2019 / 2020

Problem 1

a) in time domain

$$\nabla \cdot H = 0 \qquad \nabla \times H = j + \frac{\partial D}{\partial t}$$

$$\nabla \cdot D = 0 \qquad \nabla \times \Sigma = -\mu_0 \frac{\partial H}{\partial t}$$

in Freq domain

$$\nabla \cdot \bar{H} = 0 \qquad \nabla \times \bar{H} = \bar{J} - i w \bar{D}$$

$$\nabla \cdot \bar{D} = 0 \qquad \nabla \times \bar{\Sigma} = \mu_0 i w \bar{H}$$

ⓑ

in time domain

$$D = \epsilon_0 \Sigma + \epsilon_0 \int_{-\infty}^{t} R(r, t-t') \Sigma(r, t') \, dt'$$

in Freq domain

$$\bar{D} = \epsilon_0 \bar{\Sigma} + \epsilon_0 \chi(w) \bar{\Sigma} = \epsilon_0 \epsilon(w) \bar{\Sigma}$$

ⓒ

$$\nabla \times \nabla \times \bar{H} \doteq \nabla \times \bar{J} - i w \nabla \times \bar{D}$$

$$\nabla(\nabla \cdot H) - \Delta H = \sigma(w) \nabla \times \Sigma - i w \varepsilon_0 \epsilon(w) \nabla \times \Sigma$$

$$-\Delta H = \sigma(w) i w \mu_0 H + \frac{w^2}{c^2} \epsilon_0 \epsilon(w) \bar{H}$$

$$\Delta H + \frac{w^2}{c^2} \epsilon(w) = -\sigma i w \mu_0 H$$

Problem 2

ⓐ since $K^2 = \frac{\omega}{c}\,\varepsilon(\omega) \Rightarrow K = \frac{\omega}{c}\sqrt{\varepsilon(\omega)}$

$K = \frac{\omega}{c}\sqrt{\epsilon' + i\epsilon''}$ since $\varepsilon' >> \varepsilon''$

$K = \frac{\omega}{c}\sqrt{\epsilon'} + \frac{i\epsilon''}{2\sqrt{\epsilon'}} \Rightarrow K' = \frac{\omega}{c}\sqrt{\epsilon'}\ ,\ K'' = \frac{\epsilon''}{2\sqrt{\epsilon'}}$

ⓑ

$$\Sigma_r(r,t) = \Sigma_0 e^{-K''z}\cos(\underbrace{K'z - \omega t + \phi}_{A})\,\hat{x}$$

$$\nabla \times \Sigma_r = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \Sigma_x & 0 & 0 \end{vmatrix} = (\Sigma_0 K'' e^{-K''z}\cos A + \Sigma_0 e^{-K''z} K' \sin A)\,\hat{y}$$

$$H_y = \frac{1}{\mu_0 \omega}\left[\Sigma_0 K'' e^{-K''z}\sin A - \Sigma_0 e^{-K''} K' \cos A\right]\hat{y}$$

ⓒ $S_r(r,t) = \Sigma_r \times H_r$

ⓓ

$$\langle S_r(r,t)\rangle \Rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \Sigma_x & 0 & 0 \\ 0 & H_y & 0 \end{vmatrix} = \Sigma_x H_y\,\hat{z}$$

$$\Rightarrow \left(\frac{\Sigma_0^2 K''}{\mu_0 \omega} e^{-K''z}\cos A \sin A - \frac{\Sigma_0^2 K'}{\mu_0 \omega} e^{-2K''z}(\cos^2 A)\right)\hat{z}$$

by taking the average of both

$$\langle S_r(r,t)\rangle = \frac{-\Sigma_0^2 K'}{2\mu_0 \omega} e^{-2K''z}$$

Problem 3

(a) $u_0(x) = A\left[1+\cos\frac{2\pi x}{G}\right]$ call $\frac{2\pi}{G} = K$

$$U_0(\alpha) = \frac{A}{2\pi}\int_{-\infty}^{\infty}(1+\cos Kx)\,e^{i\alpha x}\,dx$$

$$= A\left[\delta(\alpha) + \frac{1}{2}\delta(\alpha-K) + \delta(\alpha+K)\right]$$

(b)

$$U(\alpha;z) = A\left[\delta(\alpha) + \frac{1}{2}[\delta(\alpha-K)+\delta(\alpha+K)]\right]e^{i\sqrt{K_0^2-\alpha^2}\,z}$$

$$u(x;z) = A\int_{-\infty}^{\infty} U(\alpha;z)\,e^{-i\alpha x}\,d\alpha$$

$$= A\left[e^{iK_0 z} + \frac{1}{2}\left[e^{i\sqrt{K_0^2-K^2}\,z+iKx} + e^{i\sqrt{K_0^2-K^2}\,z-iKx}\right]\right]$$

$$\Rightarrow A\left[e^{iK_0 z} + e^{i\sqrt{K_0^2-K^2}\,z}\cos Kx\right]$$

(c)
since in talbot effect $u(x, z=L_T) = u(x, z=0)\,e^{iK_0 z + 2\pi i n}$

=> if we ignored the phase term that destroy the perodicity

$$e^{i\sqrt{K_0^2-K^2}\,L_T - iK_0L_T - 2\pi n i} = 1$$

$$\Rightarrow \sqrt{K_0^2-K^2}\,L_T - K_0L_T - 2\pi n = 0$$

$$L_T = \frac{2\pi n}{\sqrt{K_0^2-K^2} - K_0}$$, The shortest repetition $n=1$

$$L_T = \frac{2\pi}{\sqrt{\frac{(2\pi)^2}{\lambda^2}-\frac{(2\pi)^2}{G^2}} - \frac{2\pi}{\lambda}} = \frac{1}{\sqrt{\frac{1}{\lambda^2}-\frac{1}{G^2}} - \frac{1}{\lambda}}$$

(d) under fersnel condition that $k_0^2 >> \alpha^2$

which cause $H_f = \underbrace{e^{ik_0 z} e^{\frac{-i\alpha^3}{2k_0}}}_{k_0\left(1 - \frac{\alpha^2}{2k_0^2}\right)}$ where $\alpha^2 = \left(\frac{2\pi}{\sigma}\right)^2$

Then

$$L_T = \frac{2\pi n}{\cancel{k_0} - \frac{k^2}{2k_0} \cancel{-k_0}} = \frac{2\pi n (2k_0)}{k^2}$$

$$= \frac{\cancel{4\pi} n \, 2\cancel{\pi}}{\lambda \frac{\cancel{4\pi^2}}{G^2}} = \frac{2nG^2}{\lambda}$$

Scanned with CamScanner

Problem 4

(a) * $\frac{1}{V_{ph}} = \frac{K_o}{w_o} = \frac{n(w_o)}{C_o} \Rightarrow V_{ph} = \frac{C_o}{n(w_o)}$

$$V_{ph} = \frac{C_o}{2+(10^{-32}\times 4\times 10^{30})} = \frac{C_o}{2+(0.04)} = \frac{C_o}{2.04}$$

* $\frac{1}{V_g} = \frac{\partial K}{\partial w} = \frac{1}{C_o}\left[n(w_o) + w\frac{\partial n}{\partial w}\Big|_{w_o}\right]$

$$\Rightarrow V_g = \frac{C_o}{2+3(10^{-32}\times 4\times 10^{30})} = \frac{C_o}{2.12}$$

where $\frac{\partial n}{\partial w} = 2wC$

(b)
* $q(0) = \frac{T_o^2 (C_o + i)}{2D(1+C_o^2)}$ since $C_o = 0$ (Flat phase)

$$q(0) = \frac{T_o^2 i}{2D}$$

* propagate distance L

$$q(L) = \frac{T_o^2 i}{2D} + L = \frac{T_o^2 i + 2LD}{2D}$$

$$\frac{1}{q(L)} = \frac{2D}{T_o^2 i + 2LD} \Rightarrow \frac{-2DT_o^2 i + 4D^2 L}{4L^2D^2 + T_o^4}$$

$$\Rightarrow \frac{4D^2 L}{4L^2D^2 + T_o^4} - \frac{i\,2DT_o^2}{4L^2D^2+T_o^4}$$

$$\frac{1}{q(\lambda)} = \frac{2\,4D^2\lambda\,(T_0^2)}{2[4\lambda^2D^2+T_0^4](T_0^2)} - \frac{i\,2D\,T_0^2\,(T_0^2)}{[4\lambda^2D^2+T_0^4](T_0^2)}$$

$$= \frac{2D}{T_0^2}\left[\underbrace{\frac{2D\lambda T_0^2}{4\lambda^2D^2+T_0^4}}_{C(z)} - i\underbrace{\frac{T_0^4}{4\lambda^2D^2+T_0^4}}_{\frac{T_0^2}{T^2(z)}}\right]$$

~~For the evolved chirp C(z)=0~~

$$\Rightarrow \frac{T_0^4}{4\lambda^2D^2+T_0^4} = \frac{T_0^2}{T^2(z)} \Rightarrow T^2(\lambda) = \frac{4\lambda^2D^2+T_0^4}{T_0^2}$$

$$T(\lambda) = \sqrt{\frac{4\lambda^2D^2+T_0^4}{T_0^2}}$$

sinc $D = \frac{\partial^2 k}{\partial\omega^2} = \frac{1}{-v_g^2}\frac{\partial v_g}{\partial\omega} = \frac{\partial}{\partial\omega}\left(\frac{1}{v_g}\right)$

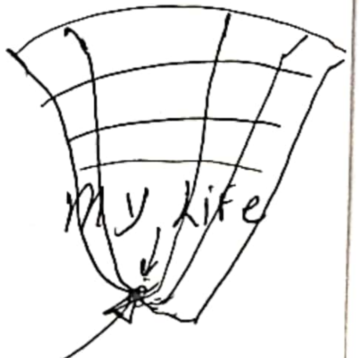
and $\frac{1}{v_g} = \frac{1}{c_0}\left[n + 2\omega^2 C\right] = \frac{1}{c_0}\left[B + 3\omega^2 C\right]\Big|_{\omega_0}$

$$\frac{\partial}{\partial\omega}\left(\frac{1}{v_g}\right) = \frac{1}{c_0}\,6\omega_0 C = D.$$

Then $D = 6 \times 2\times10^{15}\times10^{-32} = \frac{1.2\times10^{-16}}{c_0}$

Then

$$T_1(\lambda) = \sqrt{\frac{4\times400\times\left(\frac{1.2\times10^{-16}}{c_0}\right)^2+(4\times10^{-12})^4}{(4\times10^{-12})^2}} =$$



Singularity of a black hole

(c)

$C=0 \Rightarrow D=0$

$$T_2(\lambda) = \sqrt{\frac{(4\times10^{-12})^4}{(4\times10^{-12})^2}} = \cancel{\sqrt{4\times10^{-12}}} = 4\times10^{-12}\,s$$

The differenc $= |T_1 - T_2| = ?$

Problem 5

ⓐ $z_0 = \pi w_0^2/\lambda$

$$\frac{1}{q(z)} = \frac{1}{R(z)} + \frac{i}{z_0}$$ since we stat at the waist, $R=\infty$

$$\frac{1}{q(0)} = +\frac{i}{z_0} \Rightarrow q(0) = -iz_0$$

after distance z

$$q(z) = -iz_0 + z$$

$$\Rightarrow \frac{1}{q(z)} = \frac{1}{z - iz_0} \Rightarrow \frac{z}{z^2 + z_0^2} + \frac{iz_0}{z^2 + z_0^2}$$

$$\Rightarrow \underbrace{\frac{1}{z\left(1 + \frac{z_0^2}{z^2}\right)}}_{R(z)} + i\frac{1}{z_0\left(1 + \frac{z^2}{z_0^2}\right)}$$

$$\frac{\lambda}{\pi w^2(z)} = \frac{\cancel{\lambda}}{\cancel{\pi} w_0^2\left(1 + \frac{z^2}{z_0^2}\right)} \Rightarrow w^2(z) = w_0^2\left(1 + \frac{z^2}{z_0^2}\right)$$

ⓑ

$$M = \begin{pmatrix} 1 & z' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} = \begin{pmatrix} 1 - z'/f & z' \\ 1/f & 1 \end{pmatrix}$$

$$q = \frac{-iz_0(1 - z'/f) + z'}{-iz_0/f + 1} \Rightarrow \frac{1}{q} = \frac{1 - iz_0/f}{z' - iz_0(1 - z'/f)}$$

$$\frac{1}{q} = \frac{z' - iz_0(1 - z'/f) - \frac{iz_0 z'}{f} - z_0^2/f\,(1 - z'/f)}{z'^2 + z_0^2(1 - z'/f)^2}$$

$$\frac{1}{q} = \frac{z' - iz_0 + \frac{iz_0 z'}{f} - \frac{iz_0 z'}{f} - \frac{z_0^2}{f} + \frac{z_0^2 z'}{f}}{z'^2 + z_0^2 - \frac{2z_0^2 z'}{f} + \frac{z_0^2 z'^2}{f^2}}$$

$$= \frac{z' - \frac{z_0^2}{f} + \frac{z_0^2 z'}{f}}{z'^2 + z_0^2 + \frac{z_0^2 z'^2}{f^2} - \frac{2z_0^2 z'}{f}} - \frac{iz_0}{z'^2 + z_0^2 + \frac{z_0^2 z'^2}{f^2} - \frac{2z_0^2 z'}{f}}$$

at w_0'

The real part $\Rightarrow z' - \frac{z_0^2}{f} + \frac{z_0^2 z'}{f} = 0$

$$z'\left(1 + \frac{z_0^2}{f}\right) = + \frac{z_0^2}{f}$$

$$z' = \frac{\frac{z_0^2}{f}}{1 + z_0^2/f}$$

Then the Imaginary part

$$\frac{z_0}{\left(\frac{z_0^2/f}{1+z_0^2/f}\right)^2 + z_0^2 + \frac{z_0^2\left(\frac{z_0^2/f}{1+z_0^2/f}\right)^2}{f^2} - 2z_0^2\left(\frac{z_0^2/f}{1+z_0^2/f}\right)} = \frac{\lambda}{\pi w_0'^2}$$

when $z_0 \gg f$

$$\Rightarrow z' = \frac{1}{\frac{f}{z_0^2}\left(1 + z_0^2/f\right)} = \frac{1}{\frac{f}{z_0^2} + \frac{1}{f}} = f$$

Then $w_0'^2 = \frac{\lambda \quad f^2 + \cancel{z_0^2} + \cancel{z_0^2} - \cancel{2z_0^2}}{\pi\, \cancel{f^2} \cancel{+z_0}\, z_0} = \frac{\lambda f^2}{\pi z_0} = \frac{\lambda^2 f^2}{\pi^2 w_0^2}$

$$w_0' = \frac{\lambda f}{\pi w_0}$$