Friedrich Schiller University Jena
Institute of Applied Physics
Prof. Dr. Thomas Pertsch
Anastasia Romashkina, Shreyas Ramakrishna, Mostafa Abasifard, Tina (Shiu Hei Lam), Xiao Chen

WS 2022/2023

Series 9
Fundamentals of Modern Optics

to be returned on **12.01.2023**, at the beginning of the lecture

Task 1: Fraunhofer diffraction (3+1+2 points)

a) Calculate the intensity of the diffracted monochromatic (with the wavelength λ) field pattern $I(x, z_B) = |u(x, z_B)|^2$ in paraxial Fraunhofer approximation for two slits illuminated with a normally incident plane wave (prefactors are not important, the functional dependencies are important). The width of each slit is $2a$ and they are separated by a distance d ($d >> 2a$):

$$u_0(x, z=0) = \begin{cases} 1, & \text{for } |x \pm d/2| \leq a \\ 0, & \text{elsewhere.} \end{cases}$$

b) What conditions should the parameters of the initial field satisfy, for the paraxial Fraunhofer approximation to be valid?

c) Try to roughly sketch the shape of the intensity distribution, and explain how parameters a and d influence the main features of the intensity distribution.

Hint: The Fourier transform of a single slit of width $2a$ *is* $\propto \operatorname{sinc}(\alpha a)$.

Task 2: Fourier transform of gratings (3+3 points)

a) A finite periodic one-dimensional grating, with period D, has N illuminated periods, so that the transmission function of the whole grating is given by

$$t(x) = \sum_{l=0}^{N-1} \tilde{f}(x - lD),$$

where $\tilde{f}(x)$ is the grating function, which is only nonzero in the range $0 \leq x < D$.
Prove that the spatial spectrum is given by

$$T(\alpha) = \tilde{F}(\alpha) \frac{\sin(N\alpha D/2)}{\sin(\alpha D/2)} e^{i(1-N)\alpha D/2},$$

where $\tilde{F}(\alpha)$ is the Fourier transform of $\tilde{f}(x)$.

Hint: Make use of the Fourier shifting theorem.

b) Now consider an infinitely extended grating, with the transmission function:

$$t(x) = \sum_{l=-\infty}^{+\infty} \tilde{f}(x - lD).$$

Prove that the spatial spectrum is given by

$$T(\alpha) = \tilde{F}(\alpha) \frac{2\pi}{D} \sum_{n=-\infty}^{\infty} \delta\left(\alpha - \frac{2\pi n}{D}\right).$$

Hint: Make use of the fact that an infinitely extended periodic function has a Fourier series expansion.

Task 3: Fraunhofer diffraction by multiple holes (3+2+2 points)

Calculate the diffraction pattern in Fraunhofer approximation for:

a) A pinhole with radius a.
Hint: Use polar coordinates for $\mathbf{k}$ and $\mathbf{r}$ to solve the Fourier transform, which in polar coordinates looks like

$$U_0(\rho_k, \varphi_k) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^a e^{-i\rho_k \rho \cos(\varphi - \varphi_k)} \rho \, d\rho \, d\varphi.$$

b) A ring-shaped aperture bounded by two circles of radius a_1 and a_2 with $a_2 > a_1$.
c) A sequence of N pinholes with radius a placed along the x-axis with distances of $b > 2a$.

Useful formulas are:

$$\frac{\mathrm{i}^{-n}}{2\pi}\int_0^{2\pi} \exp(\mathrm{i}x\cos\alpha)\exp(\mathrm{i}n\alpha)\,\mathrm{d}\alpha = J_n(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{n+1}J_{n+1}(x)\right] = x^{n+1}J_n(x)$$

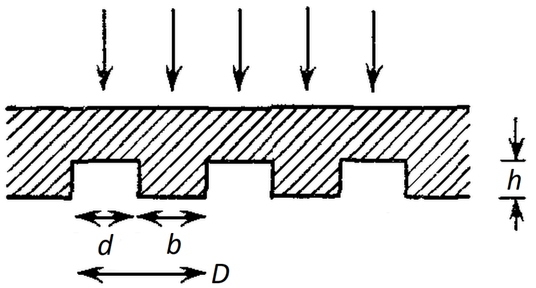
where J_i are the Bessel functions of first kind.

Task 4: Finite grating with step phase profile (3+2+2 points)

a) Consider that we have a periodic one-dimensional phase grating with the step profile as shown in the figure with N illuminated periods. Assume that the refractive index of the material of the grating is n. We can treat the grating as a phase mask with $\tilde{f}(x) = \exp(\mathrm{i}k_0 hn(x))$ within $[0, D = d + b)$, with $k_0 = 2\pi/\lambda$, where:

$$n(x) = \begin{cases} 1 & \text{for} \quad 0 \le x \le d \\ n & \text{for} \quad d < x < d + b \end{cases}$$

Calculate the intensity of the diffraction pattern in the paraxial Fraunhofer approximation using the result of Task 3.



b) Find the field amplitudes of the zeroth and first order diffraction peaks, which appear at $x_0 = 0$ and $x_1 = \frac{\lambda z}{D}$ respectively.
c) Find the values of ridge heights h_0 and h_1 that maximize the amplitudes of zeroth and first order diffraction peaks, respectively.