FoMO in a nutshell

Important things to memorize for the course »Fundamentals of Modern Optics«*

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WS 16/17

Here, we collect important relations, facts, formulas and other things that one should memorize throughout the course to give you a summary of the most important results. Of course, we cannot list every required formula and we assume that you already have a solid math background. Hence, we cannot guarantee completeness of the list.

FOMO topics

Maxwell's equations

- Macroscopic Maxwell's equations (time domain):

$$\nabla\cdot\mathbf{D}(\mathbf{r},t)=\rho(\mathbf{r},t) \qquad \nabla\times\mathbf{E}(\mathbf{r},t)=-\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t}$$

$$\nabla\cdot\mathbf{B}(\mathbf{r},t)=0 \qquad \nabla\times\mathbf{H}(\mathbf{r},t)=\mathbf{j}(\mathbf{r},t)+\frac{\partial\mathbf{D}(\mathbf{r},t)}{\partial t}$$

- Macroscopic Maxwell's equations (frequency domain):

$$\nabla\cdot\mathbf{D}(\mathbf{r},\omega)=\rho(\mathbf{r},\omega) \qquad \nabla\times\mathbf{E}(\mathbf{r},\omega)=i\omega\mathbf{B}(\mathbf{r},\omega)$$

$$\nabla\cdot\mathbf{B}(\mathbf{r},\omega)=0 \qquad \nabla\times\mathbf{H}(\mathbf{r},\omega)=\mathbf{j}(\mathbf{r},\omega)-i\omega\mathbf{D}(\mathbf{r},\omega)$$

- Constitutive relations (linear material response; in optics usually $\mu(\mathbf{r},\omega)\equiv 1$ (non-magnetizable):

$$\mathbf{D}(\mathbf{r},\omega)=\varepsilon_0\mathbf{E}(\mathbf{r},\omega)+\mathbf{P}(\mathbf{r},\omega)=\varepsilon_0\varepsilon(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega) \qquad \mathbf{B}(\mathbf{r},\omega)=\mu_0\mu(\mathbf{r},\omega)\mathbf{H}(\mathbf{r},\omega)$$

$$\mathbf{P}(\mathbf{r},\omega)=\varepsilon_0\chi(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega)$$

- Time domain material response (response function):

$$\mathbf{P}(\mathbf{r},t)=\varepsilon_0\int_{-\infty}^{t}R(\mathbf{r},t-t')\mathbf{E}(\mathbf{r},t)\,\mathrm{d}t,\ \text{where}\ R(\mathbf{r},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\chi(\mathbf{r},\omega)e^{-i\omega t}\,\mathrm{d}\omega$$

- Complex permitivity:

$$\varepsilon(\mathbf{r},\omega)=1+\chi(\mathbf{r},\omega)+i\,\frac{\sigma(\mathbf{r},\omega)}{\varepsilon_0\omega}$$

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- Continuity equation (conservation of charge):

$$\nabla\cdot\mathbf{j}(\mathbf{r},t)+\frac{\partial\rho(\mathbf{r},t)}{\partial t}=0 \qquad \underbrace{\oint\!\!\!\oint_{\partial V}\mathbf{j}\,\mathrm{d}\mathbf{S}}_{I}=-\frac{\partial}{\partial t}\underbrace{\iiint_V\rho\,\mathrm{d}V}_{Q}$$

- Time averaged Poynting vector, loss:

$$\langle\mathbf{S}(\mathbf{r})\rangle=\frac{1}{2}\mathrm{Re}\left[\mathbf{E}(\mathbf{r},\omega)\times\mathbf{H}^*(\mathbf{r},\omega)\right] \qquad \nabla\cdot\langle\mathbf{S}(\mathbf{r})\rangle<0\Leftrightarrow\text{system is lossy}$$

- Wave equations (vacuum):

$$\nabla\times\nabla\times\mathbf{E}(\mathbf{r},t)-\frac{1}{c^2}\frac{\partial^2\mathbf{E}(\mathbf{r},t)}{\partial t^2}=0 \qquad \nabla\times\nabla\times\mathbf{H}(\mathbf{r},t)-\frac{1}{c^2}\frac{\partial^2\mathbf{H}(\mathbf{r},t)}{\partial t^2}=0$$

Normal modes in homogeneous, isotropic matter, non-magnetizable

- Helmholtz equation (wave equation in temporal Fourier domain; homogeneous, isotropic matter):

$$\Delta\mathbf{E}(\mathbf{r},\omega)+\frac{\omega^2}{c^2}\varepsilon(\omega)\,\mathbf{E}(\mathbf{r},t)=0$$

- Plane waves are the eigenmodes of free space. They take the form

$$\mathbf{E}(\mathbf{r},\omega)=\mathbf{E}_0\,e^{i\mathbf{kr}}$$

  and their dispersion relation reads as:

$$k_x^2+k_y^2+k_z^2=\frac{\omega^2}{c^2}\varepsilon(\omega)$$

- Refractive index:

$$n(\omega)=\frac{k(\omega)}{k_0}=\frac{k(\omega)}{\omega}c$$

- Maxwell relation:

  $n(\omega)=\sqrt{\varepsilon(\omega)}$ which implies for real and imaginary parts: $(n'+in'')^2=n'^2-n''^2+2in'n''=\varepsilon'+i\varepsilon''$

- Finding electric or magnetic field from each other in frequency domain in regions without source:

$$\mathbf{H}(\mathbf{r},\omega)=-\frac{i}{\omega\mu_0}\nabla\times\mathbf{E}(\mathbf{r},\omega) \qquad \mathbf{E}(\mathbf{r},\omega)=\frac{i}{\omega\varepsilon_0\varepsilon(\mathbf{r},\omega)}\nabla\times\mathbf{H}(\mathbf{r},\omega)$$

- Propagating, lossy and evanescent waves:

$$\langle\mathbf{S}\rangle\neq0,\nabla\cdot\langle\mathbf{S}\rangle=0\Leftrightarrow\text{lossless propagation}$$
$$\langle\mathbf{S}\rangle\neq0,\nabla\cdot\langle\mathbf{S}\rangle<0\Leftrightarrow\text{lossy propagation}$$
$$\langle\mathbf{S}\rangle=0,\nabla\cdot\langle\mathbf{S}\rangle=0\Leftrightarrow\text{evanescence}$$

Beam propagation

- ...see beam propagation scheme handout

Gaussian beams

- Rayleigh length:

$$z_0 = \frac{k}{2} w_0^2$$

- q-parameter definition:

$$\frac{1}{q(z)} = \frac{1}{R(z)} + i\frac{2}{kw^2(z)}$$

- q-parameter at waist position:

$$q_0 = -iz_0$$

- q-parameter after propagation:

$$q(d) = q(0) + d$$

- Condition for finding waist position:

$$\mathrm{Re}[q(z)] = 0$$

- ABCD matrix formalism:

$$\hat{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \hat{M}_{\text{tot}} = \hat{M}_n \cdot \ldots \hat{M}_3 \cdot \hat{M}_2 \cdot \hat{M}_1$$

$$q_{n+1} = \frac{Aq_n + B}{Cq_n + D}$$

Mathematical tools

Miscellaneous math formulas

- Complex exponentials, trigonometric and hyperbolic functions:

$$e^{ix} = \cos x + i \sin x$$
$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$
$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$
$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right) \qquad \cosh(ix) = \cos x$$
$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right) \qquad \sinh(ix) = i \sin x$$

- Integration:

$$\int u(x)\frac{\mathrm{d}v(x)}{\mathrm{d}x}\,\mathrm{d}x = u(x)v(x) - \int \frac{\mathrm{d}u(x)}{\mathrm{d}x}v(x)\,\mathrm{d}x$$
$$\int f(x)\,\mathrm{d}x = \int f(\xi)\frac{\mathrm{d}x}{\mathrm{d}\xi}\,\mathrm{d}\xi$$

- Gaussian functions:

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$$
$$\mathrm{FT}\left\{A\exp\left[-\frac{1}{2}\frac{t^2}{t_0^2}\right]\right\} = \frac{At_0}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\frac{\omega^2}{(1/t_0)^2}\right]$$

The Fourier transform of a Gaussian function is a Gaussian function.

- Area element in radial coordinates:

$$\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\varphi$$

Field theory

- Vector identities:

$$\nabla \times \nabla \times \mathbf{a} = \nabla \cdot (\nabla \mathbf{a}) - \Delta \mathbf{a}$$
$$\Delta a = \nabla(\nabla a), \text{ but } \Delta \mathbf{a} = (\nabla \cdot \nabla)\mathbf{a} = \nabla^2 \mathbf{a} \neq \nabla(\nabla \mathbf{a})$$
$$\mathbf{a} \times \mathbf{b} \times \mathbf{c} = \mathbf{b}(\mathbf{ac}) - \mathbf{c}(\mathbf{ab})$$

- Integral theorems:

$$\iiint_V \nabla \cdot \mathbf{a}\,\mathrm{d}V = \oiint_{\partial V} \mathbf{a}\,\mathrm{d}\mathbf{S} \qquad \text{(Gauss)}$$
$$\iint_A \nabla \times \mathbf{a}\,\mathrm{d}\mathbf{S} = \oint_{\partial A} \mathbf{a}\,\mathrm{d}\mathbf{r} \qquad \text{(Stokes)}$$

Fourier transform, δ-function

- In the course, we define the one-dimensional Fourier transform as (these definitions regarding sign and prefactor conventions influence nearly every expression in this document that contains Fourier transforms):

  Forward (*going to* Fourier domain):

  $$\tilde{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, \mathrm{d}t \qquad \text{(temporal Fourier domain)}$$

  $$\tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, \mathrm{d}x \qquad \text{(spatial Fourier domain)}$$

  A Fourier transform gives the strengths of the different plane wave (space) or time harmonic (time) frequency components.

  Backward (*coming from* Fourier domain):

  $$f(t) = \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{-i\omega t}\, \mathrm{d}\omega \qquad \text{(temporal Fourier domain)}$$

  $$f(x) = \int_{-\infty}^{\infty} \tilde{f}(k)\, e^{ikx}\, \mathrm{d}k \qquad \text{(spatial Fourier domain)}$$

  An inverse Fourier transform represents a decomposition of a function into plane wave components (space) or harmonic oscillations (time).

- Fourier shifting theorem:

  $$\mathrm{FT}\left\{f(t-t_0)\right\} = e^{i\omega t_0}\,\tilde{f}(\omega) \qquad \mathrm{FT}^{-1}\left\{\tilde{f}(\omega-\omega_0)\right\} = e^{-i\omega_0 t}\, f(t)$$

  $$\mathrm{FT}\left\{f(x-x_0)\right\} = e^{-ikx_0}\,\tilde{f}(k) \qquad \mathrm{FT}^{-1}\left\{\tilde{f}(k-k_0)\right\} = e^{ik_0 x}\, f(x)$$

  The shifting of a function corresponds to a harmonic modulation in Fourier domain.

- Fourier transform of a derivative (I cannot recall an occasion where the backward relations would be needed, so I leave them out):

  $$\mathrm{FT}\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right\} = -i\omega\,\tilde{f}(\omega)$$

  $$\mathrm{FT}\left\{\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right\} = ik\,\tilde{f}(k)$$

  A derivative in the real domain corresponds to a simple multiplication in Fourier domain.

- δ-function:

  $$\delta(x-x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_0)}\,\mathrm{d}k$$

  The δ-function is the Fourier transform of a plane wave (space) or harmonic oscillation (time).

  $$f(x_0) = \int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,\mathrm{d}x$$

  The δ-function »picks out« the value of a function at a particular point. It is just meaningful under an integral.

- Fourier transform of the δ-function:

$$\mathrm{FT}\{\delta(t-t_0)\} = \frac{1}{2\pi}e^{i\omega t_0} \qquad \mathrm{FT}^{-1}\{\delta(\omega-\omega_0)\} = e^{-i\omega_0 t}$$

$$\mathrm{FT}\{\delta(x-x_0)\} = \frac{1}{2\pi}e^{-ikx_0} \qquad \mathrm{FT}^{-1}\{\delta(k-k_0)\} = e^{ik_0 x}$$

  The Fourier transform of a δ-function is a plane wave (space) of harmonic oscillation (time)

- Convolution:

$$[f\otimes g](t) = \int f(\tau)g(t-\tau)\,\mathrm{d}t \qquad [f\otimes g](x) = \int f(x')g(x-x')\,\mathrm{d}x$$

$$\mathrm{FT}\{[f\otimes g](t)\} = 2\pi\tilde{f}(\omega)\tilde{g}(\omega) \qquad \mathrm{FT}\{[f\otimes g](x)\} = 2\pi\tilde{f}(k)\tilde{g}(k)$$

  Convolution in the real domain corresponds to a simple multiplication in the Fourier domain.