# Homework 10

Wenye Xiong 2023533141

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## 1 Problem 1

(a): "Every student sends a message to some other student in DS class."
This can be translated into the following formula:
$\forall x(P(x) \rightarrow \exists y(P(y) \wedge \neg E(x, y) \wedge L(x, y)))$

(b) "There is a student who is sent a message by every other student in DS class."
This can be translated into the following formula:
$\exists x(P(x) \wedge \forall y(P(y) \wedge \neg E(y, x) \wedge L(y, x)))$

## 2 Problem 2

(a): If $D_1 = \emptyset$, then A is vacuously true because the function is in the form of universial quantification, there are no elements to falsify the formula.

(b): Let $D_2 = \{1, 2, 3\}$. We can choose $x = 1$ and $y = 2$. The formula $x \neq y$ is true, and $\forall z((z = x) \vee (z = y))$ must be true. However, letting $z = 3$ will make the formula false, thus falsifying A.

## 3 Problem 3

(a): $(\exists x P(x) \rightarrow \exists x Q(x)) \rightarrow \exists x(P(x) \rightarrow Q(x))$
This formula is logically valid. We can prove this by proving its substitue is a tautology.

$(\exists x P(x) \to \exists y Q(y)) \to \exists z(P(z) \to Q(z))$
$\equiv \neg(\exists x P(x) \to \exists y Q(y)) \vee \exists z(P(z) \to Q(z))$
$\equiv \neg(\neg\exists x P(x) \vee \exists y Q(y)) \vee \exists z(\neg P(z) \vee Q(z))$
$\equiv \exists x P(x) \wedge \neg\exists y Q(y) \vee \exists z(\neg P(z) \vee Q(z))$
$\equiv \exists x P(x) \wedge \forall y \neg Q(y) \vee \exists z(\neg P(z) \vee Q(z))$
$\equiv \exists x P(x) \wedge \forall y \neg Q(y) \vee \exists z \neg P(z) \vee \exists z Q(z)$
$\equiv \exists x P(x) \wedge \forall y \neg Q(y) \vee \exists z \neg P(z) \vee Q(z)$
$\equiv \exists x P(x) \wedge \forall y \neg Q(y) \vee \neg\forall z P(z) \vee Q(z)$
$\equiv (\exists x P(x) \vee \neg\forall z P(z) \vee \exists z Q(z)) \wedge (\forall y \neg Q(y) \vee \neg\forall z P(z) \vee \exists z Q(z))$
$\equiv (\exists x True \vee \exists z Q(z)) \wedge (True \vee \neg\forall z P(z))$
$\equiv True$
So the original formula is logically valid.

(b): $\forall x(P(x) \vee \neg\exists y(Q(y) \wedge \neg Q(y)))$
This formula is logically valid. The inner part of the formula $Q(y) \wedge \neg Q(y)$ is always false, so $\neg\exists y(Q(y) \wedge \neg Q(y))$ is always true. Thus, $P(x) \vee True$ is always true, and the formula is logically valid.

## 4 Problem 4

(a): Consider a domain with only two elements $D = \{a, b\}$. Let $P(a) = True$ and $P(b) = False$. Let $Q(a) = False$ and $Q(b) = True$. The formula $\forall x(P(x) \vee Q(x))$ is obviously true. However, the formula $\forall x P(x) \vee \forall x Q(x)$ is false, because $\forall x P(x)$ is false and $\forall x Q(x)$ is false. So the two formulas are not logically equivalent.

(b): Consider the same domain with only two elements $D = \{a, b\}$. Let $P(a) = True$ and $P(b) = False$. Let $Q(a) = False$ and $Q(b) = True$. The formula $\forall x(P(x) \wedge Q(x))$ is false. However, the formula $\forall x P(x) \wedge \forall x Q(x)$ is true, because $\forall x P(x)$ is true and $\forall x Q(x)$ is true. So the two formulas are not logically equivalent.

## 5 Problem 5

$\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$
To prove it, we have to prove $\exists x(P(x) \vee Q(x)) \to \exists x P(x) \vee \exists x Q(x)$ and

$\exists x P(x) \vee \exists x Q(x) \rightarrow \exists x(P(x) \vee Q(x))$

First we prove$\exists x(P(x) \vee Q(x)) \rightarrow \exists x P(x) \vee \exists x Q(x)$:
Assume $\exists x(P(x) \vee Q(x))$ is true. Then there exists an element $a$ in the domain such that $P(a) \vee Q(a)$ is true. If $P(a)$ is true, then $\exists x P(x)$ is true. If $Q(a)$ is true, then $\exists x Q(x)$ is true. So $\exists x P(x) \vee \exists x Q(x)$ is true.

Second we prove $\exists x P(x) \vee \exists x Q(x) \rightarrow \exists x(P(x) \vee Q(x))$:
Assume $\exists x P(x) \vee \exists x Q(x)$ is true. Then either $\exists x P(x)$ is true or $\exists x Q(x)$ is true. If $\exists x P(x)$ is true, then there exists an element $a$ in the domain such that $P(a)$ is true. So $P(a) \vee Q(a)$ is true, and $\exists x(P(x) \vee Q(x))$ is true. If $\exists x Q(x)$ is true, then there exists an element $b$ in the domain such that $Q(b)$ is true. So $P(b) \vee Q(b)$ is true, and $\exists x(P(x) \vee Q(x))$ is true.

So we have proved $\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$

# 6 Problem 6

$\forall x(P(x) \rightarrow Q(x)) \Rightarrow (\forall x P(x) \rightarrow \forall x Q(x))$
Assume $\forall x(P(x) \rightarrow Q(x))$ is true. Then for any element $a$ in the domain, $P(a) \rightarrow Q(a)$ is true. If $\forall x P(x)$ is true, then for any element $b$ in the domain, $P(b)$ is true. So $Q(b)$ is true, and $\forall x Q(x)$ is true. So $\forall x P(x) \rightarrow \forall x Q(x)$ is true.

# 7 Problem 7

Premises: $\forall x(F(x) \rightarrow (G(y) \wedge R(x)))$, $\exists x F(x)$.
Conclusion: $\exists x(F(x) \wedge R(x))$

Proof:
(1): $\forall x(F(x) \rightarrow (G(y) \wedge R(x)))$ Premise
(2): $\exists x F(x)$ Premise
(3): $F(a)$ Existential Instantiation of (2)
(4): $F(a) \rightarrow (G(y) \wedge R(a))$ Universal Instantiation of (1)
(5): $G(y) \wedge R(a)$ Modus Ponens of (3) and (4)

(6): $F(a) \wedge R(a)$ Conjunction of (3) and (5)
(7): $\exists x(F(x) \wedge R(x))$ Existential Generalization of (6)

## 8 Problem 8

Premises: $\forall x(F(x) \vee G(x))$, $\forall x(\neg G(x) \vee \neg R(x))$, $\forall x R(x)$
Conclusion: $\forall x F(x)$

Proof:
(1): $\forall x(F(x) \vee G(x))$ Premise
(2): $\forall x(\neg G(x) \vee \neg R(x))$ Premise
(3): $\forall x R(x)$ Premise
(4): $F(a) \vee G(a)$ Universal Instantiation of (1)
(5): $\neg G(a) \vee \neg R(a)$ Universal Instantiation of (2)
(6): $R(a)$ Universal Instantiation of (3)
(7): $\neg G(a)$ Disjunctive Syllogism of (5) and (6)
(8): $F(a)$ Disjunctive Syllogism of (4) and (7)
(9): $\forall x F(x)$ Universal Generalization of (8)

## 9 Problem 9

Premises: 1.All PhD students are hardworking. 2.Any person who is hardworking and smart will have a successful career. 3.Sam is a PhD student and smart.
Conclusion: Sam will have a successful career.

Let $P(x)$ be the proposition "x is a PhD student", $H(x)$ be the proposition "x is hardworking", $S(x)$ be the proposition "x is smart", and $C(x)$ be the proposition "x will have a successful career".

Building Arguments:
(1): $\forall x(P(x) \rightarrow H(x))$ Premise
(2): $\forall x((H(x) \wedge S(x)) \rightarrow C(x))$ Premise
(3): $P(Sam)$ Premise
(4): $S(Sam)$ Premise
(5): $P(Sam) \rightarrow H(Sam)$ Universal Instantiation of (1)

(6): $H(Sam)$ Modus Ponens of (3) and (5)
(7): $H(Sam) \wedge S(Sam)$ Conjunction of (6) and (4)
(8): $(H(Sam) \wedge S(Sam)) \rightarrow C(Sam)$ Universal Instantiation of (2)
(9): $C(Sam)$ Modus Ponens of (7) and (8)
So we have proved Sam will have a successful career.

## 10 Problem 10

Premises: 1.There doesn't exist an irrational number that can be expressed as a fraction. 2.All rational numbers can be expressed as fractions.
Conclusion: Rational numbers are not irrational numbers.

Let $R(x)$ be the proposition "x is a rational number", $I(x)$ be the proposition "x is an irrational number", and $F(x)$ be the proposition "x can be expressed as a fraction".

Building Arguments:
(1): $\neg\exists x(I(x) \wedge F(x))$ Premise
(2): $\forall x(R(x) \rightarrow F(x))$ Premise
(3): $\forall x(\neg(I(x) \wedge F(x)))$ De Morgan's Law of (1)
(4): $\forall x(\neg I(x) \vee \neg F(x))$ De Morgan's Law of (3)
(5): $\forall x(\neg R(x) \vee F(x))$ Equivalence of (2)
(6): $\forall x(\neg I(x) \vee \neg F(x)) \wedge (\neg R(x) \vee F(x))$ Conjunction of (4) and (5)
(7): $\forall x(\neg I(x) \vee \neg R(x))$ Resolution of (6)
(8): $\neg\exists x\neg(\neg I(x) \vee \neg R(x))$ De Morgan's Law of (7)
(9): $\neg\exists x(I(x) \wedge R(x))$ Double Negation of (8)
So we have proved Rational numbers are not irrational numbers.