Proposition: a **declarative** sentence that is either true or false. Well-formed formula =propositional formulas=formulas; recursive definition: propositional constants (T, F) and propositional variables are WFFs. If A is a WFF, then $\neg A$ is a WFF. If A and B are WFFs, then $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, and $(A \leftrightarrow B)$ are WFFs. WFFs are results of finitely many applications of these three rules.
Precedence(Priority) of Logical Operators: Formulas inside ( ) are computed firstly, Different connectives: $\neg \wedge \vee \rightarrow \leftrightarrow$ (Decreasing Precedence) Same connectives: from left to the right.
Types of WFFs: Tautology(T for all Truth Assignments), Contradiction(F for all TA), Contingency(neither tautology nor contradiction), Satifiable(T for at least one TA). substituting a propositional variable in tautology with any formula makes a tautology.
Logically Equivalent: always have the same truth value for every truth assignment. Theorem: $A \equiv B$ iff $A \leftrightarrow B$ is a tautology.

Name	Logical Equivalences	NO.
Double Negation Law 双重否定律	$\neg(\neg P) \equiv P$	1
Identity Laws 同一律	$P \wedge \mathbf{T} \equiv P$	2
	$P \vee \mathbf{F} \equiv P$	3
Idempotent Laws 等幂律	$P \vee P \equiv P$	4
	$P \wedge P \equiv P$	5
Domination Laws 零律	$P \vee \mathbf{T} \equiv \mathbf{T}$	6
	$P \wedge \mathbf{F} \equiv \mathbf{F}$	7
Negation Laws 补余律	$P \vee \neg P \equiv \mathbf{T}$	8
	$P \wedge \neg P \equiv \mathbf{F}$	9

Name	Logical Equivalences	NO.
Commutative Laws 交换律	$P \vee Q \equiv Q \vee P$	10
	$P \wedge Q \equiv Q \wedge P$	11
Associative Laws 结合律	$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$	12
	$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$	13
Distributive Laws 分配律	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	14
	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	15
De Morgan's Laws 摩根律	$\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$	16
	$\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$	17
Absorption Laws 吸收律	$P \vee (P \wedge Q) \equiv P$	18
	$P \wedge (P \vee Q) \equiv P$	19

Name	Logical Equivalences	NO.
Laws Involving Implication $\rightarrow$	$P \rightarrow Q \equiv \neg P \vee Q$	20
	$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$	21
	$(P \rightarrow R) \wedge (Q \rightarrow R) \equiv (P \vee Q) \rightarrow R$	22
	$P \rightarrow (Q \rightarrow R) \equiv (P \wedge Q) \rightarrow R$	23
	$P \rightarrow (Q \rightarrow R) \equiv Q \rightarrow (P \rightarrow R)$	24
Laws Involving Bi-Implication $\leftrightarrow$	$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$	25
	$P \leftrightarrow Q \equiv (\neg P \vee Q) \wedge (P \vee \neg Q)$	26
	$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$	27
	$P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$	28

Replacing a sub-formula in a formula F with a logically equivalent sub-formula gives a formula logically equivalent to the formula
Theorem: Let $A^{-1}(T)$ be the set of truth assignments such that A is true. Then $A \equiv B$ iff $A^{-1}(T) = B^{-1}(T)$.
Tautological Implications: Let A and B be WFFs in propositional variables $p_1, p_2, ..., p_n$. A tautologically implies B if every truth assignment that causes A to be true causes B to be true. Notation: $A \Rightarrow B$. If $A \Rightarrow B$, $A^{-1}(T) \subseteq B^{-1}(T)$; $B^{-1}(F) \subseteq A^{-1}(F)$.
Theorem: 1. $A \Rightarrow B$ iff $(A \wedge \neg B)$ is a contradiction. 2. $A \Rightarrow B$ iff $A \rightarrow B$ is a tautology.
Tautology Implication and Rules of Inferrence(relatively simple valid argument forms from tautological implications)

Name	Tautological Implication
Conjunction(合取)	$(P) \wedge (Q) \Rightarrow P \wedge Q$
Simplification(化简)	$P \wedge Q \Rightarrow P$
Addition(附加)	$P \Rightarrow P \vee Q$
Modus ponens(假言推理)	$P \wedge (P \rightarrow Q) \Rightarrow Q$
Modus tollens(拒取)	$\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$
Disjunctive syllogism(析取三段论)	$\neg P \wedge (P \vee Q) \Rightarrow Q$
Hypothetical syllogism(假言三段论)	$(P \rightarrow Q) \wedge (Q \rightarrow R) \Rightarrow (P \rightarrow R)$
Resolution (归结)	$(P \vee Q) \wedge (\neg P \vee R) \Rightarrow Q \vee R$

Rule of Inference	*Tautology*
p, $p \rightarrow q$, $\therefore q$	$(p \wedge (p \rightarrow q)) \rightarrow q$
$\neg q$, $p \rightarrow q$, $\therefore \neg p$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$
$p \rightarrow q$, $q \rightarrow r$, $\therefore p \rightarrow r$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$
$p \vee q$, $\neg p$, $\therefore q$	$((p \vee q) \wedge \neg p) \rightarrow q$
p, $\therefore p \vee q$	$p \rightarrow (p \vee q)$
$p \wedge q$, $\therefore p$	$(p \wedge q) \rightarrow p$
p, q, $\therefore p \wedge q$	$((p) \wedge (q)) \rightarrow (p \wedge q)$
$p \vee q$, $\neg p \vee r$, $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$

EXAMPLE: Show that $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S) \Rightarrow S \vee R$

(1)	$P \vee Q$	Premise
(2)	$\neg P \rightarrow Q$	Rule of replacement applied to (1)
(3)	$Q \rightarrow S$	Premise
(4)	$\neg P \rightarrow S$	Hypothetical syllogism applied to (2) and (3)
(5)	$\neg S \rightarrow P$	Rule of replacement applied to (4)
(6)	$P \rightarrow R$	Premise
(7)	$\neg S \rightarrow R$	Hypothetical syllogism applied to (5) and (6)
(8)	$S \vee R$	Rule of replacement applied to (7)

Argument: a sequence of propositions. Conclusion: the final proposition. Premises: all the other propositions. Valid: the truth of premises implies that of the conclusion. Proof: a valid argument that establishes the truth of a conclusion
Predicate and Individual: Predicate describes the property of the subject term (in a sentence). A predicate is a function from a domain of individuals to $\{T, F\}$. Individual is the object you are considering. n-ary Predicate: a predicate on n individuals.
Universal Quantifier: The universial quantifier of P(x) is "P(x) for all x in the domain"/. Notation: $\forall x P(x)$. If the domain is empty, then $\forall x P(x)$ is true for any P. Existential Quantifier: Let P(x) be a propositional function. The existential quantifier of P(x) is "P(x) for some x in the domain". Notation: $\exists x P(x)$. If the domain is empty, then $\exists x P(x)$ is false for any P.
Bound and Free: A variable is bound if it is in the scope of a quantifier. Otherwise, it is free. If A is a WFF with a free individual variable x, then $\forall x A$ and $\exists x A$ are WFFs. $\forall \exists$ have higher precedence than $\neg \wedge \vee \rightarrow \leftrightarrow$.
Types of WFFs: logically valid if it is T in every interpretation. Satisfiable if it is T in some interpretation. Unsatisfiable if it is F in every interpretation. A is logically valid iff $\neg A$ is unsatisfiable. A is satisfiable iff $\neg A$ is not logically valid.
Logically Equivalence: always have the same truth value in every interpretation. Notation: $A \equiv B$. Theorem: $A \equiv B$ iff $A \leftrightarrow B$ ($A \rightarrow B$ and $B \rightarrow A$) is logically valid.
Applying the rule of substitution to the logical equivalences in propositional logic, we get logical equivalences in predicate logic. De Morgan's Laws: $\neg \forall x P(x) \equiv \exists x \neg P(x)$, $\neg \exists x P(x) \equiv \forall x \neg P(x)$. Additional Rule: Conclusion Premise: $\frac{P(x) \Rightarrow Q(x), P(a)}{Q(a)}$.
Distributive Laws: $\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$, $\exists x (P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$.
Tautological Implication: Let A and B be WFFs in predicate logic. A tautologically implies B if every interpretation that causes A to be true causes B to be true. Notation: $A \Rightarrow B$. If $A \Rightarrow B$, $A \Rightarrow B$ iff $A \rightarrow B$ is logically valid or $A \wedge \neg B$ is unsatisfiable.
Building Argument: : Given the premises $P_1, P_2, ..., P_n$ show the conclusion Q, that is, show that $P_1 \wedge P_2 \wedge ... \wedge P_n \Rightarrow Q$.
Premise: Introduce the given formulas. **Conclusion**: Quote the intermediate formula that have been deducted. **Rule of Replacement**: Replace a formula with a logically equivalent formula. **Rule of Inference**: Deduct a new formula with a tautological implication. **Rule of substitution**: Deduct a formula from a tautology.

a. $\forall x P(x) \vee \forall x\, Q(x) \Rightarrow \forall x\,(P(x) \vee Q(x))$
b. $\exists x(P(x) \wedge Q(x)) \Rightarrow \exists x\, P(x) \wedge \exists x Q(x)$
c. $\forall x\,(P(x) \to Q(x)) \Rightarrow \forall x P(x) \to \forall x\, Q(x)$
d. $\forall x\,(P(x) \to Q(x)) \Rightarrow \exists x\, P(x) \to \exists x\, Q(x)$
e. $\forall x\,(P(x) \leftrightarrow Q(x)) \Rightarrow \forall x\, P(x) \leftrightarrow \forall x\, Q(x)$
f. $\forall x\,(P(x) \leftrightarrow Q(x)) \Rightarrow \exists x\, P(x) \leftrightarrow \exists x\, Q(x)$
g. $\forall x\,(P(x) \to Q(x)) \wedge \forall x\,(Q(x) \to R(x)) \Rightarrow \forall x\,(P(x) \to R(x))$
h. $\forall x\,(P(x) \to Q(x)) \wedge P(a) \Rightarrow Q(a)$

Name	Rules of Inference
Universal Instantiation 全称量词消去	$\forall x\, P(x) \Rightarrow P(a)$ a is any individual in the domain of x
Universal Generalization 全称量词引入	$P(a) \Rightarrow \forall x\, P(x)$ a takes any individual in the domain of x
Existential Instantiation 存在量词消去	$\exists x\, P(x) \Rightarrow P(a)$ a is a specific individual in the domain of x
Existential Generalization 存在量词引入	$P(a) \Rightarrow \exists x\, P(x)$ a is a specific individual in the domain of x

$\exists x(C(x) \wedge \neg B(x)) \wedge \forall x(C(x) \to P(x)) \Rightarrow \exists x(P(x) \wedge \neg B(x))$

(1)	$\exists x(C(x) \wedge \neg B(x))$	Premise
(2)	$C(a) \wedge \neg B(a)$	Existential instantiation from (1)
(3)	$C(a)$	Simplification from (2)
(4)	$\forall x(C(x) \to P(x))$	Premise
(5)	$C(a) \to P(a)$	Universal instantiation from (4)
(6)	$P(a)$	Modus ponens from (3) and (5)
(7)	$\neg B(a)$	Simplification from (2)
(8)	$P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
(9)	$\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8)

Graph G = (V,E): V is a set of vertices, E is a set of edges. $|V|$ is called the order of G. An edge with one endpoint is called a **loop**. A simple graph is a finite graph with no loops nor multiple edges. Each edge of a weighted graph is assigned with a strictly positive number. Directed graph: each edge is associated with an ordered pair of vertices (u,v), starts at u, ends at v. Multigraph is a graph with multiple edges(can have loop when directed). Pseudograph is an undirected graph with loops(multiple edges allowed). Mixed graph is a graph with both directed and undirected edges. Subgraph: H=(W,F) is a subgraph of G(V,E) if $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a proper subgraph if $H \neq G$. The subgraph induced by $W \subseteq V$ noted G[W]. induced by $F \subseteq E$ noted G[F] Special simple graphs: K_n: Complete graph. C_n: Cycle. W_n: Wheel(Cycle with a v_0 connecting all others). Q_n: n-Cubes $V = \{0,1\}^n, E = (x,y)$: x and y differ in exactly one bit. Adjacency list, adjaceny matrix: A=(a_{ij}), a_{ij} is the multiplicity of v_i, v_j. Incidence matrix: Let G = (V = $\{v_1,v_2,...,v_n\}$, E = $\{e_1,e_2,...,e_m\}$). The incidence matrix of G is a n x m matrix A = (a_{ij}) where a_{ij} = 1 if v_i is incident with e_j(v_i is an endpoint of e_j).

Degree(undirected): number of edges incident with v, while every loop counts twice. isolated degree 0, pendant degree 1. Degree(directed): in-degree deg- (number of edges ending at v), out-degree deg+ (number of edges starting at v).

Handshaking Theorem: undirected, then $\sum_{v \in V} deg(v) = 2|E|$. So in any undirected graph, the number of vertices of odd degree is even. Directed, then $\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v) = |E|$. Complement graph: $\overline{G} = (V, E')$, where $E' = \{\{u,v\} : u,v \in V, u \neq v, u,v \notin E\}$.

Edge Contraction: Let G be a graph and e = u,v be an edge of G. The graph obtained by contracting e is the graph G/e obtained from G by deleting u and v, adding a new vertex w, and adding an edge w,x for every vertex x that was adjacent to u or v. Graph Isomorphism: If there is a bijection σ, such that $\{u,v\} \in E_1 \Leftrightarrow \{\sigma(u), \sigma(v)\} \in E_2$, then G_1, G_2 are isomorphic and σ is an isomorphism.

Graph invariants are properties preserved by graph isomorphism, thus can be used to determine if two graphs are isomorphic or not. number of vertices, number of edges, degree sequence, number of connected components, number of cycles, etc.

Matching, maximum matching, complete matching: A matching M in a graph G is a set of edges in which no two edges have a common vertex. A matching is a maximum matching if it has the largest possible number of edges. In a bipartite graph, m is a complete matching from A to B if every vertex in A is matched.

Hall's Theorem: Let G be a bipartite graph with bipartition (A,B). G has a complete matching from A to B iff for every subset $X \subseteq A$, $|N(X)| \geq |X|$, where N(X) is the set of neighbors of X.

Connected Component: a connected subgraph that is not a proper subgraph of any other connected subgraph. v is a cut vertex if G-v has more connected components than G. e is a cut edge(bridge) if G-e has more connected components than G. vertex cut: a subset of vertices whose removal disconnects the graph. edge cut: a subset of edges whose removal disconnects the graph. Vertex connectivity $\kappa(G)$ is the minimum number of vertices that must be removed to disconnect G. G is called k-connected if $\kappa(G) \geq k$. Edge connectivity $\lambda(G)$ is the minimum number of edges that must be removed to disconnect G. $\kappa(G) \leq \lambda(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G's vertices.

A directed graph is strongly connected if there is a directed path from u to v and from v to u for every pair of vertices u and v. A directed graph is weakly connected if the underlying undirected graph is connected.

The existence of a simple circuit of length $k \geq 3$ is an isomorphism invariant. Let A be the adjacency matrix of a simple graph G. The number of walks of length k from v_i to v_j is the (i,j)-entry of A^k.

Euler Path and Circuit: A connected graph has a Euler circuit iff every vertex has even degree. A connected graph has a Euler path iff it has exactly two vertices of odd degree or all vertices have even degree. Hierholzer Algorithm: 1. Find a circuit 2. repeat finding subcircuits until all edges are used.

Hamilton Path and Circuit: A a simple path/circuit that visits every vertex exactly once. Necessary conditions on Hamilton circuit: If G has a vertex of degree 1, then G cannot have a Hamilton circuit. If G has a vertex of degree 2, then a Hamilton circuit of G traverses both edges. Sufficient conditions on Hamilton circuit: Dirac's Theorem: If G is a simple graph with n vertices and n $\geq$ 3 such that every vertex has degree $\geq n/2$, then G has a Hamilton circuit. Ore's Theorem: If G is a simple graph with n vertices and n $\geq$ 3 such that for every pair of non-adjacent vertices u and v, deg(u) + deg(v) $\geq$ n, then G has a Hamilton circuit.

Dijkstra's Algorithm: 1. Initialize S = s and $d(s) = 0$, $d(v) = \infty$ for all other vertices. 2. For each vertex v not in S, update $d(v) = min\{d(v), d(u) + w(u,v)\}$, where u is in S and (u,v) is an edge. 3. Add the vertex with the smallest d(v) to S. 4. Repeat 2 and 3 until all vertices are in S.

Planar Graph: can be drawn in the plane without any edges crossing.(Q3 is planar graph). Jordan Curve Theorem: A simple closed curve in the plane divides the plane into two regions, the inside and the outside. Regions: Let G be a planar graph, then the plane is divided into regions by the edges of G. The infinite region is exterior region, others are interior regions. The boundary of a region is a subset of E, the degree of a region is the number of edges in its boundary(Edges in the boundary are counted twice).

Euler Formula: Let G be a planar graph with p connected components, v vertices, e edges, and r regions. v - e + r = p + 1. Application: Let G be a connected planar simple graph. If every region has degree $\geq l$ then $|E(G)| \leq \frac{l}{l-2}(|V(G)| - 2)$. If $|V(G)| \geq 3$, then $|E(G)| \leq 3|V(G)| - 6$. Corollary: be a connected planar simple graph. Then has a vertex of degree ≤ 5. If $|V(G)| \geq 3$, and there is no circuits of length 3, then $|E(G)| \leq 2|V(G)| - 4$. K_5 and $K_{3,3}$ are non-planar.

Homeomorphic: if two graphs can be obtained from a same graph via elementary subdivisions. Kuratowski's Theorem: A graph is nonplanar iff it has a subgraph homeomorphic to K_5 or $K_{3,3}$.

Dual Graph: Let G be a planar graph with a planar embedding. The dual graph of G is a graph G* whose vertices correspond to the regions of G, and two vertices are adjacent in G* if their corresponding regions share an edge in G. Dual graph is always connected, planar. The dual of the dual is the original graph(not true if disconnected). A planar graph is self-dual iff it is isomorphic to its dual(e.g. W_n), and with v vertices, it has 2v-2 edges.

Graph Coloring: chromatic number$\chi(G)$. Let G be a simple graph, $1 \leq \chi(G) \leq |V|$, $\chi(G) = 1$ iff $|E| = \emptyset$, $\chi(G) = 2$ iff G is bipartite,$|E| \geq 1$. The chromatic number of a complete graph is the number of vertices. The chromatic number of a cycle is 2 if n is even, 3 if n is odd. The chromatic number of a complete bipartite graph is 2. The chromatic number of a tree is 2. The chromatic number of a simple planar graph is at most 4.

Tree: an undirected graph is a tree iff there is a unique simple path between any two of its vertices. Rooted Tree(directed): siblings: vertices with the same parent. Ancestor: a vertex is an ancestor of another if it is on the path from the other to the root. Descendant: a vertex is a descendant of another if the other is its ancestor. Leaf: a vertex with no children. Internal vertex: a vertex with at least one child. m-ary tree if every internal vertex has no more than m children. Full m-ary tree if every internal vertex has exactly m children. Balanced m-ary tree: a rooted m-ary tree of height h is balanced if all leaves are at level h or h-1.

Spanning tree: a subgraph that is a tree and contains all vertices of the original graph. A graph is connected iff it has a spanning tree.