Fundamentals of Modern Optics

series 3

03.11.2014

to be returned on 10.11.2014, at the beginning of the lecture

Problem 1 - Lorentz model (1+2+2+2* points)

With a good approximation, a dielectric medium can be modeled by an ensemble 系综 of damped harmonic oscillators, known as the Lorentz model. In the case of a homogeneous, isotropic medium, the response function reads as

$$\hat{R}_{mn}(\mathbf{r},t) = \delta_{mn} R(t) \qquad R(t) = \begin{cases} 0 & \text{for } t \le 0 \\ \frac{f}{\Omega} e^{-\gamma t} \sin \Omega t & \text{for } t > 0 \end{cases}, \qquad \Omega = \sqrt{\omega_0^2 - \gamma^2}.$$

$P(t) = \varepsilon_0 \int_{-\infty}^{\infty} R(t-t') E(t')\, dt'$

FT $\Downarrow$

$P(\omega) = \varepsilon_0 \chi(\omega) \tilde{E}(\omega)$

$\Downarrow$

$P(\omega) = \varepsilon_0\, 2\pi\, FT[R(t)]\, \tilde{E}(\omega)$

a) Calculate the susceptibility $\chi(\omega)$ of the medium. (Notice how $\chi(\omega)$ is the Fourier transform of $R(t)$, but with the $\frac{1}{2\pi}$ factor placed differently than our previous definition of a Fourier transform) FT

b) Sketch the real and imaginary part of the dielectric function $\varepsilon(\omega) = 1 + \chi(\omega) = \epsilon'(\omega) + i\epsilon''(\omega)$ for this typical insulator 绝缘体 and mark the areas of normal ($d\epsilon'(\omega)/d\omega > 0$) and anomalous ($d\epsilon'(\omega)/d\omega < 0$) 异常的 dispersion. Where does strong absorption occur?

c) Compute 计算 the polarization $\mathbf{P}(\mathbf{r},t)$ for the dielectric medium above and

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r}) \cos(\omega_{cw} t).$$

$\frac{A+A^*}{2} = Re[A]$

d*) As you may have noticed, finding $\chi(\omega)$ from $R(t)$ is easier than finding $R(t)$ from $\chi(\omega)$. The former is a simple integral, but the latter requires a complex integral. Use the residue theorem 留数定理 to solve the back Fourier transform integral to find $R(t)$ from $\chi(\omega)$, which you have already found in part a.

Problem 2 - The Continuity Equation (4 points)

Using Maxwell's equation find an expression for the time derivative of the charge density and how it is related to the current density. Find this expression in both differential, as well as in integral notation. Try to explain the results in you own words.

Problem 3 - Spherical vector wave (3+2+2+2+2* points)

Consider Maxwell's equations in frequency domain for vacuum without sources ($\rho = 0, \mathbf{j} = 0$). Often it is favorable to use an auxiliary function to find a solution, the so-called vector potential $\mathbf{A}(\mathbf{r},\omega)$, which can be used to find the magnetic and electric field

$$\mathbf{H}(\mathbf{r},\omega) = \frac{1}{\mu_0} \nabla \times \mathbf{A}(\mathbf{r},\omega)$$

$$\mathbf{E}(\mathbf{r},\omega) = \frac{1}{-i\omega\epsilon_0} \nabla \times \mathbf{H}(\mathbf{r},\omega).$$

a) Show that the equations above immediately satisfy Maxwell's divergence equations and that $\mathbf{A}(\mathbf{r},\omega)$ itself satisfies the Helmholtz equation

$$\nabla^2 \mathbf{A}(\mathbf{r},\omega) + \frac{\omega^2}{c^2} \mathbf{A}(\mathbf{r},\omega) = 0$$

if we demand $\nabla \cdot \mathbf{A}(\mathbf{r},\omega) = 0$.
(The reason that we can really "demand" such a condition is that $\mathbf{A}$ is a vector potential, and same

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as the electric potential φ, it has some degrees of freedom in it. Stated in more technical terms, this is because Maxwell's equation are "gauge invariant" 规范不变. If you are interested, you can find more about this topic online, if you look for Lorenz or Coulomb gauge.)

b) We are looking for a spherical symmetric solution, so the magnitude of $\mathbf{A}(\mathbf{r},\omega)$ should depend only on r. We make the ansatz:

$$\mathbf{A}(\mathbf{r},\omega) = A_0 U(r)\mathbf{e}_z, \qquad \text{with } \; U(r) = \frac{1}{r}\exp(-ikr).$$

with $k = \frac{\omega}{c}$. Show that this ansatz satisfies the Helmholtz equation for $\mathbf{A}$. (*Hint*: You can make your calculations easier by neglecting what happens at $r = 0$.)

c) Calculate the magnetic field of this spherical vector wave and show that it just has a component in $\mathbf{e}_\varphi$-direction.

d) Show that the electric field in the far-field ($kr \gg 1$) just has a component in $\mathbf{e}_\vartheta$ direction and that the wave can locally be approximated as an ordinary plane wave.

e*) As you have seen in part b, for proving that the ansatz for $\mathbf{A}$ really satisfies the Helmholtz equation, you have to avoid the point $r = 0$, which is singular. Meaning that you cannot get rid of fractions 分数 like $\frac{r}{r}$ exactly at the point of $r = 0$, because such fractions are undefined there. However, if a function is undefined at a point, you can talk about its properties by getting an integral containing its point of singularity. This is the same way we treat $\delta(x)$; we do not know its value at $x = 0$ but we know its integral containing $x = 0$ is equal to 1. It turns out that the equation, that $U(r) = \frac{1}{r}\exp(-ikr)$ fully satisfies it, is:

$$\nabla^2 U(r) + k^2 U(r) = B\delta(r)$$

where B is some constant. Try to find this constant, by getting a volume integral in a sphere with an infinitesimally small radius, centred around $r = 0$, in the limit that the this radius approaches zero. You will need to make use of Gauss's theorem (divergence theorem).
散度定理

Useful formulas:

- The volume element spanning from r to $r+dr$, ϑ to $\vartheta+d\vartheta$, and φ to $\varphi+d\varphi$ is $dV = r^2 \sin\vartheta\, dr\, d\vartheta\, d\varphi$.
- The surface element spanning from ϑ to $\vartheta + d\vartheta$ and φ to $\varphi + d\varphi$ on a spherical surface at constant radius r is $dS_r = r^2 \sin\vartheta\, d\vartheta\, d\varphi$.

Components of a Cartesian vector $\mathbf{V}$ in spherical coordinates:

cartesian	**spherical**
$V_x\mathbf{e}_x$	$[V_x \sin\vartheta\cos\varphi + V_y \sin\vartheta\sin\varphi + V_z\cos\vartheta]\,\mathbf{e}_r$
$V_y\mathbf{e}_y$	$[V_x \cos\vartheta\cos\varphi + V_y \cos\vartheta\sin\varphi - V_z\sin\vartheta]\,\mathbf{e}_\vartheta$
$V_z\mathbf{e}_z$	$[-V_x\sin\varphi + V_y\cos\varphi]\,\mathbf{e}_\varphi$

$$\nabla\times\mathbf{V}(r,\vartheta,\varphi) = \frac{1}{r\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}(V_\varphi\sin\vartheta) - \frac{\partial V_\vartheta}{\partial\varphi}\right]\mathbf{e}_r + \frac{1}{r}\left[\frac{1}{\sin\vartheta}\frac{\partial V_r}{\partial\varphi} - \frac{\partial}{\partial r}(rV_\varphi)\right]\mathbf{e}_\vartheta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\vartheta) - \frac{\partial V_r}{\partial\vartheta}\right]\mathbf{e}_\varphi$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\psi}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\psi}{\partial\varphi^2}$$

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1. Lorentz model

a) $\chi(\omega)=\int_{-\infty}^{\infty} k(t)\, e^{i\omega t}dt = \int_0^{\infty} \frac{f}{\Omega} e^{-\gamma t}\sin(\Omega t)\, e^{i\omega t}dt$

$$= \int_0^{\infty} \frac{f}{2i\Omega}\left(e^{-\gamma t + i\omega t + i\Omega t} - e^{-\gamma t + i\omega t - i\Omega t}\right) dt$$

$$= \frac{f}{2i\Omega}\cdot\left(\frac{-1}{-\gamma+i\omega+i\Omega} - \frac{-1}{-\gamma+i\omega-i\Omega}\right) = \frac{f}{(-\gamma+i\omega)^2+\Omega^2} = \frac{f}{\omega_0^2-\omega^2-2i\gamma\omega}$$

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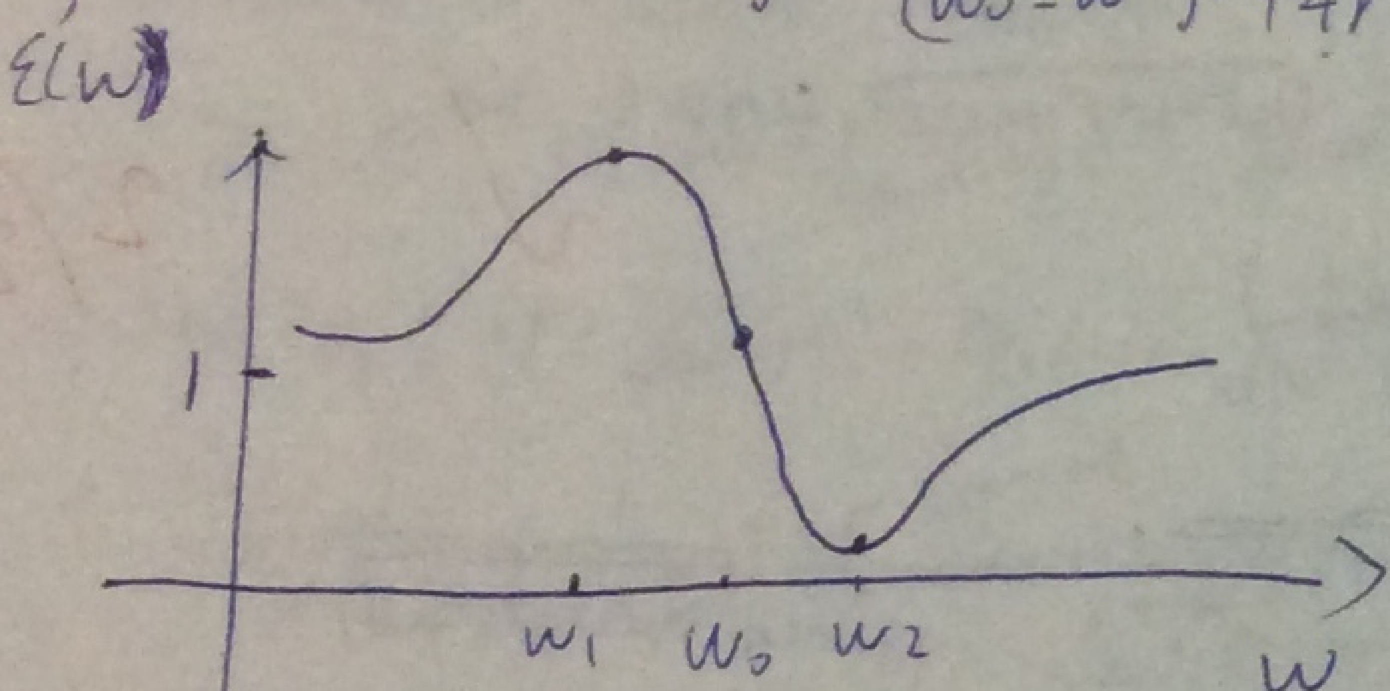
b) $\varepsilon(\omega)=1+\chi(\omega)=1+\frac{f}{(-\gamma+i\omega)^2+\Omega^2}=\varepsilon'(\omega)+i\varepsilon''(\omega)$

$\varepsilon'(\omega)=1+\mathrm{Re}\{\chi(\omega)\}=1+\frac{f(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}$

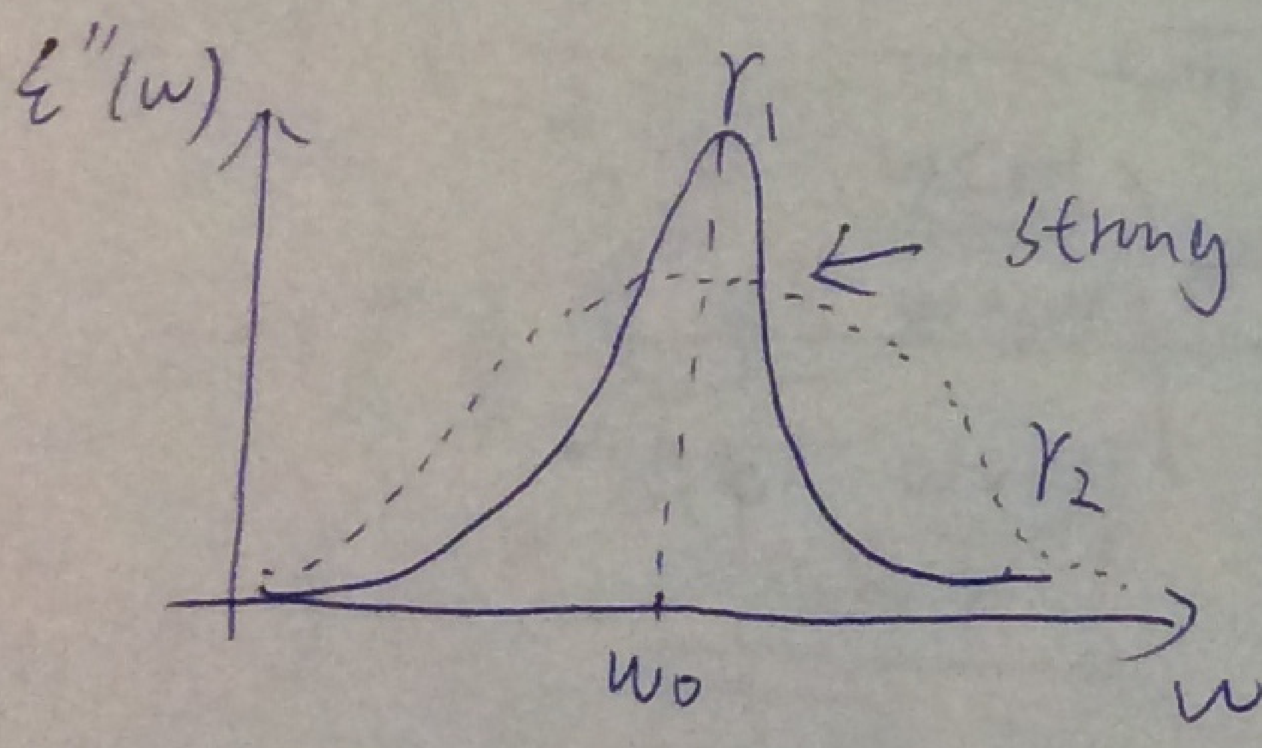
$\varepsilon''(\omega)=\mathrm{Im}\{\chi(\omega)\}=\frac{2\gamma\omega f}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}$



$\boxed{\frac{d\varepsilon'(\omega_{1,2})}{d\omega}=0}$ – extremum cond

$(0,\omega_1)\cup(\omega_2,+\infty)$ – area of normal dispersion $\left(\frac{d\varepsilon'}{d\omega}>0\right)$

(ω_1,ω_2) – area of anomalous dispersion $\left(\frac{d\varepsilon'}{d\omega}<0\right)$



strong absorpsion in vicinity of ω_0.

if $\gamma_2>\gamma_1$, the spectrum of absorption will be wider.

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c) $\vec{E}(\vec{r},t)=E(r)\cos(\omega t)$

$\vec{E}(\vec{r},\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\vec{E}(\vec{r},t)\,e^{i\omega t}dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\vec{E}(\vec{r})\cdot\cos(\omega t)\,e^{i\omega t}dt$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\vec{E}(\vec{r})\frac{1}{2}\left(e^{i\omega t}+e^{-i\omega t}\right)e^{i\omega' t}dt$$

$$= \frac{1}{2\pi}\vec{E}(\vec{r})\cdot\frac{1}{2}\left[\int_{-\infty}^{\infty} e^{i(\omega+\omega')t}\,dt + \int_{-\infty}^{\infty} e^{i(\omega'-\omega)t}\,dt\right]$$

$$= \frac{1}{2\pi}\vec{E}(\vec{r})\cdot\frac{1}{2}\cdot\left(2\pi\,\delta(\omega'+\omega) + 2\pi\,\delta(\omega'-\omega)\right]$$

$$= \frac{1}{2}\vec{E}(\vec{r})\cdot\left[\delta(\omega'+\omega) + \delta(\omega'-\omega)\right] \quad ✓$$

$$\vec{P}(\vec{r},\omega) = \varepsilon_0\chi(\omega)\,\vec{E}(\vec{r},\omega) = \varepsilon_0\frac{f}{(\omega_0^2-\omega^2)-2i\gamma\omega'}\cdot\frac{1}{2}\vec{E}(\vec{r})\cdot\left[\delta(\omega'+\omega)+\delta(\omega'-\omega)\right]$$

$$\vec{P}(\vec{r},t) = \int_{-\infty}^{\infty}\vec{P}(\vec{r},\omega)\,e^{-i\omega' t}\,d\omega' = \int_{-\infty}^{\infty}\varepsilon_0\frac{f}{(\omega_0^2-\omega^2)-2i\gamma\omega'}\cdot\frac{1}{2}\vec{E}(\vec{r})\cdot\left[\delta(\omega'+\omega)+\delta(\omega'-\omega)\right]e^{-i\omega' t}\,d\omega'$$

$$= \frac{\varepsilon_0\cdot f}{(\omega_0^2-\omega^2)-2i\gamma\omega}\cdot\frac{1}{2}\vec{E}(\vec{r})\cdot e^{-i\omega t} + \frac{\varepsilon_0\cdot f}{(\omega_0^2-\omega^2)+2i\gamma\omega}\cdot\frac{1}{2}\vec{E}(\vec{r})\,e^{i\omega t}$$

$$= \frac{\varepsilon_0 f}{2}\cdot\vec{E}(\vec{r})\cdot\left[\frac{(\omega_0^2-\omega^2)+2i\gamma\omega}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}\cdot e^{-i\omega t} + \frac{(\omega_0^2-\omega^2)-2i\gamma\omega}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}\cdot e^{i\omega t}\right]$$

$$= \frac{\varepsilon_0 f}{2}\vec{E}(\vec{r})\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}\cdot\underbrace{\left(e^{-i\omega t}+e^{i\omega t}\right)}_{2\cos\omega t} - \frac{\varepsilon_0 f}{2}\vec{E}(\vec{r})\frac{2i\gamma\omega}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}\underbrace{\left(e^{i\omega t}-e^{-i\omega t}\right)}_{2i\sin\omega t}$$

$$= \frac{\varepsilon_0 f}{2}\vec{E}(\vec{r})\cdot\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}\cdot 2\cos\omega t + \frac{\varepsilon_0 f}{2}\vec{E}(\vec{r})\cdot\frac{4\gamma\omega}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}\sin\omega t \quad ✓$$

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$$= \frac{\varepsilon_0\cdot f\,\vec{E}(\vec{r})}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}\cdot\left[(\omega_0^2-\omega^2)\cdot\cos\omega t + 2\gamma\omega\cdot\sin\omega t\right]$$

$$= \varepsilon_0\cdot f\cdot\vec{E}(\vec{r})\left[\underbrace{\frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}}_{\cos\varphi}\cdot\cos\omega t + \underbrace{\frac{2\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}}_{-\sin\varphi}\sin\omega t\right]\cdot\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}$$

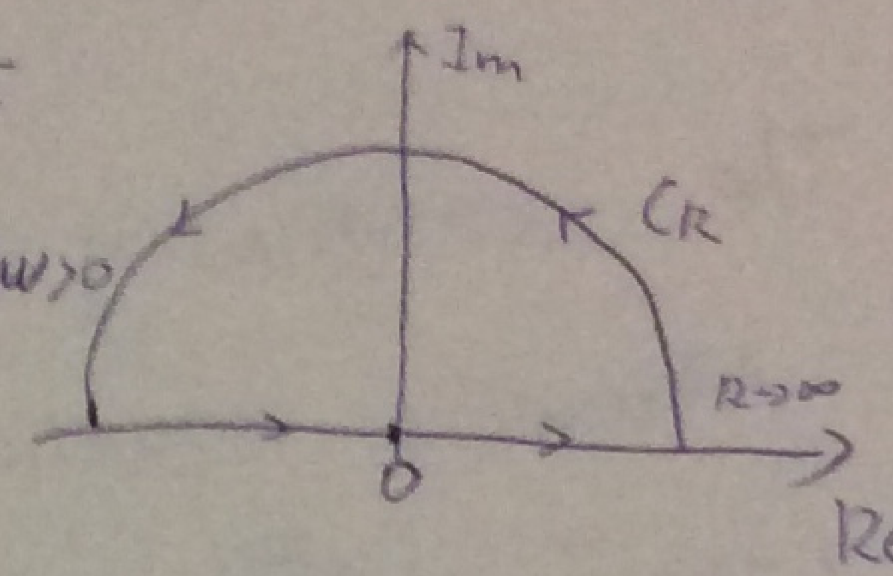
$$= \varepsilon_0\cdot f\cdot\vec{E}(\vec{r})\cdot\left(\cos\varphi\cos\omega t - \sin\varphi\sin\omega t\right)\cdot\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}$$

$$= \frac{\varepsilon_0\cdot f\cdot\vec{E}(\vec{r})\cdot\cos(\omega t+\varphi)}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}} \qquad \tan\varphi = \frac{-2\gamma\omega}{\omega_0^2-\omega^2}\quad\begin{cases}\omega_0>\omega, & \varphi<0\\ \omega_0=\omega, & \varphi=\frac{\pi}{2}\\ \omega_0<\omega, & \varphi>0\end{cases}$$

1) $R(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(w)\,e^{-iwt}dw$

$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f}{(w_0^2-w^2)-2i\gamma w}\,e^{-iwt}dw$

the singular ~~pole~~ point is $w=-i\gamma\pm\sqrt{w_0^2-\gamma^2}$



① when $t<0$, above the real axis, the imaginative part of $w>0$

when $|w|=R\to\infty$, ~~$e^{-iwt}\to 0$~~

according to Jordans lemma, $\int_{CR}X(w)\,e^{-iwt}dw\to 0$

according to residue theorum

$\int_{CR}X(w)\,e^{-iwt}\cdot\frac{1}{2\pi}dw+\int_{-\infty}^{\infty}X(w)\,e^{iwt}\frac{1}{2\pi}dw=0$

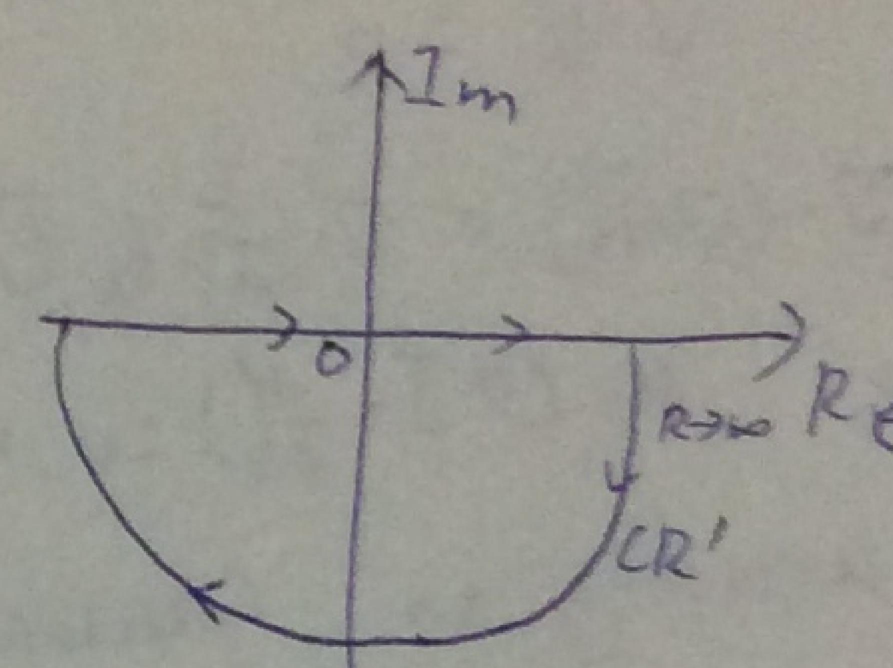
because there is no sigular point above the real axis.

② when $t>0$, under the real axis, the imagive part of $w<0$

when $|w|=R\to\infty$, ~~$e^{-iwt}\to 0$~~

according to Jordans Lemma, $\int_{CR'}X(w)e^{-iwt}dw\to 0$

according to residue theorum



$\int_{CR'}X(w)\,e^{-iwt}\frac{1}{2\pi}dw+\int_{\infty}^{-\infty}X(w)e^{-iwt}\frac{1}{2\pi}dw$

$=2\pi i[\mathrm{Res}(-i\gamma+\sqrt{w_0^2-\gamma^2})+\mathrm{Res}(-i\gamma-\sqrt{w_0^2-\gamma^2})]$

$=\frac{2\pi i}{2\pi}\cdot\left[\lim_{w\to -i\gamma-\sqrt{w_0^2-\gamma^2}}\left\{\frac{fe^{-iwt}}{w+\gamma i-\sqrt{w_0^2-\gamma^2}}\right\}+\lim_{w\to -i\gamma+\sqrt{w_0^2-\gamma^2}}\left\{\frac{fe^{-iwt}}{w+\gamma i+\sqrt{w_0^2-\gamma^2}}\right\}\right]$

$=if\cdot\left(\frac{e^{-\gamma t-i\sqrt{w_0^2-\gamma^2}\,t}}{-2\sqrt{w_0^2-\gamma^2}}+\frac{e^{-\gamma t+i\sqrt{w_0^2-\gamma^2}\,t}}{2\sqrt{w_0^2-\gamma^2}}\right)$

$=-if\,\frac{e^{-\gamma t}(e^{i\sqrt{w_0^2-\gamma^2}\,t}-e^{-i\sqrt{w_0^2-\gamma^2}\,t})}{2\sqrt{w_0^2-\gamma^2}}$

$=fe^{-\gamma t}\cdot\frac{\sin\sqrt{w_0^2-\gamma^2}\,t}{\sqrt{w_0^2-\gamma^2}}$

$=\frac{f}{\Omega}e^{-\gamma t}\sin\Omega t \qquad \Omega=\sqrt{w_0^2-\gamma^2}$

③ when $t=0$, $R(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f}{w_0^2-w^2-2i\gamma w}dw$

as $|w|=R\to\infty$, $\int_{CR}\frac{f}{w_0^2-w^2-2i\gamma w}dw=\int_{CR'}\frac{f}{w_0^2-w^2-2i\gamma w}dw\to 0$

So, $R(t)=2\pi i\,\mathrm{Res}=0$

√ $2^*/2^*$

So $R(t) = \begin{cases} 0 & t \leq 0 \\ \frac{1}{\Omega} e^{-\gamma t} \sin \Omega t & t > 0 \end{cases}$ $\quad \Omega = \sqrt{\omega_0^2 - \gamma^2}$

2. The continuity Equation

Maxwell's equation

$$\begin{cases} \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \\ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \\ \nabla \cdot \vec{B} = 0 \end{cases} \Rightarrow \nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{J} + \mu_0 \frac{\partial \rho}{\partial t}$$

$$\nabla \cdot (\nabla \times \vec{B}) = \nabla \cdot \left(\frac{\partial}{\partial j} B_k \vec{e_i} \varepsilon_{ijk}\right) = \frac{\partial}{\partial i} \frac{\partial}{\partial j} B_k \varepsilon_{ijk} = 0$$

So: $\mu_0 \left(\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t}\right) = 0 \Rightarrow \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow \int_V \nabla \cdot \vec{J} dV + \int_V \frac{\partial \rho}{\partial t} dV = 0$

according to Gaussian theorum $\oint_S \vec{J} \cdot d\vec{S} = \int_V \nabla \cdot \vec{J} dV$

we have $\oint_S \vec{J} \cdot d\vec{S} + \frac{\partial Q}{\partial t} = 0$ (assume $Q = \int_V \rho dV$)

This equation means that increasing (or decreasing) of charge is a source of currency. There is only one way that, a charge can disappear from a point. This way is just moving outward from the point.

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Spherical vector wave

a) In frequency domain, the Maxwell's equations are listed in Fourier domain:

$$\begin{cases} \nabla\times\vec{E} = -i\omega\vec{B} \\ \nabla\times\vec{B} = \mu_0\varepsilon_0 i\omega\vec{E} \\ \nabla\cdot\vec{E} = 0 \\ \nabla\cdot\vec{B} = 0 \end{cases} \quad ①$$

take $H(\vec{r},\omega) = \frac{1}{\mu_0}\nabla\times\vec{A}(\vec{r},\omega)$ ②, $\vec{E}(\vec{r},\omega) = -\frac{1}{i\omega\varepsilon_0}\nabla\times\vec{H}(\vec{r},\omega)$ ③ into the eqs.

$$\Rightarrow \begin{cases} \nabla\cdot\left(\frac{1}{-i\omega\varepsilon_0}\nabla\times\vec{H}(\vec{r},\omega)\right) = \frac{1}{-i\omega\varepsilon_0}\frac{\partial}{\partial i}\frac{\partial}{\partial j}H_k\varepsilon_{ijk} = 0 \quad \text{proved.} \\ \nabla\cdot\left(\nabla\times\vec{A}(\vec{r},\omega)\right) = \frac{\partial}{\partial i}\frac{\partial}{\partial j}A_k\varepsilon_{ijk} = 0 \quad \text{proved} \end{cases}$$

✓

combine ②. ③, we have:

$$\vec{E}(\vec{r},\omega) = \frac{1}{-i\omega\varepsilon_0}\cdot\frac{1}{\mu_0}\nabla\times\left(\nabla\times\vec{A}(\vec{r},\omega)\right)$$

$$= \frac{1}{-i\omega\varepsilon_0\mu_0}\left(-\nabla^2\vec{A} + \nabla(\nabla\cdot\vec{A})\right)$$

$$= +\frac{c^2}{i\omega}\nabla^2\vec{A}$$

$$\nabla\times\vec{E} = +i\omega B = +i\omega\,\nabla\times\vec{A}(\vec{r},\omega) \Rightarrow \vec{E} = +i\omega\vec{A}(\vec{r},\omega) + \bar{a}$$

where $\bar{a}$: $\nabla\times\bar{a} = 0$

we redefine $\vec{A}' = \vec{A} + \frac{\bar{a}}{i\omega}$, where $\nabla\times\bar{a} = 0$ (because it has some degrees of ~~f~~freedom in it)

we have $\vec{E}(\vec{r},\omega) = +i\omega\vec{A}'(\vec{r},\omega)$

$$\Rightarrow -\frac{c^2}{i\omega}\nabla^2\vec{A}' = -i\omega\vec{A}' \Rightarrow \nabla^2\vec{A} + \frac{\omega^2}{c^2}\vec{A}(\vec{r},\omega) = 0$$ ✓

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b) take $\vec{A}(\vec{r},\omega) = A_0\frac{1}{r}e^{-ikr}\vec{e_z}$ into the Helmholtz equation.

using the spherical coordinates: since $\frac{\partial}{\partial\theta}\vec{A} = \frac{\partial}{\partial\varphi}\vec{A} = 0$

$\nabla^2 U + \frac{\omega^2}{c^2}U(\vec{r}) = 0$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right)\vec{e_z} + \frac{\omega^2}{c^2}\vec{A}(\vec{r},\omega) = 0 \quad (r\neq 0)$$

$$\Rightarrow \frac{-A_0k^2e^{-ikr}}{r} + \frac{\omega^2}{c^2}\cdot A_0\frac{1}{r}e^{-ikr} = 0 \Rightarrow -k^2 + \frac{\omega^2}{c^2} = 0$$

steps missing!

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So this ansatz satisfy the Helmholtz equation.

c) $\vec B = \mu_0 \vec H = \nabla\times\vec A(\vec r,\omega) = \nabla\times(A_0 \frac{1}{r} e^{-ikr}\vec e_z)$

~~and $A_z\vec e_z = A_z\cos\theta\,\vec e_r - A_z\sin\theta\,\vec e_\theta$~~

~~using spherical coordinates we have~~

and $A_z\vec e_z = \underbrace{A_z\cos\theta\,\vec e_r}_{A_r} - \underbrace{A_z\sin\theta\,\vec e_\theta}_{A_\theta}$ $\qquad \hat e_z = \cos\theta\,\hat e_r - \sin\theta\,\hat e_\theta$

using spherical coordinates, we have $A_\varphi=0$, $\frac{\partial A_\theta}{\partial\varphi}=0$ $\frac{\partial A_r}{\partial\varphi}=0$

$$\Rightarrow \nabla\times\vec A(r,\theta) = \frac1r\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\vec e_\varphi$$
$$= \frac1r\left(-A_z\sin\theta + r\frac{\partial A_\theta}{\partial r} + \sin\theta A_z\right)\vec e_\varphi$$
$$= -\sin\theta A_0\left(-\frac{e^{-ikr}}{r^2} - \frac{ik}{r}e^{-ikr}\right)\vec e_\varphi$$
$$= A_0\frac{e^{-ikr}}{r}\sin\theta\left(ik+\frac1r\right)\vec e_\varphi$$

So it just has a component in $\vec e_\varphi$-direction.

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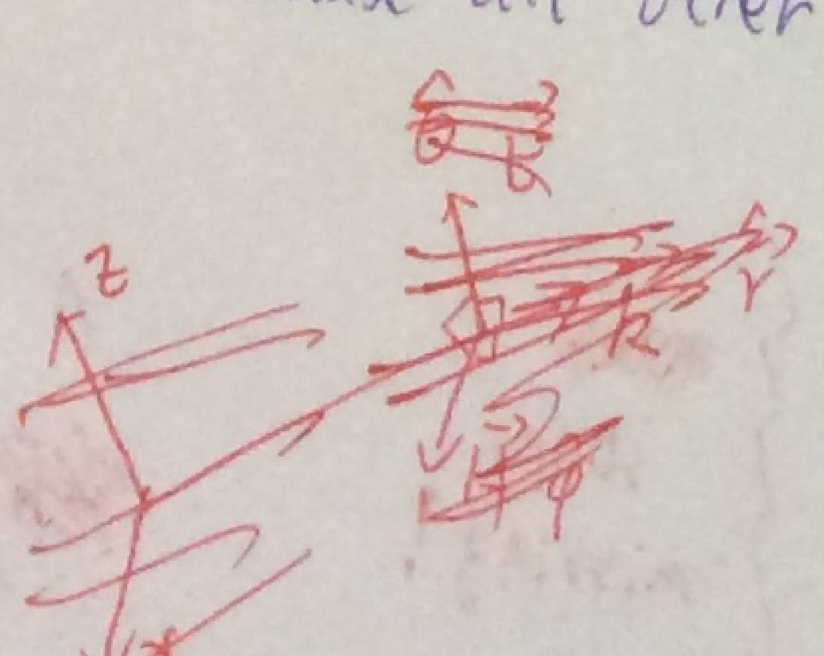
d) $$\nabla\times\vec H = \frac{1}{r\sin\theta}\left(\frac{\partial H_\varphi}{\partial\theta}\sin\theta + H_\varphi\cos\theta\right)\vec e_r + \frac1r\left(-H_\varphi - r\frac{\partial H_\varphi}{\partial r}\right)\vec e_\theta$$
$$= \frac1r\left[\frac{A_0e^{-ikr}}{\mu_0 r}\cos\theta\left(ik+\frac1r\right) + \frac{A_0e^{-ikr}}{\mu_0 r}\cos\theta\left(ik+\frac1r\right)\right]\vec e_r - \frac1r\left[\frac{A_0e^{-ikr}}{\mu_0 r}\left(ik+\frac1r\right)\sin\theta + re^{-ikr}\left(\frac{A_0\sin\theta}{\mu_0}\right)\left(\frac{-ik}{r}\left(ik+\frac1r\right) - \frac{ik}{r^2} - \frac{2}{r^3}\right)\right]\vec e_\theta$$
$$= \frac{2A_0}{\mu_0 r^2}e^{-ikr}\cos\theta\left(ik+\frac1r\right)\vec e_r + \frac{A_0}{\mu_0}e^{-ikr}\sin\theta\left(\frac{1}{r^3}+\frac{ik}{r^2}-\frac{k^2}{r}\right)\vec e_\theta \quad ✓$$

$$\vec E(\vec r,\omega) = \frac{1}{-i\omega\varepsilon_0}\nabla\times\vec H = \frac{i}{\omega\varepsilon_0\mu_0}\frac{A_0}{r}e^{-ikr}\left(\frac{2\cos\theta}{r}\left(ik+\frac1r\right)\vec e_r + \sin\theta\left(-k^2+\frac{ik}{r}+\frac{1}{r^2}\right)\vec e_\theta\right)$$

when $r\gg\frac1k$, $\vec E \sim \frac{iA_0e^{-ikr}\sin\theta}{\omega\varepsilon_0\mu_0 r}(-k^2)\vec e_\theta$

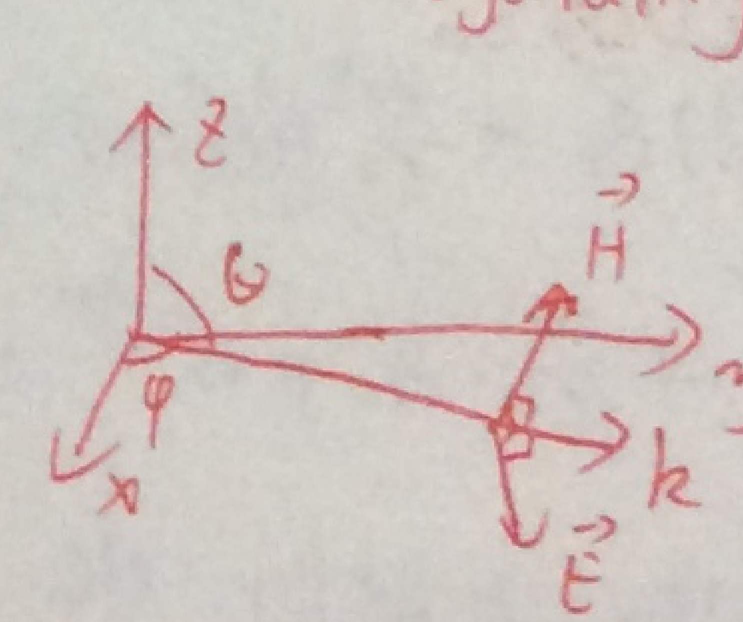
$r\gg\lambda$ $\qquad H_{far}\approx\frac{iA_0}{\mu_0}\frac{e^{-ikr}}{r}k\sin\theta\,\vec e_\varphi$, $E_{far}\approx\frac{-iA_0}{\omega\mu_0\varepsilon_0}\cdot\frac{e^{-ikr}}{r}\cdot k^2\sin\theta\,\vec e_\theta$

because all other terms can be neglected in comparison with the leading term.



in the far-field, ✓ $\vec E, \vec k, \vec H$ is orthogonality, so it looks like a plane wave

but why is this locally a plane wave in the far-field?



1.5/2

when $r\neq 0$, we have proved that $\nabla^2 U(r) + k^2 U(r) = 0$

when $r\to 0$, $\frac{1}{r}e^{-ikr} = \frac{1}{r}(1 - ikr + \frac{1}{2!}(-ikr)^2 + \frac{1}{3!}(-ikr)^3 + \cdots)$

$= \frac{1}{r} - ik + \frac{1}{2!}(-k^2 r) + \cdots$

except for the first term, $\frac{1}{r}$, the rest terms are analysist

so $\int_V \nabla^2 U(r) dv = \int_V \nabla^2 \frac{1}{r} dv$ V is very small around the origin

same situation $\int_V U(r) dv = \int_V \frac{1}{r} dv$

$\int \nabla^2 \frac{1}{r} dv = \lim_{a\to 0} \int \nabla^2 \frac{1}{(r^2+a^2)^{\frac{1}{2}}} dv = \lim_{a\to 0} \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\varphi \int_0^{\infty} \frac{-3a^2 r^2}{(r^2+a^2)^{\frac{5}{2}}} dr$

$\overset{r=a\rho}{=} -12\pi \int_0^{\infty} \frac{\rho^2 d\rho}{(\rho^2+1)^{\frac{5}{2}}} = -4\pi \left.\frac{\rho^3}{(\rho^2+1)^{\frac{3}{2}}}\right|_0^{\infty} = -4\pi$

$\int_V \frac{1}{r} dv = \int_V \frac{1}{r}\cdot r^2 \sin\theta\, d\theta\, d\varphi\, dr = \int_V r\, dr \sin\theta\, d\theta\, d\varphi \to 0$

$\int B\delta(r) dv = B$

$\Rightarrow \nabla^2 U(r) + k^2 U(r) = B\delta(r)$

$\Rightarrow \int [\nabla^2 U(r) + k^2 U(r)] dv = \int B\delta(r) dv$

$\Rightarrow -4\pi = B \Rightarrow B = -4\pi$ ✓

$2^*/2^*$

$\vec{A}(\vec{r}) = A_0 u(\vec{r})\vec{e_z}$ $\quad u(r) = e^{-ikr}/r$

$\delta(\vec{r}) = \delta(x)\delta(y)\delta(z)$

$\nabla^2\vec{A} + \frac{\omega^2}{c^2}\vec{A} = A_0 B\delta(\vec{r})\vec{e_z}$

$\nabla^2\vec{A} + \frac{\omega^2}{c^2}\vec{A} = -\mu_0\vec{J} \Rightarrow \nabla^2 U(r) + k^2 U = B\delta(\vec{r})$

$\int(\nabla^2 U(r) + k^2 U) dv = \int B\delta(\vec{r}) dV$

$\int \nabla\cdot\nabla U\, dv = \oint_{r=R} \nabla U\cdot d\vec{s}$, $d\vec{s} = R^2 \sin\theta\, d\theta\, d\hat{r}\, d\varphi$

$\int_0^{\pi}\int_0^{\pi} R^2 \left.\frac{\partial U}{\partial r}\right|_{r=R} \underbrace{(\hat{r}\cdot\hat{r})}_{1} \cdot \sin\theta\, d\theta\, d\varphi$

$\int k^2 U\, dV = \int_0^R \int_0^{2\pi}\int_0^{\pi} \frac{e^{-ikr}}{r} r^2 \sin\theta\, d\theta\, d\varphi = 4\pi(ikR e^{-ikR} + e^{-ikR} - 1)$

$-4\pi e^{-ikR}(1 + ikR)$

$\Rightarrow B = -4\pi \quad \vec{J} = \frac{4\pi A_0}{\mu_0}\delta(\vec{r})\vec{e_z}$

$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\frac{e^{-ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} dV$