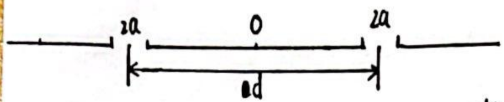
Task 1 Jinsong Liu

a)

Solution:

$$U_0(x, z=0)=\begin{cases}1, & \text{for } |x\pm d/2|\le a\\ 0, & \text{elsewhere}\end{cases}$$




$$U_+\left(k\frac{x}{z_B}\right)=\int_{-d/2-a}^{-d/2+a}\exp\left(-i\frac{kx}{z_B}x'\right)dx'+\int_{d/2-a}^{d/2+a}\exp\left(-i\frac{kx}{z_B}x'\right)dx'$$

$$=\frac{z_B}{kx}\left\{\sin\left[\frac{kx}{z_B}(-d/2+a)\right]-\sin\left[\frac{kx}{z_B}(-d/2-a)\right]+i\cos\left[\frac{kx}{z_B}(-d/2+a)\right]-i\cos\left[\frac{kx}{z_B}(-d/2-a)\right]\right\}$$

$$+\frac{z_B}{kx}\left\{\sin\left[\frac{kx}{z_B}(d/2+a)\right]-\sin\left[\frac{kx}{z_B}(d/2-a)\right]+i\cos\left[\frac{kx}{z_B}(d/2+a)\right]-i\cos\left[\frac{kx}{z_B}(d/2-a)\right]\right\}$$

$$=\frac{z_B}{kx}\left[2\cos\left(-\frac{kxd}{2z_B}\right)\sin\left(\frac{kxa}{z_B}\right)-2i\sin\left(-\frac{kxd}{2z_B}\right)\sin\left(\frac{kxa}{z_B}\right)\right]+\frac{z_B}{kx}\left[2\cos\left(\frac{kxd}{2z_B}\right)\sin\left(\frac{kxa}{z_B}\right)-2i\sin\left(\frac{kxd}{2z_B}\right)\sin\left(\frac{kxa}{z_B}\right)\right]$$

$$=\frac{2\sin\left(\frac{kxa}{z_B}\right)}{\frac{kx}{z_B}}\exp\left(-i\frac{kxd}{2z_B}\right)+\frac{2\sin\left(\frac{kxa}{z_B}\right)}{\frac{kx}{z_B}}\exp\left(i\frac{kxd}{2z_B}\right)$$

$$=4a\frac{\sin\left(\frac{kxa}{z_B}\right)}{\frac{kxa}{z_B}}\cos\left(\frac{kxd}{2z_B}\right)$$

$$=4a\,\mathrm{sinc}(\alpha a)\cos\left(\frac{kxd}{2z_B}\right)$$

(Let $\alpha=\frac{kx}{z_B}$)

$$\therefore I=\frac{1}{(\lambda z_B)^2}16a^2\,\mathrm{sinc}^2(\alpha a)\cos^2\left(\frac{kxd}{2z_B}\right)$$

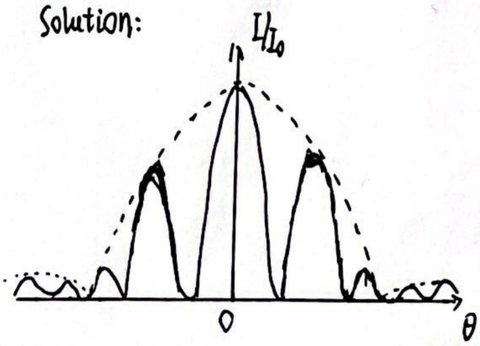
b)

Solution:

$$\frac{(2a)^2}{\lambda z_B}<0.1$$

c)

Solution:




d can decide the number of peak values which are included in the envelope.

a can decide the width of the fringes

Task 2 Jinsong Liu

a)

Solution:

$$t(x)=\sum_{l=0}^{N-1}\tilde{f}(x-lD)$$

$$\alpha=\frac{kx}{z_B} \quad \text{let } x'=x'-lD$$

$$\therefore T(\alpha)=\sum_{l=0}^{N-1}\int_{-\infty}^{\infty}\tilde{f}(x'-lD)\exp(-i\alpha x')dx'$$

$$=\sum_{l=0}^{N-1}\int_{-\infty}^{\infty}\tilde{f}(x')\exp(-i\alpha x')\exp(-i\alpha lD)dx'$$

$$=\tilde{F}(\alpha)\sum_{l=0}^{N-1}\exp(-i\alpha Dl)$$

$$=\tilde{F}(\alpha)\frac{1-\exp[-iN\alpha D]}{1-\exp(-i\alpha D)}$$

$$=\tilde{F}(\alpha)\frac{\exp(-iN\alpha D/2)\left[\exp(iN\alpha D/2)-\exp(-iN\alpha D/2)\right]}{\exp(-i\frac{\alpha D}{2})\left[\exp(i\frac{\alpha D}{2})-\exp(-i\frac{\alpha D}{2})\right]}$$

$$=\tilde{F}(\alpha)\frac{\sin(N\alpha D/2)}{\sin(\alpha D/2)}e^{i(1-N)\alpha D/2}$$

b)

Solution:

$$F\left[\sum_{l=-\infty}^{\infty}\tilde{f}(x-lD)\right]=F\left[\sum_{n=-\infty}^{\infty}F_n e^{in\omega_0 t}\right]=2\pi\sum_{n=-\infty}^{\infty}F_n\delta(\alpha-\frac{2n\pi}{D})$$

$$\omega_0=\frac{2\pi}{D}$$

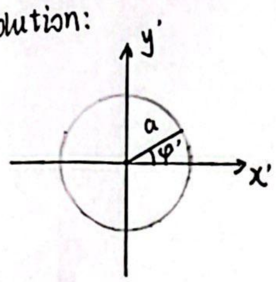
$$F_n=\frac{1}{D}\tilde{F}(\alpha)$$

$$\therefore T(\alpha)=\frac{2\pi}{D}\tilde{F}(\alpha)\sum_{n=-\infty}^{\infty}\delta(\alpha-\frac{2n\pi}{D})$$

Task 3 Jinsong Liu

a)

solution:




Solution:

coordinates:

$$\begin{cases} x' = r'\cos\varphi' \\ y' = r'\sin\varphi' \end{cases}, \quad \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

$$dx'dy' = r'dr'd\varphi'$$

$$U_+\left(k\frac{x}{z_B}, k\frac{y}{z_B}\right) = \iint_{-\infty}^{\infty} \exp\left[-i\left(\frac{kx}{z_B}x' + \frac{ky}{z_B}y'\right)\right] dx'dy'$$

$$= \int_0^{2\pi}\int_0^a \exp\left[-ik\frac{rr'}{z_B}(\cos\varphi'\cos\varphi + \sin\varphi'\sin\varphi)\right] r'dr'd\varphi'$$

Let $r/z_B = \theta$

$$U_+(\theta, \varphi) = \int_0^{2\pi}\int_0^a \exp\left[-ik\theta r'\cos(\varphi'-\varphi)\right] r'dr'd\varphi'$$

$$\because \int_0^{2\pi} \exp[-ik\theta r'\cos(\varphi'-\varphi)]d\varphi' = 2\pi \underline{J_0(kr'\theta)} \text{ (Bessel function)}$$

$$\therefore U_+(\theta, \varphi) = 2\pi \int_0^{k\theta a} (kr'\theta) J_0(kr'\theta)\, d(kr'\theta) \cdot \frac{1}{(k\theta)^2}$$

$$\because \frac{d}{dx}\left[x^{n+1} J_{n+1}(x)\right] = x^{n+1} J_n(x) \qquad \therefore \int_0^t x J_0(x) dx = t J_1(t)$$

$$\therefore \int_0^{k\theta a} (kr'\theta) J_0(kr'\theta)\, d(kr'\theta) = \int_0^{k\theta a} x J_0(x) dx = ka\theta J_1(ka\theta)$$

$$\therefore U_+(\theta, \varphi) = \frac{2\pi}{(k\theta)^2} ka\theta J_1(ka\theta) = \frac{2\pi a J_1(ka\theta)}{k\theta}$$

$$I = \frac{4\pi^2 a^2 J_1^2(ka\theta)}{k^2\theta^2}$$

b)

Solution:

From problem a)

$$U_t\left(k\frac{x}{z_B}, k\frac{y}{z_B}\right) = \int_0^{2\pi}\int_{a_1}^{a_2} \exp\left[-ik\frac{rr'}{z_B}(\cos\varphi'\cos\varphi + \sin\varphi'\sin\varphi)\right] r'dr'd\varphi'$$

Let $r/z_B = \theta$

$$U_t(\theta,\varphi) = \int_0^{2\pi}\int_{a_1}^{a_2} \exp\left[-ik\theta r'\cos(\varphi'-\varphi)\right] r'dr'd\varphi'$$

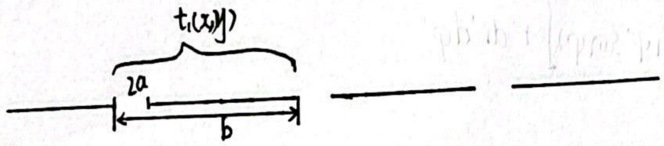
$$= 2\pi\int_{a_1k\theta}^{a_2k\theta} (kr'\theta) J_0(kr'\theta)\, d(kr'\theta)\frac{1}{(k\theta)^2}$$

$$= \frac{2\pi a_2 J_1(ka_2\theta)}{k\theta} - \frac{2\pi a_1 J_1(ka_1\theta)}{k\theta}$$

$$I = \left[\frac{2\pi a_2 J_1(ka_2\theta)}{k\theta} - \frac{2\pi a_1 J_1(ka_1\theta)}{k\theta}\right]^2$$

c)

Solution:




$\cancel{t(x) = \sum_{n=0}^{N-1} t_1(x-nb)}$

$$t(x) = \sum_{n=0}^{N-1} t_1(x-nb, y) \quad \text{with } t_1(x,y) = \begin{cases} 1 & \text{for } x^2+y^2 \leq a^2 \\ 0 & \text{elsewhere} \end{cases}$$

$$T\left(k\frac{x}{z_B}\right) = \sum_{n=0}^{N-1}\int_{-\infty}^{\infty} t_1(x'-nb, y) \exp\left[-i\left(k\frac{x}{z_B}x' + k\frac{y}{z_B}y'\right)\right] dx'dy'$$

$$\begin{pmatrix}\text{let} \\ x' = x - nb\end{pmatrix} = \sum_{n=0}^{N-1}\int_{-a}^{a} t_s(x', y') \exp\left[-i\left(k\frac{x}{z_B}x' + k\frac{y}{z_B}y'\right)\right] \exp\left(-ik\frac{x}{z_B}nb\right) dx'dy'$$

$$= T_s\left(k\frac{x}{z_B}, k\frac{y}{z_B}\right) \sum_{n=0}^{N-1} \exp\left(-ik\frac{x}{z_B}nb\right)$$

From lecture script

$$\left|\sum_{n=0}^{N-1} \exp(-i\delta n)\right| = \left|\frac{\sin(N\frac{\delta}{2})}{\sin(\frac{\delta}{2})}\right|$$

From Problem a)

we know that $T_s\left(k\frac{x}{z_B}, k\frac{y}{z_B}\right) = \dfrac{2\pi a J_1(ka\theta)}{k\theta}$

$$\therefore T = \frac{2\pi a J_1(ka\theta)}{k\theta}\left|\frac{\sin(N\frac{kxb}{2z_B})}{\sin(\frac{kxb}{2z_B})}\right|$$

$$I = \frac{4\pi a^2 J_1^2(ka\theta)}{k^2\theta^2}\left|\frac{\sin(N\frac{kxb}{2z_B})}{\sin(\frac{kxb}{2z_B})}\right|^2$$