

Task 1

Solution:

$$\begin{aligned}
 h_F(x, y, z) &= \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} H_F(\alpha, \beta; z) \exp[i(\alpha x + \beta y)] d\alpha d\beta \\
 &= \frac{1}{4\pi^2} \iint_{-\infty}^{+\infty} \exp[ikz] \exp[-i \frac{\alpha^2 + \beta^2}{2k} z] \exp[i(\alpha x + \beta y)] d\alpha d\beta \\
 &= \frac{e^{ikz}}{4\pi^2} \iint_{-\infty}^{+\infty} e^{-i \frac{\alpha^2}{2k} z + i\alpha x} \cdot e^{-i \frac{\beta^2}{2k} z + i\beta y} d\alpha d\beta \\
 &= \frac{e^{ikz}}{4\pi^2} \int_{-\infty}^{+\infty} e^{-i \frac{\alpha^2}{2k} z + i\alpha x} d\alpha \int_{-\infty}^{+\infty} e^{-i \frac{\beta^2}{2k} z + i\beta y} d\beta \\
 &= \frac{e^{ikz}}{4\pi^2} \int_{-\infty}^{+\infty} e^{-i(\frac{\sqrt{2k}}{2k} \alpha - \frac{1}{2} \frac{\sqrt{2k}}{2k} x)^2} \cdot e^{i \frac{k}{2z} x^2} d\alpha \int_{-\infty}^{+\infty} e^{-i(\frac{\sqrt{2k}}{2k} \beta - \frac{1}{2} \frac{\sqrt{2k}}{2k} y)^2} \cdot e^{i \frac{k}{2z} y^2} d\beta \\
 &= \frac{e^{ikz}}{4\pi^2} e^{i \frac{k}{2z} (x^2 + y^2)} \cdot \left(\sqrt{\frac{2k}{z}}\right)^2 \cdot \left(\sqrt{\frac{\pi}{i}}\right)^2 \\
 &= -i \frac{k}{2\pi z} e^{ikz} e^{i \frac{k}{2z} (x^2 + y^2)}
 \end{aligned}$$

$$\therefore k = \frac{2\pi}{\lambda}$$

$$\therefore h_F(x, y, z) = -i \frac{1}{\lambda z} e^{ikz} e^{i \frac{k}{2z} (x^2 + y^2)}$$

Task 2

2+1

a)

Solution:

From Helmholtz' equation

We can get the expression of spherical wave ~~U(r) = \frac{A}{r} e^{ikr}~~

$$U(\vec{r}) = \frac{A}{r} e^{ikr}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore U(\vec{r}) = \frac{A}{\sqrt{x^2 + y^2 + z^2}} e^{ik\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{A}{z \sqrt{1 + \frac{x^2 + y^2}{z^2}}} e^{ikz \sqrt{1 + \frac{x^2 + y^2}{z^2}}}$$

In paraxial approximation $x^2 + y^2 \ll z^2$

$$U(\vec{r}) \xrightarrow{\text{Taylor expansion}} \frac{A}{z} \left(1 - \frac{x^2 + y^2}{2z^2} + \frac{-\frac{1}{2} \cdot (-\frac{1}{2}) [(x^2 + y^2)^2]}{2! z^4} + \dots\right) e^{ikz \left(1 + \frac{x^2 + y^2}{2z^2} + \frac{\frac{1}{2} \cdot (-\frac{1}{2}) (x^2 + y^2)^2}{2! z^4} + \dots\right)}$$

$$\text{let } x^2 + y^2 = \rho^2$$

$$\therefore U(\vec{r}) \approx \frac{A}{z} e^{ikz \left(1 + \frac{\rho^2}{2z^2} - \frac{\rho^4}{8z^4}\right)} = \frac{A}{z} e^{ikz \left(z + \frac{\rho^2}{2z} - \frac{\rho^4}{8z^3}\right)}$$

If the wavefront of a Gaussian beam is the same as a wavefront of the spherical wave

The phase part ~~is~~ must be the same

$$\therefore ik\left(z + \frac{x^2+y^2}{2R(z)}\right) = ik\left(z + \frac{x^2+y^2}{2z} - \frac{(x^2+y^2)^2}{8z^3}\right)$$

$$\frac{\rho^2}{2R(z)} = \frac{\rho^2}{2z} - \frac{\rho^4}{8z^3}$$

$$\frac{1}{2R(z)} = \frac{1}{2z} - \frac{\rho^2}{8z^3}$$

$$4z^3 = 4z^2 R(z) - \rho^2 R(z)$$

$$R(z) = \frac{4z^3}{4z^2 - \rho^2} = \frac{\cancel{4z^3}}{\cancel{4z^2} - \frac{\rho^2}{z^2}} = \frac{4z}{4 - \frac{\rho^2}{z^2}}$$

$$\therefore \rho^2 \ll z^2$$

$$\therefore R(z) \approx z$$

The condition is $R(z) \approx z$

b)

Solution:

When $w = 1.1w_0$

$$\therefore \begin{cases} w = w_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2} \\ z_0 = \frac{\pi}{\lambda} w_0^2 \end{cases}$$

$$\therefore z \approx 2.15 \times 10^4 \text{ m}$$

Answer is not right!

Task 3

a)

Solution:

The ABCD matrix of the lens is $\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$

$$q_0 = iZ_0 = i \frac{\pi w_0^2}{\lambda} = q_0 \quad (1)$$

after lens

$$q_L = \frac{Aq_0 + B}{Cq_0 + D} = \frac{q_0 + 0}{-1/f q_0 + 1}$$

$$\therefore \frac{1}{q_L} = \frac{1}{q_0} - \frac{1}{f} \quad (2)$$

At the waist w_1

$$q_1 = q_L + d \quad (3)$$

From these three equations, we can get:

$$\frac{1}{q_L} = \frac{1}{i \frac{\pi w_0^2}{\lambda}} - \frac{1}{f} = \frac{\lambda f - i \pi w_0^2}{i \pi w_0^2 f}$$

$$q_1 = \frac{i \pi w_0^2 f}{\lambda f - i \pi w_0^2} + d$$

$$= d - f \frac{(\frac{\pi w_0^2}{\lambda})^2}{f^2 + (\frac{\pi w_0^2}{\lambda})^2} + i \frac{f^2 (\frac{\pi w_0^2}{\lambda})}{f^2 + (\frac{\pi w_0^2}{\lambda})^2}$$

$\therefore q_1$ is the q -parameter of the waist ($\text{Re}[\frac{1}{q_1}] = 0$)

$$\therefore d - f \frac{(\frac{\pi w_0^2}{\lambda})^2}{f^2 + (\frac{\pi w_0^2}{\lambda})^2} = 0$$

$$q_1 = i \frac{f^2 (\frac{\pi w_0^2}{\lambda})}{f^2 + (\frac{\pi w_0^2}{\lambda})^2}$$

$$\therefore \frac{i}{w_1^2} = -\frac{\pi}{\lambda} \frac{1}{q_1} = -\frac{\pi}{\lambda} \frac{f^2 + (\frac{\pi w_0^2}{\lambda})^2}{f^2 (\frac{\pi w_0^2}{\lambda})} \cdot \frac{\pi}{\lambda}$$

$$\therefore \begin{cases} d = f \frac{(\frac{\pi w_0^2}{\lambda})^2}{f^2 + (\frac{\pi w_0^2}{\lambda})^2} \\ \frac{1}{w_1^2} = -\frac{\pi}{\lambda} \frac{f^2 + (\frac{\pi w_0^2}{\lambda})^2}{f^2 (\frac{\pi w_0^2}{\lambda})} \end{cases} \quad \because \frac{\pi w_0^2}{\lambda} \gg f \quad \therefore d \approx f$$

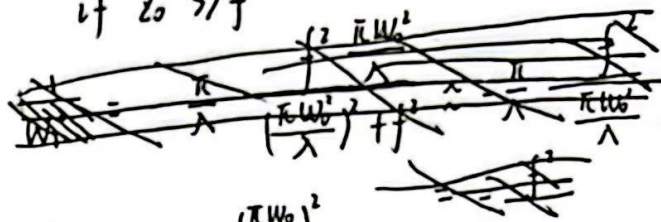
$$\frac{1}{w_1^2} = -\frac{\pi}{\lambda} \frac{f^2 + (\frac{\pi w_0^2}{\lambda})^2}{f^2 (\frac{\pi w_0^2}{\lambda})} = \frac{1}{w_0^2} + \frac{1}{f^2} \left(\frac{\pi w_0^2}{\lambda} \right)^2 \quad \because \frac{\pi w_0^2}{\lambda} \gg f$$

$$\therefore \frac{1}{w_1^2} \approx \frac{1}{w_0^2} + \frac{1}{f^2}$$

b)

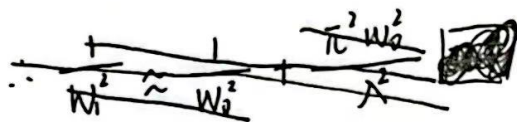
Solution:

if $Z_0 \gg f$



$$\frac{1}{w_1^2} = \frac{1}{w_0^2} + \frac{(\frac{\pi w_0}{\lambda})^2}{f^2}$$

$$\therefore \frac{\pi w_0^2}{\lambda} \gg f$$



b) (0)

a) (3.5)

approx w_1 ?
(-0.5)