Lesson 5: Uncertainty principle

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Uncertainty (or **indeterminacy**) of an observable $\hat{Q}$ is related to fluctuations or dispersion around $< \hat{Q} >$

Uncertainty

$$\Delta \hat{Q}_\Psi \; = \; \sqrt{< \hat{Q}^2 >_\Psi - < \hat{Q} >_\Psi^2} \; = \; \sqrt{< (\hat{Q} - < \hat{Q} >_\Psi)^2 >_\Psi}$$

- If Ψ is eigenstate of $\hat{Q} \; \rightarrow \; \hat{Q}\Psi \; = \; q \, \Psi$

$$< \hat{Q} >_\Psi \; = \; q \; ; \; < \hat{Q} >_\Psi^2 \; = \; q^2 \; \rightarrow \; \Delta \hat{Q}_\Psi \; = \; 0$$

- Uncertainties of a pair of Hermitian operators $\hat{P}$ and $\hat{Q}$ are correlated by uncertainty (or indeterminacy) relations

$$\left(\Delta \hat{Q}_\Psi\right)^2 \; \left(\Delta \hat{P}_\Psi\right)^2 \geq \frac{1}{4} < i[\hat{Q}, \hat{P}] >_\Psi^2$$

$\Downarrow$ **Heisenberg uncertainty principle**

- $[\hat{x}, \hat{p}_x] \; = \; i\hbar \; \rightarrow \; \Delta \hat{x} \Delta \hat{p}_x \; \geq \frac{\hbar}{2}$

- $[\hat{H}, t] \; = \; i\hbar \; \rightarrow \; \Delta \hat{H} \Delta t \; \geq \frac{\hbar}{2}$

Δt characteristic time of the system

The two previous inequalities are the ones of Heisenberg

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- $\Delta \hat{x} = 0 \rightarrow \Delta \hat{p}_x = \infty$ Confinement in space $\rightarrow$ indeterminacy of associated momentum and vice versa

- $\Delta \hat{p}_x = 0$ (plane wave) $\rightarrow \Delta \hat{x} = \infty$

$$\hat{p}_x \left(e^{i(k_x x - \omega t)} \right) = \hbar k_x \, e^{i(k_x x - \omega t)}$$

  eigenstate of $\hat{p}_x \rightarrow \Delta \hat{p}_x = 0$

- Wave-particle duality $\rightarrow$ leaves the idea of having position and associated momentum accurately defined simultaneously

Implications and applications of the uncertainty principle

- 1) **Estimate** of radius and energy of the ground state of a hydrogen atom (p-e$^-$) μ is the reduced mass of the system ; r is the p-e$^-$ distance

  $E = \frac{p^2}{2\mu} - \frac{e^2}{r}$ (using the Gaussian system of units,where $4\pi \, \epsilon_0 = 1$)

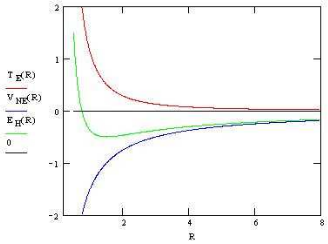
$$\Delta r \, \Delta p \sim \hbar$$

$\Downarrow$ approximation

$$r \, p = \hbar \rightarrow p = \frac{\hbar}{r} \rightarrow E = \frac{\hbar^2}{2\mu r^2} - \frac{e^2}{r}$$

$r \rightarrow \infty \implies E \rightarrow 0$

bound system $\implies E < 0$



We seek $r = r_0$ that minimizes E

$$\left.\frac{dE}{dr}\right|_{r_0} = -\frac{\hbar^2}{\mu r_0^3} + \frac{e^2}{r_0^2} = 0$$

$r_0 = \frac{\hbar^2}{\mu e^2}$ **Bohr radius** (most probable radius in the G.S.)

$r_0 \approx 0.5$ Å

- cont. 1)

$$E_{min} = \frac{\hbar^2}{2\mu}\frac{\mu^2 e^4}{\hbar^4} - e^2\frac{\mu e^2}{\hbar^2} = -\frac{\mu e^4}{2\hbar^2} = -13.6 \text{ eV} \rightarrow$$

G.S. energy of the hydrogen atom

  - atoms do not collapse $\rightarrow$ $E_c + V +$ uncertainty relations They collapse clasically; r for minimum energy is 0 and binding energy is ∞
  - classical orbits are incompatible with the wave theory $\Delta r \sim r$

Fine-structure constant: $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} = \frac{1}{137}$

In gaussian system of units $e^2 = \frac{\hbar c}{137}$

- 2) Nuclear radius $r \sim 1.4 \times 10^{-13}$ cm $= 1.4$ fm

$$p \sim \frac{\hbar}{r} \rightarrow \frac{E_{cin}}{\text{nucleon}} \approx \frac{1}{2M_p}\left(\frac{\hbar}{r}\right)^2 \approx 10 \text{ MeV / nucleon}$$

Nucleons are bound in the nucleus $\rightarrow E_T < 0$

$$| < U > | \geq 10 \text{ MeV} / \text{nucleon}$$

(coincides with known data)

10 / 24

- 3) Determination of the mass of the **pion**

Yukawa $\rightarrow$ nuclear (strong) force $\rightarrow$ pions exchange between nucleons

While there is the pion energy is not conserved

$$\Delta E \sim \mu c^2$$

The uncertainty principle allows non-conservation of energy in time smaller than

$$\Delta t \approx \frac{\hbar}{\Delta E} \approx \frac{\hbar}{\mu c^2}$$

it is not detectable the violation of the law of conservation of energy. It can not be measured at a time with greater precision than ΔE

11 / 24

Range of the particle assuming $v = v_{max} = c$

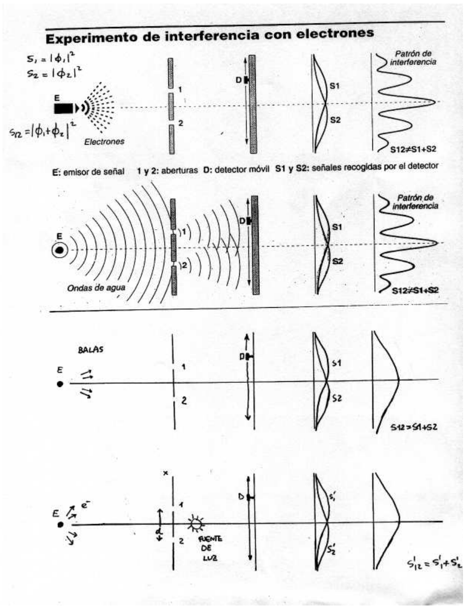
$$c\,\Delta t \approx \frac{\hbar c}{\mu c^2} = r_0 = 1.4\,fm$$

(known range of the nuclear force)

$$\mu c^2 = \frac{\hbar c}{r_0} = 141 \text{ MeV} \quad (m_\pi \approx 141 \text{ MeV/c}^2)$$ 12 / 24

The two-slit experiment. Principle of complementarity

13 / 24



14 / 24

Any attempt to **observe** which of the two slits the electron passes through results in the **disappearance of the interference pattern**. The electron is perturbed by the act of observation.

If we do not try to “see” which of the two slits the electron passes through $\rightarrow$ appearance of the interference pattern

Because of the uncertainty principle

- 1) Particle-like behaviour $\Longrightarrow$ knowing which of the two slits the electron passes through

  $\Delta x < d$
- 2) Wave-like behaviour (interference)

  $\theta' \rightarrow$ outgoing angle for the e$^-$

  $\Delta\theta' <$ angular distance between consecutive maximum and minimum

From central maximum to first minimum $\theta \approx \sin\theta = \frac{\lambda}{2d}$

$\Delta\theta' \sim \Delta\sin\theta' << \frac{\lambda}{2d}$

$\frac{\Delta p_x}{p} << \frac{\lambda}{2d} = \frac{h}{2dp}$

$\Delta p_x << \frac{h}{2d}$

1) and 2) lead to $\Delta x \Delta p_x << \frac{h}{2}$ contrary to the uncertainty principle

Bohr’s principle of complementarity $\rightarrow$ The wave and particle models are complementary. If a measurement proves the **wave character** of radiation or matter , then it is impossible to prove the **particle character** in the same measurement, and conversely

- Which model to use? $\longrightarrow$ determined by the nature of the experiment
- Knowledge of radiation or matter (classical) is incomplete without measurements that reveal aspects of wave and particle
- We can not make measurements more precise than the uncertainty principle allows ($\neq$ classically)
- Measurements disturb “now” the system being observed, but this perturbation can be calculated and taken into account
- When trying to accurately measure a variable belonging to a pair of canonical variables, the other changes in an amount that does not allow to be measured very accurately without interfering with the first trial

17 / 24

Demonstration that $\Delta A \Delta B \; \geq \; \frac{1}{2} \, | < [A,B] > |$

18 / 24

If A and B are Hermitian operators

$A \; = \; A^{\dagger} \;\; ; \;\; B \; = \; B^{\dagger}$

$[A,B]^{\dagger} \; = \; (AB - BA)^{\dagger} \; = \; B^{\dagger}A^{\dagger} \; - \; A^{\dagger}B^{\dagger} \; =$
$BA \; - \; AB \; = \; -[A,B]$

$(i[A,B])^{\dagger} \; = \; -i[A,B]^{\dagger} \; = \; i[A,B] \; = \; M \; \rightarrow$ Hermitian

We define

- $\widetilde{A} \; = \; A - < A >$

  $\widetilde{A}^{\dagger} \; = \; A^{\dagger} - < A >^{*} \; = \; A - < A > \; = \; \widetilde{A}$

  $< \widetilde{A}^2 > \; = \; \langle A^2 - 2 < A > A \; + \; < A >^2 \rangle \; =$

  $< A^2 > -2 < A >^2 + < A >^2 = < A^2 > - < A >^2 = (\Delta A)^2$

19 / 24

- $\widetilde{B} \;=\; B- < B >$ $\qquad\qquad < \widetilde{B}^2 > \;=\; (\Delta B)^2$
- $[\widetilde{A}, \widetilde{B}] \;=\; [A- < A >, B- < B >] \;=\; [A, B] \;=\; \frac{M}{i}$

we choose

$\Psi \;=\; (\alpha\widetilde{A} \;+\; i\widetilde{B})\phi$ with $\alpha \in \mathcal{R}$

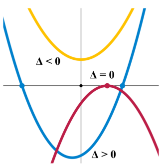
$\int d\tau|\Psi|^2 \geq 0$ and real $\forall\alpha$

$$\begin{aligned}
\int d\tau|(\alpha\widetilde{A} + i\widetilde{B})\phi|^2 &= \int d\tau \left[(\alpha\widetilde{A} \;+\; i\widetilde{B})\phi\right]^* (\alpha\widetilde{A} \;+\; i\widetilde{B})\phi \\
&= \int d\tau\phi^*(\alpha\widetilde{A} \;+\; i\widetilde{B})^\dagger(\alpha\widetilde{A} \;+\; i\widetilde{B})\phi \\
&= \int d\tau\phi^*(\alpha\widetilde{A} \;-\; i\widetilde{B})(\alpha\widetilde{A} \;+\; i\widetilde{B})\phi \\
&= \int d\tau\phi^*(\alpha^2\widetilde{A}^2 + i\alpha\widetilde{A}\widetilde{B} - i\alpha\widetilde{B}\widetilde{A} + \widetilde{B}^2)\phi
\end{aligned}$$

20 / 24

$$\begin{aligned}
&= \int d\tau\phi^*(\alpha^2\widetilde{A}^2 \;+\; i\alpha[\widetilde{A}, \widetilde{B}] \;+\; \widetilde{B}^2)\phi \\
&= \int d\tau\phi^*(\alpha^2\widetilde{A}^2 \;+\; i\alpha\frac{M}{i} \;+\; \widetilde{B}^2)\phi \\
&= \alpha^2 < \widetilde{A}^2 >_\phi \;+\; \alpha < M >_\phi \;+\; < \widetilde{B}^2 >_\phi \;\geq\; 0 \quad \forall\alpha
\end{aligned}$$

The discriminant of the quadratic equation in α must be $\Delta \leq 0$ so as not to have two real roots $\longrightarrow$ in this case the expression would take negative values for a range of values of α



21 / 24

$$\Delta \;=\; <M>^2 \;-\; 4 <\widetilde{A}^2><\widetilde{B}^2> \;\le\; 0$$

$$<M>^2 \;\le\; 4 <\widetilde{A}^2><\widetilde{B}^2>$$

$$<\widetilde{A}^2> \;=\; (\Delta A)^2 \quad ; \quad <\widetilde{B}^2> \;=\; (\Delta B)^2$$

$$(\Delta A)^2\,(\Delta B)^2 \ge \tfrac{1}{4} < i[A,B] >^2$$

$i[A,B]$ is Hermitian $\rightarrow$ $< i[A,B] >$ is real

$\downarrow$

$$\Delta A\,\Delta B \;\ge\; \tfrac{1}{2}|< i[A,B] >| \;=\; \tfrac{1}{2}|< [A,B] >|$$

22 / 24

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23 / 24

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