19.12.2014

to be returned: 09.01.2015, at the beginning of the lecture

Fundamentals of Modern Optics
Exercise 7
01.12.2014

to be returned: 08.12.2014, at the beginning of the lecture

Problem 1 - Paraxial optical cloak (2+3+3+2+2+1* for each drawbacks +1* for each methods)

An ABCD matrix for a system is given as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & L/n \\ 0 & 1 \end{pmatrix},$$

where L is the length of an optical system and n is the refractive index of the environment around the system ($n = 1$ in our case). The matrix represents an optical cloak in paraxial limit so that when an object is layed within the system, one will not see it from the other side. The matrix is as same as the simple translation matrix meaning that when the light passes through the system, the light simply translates by the distance L of the system regardless of what is inside the system (as long as they are away from the optical axis).

a) Assume a system with two thin lenses with a focal length of f_1 and f_2 where the distance between them is given as t (see figure 1a). In what condition, does this system act like an optical cloak?

b) Assume a system with four thin lenses which are placed with mirror symmetry (see figure 1b). In what condition, does this system act like an optical cloak? Express t_1 and t_2 as a function of f_1 and f_2. *You may use Mathmatica or Mathcad to solve this long and tedious calculation; if so, submit your code along with your solution.*

c) In b), what are the waists W_1, W_2, and W_3 (see figure 1b)? Remember $f_1, f_2 \ll 1/\lambda$. [illegible] $\Rightarrow \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

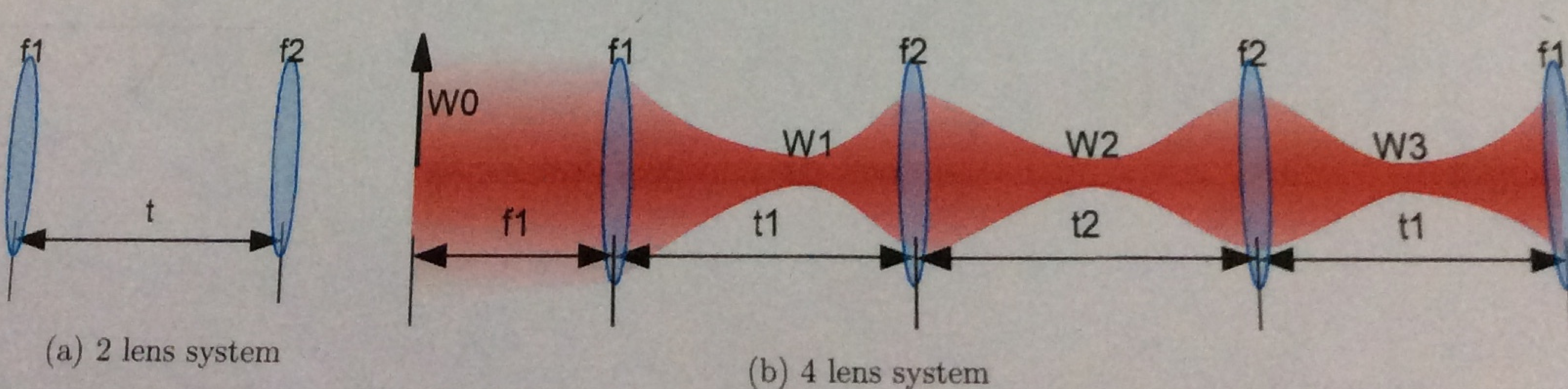
$t \to 0$, [illegible] very short

d) If $f_1 = 200$ mm, $f_2 = 100$ mm, $W_0 = 25$ mm, and $\lambda = 500$ nm, estimate W_1, W_2, and W_3 which you find in c). impossible in physics, [illegible]

e) Consider c). Suppose we have an opaque (不透明) ring with the radii of $R_{min} = 2W_3$ and $R_{max} = 5W_3$ and an opaque disk with a radius of $R = 0.1W_3$, and these are placed (not simulaneously) at position W_3 (in the middle of 3rd and 4th lenses) concentric with the opical axis. When a bright screen with images is placed at the left hand side of the system, explain what you will see at the other side of the system in each cases. 3 [illegible]

f*) What are the drawbacks of the system (figure 1b) as an optical cloak? Explain why they happen.

g*) Recommend and explain other methods for optical cloaking.



(a) 2 lens system

(b) 4 lens system

Figure 1: Paraxial optical cloak

17.5

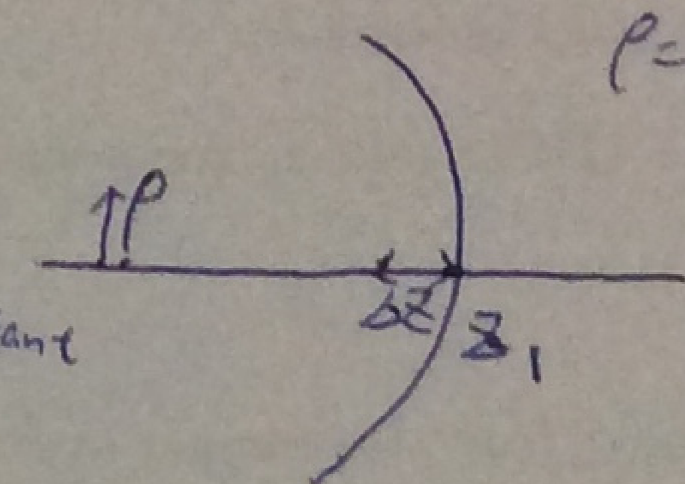
Problem 2 - Gaussian beam (2+2 points)

In the lecture we defined the Gaussian beam as

$$v(x,y,z) = A(z)\exp\left[-\frac{x^2+y^2}{w(z)^2}\right]\exp\left[ikz + i\frac{k}{2}\frac{x^2+y^2}{R(z)} + i\phi(z)\right].$$

a) Confirm that $R(z)$ in the expression $\exp\left[i\frac{k}{2}\frac{x^2+y^2}{R(z)}\right]$ is indeed the radius of the curvature 曲率 of a spherical wavefront.
Hint: Use paraxial approximation ($x^2/R^2 \ll 1$; $y^2/R^2 \ll 1$) and neglect Guoy phase shift.
Hint: Start with an arbitrary sphere in cartesian coordinate.

b) In which limit, the center of this sphere coincides with the position of the focus or the origin of the coordinate system ([x,y,z]=[0,0,0])? 一致



$\rho = \sqrt{x^2+y^2}$

$\phi(z) = 0$

$\Rightarrow e^{ik[z+\frac{x^2+y^2}{2R(z)}]} = e^{i\phi_0}$ make it as a constant

$\phi_0 = kz_1$

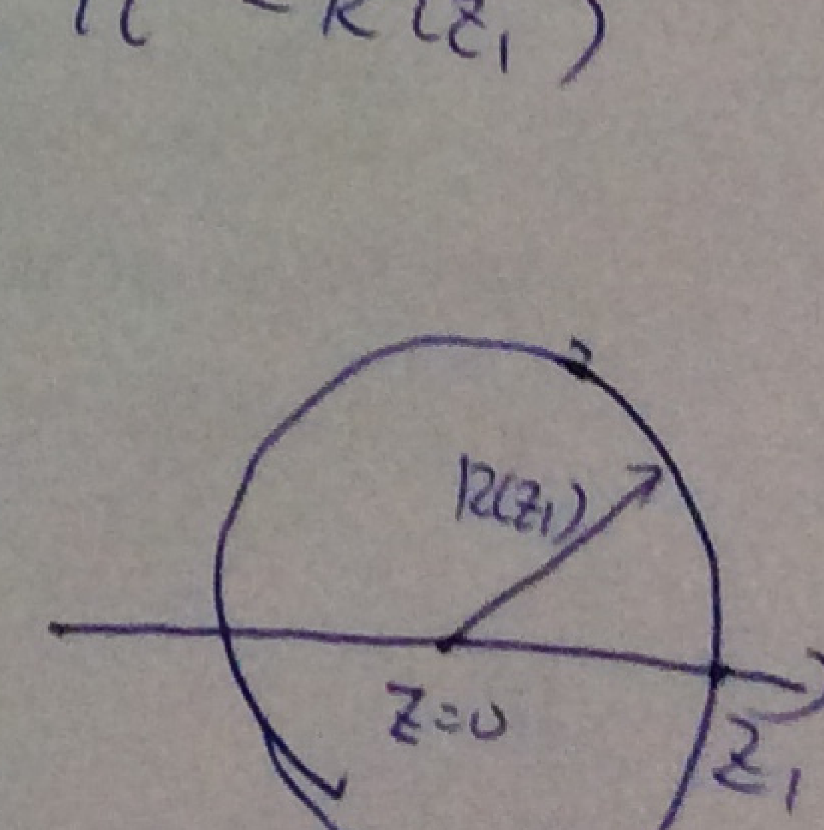
$z + \frac{x^2+y^2}{2R(z)} = z_1 \Rightarrow \frac{\rho^2}{2R(z)} = z_1 - z$

$R(z) = z + \frac{z_0^2}{z}$ (from lecture) $\Rightarrow 2(z+\frac{z_0^2}{z})z + \rho^2 = 2z_1(z+\frac{z_0^2}{z})$

$\Rightarrow 2z^2 + 2z_0^2 + \rho^2 = 2zz_1 + \frac{2z_0^2 z_1}{z} \xrightarrow[\substack{\Delta z \ll z_1 - z \\ \Delta z^2 \approx 0}]{z = z_1 + \Delta z} 2z_1\Delta z + \rho^2 \approx -\Delta z\frac{2z_0^2}{z_1}$

$\Rightarrow \rho^2 = -\Delta z(\frac{2z_0^2}{z_1} + 2z_1) \Rightarrow \Delta z = \frac{-\rho^2}{2(\frac{z_0^2}{z_1}+z_1)} = \frac{-\rho^2}{2R(z_1)}$ $R(z) \to$ [illegible] when $z_1 \gg z_0$

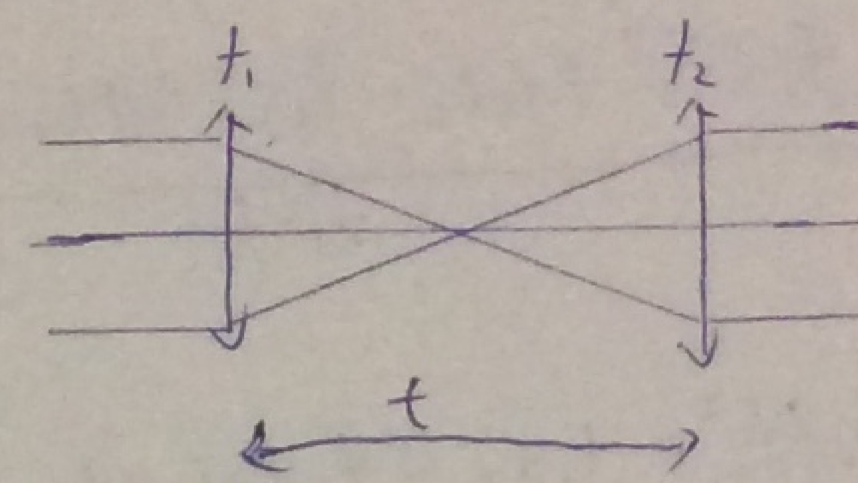
$(\Delta z)^2 + R^2(z_1) + 2R(z_1)\Delta z + \rho^2 = R^2(z_1) \Rightarrow [z - (z_1 - R(z_1))]^2 + \rho^2 = R^2(z_1)$



Yining Li 149824 FOMO Exercise 7 Monday

17.5

Problem 1 Paraxial optical cloak



a) $M = M_{f_2} M_t M_{f_1}$

$$= \begin{bmatrix} 1 & 0 \\ \frac{-1}{f_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{-1}{f_1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t \\ -\frac{1}{f_2} & \frac{-t}{f_2}+1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{-1}{f_1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\frac{t}{f_1} & t \\ \frac{t}{f_1 f_2} - \frac{1}{f_1} - \frac{1}{f_2} & 1-\frac{t}{f_2} \end{bmatrix}$$, with n=1

and we require M to be in $\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$ form

so $\begin{cases} 1-\frac{t}{f_1} = 1-\frac{t}{f_2} \\ \frac{t}{f_1 f_2} - \frac{1}{f_1} - \frac{1}{f_2} = 0 \end{cases} \Rightarrow \begin{cases} f_1 = f_2 = f \\ t = 2f \end{cases}$, $M = \begin{bmatrix} -1 & -t \\ 0 & -1 \end{bmatrix}$

$f_1 \to \infty$

$f_2 \to \infty$

1.5

$M = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

In this condition, the system act like an optical cloak.

b) $M = M_{f_1} M_{t_1} M_{f_2} M_{t_2} M_{f_2} M_{t_1} M_{f_1} M_d$

$$= \begin{bmatrix} 1 & 0 \\ \frac{-1}{f_1} & 1 \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{-1}{f_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & t_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{-1}{f_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{-1}{f_1} & 1 \end{bmatrix} \cancel{\begin{bmatrix} 1 & t_1 \\ 0 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} M_A & M_B \\ M_C & M_D \end{bmatrix}$$

use Mathematica to calculate the Martix, we get the simplify form ~~(The code is as follow)~~ of M, it is as follow:

$$\cancel{M_A = 1 - \frac{t_1}{f_2} - \frac{t_1+(1-\frac{t_1}{f_2})t_2}{f_2} - \frac{t_1+(1-\frac{t_1}{f_2})t_2 + t_1\left(1-\frac{t_1}{f_2} - \frac{t_1+(1-\frac{t_1}{f_2})t_2}{f_2}\right)}{f_1}}$$

$$M_A = \frac{f_1(f_2^2 + t_1 t_2 - f_2(2t_1+t_2)) - (f_2 - t_1)(-t_1 t_2 + f_2(2t_1+t_2))}{f_1^2 f_2^2}$$

$$M_B = \frac{f_1(f_2^2 + t_1 t_2 - f_2(2t_1+t_2))}{f_2^2}$$

$$M_C = -\frac{(f_1+f_2-t_1)(f_1(2f_2-t_2)+t_1 t_2 - f_2(2t_1+t_2))}{f_1^2 f_2^2}$$

$$M_D = \frac{-f_2^2 - t_1 t_2 + f_1(-2f_2+t_2) + f_2(2t_1+t_2)}{f_2^2}$$

If the system act like an optical clock, then

$$M = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} M_A & M_B \\ M_C & M_D \end{bmatrix} \text{ with } L = f_1 + t_1 + t_2 + t_1 = f_1 + 2t_1 + t_2$$

use Mathematica to calculate it, we get

$$\begin{cases} t_1 = f_1 + f_2 \\ t_2 = \dfrac{2(f_1 f_2 + f_2^2)}{f_1 - f_2}. \end{cases}$$

3

The code is in the last page.

c) For a Gaussian Beam

$$z_0 = \frac{\pi}{\lambda} w_0^2 \Rightarrow z_n = \frac{\pi}{\lambda} w_n^2$$

and $q_n = z - i z_n \Rightarrow z_n = -\mathrm{Im}\{q_n\}$

so $w_n = \sqrt{\frac{-\mathrm{Im}\{q\}\cdot\lambda}{\pi}}$

Before the first lens : $q_0 = -\frac{i\pi w_0^2}{\lambda}$

For w_1, the beam pass distance f_1 and lens f_1 ✓

$$M_1 = \begin{bmatrix} 1 & 0 \\ \frac{-1}{f_1} & 1 \end{bmatrix}\begin{bmatrix} 1 & f_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & f_1 \\ \frac{-1}{f_1} & 0 \end{bmatrix}$$

$$q_1 = \frac{Aq_0 + B}{Cq_0 + D} = \frac{q_0 + f_1}{-q_0/f_1} = -f_1 - \frac{f_1^2}{q_0} = -f_1 + i\frac{\lambda f_1^2}{\pi w_0^2}$$

$$\Rightarrow w_1 = \sqrt{\frac{-\mathrm{Im}\{q\}\lambda}{\pi}} = \left|\frac{f_1 \lambda}{\pi w_0}\right|$$ ✓

For W_2, the beam pass distance $t = f_1 + f_2$ and lens f_2 ✓

$$M_2 = \begin{bmatrix} 1 & 0 \\ \frac{-1}{f_2} & 1 \end{bmatrix}\begin{bmatrix} 1 & f_1+f_2 \\ 0 & 1 \end{bmatrix} M_1 = \begin{bmatrix} 1 & f_1+f_2 \\ \frac{-1}{f_2} & -\frac{f_1}{f_2} \end{bmatrix}\begin{bmatrix} 1 & f_1 \\ \frac{-1}{f_1} & 0 \end{bmatrix} = \begin{bmatrix} -\frac{f_2}{f_1} & f_1 \\ 0 & -\frac{f_1}{f_2} \end{bmatrix}$$

$$q_2 = \frac{Aq_0 + B}{Cq_0 + D} = \frac{-\frac{f_2}{f_1}q_0 + f_1}{-\frac{f_1}{f_2}} = -f_2 + \frac{f_2^2}{f_1^2} q_0 = \cancel{\frac{f_2^2}{f_1^2}} - f_2 - i\frac{f_2^2}{f_1^2}\cdot\frac{\pi w_0^2}{\lambda}$$

$$\Rightarrow w_2 = \sqrt{\frac{-\mathrm{Im}\{q\}\lambda}{\pi}} = \left|\frac{f_2}{f_1} w_0\right|$$ ✓

For W_3, the beam pass distance $t_2 = \frac{2(f_1 f_2 + f_2^2)}{f_1 - f_2}$ and a lens f_2

$$M_3 = \begin{bmatrix} 1 & 0 \\ \frac{-1}{f_2} & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{2(f_1 f_2 + f_2^2)}{f_1 - f_2} \\ 0 & 1 \end{bmatrix} M_2 = \begin{bmatrix} 1 & \frac{2(f_1 f_2 + f_2^2)}{f_1 - f_2} \\ \frac{-1}{f_2} & 1 - \frac{2(f_1 f_2 + f_2^2)}{(f_1 - f_2)\cdot f_2} \end{bmatrix}\cdot\begin{bmatrix} -\frac{f_2}{f_1} & f_1 \\ 0 & -\frac{f_1}{f_2} \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} -\frac{f_2}{f_1} & f_1 - \frac{2(f_1 f_2 + f_2^2)}{(f_1 - f_2)\cdot f_2} \\ \frac{1}{f_1} & -\frac{f_1}{f_2} - \frac{f_1\left(1 - \frac{2(f_1 f_2 + f_2^2)}{(f_1 - f_2)\cdot f_2}\right)}{f_2} \end{bmatrix}}$$

$$M_3 = \begin{bmatrix} -\frac{f_2}{f_1} & f_1 - \frac{2f_1(f_1f_2+f_2^2)}{(f_1-f_2)\cdot f_2} \\ \frac{1}{f_1} & -\frac{f_1}{f_2} - \frac{f_1\left(1-\frac{2(f_1f_2+f_2^2)}{(f_1-f_2)f_2}\right)}{f_2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{f_2}{f_1} & -\frac{f_1(f_1+3f_2)}{f_1-f_2} \\ \frac{1}{f_1} & \frac{4f_1}{f_1-f_2} \end{bmatrix}$$

$$q_3 = \frac{Aq_0+B}{Cq_0+D} = \frac{(f_1-f_2)\cdot q_0\cdot\left(-\frac{f_2}{f_1}\right) - f_1(f_1+3f_2)}{(f_1-f_2)\cdot q_0\cdot\frac{1}{f_1} + 4f_1} = \frac{-f_2\cdot(f_1-f_2)\cdot q_0 - f_1^2(f_1+3f_2)}{\underbrace{(f_1-f_2)\cdot q_0}_{a} + \underbrace{4f_1^2}_{b}}$$

$$= \frac{-f_2\cdot a - \frac{b}{4}\cdot(f_1+3f_2)}{a+b} = \frac{[-f_2 a - \frac{b}{4}(f_1+3f_2)][a-b]}{a^2-b^2} = \frac{-f_2a^2 + \frac{b^2}{4}(f_1+3f_2) - \frac{ab}{4}(f_1+3f_2) \cancel{\frac{b^2}{4}(f_1+3f_2)} + f_2 a\cdot b}{a^2-b^2}$$

since $f_1, f_2 \ll \frac{1}{\lambda}$, $q_0 = -i\frac{\pi w_0^2}{\lambda}$ $\Rightarrow q_0^2 \gg f_1^2, f_2^2 \Rightarrow a^2 \gg b^2$.

so $a^2-b^2 \simeq a^2$, and only $\underset{\text{term}}{ab}$ is related to Imagenary part. ($\underset{\text{terms}}{a^2, b^2}$ is real part)

$$\text{so } \underset{q_3-Re(q_3)}{\cancel{Im(q_3)}} = \frac{f_2\cdot a\cdot b - \frac{ab}{4}(f_1+3f_2)}{\underbrace{a^2-b^2}_{\simeq a^2}} \simeq \frac{f_2\cdot b}{a} - \frac{b}{4a}(f_1+3f_2) = \frac{f_2\cdot 4f_1^2}{(f_1-f_2)\cdot q_0} - \frac{f_1^2\cdot(f_1+3f_2)}{(f_1-f_2)\cdot q_0}$$

$$= -\frac{f_1^2(f_1-f_2)}{q_0(f_1-f_2)} = \frac{-f_1^2}{q_0} = -i f_1^2\cdot\frac{\lambda}{\pi w_0^2}$$

$$\Rightarrow Im\{q_3\} = -f_1^2\frac{\lambda}{\pi w_0^2}$$

$$\Rightarrow w_3 = \sqrt{\frac{-Im(q_3)\cdot\lambda}{\pi}} = \left|\frac{f_1\lambda}{\pi w_0}\right| = w_1$$

3

d) $W_1 = W_3 = \left|\frac{f_1 \lambda}{\pi W_0}\right| = \left|\frac{200 \times 500}{25\pi}\right| nm = 1273.2\, nm$

$W_2 = \left|\frac{f_2}{f_1} W_0\right| = \left|\frac{100}{200} \times 25\right| (nm) = 12.5\, nm$ m

2

e) ① opaque ring

Because the radii of $R_{min} = 2W_3$
so the light can go through the ring completely, that means the ring is under the "cloak". we can see the same image of the other side.

② opaque disk

Because the radii of $R = 0.1 W_3$

2

~~So the light will be stop by the disk partly.~~
So the light will be blocked by the disk partly.
We can see same obstacle in the other side or the image will become dark.

~~f) the lenses are not thin lens, they have thickness~~
~~so it is not ideal situation.~~

f) ① the lens is not thin lens, it has thickness. So it is not ideal situation.
② Only the object is near the waist, will it be ~~under~~ cloak, if it moves a little, it will ~~appare~~ appear.

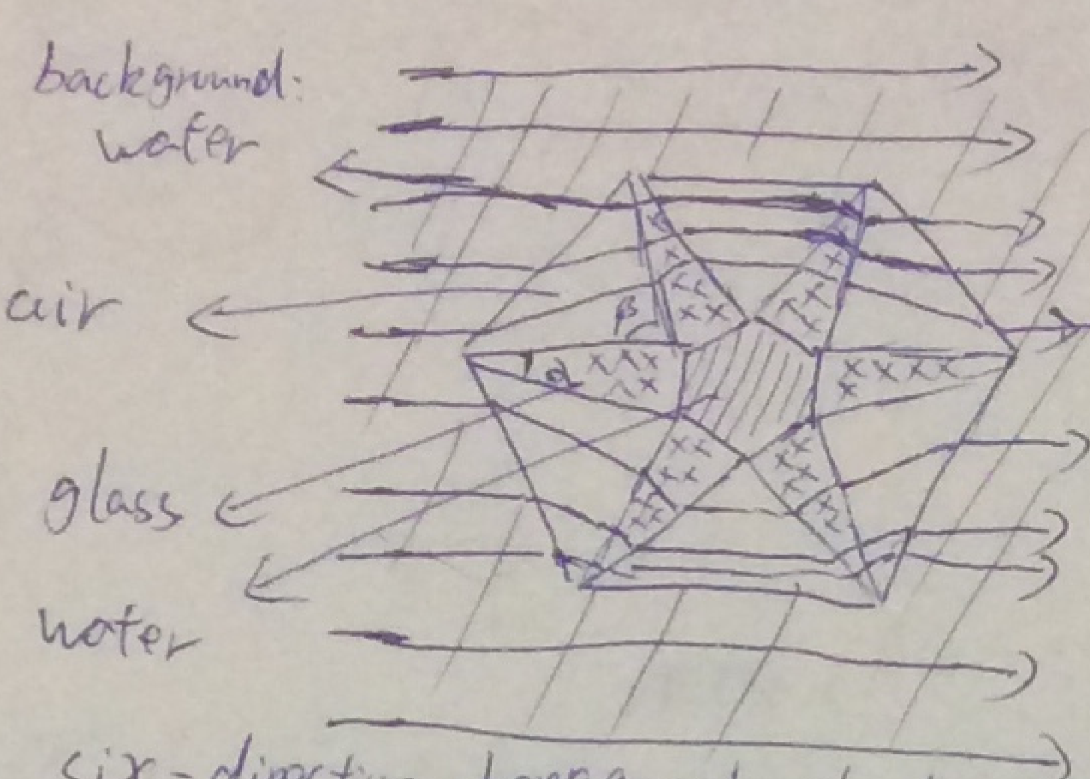
+1

① limited aperture

③ looking angle

② aberration

9) the other way is using the six-direction hexagonal cloak



//////// → water $n_b = 1.33$

xxxxx xxxx → glass $n_1 = 1.78$

white region → air $n_2 = 1$

$\alpha = 13\%$

figure 1: six-direction hexagonal cloak.

→: light rays

As in the figure, Ray diagram of light passing through the cloak in an aquatic environment. The background and region in the center contain water ($n_b = 1.33$), the white region is air ($n_2 = 1$). The region of is glass ($n_1 = 1.78$) with $\alpha = 13\%$.

So the optical path length of the rays travel through the cloak and those travel straight in water. That is, the region is under the cloak.

+1

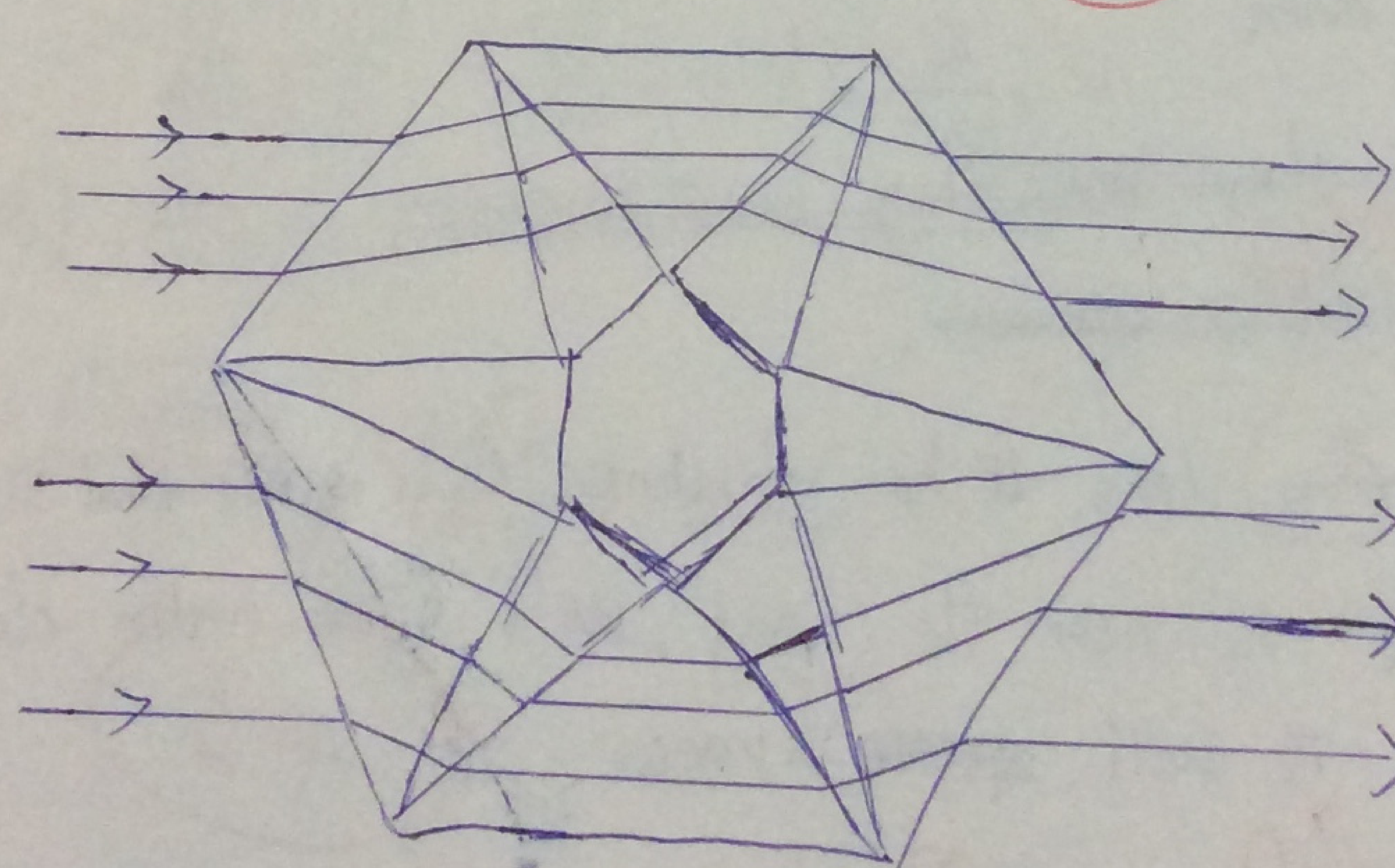
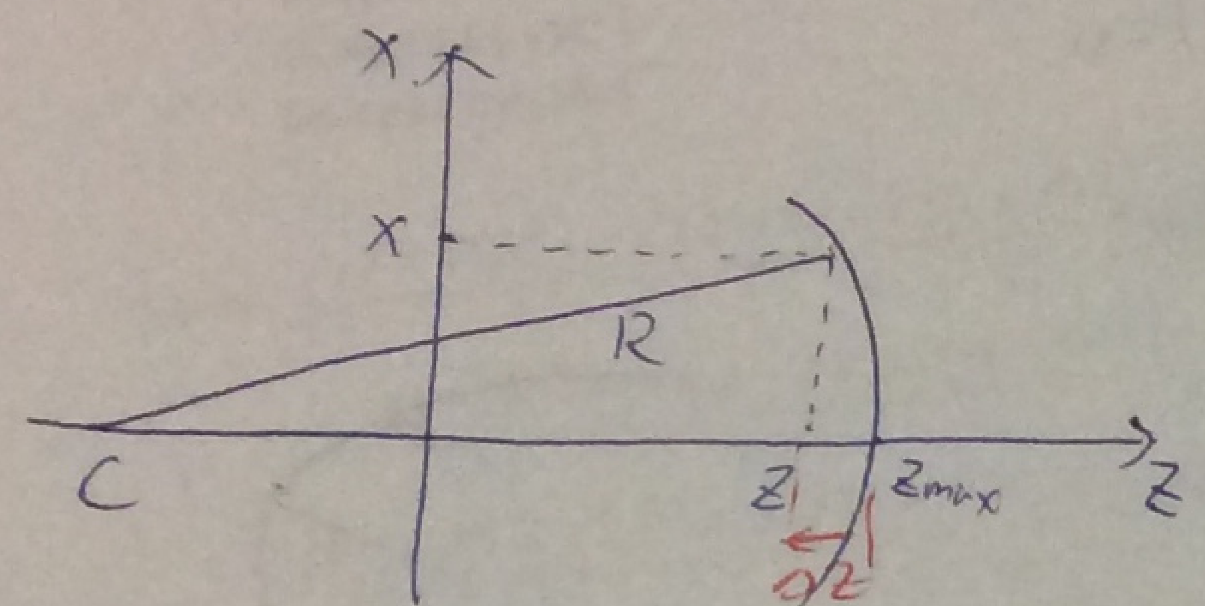


figure 2: rays travel through the cloak.

Problem 2 – Gaussian Beam

$$V(x,y,z) = A(z)\exp\left[-\frac{x^2+y^2}{w(z)^2}\right]\exp\left[ikz + i\frac{k}{2}\frac{x^2+y^2}{R(z)} + i\varphi(z)\right]$$

a) $x^2+y^2+(z-c)^2 = R^2$ ✓



$z - z_{max} = \Delta z$

$R = R(z_{max}) = z_{max} - C$

since $\Delta z = z - z_{max} < 0$.

$\Rightarrow x^2+y^2+(\Delta z+R)^2 = R^2$

In paraxial approximation $\frac{x^2}{R^2} \ll 1$, $\frac{y^2}{R^2} \ll 1$

$\Rightarrow \frac{x^2+y^2}{R^2} = \frac{R^2-(\Delta z+R)^2}{R^2} \ll 1$

$\Rightarrow 0 \ll \frac{(\Delta z+R)^2}{R^2} \leq 1$

and $\frac{(\Delta z+R)^2}{R^2} = \frac{R^2(\frac{\Delta z}{R}+1)^2}{R^2} = (\frac{\Delta z}{R}+1)^2 = 1+2\frac{\Delta z}{R}+\frac{\Delta z^2}{R^2} \gtrsim 1+2\frac{\Delta z}{R}$

So $\frac{x^2+y^2}{R^2} + (1+2\frac{\Delta z}{R}) = 1$

$\Rightarrow \frac{x^2+y^2}{R} + 2\Delta z = 0 \Rightarrow \frac{x^2+y^2}{2R} + \Delta z = 0$ ✓

for $\Delta z = z - z_{max}$

so $\frac{x^2+y^2}{2R} + z = z_{max}$ ✓

~~neglect Guoy phase shift.~~ ~~$\varphi(z)=0$~~

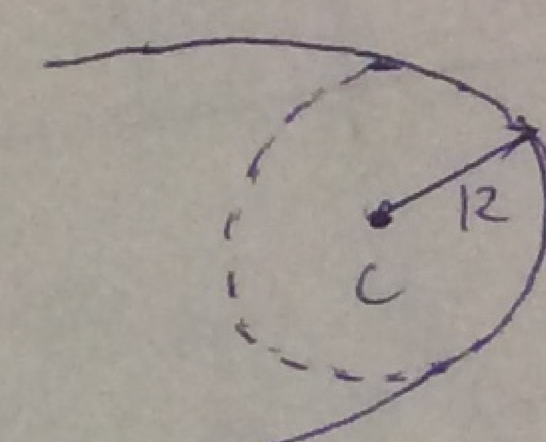
~~then~~ ~~$\frac{x^2+y^2}{2R(z)} + z = z_{max} = const$~~

In our case, the wave front is given by the phase of the beam

$$k\left(z + \frac{1}{2}\frac{x^2+y^2}{R(z)}\right) + i\varphi(z) = \text{constant}$$

we neglect Guoy phase shift, $\varphi(z)=0$

So $\frac{x^2+y^2}{2R(z)} + z$ should be constant C



Therefore, if the constant C is z_{max}

$$\Rightarrow \frac{x^2+y^2}{2R(z)} + z = \frac{x^2+y^2}{2R} + z = z_{max}$$

2

So $R(z)$ is the radius of curvature of a spherical wavefront, which approach the original parabolic surface in the paraxical approximation.

b) $R(z) = z\left(1+\left(\frac{z_0}{z}\right)^2\right)$

$R(z) \simeq z$ when $z_0 \ll z$

2

$\Rightarrow R(z) = z$ at $z \gg z_0$. ✓

$\left|\frac{z}{z_0}\right| \gg 1$ — limit of the sphere center and the focus coincidence.

In[316]:= $\texttt{Mf1} = \begin{pmatrix} 1 & 0 \\ -1/\texttt{f1} & 1 \end{pmatrix}$; $\texttt{Mf2} = \begin{pmatrix} 1 & 0 \\ -1/\texttt{f2} & 1 \end{pmatrix}$; $\texttt{Mt1} = \begin{pmatrix} 1 & \texttt{t1} \\ 0 & 1 \end{pmatrix}$; $\texttt{Mt2} = \begin{pmatrix} 1 & \texttt{t2} \\ 0 & 1 \end{pmatrix}$; $\texttt{Md} = \begin{pmatrix} 1 & \texttt{f1} \\ 0 & 1 \end{pmatrix}$;

```
M = Mf1.Mt1.Mf2.Mt2.Mf2.Mt1.Mf1.Md;
Simplify[M]
MA = M[[1, 1]];
MB = M[[1, 2]];
MC = M[[2, 1]];
MD = M[[2, 2]];
Simplify[MA]
Simplify[MB]
Simplify[MC]
Simplify[MD]
```

Out[318]= $$\left\{\left\{\frac{f1\,(f2^2 + t1\,t2 - f2\,(2\,t1 + t2)) - (f2 - t1)\,(-t1\,t2 + f2\,(2\,t1 + t2))}{f1\,f2^2}, \frac{f1\,(f2^2 + t1\,t2 - f2\,(2\,t1 + t2))}{f2^2}\right\}, \left\{-\frac{(f1 + f2 - t1)\,(f1\,(2\,f2 - t2) + t1\,t2 - f2\,(2\,t1 + t2))}{f1^2\,f2^2}, \frac{-f2^2 - t1\,t2 + f1\,(-2\,f2 + t2) + f2\,(2\,t1 + t2)}{f2^2}\right\}\right\}$$

M_A Out[323]= $$\frac{f1\,(f2^2 + t1\,t2 - f2\,(2\,t1 + t2)) - (f2 - t1)\,(-t1\,t2 + f2\,(2\,t1 + t2))}{f1\,f2^2}$$

M_B Out[324]= $$\frac{f1\,(f2^2 + t1\,t2 - f2\,(2\,t1 + t2))}{f2^2}$$

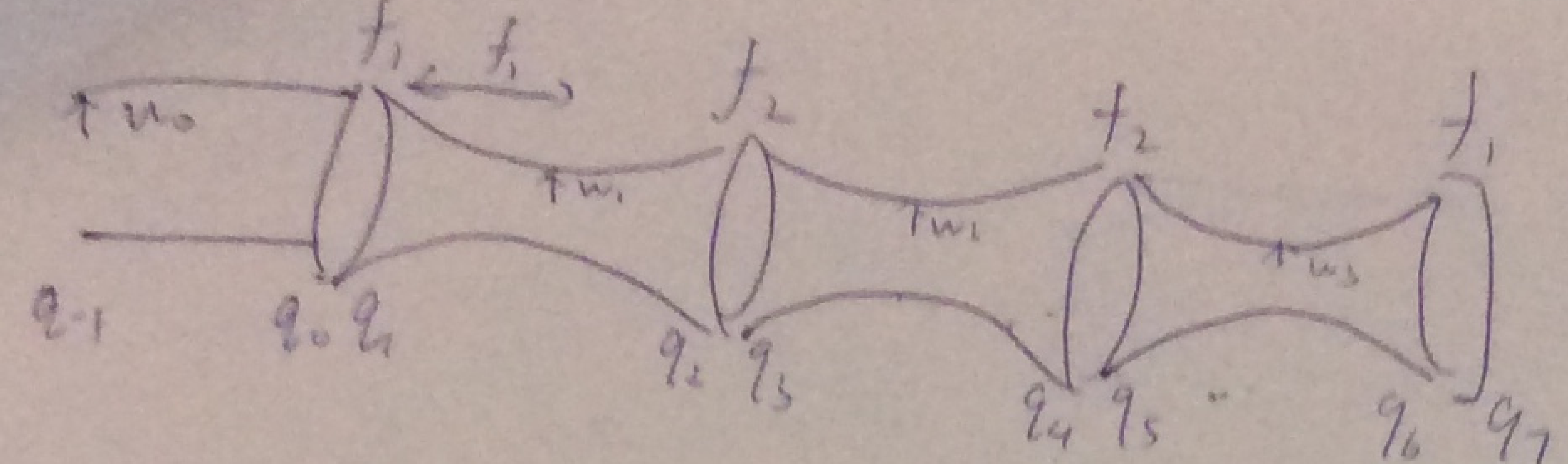
M_C Out[325]= $$-\frac{(f1 + f2 - t1)\,(f1\,(2\,f2 - t2) + t1\,t2 - f2\,(2\,t1 + t2))}{f1^2\,f2^2}$$

M_D Out[326]= $$\frac{-f2^2 - t1\,t2 + f1\,(-2\,f2 + t2) + f2\,(2\,t1 + t2)}{f2^2}$$

In[304]:=
```
Solve[M == {{1, f1 + 2 * t1 + t2}, {0, 1}}, {t1, t2}]
```

Out[304]= $$\left\{\left\{t1 \to f1 + f2,\ t2 \to \frac{2\,(f1\,f2 + f2^2)}{f1 - f2}\right\}\right\}$$

another way. Problem 1. c)



$$q_{-1} = iz_0 \Rightarrow q_0 = f_1 + iz_0 \Rightarrow q_1 = \frac{Aq_0 + B}{Cq_0 + D} = \frac{q_0}{-\frac{q_0}{f_1}+1} = \frac{f_1 + iz_0}{\frac{-iz_0}{f_1}} = -f_1 + i\frac{f_1^2}{z_0}$$

$$z_0 = \frac{\pi w_0^2}{\lambda} \qquad z_1 = \frac{\pi w_1^2}{\lambda} = \frac{f_1^2}{z_0} = \frac{f_1^2\lambda}{\pi w_0^2} \Rightarrow \boxed{w_1 = \frac{f_1\lambda}{\pi w_0}}$$

$$\Rightarrow q_2 = q_1 + f_1 + f_2 = f_2 + i\frac{f_1^2}{z_0} \Rightarrow q_3 = \frac{q_2}{-\frac{q_2}{f_2}+1} = \frac{f_2 + i\frac{f_1^2}{z_0}}{-i\frac{f_1^2}{f_2 z_0}}$$

$$q_3 = -f_2 + i\underbrace{\left(\frac{f_2}{f_1}\right)^2 z_0}_{z_2} \qquad z_2 = \frac{\pi w_2^2}{\lambda} \Rightarrow \boxed{w_2 = \frac{f_2}{f_1}w_0}$$

$$q_4 = f_2 + q_3 = 2f_2\frac{f_1+f_2}{f_1-f_2} - f_2 + iz_2 = \underbrace{f_2\frac{f_1+3f_2}{f_1-f_2}}_{\varepsilon} + iz_2$$

$$q_5 = \frac{q_4}{-\frac{q_4}{f_2}+1} = \frac{-(\varepsilon + iz)f_2}{\varepsilon + iz_2 - f_2} = \frac{-(\varepsilon + iz_2)f_2}{iz_2\left(1 + \frac{\varepsilon - f_2}{iz_2}\right)}$$

$$\approx \frac{-(\varepsilon + iz_2)f_2}{iz_2}\left(1 - \frac{\varepsilon - f_2}{iz_2}\right)$$

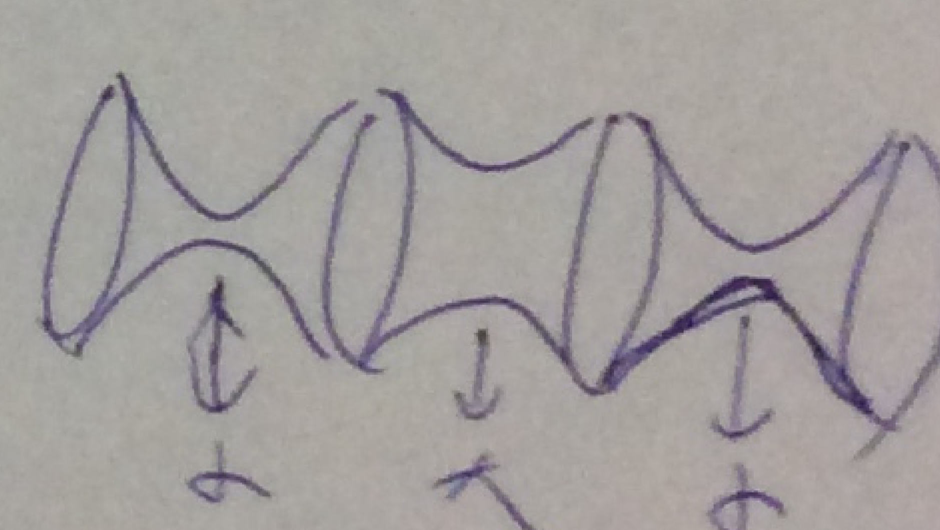
$$= \left(-f_2 - \frac{\varepsilon f_2}{iz_2}\right)\left(1 - \frac{(\varepsilon - f_2)}{iz_2}\right)$$

$$= -f_2 + \frac{f_2(\varepsilon - f_2)}{iz_2} - \frac{\varepsilon f_2}{iz_2} + \underbrace{\frac{\varepsilon f_2(\varepsilon - f_2)}{(iz_2)^2}}_{\approx 0}$$

$$\approx -f_2 - \frac{f_2^2}{iz_2}$$

$$= -f_2 + i\frac{f_2^2}{z_2}$$

$$\underbrace{}_{z_3} = \frac{\pi w_3^2}{\lambda} \Rightarrow w_3 = \frac{f_1\lambda}{\pi w_0}$$



$$q_6 = f_1 + iz_3 \Rightarrow q_7 = -f_1 + iz_0$$