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Problem 1

(a)

$$F_L(\nu) = \frac{C_1}{(\nu-\nu_{21})^2 + (B_1 \Delta\nu)^2}$$

when $\nu = \nu_{21}$, $F_L(\nu)$ equals to the maximum value.

suppose when $\nu = \nu'$, $F_L(\nu') = \frac{1}{2} F_L(\nu_{21})$

$$\nu' - \nu_{21} = \pm \frac{1}{2}\Delta\nu$$

$$\therefore \frac{\dfrac{C_1}{\frac{1}{4}\Delta\nu^2 + (B_1\Delta\nu)^2}}{\dfrac{C_1}{(B_1\Delta\nu)^2}} = \frac{1}{2}$$

$$2B_1^2 = \frac{1}{4} + B_1^2$$

$$B_1 = \frac{1}{2}$$

$$F_G(\nu) = C_2 \exp\left[-B_2 \frac{(\nu-\nu_{21})^2}{\Delta\nu^2}\right]$$

when $\nu = \nu_{21}$, $F_G(\nu)$ equals to the maximum value

suppose when $\nu = \nu'$, $F_G(\nu') = \frac{1}{2} F_G(\nu_{21})$

$$\nu' - \nu_{21} = \pm \frac{1}{2}\Delta\nu$$

$$\therefore \frac{C_2 \exp\left[-B_2 \frac{\frac{1}{4}\Delta\nu^2}{\Delta\nu^2}\right]}{C_2} = \frac{1}{2}$$

$$\therefore \exp\left[-\frac{B_2}{4}\right] = \frac{1}{2}$$

$$-\frac{B_2}{4} = -\ln 2$$

$$B_2 = 4\ln 2$$

(b)

$$\because F_L(\nu_{21}) = \frac{C_1}{(B_1\Delta\nu)^2} = 1$$

$$\because B_1 = \frac{1}{2} \quad \therefore C_1 = \frac{\Delta\nu^2}{4}$$

$$\because F_G(\nu_{21}) = C_2 = 1 \quad \therefore C_2 = 1$$

(c)

$$\int_{-\infty}^{\infty} F_L(\nu)\, d\nu = 1$$

$$\int_{-\infty}^{\infty} \frac{C_1}{(\nu-\nu_{21})^2 + (B_1\Delta\nu)^2}\, d\nu = 1$$

$$\int_{-\infty}^{\infty} \frac{C_1}{(\nu-\nu_{21})^2 + (\frac{\Delta\nu}{2})^2}\, d\nu \qquad (\nu-\nu_{21})\cdot\frac{2}{\Delta\nu} = t$$

$$= C_1 \int_{-\infty}^{\infty} \left(\frac{2}{\Delta\nu}\right)^2 \frac{1}{t^2+1}\, dt\ \frac{\Delta\nu}{2}$$

$$= C_1 \cdot \frac{2}{\Delta\nu} \arctan\left[(\nu-\nu_{21})\cdot\frac{2}{\Delta\nu}\right]\Big|_{-\infty}^{\infty}$$

$$= C_1 \cdot \frac{2}{\Delta\nu}\cdot\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 1$$

$$\therefore C_1 = \frac{\Delta\nu}{2\pi}$$

$$\int_{-\infty}^{\infty} F_G(\nu)\, d\nu = 1$$

$$\int_{-\infty}^{\infty} C_2 \exp\left[-B_2 \frac{(\nu-\nu_{21})^2}{\Delta\nu}\right] d\nu = 1$$

$$\int_{-\infty}^{\infty} C_2 \exp\left[-4\ln 2 \cdot \frac{(\nu-\nu_{21})^2}{\Delta\nu^2}\right] d\nu$$

$$= C_2 \sqrt{2\pi}\frac{\Delta\nu}{\sqrt{8\ln 2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\cdot\frac{\Delta\nu}{\sqrt{8\ln 2}}} \exp\left(-\frac{(\nu-\nu_{21})^2}{2(\frac{\Delta\nu}{\sqrt{8\ln 2}})^2}\right) d\nu$$

$$\therefore C_2 = \frac{2}{\Delta\nu}\sqrt{\frac{\ln 2}{\pi}}$$

Problem 2 Jinsong Liu 206216

(a)

suppose the radius of the circular aperture is a

$x^2+y^2=r^2$

$$T=\frac{P_a}{P_\infty}=\frac{\int_0^a\int_0^{2\pi}\frac{2}{\pi W^2}\exp(-2\frac{r^2}{W^2})\,2\pi r\,dr\,d\theta}{\int_0^\infty\int_0^{2\pi}\frac{2}{\pi W^2}\exp(-2\frac{r^2}{W^2})\,2\pi r\,dr\,d\theta}=1-\exp(-\frac{2a^2}{W^2})=0.99$$

$\therefore\ a\approx 1.52W$

(b)

$$I_P=P_P\cdot\frac{2}{\pi W^2}$$

$$=\frac{2}{\pi\times(5\times10^{-6})^2}\times10^{15}\ W/m^2$$

$$=2.55\times10^{25}\ W/m^2$$

$$\because\ I=\frac{1}{2}c\varepsilon_0E^2$$

$$E=\sqrt{\frac{2I}{c\varepsilon_0}}=\sqrt{\frac{2\times2.55\times10^{25}}{3\times10^8\times8.854\times10^{-12}}}\ V/m=1.39\times10^{14}\ V/m$$

(c)

$$E_{pulse} = \int_{-\infty}^{\infty} P(t)\, dt = \int_{-\infty}^{\infty} P_p \exp\left[-4\ln 2 \left(\frac{t}{\tau}\right)^2\right] dt$$

$$= 25 \times \sqrt{\frac{\pi}{4\ln 2}}\ J$$

$$\approx 25 \times 1.064\ J = 26.6\ J$$

$$P_{avg} = E_{pulse} \cdot f = 26.6\ W$$

(d)

We can use a photodiode to measure the average power, then we can calculate the peak power

Problem 3

(a)

$$\frac{dN_2}{dt} = -A_{21} \cdot N_2 - B_{21}\,\rho(\nu)\,N_2 + B_{12}\,\rho(\nu)\,N_1$$

(b)

$\because$ thermal equilibrium and that the atoms are enclosed in a conducting cavity

$$\therefore \quad \frac{dN_2}{dt} = 0 \qquad \frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-\frac{h\nu}{k_B T}} \quad ①$$

From Planck's formula, we can get

$$\rho(\nu) = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1} \quad ②$$

$$\therefore \quad A_{21} \cdot N_2 = -B_{21}\,\rho(\nu) N_2 + B_{12}\,\rho(\nu)\,N_1 \quad ③$$

$$\therefore \quad A_{21} \cdot N_2 = N_2 \frac{8\pi h\nu^3}{c^3} \left[e^{\frac{h\nu}{k_B T}} - 1 \right]^{-1} \cdot \left[B_{12} \frac{g_1}{g_2} e^{\frac{h\nu}{k_B T}} - B_{21} \right]$$

(c)

From the equation ③ $A_{21} \cdot N_2 = -B_{21}\,\rho(\nu) N_2 + B_{12}\,\rho(\nu)\,N_1$

and the equation ① $\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-\frac{h\nu}{k_B T}}$

we can get

$$\rho(\nu) = \frac{A_{21} / B_{21}}{\frac{g_1}{g_2} \frac{B_{12}}{B_{21}} e^{h\nu/k_B T} - 1} \quad ④$$

compare to the equation ②

we can get

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3}{c^3} \qquad A_{21} = \frac{8\pi h\nu^3}{c^3} B_{21}$$

$$\frac{g_1}{g_2} \frac{B_{12}}{B_{21}} = 1 \qquad B_{21} = \frac{g_1}{g_2} B_{12}$$

Problem 3

(d)

$$B_{21} = \frac{g_1}{g_2} B_{12}$$

physical meaning: For the same degeneracy factors ($g_1 = g_2$), the probability of stimulated emission and absorption are equal. The ratio of B_{21} and B_{12} depends on the degeneration of two levels.

$$A_{21} = \frac{8\pi h\nu^3}{c^3} B_{21}$$

physical meaning: the number of modes per unit volume at frequency ν is $n(\nu) = \frac{8\pi\nu^2}{c^3}$

$$\therefore \frac{A_{21}}{n(\nu)} = B_{21} h\nu$$

which means the probability of spontaneous emission in each mode is equal to the proability of stimulated emission induced by a photon.

For a 2-level system with $g_2 = 2$ and $g_1 = 1$ when $T \to \infty$ $\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{\frac{-h\nu}{k_B T}} = \frac{g_2}{g_1} = 2$

$\therefore$ inversion is possible

$$\because \frac{\left.\frac{dN_2}{dt}\right|_{induced}}{\left.\frac{dN_1}{dt}\right|_{absorption}} = \frac{B_{21}}{B_{12}} \cdot \frac{N_2}{N_1} = \frac{g_1}{g_2} \frac{g_2}{g_1} e^{-\frac{h\nu}{k_B T}} = e^{-\frac{h\nu}{k_B T}} < 1$$

$$\therefore \left.\frac{dN_2}{dt}\right|_{induced} < \left.\frac{dN_1}{dt}\right|_{absorption}$$

$\therefore$ gain is impossible.