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Series 5
Fundamentals of Modern Optics

to be returned on , at the beginning of the lecture

Task 1: Talbot Effect, with and without the Fresnel approximation (5+2+5+3 * points)

Assume an initial field $f(x,z=0)$ (with full translational symmetry in the y-direction), which is periodic along the x-direction with a period of a, such that $f(x+a,z=0)=f(x,z=0)$. We want to calculate the field $f(x,z)$ after propagation along the z-direction, in vacuum, where the vacuum wavelength of the field is λ. If we treat this diffraction problem in the Fresnel (paraxial) approximation, we will find that after a certain length L_T the initial field reappears except for an extra phase, such that

$$f(x,z=L_T)=f(x,z=0)\exp(ikL_T+i2\pi m_l) \quad \text{with } m_l\in\mathbb{Z}$$

This is known as the Talbot effect and L_T is known as the Talbot length.

a) Find the expression for L_T under the assumptions specified above. Hint: You do not need to know the specific expression for the function $f(x)$. Express $f(x)$ as a Fourier series, and then follow through with the standard approach for calculating beam diffraction. Keep in mind that we assume the paraxial approximation to be valid.

b) Which colour light field should be used to have the Talbot length as 25.71 m, given the period to be $a=3$ mm?

The Talbot effect always holds true in the Fresnel approximation. In contrast, if the Fresnel approximation is not valid, for example when the period a is comparable to the wavelength λ, the Talbot effect does not necessarily take place. However, it can still occur for certain field patterns.

c) Show that for an initial field distribution of the form

$$f(x,z=0)=A\cos(x2\pi/a_1)\cos(x2\pi/a_2)$$

the Talbot effect still takes place outside the paraxial regime and calculate the Talbot length. Find the value of L_T for the wavelength of $\lambda=800$ nm and periods $a_1=4\mu\text{m}, a_2=5\mu\text{m}$.

d*) Consider now an initial field, which is formed as the superposition of three periodic components

$$f(x,z=0)=A_1\cos(x2\pi/a_1)+A_2\cos(x2\pi/a_2)+A_3\cos(x2\pi/a_3).$$

Show that the Talbot effect in this case will only take place if a certain relation between λ,a_1,a_2,a_3 is satisfied and find this relation.

Solution Task 1:

a) Periodic field $\implies$ $f(x+a)=f(x)$
Expressed in Fourier series: $\Rightarrow$ $f(x)=\sum_l c_l\exp\left(il\frac{2\pi}{a}x\right)$
Fourier transform: $\implies$ $f(\alpha)=\sum_l c_l\delta\left(l\frac{2\pi}{a}-\alpha\right)$
Field at z :

$$\begin{aligned}f(\alpha,z)&=f(\alpha)\exp(i\gamma z)=f(\alpha)\exp\left(i\left[k-\frac{\alpha^2}{2k}\right]z\right)\\&=\sum_l c_l\delta\left(l\frac{2\pi}{a}-\alpha\right)\exp\left(i\left[k-\frac{\alpha^2}{2k}\right]z\right)\end{aligned}$$

Fourier back transform

$$\begin{aligned}f(x,z)&=\int_{\infty}^{\infty}f(\alpha,z)\exp(i\alpha x)d\alpha\\&=\sum_l c_l\exp\left(i\left[k-\frac{1}{2k}\left(\frac{2\pi l}{a}\right)^2\right]z\right)\exp\left(i\frac{2\pi l}{a}x\right)\end{aligned}$$

$$f(x,z) \stackrel{!}{=} f(x)\exp(ikz+i\varphi) = \sum_l c_l \exp(ikz+i\varphi)\exp\left(i\frac{2\pi l}{a}x\right)$$
$$\stackrel{!}{=} \sum_l c_l \exp\left(i\left[k-\frac{1}{2k}\left(\frac{2\pi l}{a}\right)^2\right]z\right)\exp\left(i\frac{2\pi l}{a}x\right)$$

If we find φ now from above equation,

$$\exp(i\varphi) = \exp\left(i\left[k-\frac{1}{2k}\left(\frac{2\pi l}{a}\right)^2\right]z - ikz\right) \quad \forall l \in \mathbb{Z}$$
$$\exp(i\varphi) = \exp\left(-i\frac{1}{2k}\left(\frac{2\pi l}{a}\right)^2 z\right)$$

To have the reconstructed field, the phase needs to be a multiple of 2π.

$$\Longrightarrow 2\pi m_l = \frac{\left(\frac{2\pi l}{a}\right)^2}{2k} z \quad \forall l, \text{ with } m_l \in \mathbb{Z}$$
$$(\simeq) \quad m_l = l^2 z\frac{2\pi}{2a^2k} \quad \text{ is fullfilled if } \frac{z\pi}{a^2k} \in \mathbb{N}$$
$$\text{is fullfilled for the first time if : } \quad z = L_T = \frac{a^2k}{\pi} = \frac{2a^2}{\lambda}$$

b) From the Talbot length L_T formula we have:

$$\lambda = \frac{2a^2}{L_T}$$
$$= \frac{2\times 9\ \text{mm}^2}{25.71\text{m}} = 0.700\times 10^{-6}\ \text{m} = 700\text{nm}$$

The wavelength 700 nm corresponds to the red colour light field.

c) We can rewrite the field from the task description using $\cos(x)\cos(y) = \frac{1}{2}(\cos(x-y)+\cos(x+y))$:

$$f(x,z=0) = A\cos\left(\frac{2\pi}{a_1}x\right)\cos\left(\frac{2\pi}{a_2}x\right) = \frac{A}{2}\cos\left[2\pi\left(\frac{1}{a_1}+\frac{1}{a_2}\right)x\right] + \frac{A}{2}\cos\left[2\pi\left(\frac{1}{a_1}-\frac{1}{a_2}\right)x\right].$$

By introducing new variables as
we finally obtain

$$\frac{1}{\Lambda_1} = \frac{1}{a_1}+\frac{1}{a_2}, \frac{1}{\Lambda_2} = \frac{1}{a_1}-\frac{1}{a_2}$$
$$f(x,z=0) = \frac{A}{4}e^{i\frac{2\pi}{\Lambda_1}x} + \frac{A}{4}e^{-i\frac{2\pi}{\Lambda_1}x} + \frac{A}{4}e^{i\frac{2\pi}{\Lambda_2}x} + \frac{A}{4}e^{-i\frac{2\pi}{\Lambda_2}x}$$

Using $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(i\alpha x)\mathrm{d}\alpha$ we get for the spectrum:

$$U_0(\alpha) = \frac{1}{2\pi}\int f(x,z=0)e^{-i\alpha x}\,\mathrm{d}x$$
$$= \frac{A}{4}\left[\delta\left(\frac{2\pi}{\Lambda_1}-\alpha\right)+\delta\left(\frac{2\pi}{\Lambda_1}+\alpha\right)+\delta\left(\frac{2\pi}{\Lambda_2}-\alpha\right)+\delta\left(\frac{2\pi}{\Lambda_2}-\alpha\right)\right]$$

Spectrum propagation:

$$U(\alpha,z) = U_0(\alpha)H(\alpha,z)$$
$$= \frac{A}{4}\left[\delta\left(\frac{2\pi}{\Lambda_1}-\alpha\right)+\delta\left(\frac{2\pi}{\Lambda_1}+\alpha\right)+\delta\left(\frac{2\pi}{\Lambda_2}-\alpha\right)+\delta\left(\frac{2\pi}{\Lambda_2}-\alpha\right)\right]e^{i\sqrt{k^2-\alpha^2 z}}$$

Back transform:

$$
\begin{aligned}
u(x,z) &= \int U(\alpha,z)e^{i\alpha x}\,\mathrm{d}\alpha \\
&= \frac{A}{4}\left(e^{i\frac{2\pi}{\Lambda_1}x}+e^{-i\frac{2\pi}{\Lambda_1}x}\right)e^{i\sqrt{k^2-\left(\frac{2\pi}{\Lambda_1}\right)^2}z}+\frac{A}{4}\left(e^{i\frac{2\pi}{\Lambda_2}x}+e^{-i\frac{2\pi}{\Lambda_2}x}\right)e^{i\sqrt{k^2-\left(\frac{2\pi}{\Lambda_2}\right)^2}z} \\
&= \frac{A}{2}\cos\left(\frac{2\pi}{\Lambda_1}x\right)e^{i\sqrt{k^2-\left(\frac{2\pi}{\Lambda_1}\right)^2}z}+\frac{A}{2}\cos\left(\frac{2\pi}{\Lambda_2}x\right)e^{i\sqrt{k^2-\left(\frac{2\pi}{\Lambda_2}\right)^2}z} \\
&= \frac{A}{2}e^{i\sqrt{k^2-\left(\frac{2\pi}{\Lambda_1}\right)^2}}z\left[\cos\left(\frac{2\pi}{\Lambda_1}x\right)+\cos\left(\frac{2\pi}{\Lambda_2}x\right)e^{i}\left[\sqrt{k^2-\left(\frac{2\pi}{\Lambda_2}\right)^2}-\sqrt{k^2-\left(\frac{2\pi}{\Lambda_1}\right)^2}z^2\right]\right]
\end{aligned}
$$

To cancel the extra factor we need:

$$
e^{i}\left[\sqrt{k^2-\left(\frac{2\pi}{\Lambda_2}\right)^2}-\sqrt{k^2-\left(\frac{2\pi}{\Lambda_1}\right)^2}\right]^{L_T}\stackrel{!}{=}1 \quad\Rightarrow\quad \sqrt{k^2-\left(\frac{2\pi}{\Lambda_2}\right)^2}-\sqrt{k^2-\left(\frac{2\pi}{\Lambda_1}\right)^2}\mid L_T=2\pi
$$

$$
L_T=\frac{2\pi}{\left|\sqrt{k^2-\left(\frac{2\pi}{\Lambda_2}\right)^2}-\sqrt{k^2-\left(\frac{2\pi}{\Lambda_1}\right)^2}\right|}.
$$

$$
L_T=\frac{1}{\left|\sqrt{\left(\frac{1}{\lambda}\right)^2-\left(\frac{1}{\Lambda_2}\right)^2}-\sqrt{\left(\frac{1}{\lambda}\right)^2-\left(\frac{1}{\Lambda_1}\right)^2}\right|}
$$

Substituting here $\lambda = 800$ nm, $a_1 = 4\mu$m, $a_2 = 5\mu$m, we get $L_T = 12.076\mu$m.

d) Following the calculations in the previous part we can obtain the field for the case of three periodic components in the initial field:

$$
\begin{aligned}
u(x,z) &= \int U(\alpha,z)e^{i\alpha x}\,\mathrm{d}\alpha \\
&= A_1\cos\left(\frac{2\pi}{a_1}x\right)e^{i\sqrt{k^2-\left(\frac{2\pi}{a_1}\right)^2z}}+A_2\cos\left(\frac{2\pi}{a_2}x\right)e^{i\sqrt{k^2-\left(\frac{2\pi}{a_2}\right)^2}}+A_3\cos\left(\frac{2\pi}{a_3}x\right)e^{i\sqrt{k^2-\left(\frac{2\pi}{a_3}\right)^2z}} \\
&= e^{i\sqrt{k^2-\left(\frac{2\pi}{a_1}\right)^2z}}\left[A_1\cos\left(\frac{2\pi}{a_1}x\right)+A_2\cos\left(\frac{2\pi}{a_2}x\right)e^{i\left(\sqrt{k^2-\left(\frac{2\pi}{a_2}\right)^2}-\sqrt{k^2-\left(\frac{2\pi}{a_1}\right)z}\right)}+A_3\cos\left(\frac{2\pi}{a_3}x\right)e^{i\left(\sqrt{k^2-\left(\frac{2\pi}{a_3}\right)^2}-\sqrt{k^2-\left(\frac{2\pi}{a_1}\right)z}\right)}\right.
\end{aligned}
$$

To get a field that repeats itself after distance L_T we now obtain two conditions:

$$
\left[\sqrt{k^2-\left(\frac{2\pi}{a_1}\right)^2}-\sqrt{k^2-\left(\frac{2\pi}{a_2}\right)^2}\right]L_T=2\pi m_1
$$

and

$$
\left[\sqrt{k^2-\left(\frac{2\pi}{a_1}\right)^2}-\sqrt{k^2-\left(\frac{2\pi}{a_3}\right)^2}\right]L_T=2\pi m_2
$$

for some INTEGER m_1 and m_2.

So in order to have L_T that will satisfy both of these equations simultaneously we need the following relation to be true:

$$
\frac{\sqrt{k^2-\left(\frac{2\pi}{a_1}\right)^2}-\sqrt{k^2-\left(\frac{2\pi}{a_2}\right)^2}}{\sqrt{k^2-\left(\frac{2\pi}{a_1}\right)^2}-\sqrt{k^2-\left(\frac{2\pi}{a_3}\right)^2}}=\frac{m_1}{m_2}
$$

for some INTEGER m_1 and m_2. This equation can be rewritten in the form:

$$
\frac{\sqrt{\left(\frac{1}{\lambda}\right)^2-\left(\frac{1}{a_1}\right)^2}-\sqrt{\left(\frac{1}{\lambda}\right)^2-\left(\frac{1}{a_2}\right)^2}}{\sqrt{\left(\frac{1}{\lambda}\right)^2-\left(\frac{1}{a_1}\right)^2}-\sqrt{\left(\frac{1}{\lambda}\right)^2-\left(\frac{1}{a_3}\right)^2}}=\frac{m_1}{m_2}.
$$