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Problem 1

(a)

$$\begin{cases} PV = k_1(T,n) \\ V = k_2(P,n)T \\ V = k_3(P,T)n \\ P = k_4(V,n)T \end{cases}$$

From these 4 equations, we can get

$$\begin{cases} PV = k_1(T,n) & ① \\ PV = k_2(P,n)PT & ② \\ PV = k_3(P,T)Pn & ③ \\ PV = k_4(V,n)VT & ④ \end{cases}$$

$$\therefore \frac{k_1(T,n)}{T} = k_2(P,n)P = \frac{k_3(P,T)Pn}{T} = k_4(V,n)V = \text{constant}$$

$\because$ n is a constant

$\therefore \frac{k_3(P,T)P}{T}$ is a constant

$\therefore$ The equation ③ can be written as : $PV = \frac{k_3(P,T)P}{T} nT$

$$\therefore R = \frac{k_3(P,T)P}{T} \qquad PV = nRT$$

(b)

From Maxwell Distribution, the proability density function of speeds is :

$$f(v) = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi v^2 e^{-\frac{mv^2}{2kT}}$$

the proability of the molecules in $(v, v+dv)$: $f(v)\,dv$

suppose the total number of the molecules is N

the number of the molecules in $(v, v+dv)$: $Nf(v)\,dv$

$$V_{rms} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{\int_0^\infty v^2 N f(v)dv}{N}} = \sqrt{\int_0^\infty v^2 f(v)dv} = \sqrt{\int_0^\infty 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-\frac{mv^2}{2kT}} v^4\,dv} = \sqrt{4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int_0^\infty e^{-\frac{mv^2}{2kT}} v^4 dv}$$

suppose $I = 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int_0^\infty e^{-\frac{mv^2}{2kT}} v^4 dv \qquad I_1 = \int_0^\infty e^{-\frac{mv^2}{2kT}} v^4\,dv \qquad I = 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \cdot I_1$

$$I_1 = -\int_0^\infty \frac{kTv^3}{m}\, de^{-\frac{mv^2}{2kT}} = -\frac{kTv^3}{m} \cdot e^{-\frac{mv^2}{2kT}}\Big|_0^\infty + \frac{kT}{m}\int_0^\infty e^{-\frac{mv^2}{2kT}}\,dv^3$$

$$= \frac{3kT}{m}\int_0^\infty v^2 e^{-\frac{mv^2}{2kT}}\,dv = -\frac{3k^2T^2}{m^2}\int_0^\infty v\,de^{-\frac{mv^2}{2kT}} = -\frac{3k^2T^2 v}{m^2} e^{-\frac{mv^2}{2kT}}\Big|_0^\infty + \frac{3k^2T^2}{m^2}\int_0^\infty e^{-\frac{mv^2}{2kT}}dv$$

$$= \frac{3\sqrt{2}\,k^{\frac{5}{2}}T^{\frac{5}{2}}}{m^{\frac{5}{2}}}\int_0^\infty e^{-\frac{mv^2}{2kT}}\,d\sqrt{\frac{mv^2}{2kT}}$$

$\because \int_0^\infty e^{-x^2}dx = \frac{\pi}{2} \qquad \therefore I_1 = \frac{3\sqrt{2\pi}\,k^{\frac{5}{2}}T^{\frac{5}{2}}}{2m^{\frac{5}{2}}} \qquad I = 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \cdot \frac{3\sqrt{2\pi}\,k^{\frac{5}{2}}T^{\frac{5}{2}}}{2m^{\frac{5}{2}}} = \frac{3kT}{m} \qquad \therefore V_{rms} = \sqrt{\frac{3kT}{m}}$

(C)

From (b) we know that $\bar{v} = \sqrt{\frac{3k_BT}{m}}$

$$\tau_{col} = \frac{l}{\bar{v}} = \frac{V}{N\sigma\sqrt{\frac{3k_BT}{m}}}$$

$\because \Delta\nu = (2\pi \cdot \tau_{col})^{-1}$

$$\therefore \Delta\nu = \frac{1}{2\pi} \cdot \frac{N\pi d^2\sqrt{\frac{3k_BT}{m}}}{V}$$

$\because PV = nRT = k_BNT$

$$\therefore \Delta\nu = \frac{1}{2\pi} \frac{N\pi d^2 P\sqrt{\frac{3k_BT}{m}}}{k_BNT} = \sqrt{\frac{3}{4mk_BT}}\, d^2 P$$

Problem 2 Jinsong Liu 206216

(a)

suppose a 2-level system with degenerated energy levels

α is the absorption coefficient

g is the gain coefficient

$$dI_\nu = g \cdot I_\nu \cdot dx$$

$$g = \sigma_{e\nu}\left(N_2 - N_1\frac{g_2}{g_1}\right)$$

$$dI_\nu = \sigma_{e\nu}\left(N_2 - N_1\frac{g_2}{g_1}\right)\cdot I_\nu \cdot dx$$

$$\because \sigma_{e\nu} = \sigma_0 \cdot F(\nu)$$

$$dI_\nu = \left(N_2 - N_1\frac{g_2}{g_1}\right)\sigma_0 \cdot F(\nu)\cdot I_\nu \cdot dx$$

$$\because \int_{-\infty}^{\infty} F(\nu)\,d\nu = \frac{\pi\Delta\nu}{2}, \int \sigma_\nu\, d\nu = \frac{B_{21}\cdot h\nu_0}{c}$$

$$\therefore dI_\nu = \left(N_2 - N_1\frac{g_2}{g_1}\right)\frac{2}{\pi\Delta\nu}\cdot\frac{B_{21}h\nu_0}{c} I_\nu \cdot F(\nu)\,dx$$

$$\cancel{dI_\nu =} \quad \because \frac{A_{21}}{B_{21}} = \frac{8\pi h\nu_0^3}{c^3}$$

$$\therefore dI_\nu = \left(N_2 - N_1\frac{g_2}{g_1}\right)\frac{2}{\pi\Delta\nu}\cdot\frac{c^3}{8\pi h\nu_0^3}\cdot\frac{h\nu_0}{c}A_{21}\cdot I_\nu\, F(\nu)\,dx$$

$$dI_\nu = \left(N_2 - N_1\frac{g_2}{g_1}\right)\frac{c^2}{4\pi^2\nu_0^2}\frac{A_{21}}{\Delta\nu}\cdot I_\nu \cdot F(\nu)\,dx$$

$$\because \phi_\nu = \frac{I_\nu}{h\nu}$$

$$\therefore d\phi_\nu = \left(N_2 - N_1\frac{g_2}{g_1}\right)\cdot F(\nu)\cdot\left(\frac{c}{2\pi\nu_0}\right)^2\cdot\frac{A_{21}}{\Delta\nu}\,\phi_\nu\,dx$$

$$\therefore \frac{\phi_\nu(x)}{\phi_\nu(0)} = \exp\left[\left(N_2 - N_1\frac{g_2}{g_1}\right)\cdot F(\nu)\cdot\left(\frac{c}{2\pi\nu_0}\right)^2\cdot\frac{A_{21}}{\Delta\nu}\cdot x\right]$$

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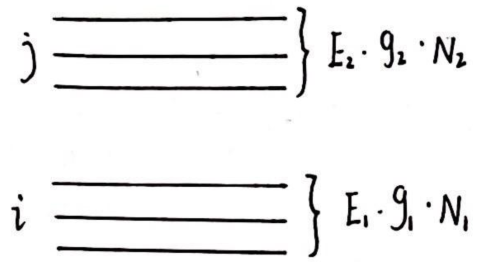
(b)



A two level system with degenerated levels

We calculate the rate of change of the population N_2 between sublevels j and i

$$\frac{dN_2}{dt} = -\sum_{i=1}^{g_1}\sum_{j=1}^{g_2}(W_{ji}N_{2j} - W_{ij}N_{1i})$$ (Neglect the spontaneous emission)

W_{ji} is the rate of stimulated transition between j and i levels

W_{ij} is the rate of absorption

$W_{ij} = W_{ji}$ $\quad g_1 B_{12} = g_2 B_{21}$

which means the proability of an electron absorbing a photon to transit to a higher energy level equals to the proability of an electron releasing a photon to transit to a lower energy level.

$\therefore$ There is no photon flux gain in this 2-level system.

Problem 3 Jinsong Liu 206216

(a)

Homogeneously-broadened
All particles have the same cross-section

Inhomogeneously-broadened
Different particles have different cross-sections

(b)

Homogeneously-broadened: the cross-sections of different particles change equally when the λ changes

Inhomogeneously-broadened: the cross-sections of different particles change differently when the λ changes

(c)

The volume is $V = 10^{-2} \times 10^{-2} \times 10^{-6} m^{-3} = 10^{-10} m^3$

The number of absorbing particles is $N = 1.5 \times 10^{20} \times 10^{-10} = 1.5 \times 10^{16}$

$\therefore$ The proability is $\frac{N}{dA} \sigma_Y = \frac{1.5 \times 10^{16} \times 2 \times 10^{-21}}{10^{-4}} = 0.3 = 30\%$

(d)

With $1 \mu W$ $\quad n_{ph} = \frac{P \cdot t}{h \cdot \frac{c}{\lambda}} = \frac{10^{-6} \times 1 \times 633 \times 10^{-9}}{6.62 \times 10^{-34} \times 3 \times 10^{8}} \approx 3.19 \times 10^{12}$

$\because n_{ph} < N$

$n_{absorbed} = n_{ph} \cdot 30\% = 9.57 \times 10^{11}$

With $100 W$ $\quad n'_{ph} = \frac{P' \cdot t}{h \cdot \frac{c}{\lambda}} = \frac{100 \times 1 \times 633 \times 10^{-9}}{6.62 \times 10^{-34} \times 3 \times 10^{8}} \approx 3.19 \times 10^{20}$

$\because n'_{ph} > N$ $\quad \therefore$ the absorption is saturated.

$n_{absorbed} = N \cdot 30\% = 1.5 \times 10^{16} \times 0.3 = 4.5 \times 10^{15}$