha + \alpha\nabla_j a_k)\hat{e_i} = (\varepsilon_{ijk}\alpha\nabla_j a_k - \varepsilon_{ikj}a_k\nabla_j\alpha)\hat{e_i} = \alpha(\nabla\times\vec{a}) - \vec{a}\times\nabla\alpha$

(c) $\nabla\cdot(\vec{a}\times\vec{b}) = \nabla_i\cdot(\vec{a}\times\vec{b})_i = \nabla_i(\varepsilon_{ijk}a_j b_k)\hat{e_i} = (\varepsilon_{ijk}b_k\nabla_i a_j + \varepsilon_{ijk}a_j\nabla_i b_k)\hat{e_i} = (\varepsilon_{kij}b_k\nabla_i a_j - \varepsilon_{jik}a_j\nabla_i b_k)\hat{e_i}$

$= \vec{b}\cdot(\nabla\times\vec{a}) - \vec{a}\cdot(\nabla\times\vec{b})$

(d) $\nabla\cdot(\nabla\times\vec{a}) = \nabla_i(\nabla\times\vec{a})_i\hat{e_i} = \nabla_i(\varepsilon_{ijk}\nabla_j a_k) = (\varepsilon_{ijk}\nabla_i\nabla_j a_k + \varepsilon_{ijk}\nabla_j\nabla_i a_k)\hat{e_i} = (\varepsilon_{ijk}\nabla_i\nabla_j a_k - \varepsilon_{jik}\nabla_j\nabla_i a_k)\hat{e_i}$

$= \nabla\cdot(\nabla\times\vec a) - \nabla(\nabla\times\vec a) = 0$

Task 2: Conducting wire (4 pts.)

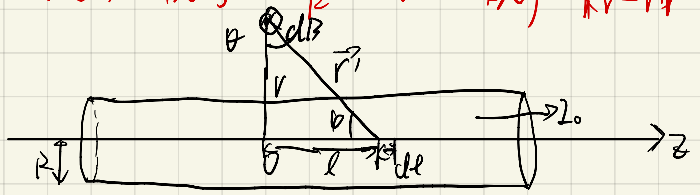
Consider the magnetostatic case of Maxwell's equations. Given a constant current density *inside* a cylindrically symmetric conducting wire of radius R that is oriented along the z-direction,

$$\mathbf{j}(\mathbf{r}) = \begin{cases} j_0\mathbf{e}_z & , r \le R \\ 0 & , r > R, \end{cases}$$

find the magnetic field $\mathbf{H}(\mathbf{r})$ inside ($r \le R$) and outside ($r > R$) the wire.
Hint: Make use of the symmetry properties of an infinitely extended cylindrical wire and their consequences for the components of $\mathbf{H}(\mathbf{r})$. Solve the resulting differential equations either directly or by the help of Stoke's theorem.)

$\vec B(\vec r) = \frac{\mu_0}{4\pi}\oint \frac{I\,d\vec l'\times\hat R}{R^2} = \frac{\mu_0}{4\pi}\oint\frac{I\,d\vec l'\times(\vec r-\vec r')}{|\vec r-\vec r'|^3} = \frac{\mu_0}{4\pi}\nabla\times\oint\frac{I}{|\vec r-\vec r'|}d\vec l'$ $\hat R = \frac{\vec R}{|R|}$ unit vector

Also $\vec B(\vec r) = \frac{\mu_0}{4\pi}\int\frac{\vec j(\vec r')\times\hat R}{R^2}dV' = \frac{\mu_0}{4\pi}\int\frac{\vec j(\vec r')\times(\vec r-\vec r')}{|\vec r-\vec r'|^3}dV' = \frac{\mu_0}{4\pi}\nabla\times\int\frac{\vec j(\vec r')}{|\vec r-\vec r'|}dV'$ $\vec R = \vec r-\vec r'$



$I(\vec r) = \begin{cases}\pi R^2 j_0 \vec e_z & r\le R\\ 0 & r>R\end{cases}$

$d\vec l\times\hat r' = dl\cdot|\hat r'|\cdot\sin\varphi = dl\sin\varphi = dl\cos\theta$

$l = r'\sin\theta$, $r = r'\cos\theta$ $\Rightarrow \frac{l}{r} = \tan\theta \Rightarrow l = r\tan\theta$

$\vec H(\vec r) = \frac{1}{\mu_0}\vec B(\vec r) = \frac{1}{4\pi}\oint\frac{I\,d\vec l\times\hat r'}{r'^2} = \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{I\,dl\cos\theta}{r'^2}$

$dl = r\,d\tan\theta = \frac{r\,d\theta}{\cos^2\theta} \Rightarrow \vec H = \frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{Ir\,d\theta\cos\theta}{\cos^2\theta\cdot\frac{r^2}{\cos^2\theta}} = \frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{I\cos\theta}{r}d\theta$

$r_0 \ge R$ $\vec H = \frac{I}{4\pi r}\sin\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{I}{2\pi r} = \frac{R^2 j_0}{2r}\vec e_z$

$r_0\le R$ $I' = \frac{\pi r^2}{\pi R^2}I \Rightarrow \vec H = \frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\pi r I\cos\theta}{\pi R^2}d\theta = \frac{rI}{4\pi R^2}\sin\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{rI}{2\pi R^2} = \frac{r\pi R^2 j_0}{2\pi R^2} = \frac{j_0}{2}r\vec e_z$

Task 3: Maxwell's equations (a=2, b=2, c=2, d=1 pts.)

Consider Maxwell's equations for the electric field $\mathbf{E}(\mathbf{r},t)$ and the magnetic field $\mathbf{H}(\mathbf{r},t)$ in empty space with sources $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t) = \sigma\mathbf{E}(\mathbf{r},t)$.

a) Derive the wave equation for the magnetic field $\mathbf{H}(\mathbf{r},t)$ for that case.
b) Transform this equation in the frequency domain to obtain the wave equation for the spectrum $\overline{\mathbf{H}}(\mathbf{r},\omega)$.
c) In lecture script section 2.1.5, it is demonstrated that the vectorial wave equation for $\overline{\mathbf{E}}(\mathbf{r},\omega)$ can be separated into two wave equations for $\overline{\mathbf{E}}_\perp$ and $\overline{\mathbf{E}}_\parallel$, assuming translational invariance along one direction (e.g y-direction). Starting from the wave equation obtained in (b), derive the separated equations for $\overline{\mathbf{H}}_\perp$ and $\overline{\mathbf{H}}_\parallel$, respectively.
d) The derivation of the decoupled equations for the magnetic field is simpler than that for the electric field. Observe the difference in the derivation and explain it briefly.

Maxwell equations: $\nabla\times\vec E(\vec r,t) = -\frac{\partial\vec B(\vec r,t)}{\partial t}$ $\nabla\cdot\vec D(\vec r,t) = \rho(\vec r,t)$

$\nabla\times\vec H(\vec r,t) = \vec j(\vec r,t) + \frac{\partial\vec D(\vec r,t)}{\partial t}$ $\nabla\cdot\vec B = 0$

(a) $\nabla\times(\nabla\times\vec H) = \nabla(\nabla\cdot\vec H) - \nabla^2\vec H$ $\nabla\cdot\vec H = \nabla\cdot\frac{1}{\mu_0}\vec B = 0$ thus $\nabla\times(\nabla\times\vec H) = -\nabla^2\vec H$

$\vec j(\vec r,t) = \sigma\vec E(\vec r,t)$ $\vec D(\vec r,t) = \varepsilon_0\vec E(\vec r,t) \Rightarrow \nabla\times\vec H = \sigma\vec E(\vec r,t) + \varepsilon_0\frac{\partial\vec E(\vec r,t)}{\partial t}$

$\nabla\times\nabla\times\vec H = \sigma\nabla\times\vec E(\vec r,t) + \varepsilon_0\frac{\partial\,\nabla\times\vec E(\vec r,t)}{\partial t}$ $\nabla\times\vec E(\vec r,t) = -\frac{\partial\vec B(\vec r,t)}{\partial t} = -\mu_0\frac{\partial\vec H(\vec r,t)}{\partial t}$

Then $\nabla\times\nabla\times\vec H = -\nabla^2\vec H = -\mu_0\sigma\frac{\partial\vec H(\vec r,t)}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2\vec H(\vec r,t)}{\partial t^2}$

$\Rightarrow$ wave equation: $\nabla^2\vec H(\vec r,t) - \mu_0\sigma\frac{\partial\vec H(\vec r,t)}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2\vec H(\vec r,t)}{\partial t^2} = 0$

(b) $f(t) = \int_{-\infty}^{\infty} f(\omega)e^{-i\omega t}d\omega$ $\frac{df(t)}{dt} = \int_{-\infty}^{\infty}f(\omega)\frac{d}{dt}e^{-i\omega t}d\omega = \int_{-\infty}^{\infty}-i\omega f(\omega)e^{-i\omega t}d\omega$

$\Rightarrow FT[\frac{df(t)}{dt}] = -i\omega$ similarly $FT[\frac{\partial^2 f(t)}{\partial t^2}] = (-i\omega)^2 = -\omega^2$

Thus in frequency domain: $\nabla^2\vec H(\vec r,\omega) + i\omega\mu_0\sigma\vec H(\vec r,\omega) + \omega^2\mu_0\varepsilon_0\vec H(\vec r,\omega) = 0$

(c) if $\vec H_\perp = \begin{bmatrix}0\\H_y\\0\end{bmatrix}$ $\vec H_\parallel = \begin{bmatrix}H_x\\0\\H_z\end{bmatrix}$ then $\frac{\partial}{\partial y}=0$ $\nabla' = \frac{\partial}{\partial x}+\frac{\partial}{\partial z}$ $\Delta' = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}$

$$\nabla\times\nabla\times\vec{H} = \nabla(\nabla\cdot\vec{H}) - \Delta\vec{H} = \nabla(\nabla\cdot\vec{H}) - \nabla^2\vec{H} \qquad \nabla(\nabla\cdot\vec{H}) = \text{grad}\cdot\text{div}\,\vec{H}$$

$$\nabla\cdot\vec{H} = \frac{\partial}{\partial x}\vec{H}_x + \frac{\partial}{\partial y}\vec{H}_y + \frac{\partial}{\partial z}\vec{H}_z \quad \left(\text{div}\cdot\vec{F} = \frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}F_y + \frac{\partial}{\partial z}F_z,\ \text{grad}\cdot\vec{F} = \frac{\partial}{\partial x}\vec{F} + \frac{\partial}{\partial y}\vec{F} + \frac{\partial}{\partial z}\vec{F}\right.$$

$$\nabla(\nabla\cdot\vec{H}) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}\bar{H}_x + \frac{\partial}{\partial y}\bar{H}_y + \frac{\partial}{\partial z}\bar{H}_z\right) =$$

$$= \left(\frac{\partial^2}{\partial x^2}\bar{H}_x + \frac{\partial}{\partial x}\frac{\partial}{\partial y}\bar{H}_y + \frac{\partial}{\partial x}\frac{\partial}{\partial z}\bar{H}_z\right) + \left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}\bar{H}_x + \frac{\partial^2}{\partial y^2}\bar{H}_y + \frac{\partial}{\partial y}\frac{\partial}{\partial z}\bar{H}_z\right) + \left(\frac{\partial}{\partial x}\frac{\partial}{\partial z}\bar{H}_x + \frac{\partial}{\partial y}\frac{\partial}{\partial z}\bar{H}_y + \frac{\partial^2}{\partial z^2}\bar{H}_z\right)$$

$$\nabla\cdot\vec{H} = 0$$

$$\nabla(\nabla\cdot\vec{H}) = \begin{bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{bmatrix}\cdot\begin{bmatrix}\partial/\partial x\,\bar{H}_x\\ \partial/\partial y\,\bar{H}_y\\ \partial/\partial z\,\bar{H}_z\end{bmatrix} = \begin{bmatrix}\frac{\partial^2}{\partial x^2}\bar{H}_x + \frac{\partial}{\partial x}\frac{\partial}{\partial y}\bar{H}_y + \frac{\partial}{\partial x}\frac{\partial}{\partial z}\bar{H}_z\\ \frac{\partial}{\partial y}\frac{\partial}{\partial x}\bar{H}_x + \frac{\partial^2}{\partial y^2}\bar{H}_y + \frac{\partial}{\partial y}\frac{\partial}{\partial z}\bar{H}_z\\ \frac{\partial}{\partial z}\frac{\partial}{\partial x}\bar{H}_x + \frac{\partial}{\partial z}\frac{\partial}{\partial y}\bar{H}_y + \frac{\partial^2}{\partial z^2}\bar{H}_z\end{bmatrix} \xrightarrow{\frac{\partial}{\partial y}=0} \begin{bmatrix}\frac{\partial^2}{\partial x^2}\bar{H}_x + \frac{\partial}{\partial x}\frac{\partial}{\partial z}\bar{H}_z\\ 0\\ \frac{\partial}{\partial z}\frac{\partial}{\partial x}\bar{H}_x + \frac{\partial^2}{\partial z^2}\bar{H}_z\end{bmatrix} = \nabla'(\nabla'\cdot\vec{H}) = 0$$

$$\nabla^2\vec{H} = \frac{\partial^2}{\partial x^2}\bar{H}_x + \frac{\partial^2}{\partial y^2}\bar{H}_y + \frac{\partial^2}{\partial z^2}\bar{H}_z \xrightarrow{\partial/\partial y=0} \frac{\partial^2}{\partial x^2}\bar{H}_x + \frac{\partial^2}{\partial z^2}\bar{H}_z = \Delta'\vec{H} \qquad \Delta' = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \qquad \vec{H}_\perp = \begin{bmatrix}0\\ \bar{H}_y\\ 0\end{bmatrix} \quad \vec{H}_\parallel = \begin{bmatrix}\bar{H}_x\\ 0\\ \bar{H}_z\end{bmatrix}$$

$$\vec{H}_\perp: \frac{\partial}{\partial y} = 0 \Rightarrow \nabla'(\nabla'\cdot\vec{H}_\perp) - \Delta'\vec{H}_\perp = -\Delta'\vec{H}_\perp \Rightarrow \underline{\Delta'\vec{H}_\perp(\vec{r},\omega) + i\omega\mu_0\sigma\vec{H}_\perp(\vec{r},\omega) + \frac{\omega^2}{c^2}\vec{H}_\perp(\vec{r},\omega) = 0} \qquad \nabla\cdot\vec{H}_\parallel = \nabla\cdot\vec{H}_\perp = 0$$

$$\vec{H}_\parallel: \frac{\partial}{\partial y} = 0 \Rightarrow \nabla'(\nabla'\cdot\vec{H}_\parallel) - \Delta'\vec{H}_\parallel = i\omega\mu_0\sigma\vec{H}_\parallel + \omega^2\mu_0\varepsilon_0\vec{H}_\parallel \Rightarrow \Delta'\vec{H}_\parallel + \frac{\omega^2}{c^2}\vec{H}_\parallel + i\omega\mu_0\sigma\vec{H}_\parallel = 0$$

$$\text{if } \vec{E}_\perp = \begin{bmatrix}0\\ E_y\\ 0\end{bmatrix} \text{ and } \frac{\partial}{\partial y} = 0 \text{ then } \vec{H}_\perp = \begin{bmatrix}\bar{H}_x\\ 0\\ \bar{H}_z\end{bmatrix} \quad \vec{H}_\parallel = \begin{bmatrix}0\\ \bar{H}_y\\ 0\end{bmatrix} \text{ and } \frac{\partial}{\partial x} = 0\ \frac{\partial}{\partial z} = 0 \Rightarrow \nabla' = \frac{\partial}{\partial y} \qquad \Delta' = \frac{\partial^2}{\partial y^2}$$

$$\nabla(\nabla\cdot\vec{H}) = \begin{bmatrix}\frac{\partial^2}{\partial x^2}\bar{H}_x + \frac{\partial}{\partial x}\frac{\partial}{\partial y}\bar{H}_y + \frac{\partial}{\partial x}\frac{\partial}{\partial z}\bar{H}_z\\ \frac{\partial}{\partial y}\frac{\partial}{\partial x}\bar{H}_x + \frac{\partial^2}{\partial y^2}\bar{H}_y + \frac{\partial}{\partial y}\frac{\partial}{\partial z}\bar{H}_z\\ \frac{\partial}{\partial z}\frac{\partial}{\partial x}\bar{H}_x + \frac{\partial}{\partial z}\frac{\partial}{\partial y}\bar{H}_y + \frac{\partial^2}{\partial z^2}\bar{H}_z\end{bmatrix} \xrightarrow{\frac{\partial}{\partial x}=\frac{\partial}{\partial z}=0} \begin{bmatrix}0\\ \frac{\partial^2}{\partial y^2}\bar{H}_y\\ 0\end{bmatrix} = \nabla'(\nabla'\cdot\vec{H}) \Rightarrow \nabla'(\nabla'\cdot\vec{H}_\parallel) = \frac{\partial^2}{\partial y^2}\bar{H}_y$$

$$\nabla'(\nabla'\cdot\vec{H}_\perp) = 0$$

$$\vec{H}_\parallel \Rightarrow \nabla'(\nabla'\cdot\vec{H}_\parallel) - \Delta'\vec{H}_\parallel = -\Delta'\vec{H}_\parallel \Rightarrow \underline{\Delta'\vec{H}_\parallel(\vec{r},\omega) + i\omega\mu_0\sigma\vec{H}_\parallel(\vec{r},\omega) + \frac{\omega^2}{c^2}\vec{H}_\parallel(\vec{r},\omega) = 0}$$

$$\vec{H}_\perp \Rightarrow \nabla'(\nabla'\cdot\vec{H}_\perp) - \Delta'\vec{H}_\perp = i\omega\mu_0\sigma\vec{H}_\perp + \frac{\omega^2}{c^2}\vec{H}_\perp \Rightarrow \underline{\Delta'\vec{H}_\perp + i\omega\mu_0\sigma\vec{H}_\perp + \frac{\omega^2}{c^2}\vec{H}_\perp = 0}$$

(d) In terms of $\vec{E}$: $\nabla\times\nabla\times\vec{E} = \nabla(\nabla\cdot\vec{E}) - \Delta\vec{E}$ and $\nabla\cdot\vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0}\rho(\vec{r},t) = \frac{\sigma}{\varepsilon_0}\vec{E}(\vec{r},t) \neq 0$

But in terms of $\vec{H}$: $\nabla\times\nabla\times\vec{H} = \nabla(\nabla\cdot\vec{H}) - \Delta\vec{H}$ $\qquad \nabla\cdot\vec{H} = \mu_0\nabla\vec{B} = 0$

Thus the derivation of the decouple equations for the magnetic field is simpler.

Task 4: Polarization and Symmetry (3 pts.)

In the source-free case, the general wave equation for the electric field in a dispersionless and isotropic medium of relative permittivity $\varepsilon(\mathbf{r})$ reads as

$$\nabla\times\nabla\times\mathbf{E}(\mathbf{r},t) + \frac{\varepsilon(\mathbf{r})}{c^2}\frac{\partial^2\mathbf{E}(\mathbf{r},t)}{\partial t^2} = 0.$$

Now assume translational symmetry in the x-direction. In other words, we have $\varepsilon(\mathbf{r}) = \varepsilon(y,z)$ and we are only interested in x-invariant solutions for the electric field. Show that under this assumption the wave equation can be split into two decoupled equations, one for each of two independent orthogonal polarization contributions.

source free $\rightarrow \nabla\cdot\vec{D} = \nabla\cdot\varepsilon\vec{E} = 0$

$\Rightarrow \nabla\cdot\vec{E} = 0$

According to the assumption, $\frac{\partial}{\partial x} = 0$

$$\vec{E}(\vec{r},t) = \vec{E}_\perp + \vec{E}_\parallel \qquad \vec{E}_\perp = \begin{bmatrix}E_x\\ 0\\ 0\end{bmatrix} \quad \vec{E}_\parallel = \begin{bmatrix}0\\ E_y\\ E_z\end{bmatrix} \qquad \nabla' = \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \qquad \Delta' = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla\times\nabla\times\vec{E}(\vec{r},t) = \nabla'(\nabla'\cdot\vec{E}(\vec{r},t)) - \Delta'\vec{E}(\vec{r},t) \xrightarrow{s.f} \nabla\times\nabla\times\vec{E}(\vec{r},t) = -\Delta'\vec{E}(\vec{r},t)$$

$$\nabla(\nabla\cdot\vec{E}) = \begin{bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{bmatrix}\begin{bmatrix}\partial/\partial x\,E_x\\ \partial/\partial y\,E_y\\ \partial/\partial z\,E_z\end{bmatrix} = \begin{bmatrix}\frac{\partial^2}{\partial x^2}E_x + \frac{\partial}{\partial x}\frac{\partial}{\partial y}E_y + \frac{\partial}{\partial x}\frac{\partial}{\partial z}E_z\\ \frac{\partial}{\partial y}\frac{\partial}{\partial x}E_x + \frac{\partial^2}{\partial y^2}E_y + \frac{\partial}{\partial y}\frac{\partial}{\partial z}E_z\\ \frac{\partial}{\partial z}\frac{\partial}{\partial x}E_x + \frac{\partial}{\partial z}\frac{\partial}{\partial y}E_y + \frac{\partial^2}{\partial z^2}E_z\end{bmatrix} \xrightarrow{\frac{\partial}{\partial x}=0} \begin{bmatrix}0\\ \frac{\partial^2}{\partial y^2}E_y + \frac{\partial}{\partial y}\frac{\partial}{\partial z}E_z\\ \frac{\partial}{\partial z}\frac{\partial}{\partial y}E_y + \frac{\partial^2}{\partial z^2}E_z\end{bmatrix} = \nabla'(\nabla'\cdot\vec{E}) = 0$$

$$\vec{E}_\perp: \nabla'(\nabla'\cdot\vec{E}_\perp) = \nabla'(\nabla'\cdot E_x) = \frac{\partial^2}{\partial x^2}E_x + \frac{\partial}{\partial x}\frac{\partial}{\partial y}E_y + \frac{\partial}{\partial x}\frac{\partial}{\partial z}E_z \xrightarrow{E_z=E_y=0\ \frac{\partial}{\partial x}=0} \nabla(\nabla\cdot\vec{E}_\perp) = 0$$

$$\text{Thus } \nabla'\times\nabla'\times\vec{E}_\perp(\vec{r},t) = \nabla'(\nabla'\cdot\vec{E}_\perp) - \Delta'\vec{E}_\perp = -\Delta'\vec{E}_\perp = -\frac{\varepsilon(\vec{r},t)}{c^2}\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2} \Rightarrow \underline{\Delta'\vec{E}_\perp(\vec{r},t) - \frac{\varepsilon(y,z)}{c^2}\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2} = 0}$$

$\vec{E}_{\parallel}$: $\nabla(\nabla\cdot\vec{E}_{\parallel}) = \begin{bmatrix} \frac{\partial^2}{\partial y^2}E_y + \frac{\partial}{\partial y\partial z}E_z \\ \frac{\partial}{\partial y\partial z}E_y + \frac{\partial^2}{\partial z^2}E_z \end{bmatrix} = 0 \Rightarrow -\Delta'\vec{E}_{\parallel}(\vec{r},t) + \frac{\varepsilon(y,z)}{c^2}\frac{\partial^2\vec{E}_{\parallel}(\vec{r},t)}{\partial t^2} = 0$

$$\Rightarrow \Delta'\vec{E}_{\parallel}(\vec{r},t) - \frac{\varepsilon(y,z)}{c^2}\frac{\partial^2\vec{E}_{\parallel}(\vec{r},t)}{\partial t^2} = 0$$

Task 5: Heaviside Step Function (4* pts.)

The Heaviside step function,

$$\Theta(t) = \begin{cases} 0 & ,t<0 \\ 1 & ,t>0 \end{cases}$$

has many applications in physics and engineering. Similarly, knowing its Fourier transform can be useful, e.g., when you try to prove the properties of the Kramers-Kronig relation. However, the standard (Riemann) integral to obtain its Fourier transform

$$\overline{\Theta}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Theta(t)e^{i\omega t}dt,$$

does not converge as Θ does not vanish at $+\infty$. Still, in the context of distributions, i.e. generalized functions, the Fourier transform exists and is given by

广义函数

$$\overline{\Theta}(\omega) = \frac{1}{2\pi}\left(\text{p.v.}\frac{i}{\omega} + \pi\delta(\omega)\right).$$

The above expression like the δ-distribution itself has to be interpreted inside an integral. p.v. stands for Cauchy's principal value and allows to assign values to integrals that otherwise would be undefined. If you have a function $f(x)$ that is unbounded at some point c, the integral over f is similarly unbounded. However, the integral over p.v.$f(x)$ that is defined by

$$\text{p.v.}\int_a^b f(x)dx \equiv \lim_{\varepsilon\to 0}\left(\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x)dx\right).$$

may be bounded. Use this definition to show that the inverse Fourier transform of $\overline{\Theta}(\omega)$ as given above indeed leads to the Heaviside function. Make use of the relation $\int_0^\infty \frac{\sin(ax)}{x}dx = \frac{\pi}{2}$ for $a>0$.

Recommendation: Use this chance to take a look at distribution theory in physics. You may look at how to prove $\int_0^\infty \frac{\sin(ax)}{x}dx = \frac{\pi}{2}$ for $a>0$ using Cauchy's integral theorem and Jordan's lemma. If you also want to study in more details, please go to mathematical physics literatures.

$$\int_0^\infty \frac{\sin(ax)}{x}dx = -\frac{1}{a}\int\frac{1}{x}\,d\cos ax$$

$$= -\frac{1}{a}\left[\frac{\cos ax}{x} - \int \cos ax\, d\frac{1}{x}\right.$$

$$FT^{-1}[\bar{\theta}(\omega)] = \int_{-\infty}^{\infty}\bar{\theta}(\omega)e^{-i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\text{P.V.}\frac{i}{\omega} + \pi\delta(\omega)]e^{-i\omega t}d\omega$$

$$= \frac{i}{2\pi}\int_{-\infty}^{\infty}\text{P.V.}\frac{1}{\omega}e^{-i\omega t}d\omega + \frac{1}{2}\int_{-\infty}^{\infty}\delta(\omega)e^{-i\omega t}d\omega$$

Since $\omega \neq 0 \Rightarrow = \frac{i}{2\pi}\left(\lim_{\varepsilon\to 0}\int_{-\infty}^{0-\varepsilon}\frac{1}{\omega}e^{-i\omega t}d\omega + \int_{0+\varepsilon}^{\infty}\frac{1}{\omega}e^{-i\omega t}d\omega\right) + \frac{1}{2}$

$$= \frac{i}{2\pi}\left(\lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}\frac{1}{\omega}e^{-i\omega t}d\omega + \int_{\varepsilon}^{\infty}\frac{1}{\omega}e^{-i\omega t}d\omega\right) + \frac{1}{2}$$

let $\omega = -\omega$

$\int_{-\infty}^{-\varepsilon} \Rightarrow -\int_{\varepsilon}^{\infty}$

$$= \frac{i}{2\pi}\left(\lim_{\varepsilon\to 0} -\int_{\varepsilon}^{\infty}\frac{1}{-\omega}e^{-i(-\omega)t}d(-\omega) + \int_{\varepsilon}^{\infty}\frac{1}{\omega}e^{-i\omega t}d\omega\right) + \frac{1}{2}$$

$$= \frac{i}{2\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\left(\frac{1}{\omega}e^{-i\omega t} - \frac{1}{\omega}e^{i\omega t}\right)d\omega + \frac{1}{2}$$

$$= \frac{i}{2\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\frac{-2i\sin(\omega t)}{\omega}d\omega + \frac{1}{2}$$

$$= \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\frac{\sin\omega t}{\omega}d\omega + \frac{1}{2}$$

According to $\int_0^\infty \frac{\sin ax}{x}dx = \frac{\pi}{2}$ for $a>0$

Then for $t>0$ $\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\frac{\sin t\omega}{\omega}d\omega = \frac{\pi}{2}$ $FT^{-1}[\bar{\theta}(\omega)] = \frac{1}{\pi}\cdot\frac{\pi}{2} + \frac{1}{2} = 1$

for $t=0$, $\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\frac{\sin(0)}{\omega}d\omega = 0$ $FT^{-1}(\bar{\theta}(\omega)] = 0\cdot\frac{\pi}{2} + \frac{1}{2} = \frac{1}{2}$

for $t<0$, $-t>0 \Rightarrow \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\frac{\sin(t\omega)}{\omega}d\omega = -\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\frac{\sin(-t\omega)}{\omega} = -\frac{\pi}{2}$

Thus $FT^{-1}[\bar{\theta}(\omega)] = \frac{1}{\pi}\cdot(-\frac{\pi}{2}) + \frac{1}{2} = 0$

$$\text{Thus } FT^{-1}(\bar{\theta}(\omega) = \theta(t) = \begin{cases} 1 & t>0 \\ \frac{1}{2} & t=0 \\ 0 & t<0 \end{cases}$$