Homework 7

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April 14, 2024

1 Problem 1

To prove that $S_2(n, n-2) = \binom{n}{3} + 3 \cdot \binom{n}{4}$ for $n \geq 3$, we can use the definition of Stirling number of the second kind. We know that $S_2(n, n-2)$ is the number of ways to partition a set of n labeled elements into $n-2$ unlabeled and nonempty subsets.

To partition n labeled elements into n-2 unlabeled and nonempty subsets, we can consider two ways of partition:

1. Choose 3 elements from n elements and put them into one subset, then partition the rest n-3 elements into n-3 subsets. There are $\binom{n}{3}$ ways to choose 3 elements from n elements.

2. Choose 2 elements from n elements and put them into one subset, and then another 2 elements into one subset. Finally, partition the rest n-4 elements into n-4 subsets. There are $\binom{n}{4}$ ways to choose 4 elements from n elements. Then we consider dividing the 4 elements into two pairs: For a certain element, it can choose any of the rest three elements to form a pair. Once the pair is determined, the case is settled. So there are $\binom{3}{1}$ ways to form two pairs. And the total number of ways in this situation is $3 \cdot \binom{n}{4}$.

Hence, $S_2(n, n-2) = \binom{n}{3} + 3 \cdot \binom{n}{4}$ for $n \geq 3$.

2 Problem 2

Consider the first n elements, we only have two cases: Firstly, if these n elements are partitioned into k-1 nonempty subsets, then the last element can only be put into the last subset. So in this case, we have $S_2(n, k-1)$ ways.

Or if the first n elements are partitioned into k nonempty subsets, then the last element can be put into any of the k subsets. So in this case, we have $k \cdot S_2(n, k)$ ways.

Hence, we have $S_2(n+1, k) = S_2(n, k-1) + k \cdot S_2(n, k)$.

3 Problem 3

$p_3(n) = p_3(3+n-3) = p_1(n-3) + p_2(n-3) + p_3(n-3)$

$p_1(n-3) = 1$
$p_2(n-3) = \frac{n-3}{2}$ if n is an odd number, $\frac{n-4}{2}$ if n-3 is an even number.
$p_3(n-3) = p_1(n-6) + p_2(n-6) + p_3(n-6)$

$p_1(n-6) = 1$
$p_2(n-6) = \frac{n-6}{2}$ if n is an even number, $\frac{n-7}{2}$ if n is an odd number.

If n is an odd number, $p_3(n) = 1 + \frac{n-3}{2} + 1 + \frac{n-7}{2} + p_3(n-6) = p_3(n-6) + n - 3$
If n is an even number, $p_3(n) = 1 + \frac{n-4}{2} + 1 + \frac{n-6}{2} + p_3(n-6) = p_3(n-6) + n - 3$

Hence, $p_3(n) = p_3(n-6) + n - 3$.

4 Problem 4

Consider n=2k and n=2k+1 separately, where k is an integer no smaller than 2.

For n=2k, we have $a_{2k} = 8a_{2k-1} - 16a_{2(k-2)}$

Consider the characteristic equation $r^2 - 8r + 16 = 0$, we have $r1 = 4$

So $a_{2k} = a_{1,0}4^k + a_{1,1}k4^k$

$a_0 = 3 = a_{1,0}$
$a_2 = 44 = 4a_{1,0} + 4a_{1,1}$

So $a_{1,0} = 3$ and $a_{1,1} = 8$

Hence, $a_{2k} = 3 \cdot 4^k + 8k \cdot 4^k$, and so $a_n = (4n + 3)2^n$ when n is an even number

For n=2k+1, we have $a_{2k+1} = 8a_{2(k-1)+1} - 16a_{2(k-2)+1}$

Consider the characteristic equation $r^2 - 8r + 16 = 0$, we have $r2 = 4$

So $a_{2k+1} = a_{2,0}4^k + a_{2,1}k4^k$

$a_1 = 6 = a_{2,0}$
$a_3 = 56 = 4a_{2,0} + 4a_{2,1}$

So $a_{2,0} = 6$ and $a_{2,1} = 8$

Hence, $a_{2k+1} = 6 \cdot 4^k + 8k \cdot 4^k$, and so $a_n = (4n+2)2^{n-1}$ when n is an odd number

Hence, $a_n = (4n+3)2^n$ when n is an even number, and $a_n = (4n+2)2^{n-1}$ when n is an odd number.

5 Problem 5

For the LNRR $a_n = 3a_{n-1} - 2a_{n-2} + n \cdot 2^n (n \geq 2)$, consider the characteristic equation of associated LHRR $r^2 - 3r + 2 = 0$, we have $r1 = 1$ and $r2 = 2$.

The particular solution of the LNRR is $x_n = (p_1 n + p_0)2^n n$

General solution of the associated LHRR is $y_n = a_{1,0} + a_{1,1}2^n$

So $a_n = a_{1,0} + (p_1 n^2 + p_0 n + a_{1,1})2^n$

$a_0 = 1 = a_{1,0} + a_{1,1}$
$a_1 = -1 = a_{1,0} + 2a_{1,1} + 2p_1 + 2p_0$
$a_2 = -3 - 2 + 8 = 3 = a_{1,0} + 4a_{1,1} + 16p_1 + 8p_0$
$a_3 = 9 + 2 + 24 = 35 = a_{1,0} + 8a_{1,1} + 72p_1 + 24p_0$

So $a_{1,0} = 3, a_{1,1} = -2, p_0 = -1, p_1 = 1$

Hence, $a_n = 3 + (n^2 - n - 2)2^n$.

6 Problem 6

Let $S(n,i)=\sum_{k=0}^{n} k^i$

First, we have $(n+1)^6=\sum_{i=0}^{6}\binom{6}{i}n^i$

So $(n+1)^6-n^6=\binom{6}{5}n^5+\binom{6}{4}n^4+\binom{6}{3}n^3+\binom{6}{2}n^2+\binom{6}{1}n+1$

$n^6-(n-1)^6=\binom{6}{5}(n-1)^5+\binom{6}{4}(n-1)^4+\binom{6}{3}(n-1)^3+\binom{6}{2}(n-1)^2+\binom{6}{1}(n-1)+1$

....

$2^6-1=\binom{6}{5}+\binom{6}{4}+\binom{6}{3}+\binom{6}{2}+\binom{6}{1}+1$

Hence, $(n+1)^6-1=6S(n,5)+\sum_{i=0}^{4}\binom{6}{i}S(n,i)$

So $S(n,5)=\frac{1}{6}((n+1)^6-1-\sum_{i=0}^{4}\binom{6}{i}S(n,i))$

$S(n,1)=\frac{n(n+1)}{2}=\frac{1}{2}n^2+\frac{1}{2}n$

$S(n,2)=\frac{1}{3}((n+1)^3-1-n-3S(n,1))=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$

$S(n,3)=\frac{1}{4}((n+1)^4-1-n-4S(n,1)-6S(n,2))=\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2$

$S(n,4)=\frac{1}{5}((n+1)^5-1-n-5S(n,1)-10S(n,2)-10S(n,3))=\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n$

$S(n,5)=\frac{1}{6}((n+1)^6-1-n-6S(n,1)-15S(n,2)-20S(n,3)-15S(n,4))=\frac{1}{6}n^6+\frac{1}{2}n^5+\frac{5}{12}n^4-\frac{1}{12}n^2$

Hence, $S(n,5) = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$.