Problem 1 (5 points)

Some telescopes for astronomical observations employ adaptive optics to compensate the distortions that are introduced by the earth's atmosphere. To calibrate such systems artificial laser guide stars are used. To generate those guide stars, sodium atoms are excited with a laser beam in the mesosphere at an altitude of around 90km, which afterwards emit spontaneously. The wavelength of the sodium transition is 589.2 nm and it has a natural linewidth of only 10MHz.

a) Assume that the sodium atoms are excited by a single frequency laser and that the whole laser power is absorbed if its frequency coincides with the central frequency of the transition ν_{21}. How much does the excitation efficiency drop if the frequency of the laser is shifted by 10 MHz with respect to the central frequency of the transition? (1 point)

If only considering the natural broadening, the absorbed power should have a Lorentzian shape:

$$P_a(\nu) = P_0 \frac{\left(\frac{\Delta\nu}{2}\right)^2}{(\nu - \nu_{21})^2 + \left(\frac{\Delta\nu}{2}\right)^2}$$

$\Delta\nu$ = 10 MHz

Before frequency shift:

$$P_a(\nu_{21}) = P_0$$

After frequency shift:

$$P_a(\nu_{21} + \Delta\nu) = P_0 \frac{\left(\frac{\Delta\nu}{2}\right)^2}{(\Delta\nu)^2 + \left(\frac{\Delta\nu}{2}\right)^2} = \frac{1}{5} P_0$$

$$\eta_{abs} = \frac{P_a(\nu_{21})}{P_a(\nu_{21} + \Delta\nu)} = 20\%$$

Excitation efficiency drops 80%

b) Now assume that the frequency of the laser shifts in time in the following manner: $\nu = \nu_{21} + 10MHz/s * t$.
What is the average excitation efficiency during the first ten seconds of the measurement? (1 point)

Average absorbed power:

$$\overline{P_a} = \frac{1}{T}\int_T P_a(\nu)dt = \frac{1}{T}\int_T P_0 \frac{\left(\frac{\Delta\nu}{2}\right)^2}{(\nu-\nu_{21})^2 + \left(\frac{\Delta\nu}{2}\right)^2} dt$$

$$\boxed{\begin{gathered}\nu = \nu_{21} + \frac{10MHz}{s}t \\ T = 10\ s \\ \Delta\nu = 10\text{ MHz}\end{gathered}}$$

$$= \frac{1}{10}\int_0^{10} P_0 \frac{25}{100t^2 + 25} dt$$

$$= \frac{P_0}{10}\int_0^{10} \frac{1}{4t^2+1} dt$$

$$\boxed{\int \frac{1}{(ax)^2+1} dx = \frac{\tan^{-1}(ax)}{a}}$$

$$= \frac{P_0}{10}\frac{1}{2}\left[\tan^{-1}(20) - \tan^{-1}(0)\right] = 0.076P_0$$

$$\overline{\eta_{abs}} = \frac{\overline{P_a}}{P_0} = 7.6\%$$

c) Now assume an inhomogeneously broadened laser with a spectral width of 1 GHz but the same average power as before. Assume spectral emission of the laser to be centered exactly at ν_{21}. How much does the excitation efficiency drop compared to the case when the laser is perfectly monochromatic? (1 point)

Inhomogeneously broadened laser → **Emission spectrum is a Gaussian function**

$$P_e(v) = A\exp\left(-4\ln 2\left(\frac{\nu - \nu_{21}}{\Delta\nu_L}\right)^2\right)$$

$\Delta\nu_L = 1GHz$
A is undetermined coefficient (related to power)

Monochromatic laser → **Emission spectrum is a Delta-function**

$$P_{emono}(v) = \delta(\nu - \nu_{21})$$

Assuming the average power is 1

They have the same average power:

$$\int_{-\infty}^{+\infty} P_e(v)\,d\nu = \int_{-\infty}^{+\infty} P_{emono}(v)\,d\nu = 1$$

$$\int_{-\infty}^{+\infty} P_e(v)\,d\nu = A\sqrt{\frac{\pi\Delta\nu_L^2}{4\ln 2}} = 1$$

$$\int_{-\infty}^{+\infty} exp(-ax^\wedge 2) = \sqrt{\frac{\pi}{a}}$$

$$A = \sqrt{\frac{4\ln 2}{\pi\Delta\nu_L^2}}$$

c) Now assume an inhomogeneously broadened laser with a spectral width of 1 GHz but the same average power as before. Assume spectral emission of the laser to be centered exactly at ν_{21}. How much does the excitation efficiency drop compared to the case when the laser is perfectly monochromatic? (1 point)

Inhomogeneously broadened laser

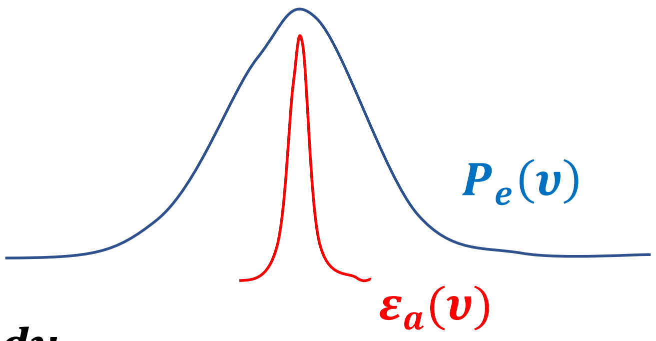
$$P_e(\nu) = A\exp\left(-4\ln 2\left(\frac{\nu-\nu_{21}}{\Delta\nu_L}\right)^2\right) \qquad A = \sqrt{\frac{4\ln 2}{\pi\Delta\nu_L^2}}$$

Monochromatic laser

$$P_{emono}(\nu) = \delta(\nu-\nu_{21})$$

Absorption of the medium:

$$\varepsilon_a(\nu) = \frac{P_a(\nu)}{P_0} = \frac{\left(\frac{\Delta\nu}{2}\right)^2}{(\nu-\nu_{21})^2+\left(\frac{\Delta\nu}{2}\right)^2}$$



$$\eta_{abs} = \frac{\int_{-\infty}^{+\infty} P_e(\nu)\varepsilon_a(\nu)\,d\nu}{\int_{-\infty}^{+\infty} P_{emono}(\nu)\varepsilon_a(\nu)\,d\nu} = \int_{-\infty}^{+\infty} P_e(\nu)\varepsilon_a(\nu)\,d\nu = P_e(\nu_{21})\int_{-\infty}^{+\infty}\varepsilon_a(\nu)\,d\nu$$

$$= A\int_{-\infty}^{+\infty}\frac{\left(\frac{\Delta\nu}{2}\right)^2}{(\nu-\nu_{21})^2+\left(\frac{\Delta\nu}{2}\right)^2}d\nu = A\frac{\pi\Delta\nu}{2} = \sqrt{\frac{4\ln 2}{\pi\Delta\nu_L^2}}\frac{\pi\Delta\nu}{2} = \sqrt{\pi\ln 2}\,\frac{\Delta\nu}{\Delta\nu_L} = 0.015$$

Excitation efficiency drops 98.5%

d) The temperature of the sodium atoms in the mesosphere is 200 K. Calculate the line width of the transition taking the Doppler broadening into account. Again assume that all the laser radiation would be absorbed if the atoms would be excited with single frequency radiation at the central frequency of the transition ν_{21}. (1 point)

From lecture eq(2.7c):

$$\Delta\nu_D = \frac{2\nu_{21}}{c}\sqrt{\frac{2kT\ln 2}{m}} = \frac{2}{\lambda_{21}}\sqrt{\frac{2kT\ln 2}{23u}} = 1.07GHz \qquad u = 1.66\times 10^{-27}kg$$

e) Repeat c) using the Doppler broadened absorption line of the sodium transition obtained in d). (1 point)

Now the absorbed power:

$$P_{aD}(\upsilon) = A_{aD}\exp\left(-4\ln 2\left(\frac{\nu-\nu_{21}}{\Delta\nu_D}\right)^2\right)$$

Absorption of the medium:

$$\varepsilon_{aD}(\nu) = \exp\left(-4\ln 2\left(\frac{\nu-\nu_{21}}{\Delta\nu_D}\right)^2\right)$$

Inhomogeneously broadened laser

$$P_e(\upsilon) = A\exp\left(-4\ln 2\left(\frac{\nu-\nu_{21}}{\Delta\nu_L}\right)^2\right) \qquad A = \sqrt{\frac{4\ln 2}{\pi\Delta\nu_L^2}}$$

Monochromatic laser

$$P_{emono}(\upsilon) = \delta(\nu-\nu_{21})$$

$$\eta_{abs} = \frac{\int_{-\infty}^{+\infty} P_e(\upsilon)\varepsilon_{aD}(\upsilon)\,d\nu}{\int_{-\infty}^{+\infty} P_{emono}(\upsilon)\varepsilon_{aD}(\upsilon)\,d\nu}$$

$$\eta_{abs} = \frac{\int_{-\infty}^{+\infty} P_e(v)\varepsilon_{aD}(v)\,d\nu}{\int_{-\infty}^{+\infty} P_{emono}(v)\varepsilon_{aD}(v)\,d\nu} = \int_{-\infty}^{+\infty} P_e(v)\varepsilon_{aD}(v)\,d\nu$$

$$= A\int_{-\infty}^{+\infty} \exp\left(-4\ln 2\left(\frac{\nu-\nu_{21}}{\Delta\nu_L}\right)^2\right)\exp\left(-4\ln 2\left(\frac{\nu-\nu_{21}}{\Delta\nu_D}\right)^2\right)d\nu$$

$$= A\int_{-\infty}^{+\infty} \exp\left(-4\ln 2\left(\frac{1}{\Delta\nu_L^2}+\frac{1}{\Delta\nu_D^2}\right)(\nu-\nu_{21})^2\right)d\nu$$

$$= A\sqrt{\frac{\pi}{4\ln 2}}\sqrt{\frac{1}{\frac{1}{\Delta\nu_L^2}+\frac{1}{\Delta\nu_D^2}}} = \sqrt{\frac{4\ln 2}{\pi\Delta\nu_L^2}}\sqrt{\frac{\pi}{4\ln 2}}\sqrt{\frac{1}{\frac{1}{\Delta\nu_L^2}+\frac{1}{\Delta\nu_D^2}}}$$

$$\boxed{A = \sqrt{\frac{4\ln 2}{\pi\Delta\nu_L^2}}}$$

$$= \sqrt{\frac{\frac{1}{\Delta\nu_L^2}}{\frac{1}{\Delta\nu_L^2}+\frac{1}{\Delta\nu_D^2}}} = \sqrt{\frac{\Delta\nu_D^2}{\Delta\nu_L^2+\Delta\nu_D^2}} \approx \sqrt{\frac{1}{2}} = 0.71$$

$$\boxed{\begin{array}{c}\Delta\nu_L = 1GHz\\ \Delta\nu_D = 1.07GHz\end{array}}$$

Excitation efficiency drops 29%

Problem 3 (6 Points)

A glass rod with a length of 10 cm and a diameter of 1 cm is doped with Ytterbium in a concentration of $1 \cdot 10^{19}\ cm^{-3}$. The absorption cross-section has a parabolic shape

$$\sigma_\nu = 2.3 \cdot 10^{-24} \cdot \left[1 - \left(\frac{\nu - \nu_0}{\frac{1}{2}\Delta\nu}\right)^2\right] \mathrm{m}^2$$

with its peak at 976 nm and a base width of 10 nm (this is the bandwidth where $\sigma_\nu \geq 0$. Also consider $\sigma_\nu = 0$ outside of this region).

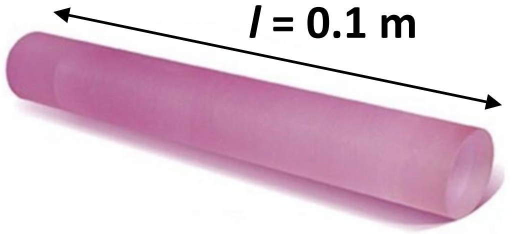
Taking into account that the rod is pumped from one of its 1 cm end facets:

a) Calculate the absorption coefficient. (1 Point)

We know:

$$N = 1 \cdot 10^{19}\ \mathrm{cm}^{-3} = 1 \cdot 10^{25}\ \mathrm{m}^{-3}$$

$$\sigma_\nu = 2.3 \cdot 10^{-24} \cdot \left(1 - \left(\frac{\nu - \nu_0}{\frac{1}{2}\Delta\nu}\right)^2\right)$$



First convert wavelength to frequency:

$$\nu_0 = \frac{c}{\lambda_0}$$

$$\frac{\Delta\nu}{\nu_0} = \frac{\Delta\lambda}{\lambda_0},$$

$$\Rightarrow \Delta\nu = \frac{c \cdot \Delta\lambda}{\lambda_0^2}$$

$$\nu_0 = 3.07 \cdot 10^{14}\ \mathrm{Hz}$$

$$\Delta\nu = 3.15 \cdot 10^{12}\ \mathrm{Hz}$$

From the script:

$$\alpha_\nu = N \cdot \sigma_\nu = 23 \cdot \left(1 - \left(\frac{\nu - 3.07 \cdot 10^{14}\ \mathrm{Hz}}{1.57 \cdot 10^{12}\ \mathrm{Hz}}\right)^2\right) \mathrm{m}^{-1}$$

Two different light sources are available for pumping the rod:

- a laser pump diode with a negligibly narrow bandwith centered at 976 nm.
- a regular LED with a rectangular output spectrum centered at 976 nm but with a bandwidth of 50 nm.

b) Calculate the fraction of light being absorbed when using the laser pump diode. (1 Point)

c) Calculate the fraction of light being absorbed when using the LED pump. (2 Points)

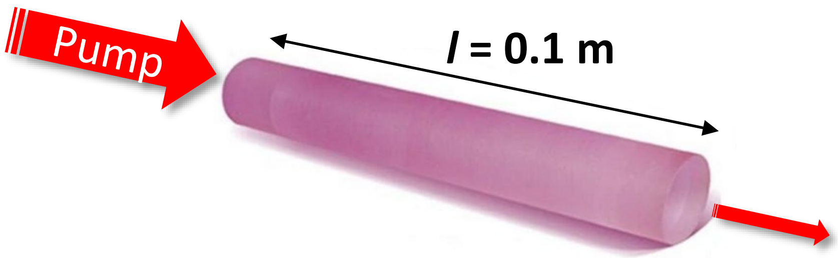
We know:

$$\alpha_\nu = 23 \cdot \left(1 - \left(\frac{\nu - 3.07 \cdot 10^{14}\ \mathrm{Hz}}{1.57 \cdot 10^{12}\ \mathrm{Hz}}\right)^2\right) \mathrm{m}^{-1}$$

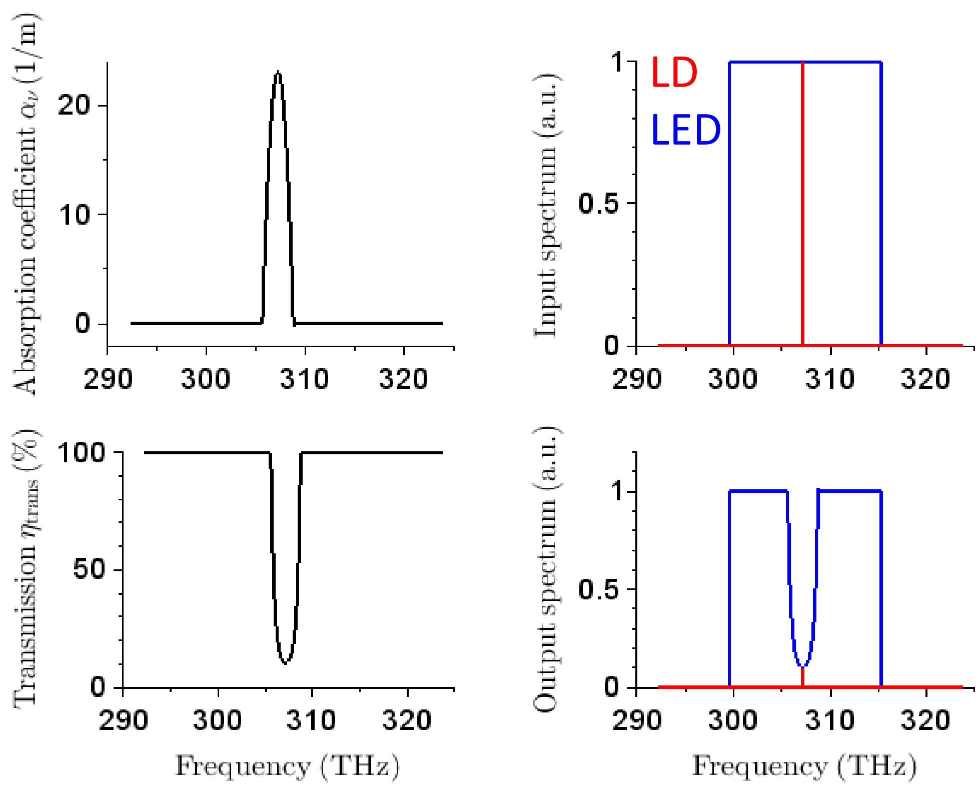
From script:

$$\frac{I_\nu(l)}{I_\nu(0)} = \exp(-\alpha_\nu \cdot l)$$

$$\eta_{abs} = 1 - \frac{I(l)}{I(0)} = 1 - \frac{\int I_\nu(l) d\nu}{\int I_\nu(0) d\nu}$$



What we deal with:



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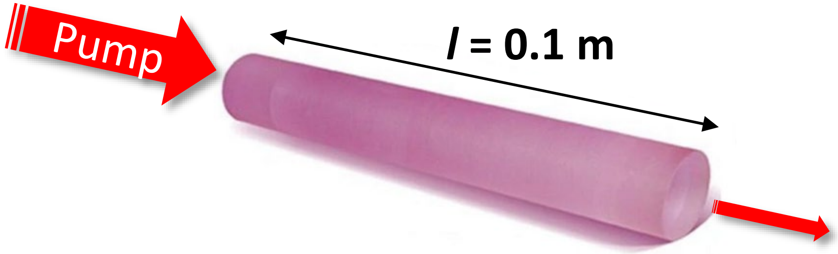

First Laser diode:

$$\eta_{abs,LD} = 1 - \frac{I(l)}{I(0)} = 1 - \frac{\int \delta(\nu - \nu_0) e^{(-\alpha_\nu \cdot l)}\, d\nu}{\int \delta(\nu - \nu_0)\, d\nu}$$

$$= 1 - e^{-\alpha_\nu(\nu_0) \cdot l}$$

$$= 1 - e^{-23 m^{-1} \cdot 0.1 m} = 0.9 = 90.0\ \%$$

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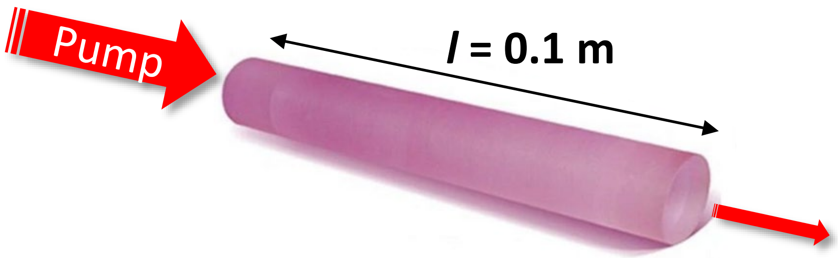
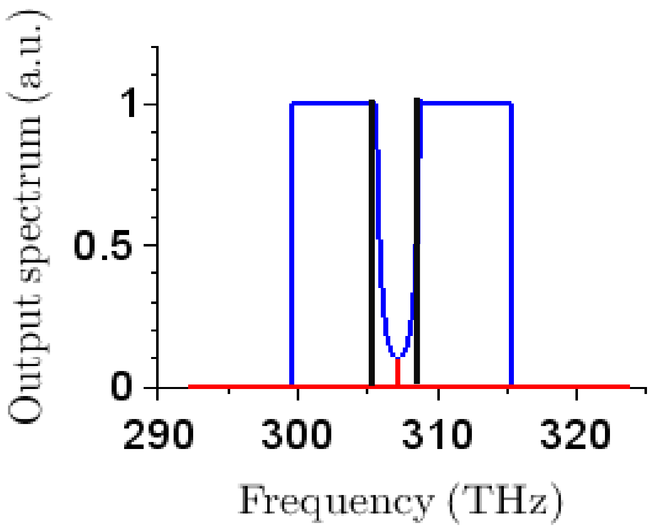


Second LED:

Integration over three regions:



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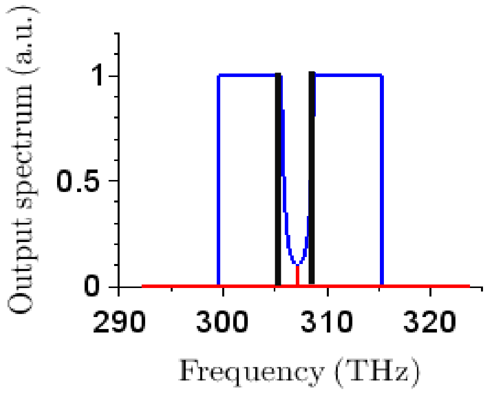

Second LED:

$$I(0) = \Delta\nu_{\mathrm{LED}} = \frac{c\Delta\lambda}{\lambda_0^2} = 1.57 \cdot 10^{13}\,\mathrm{Hz}$$

$$\eta_{\mathrm{abs,LED}} = 1 - \frac{1}{\Delta\nu_{\mathrm{LED}}}\left(\int_{\nu_0-\frac{\Delta\nu_{\mathrm{LED}}}{2}}^{\nu_0-\frac{\Delta\nu}{2}} \exp(-0\cdot l)\,\mathrm{d}\nu + \int_{\nu_0-\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu}{2}} \exp(-\alpha_\nu(\nu)l)\,\mathrm{d}\nu + \int_{\nu_0+\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu_{LED}}{2}} \exp(-0\cdot l)\,\mathrm{d}\nu\right)$$

$$= 1 - \frac{1}{\Delta\nu_{\mathrm{LED}}}\left(\int_{\nu_0-\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu}{2}} \exp(-\alpha_\nu(\nu)l)\,\mathrm{d}\nu + \Delta\nu_{\mathrm{LED}} - \Delta\nu\right)$$

$$= 1 - \frac{1}{\Delta\nu_{\mathrm{LED}}}\left(\int_{\nu_0-\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu}{2}} \exp\left(-23\cdot\left(1-\left(\frac{\nu-\nu_0}{\Delta\nu/2}\right)^2\right)\mathrm{m}^{-1}\ l\right)\mathrm{d}\nu + \Delta\nu_{\mathrm{LED}} - \Delta\nu\right)$$

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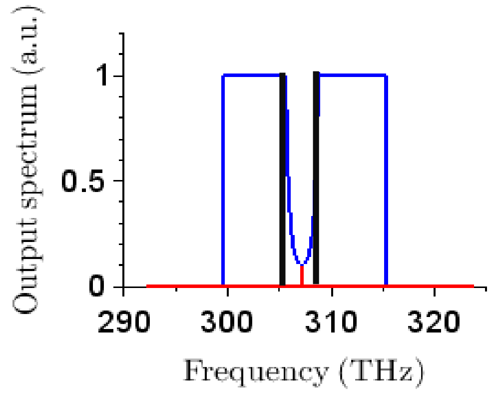


Second LED:

$$\eta_{\text{abs,LED}} = 1 - \frac{1}{\Delta\nu_{\text{LED}}}\left(\int_{\nu_0-\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu}{2}} \exp\left(-23\cdot\left(1-\left(\frac{\nu-\nu_0}{\Delta\nu/2}\right)^2\right)\ l\right)\mathrm{d}\nu + \Delta\nu_{\text{LED}} - \Delta\nu\right)$$

$$\eta_{\text{abs,LED}} = 1 - \frac{1}{\Delta\nu_{\text{LED}}}\left(\int_{-\frac{\Delta\nu}{2}}^{\frac{\Delta\nu}{2}} \exp\left(-23\cdot\left(1-\left(\frac{\nu}{\Delta\nu/2}\right)^2\right)\ l\right)\mathrm{d}\nu + \Delta\nu_{\text{LED}} - \Delta\nu\right)$$

$$\eta_{\text{abs,LED}} = 1 - \frac{1}{\Delta\nu_{\text{LED}}}\left(\exp(-2.3)\frac{\sqrt{\pi}\Delta\nu\,\mathrm{erfi}(\sqrt{2.3})}{2\sqrt{2.3}} + \Delta\nu_{\text{LED}} - \Delta\nu\right)$$

$\Rightarrow \eta_{\text{abs,LED}} = 14.4\,\%$

Error function

$$\boxed{erf(x) = \sqrt{\frac{1}{\pi}\int_{-x}^{x} e^{-t^2}\,dt}}$$

$$\boxed{erfi(x) = -i\,erf(ix)}$$

d) What are the advantages and disadvantages of using narrow-bandwidth laser diodes for pumping instead of broadband light sources (e.g. flashlights, LEDs, ...)? (1 Point)

- **Advantages:**
  - **The narrower bandwidth allows to accurately pump certain transitions / spectral regions, which allows efficient pumping and reduction of heating of the active medium**
  - **Higher electrical to optical efficiency of the pump source**
- **Disadvantages:**
  - **Relatively expensive**
  - **Prone to electrical / electro-static damage and optical feedback**

e) Assume that both pump sources generate 1 mW of average power and that every absorbed photon results in the generation of a signal photon with a wavelength of 1030 nm. Assuming that there are no saturation effects, calculate the signal output power for both pump sources and compare them. (1 Point)

Laser Diode:

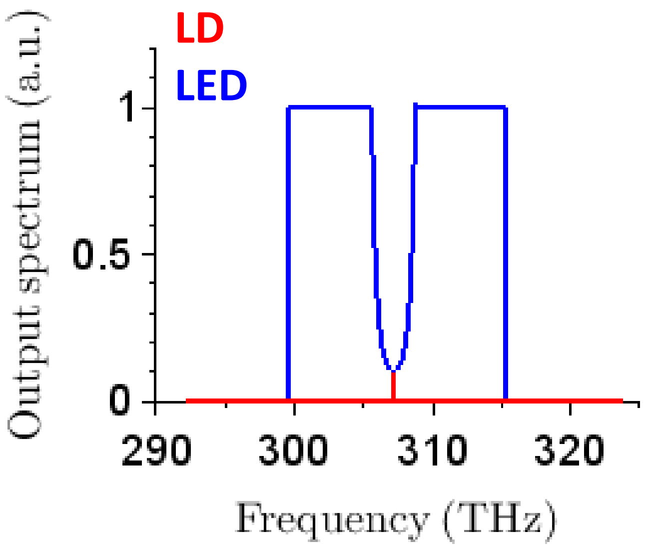
$$P_{\text{out}} = \frac{P_{\text{in}} \cdot \eta_{\text{abs.LD}}}{E_{\text{pump}}} \cdot E_{\text{signal}}$$

$$= P_{\text{in}} \cdot \eta_{\text{abs.LD}} \cdot \frac{\lambda_{\text{pump}}}{\lambda_{\text{signal}}}$$

$$= 850\ \mu\text{W}$$

Photon energy:

$$\boxed{E_{\text{pump}} = h\frac{c}{\lambda_{\text{pump}}} \qquad E_{\text{signal}} = h\frac{c}{\lambda_{\text{signal}}}}$$



LED:

$$P_{\text{out}} = P_{\text{in}} \cdot \eta_{\text{abs.LED}} \cdot \frac{\lambda_{\text{pump}}}{\lambda_{\text{signal}}}$$

$$= 137\ \mu\text{W}$$

Laserdiode has higher overlap with absorption crossection

⇒ Higher efficiency

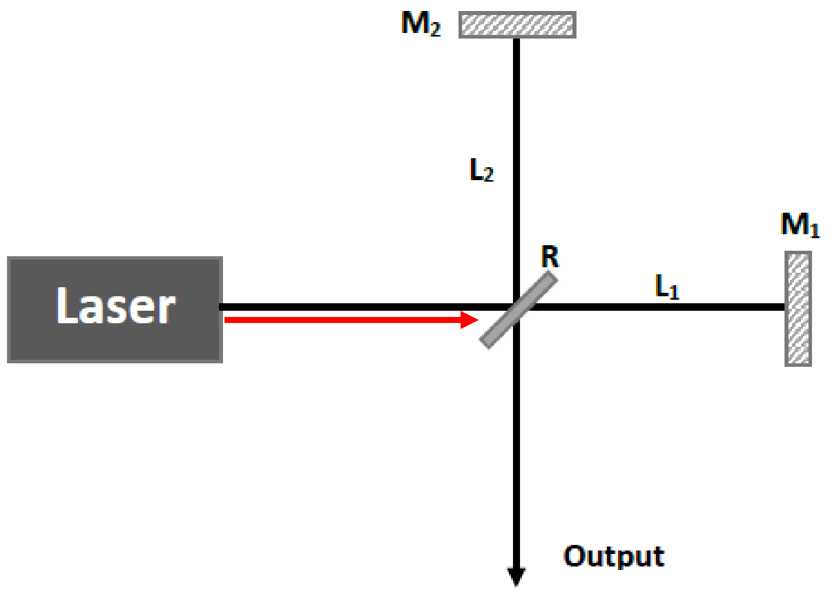
Problem 1 (8 points)

Imagine a Michelson interferometer as depicted below. R is the (power) reflectivity of the beam splitter, L_1 and L_2 are the lengths of the interferometer arms and, finally, M_1 and M_2 are the (power) reflectivities of the end mirrors in each arm.

a) Calculate the output intensity I_{out} of the interferometer as a function of the optical path difference in the interferometer arms when using a monochromatic light source (e.g. a cw-laser). (1 point)

First calculate electric field of every arm:

$$E_1 = E_{in}\sqrt{(1-R)}$$



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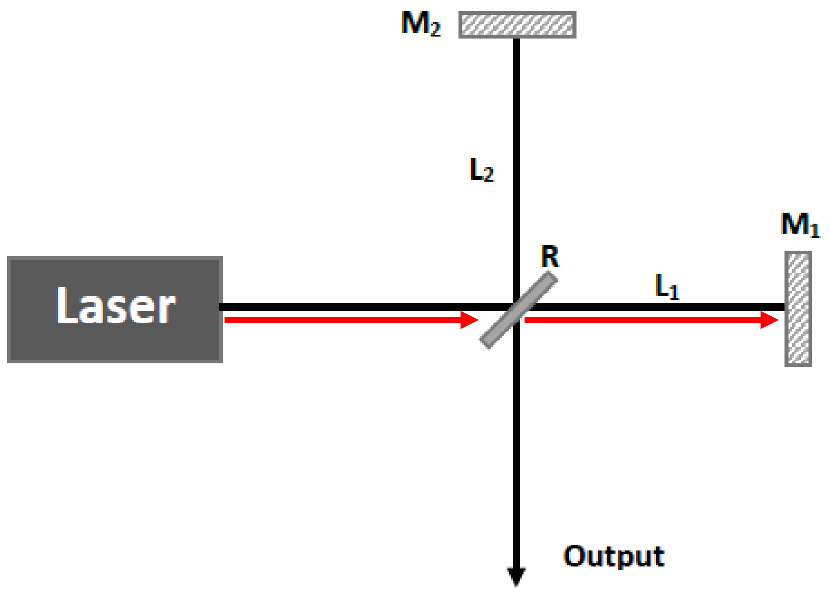

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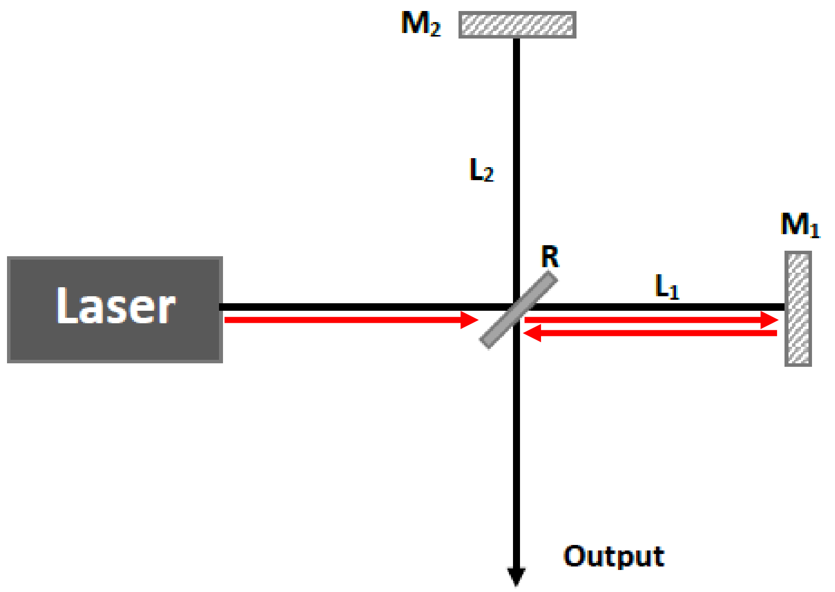

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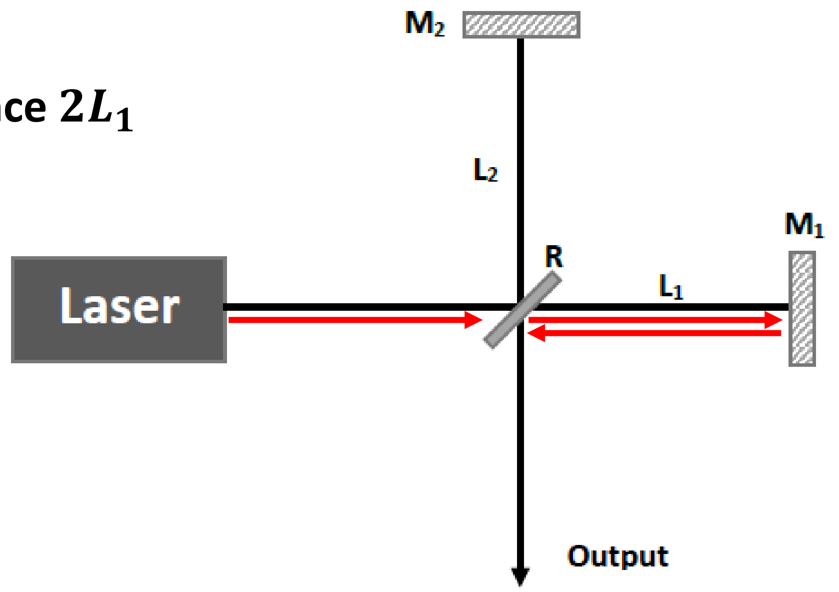
Propagation over distance $2L_1$

$$E_1 = E_{in}\sqrt{(1-R)\cdot M_1 \cdot R} \cdot \exp(-ik \cdot 2L_1)$$



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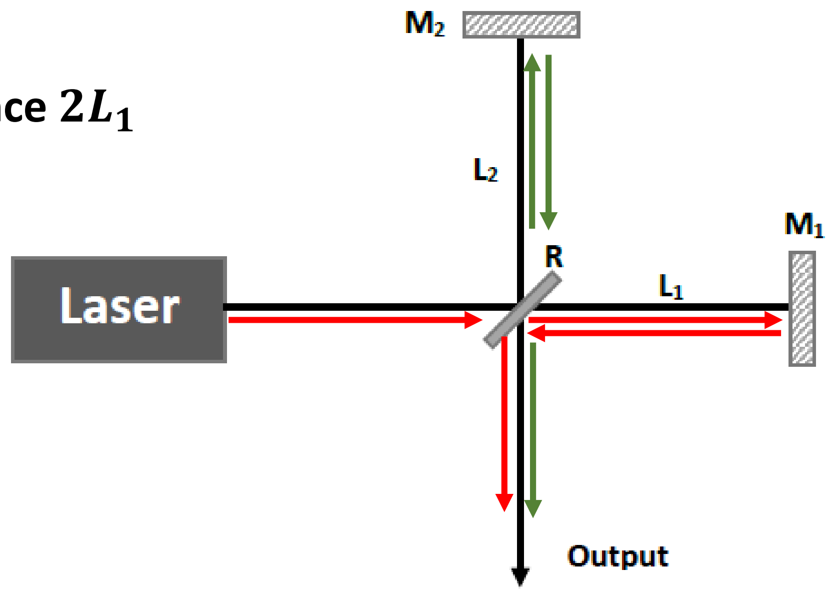

Propagation over distance $2L_1$

$$E_1 = E_{\text{in}}\sqrt{(1-R)\cdot M_1 \cdot R}\cdot \exp(-ik\cdot 2L_1)$$

$$E_2 = E_{\text{in}}\sqrt{(1-R)\cdot M_2 \cdot R}\cdot \exp(-ik\cdot 2L_2)$$

Calculate the total electric field

$$\Rightarrow E_{\text{tot}} = E_1 + E_2 = E_{\text{in}}\sqrt{(1-R)\cdot R}\left(\sqrt{M_1}\exp(-2ik\,L_1) + \sqrt{M_2}\exp(-2ik\,L_2)\right)$$

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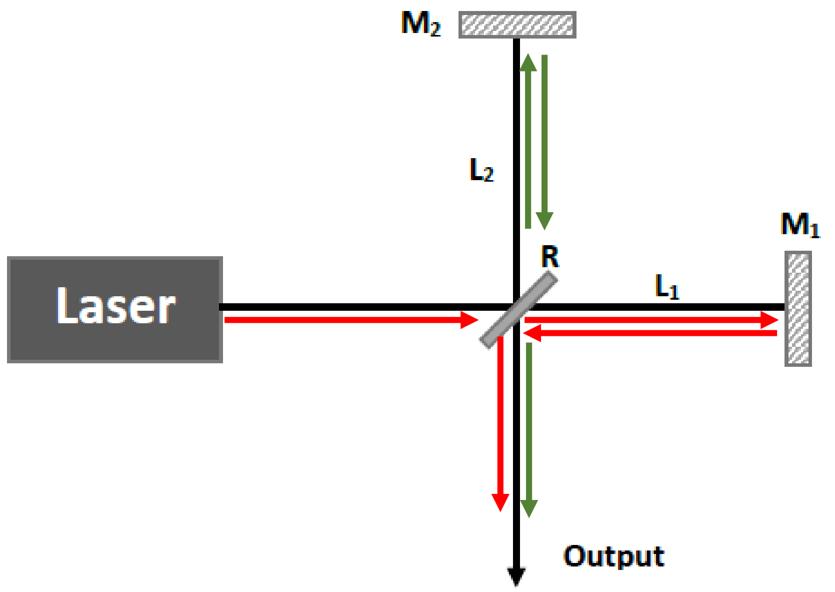
Calculate the total electric field:



$$E_{\text{tot}} = E_1 + E_2 = E_{\text{in}}\sqrt{(1-R)\cdot R}\,\left(\sqrt{M_1}\exp(-2ik\,L_1) + \sqrt{M_2}\exp(-2ik\,L_2)\right)$$

On the Detector we measure intensity:

$$I_{\text{out}} = \frac{1}{2\eta}E^*_{\text{tot}}E_{\text{tot}}$$

$\eta = \frac{1}{\epsilon_0 c}$: *impedance of free space*

$$= \frac{1}{2\eta}E^2_{\text{in}}\,(1-R)\,R\left[\left(\sqrt{M_1}\exp(2ik\,L_1) + \sqrt{M_2}\exp(2ik\,L_2)\right)\cdot\left(\sqrt{M_1}\exp(-2ik\,L_1) + \sqrt{M_2}\exp(-2ik\,L_2)\right)\right]$$

$$= \frac{1}{2\eta}E^2_{\text{in}}\,(1-R)\,R\left[\mathrm{M}_1 + \mathrm{M}_2 + \sqrt{M_1M_2}\,\left(\exp\left(2ik\,(L_1-L_2)\right) + \exp\left(-2ik\,(L_1-L_2)\right)\right)\right]$$

Problem 1 (8 points)

Imagine a Michelson interferometer as depicted below. R is the (power) reflectivity of the beam splitter, L_1 and L_2 are the lengths of the interferometer arms and, finally, M_1 and M_2 are the (power) reflectivities of the end mirrors in each arm.

a) Calculate the output intensity I_{out} of the interferometer as a function of the optical path difference in the interferometer arms when using a monochromatic light source (e.g. a cw-laser). (1 point)

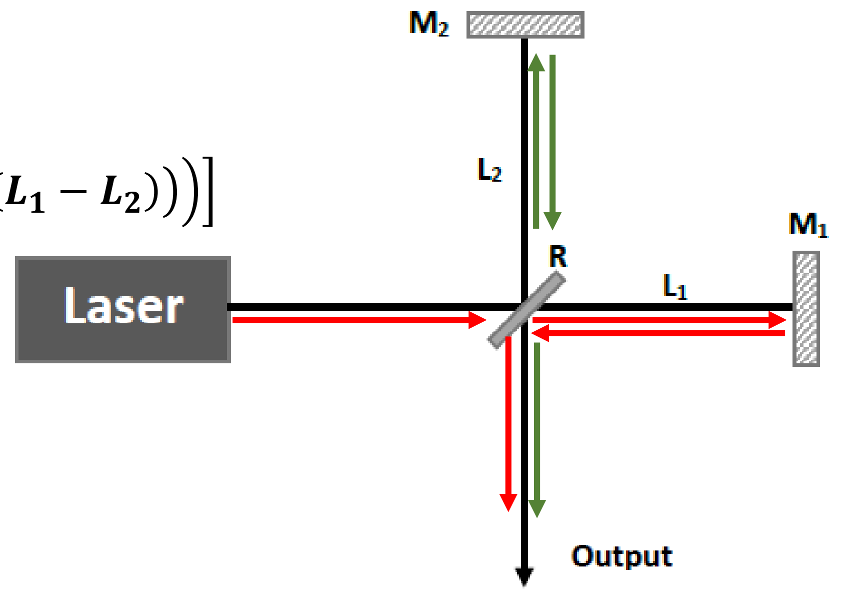
On the Detector we measure intensity:

$$I_{\text{out}} = \frac{1}{2\eta} E^*_{\text{tot}} E_{\text{tot}}$$

$$= \frac{1}{2\eta} E^2_{\text{in}} (1-R)\, R\left[\mathrm{M}_1 + \mathrm{M}_2 + \sqrt{M_1 M_2}\left(\exp\left(2ik\,(L_1 - L_2)\right) + \exp\left(-2ik\,(L_1 - L_2)\right)\right)\right]$$

$$\boxed{= \frac{1}{2\eta} E^2_{\text{in}} (1-R)\, R\left[\mathrm{M}_1 + \mathrm{M}_2 + 2\sqrt{M_1 M_2}\cos\left(2k(L_1 - L_2)\right)\right]}$$

$$= \frac{1}{2\eta} E^2_{\text{in}} (1-R)\, R\left[\mathrm{M}_1 + \mathrm{M}_2 + 2\sqrt{M_1 M_2}\cos(\omega_0\, \Delta \mathrm{t})\right], \qquad \Delta \mathrm{t} = \frac{2(\mathrm{L}_1 - \mathrm{L}_2)}{c}$$

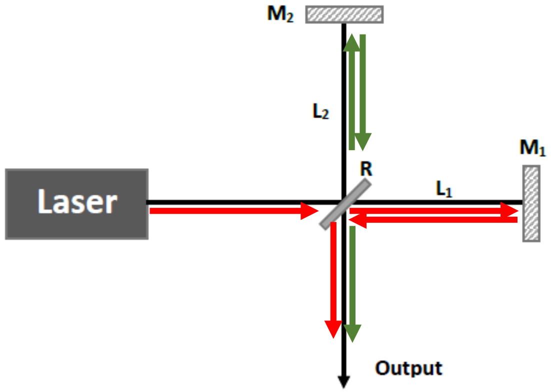


Problem 1 (8 points)

Imagine a Michelson interferometer as depicted below. R is the (power) reflectivity of the beam splitter, L_1 and L_2 are the lengths of the interferometer arms and, finally, M_1 and M_2 are the (power) reflectivities of the end mirrors in each arm.

b) Calculate the visibility of the interference given by:

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$



Visibility defines ratio between lowest and highest Intensity at Interference:

$$V = \frac{I_{\text{max}} - I_{\text{min}}}{I_{max} + I_{min}}$$

Remember Result from 1a)

$$I_{\text{out}} = \frac{1}{2\eta} E_{\text{in}}^2 (1-R)\, R\left[\mathrm{M}_1 + \mathrm{M}_2 + 2\sqrt{M_1 M_2}\cos(\omega_0\, \Delta\mathrm{t})\right], \qquad \Delta\mathrm{t} = \frac{2(\mathrm{L}_1 - \mathrm{L}_2)}{c}$$

$$\max(I_{\text{out}})\ @\ \cos(\omega_0 t) = 1:$$

$$\Rightarrow I_{\text{max}} = \frac{1}{2\eta} E_{\text{in}}^2 (1-R)\, R\left[\mathrm{M}_1 + \mathrm{M}_2 + 2\sqrt{M_1 M_2}\,\right]$$

$$\min(I_{\text{out}})\ @\ \cos(\omega_0 t) = -1:$$

$$\Rightarrow I_{min} = \frac{1}{2\eta} E_{\text{in}}^2 (1-R)\, R\left[\mathrm{M}_1 + \mathrm{M}_2 - 2\sqrt{M_1 M_2}\,\right]$$

$$\boxed{\Rightarrow V = \frac{2\sqrt{M_1 M_2}}{M_1 + M_2}}$$

c) Calculate the interferometer output intensity for the case of a broadband input source (e.g. an ultrashort pulse laser). Assume that $M_1 = M_2 = 1$ and $R = 0.5$. The input electric field is given by:

$$E_{in} = \frac{E_o}{2} e^{-2ln2 \cdot \frac{t^2}{\tau^2} + i\omega t} + cc.$$

where E_o is the peak electric field, t is the time, τ is the FWHM pulse duration (of the pulse intensity) and ω is the angular frequency of the carrier wave. The output intensity is given by the time averaged product of the electric field and its complex conjugate divided by the impedance of free space η.

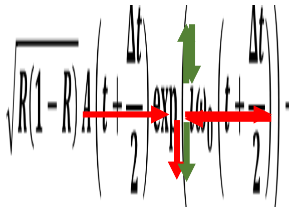
$$I_{out} = \frac{1}{2\eta} \langle E_{out} E^*_{out} \rangle_t$$

Plot the output intensity of c) as a function of increasing optical path difference and mark distinct features. (2 points)

Given:

$$E_{\text{in}} = \frac{E_0}{2} \exp\left(-2\log(2)\left(\frac{t}{\tau}\right)^2 + i\omega_0 t\right) + \text{c.c.}$$

$$= A(t)\exp(i\omega_0 t) + \text{c.c.}$$



From a) :

$$\Delta t = \frac{2(L_1 - L_2)}{c}$$

Now look at the E-Field on the output:

Symmetric movement of the two arm around Central time frame

$$E_{1,\text{out}} = \sqrt{R(1-R)}\, A\left(t - \frac{\Delta t}{2}\right) \exp\left(i\omega_0\left(t - \frac{\Delta t}{2}\right)\right) + \text{c.c.}$$

$$E_{2,\text{out}} = \sqrt{R(1-R)}\, A\left(t + \frac{\Delta t}{2}\right) \exp\left(i\omega_0\left(t + \frac{\Delta t}{2}\right)\right) + \text{c.c.}$$

c) Calculate the interferometer output intensity for the case of a broadband input source (e.g. an ultrashort pulse laser). Assume that $M_1 = M_2 = 1$ and $R = 0.5$. The input electric field is given by:

$$E_{in} = \frac{E_o}{2} e^{-2\ln 2\cdot\frac{t^2}{\tau^2} + i\omega t} + cc.$$

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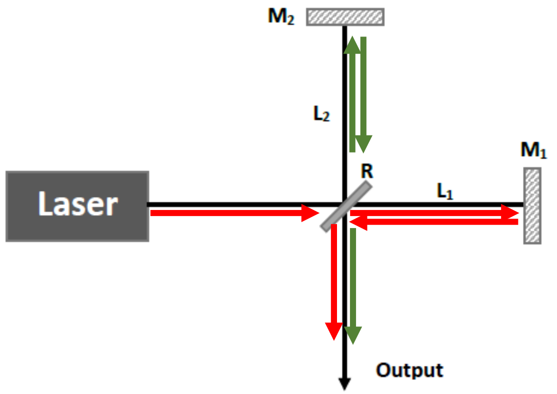
Plot the output intensity of c) as a function of increasing optical path difference and mark distinct features. (2 points)



$$\Rightarrow E_{tot} = E_{1,out} + E_{2,out}$$

$$= \sqrt{R(1-R)}\left(A\left(t-\frac{\Delta t}{2}\right)\exp\left(i\omega_0\left(t-\frac{\Delta t}{2}\right)\right) + A\left(t+\frac{\Delta t}{2}\right)\exp\left(i\omega_0\left(t+\frac{\Delta t}{2}\right)\right) + \text{c.c.}\right)$$

Now evaluate Intensity:

$$\boxed{\langle f(t)\rangle = \frac{1}{\Delta T}\int_t^{t+\Delta T} f(t')dt'}$$

$$I_{out} = \frac{1}{2\eta}\frac{1}{T}\int_T dt\, E^*_{tot}\, E_{tot}$$

$$= \frac{1}{8\eta}\frac{1}{T}\int_T dt\,\left\{2A^2\left(t-\frac{\Delta t}{2}\right) + 2A^2\left(t+\frac{\Delta t}{2}\right) + A^2\left(t-\frac{\Delta t}{2}\right)\left[\exp\left(2i\omega_0\left(t-\frac{\Delta t}{2}\right)\right) + \text{c.c.}\right]\right.$$

$$\left. + A^2\left(t+\frac{\Delta t}{2}\right)\left[\exp\left(2i\omega_0\left(t+\frac{\Delta t}{2}\right)\right) + \text{c.c.}\right] + 2A\left(t-\frac{\Delta t}{2}\right)A\left(t+\frac{\Delta t}{2}\right)\left[\exp(2i\omega_0 t) + \exp(i\omega_0\Delta t) + \text{c.c.}\right]\right\}$$

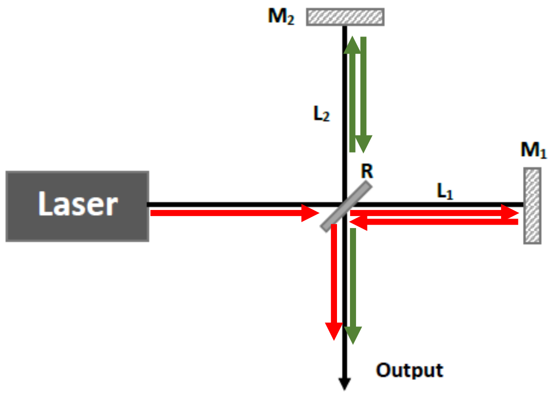
c) Calculate the interferometer output intensity for the case of a broadband input source (e.g. an ultrashort pulse laser). Assume that $M_1 = M_2 = 1$ and $R = 0.5$. The input electric field is given by:

$$E_{in} = \frac{E_o}{2} e^{-2ln2 \cdot \frac{t^2}{\tau^2} + i\omega t} + cc.$$

where E_o is the peak electric field, t is the time, τ is the FWHM pulse duration (of the pulse intensity) and ω is the angular frequency of the carrier wave. The output intensity is given by the time averaged product of the electric field and its complex conjugate divided by the impedance of free space η.

$$I_{out} = \frac{1}{2\eta} \langle E_{out} E^*_{out} \rangle_t$$

Plot the output intensity of c) as a function of increasing optical path difference and mark distinct features. (2 points)



Now simplify it:

$$I_{\text{out}} = \frac{1}{8\eta}\frac{1}{T}\int_T dt \left\{2A^2\left(t-\frac{\Delta t}{2}\right) + 2A^2\left(t+\frac{\Delta t}{2}\right) + A^2\left(t-\frac{\Delta t}{2}\right)\cancelto{0}{\left[\exp\left(2i\omega_0\left(t-\frac{\Delta t}{2}\right)\right) + \text{c.c.}\right]}\right.$$

$$\left. + A^2\left(t+\frac{\Delta t}{2}\right)\cancelto{0}{\left[\exp\left(2i\omega_0\left(t+\frac{\Delta t}{2}\right)\right) + \text{c.c.}\right]} + 2A\left(t-\frac{\Delta t}{2}\right)A\left(t+\frac{\Delta t}{2}\right)\left[\cancelto{0}{\exp(2i\omega_0 t)} + \exp(i\omega_0\Delta t) + \text{c.c.}\right]\right\}$$

$$\boxed{\int_{-n\pi}^{n\pi} \cos(x)\, dx = 0 \qquad \Rightarrow \text{Oscillating terms vanish}}$$

$$\Rightarrow = \frac{1}{8\eta}\frac{1}{T}\int_T dt \left\{2A^2\left(t-\frac{\Delta t}{2}\right) + 2A^2\left(t+\frac{\Delta t}{2}\right) + 4A\left(t-\frac{\Delta t}{2}\right)A\left(t+\frac{\Delta t}{2}\right)\cos(\omega_0\Delta t)\right\}$$

$$\Rightarrow = \frac{1}{2\eta}\frac{1}{T}\int_T dt\, A^2(t) + \frac{1}{2\eta}\frac{1}{T}\int_T dt\, A\left(t-\frac{\Delta t}{2}\right)A\left(t+\frac{\Delta t}{2}\right)\cos(\omega_0\Delta t)$$

$$= \frac{1}{2\eta}\frac{1}{T}\int_T dt\, A^2(t) + \frac{1}{2\eta}\frac{1}{T}\int_T dt\, \frac{E_0^2}{4}\exp\left(-2\log(2)\left(\frac{t-\Delta t/2}{\tau}\right)^2\right)\exp\left(-2\log(2)\left(\frac{t+\Delta t/2}{\tau}\right)^2\right)\cos(\omega_0\Delta t)$$

c) Calculate the interferometer output intensity for the case of a broadband input source (e.g. an ultrashort pulse laser). Assume that $M_1 = M_2 = 1$ and $R = 0.5$. The input electric field is given by:

$$E_{in} = \frac{E_o}{2} e^{-2ln2 \cdot \frac{t^2}{\tau^2} + i\omega t} + cc.$$

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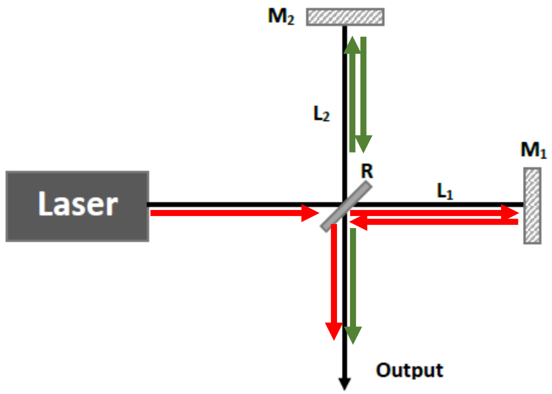
$$I_{out} = \frac{1}{2\eta} \langle E_{out} E^*_{out} \rangle_t$$

Plot the output intensity of c) as a function of increasing optical path difference and mark distinct features. (2 points)



Now simplify it:

$$I_{\text{out}} = \frac{1}{2\eta}\frac{1}{T}\int_T dt\, A^2(t) + \frac{1}{2\eta}\frac{1}{T}\int_T dt\, \frac{E_0^2}{4} \exp\left(-2\log(2)\left(\frac{t-\Delta t/2}{\tau}\right)^2\right) \exp\left(-2\log(2)\left(\frac{t+\Delta t/2}{\tau}\right)^2\right) \cos(\omega_0 \Delta t)$$

$$= \frac{1}{2\eta}\frac{1}{T}\int_T dt\, A^2(t) + \frac{1}{2\eta}\frac{1}{T}\int_T dt\, \frac{E_0^2}{4} \exp\left(-2\log(2)\left(\frac{t^2 - t\Delta t + \Delta t^2/4 + t^2 + t\Delta t + \Delta t^2/4}{\tau^2}\right)\right) \cos(\omega_0 \Delta t)$$

$$= \frac{1}{2\eta}\frac{1}{T}\int_T dt\, A^2(t) + \frac{1}{2\eta}\frac{1}{T}\int_T dt\, \frac{E_0^2}{4} \exp\left(-4\log(2)\left(\frac{t^2}{\tau^2}\right)\right) \exp\left(-2\log(2)\left(\frac{\Delta t^2/2}{\tau^2}\right)\right) \cos(\omega_0 \Delta t)$$

$$= \frac{1}{2\eta}\frac{1}{T}\int_T dt\, A^2(t) + \frac{1}{2\eta}\frac{1}{T}\int_T dt\, A^2(t) \exp\left(-4\log(2)\left(\frac{\Delta t^2/4}{\tau^2}\right)\right) \cos(\omega_0 \Delta t)$$

$$\Rightarrow I_{\text{out}} = \frac{1}{2\eta}\frac{1}{T}\int_T dt\, A^2(t) \cdot \left(1 + \exp\left(-4\log(2)\left(\frac{\Delta t^2/4}{\tau^2}\right)\right) \cos(\omega_0 \Delta t)\right)$$

c) Calculate the interferometer output intensity for the case of a broadband input source (e.g. an ultrashort pulse laser). Assume that $M_1 = M_2 = 1$ and $R = 0.5$. The input electric field is given by:

$$E_{in} = \frac{E_o}{2} e^{-2ln2 \cdot \frac{t^2}{\tau^2} + i\omega t} + cc.$$

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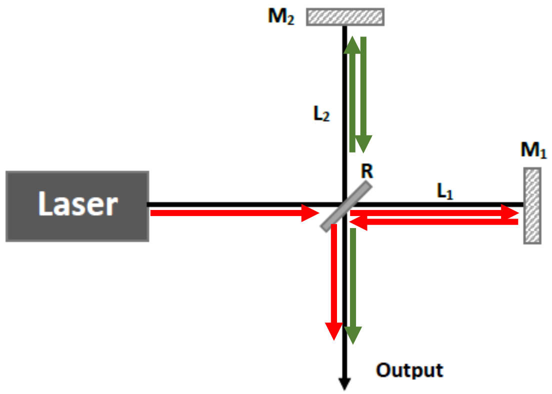
Plot the output intensity of c) as a function of increasing optical path difference and mark distinct features. (2 points)



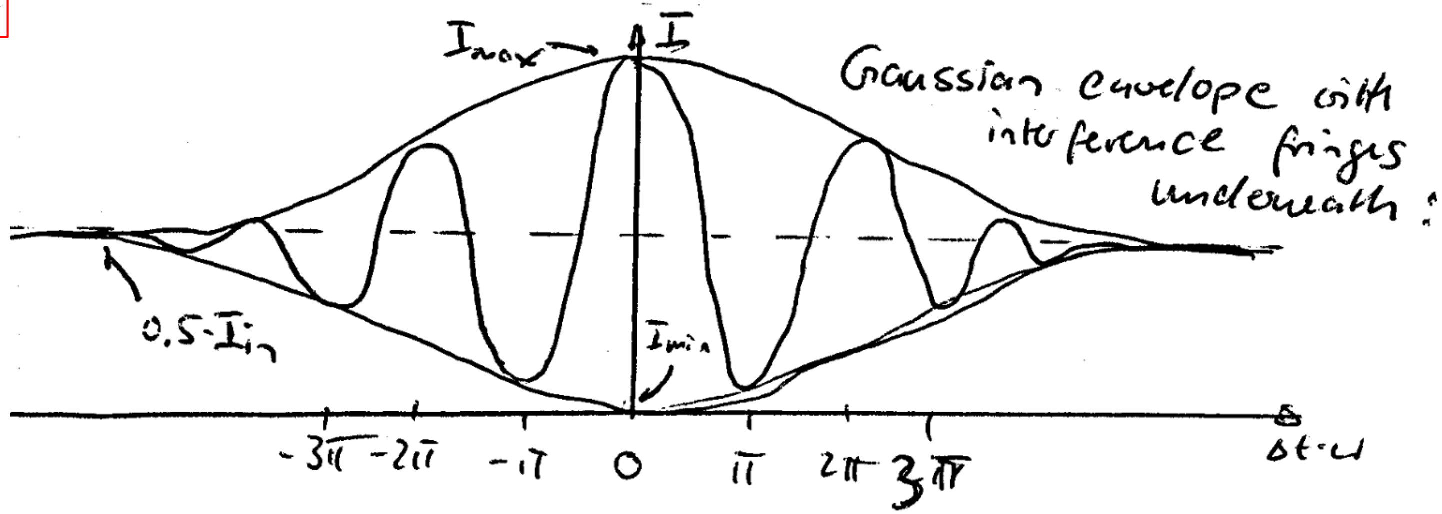
$$I_{\text{out}} = \underbrace{\frac{1}{2\eta}\frac{1}{T}\int_T dt\, A^2(t)}_{\frac{1}{2} I_{in}} \cdot \left(1 + \underbrace{\exp\left(-\log(2)\left(\frac{\Delta t^2}{\tau^2}\right)\right)}_{\text{Contrast term}} \underbrace{\cos(\omega_0 \Delta t)}_{\text{Interference term}}\right)$$

Contrast term
decreasing contrast
as pulses are delayed

Interference term
depends on temporal/
arm length missmatch

$$\boxed{I_{in} = \frac{1}{2\eta}\frac{1}{T}\int_T \mathrm{dt}\, E_{in}^* E_{in}}$$

$$\boxed{= \frac{1}{\eta}\frac{1}{T}\int_T \mathrm{dt}\, A^2(t)}$$



d) Now you are given a laser that emits ultrashort gaussian pulses (as in c)) at a central wavelength $\lambda_o = 1\mu m$ with a spectral FWHM of $\Delta\lambda = 10nm$. Assume an optical path mismatch of $0.2mm$ between the interferometer arms. Calculate the optical spectrum $I(\lambda)$ at the interferometer output and draw a plot. (2 points)

From c:

$$\mathbf{E_{in}(t) = E_0 \exp\left(-2\ln(2)\cdot\left(\frac{t}{\tau}\right)^2\right)\cdot \exp(i\omega t)}$$

$$\mathbf{E_{out}(t) = \frac{1}{2}\left\{E_{in}\left(t-\frac{\Delta t}{2}\right)+E_{in}\left(t+\frac{\Delta t}{2}\right)\right\}}, \qquad \mathbf{\Delta t = \frac{2\cdot(L_1-L_2)}{c} = \frac{\Delta OPL}{c}}$$

$$\mathbf{\tilde{E}_{out}(\omega) = FT\{E_{out}(t)\} = \frac{1}{2}\left\{FT\left\{E_{in}\left(t-\frac{\Delta t}{2}\right)\right\}+FT\left\{E_{in}\left(t+\frac{\Delta t}{2}\right)\right\}\right\}}$$

$$\mathbf{= \frac{1}{2}\left\{\tilde{E}_{in}(\omega)\exp\left(-i\omega\frac{\Delta t}{2}\right)+\tilde{E}_{in}(\omega)\exp\left(i\omega\frac{\Delta t}{2}\right)\right\}}$$

$$\mathbf{= \frac{1}{2}\tilde{E}_{in}(\omega)\left\{\exp\left(i\omega\frac{\Delta t}{2}\right)+\exp\left(-i\omega\frac{\Delta t}{2}\right)\right\}}$$

Translation/ time shifting property:

$$\mathbf{FT\{E_{out}(t-\Delta t)\} = \tilde{E}_{in}(\omega)e^{-i\omega\Delta t}}$$

$$\mathbf{\Rightarrow I_{out}(\omega) \sim \tilde{E}_{out}(\omega)\cdot\tilde{E}^*_{out}(\omega)}$$

$$\mathbf{= \frac{1}{4}I_{in}(\omega)\left[\exp\left(i\omega\frac{\Delta t}{2}\right)+\exp\left(-i\omega\frac{\Delta t}{2}\right)\right]\cdot\left[\exp\left(i\omega\frac{\Delta t}{2}\right)+\exp\left(-i\omega\frac{\Delta t}{2}\right)\right]}$$

$$\mathbf{= I_{in}(\omega)\cdot\cos^2\left(\frac{\omega\Delta t}{2}\right)}$$

d) Now you are given a laser that emits ultrashort gaussian pulses (as in c)) at a central wavelength $\lambda_o = 1\mu m$ with a spectral FWHM of $\Delta\lambda = 10nm$. Assume an optical path mismatch of $0.2mm$ between the interferometer arms. Calculate the optical spectrum $I(\lambda)$ at the interferometer output and draw a plot. (2 points)

$$I_{\text{out}}(\omega) = I_{\text{in}}(\omega) \cdot \cos^2\left(\frac{\omega \Delta t}{2}\right)$$

Now translation from angular frequency to wavelength:

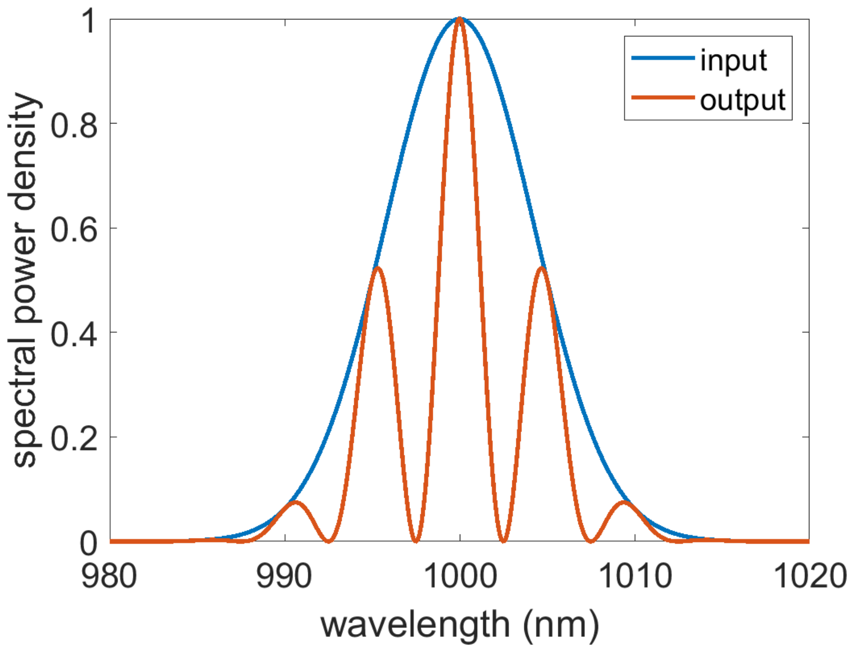
$$\omega = \omega_0 + \delta\omega, \qquad \delta\omega = \frac{2\pi c}{\lambda_0^2} \cdot \delta\lambda$$

$$I_{\text{out}}(\lambda) = I_{\text{in}}(\lambda) \cdot \cos^2\left[\frac{\Delta t}{2}\left(\frac{2\pi c}{\lambda_0} + \frac{2\pi c}{\lambda_0^2} \cdot \delta\lambda\right)\right], \qquad \Delta t = \frac{\Delta \text{OPL}}{c}$$

$$= I_{\text{in}}(\lambda) \cdot \cos^2\left[\frac{\pi}{\lambda_0}\Delta\text{OPL} + \frac{\pi}{\lambda_0^2}\Delta\text{OPL} \cdot \delta\lambda\right]$$

$$\Delta\text{OPL} = 2e^{-4}\, m$$
$$\lambda_0 = 1e^{-6}\, m$$

$$= \underbrace{I_{\text{in}}(\lambda)}_{\text{Gaussian envelope}} \cdot \cos^2[100\pi + \underbrace{2 \cdot 10^8 \pi \cdot \delta\lambda}_{\text{Spectral interference fringe}}]$$



One fringe period: $2 \cdot 10^8 \pi \cdot \delta\lambda = \pi$ $\qquad \boxed{\Rightarrow \delta\lambda = 5\ \text{nm}}$

e) For the case of d), calculate the interferometric visibility and give a physical explanation for the result. (1 point)

From c:

$$I_{\text{out}} = \frac{1}{2} I_{\text{in}} \left[1 + \exp\left(-\log(2) \cdot \frac{\Delta t^2}{\tau^2} \right) \cdot \cos(\omega \Delta t) \right]$$

$$(I_{\text{out}})_{\max} = \frac{1}{2} I_{\text{in}} \left[1 + \exp\left(-\log(2) \cdot \frac{\Delta t^2}{\tau^2} \right) \right]$$

$$(I_{\text{out}})_{\min} = \frac{1}{2} I_{\text{in}} \left[1 - \exp\left(-\log(2) \cdot \frac{\Delta t^2}{\tau^2} \right) \right]$$

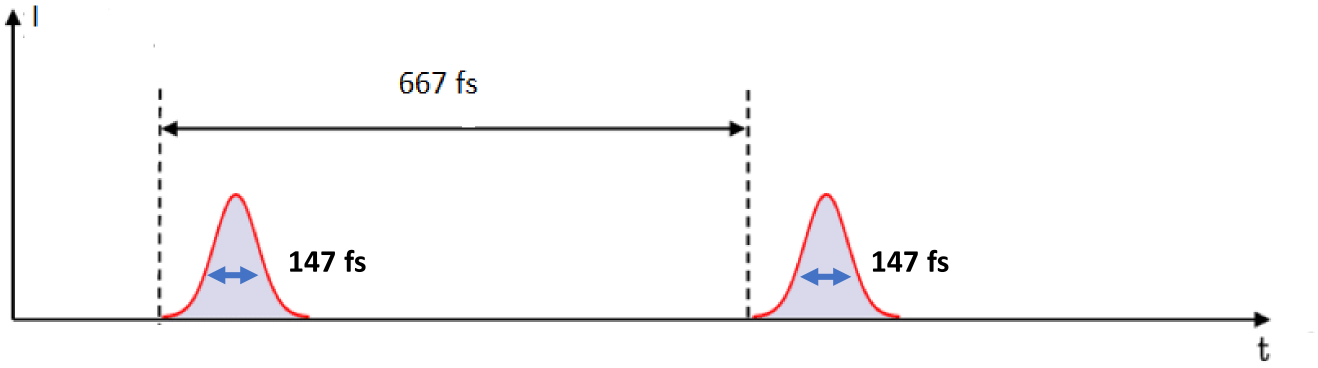
$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \exp\left(-\log(2) \cdot \frac{\Delta t^2}{\tau^2} \right)$$

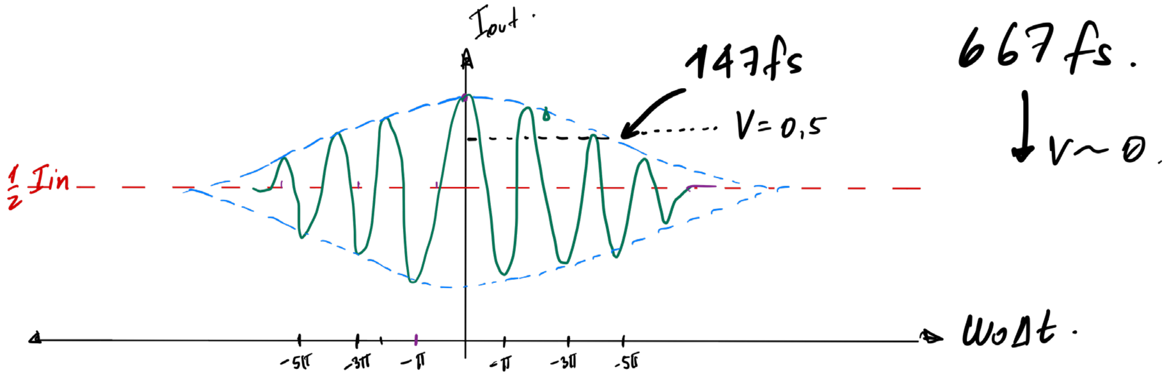
$$\Delta t = \frac{\Delta OPL}{c} = 667 \text{ fs}$$

$$\tau = \frac{2\, ln(2)}{\pi \Delta \nu} = 147 \text{ fs}$$

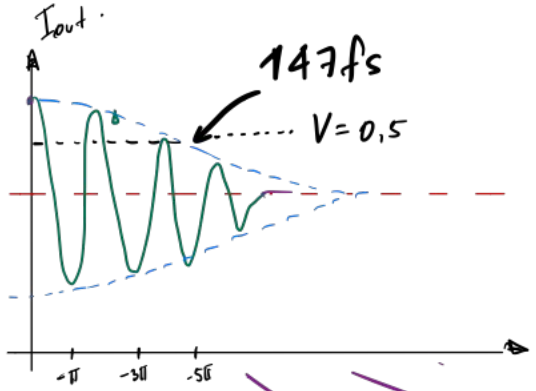
$$\boxed{\Rightarrow V = 6.4 \times 10^{-7} \sim 0}$$

<u>Reason: Pulses don‘t overlap in time domain!</u>





No interferometric fringe contrast due to exceeded Temporal coherence time $\Delta t \gg \tau$



667 fs.

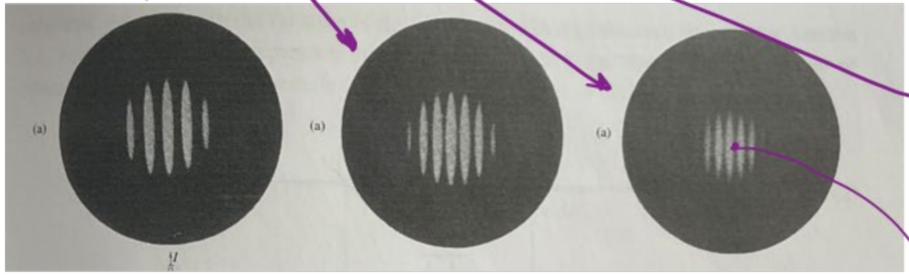
$\downarrow$ $V \sim 0$.

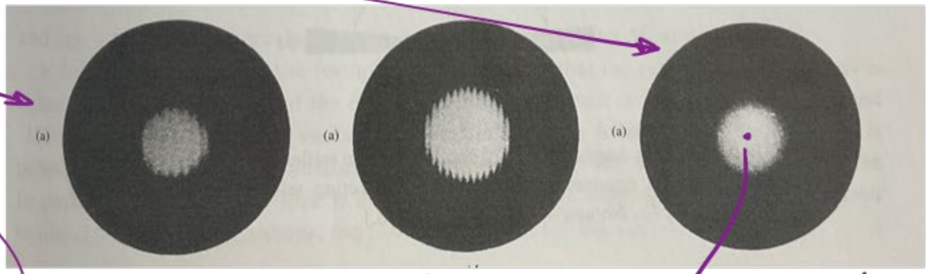
Fringes in Time-domain $\rightarrow$ No Spectral fringes.

and vice versa $\leftrightarrow$.

Visibility ~ 0.

$\Delta t = 667$fs.





fringe Period $\sim \frac{\lambda_0^2}{\Delta t} \cdot c$

$\Delta t = 147$fs

$V \sim 0,5$.

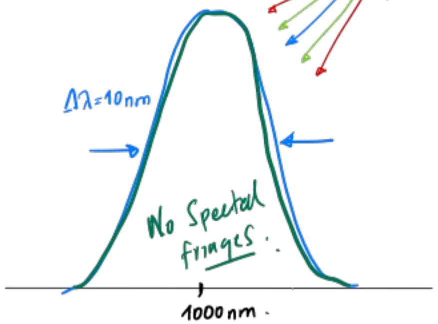
$\Delta t \sim 300$fs.

fringe Period ~11nm

$\Delta t = 667$fs.

fringe Period ~5nm.

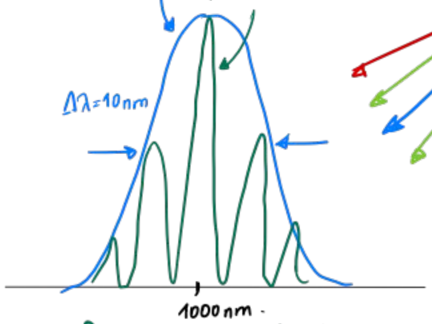
Spectrometer at center pixel.



fringe Period 22,7nm.

Input Spectra

Output Spectra



fringes period 5nm.

Spectrometer

at center pixel.

f) What is the reason for the difference between the results of d) and e) in terms of visibility of the interference? (1 point)

Theoretical answer:

Fourier transform is integrating over the whole time domain, therefore interaction of pulses is visible in frequency space

Experimental answer:

Spectrometer is a narrow spectral filter, artifically increasing the pulse duration (and hence the coherence time)
Interference fringes in time-domain = no inteference fringes in spectral-domain