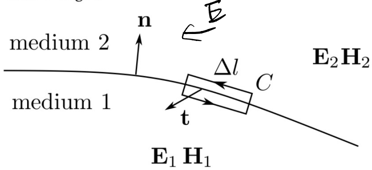
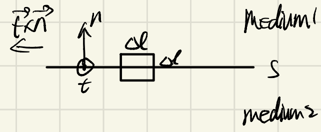
Task 1: Interface conditions (a=2, b=2 pts.)

Consider an interface $\mathcal{S}$ between two homogeneous and isotropic media (refractive indices n_1 and n_2). Do not consider surface current or surface charges.





$\vec{t}\times\vec{n}$

$\vec n\times(\vec E_2-\vec E_1)=0 \quad (\vec D_2-\vec D_1)\cdot\vec n=\sigma_s$

$\vec n\times(\vec H_2-\vec H_1)=\vec j_s \quad (\vec B_2-\vec B_1)\cdot\vec n=0$

a) Starting from the differential form of Maxwell's equations, derive the conditions at such an interface for the tangential electric and magnetic field $\mathbf{n}\times\mathbf{E}(\mathbf{r},t)$ and $\mathbf{n}\times\mathbf{H}(\mathbf{r},t)$, respectively. $\mathbf{n}$ is the normal vector to the surface. *Hint:* consider an infinitesimal, closed loop, which contains the surface and use Stokes' theorem to write the integral form of Maxwell's equations (see figure above). $\mathbf{t}$ is the tangential vector to the surface between medium 1 and 2, and is normal to the surface of the infinitesimal closed loop).

b) Similarly, derive the interface conditions for the normal components of the electric flux density $\mathbf{D}$ and the magnetic flux density $\mathbf{B}$. *Hint:* use an infinitesimal volume and Gauss' theorem.

a) $\nabla\times\vec E=-\frac{\partial B}{\partial t}$ $\qquad \nabla\cdot\vec D=0$

$\nabla\times\vec H=\vec j+\frac{\partial D}{\partial t}=\frac{\partial D}{\partial t}$ $\qquad \nabla\cdot\vec H=0$

$\vec D=\varepsilon_0\vec E+\vec P=\varepsilon_0\vec E+\varepsilon_0\chi(\omega)\vec E=\varepsilon_0[1+\chi(\omega)]\vec E=\varepsilon_0\varepsilon(\omega)\vec E$

$\iint\nabla\times\vec E\,ds=\oint_C\vec E\cdot d\vec l=\Delta l[\vec E_1(\vec t\times\vec n)-\vec E_2(\vec t\times\vec n)]$

$\iint_S\nabla\times\vec E\,ds=-\iint_S\frac{\partial B}{\partial t}ds=-\frac{\partial}{\partial t}\iint_S\vec B\cdot d\vec s=-\frac{\partial}{\partial t}\vec B\cdot\vec t\,\Delta l^2$

$\vec a\cdot(\vec b\times\vec c)=a_i(\vec b\times\vec c)_i=a_i\varepsilon_{ijk}b_jc_k=\varepsilon_{jki}b_jc_ka_i=\vec b(\vec c\times\vec a)$

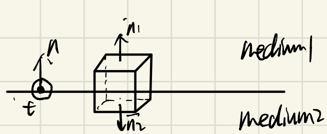
$\Delta l[\vec E_1(\vec t\times\vec n)-\vec E_2(\vec t\times\vec n)]=\Delta l[\vec t(\vec n\times\vec E_1)-\vec t(\vec n\times\vec E_2)]=\Delta l[(\vec n\times\vec E_1)_t-(\vec n\times\vec E_2)_t]$

$\Rightarrow(\vec n\times\vec E_1)_t-(\vec n\times\vec E_2)_t=-\frac{\partial(\vec B\cdot\vec t)}{\partial t}\Delta l \qquad \Delta l\to0\Rightarrow-\frac{\partial(\vec B\cdot\vec t)}{\partial t}\Delta l=0\Rightarrow(\vec n\times\vec E_1)_t-(\vec n\times\vec E_2)_t=0$

$\iint_S\nabla\times\vec H=\oint\vec H\cdot dl=\iint_S\frac{\partial D}{\partial t}ds\Rightarrow\Delta l[\vec H_1(\vec t\times\vec n)-\vec H_2(\vec t\times\vec n)]=\frac{\partial}{\partial t}\vec D\cdot\vec t\,\Delta l^2$

$\Delta l[\vec H_1(\vec t\times\vec n)-\vec H_2(\vec t\times\vec n)]=\Delta l[\vec t\cdot(\vec n\times\vec H_1)-\vec t(\vec n\times\vec H_2)]=\Delta l[(\vec n\times\vec H_1)_t-(\vec n\times\vec H_2)_t]$

$\Delta l\to0\Rightarrow\frac{\partial}{\partial t}\vec D\cdot\vec t\,\Delta l^2=0\Rightarrow\Delta l[(\vec n\times\vec H_1)_t-(\vec n\times\vec H_2)_t]=0\Rightarrow(\vec n\times\vec H_1)_t=(\vec n\times\vec H_2)_t$



b) $\iiint_V\nabla\cdot\vec D\,dV=\iint_S\vec D\,d\vec s$

$=\vec D_1\cdot\vec n_1\cdot\Delta s+\vec D_2\cdot\vec n_2\Delta s \qquad \vec n_1=\vec n\ \ \vec n_2=-\vec n\Rightarrow\iint_S\vec D\,d\vec s=\vec D_1\cdot\vec n\cdot\Delta s-\vec D_2\cdot\vec n\cdot\Delta s=0$

$\Rightarrow\vec D_1\vec n=\vec D_2\vec n_n\Rightarrow\vec D_{1n}=\vec D_{2n}$

$\iiint_V\nabla\cdot\vec H\,dV=\iint_S\vec H\,d\vec s=\frac{1}{\mu_0}\iint_S\vec B\,d\vec s=\frac{1}{\mu_0}(\vec B_1\vec n_1\Delta s+\vec B_2\vec n_2\Delta s)=\frac{1}{\mu_0}(\vec B_1\vec n\Delta s-\vec B_2\vec n\Delta s)=0$

$\Rightarrow\vec B_{1n}-\vec B_{2n}=0\Rightarrow\vec B_{1n}=\vec B_{2n}$

Task 2: Lorentz model (a=1, b=2, c=2, d=2* pts.)

With a good approximation, a dielectric medium can be modeled by an ensemble of damped harmonic oscillators, known as the Lorentz model. In the case of a homogeneous, isotropic medium, the response function reads as

$$\hat{R}_{mn}(\mathbf{r},t) = \delta_{mn} R(t) \qquad R(t) = \begin{cases} 0 & \text{for } t \le 0 \\ \frac{f}{\Omega} e^{-\gamma t} \sin \Omega t & \text{for } t > 0 \end{cases}, \qquad \Omega = \sqrt{\omega_0^2 - \gamma^2}.$$

a) Calculate the susceptibility $\chi(\omega)$ of the medium. Notice how $\chi(\omega)$ is the Fourier transform of $R(t)$, but with a different normalization convention than our usual definition of the Fourier transform.

b) Sketch the real and imaginary part of the dielectric function $\varepsilon(\omega) = 1 + \chi(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$ for this typical insulator and mark the areas of normal ($\mathrm{d}\varepsilon'(\omega)/\mathrm{d}\omega > 0$) and anomalous ($\mathrm{d}\varepsilon'(\omega)/\mathrm{d}\omega < 0$) dispersion. Where does strong absorption occur?

c) Compute the polarization $\mathbf{P}(\mathbf{r},t)$ for the dielectric medium above with an electric field excitation of

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r}) \cos(\omega_{\mathrm{cw}} t).$$

d*) As you may have noticed, finding $\chi(\omega)$ from $R(t)$ is easier than finding $R(t)$ from $\chi(\omega)$. The former is a simple integral, but the latter requires a complex integral. Use the residue theorem to calculate the inverse Fourier transform of $\chi(\omega)$ to obtain the $R(t)$ from part a.

(a) $R(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi_{ij}(\omega) e^{-i\omega t} d\omega \Rightarrow \chi_{ij}(\omega) = \int_{-\infty}^{\infty} R(t) e^{i\omega t} dt$

$\Rightarrow \chi(\omega) = \int_0^\infty \frac{f}{\Omega} e^{-\gamma t} \sin \Omega t \, e^{i\omega t} dt = \frac{f}{\Omega 2i}\int_0^\infty e^{-\gamma t}(e^{i\Omega t} - e^{-i\Omega t}) e^{i\omega t} dt$

$= \frac{-if}{2\Omega}\left(\int_0^\infty \exp[(-\gamma + i\Omega + i\omega)t] dt - \int_0^\infty \exp[(-\gamma - i\Omega + i\omega)t] dt\right)$

$= \frac{-if}{2\Omega}\left(\frac{\exp[i(\Omega+\omega)t] e^{-\gamma t}}{-\gamma + i\Omega + i\omega}\Big|_0^\infty - \frac{\exp[(-i\Omega + i\omega)t] e^{-\gamma t}}{-\gamma - i\Omega + i\omega}\Big|_0^\infty\right)$

$= \frac{-if}{2\Omega}\left(\frac{1}{(i\omega - \gamma) - i\Omega} - \frac{1}{(i\omega-\gamma) + i\Omega}\right) = \frac{2i\Omega}{(i\omega - \gamma)^2 + \Omega^2} \cdot \frac{-if}{2\Omega} = \frac{f}{(i\omega - \gamma)^2 + \Omega^2}$

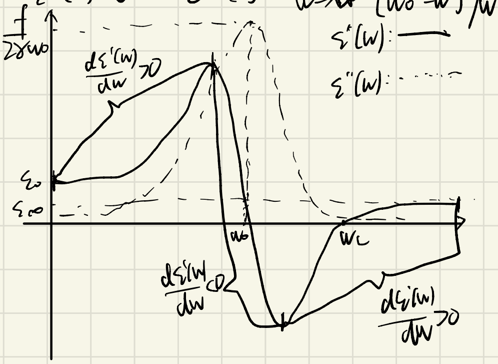
$\Omega = \sqrt{\omega_0^2 - \gamma^2} \Rightarrow \chi(\omega) = \frac{f}{-\omega^2 + \gamma^2 - 2i\gamma\omega + \omega_0^2 - \gamma^2} = \frac{f}{\omega_0^2 - 2i\gamma\omega - \omega^2} = \frac{f}{(\omega_0^2 - \omega^2) - 2i\gamma\omega}$

(b) $\varepsilon(\omega) = 1 + \chi(\omega) = 1 + \frac{f}{(\omega_0^2 - \omega^2) - 2i\gamma\omega} = 1 + \frac{f[(\omega_0^2 - \omega^2) + 2i\gamma\omega]}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2} = \varepsilon'(\omega) + i\varepsilon''(\omega)$

$\varepsilon'(\omega) = 1 + \frac{f(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2} \qquad \varepsilon''(\omega) = \frac{2f\gamma\omega}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}$

$\varepsilon'(0) = \varepsilon_0 = 1 + \frac{f}{\omega_0^2} \qquad \varepsilon'(\infty) = \lim_{\omega\to\infty} \frac{f}{(\omega_0^2 - \omega^2) + \frac{4\gamma^2}{\omega_0^2/\omega^2 - 1}} + 1 = 1 = \varepsilon_\infty \qquad \varepsilon'(\omega_0) = 1 \qquad \varepsilon''(\omega_0) = \frac{f}{2\gamma\omega_0}$

$\varepsilon''(0) = 0 \qquad \varepsilon''(\infty) = \lim_{\omega\to\infty} \frac{2f\gamma}{(\omega_0^2 - \omega^2)^2/\omega + 4\gamma^2\omega} = 0$



$\omega_c = \omega_0 \sqrt{\frac{\varepsilon_0}{\varepsilon_\infty}} = \omega_0 \sqrt{1 + \frac{f}{\omega_0^2}} = \sqrt{\omega_0^2 + f}$

Strong absorption occurs when $\omega = \omega_0$

(c): $P(\vec{r},t)=\varepsilon_0\int_{-\infty}^{\infty} R(\vec{r},t-t')E(\vec{r},t')dt'$

$\vec{P}(\vec{r},\omega)=\varepsilon_0\chi(\omega)\vec{E}(\vec{r},\omega) \qquad \vec{E}(\vec{r},\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\vec{E}(\vec{r})e^{i\omega t}dt=\frac{\vec{E}(\vec{r})}{2\pi}\int_{-\infty}^{\infty}\cos(\omega_{cw}t)e^{i\omega t}dt$

$$\Rightarrow \int_{-\infty}^{\infty}\cos(\omega_{cw}t)e^{i\omega t}dt=\frac{1}{2}\int_{-\infty}^{\infty}(e^{i\omega_{cw}t}+e^{-i\omega_{cw}t})e^{i\omega t}dt=\frac{1}{2}\int_{-\infty}^{\infty}e^{i(\omega_{cw}+\omega)t}+e^{i(\omega-\omega_{cw})t}dt$$

$$FT^{-1}(\delta(\omega))=\int_{-\infty}^{\infty}\delta(\omega)e^{-i\omega t}d\omega=1 \Rightarrow FT(1)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega t}dt=\delta(\omega)$$

$$\Rightarrow \int_{-\infty}^{\infty}e^{i\omega t}dt=2\pi\delta(\omega) \Rightarrow \int_{-\infty}^{\infty}e^{i(\omega+\omega_{cw})t}dt=2\pi\delta(\omega+\omega_{cw}) \Big/ \int_{-\infty}^{\infty}e^{i(\omega-\omega_{cw})t}dt=2\pi\delta(\omega-\omega_{cw})$$

$$\Rightarrow \frac{1}{2}\int_{-\infty}^{\infty}e^{i(\omega+\omega_{cw})t}+e^{i(\omega-\omega_{cw})t}dt=\pi[\delta(\omega+\omega_{cw})+\delta(\omega-\omega_{cw})]$$

$$\Rightarrow \vec{E}(\vec{r},\omega)=\frac{1}{2}\vec{E}(\vec{r})[\delta(\omega+\omega_{cw})+\delta(\omega-\omega_{cw})]$$

$$\Rightarrow \vec{P}(\vec{r},\omega)=\frac{\varepsilon_0\vec{E}(\vec{r})f}{2}\frac{\delta(\omega+\omega_{cw})+\delta(\omega-\omega_{cw})}{(\omega_0^2-\omega^2)-2i\gamma\omega}$$

$$\vec{P}(\vec{r},t)=\int_{-\infty}^{\infty}\vec{P}(\vec{r},\omega)e^{-i\omega t}d\omega=\frac{\varepsilon_0\vec{E}(\vec{r})f}{2}\int_{-\infty}^{\infty}[\delta(\omega+\omega_{cw})+\delta(\omega-\omega_{cw})]\frac{e^{-i\omega t}}{(\omega_0^2-\omega^2)-2i\gamma\omega}d\omega$$

$$=\frac{\varepsilon_0\vec{E}(\vec{r})f}{2}\left(\frac{e^{i\omega_{cw}t}}{(\omega_0^2-\omega_{cw}^2)+2i\gamma\omega_{cw}}+\frac{e^{-i\omega_{cw}t}}{(\omega_0^2-\omega_{cw}^2)-2i\gamma\omega_{cw}}\right)$$

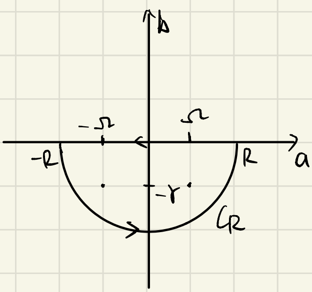
$$=\frac{\varepsilon_0\vec{E}(\vec{r})f}{2}\frac{(\omega_0^2-\omega_{cw}^2)(e^{i\omega_{cw}t}+e^{-i\omega_{cw}t})-2i\gamma\omega_{cw}(e^{i\omega_{cw}t}-e^{-i\omega_{cw}t})}{(\omega_0^2-\omega_{cw}^2)^2+4\gamma^2\omega_{cw}^2}=\frac{2(\omega_0^2-\omega_{cw}^2)\cos(\omega_{cw}t)+4\gamma\omega_{cw}\sin(\omega_{cw}t)}{(\omega_0^2-\omega_{cw}^2)^2+4\gamma^2\omega_{cw}^2}$$

$$\Rightarrow \vec{P}(\vec{r},t)=\varepsilon_0\vec{E}(\vec{r})f\frac{(\omega_0^2-\omega_{cw}^2)\cos(\omega_{cw}t)+2\gamma\omega_{cw}\sin(\omega_{cw}t)}{(\omega_0^2-\omega_{cw}^2)^2+4\gamma^2\omega_{cw}^2}$$

(d*) $R(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\chi(\omega)e^{-i\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{fe^{-i\omega t}d\omega}{(i\omega-\gamma)^2+\Omega^2}=\frac{-f}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega t}d\omega}{\omega^2-\gamma^2+2i\gamma\omega-\Omega^2}$

Residue theorem: $\int_{-\infty}^{\infty}\frac{e^{-i\omega t}d\omega}{\omega^2+2i\gamma\omega-\gamma^2-\Omega^2} \Rightarrow \int_C\frac{e^{-itz}dz}{z^2+2i\gamma z-\gamma^2-\Omega^2}$

$$z^2+2i\gamma z-(\gamma^2+\Omega^2)=0 \Rightarrow z=\frac{-2i\gamma\pm\sqrt{-4\gamma^2+4(\gamma^2+\Omega^2)}}{2}=-i\gamma\pm\Omega \Rightarrow \begin{array}{l} z_1=-i\gamma-\Omega \\ z_2=-i\gamma+\Omega \end{array}$$



$$\int_C\frac{e^{-itz}dz}{z^2+2i\gamma z-\gamma^2-\Omega^2}=\int_C\frac{e^{-itz}dz}{[z-(-i\gamma-\Omega)][z-(-i\gamma+\Omega)]}=\int_C f(z)dz$$

$$\text{Res}[f(z),-i\gamma-\Omega]=\lim_{z\to-i\gamma-\Omega}[z-(-i\gamma-\Omega)]\frac{e^{-itz}}{[z-(-i\gamma-\Omega)][z-(-i\gamma+\Omega)]}$$

$$=\frac{e^{-it(-i\gamma-\Omega)}}{-i\gamma-\Omega+i\gamma-\Omega}=-\frac{e^{it\Omega}}{2\Omega}e^{-\gamma t}$$

$$\text{Res}[f(z),-i\gamma+\Omega]=\lim_{z\to-i\gamma+\Omega}[z-(-i\gamma+\Omega)]\frac{e^{-itz}}{[z-(-i\gamma-\Omega)][z-(-i\gamma+\Omega)]}$$

According to Residue theorem

$$= \frac{e^{-it(-i\gamma+\Omega)}}{-i\gamma+\Omega+i\gamma+\Omega} = \frac{e^{-i\Omega t}}{2\Omega} e^{-\gamma t}$$

$$\int_C \frac{e^{-itz}\,dz}{z^2+2i\gamma z-\gamma^2-\Omega^2} = 2\pi i\left[Res[f(z),-i\gamma-\Omega]+Res[f(z),-i\gamma+\Omega]\right] = 2\pi i\left(\frac{e^{i\Omega t}}{2\Omega}e^{-\gamma t} - \frac{e^{it\Omega}}{2\Omega}e^{-\gamma t}\right)$$

$$= \frac{\pi i}{\Omega}e^{-\gamma t}\left(e^{-i\Omega t}+e^{it\Omega}\right) = \frac{2\pi}{\Omega}e^{-\gamma t}\sin\Omega t$$

$$\int_C \frac{e^{-itz}\,dz}{z^2+2i\gamma z-\gamma^2-\Omega^2} = \int_{arc}\frac{e^{-itz}\,dz}{z^2+2i\gamma z-\gamma^2-\Omega^2} + \int_R^{-R}\frac{e^{-itz}\,dz}{z^2+2i\gamma z-\gamma^2-\Omega^2} = \frac{2\pi}{\Omega}e^{-\gamma t}\sin\Omega t$$

Because $f(z)$ is continuous on the semicircular contour C_R for all large R, then

by Jordan's Lemma $\int_{arc}\frac{e^{-itz}}{z^2+2i\gamma z-\gamma^2-\Omega^2} = 0$

Then $\int_R^{-R}\frac{e^{-itz}}{z^2+2i\gamma z-\gamma^2-\Omega^2} = -\int_{-R}^{R}\frac{e^{-itz}\,dz}{z^2+2i\gamma z-\gamma^2-\Omega^2} = \frac{2\pi}{\Omega}e^{-\gamma t}\sin\Omega t$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{-iwt}\,dw}{w^2+2i\gamma w-\gamma^2-\Omega^2} = \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{-itz}}{z^2+2i\gamma z-\gamma^2-\Omega^2} = -\frac{2\pi}{\Omega}e^{-\gamma t}\sin\Omega t$$

Thus $R(t) = \frac{-f}{2\pi}\cdot -\frac{2\pi}{\Omega}e^{-\gamma t}\sin\Omega t = \frac{f}{\Omega}e^{-\gamma t}\sin\Omega t$

$Res(f, c) = \frac{1}{(n-1)!}\lim_{z\to c}\frac{d^{n-1}}{dz^{n-1}}\left[(z-c)^n f(z)\right]$

Task 3: Debye relaxation (a=2, b=3, c=2 pts.)

Dielectric permeability of a sample is determined by the molecular properties of its material. The applied electric field induces molecular dipole moments which align with the field. However, this process is not instantaneous as the alignment of the molecular dipoles is rather slow, which results in a phase delay between the electric field and polarization. In the linear case, the polarization can be represented as the integral of the responses to the electric field over time from which the following expression for the electric displacement field can be defined:

$$\mathbf{D}(t) = \varepsilon_\infty \mathbf{E}(t) + (\varepsilon_s - \varepsilon_\infty) \int_{-\infty}^{t} \left(1 - e^{-(t-t')/\tau}\right) \frac{\mathrm{d}\mathbf{E}(t')}{\mathrm{d}t'} \mathrm{d}t',$$

where t' is the time variable. The first term of this equation ($\varepsilon_\infty \mathbf{E}(t)$) represents the instantaneous response of the media, associated with the response of the electrons and nuclei. The corresponding value ε_∞ is called the instantaneous dielectric permittivity. The value of ε_s is called the static permittivity, which corresponds to the maximal polarization and established equilibrium configuration of the molecular dipoles. The function $1 - e^{-(t-t')/\tau}$ describes the time dependence of the polarization process which is characterized by the relaxation rate τ. In this task, you will derive the dielectric permittivity for which the finite-time polarization is taken into account.

a) Transform the integral on the right-hand side to get rid of the derivative. You might use the expression $\mathbf{E}(-\infty) \to 0$, as it is considered that the field is applied at $t = 0$.

b) Differentiate both sides of the obtained equation over time t and transform the equation to the differential one.
Hint: After the differentiation, the integral on the right-hand side will be transformed again. You actually do not need to calculate this integral, just think about recursions...

c) Using the time dependencies $\mathbf{E}(t) = \mathbf{E}_0 e^{i\omega t}$ and $\mathbf{D}(t) = \mathbf{D}_0 e^{i\omega t}$, derive the dielectric permittivity, defined as $\varepsilon = \mathbf{D}_0/\mathbf{E}_0$. Write down explicitly the real and imaginary parts of ε.

$\vec{D} = \vec{D}_0 \cos(\omega t - \delta)$

$\vec{D} = \vec{D}_1 \cos\omega t + \vec{D}_2 \sin\omega t$

$\vec{D}_1 = \vec{D}_0 \cos\delta \quad \vec{D}_2 = \vec{D}_0 \sin\delta$

$\varepsilon' = \dfrac{\vec{D}_1}{\varepsilon_0 \vec{E}_0} \quad \varepsilon'' = \dfrac{\vec{D}_2}{\varepsilon_0 \vec{E}_0}$

$\vec{D} = \vec{D}_0 e^{i\omega t} = D_0 \cos\omega t + i D_0 \sin(\omega t)$

$\varepsilon' \quad \varepsilon''$

(a) $\displaystyle\int_{-\infty}^{t} (1 - e^{-\frac{t-t'}{\tau}}) \frac{d\vec{E}(t')}{dt'} dt' = \int_{-\infty}^{t} d\vec{E}(t') - \int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} \frac{d\vec{E}(t')}{dt'} dt' = \vec{E}(t) - \int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} d\vec{E}(t')$

$\displaystyle\Rightarrow \int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} d\vec{E}(t') = \vec{E}(t') e^{-\frac{t-t'}{\tau}} \Big|_{-\infty}^{t} - \int_{-\infty}^{t} \vec{E}(t') \, d e^{-\frac{t-t'}{\tau}} = \vec{E}(t) - \int_{-\infty}^{t} \frac{1}{\tau} e^{-\frac{t-t'}{\tau}} \vec{E}(t') dt'$

$\displaystyle\Rightarrow \int_{-\infty}^{t} (1 - e^{-\frac{t-t'}{\tau}}) \frac{d\vec{E}(t')}{dt'} dt' = \frac{1}{\tau}\int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} \vec{E}(t') dt' = \frac{e^{-\frac{t}{\tau}}}{\tau} \int_{-\infty}^{t} e^{\frac{t'}{\tau}} \vec{E}(t') dt'$

Thus $\displaystyle\vec{D}(t) = \varepsilon_\infty \vec{E}(t) + (\varepsilon_s - \varepsilon_\infty) \frac{e^{-\frac{t}{\tau}}}{\tau} \int_{-\infty}^{t} e^{\frac{t'}{\tau}} \vec{E}(t') dt'$

(b) $\displaystyle\frac{d\vec{D}(t)}{dt} = \varepsilon_\infty \frac{d\vec{E}(t)}{dt} + (\varepsilon_s - \varepsilon_\infty) \frac{d}{dt}\left[\frac{e^{-\frac{t}{\tau}}}{\tau} \int_{-\infty}^{t} e^{\frac{t'}{\tau}} \vec{E}(t') dt'\right]$

$\displaystyle\Rightarrow \frac{d\vec{D}(t)}{dt} = \varepsilon_\infty \frac{d\vec{E}(t)}{dt} + (\varepsilon_s - \varepsilon_\infty)\left[-\frac{e^{-\frac{t}{\tau}}}{\tau^2} \int_{-\infty}^{t} e^{\frac{t'}{\tau}} \vec{E}(t') dt' + \frac{e^{-\frac{t}{\tau}}}{\tau} \frac{d}{dt}\left[\int_{-\infty}^{t} e^{\frac{t'}{\tau}} \vec{E}(t') dt'\right]\right]$

$\displaystyle\Rightarrow \frac{d\vec{D}(t)}{dt} = \varepsilon_\infty \frac{d\vec{E}(t)}{dt} - \frac{e^{-\frac{t}{\tau}}}{\tau} (\varepsilon_s + \varepsilon_\infty)\left[\frac{1}{\tau}\int_{-\infty}^{t} e^{\frac{t'}{\tau}} \vec{E}(t') dt' - \frac{d}{dt}\int_{-\infty}^{t} e^{\frac{t'}{\tau}} \vec{E}(t') dt'\right]$

$e^{\frac{t}{\tau}} \vec{E}(t)$

(c) $\vec{E}(t) = \vec{E}_0 e^{i\omega t}$, $\vec{D}(t) = \vec{D}_0 e^{i\omega t}$ $\displaystyle\Rightarrow \vec{D}_0 e^{i\omega t} = \varepsilon_\infty \vec{E}_0 e^{i\omega t} + (\varepsilon_s - \varepsilon_\infty) \frac{e^{-\frac{t}{\tau}}}{\tau} \vec{E}_0 \int_{-\infty}^{t} e^{\frac{t'}{\tau}} e^{i\omega t'} dt'$

$\displaystyle\int_{-\infty}^{t} e^{\frac{t'}{\tau}} e^{i\omega t'} dt' = \int_{-\infty}^{t} e^{(\frac{1}{\tau} + i\omega)t'} dt' = \frac{e^{\frac{t}{\tau}} e^{i\omega t}}{\frac{1}{\tau} + i\omega} \Rightarrow \vec{D}_0 e^{i\omega t} = \varepsilon_\infty \vec{E}_0 e^{i\omega t} + (\varepsilon_s - \varepsilon_\infty) \frac{\vec{E}_0 e^{i\omega t}}{1 + i\omega\tau}$

$\displaystyle\Rightarrow \varepsilon = \frac{\vec{D}_0}{\vec{E}_0} = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + i\omega\tau} = \varepsilon_\infty + \frac{(\varepsilon_s - \varepsilon_\infty)(1 - i\omega\tau)}{1 + \omega^2\tau^2} = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + \omega^2\tau^2} - i\frac{\omega\tau(\varepsilon_s - \varepsilon_\infty)}{1 + \omega^2\tau^2}$

$\displaystyle\Rightarrow \varepsilon = \varepsilon' + i\varepsilon'' \quad \varepsilon' = \varepsilon_\infty + \frac{\varepsilon_s \ \varepsilon_\infty}{1 + \omega^2\tau^2} \quad \varepsilon'' = \frac{-\omega\tau(\varepsilon_s + \varepsilon_\infty)}{1 + \omega^2\tau^2}$

$$\int_{-\infty}^{t} e^{\frac{t'}{\tau}} E_0 e^{i\omega t'}\, dt'$$

$$E_0 \int_{-\infty}^{\infty} e^{(\frac{1}{\tau}+i\omega)t'}\, dt'$$

$$\frac{E_0 e^{(\frac{1}{\tau}+i\omega)t'}}{\frac{1}{\tau}+i\omega}\Big|_{-\infty}^{t}$$

$$\frac{E_0}{\frac{1}{\tau}+i\omega} e^{(\frac{1}{\tau}+i\omega)t}$$

$$\tau$$

$$E_0 e^{\frac{t}{\tau}} e^{i\omega t}$$

$$\Rightarrow \frac{d\vec{D}(t)}{dt} = \varepsilon_\infty \frac{d\vec{E}(t)}{dt} - \frac{e^{-\frac{t}{\tau}}}{\tau}(\varepsilon_s + \varepsilon_\infty)\Big[\underbrace{\frac{1}{\tau}\int_{-\infty}^{t} e^{\frac{t'}{\tau}} \vec{E}(t')\,dt'}_{t} - \frac{d}{dt}\int_{-\infty}^{t} e^{\frac{t'}{\tau}} \vec{E}(t')\,dt'\Big]$$

$$\frac{1}{\tau}\int_{-\infty}^{t} e^{\frac{t'}{\tau}} E(t')\,dt'$$

$$= \int_{-\infty}^{t} E(t')\, d(e^{\frac{t'}{\tau}}) = E(t')e^{\frac{t'}{\tau}}\Big|_{-\infty}^{t} - \int_{-\infty}^{t} e^{\frac{t'}{\tau}} \frac{dE(t')}{dt'}\,dt'$$

$$= \cancel{E(t)e^{\frac{t}{\tau}}}$$

$$\frac{dD(t)}{dt} = \varepsilon_\infty \frac{dE(t)}{dt} + (\varepsilon_s + \varepsilon_\infty)\cdot \frac{1}{\tau}\int e^{-\frac{t-t'}{\tau}} \frac{dE(t')}{dt'}\,dt'$$

$$D(t) = \varepsilon_s E(t) - (\varepsilon_s - \varepsilon_\infty)\cdot \int e^{-\frac{t-t'}{\tau}} \frac{dE(t')}{dt'}\,dt'$$

$$\tau \frac{dD(t)}{dt} + D(t) = \tau \cdot \varepsilon_\infty \cdot \frac{dE(t)}{dt} + \varepsilon_s E(t)$$

$$\tau \cdot i\omega D_0 + D_0 = \tau \cdot \varepsilon_\infty \cdot i\omega \cdot E_0 + \varepsilon_s \cdot E_0$$

$$(1+\tau i\omega)D_0 = (\varepsilon_s + \tau \cdot \varepsilon_\infty \cdot i\omega)\cdot E_0$$

$$\varepsilon_0 = \frac{\varepsilon_s + \varepsilon_\infty \cdot \tau \cdot i\omega}{1+\tau i\omega} = \frac{\varepsilon_s + \varepsilon_\infty(1+\tau i\omega) - \varepsilon_\infty}{1+\tau i\omega} = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1+\tau i\omega}$$

$$e^{-i(w'+iw'')t}$$

$$= e^{w''t} e^{-iw't}$$

$$\frac{d\vec{D}(t)}{dt} = \varepsilon_\infty \frac{d\vec{E}}{dt} + \frac{\varepsilon_s - \varepsilon_\infty}{\tau} \int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} \frac{d\vec{E}(t')}{dt'} dt'$$

$$\vec{D}(t) = \varepsilon_\infty \vec{E} + \frac{\varepsilon_s - \varepsilon_\infty}{\tau} \int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} \vec{E}(t') dt'$$

$$e^{i(w'+iw'')t}$$

$$\vec{D}_t = \varepsilon_\infty \vec{E} + (\varepsilon_s - \varepsilon_\infty) \int_{-\infty}^{t} (1 - e^{-\frac{t-t'}{\tau}}) \frac{d\vec{E}(t')}{dt'} dt'$$

$$e^{-w''t} e^{iw't}$$

$$t > 0$$

$$= \varepsilon_\infty \vec{E} + (\varepsilon_s - \varepsilon_\infty) \int_{-\infty}^{t} d\vec{E}(t') - (\varepsilon_s - \varepsilon_\infty) \int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} \frac{d\vec{E}}{dt'} dt'$$

$$\mathrm{I}\, w'' > 0$$

$$= \varepsilon_s \vec{E} - ($$

$$\frac{d}{dy} \int_{\varphi(y)}^{\psi(y)} f(x,y) dx = f(\psi(y), y) \frac{d\psi(y)}{dy} - f(\varphi(y), y) \frac{d\varphi(y)}{dy} + \int_{\varphi(y)}^{\psi(y)} \frac{\partial}{\partial y} f(x,y) dx$$

$\psi(y)$ $\varphi(y)$