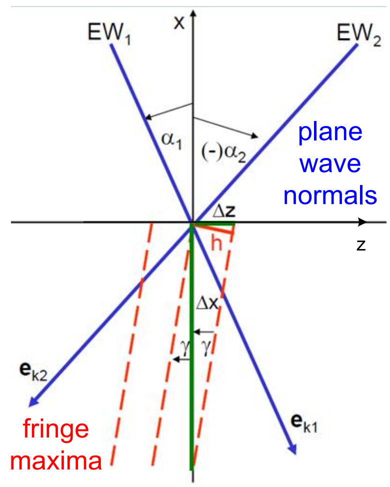
Optical Metrology and Sensing – Seminar 3

Task 1: Two-Beam Interference

Consider Two plane waves with same wavelength λ, normals $e_{k1,2}$, angles α_1, α_2 against x-axis:

$U_1(\vec{r})=A_1\exp(-j\overrightarrow{k_1}\vec{r})$ & $U_2(\vec{r})=A_2\exp(-j\overrightarrow{k_2}\vec{r})$



a) Write down the equation of interference between $U_1(\vec{r})$ and $U_2(\vec{r})$.
b) Find location and form of the maxima.
c) Find the distance of maxima along x, z and the angle γ
d) Find the fringe distance.

Task 2: Interference and polarization

Consider two monochromatic plane waves propagating in x-direction:

$$\vec{E}_1 = \vec{e}_1\sqrt{I_1}\exp\big(i(kx+\varphi_1)\big)$$
$$\vec{E}_2 = \vec{e}_2\sqrt{I_2}\exp\big(i(kx+\varphi_2)\big)$$

with real-valued intensities I_1, I_2 and complex-valued polarization vectors $\vec{e}_1, \vec{e}_2$ such that $\vec{e}_1 \neq \vec{e}_1^*$, $\vec{e}_2 \neq \vec{e}_2^*$. Assume $|\vec{e}_1| = |\vec{e}_2| = 1$. The intensity distribution as $I_{\text{tot}} = \vec{E}_{\text{tot}} \cdot (\vec{E}_{\text{tot}})^*$ for superimposed fields $\vec{E}_1$ and $\vec{E}_2$ can be written as

$$I_{\text{tot}} = I_1 + I_2 + 2\sqrt{I_1 I_2} \cdot Real\left\{ \vec{e}_1\vec{e}_2^*\exp\big(i(\varphi_1-\varphi_2)\big)\right\}$$

If we have two circularly polarized plane waves with opposite handedness i.e. with $\vec{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$ and $\vec{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$, prove that they do not interfere.

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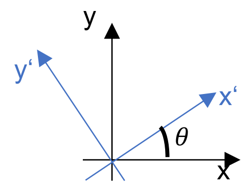
Task 3: Polarization

a) Using coordinate-transformation (rotation) matrix

$$R(\theta) = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$



where θ is an angle between the x-axis and the new x'-axis, which is rotated by θ (see image), derive the Jones matrix of a wave retarder with the fast axis rotated by α with respect to the x-axis.

b) Similar to a), derive the Jones matrix of a linear polarizer with the fast axis rotated by α with respect to the x-axis.

c) Imagine you need to attenuate a linearly polarized light source but preserve its polarization at the same time. What two polarization devices can one use to do so? How should they be placed in the optical setup?

d) Derive the resulting Jones matrix of these 2 devices for a general case (when the fast axis of the first device has angle α with the x-axis and the fast axis of the second device has β with the x-axis). Which angles should their fast axes have with the x-axis to block the beam when assuming incoming light is x-polarized?

Hint: use your derivations from a) and b)

Task 1: Two-Beam Interference

Consider Two plane waves with same wavelength λ, normals $e_{k1,2}$, angles α_1, α_2 against x-axis:

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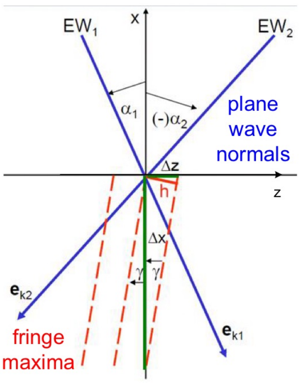

a) Write down the equation of interference between $U_1(\vec{r})$ and $U_2(\vec{r})$.
b) Find location and form of the maxima.
c) Find the distance of maxima along x, z and the angle γ
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(a) $U = U_1 + U_2 = A_1\exp(-j\vec{k}_1\vec{r}) + A_2\exp(-j\vec{k}_2\vec{r})$

$I = U^2 = [A_1\exp(-j\vec{k}_1\vec{r}) + A_2\exp(-j\vec{k}_2\vec{r})][A_1\exp(j\vec{k}_1\vec{r}) + A_2\exp(j\vec{k}_2\vec{r})]$

$= A_1^2 + A_2^2 + A_1A_2\exp[j(k_1-k_2)\vec{r}] + A_1A_2\exp[-j(k_1-k_2)\vec{r}]$

$= A_1^2 + A_2^2 + 2A_1A_2\cos[(k_1-k_2)\vec{r}]$ $\quad \varphi = \frac{2\pi}{\lambda}(x\cos\alpha_1 + z\sin\alpha_2) - \frac{2\pi}{\lambda}(x\cos\alpha_2 + z\sin\alpha_2)$

$\varphi = [k_1\vec{r}_1 - k_2\vec{r}_2]$

(b)

$= \frac{2\pi}{\lambda}[x(\cos\alpha_1 - \cos\alpha_2) + z(\sin\alpha_1 - \sin\alpha_2)]$

b) $\varphi = 2\pi m$ m is integer $\Rightarrow \frac{2\pi}{\lambda}[x(\cos\alpha_1 - \cos\alpha_2) + z(\sin\alpha_1 - \sin\alpha_2)] = 2\pi m$

$\Rightarrow x = \frac{-z(\sin\alpha_1 - \sin\alpha_2)}{\cos\alpha_1 - \cos\alpha_2} + \frac{m\lambda}{(\cos\alpha_1 - \cos\alpha_2)}$

-form = straight lines

c) $\Delta x = x(m) - x(m-1) = \frac{m\lambda}{\cos\alpha_1 - \cos\alpha_2} - \frac{(m-1)\lambda}{\cos\alpha_1 - \cos\alpha_2} = \frac{\lambda}{\cos\alpha_1 - \cos\alpha_2}$

$\Delta(z) = z(m) - z(m-1) = \frac{\lambda}{\sin\alpha_1 - \sin\alpha_2}$

$\tan\gamma = \frac{\Delta z}{\Delta x} = \frac{\cos\alpha_1 - \cos\alpha_2}{\sin\alpha_1 - \sin\alpha_2} = \frac{-2\sin\frac{\alpha_1+\alpha_2}{2}\sin\frac{\alpha_1-\alpha_2}{2}}{2\cos(\frac{\alpha_1+\alpha_2}{2})\sin\frac{\alpha_1-\alpha_2}{2}} = -\tan\frac{\alpha_1+\alpha_2}{2}$

$\Downarrow$

$\gamma = -\frac{\alpha_1+\alpha_2}{2}$

d) $\cos\alpha = \frac{h}{\Delta z}$ $\quad h = \Delta z\cos\gamma = \frac{\lambda}{\sin\alpha_1 - \sin\alpha_2}\cos\gamma$

$= \frac{\lambda\cos\frac{\alpha_1+\alpha_2}{2}}{2\cos(\frac{\alpha_1+\alpha_2}{2})\sin(\frac{\alpha_1-\alpha_2}{2})} = \frac{\lambda}{2\sin\frac{\alpha_1-\alpha_2}{2}}$

Task 2: Interference and polarization

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$\vec{e}_2 = \frac{1}{\sqrt{2}}[1, -i]$ $\quad \vec{e}_2^{\,*} = \frac{1}{\sqrt{2}}[1, i]$

If we have two circularly polarized plane waves with opposite handedness i.e. with $\vec{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$ and $\vec{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$, prove that they do not interfere.

$$I_{tot} = I_1 + I_2 + 2\sqrt{I_1 I_2}\, Real\Big\{\underbrace{\left[\tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix} \cdot \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\end{bmatrix}\right.}_{\downarrow}\exp[i(\varphi_1-\varphi_2)]\Big\} \qquad \begin{bmatrix}1\\ i\end{bmatrix}[1, i]$$

$$\frac{1}{2}(1\cdot 1 + i\cdot i) = 0$$

$\Rightarrow I_{tot} = I_1 + I_2$ $\quad$ I_{tot} is given by the incoherent sum of two wave

Task 3: Polarization

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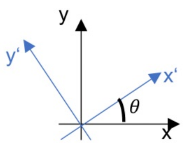

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c) Imagine you need to attenuate a linearly polarized light source but preserve its polarization at the same time. What two polarization devices can one use to do so? How should they be placed in the optical setup?

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Hint: use your derivations from a) and b)

(a) $T' = \begin{bmatrix}1 & 0\\ 0 & e^{-j\Gamma}\end{bmatrix}$ $\quad T = \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & e^{-j\Gamma}\end{bmatrix}\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}A_x\\ A_y\end{bmatrix}$

$$= \begin{bmatrix}\cos\alpha & -\sin\alpha\, e^{-j\Gamma}\\ \sin\alpha & \cos\alpha\, e^{-j\Gamma}\end{bmatrix}\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$$

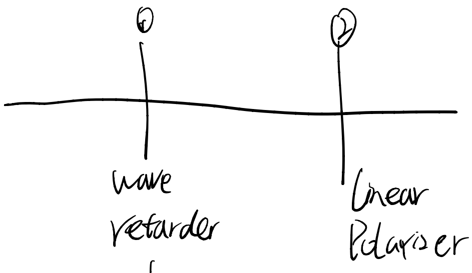
$$= \begin{bmatrix}\cos^2\alpha + \sin^2\alpha\, e^{-j\Gamma} & \cos\alpha\sin\alpha - \cos\alpha\sin\alpha\, e^{-j\Gamma}\\ \sin\alpha\cos\alpha - \sin\alpha\cos\alpha\, e^{-j\Gamma} & \sin^2\alpha + \cos^2\alpha\, e^{-j\Gamma}\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2\alpha + \sin^2\alpha\, e^{-j\Gamma} & \frac{1}{2}\sin 2\alpha(1-e^{-j\Gamma})\\ \frac{1}{2}\sin 2\alpha(1-e^{-j\Gamma}) & \sin^2\alpha + \cos^2\alpha\, e^{-j\Gamma}\end{bmatrix}$$

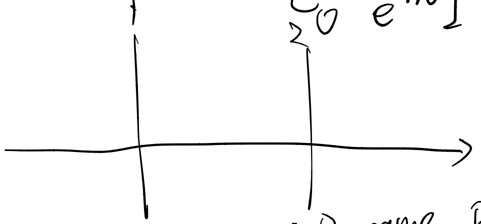
(b) $T' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad T = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$$= \begin{bmatrix} \cos\alpha & 0 \\ \sin\alpha & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix}$$

(c) light →



$\hookrightarrow \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi} \end{bmatrix}$



2P same Polarization as input light

1P

↳ retard (different Polarization compared to Polarization of input light)

(d) first device: half waveplate α with x-axis

Second device: Polarizer: x': β with respect x-axis

First: wave retarder $T = \begin{bmatrix} \cos^2\alpha + \sin^2\alpha e^{-i\Gamma} & \frac{1}{2}\sin 2\alpha (1-e^{-i\Gamma}) \\ \frac{1}{2}\sin 2\alpha (1-e^{-i\Gamma}) & \sin^2\alpha + \cos^2\alpha e^{-i\Gamma} \end{bmatrix}$ $\quad \Gamma = \pi$ half waveplate (HW)

$$T_{HW} = \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & \sin 2\alpha \\ \sin 2\alpha & \sin^2\alpha - \cos^2\alpha \end{bmatrix} = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$$

$$T_{pol} T_{HW} = \begin{bmatrix} \cos^2\beta & \cos\beta\sin\beta \\ \sin\beta\cos\beta & \sin^2\beta \end{bmatrix} \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\beta\cos 2\alpha + \cos\beta\sin\beta\sin 2\alpha & \cos^2\beta\sin 2\alpha - \cos\beta\sin\beta\cos 2\alpha \\ \sin\beta\cos\beta\cos 2\alpha + \sin^2\beta\sin 2\alpha & \sin\beta\cos\beta\sin 2\alpha - \sin^2\beta\cos 2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta(\cos\beta\cos2\alpha + \sin\beta\sin2\alpha) & \cos\beta(\cos\beta\sin2\alpha + \sin\beta\cos2\alpha) \\ \sin\beta(\cos\beta\cos2\alpha + \sin\beta\sin2\alpha) & \sin\beta(\cos\beta\sin2\alpha - \sin\beta\cos2\alpha) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos\beta\cos(2\alpha-\beta) & \cos\beta\sin(2\alpha-\beta) \\ \sin\beta\cos(2\alpha-\beta) & \sin\beta\sin(2\alpha-\beta) \end{bmatrix} = T_d T_{4w}$$

$$A_1 = \begin{bmatrix} A_1x \\ 0 \end{bmatrix} \Rightarrow A_2\begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} A_{2x} \\ A_{2y} \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos(2\alpha-\beta) & \cos\beta\sin(2\alpha-\beta) \\ \sin\beta(\cos(2\alpha-\beta) & \sin\beta(\sin(2\alpha-\beta) \end{bmatrix} \begin{bmatrix} A_1x \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} A_1x\cos\beta\cos(2\alpha-\beta) \\ A_1x\sin\beta\cos(2\alpha-\beta) \end{bmatrix} = \underline{\cos(2\alpha-\beta)}\begin{bmatrix} A_1x\cos\beta \\ A_1x\sin\beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \cos(2\alpha-\beta) = 0 \Rightarrow 2\alpha-\beta = k\pi + \frac{\pi}{2}$$

$$\cos\beta = 0 \Rightarrow \beta = \frac{\pi}{2} \pm n\pi$$

$$\cos(2\alpha-\beta) = 0 \Rightarrow 2\alpha - \frac{\pi}{2} \pm n\pi = \frac{\pi}{2} \pm m\pi$$

$$\Rightarrow 2\alpha = \pi \mp n\pi \pm m\pi = \pi(1 \mp n \pm m) = \alpha = \frac{\pi}{2}(1 \mp n \pm m) = \frac{\pi}{2}(1 \pm k)$$