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Stigmatic aspherical refracting surfaces from Cartesian ovoids

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A theory for Descartes ovoids has been developed in terms of four form parameters, (GOTS). This theory allows the design of optical imaging systems that, in addition to a rigorous stigmatism, exhibit the property of aplanatism, necessary for the proper imaging of extended objects. As a decisive step for the production of these systems, in this work, we propose a formulation of Descartes ovoids in the form of standard aspherical surfaces (ISO 10110-12: 2019), by means of explicit formulas for the corresponding aspheric coefficients. Thus, with these results, the designs developed with Descartes ovoids are finally translated into the language of aspherical surfaces for their production, inheriting the aspherical surfaces of all optical properties of Cartesian surfaces. Consequently, these results make this optical design methodology viable for the development of technological solutions using the current optical fabrication capabilities of the industry.

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1. INTRODUCTION

Descartes ovoids or Cartesian surfaces are stigmatic surfaces in mathematical rigor, capable of forming a perfect point image, in geometrical optics, from a point object located on the optical axis. These surfaces can be used for the design of spherical-aberration-free systems [1–5]. There are different formulations of Cartesian surfaces in the literature [4–8], used to express these surfaces explicitly or parametrically. In [9] the relationship between the parameters of a particular formulation with aspherical coefficients of the standard ISO 10110-12:2007 is studied, and the design of aspherical lenses with minimal spherical aberration is obtained. Usually, aspherical lenses are obtained by applying optimization processes that result in the calculation of aspheric coefficients for aspherical surfaces in the standard ISO 10110-12:2019, but in this work, we present a different approach.

Recently, a new formulation of Cartesian surfaces has been proposed that allows expressing them through four form parameters, (GOTS) [5,8]. From this approach, spherical-aberration-free optical systems can be derived that, additionally, could fulfill an aplanatism condition [10,11]. In other words, this formulation allows us to design imaging optical systems with great practical interest given that, in theory, these systems are free of spherical aberration, with a correction to third-order of the coma. Therefore, the resulting systems could be diffraction limited for point images of certain regions of object space.

However, the production of these surfaces presents drawbacks of a technological nature, since the entire industry is focused on the production of standard conical and aspherical surfaces. Therefore, in this work, we have developed a formulation of the aspheric coefficients of the ISO 10110-12:2019 standard formulation as a function of the (GOTS) parameters of Cartesian surfaces, resulting in a complete aspheric optical design based on Descartes ovoids. Thus, Descartes ovoids make it possible to achieve high-performance systems free from spherical aberration, with coma corrected, which can finally be formulated in the form of aspherical surfaces for their production, allowing considering this procedure as a new methodology for optical design.

2. STIGMATIC SYSTEMS USING CARTESIAN SURFACES

Descartes ovoids are surfaces capable of forming a perfect point image from a point object. These surfaces arise from the condition that all rays that start from a point object travel the same optical path length until they converge to a perfect point image. Figure 1 shows the Cartesian surface Σ_k, with vertex at $z = \zeta_k$ and separating the media with refractive indices n_k and n_{k+1}, forming a point image at A' from a point object located at A. Points A and A' are known as a stigmatic pair, located at distances d_k and d_{k+1} and measured from the origin of coordinates $\mathcal{O}$.

As can be seen in [5], Cartesian surfaces can be implicitly expressed as

$$O_k G_k((z_k)_q - \zeta_k)^2 - 2(1 + S_k \rho_k^2)((z_k)_q - \zeta_k) + (O_k + T_k \rho_k^2)\rho_k^2 = 0, \quad (1)$$

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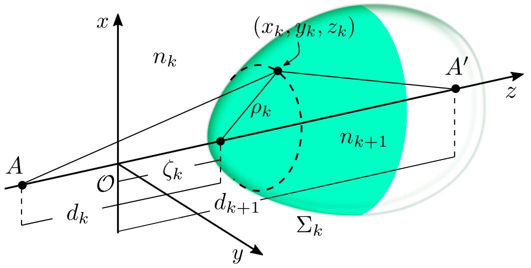

Fig. 1. Cartesian oval.

where $\rho_k = \sqrt{((z_k)_q - \zeta_k)^2 + r_k^2}$ is the vertex–surface distance, and $(z_k)_q$ and $r_k = \sqrt{x_k^2 + y_k^2}$ are axial and transversal coordinates of a Cartesian coordinate system, respectively. $(z_k)_q$ is the dependent variable, and its subscript q is related to Cartesian surfaces. The $(\mathrm{GOTS})_\mathrm{k}$ parameters are called form parameters and are written as

$$G_k = \frac{\left(\frac{n_{k+1}^2}{d_k - \zeta_k} - \frac{n_k^2}{d_{k+1} - \zeta_k}\right)^2}{n_k n_{k+1}\left(\frac{n_{k+1}}{d_{k+1} - \zeta_k} - \frac{n_k}{d_k - \zeta_k}\right)\left(\frac{n_{k+1}}{d_k - \zeta_k} - \frac{n_k}{d_{k+1} - \zeta_k}\right)}, \quad \text{(2)}$$

$$O_k = \frac{\frac{n_{k+1}}{d_{k+1} - \zeta_k} - \frac{n_k}{d_k - \zeta_k}}{n_{k+1} - n_k}, \quad \text{(3)}$$

$$T_k = \frac{\left(\frac{n_{k+1} + n_k}{(d_{k+1} - \zeta_k)(d_k - \zeta_k)}\right)^2 (n_{k+1} - n_k)}{4 n_{k+1} n_k \left(\frac{n_{k+1}}{d_k - \zeta_k} - \frac{n_k}{d_{k+1} - \zeta_k}\right)}, \quad \text{(4)}$$

$$S_k = \frac{\frac{(n_{k+1} + n_k)}{(d_{k+1} - \zeta_k)(d_k - \zeta_k)}\left(\frac{n_{k+1}^2}{d_k - \zeta_k} - \frac{n_k^2}{d_{k+1} - \zeta_k}\right)}{2 n_{k+1} n_k \left(\frac{n_{k+1}}{d_k - \zeta_k} - \frac{n_k}{d_{k+1} - \zeta_k}\right)}, \quad \text{(5)}$$

used to characterize this type of surface.

In addition to Expression (1), the coordinates of Cartesian surfaces can be parametrically expressed as [5,8]

$$(z_k)_q = \zeta_k + \frac{(O_k + T_k \rho_k^2)\rho_k^2}{1 + S_k \rho_k^2 + \sqrt{1 + (2S_k - O_k^2 G_k)\rho_k^2}}, \quad \text{(6)}$$

$$r_k = \pm\sqrt{\rho_k^2 - ((z_k)_q - \zeta_k)^2}. \quad \text{(7)}$$

It is worth mentioning that the form parameters, Expressions (2)–(5), are related by the constraint $S_k^2 = G_k O_k T_k$ [8], and their handling allows the design of optical systems with desired characteristics. Although the use of these parameters offers the advantage of achieving highly complex optical designs with specific properties, such as aplanatism, in the industry, there are no instruments adapted to this formulation for the manufacture of such surfaces. Therefore, we present here how these form parameters of Cartesian surfaces are related to the aspheric coefficients of aspherical surfaces written in the standard formulation ISO 10110-12: 2019.

3. ASPHERICAL COEFFICIENTS AS A FUNCTION OF SHAPE PARAMETERS

Since conic surfaces can be obtained from Cartesian surfaces, for the case of $S_k = T_k = 0$ [5], the implicit expression, Eq. (1), is reduced to

$$O_k(G_k + 1)((z_k)_c - \zeta_k)^2 - 2((z_k)_c - \zeta_k) + O_k r_k^2 = 0, \quad \text{(8)}$$

and the explicit expression, Eq. (6), is reduced to

$$(z_k)_c = \zeta_k + \frac{O_k r_k^2}{1 + \sqrt{1 - (G_k + 1) O_k^2 r_k^2}}, \quad \text{(9)}$$

where O_k is the axial curvature, and G_k corresponds to the conic constant. Subscript c indicates that it is related to conic surfaces. Considering this conic base, Expression (9), the standard aspherical surfaces are formulated as

$$(z_k)_a = \zeta_k + \frac{O_k r_k^2}{1 + \sqrt{1 - (G_k + 1) O_k^2 r_k^2}} + \sum_{m=2}^{\infty} [A_{2m}]_k r_k^{2m}, \quad \text{(10)}$$

where coefficients $[A_{2m}]_k$ are known as aspheric coefficients, which indicates the deviation of aspherical surfaces from conical surfaces (ISO 10110-12: 2019). Subscript a indicates that this is the z coordinate for an aspherical surface. If coefficients $[A_{2m}]_k$ are zero, Expression (10) is reduced to Expression (9).

Expanding the first term of Expression (10) using Taylor series (see Appendix A), that is, expanding Eq. (9), then Eq. (10) is written as follows:

$$\begin{aligned}(z_k)_a &= \zeta_k + \sum_{m=1}^{\infty} \frac{a_m O_k^{2m-1}(G_k + 1)^{m-1}}{(2m)!} r_k^{2m} + \sum_{m=2}^{\infty} [A_{2m}]_k r_k^{2m} \\ &= \zeta_k + \frac{a_1 O_k}{2} r_k^2 + \sum_{m=2}^{\infty} \left(\frac{a_m O_k^{2m-1}(G_k + 1)^{m-1}}{(2m)!} + [A_{2m}]_k\right) r_k^{2m}, \end{aligned} \quad \text{(11)}$$

with

$$a_m = (-1)^{m+1} \binom{1/2}{m} (2m)!. \quad \text{(12)}$$

Similar to conical surfaces, Cartesian surfaces expressed by Eq. (6) can be written as a power series. This equation can also be rewritten using Taylor series expansion as

$$(z_k)_q = \zeta_k + \sum_{m=0}^{\infty} \frac{1}{(2m)!} \left.\frac{d^m (z_k)_q}{dr_k^m}\right|_{r_k=0} r_k^m. \quad \text{(13)}$$

Developing the derivatives of $(z_k)_q$ concerning r_k, by using the implicit expression of Cartesian surfaces, Eq. (1), the following expression in power series of r_k is obtained (see Appendix B):

$$(z_k)_q = \zeta_k + \sum_{m=1}^{\infty} \frac{1}{(2m)!} \left(a_m O_k^{2m-1}(G_k + 1)^{m-1} + [B_m]_k\right) r_k^{2m}, \quad \text{(14)}$$

where a_m are coefficients given by Eq. (12). The first term of this expression, found within the parentheses, is associated with conic surfaces, as can be seen in Eq. (11), given as a function of

O_k and G_k. The second term, $[B_m]_k$ (see Appendix B), which contains the contribution of the other form parameters (S_k and T_k), measures the deviation that exists between Cartesian surfaces and conics.

From here, we can write Eq. (14) as

$$(z_k)_q = \zeta_k + \frac{a_1 O_k}{2} r_k^2 + \sum_{m=2}^{\infty} \left(\frac{a_m O_k^{2m-1} (G_k + 1)^{m-1}}{(2m)!} + \frac{[B_m]_k}{(2m)!} \right) r_k^{2m}. \tag{15}$$

If this expression is compared with Expression (11), it can be seen that the aspheric coefficients are given by the following expression:

$$[A_{2m}]_k = \frac{[B_m]_k}{(2m)!}. \tag{16}$$

Thus, aspheric coefficients are determined by expansion coefficients $[B_m]_k$, resulting in aspherical surfaces that inherit all optical properties of the Cartesian surfaces' design. Therefore, here we consider that any other scheme to achieve optical systems composed of aspherical surfaces, after an exhaustive optimization process, should converge to the solutions proposed here based on Cartesian surfaces.

In Fig. 2 are shown several curves associated with the difference between Expression (6), the Cartesian surface expression, and Expression (10), the aspherical expression with aspherical coefficients given by Expression (16). The legend in the figure indicates the maximum degree of approximation taken into account. It can be observed that as fewer coefficients are taken, the aspherical formulation is less approximated. It is also observed that after taking at least three coefficients ($m = 4$ or 8th degree), with a maximum value of the transversal coordinate of 10 mm, the resulting approximation is improved. As long as surfaces with larger diameters are taken into account, a less approximation of Expression (10) at the periphery of the surface is obtained. Therefore, the number of coefficients taken into account in the approximation is linked with the maximum value of the diameter of the surface, which means that there exists a maximum value of the diameter where the aspherical approximation fits the Cartesian surface. Hence, this characteristic of the aspherical formulation is relevant to consider when replacing a Cartesian surface with an aspherical one. Thus, using the standard aspherical formulation (ISO 10110-12: 2019) can lead to a fast convergence to Cartesian surfaces.

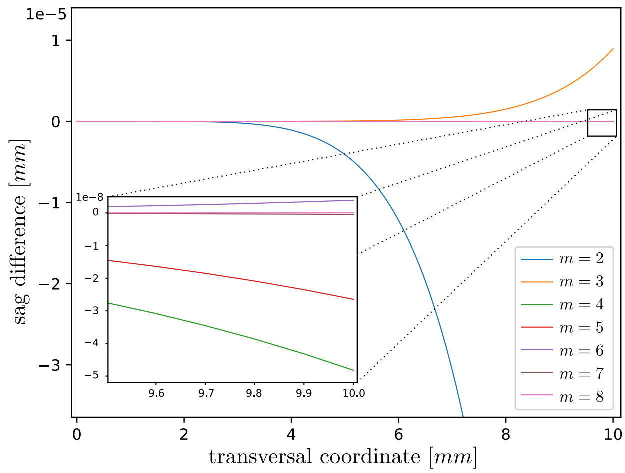

Fig. 2. Difference between the value of the sag for the Cartesian oval and the sag resulting from the approximation using aspherical formulation. The legend of the figure indicates the maximum degree of approximation used in the aspherical formulation.

4. EXAMPLES

In the design of an aplanatic singlet, using the methods developed in [5,10], composed of Cartesian surfaces, the form parameters shown in the Table 1 are obtained. The distance between the first surface of the singlet and the object's axial position is 800 mm, and the object's maximum height is 200 mm. The thickness of the singlet is 1.3 mm, the distance between the second surface of the singlet and the aperture stop is 1.68 mm, and the aperture radius is 0.2 mm. The resulting simulation considering these Cartesian surfaces is shown in Fig. 3. The corresponding asphericity coefficients, calculated from the methods developed in this work, are in Table 2. The simulation considering aspherical surfaces, carried out using OSLO EDU software, is shown in Figs. 4 and 5.

As shown in Figs. 3 and 4, the behavior of the system formed by aspherical surfaces is in practice the same as that of the system formed by Cartesian surfaces, the differences of which depend on the number of asphericity coefficients used. This makes it possible to make use of the ease offered by Cartesian surfaces to achieve designs of optical imaging systems with high performance, since these can be free from spherical aberration, with coma corrected.

The size of the spot diagram of an optical system depends on a series of factors, such as the diameter of the aperture, the transversal distance of the object position, and astigmatism. The optical system shown in the previous example can be diffraction limited for a smaller object size or a reduced diameter of the aperture stop. But due to limitations of the free license of OSLO EDU, it was not possible to use a higher degree of approximation. The higher the degree of asphericity, the results can only get closer to those shown in Fig. 3, whose surfaces are formulated by the exact expression of Cartesian ovoids.

Table 1. Parameters for a Singlet Composed of Cartesian Surfaces

k	ζ_k	G_k	$1/O_k$	T_k	S_k	n_k	n_{k+1}
0	0	−0.301543	2.654885	−2.481000e-7	−0.000168	AIR	N-LASF9
1	1.3	8.033421	6.777474	0.012278	0.120639	N-LASF9	AIR

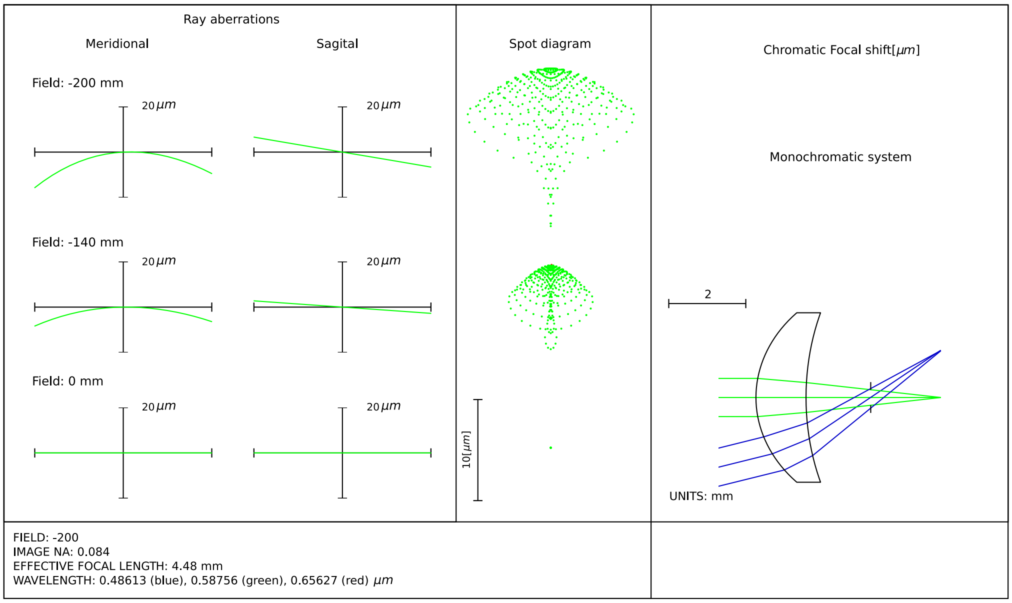

Fig. 3. Aplanatic singlet lens composed of Cartesian surfaces. The figure shows the behavior of this lens for point objects on and off axis. A good performance for forming extended images can be seen.

Table 2. Aspheric Coefficients for an Aplanatic Singlet

k	ζ_k	$1/O_k$	G_k	A_4	A_6	A_8	A_{10}	n_k	n_{k+1}
0	0	2.654	−0.301	3.149e-5	3.461e-6	3.329e-7	3.139e-8	AIR	N-LASF9
1	1.3	6.777	8.033	−2.761e-3	−3.576e-4	−4.327e-5	−6.050e-6	N-LASF9	AIR

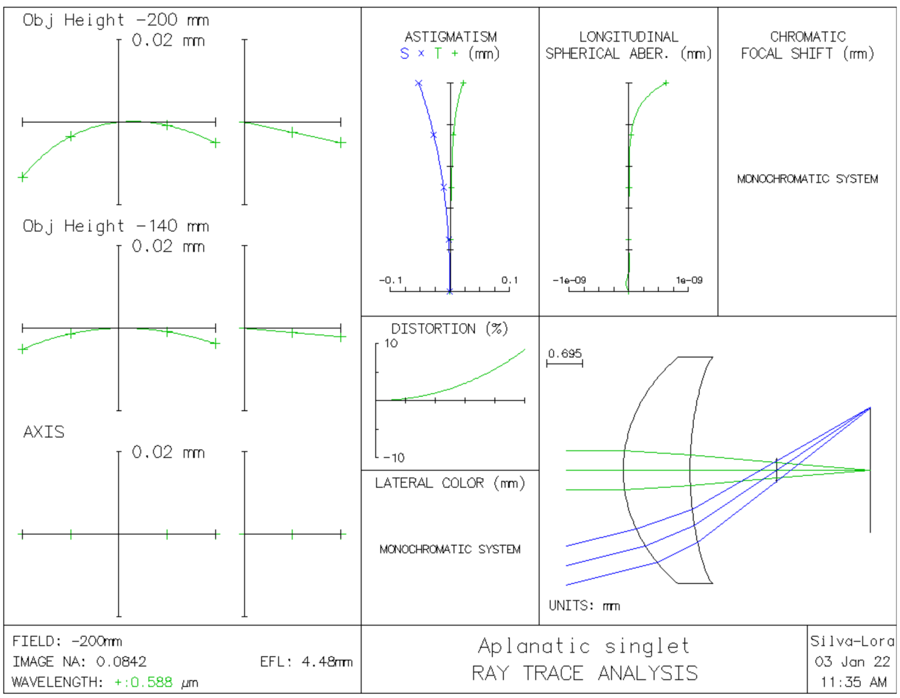

Fig. 4. Aplanatic singlet lens composed of aspherical surfaces. The figure shows the behavior of this lens for point objects on and off axis. It is a performance similar to that obtained when using the Cartesian surface. The higher order of approximation used in the simulation for both surfaces was the maximum allowed by the software, 10th order or $m = 5$.

Fig. 5. Spot diagram produced by an aplanatic lens composed of aspherical surfaces.

5. CONCLUSIONS

As a conclusion, the exact expressions for aspheric coefficients have been developed in terms of the form parameters of Cartesian surfaces, Expression (16). This expression allows us to design optical imaging systems composed of Cartesian surfaces, to later be formulated by aspherical surfaces. These results allow to carry out the polishing of these surfaces employing traditional methods of polishing aspherical surfaces, which paves the way for the industrial development of applications of Cartesian systems. With these results, we can affirm that the Cartesian surfaces' design is essentially a design with aspherical surfaces, with the advantage that the Cartesian surfaces have well-established optical conjugation rules, and these surfaces are characterized by only three form parameters that substantially simplify the manipulation of the optical properties of these systems. Thus, the optical properties of these Cartesian systems, such as stigmatism and aplanatism, are inherited by the translated system into the formulation of aspherical surfaces, which makes Cartesian optical design a new paradigm.

APPENDIX A: POWER SERIES EXPANSION FOR ASPHERICAL SURFACES

Aspherical surfaces, written in the ISO 10110-12: 2019 standard formulation, are formulated from a conical base followed by a series of even powers from the fourth-degree power onward, as can be seen in Expression (10). This expression can be written as a series expansion if the first term, associated with the conical base, is expanded in terms of r_k.

Expanding Eq. (9) in terms of r_k gives

$$(z_k)_c = \zeta_k + \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n (z_k)_c}{dr_k^n}\right|_{r_k=0} r_k^n, \tag{A1}$$

where the expression of the derivatives of the conics, for $r_k = 0$, can be obtained by differentiating Expression (8), from which it follows that

$$\left.\frac{d(z_k)_c}{dr_k}\right|_{r_k=0} = -\left.\frac{O_k r_k}{O_k(G_k+1)[(z_k)_c - \zeta_k] - 1}\right|_{r_k=0} = 0, \tag{A2}$$

$$\left.\frac{d^2(z_k)_c}{dr_k^2}\right|_{r_k=0} = -\left.\frac{O_k\left[(G_k+1)\left(\frac{d(z_k)_c}{dr_k}\right)^2 + 1\right]}{O_k(G_k+1)[(z_k)_c - \zeta_k] - 1}\right|_{r_k=0} = O_k, \tag{A3}$$

$$\left.\frac{d^3(z_k)_c}{dr_k^3}\right|_{r_k=0} = -\left.\frac{3O_k(G_k+1)\frac{d(z_k)_c}{dr_k}\frac{d^2(z_k)_c}{dr_k^2}}{O_k(G_k+1)[(z_k)_c - \zeta_k] - 1}\right|_{r_k=0} = 0, \tag{A4}$$

$$\begin{aligned}\left.\frac{d^4(z_k)_c}{dr_k^4}\right|_{r_k=0} &= -\left.\frac{O_k(G_k+1)\left[4\frac{d(z_k)_c}{dr_k}\frac{d^3(z_k)_c}{dr_k^3} + 3\left(\frac{d^2(z_k)_c}{dr_k^2}\right)^2\right]}{O_k(G_k+1)[(z_k)_c - \zeta_k] - 1}\right|_{r_k=0} \\ &= 3O_k^3(G_k+1),\end{aligned} \tag{A5}$$

$$\left.\frac{d^5(z_k)_c}{dr_k^5}\right|_{r_k=0} = -\left.\frac{O_k(G_k+1)\left[10\frac{d^2(z_k)_c}{dr_k^2}\frac{d^3(z_k)_c}{dr_k^3} + 5\frac{d(z_k)_c}{dr_k}\frac{d^4(z_k)_c}{dr_k^4}\right]}{O_k(G_k+1)[(z_k)_c - \zeta_k] - 1}\right|_{r_k=0} = 0, \tag{A6}$$

$$\begin{aligned}\left.\frac{d^6(z_k)_c}{dr_k^6}\right|_{r_k=0} &= -\left.\frac{O_k(G_k+1)\left[15\frac{d^2(z_k)_c}{dr_k^2}\frac{d^4(z_k)_c}{dr_k^4} + 6\frac{d(z_k)_c}{dr_k}\frac{d^5(z_k)_c}{dr_k^5} + 10\left(\frac{d^3(z_k)_c}{dr_k^3}\right)^2\right]}{O_k(G_k+1)[(z_k)_c - \zeta_k] - 1}\right|_{r_k=0} \\ &= 45\,O_k^4(G_k+1)^2 O_k^2,\end{aligned} \tag{A7}$$

and in this way, all the derivative expressions can be obtained. These expressions of the derivatives can be synthesized as follows:

$$\left.\frac{d^{2m}(z_k)_c}{dr_k^{2m}}\right|_{r_k=0} = a_m\, O_k^{2m-1}(G_k+1)^{m-1}, \tag{A8}$$

where

$$a_m = 4^{m-1}\frac{\Gamma\left(m-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} = (-1)^{m+1}\binom{1/2}{m}(2m)!, \tag{A9}$$

with Γ being the gamma function. Then, from Eq. (A8), we have that the expansion for conic surfaces can be written as

$$(z_k)_c = \zeta_k + \sum_{m=1}^{\infty} \frac{a_m O_k^{2m-1}(G_k+1)^{m-1}}{(2m)!} r_k^{2m}, \tag{A10}$$

and from this expansion, we can write the expression for aspherical surfaces, Eq. (10), as

$$(z_k)_a = \zeta_k + \sum_{m=1}^{\infty} \frac{a_m O_k^{2m-1}(G_k+1)^{m-1}}{(2m)!} r_k^{2m} + \sum_{m=2}^{\infty} [A_{2m}]_k r_k^{2m}$$
$$= \frac{a_1 O_k}{2} r_k^2 + \sum_{m=2}^{\infty} \left(\frac{a_m O_k^{2m-1}(G_k+1)^{m-1}}{(2m)!} + [A_{2m}]_k \right) r_k^{2m}. \tag{A11}$$

APPENDIX B: POWER SERIES EXPANSION FOR CARTESIAN SURFACES

In addition to aspherical surfaces, Cartesian surfaces also have their expression in power series, obtained from a Taylor expansion, where Eq. (6) can be expressed in terms of r_k as

$$(z_k)_q = \zeta_k + \sum_{n=0}^{\infty} \frac{1}{(2m)!} \frac{d^m z_k}{dr_k^m}\bigg|_{r_k=0} r_k^m, \tag{B1}$$

where the derivatives of $(z_k)_q$ as a function of r_k are obtained from Expression (1) in the same way as for the expansion of conic surfaces. These derivatives can be written as

$$\frac{d(z_k)_q}{dr_k}\bigg|_{r_k=0} = 0, \tag{B2}$$

$$\frac{d^2(z_k)_q}{dr_k^2}\bigg|_{r_k=0} = O_k, \tag{B3}$$

$$\frac{d^3(z_k)_q}{dr_k^3}\bigg|_{r_k=0} = 0, \tag{B4}$$

$$\frac{d^4(z_k)_q}{dr_k^4}\bigg|_{r_k=0} = 3O_k^3(G_k+1) - 12(O_k S_k - T_k), \tag{B5}$$

$$\frac{d^5(z_k)_q}{dr_k^5}\bigg|_{r_k=0} = 0, \tag{B6}$$

$$\frac{d^6(z_k)_q}{dr_k^6}\bigg|_{r_k=0} = 45O_k^5(G_k+1)^2 - 90(3G_k O_k^3 S_k - 2G_k O_k^2 T_k$$
$$+ 4O_k^3 S_k - 4O_k^2 T_k - 4O_k S_k^2 + 4S_k T_k), \tag{B7}$$

from which it is observed that these expressions of the derivatives are the expressions of the derivatives of the conics, Expression (24), plus an additional term that can be easily identified. Then the Cartesian surfaces can be written as follows:

$$(z_k)_q = \zeta_k + \sum_{m=1}^{\infty} \frac{1}{(2m)!} \left(a_m O_k^{2m-1}(G_k+1)^{m-1} + [B_m]_k \right) r_k^{2m}, \tag{B8}$$

where B_m, whose amount depends on the form parameters of the Cartesian surfaces, can be determined, and its first five terms are

$$[B_1]_k = 0, \tag{B9}$$

$$[B_2]_k = -12(O_k S_k - T_k), \tag{B10}$$

$$[B_3]_k = -90(3G_k O_k^3 S_k - 2G_k O_k^2 T_k + 4O_k^3 S_k$$
$$- 4O_k^2 T_k - 4O_k S_k^2 + 4S_k T_k), \tag{B11}$$

$$[B_4]_k = -1260(10G_k^2 O_k^5 S_k - 6G_k^2 O_k^4 T_k + 25G_k O_k^5 S_k$$
$$- 20G_k O_k^4 T_k - 24G_k O_k^3 S_k^2 + 24G_k O_k^2 S_k T_k$$
$$- 4G_k O_k T_k^2 + 15O_k^5 S_k - 15O_k^4 T_k - 40O_k^3 S_k^2$$
$$+ 60O_k^2 S_k T_k + 16O_k S_k^3 - 20O_k T_k^2 - 16S_k^2 T_k), \tag{B12}$$

$$[B_5]_k = -28350(35G_k^3 O_k^7 S_k - 20G_k^3 O_k^6 T_k + 126G_k^2 O_k^7 S_k$$
$$- 90G_k^2 O_k^6 T_k - 120G_k^2 O_k^5 S_k^2 + 120G_k^2 O_k^4 S_k T_k$$
$$- 24G_k^2 O_k^3 T_k^2 + 147G_k O_k^7 S_k - 126G_k O_k^6 T_k$$
$$- 360G_k O_k^5 S_k^2 + 480G_k O_k^4 S_k T_k + 160G_k O_k^3 S_k^3$$
$$- 144G_k O_k^3 T_k^2 - 192G_k O_k^2 S_k^2 T_k + 48G_k O_k S_k T_k^2$$
$$+ 56O_k^7 S_k - 56O_k^6 T_k - 252O_k^5 S_k^2 + 420O_k^4 S_k T_k$$
$$+ 320O_k^3 S_k^3 - 168O_k^3 T_k^2 - 576O_k^2 S_k^2 T_k - 64O_k S_k^4$$
$$+ 288O_k S_k T_k^2 + 64S_k^3 T_k - 32T_k^3). \tag{B13}$$

From this, we can write Eq. (B8) as

$$(z_k)_q = \frac{a_1 O_k}{2} r_k^2 + \sum_{m=2}^{\infty} \left(\frac{a_m O_k^{2m-1}(G_k+1)^{m-1}}{(2m)!} + \frac{[B_m]_k}{(2m)!} \right) r_k^{2m}, \tag{B14}$$

where we set $(z_k)_a = (z_k)_q$, which means that Expression (A11) is equal to Expression (B14), from which the coefficients $[A_{2m}]_k$ that best adapt to Cartesian surfaces can be obtained. Therefore, comparing these two expressions, it can be seen that

$$[A_{2m}]_k = \frac{[B_m]_k}{(2m)!}, \tag{B15}$$

whose expression allows to obtain the aspheric coefficients as a function of the form parameters of the Cartesian surfaces.

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Data availability. Data underlying the results presented in this paper are not publicly available at this time but may be obtained from the authors upon reasonable request.

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