Name: ______________________ Date of birth: ______________ Student ID No.: ______________

FRIEDRICH SCHILLER UNIVERSITY JENA
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WS 2018/2019

Final Exam

FUNDAMENTALS OF MODERN OPTICS

to be written on February 15, 10:00 – 12:00

Problem 1: Maxwell's Equations — 2 + 3 + 3 = 8 points

a) Write down Maxwell's equations (MWE) in the frequency domain in a linear, isotropic, dispersive and *inhomogeneous* medium without sources and currents. Use only the fields $\mathbf{E}(\mathbf{r},\omega)$ and $\mathbf{H}(\mathbf{r},\omega)$ as well as the permittivity $\varepsilon(\mathbf{r},\omega)$.

b) Derive the wave equation for $\mathbf{E}(\mathbf{r},\omega)$ from MWE in this medium. Simplify it for the case $\nabla\varepsilon(\mathbf{r},\omega) \perp \bar{\mathbf{E}}(\mathbf{r},\omega)$.

c) In the Drude model the polarization $\mathbf{P}(\mathbf{r},t)$ is determined by the following differential equation

$$\frac{\partial^2\mathbf{P}(\mathbf{r},t)}{\partial t^2} + g\frac{\partial\mathbf{P}(\mathbf{r},t)}{\partial t} + \omega_0^2\mathbf{P}(\mathbf{r},t) = \varepsilon_0 f\mathbf{E}(\mathbf{r},t),$$

with the damping factor g and the oscillator strength f. Calculate the expression for the permittivity $\varepsilon(\omega)$ in this material and decompose it in real and imaginary part.

Problem 2: Normal modes — 2 + 3 + 2 + 3 = 10 points

Consider the complex representation of a plane wave:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0\exp\left[\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \omega t)\right].$$

a) Name the variables $\mathbf{k}$ and ω, and state their physical units. What is the connection between $\mathbf{k}$ and the wavelength λ in vacuum?

b) Assume a linear, isotropic, dispersive, and homogeneous medium with $\varepsilon(\omega) \neq 0$. First show that $\mathbf{k}$ is orthogonal to $\mathbf{E}_0$. Now, derive the dispersion relation of the plane wave.

c) Consider a wave with $\mathbf{k} = (a, 0, ib)$ with $a, b \in \mathbb{R}$. Derive the expressions that define the planes of constant amplitude and constant phase for this wave. Show that these planes are orthogonal. What is such a wave called? Name one situation in which this type of wave is generated.

d) Show that in a lossless, isotropic material the time-averaged Poynting vector of a plane wave has the direction of $\mathbf{k}$, and that its magnitude is proportional to $|\mathbf{E}_0|^2$.

Problem 3: Diffraction — 2 + 1 + 3 + 4 = 10 points

Note that each task can be solved independently.

a) Write down the conditions where 1) the Fresnel approximation, 2) the paraxial Fraunhofer approximation, and 3) the non-paraxial Fraunhofer approximation are valid for calculating a diffraction pattern. Specify the conditions depending on the angular spectrum, the Fresnel number N_F, the aperture size a, the wavelength λ, and the observation distance z_B.

b) Assume that some aperture is illuminated with a plane wave that is inclined as

$$u_0(x,y,z=0) = A_0\exp(ik_x x).$$

How does the diffraction pattern in the paraxial Fraunhofer approximation depend on k_x?

c) Assume that some aperture of size b is repeated N times along the x-axis with a constant period of $d > 2b$. How does the diffraction pattern in the paraxial Fraunhofer approximation change as compared to the single aperture? How do the parameters N, d, and b influence the position of local diffraction orders and the global width of the diffraction pattern?

d) Calculate the diffraction pattern (intensity) in the paraxial Fraunhofer approximation at distance z_B when a plane wave is normally incident on an aperture with the following transmission function at $z = 0$

$$t(x) = \begin{cases} \frac{1}{2} + \frac{x}{b} & \text{for } |x| \leq b/2 \\ 0 & \text{otherwise} \end{cases}.$$

Problem 4: Pulses **2 + 2 + 4 = 8 points**

a) Define the phase and group velocity in terms of the frequency-dependent wavenumber $k(\omega)$. Explain their physical meaning for the propagation of an optical pulse in a dispersive material.

b) Consider a material with the refractive index

$$n(\lambda) = A + \frac{B}{\lambda^2}$$

How long will it take for a Gaussian pulse with central wavelength λ_0 to travel through a distance L of this material? Express your result in terms of L, λ_0, A, and B.

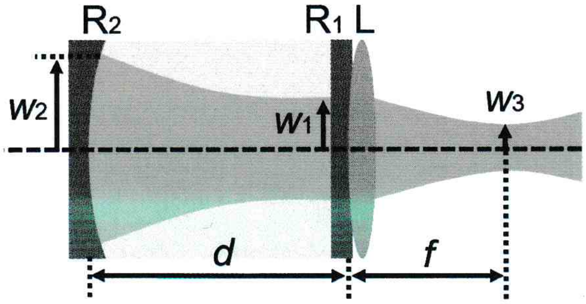
c) Starting from the transfer function of a pulsed beam in Fresnel approximation

$$H_{\mathrm{F}}(\alpha, \beta, \omega; z) = \exp[\mathrm{i}k(\omega)z] \exp\left[-\mathrm{i}\frac{\alpha^2+\beta^2}{2k(\omega)}z\right]$$

derive the transfer function of the slowly varying envelope $v(x, y, t; z)$ in the parabolic approximation at frequency ω_0.

Problem 5: Gaussian Beams **3 + 1 + 4 = 8 points**

A mirror with a radius of curvature $R_2 = 2d$ and a mirror with a radius of curvature $R_1 = \infty$ form a resonator (see figure below). The resulting Gaussian beam of wavelength λ has a width w_1 and a radius of curvature $R_1 = \infty$ at the output mirror and is then focused by a thin-lens with a focal length f.



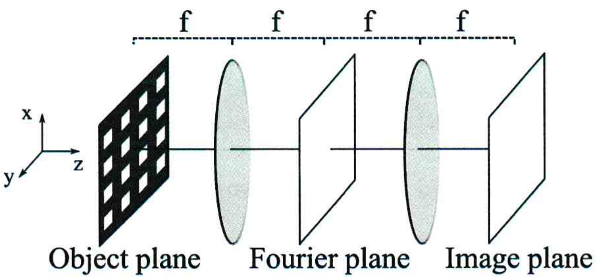
a) Derive expressions for the beam widths w_1 and w_2, using only the resonator length d and the wavelength λ.

b) Define the resonator as stable or unstable. Give the stability criterion, which justifies your answer.

c) Derive the expression for the new waist w_3 at the focal position of the lens using the beam width w_1.

Problem 6: Fourier Optics **2 + 2 + 2 + 2 + 2 = 10 points**

Consider a 4f-setup as shown in the figure below. The object in the object plane is illuminated with a monochromatic plane wave of wavelength λ.



a) Describe how to derive an expression to calculate the field $u(x, y, z = 2f)$ in the Fourier plane from the field $u_0(x, y, z = 0)$ in the object plane (in words, no calculation necessary). Write down this expression.

b) We now put a transmission mask in the Fourier plane and block all the light not on the optical axis. Argue how the field distribution will look in the image plane. Explain your answer. *Hint:* No calculation is needed here.

c) Which transmission mask can we use if we want to see only vertical lines in the image plane? Describe it in one sentence.

d) Is it possible to obtain a perfect image of the object after 4f-setup? Shortly explain your answer.

e) Estimate the limit of optical resolution of this system, if $\lambda = 1000\,\mathrm{nm}$, $f = 10\,\mathrm{cm}$, and the diameter of the transmission mask placed into the center of the setup is $D = 2\,\mathrm{cm}$.

Problem 7: Anisotropy **2 + 2 + 3 + 3 = 10 points**

Consider a general homogeneous, transparent, and *anisotropic* medium.

a) What are the normal modes in this medium? How do they differ from the isotropic case?

b) Show that for a linearly polarized plane wave the Poynting vector $\mathbf{S}$ is in general *not* parallel to the wave vector $\mathbf{k}$, i.e., $\mathbf{k} \nparallel \mathbf{S}$.

Now assume the special case of a uniaxial crystal with ordinary refractive index n_o and extraordinary refractive index n_e. The planar interface of the crystal to air lies in the x-y-plane.

c) The figure below shows the normal surfaces for a specific crystal orientation and non-normal incidence. Draw the optical axis and construct the wavevectors and Poynting vectors for the ordinary and extraordinary beam in the crystal. Draw your solution directly in the figure below!

d) Now assume that the optical axis of the crystal lies in the x-y plane and forms a 45° angle with the x axis. An x-polarized incident beam propagates in z direction. Calculate the thickness L for which this crystal acts as a half-wave plate at wavelength λ.

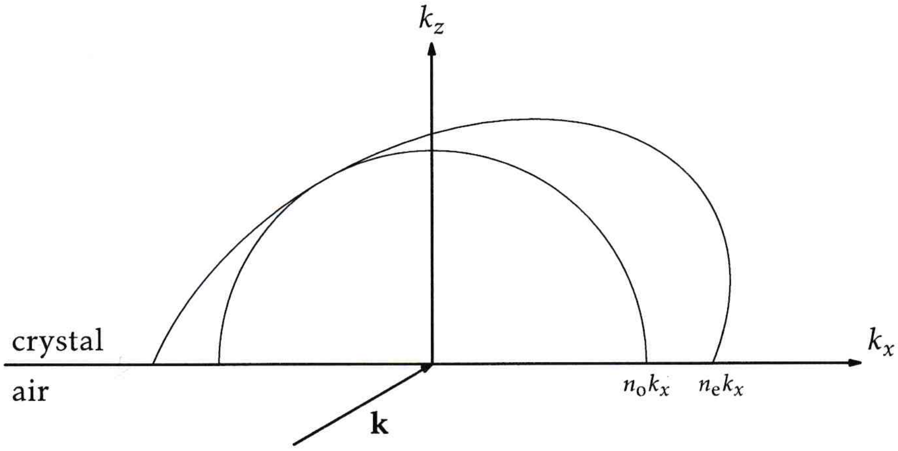


Figure to part c)

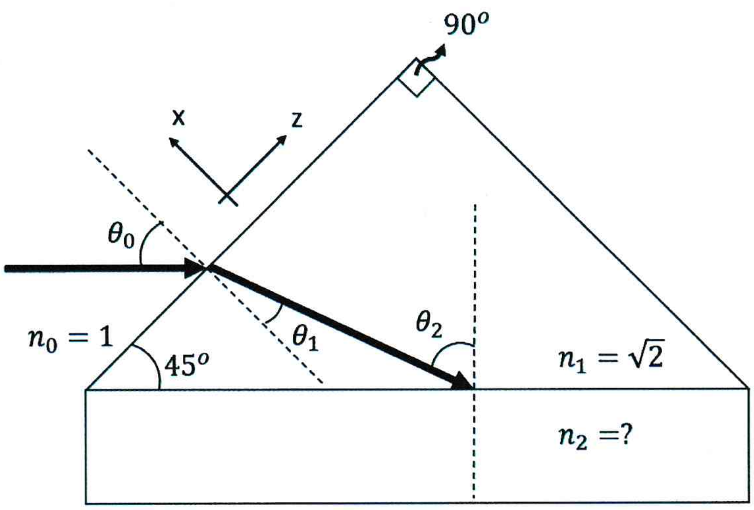
Problem 8: Interfaces **1 + 2 + 2 = 5 points**

A prism with refractive index $n_1 = \sqrt{2}$ is connected to a substrate with refractive index n_2. An incoming beam has TE polarization and an incidence angle to the surface of the prism of $\theta_0 = 45°$.

a) State the law that connects θ_0 and θ_1. Calculate the angle θ_1.

b) Draw the direction of electric and magnetic field of the incident field before the prism (draw your solution directly in the figure below). Which components of $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{k}$ are continuous (use the local coordinate system shown on the picture)?

c) What is the maximum value for n_2 to have total internal reflection inside the prism?



Problem 1: Maxwell's Equations

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with the damping factor g and the oscillator strength f. Calculate the expression for the permittivity $\varepsilon(\omega)$ in this material and decompose it in real and imaginary part.

Solution:

(a) $\nabla \times E = -\mu_0 \frac{\partial H(r,t)}{\partial t}$

$\nabla \times H = \varepsilon_0 \frac{\partial E(r,t)}{\partial t}$

$\varepsilon_0 \nabla \cdot E = 0$

$\nabla \cdot H = 0$

(b) $\nabla \times \nabla \times E = -\mu_0 \frac{\partial}{\partial t} \nabla \times H$

$= -\mu_0 \frac{\partial}{\partial t} \varepsilon_0 \frac{\partial}{\partial t} E(r,t)$

$= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} E(r,t)$

$\nabla \times \nabla \times E = \frac{\omega^2}{c^2} E(r,\omega)$

$\nabla \times \nabla \times E - \frac{\omega^2}{c^2} E(r,\omega) = 0$

(c) $\Rightarrow$ fourier transform

$-\omega^2 P(r,\omega) - i\omega g P(r,\omega) + \omega_0^2 P = \varepsilon_0 f E(r,\omega)$

$P(r,\omega) = \frac{\varepsilon_0 f}{(\omega_0^2-\omega^2)-ig\omega} E(r,\omega)$ $P(r,\omega) = \varepsilon_0 (\chi(\omega)) E(r,\omega)$

$\nabla \times P = -D$

$D = P + \varepsilon E = \varepsilon_0 \chi(\omega) E + \varepsilon E$

$= (1+\chi)\varepsilon_0 E$

$\varepsilon(\omega) = 1 + \chi(\omega)$

$\chi(\omega) = \varepsilon(\omega) - 1$

$\frac{f}{(\omega_0^2-\omega^2)-ig\omega} = \varepsilon(\omega) - 1$

$\varepsilon(\omega) = \frac{f}{(\omega_0^2-\omega^2)-ig\omega} + 1$

$= \frac{f[(\omega_0^2-\omega^2)+ig\omega]}{(\omega_0^2-\omega^2)^2 + g^2\omega^2} + 1$

$= \frac{f(\omega_0^2-\omega^2) + ig\omega f}{(\omega_0^2-\omega^2)^2 + g^2\omega^2} + 1$

Real $\Rightarrow \varepsilon'(\omega) = \frac{f(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2 + g^2\omega^2} + 1$

imaginary $\Rightarrow \varepsilon''(\omega) = \frac{g\omega f}{(\omega_0^2-\omega^2)^2 + g^2\omega^2}$

Problem 2: Normal modes **2 + 3 + 2 + 3 = 10 points**

Consider the complex representation of a plane wave:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \exp\left[\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \omega t)\right].$$

a) Name the variables $\mathbf{k}$ and ω, and state their physical units. What is the connection between $\mathbf{k}$ and the wavelength λ in vacuum?

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c) Consider a wave with $\mathbf{k} = (a, 0, ib)$ with $a, b \in \mathbb{R}$. Derive the expressions that define the planes of constant amplitude and constant phase for this wave. Show that these planes are orthogonal. What is such a wave called? Name one situation in which this type of wave is generated.

d) Show that in a lossless, isotropic material the time-averaged Poynting vector of a plane wave has the direction of $\mathbf{k}$, and that its magnitude is proportional to $|\mathbf{E}_0|^2$.

Solution: (a) k: wave number

ω: angular frequency

$k = \frac{2\pi c}{\lambda} = \frac{2\pi}{\omega}$

(b) Since $\nabla \cdot E = 0$ in this condition.

$\rightarrow ik_0 \vec{E}(r,\omega) = 0$ $\quad \vec{E} \perp k$

$\Delta E(r,\omega) + \frac{\omega^2}{c^2}\varepsilon(\omega) E(r,\omega) = 0$

$\left[-k^2 + \frac{\omega^2}{c^2}\varepsilon(\omega)\right]\vec{E}(\omega) = 0$.

$k^2 = \frac{\omega^2}{c^2}\varepsilon(\omega)$

(c) $k = (a, 0, ib)$

$$E(r,t) = E_0 e^{-i\omega t}\begin{pmatrix} e^{iax} \\ 0 \\ e^{-bz} \end{pmatrix}$$

plane of constant amplitude: $bz = $ constant.

plane of constant phase. $\quad z = \frac{c}{b}$.

$ax = $ constant. $\quad x = \frac{c}{a}$

normal vector of constant amplitude. $\hat{e}_z$.

normal vector of constant phase. $\hat{e}_x$

$\hat{e}_x \perp \hat{e}_z$ $\rightarrow$ these planes are orthogonal.

$k' \perp k''$ $\rightarrow$ evanescent wave.

(d) $H(r,\omega) = -\frac{1}{i\omega\mu_0}\nabla \times E(r,\omega)$

$= \frac{1}{\omega\mu_0} k\, E(r,\omega)$

$\langle S(r,t)\rangle = \frac{1}{2}\mathrm{Re}\left[E \times H^*\right] = \frac{1}{2}\mathrm{Re}\left[E \times \frac{k^*}{\omega\mu_0} \times E(r,\omega)\right]$

since lossless. $\quad k^* = k$. , therefore it has direction of k

$\langle S(r,t)\rangle = \frac{\vec{k}}{2\omega\mu_0}|E_0|^2$

Problem 3: Diffraction **2 + 1 + 3 + 4 = 10 points**

Note that each task can be solved independently.

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How does the diffraction pattern in the paraxial Fraunhofer approximation depend on k_x?

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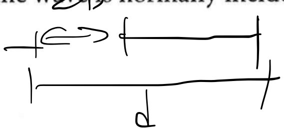


$$t(x) = \begin{cases} \frac{1}{2} + \frac{x}{b} & \text{for } |x| \le b/2 \\ 0 & \text{otherwise} \end{cases}.$$

Solution: (a) (1) Fresnel approximation $N_F = \frac{a^2}{\lambda z_B} \le 10$

$\alpha^2 + \beta^2 << k^2$ $\qquad H_F = \exp(ikz_B)\exp\left[-i\frac{z_B}{2k}(\alpha^2+\beta^2)\right]$

2) paraxial Fraunhofer approximation:

$\alpha^2 + \beta^2 << k^2 \quad N_F \le 0.1 \qquad N_F = \frac{a}{\lambda}\frac{a}{z_B}$

→ large apertune, $\qquad I_{FR}(x, y, z_B) \sim \frac{1}{(\lambda z_B)^2}\left|U_f\left(k\frac{x}{z_B}, k\frac{y}{z_B}, z_A\right)\right|^2$

3) Non paraxial Fraunhofer approximation

$N_F \lesssim 0.1$

(b) $u_0(x, y, z = 0) = A_0 \exp(ik_0 x)$ $\qquad U_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_0 \exp(ik_0 x)\exp(-i\alpha x)\,dx$

$u_{FR}(x, y, z) \sim U_0\left(\frac{kx}{z_B}, \frac{ky}{z_B}\right)$

$= A\,\delta(\alpha - k_0)$.

$I_{FR}(x, y, z_B) \sim \frac{1}{(\lambda z_B)^2} A^2 \left|\delta^2\left(\frac{x}{z_B} - k_0, \frac{y}{z_B} - k_0, z_A - k_0\right)\right|$

(c) $t(x) = \sum_{n=0}^{N-1}(x - nd)$ with $\begin{cases} t(x) & \text{for } x \le b \\ 0 & \text{elsewhere} \end{cases}$

$T\left(k\frac{x}{z_B}\right) \sim \sum_{n=0}^{N-1}\int_{-\infty}^{\infty} t_r(x' - nd)\exp\left(-ik\frac{x}{z_B}x'\right)dx'$

variable substitution: $x' + nb = x$

$$T\left(k\frac{x}{z_B}\right) \sim \sum_{n=0}^{N-1} t(x')\exp\left(-ik\frac{x}{z_B}x'\right)\exp\left(-ik\frac{x}{z_B}nb\right)dx'$$

$$T\left(k\frac{x}{z_B}\right) \sim T_s\left(\quad x' \right)\sum_{n=0}^{N-1}\exp\left(-ik\frac{x}{z_B}nb\right)$$

$$T\left(k\frac{x}{z_B}\right) \sim T_s\left(k\frac{x}{z_B}\right) \qquad \left|\sum_{n=0}^{N-1}\exp(-i\delta n)\right| = \left|\frac{\sin\left(N\frac{\delta}{2}\right)}{\sin\left(\frac{\delta}{2}\right)}\right|$$

$$\sum_{n=0}^{N-1}\exp\left[-i\frac{kx}{z_B}nd\right]$$

$$\rightarrow I \sim \sin c^2\left(k\frac{x}{z_B}b\right)\frac{\sin^2\left(N\cdot\frac{k}{2}\cdot\frac{x}{z_B}d\right)}{\sin^2\left(\frac{k}{2}\frac{x}{z_B}d\right)}$$

$$= \frac{\sin\left(Nx\frac{kx}{2z_B}\right)d}{\sin\left(\frac{kx}{2z_B}\right)d}$$

① Global width $k\frac{x}{z_B}\cdot b = \pi \rightarrow x = \frac{\pi\cdot z}{bk} = \frac{\lambda z_B}{2b}$

② position of local maxima

$$\frac{k}{2}\cdot\frac{x_p}{z_B}\cdot d = n\pi$$

$$x_p = n\pi\cdot\frac{z_B\,2}{d\,k} = \frac{\lambda z n}{d}$$

③ width of local maxima.

$$N\cdot\frac{k}{2}\cdot\frac{x}{z_B}\cdot d = \pi \rightarrow x = \frac{\pi\cdot z\cdot 2}{N\cdot k\cdot d} = \frac{\lambda z}{Nd}$$

(d) $f(x) = \begin{cases} \frac{1}{2}+\frac{x}{b} & |x| \le \frac{b}{2} \\ 0 & \text{otherwise.} \end{cases}$

$$T(\alpha) = \frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left(\frac{1}{2}+\frac{x}{b}\right)\exp(-i\alpha x)\,dx,$$

$$= \frac{1}{2\pi}\left[\frac{1}{2}\delta(\alpha) + \frac{1}{b}\left[\int_{-\frac{b}{2}}^{\frac{b}{2}} x e^{-i\alpha x}dx\right.\right.$$

$$= \frac{1}{2\pi}\left[\frac{1}{2}\delta(\alpha)+\frac{1}{b}\left[-\frac{1}{i\alpha}xe^{-i\alpha x}\Big|_{-\frac{b}{2}}^{\frac{b}{2}} - \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{-i\alpha x}dx\right]\right.$$

$$= \frac{1}{4\pi}\delta(\alpha) + \frac{1}{b}\frac{i}{\alpha}\left(\frac{b}{2}e^{-i\frac{b}{2}\alpha} + \frac{b}{2}e^{i\frac{b}{2}\alpha}\right) + \frac{1}{\alpha i}\left[e^{-i\frac{b}{2}\alpha} - e^{i\frac{b}{2}\alpha}\right]$$

$$= \frac{1}{4\pi}\delta(\alpha) + \frac{1}{b}\frac{i}{\alpha}b\cos\left(\frac{\alpha b}{2}\right) - \frac{i}{\alpha}\left[2\sin\frac{\alpha b}{2}\right]$$

Problem 4: Pulses **2 + 2 + 4 = 8 points**

a) Define the phase and group velocity in terms of the frequency-dependent wavenumber $k(\omega)$. Explain their physical meaning for the propagation of an optical pulse in a dispersive material.

b) Consider a material with the refractive index

$$n(\lambda) = A + \frac{B}{\lambda^2}$$

How long will it take for a Gaussian pulse with central wavelength λ_0 to travel through a distance L of this material? Express your result in terms of L, λ_0, A, and B.

c) Starting from the transfer function of a pulsed beam in Fresnel approximation

$$H_F(\alpha, \beta, \omega; z) = \exp[ik(\omega)z] \exp\left[-i\frac{\alpha^2 + \beta^2}{2k(\omega)}z\right]$$

derive the transfer function of the slowly varying envelope $v(x, y, t; z)$ in the parabolic approximation at frequency ω_0.

Solution: phase velocity: is the velocity of the phase fronts for light at the central frequency $w = w_0$

Group velocity is the velocity of the center of the pulse with central frequency at $w = w_0$.

$$V_{ph} = \frac{w_0}{k_0} \qquad V_{gr} = \frac{dw}{dk}$$

(b) $$n(\lambda) = A + \frac{B}{\lambda^2} = A + \frac{Bw^2}{4\pi^2 c^2}$$

$$\frac{1}{v_{gr}} = \frac{dk}{dw} = \partial C$$

$$k = \frac{w}{c} n$$

$$k(w) = \frac{w}{c}A + \frac{wB}{c^3\lambda^2}$$

$$= \frac{w_0}{c}A + \frac{w_0^3 B}{c^3 4\pi^2}$$

$$V_{gr} = \frac{dk}{dw} C = \frac{A}{c} + \frac{3w_0^2 B}{4\pi^2 c^3}$$

$$t = \frac{L}{v_{gr}} = \frac{L}{c}\left(A + \frac{3Bw_0^2}{4\pi^2 c^2}\right) = \frac{L}{c}\left(A + \frac{3Bw_0^2}{4\pi^2 c^2}\right),$$

(c) $H_{F}(\alpha, \beta, \omega, z) = \exp\left[-i\frac{\alpha^2+\beta^2}{2k(\omega)}z\right]\exp[ik(\omega)z]$

Taylor expansion of $k(\omega)$

$$k(\omega) = k(\omega_0) + \frac{dk}{d\omega}\Big|(\omega-\omega_0) + \frac{1}{2}\frac{d^2k}{d\omega^2}\Big|(\omega-\omega_0)^2$$

For parabolic approximation

$$k(\omega) = k(\omega_0) = k_0$$

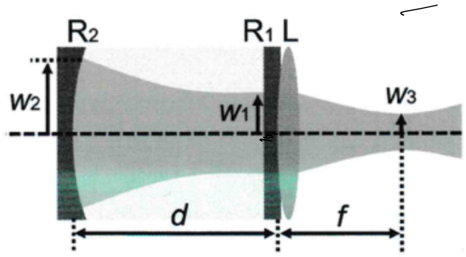
$$H_{FP} \simeq \exp[ik_0 z]\exp\left[i\frac{1}{V_g}(\omega-\omega_0)\right]\exp\left[i\frac{D}{2}(\omega-\omega_0)^2 z\right]\exp\left[-i\frac{\alpha^2+\beta^2}{2k_0}z\right]$$

Let $\bar{\omega} = \omega-\omega_0$

$$H_{FP}(\alpha, \beta, \omega, z) = \exp\left[iz\left(\frac{\bar{\omega}}{v_g} + \frac{1}{2}D\bar{\omega}^2 - \frac{1}{2}\frac{1}{k_0}(\alpha^2+\beta^2)\right)\right]$$

Problem 5: Gaussian Beams **3 + 1 + 4 = 8 points**

A mirror with a radius of curvature $R_2 = 2d$ and a mirror with a radius of curvature $R_1 = \infty$ form a resonator (see figure below). The resulting Gaussian beam of wavelength λ has a width w_1 and a radius of curvature $R_1 = \infty$ at the output mirror and is then focused by a thin-lens with a focal length f.



a) Derive expressions for the beam widths w_1 and w_2, using only the resonator length d and the wavelength λ.
b) Define the resonator as stable or unstable. Give the stability criterion, which justifies your answer.
c) Derive the expression for the new waist w_3 at the focal position of the lens using the beam width w_1.

Solution (a) $R_2(z) = z\left[1+\left(\frac{z_0}{z}\right)^2\right]$

$R_z = z + \frac{z_0^2}{z}$ → $R_z = 2d,\quad z = d,$

↓

$2d = d + \frac{z_0^2}{d}$

$2d^2 = d^2 + z_0^2$ $\quad d = z_0$

$z_0 = \frac{kw^2}{2} = \frac{\pi w_1^2}{\lambda}$

↓

$w_1 = \sqrt{\frac{\lambda d}{\pi}}$ $\quad q_0 = -iz_0$

$w_z = w_0\sqrt{1+\left(\frac{z}{z_0}\right)^2} = w_0\sqrt{1+\left(\frac{d}{d}\right)^2} = \sqrt{2}w_0 = \sqrt{\frac{2\lambda d}{\pi}}$

$q_2 = \frac{f}{-\frac{q_0}{f}+1} = \frac{f}{\frac{f-q_0}{f}} = \frac{f^2}{f-q_0} = \frac{f^2}{f+iz_0}$

(b) $g_1g_2 = \left(1+\frac{d}{R_1}\right)\left(1-\frac{d}{2d}\right) = \frac{1}{2}$ Stable.

(c) $\begin{vmatrix} 1 & f \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{vmatrix}$ $\quad q_0 = -iz_0 = -i\frac{\pi w_1^2}{\lambda}.$

$= \begin{vmatrix} 1-1, & f \\ -\frac{1}{f}, & 1 \end{vmatrix} = \begin{vmatrix} 0, & f \\ -\frac{1}{f}, & 1 \end{vmatrix}$ $\quad q_1 = \frac{Aq_0 + B}{Cq_0 + D}$

$\frac{1}{q_2} = \frac{1}{f} + i\frac{z_0}{f^2}$ $\quad \frac{z_0}{f^2} = \frac{\lambda}{\pi w^2}$ $\quad w^2 = \frac{\lambda f^2}{\pi z_0}$

$\frac{1}{q(z)} = \frac{1}{R(z)} + i\frac{\lambda n}{\ldots}$ $\quad w_3 = f\sqrt{\frac{\lambda}{\ldots}}$

P119

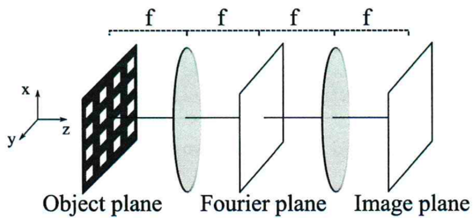
Problem 6: Fourier Optics $q(z) = R(z) + 1$ **2 + 2 + 2 + 2 + 2 = 10 points**

Consider a 4f-setup as shown in the figure below. The object in the object plane is illuminated with a monochromatic plane wave of wavelength λ.

$= f\sqrt{\dfrac{\lambda}{\pi \frac{\pi w^2}{\lambda}}}$

$= f\sqrt{\dfrac{\lambda^2}{\pi^2 w^2}}$

$= \dfrac{f\lambda}{\pi w_1}$



a) Describe how to derive an expression to calculate the field $u(x,y,z=2f)$ in the Fourier plane from the field $u_0(x,y,z=0)$ in the object plane (in words, no calculation necessary). Write down this expression.

b) We now put a transmission mask in the Fourier plane and block all the light not on the optical axis. Argue how the field distribution will look in the image plane. Explain your answer. *Hint:* No calculation is needed here.

c) Which transmission mask can we use if we want to see only vertical lines in the image plane? Describe it in one sentence.

d) Is it possible to obtain a perfect image of the object after 4f-setup? Shortly explain your answer.

e) Estimate the limit of optical resolution of this system, if $\lambda = 1000\,\text{nm}$, $f = 10\,\text{cm}$, and the diameter of the transmission mask placed into the center of the setup is $D = 2\,\text{cm}$.

solution (a) ① fourier transform $U(\alpha,\beta) = FT[U_0(x,y)]$

② propagation of distance,

$U_- = U_0(\alpha,\beta)\exp(ikf)\exp[-\frac{i}{2k}(\alpha^2+\beta^2)f]$

③ interaction with lens

$u_f = t_L \cdot u_- \Rightarrow U_f = T_L * U_-$

④ f propagation,

⑤ Fourier back transform

~~expression~~ $u(x,y,2f) = -i\dfrac{(2\pi)^2}{\lambda f}\exp(2ikf)\,U_0(\frac{k}{f}x, \frac{k}{f}y)$

(b) The image is a perfect replica of the object (in fresnel approximation)

since The 4f-system performs a fourier transform followed by an inverse fourier transform,

(c) The mask can filter horizontal spatial frequency.
[α is filtered out]

(d) No, since the image is only The edge of lens perfect in the Fresnel approximation. Will limit resolution of this set-up

$D = 2\,cm$

$f = 10\,cm \quad \lambda = 1000\,nm$

(e) $\Delta r = \frac{1.22\lambda f}{nD} = \frac{1.22 \times 1000 \times 10^{-9}\,m \times 10 \times 10^{-2}\,m}{2 \times 10^{-2}\,m} = 6.1 \times 10^{-6}\,m$

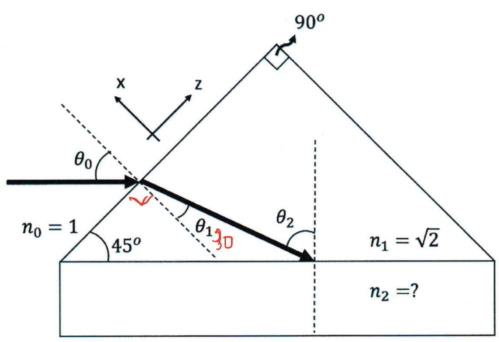
Problem 8: Interfaces **1 + 2 + 2 = 5 points**

A prism with refractive index $n_1 = \sqrt{2}$ is connected to a substrate with refractive index n_2. An incoming beam has TE polarization and an incidence angle to the surface of the prism of $\theta_0 = 45°$.

a) State the law that connects θ_0 and θ_1. Calculate the angle θ_1.

b) Draw the direction of electric and magnetic field of the incident field before the prism (draw your solution directly in the figure below). Which components of **E**, **H**, and **k** are continuous (use the local coordinate system shown on the picture)?

c) What is the maximum value for n_2 to have total internal reflection inside the prism?



(a) $n_0 \sin\theta_0 = n_1 \sin\theta_1$

$\frac{\sqrt{2}}{2} = \sqrt{2}\sin\theta_1$

$\sin\theta_1 = \frac{1}{2} \quad \theta_1 = 30°$

(b) E⊙ k, H

$E_0 = E_1 \quad H_{z_1} = H_{z_2}$

(c) $\theta_2 = 70°$. For TIR, $Re(k_{2x}) < 0$.

$\sqrt{\frac{\omega^2 n_2^2}{c^2} - \frac{\omega^2}{c^2} 2\sin\theta_2} < 0 \quad n_2^2 < 2\sin 75°$.

$n_{2max} = 1.39$

Problem 7: Anisotropy **2 + 2 + 3 + 3 = 10 points**

Consider a general homogeneous, transparent, and *anisotropic* medium.

a) What are the normal modes in this medium? How do they differ from the isotropic case? $D \times E$ $S = E \times H$,

b) Show that for a linearly polarized plane wave the Poynting vector **S** is in general *not* parallel to the wave vector **k**, i.e., $\mathbf{k} \nparallel \mathbf{S}$. $k \times S$ $S \perp E$,

Now assume the special case of a uniaxial crystal with ordinary refractive index n_o and extraordinary refractive index n_e. The planar interface of the crystal to air lies in the x-y-plane.

c) The figure below shows the normal surfaces for a specific crystal orientation and non-normal incidence. Draw the optical axis and construct the wavevectors and Poynting vectors for the ordinary and extraordinary beam in the crystal. Draw your solution directly in the figure below!

d) Now assume that the optical axis of the crystal lies in the x-y plane and forms a 45° angle with the x axis. $D \perp k$ An x-polarized incident beam propagates in z direction. Calculate the thickness L for which this crystal acts as a half-wave plate at wavelength λ.

(a) $D^{(i)} = D \cdot \exp[i(k_i r - \omega t)]$

$k_i = \frac{\omega}{c} \varepsilon_i(\omega)$

$E^{(i)} = \frac{D^{(i)}}{\varepsilon_0 \varepsilon^{(i)}}$

E do not parallel with D

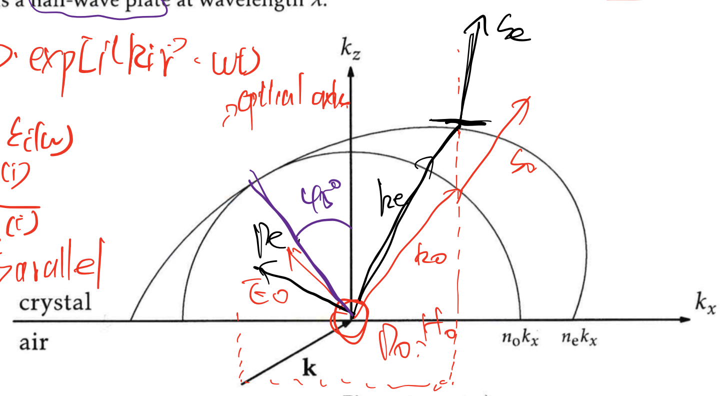


Figure to part c)

Solution (a) ① ordinary normal mode:

n is independent on direction,

$D_o \perp k$ and z axis

② extraordinary normal mode:

n depend on direction

Isotropic case;

The normal mode are monochromatic plane waves

$E = E \exp(i(k(\omega) r - \omega t))$ $k^2 = \frac{\omega^2}{c^2} \varepsilon(\omega)$

D and E are not parallel

$E^{(i)} = \frac{D^{(i)}}{\varepsilon_0 \varepsilon^{(i)}}$

Anisotropic case;

Difference: In this case, normal mode can't be elliptically polarized,

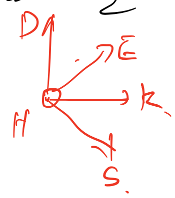
$D^{(a)} = \{D_x \exp[i(k_a r - \omega t)]\} e_x \rightarrow k_a^2 = \frac{\omega^2}{c^2} \varepsilon_x$ normal mode a

$D^{b} = \{D_n \exp[i(k_b r - \omega t)]\} e_y \rightarrow k_b^2 = \frac{\omega^2}{c^2} \varepsilon_y$ normal mode b

(b) $E_a = \frac{D_a}{\varepsilon_0 \varepsilon_r}$ $\quad E_b = \frac{D_b}{\varepsilon_0 \varepsilon_r}$ $\quad$ For poynting vector,

$D \not\parallel E \qquad E \not\perp k \qquad \langle S \rangle = \frac{1}{2} \mathrm{Re}(E \times H^*)$

$\langle S \rangle \perp E \rightarrow k \not\parallel \langle S \rangle$

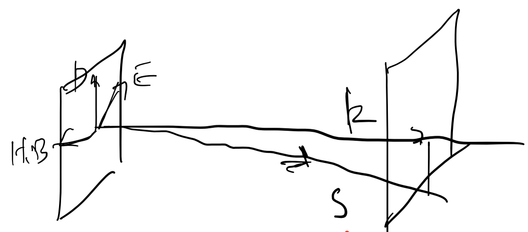


$S \not\parallel k$,

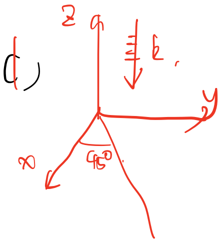
$k \times H = -i\omega\mu_0 D$

$\downarrow$

$D \perp k$, $D \perp H$



(c)



TM: $E_\parallel = E\cos 45° = \frac{\sqrt{2}}{2} E$, $\qquad k_e = n_e k = \frac{\omega}{c} n_e = \frac{2\pi n_e}{\lambda}$

TE: $E_\perp = E\sin 45° = \frac{\sqrt{2}}{2} E$ $\qquad k_o = n_o k = \frac{2\pi n_o}{\lambda}$

after:

TM: $E_\parallel = \frac{\sqrt{2}}{2} E \exp(ik_e L)$ $\quad$ TE: $E_\perp = \frac{\sqrt{2}}{2} E \exp(ik_o L)$

$\Delta\varphi = (k_e - k_o)L = \frac{2\pi L}{\lambda}(n_e - n_o) = \pi$

$L = \frac{\lambda}{2(n_e - n_o)}$

for half wave: $\quad d = \frac{\lambda}{2(n_e - n_o)}$,

quarter-wave plate. $\quad d = \frac{\lambda}{4(n_e - n_o)}$