Homework 5

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1 Problem 1

First, we will find the number of T-Routes from $A(0,0)$ to $B(7,5)$

This number is $\frac{(b-a)!}{(\frac{b-a}{2}+\frac{\beta-\alpha}{2})!(\frac{b-a}{2}-\frac{\beta-\alpha}{2})!}$, where a=0, b=7, α=0, β=5. And we can get 7 is the number of T-Routes.

Then, we will list all the T-Routes.

$\overline{(0,0),(1,-1),(2,0),(3,1),(4,2),(5,3),(6,4),(7,5)}$ $\overline{(0,0),(1,1),(2,0),(3,1),(4,2),(5,3),(6,4),(7,5)}$
$\overline{(0,0),(1,1),(2,2),(3,1),(4,2),(5,3),(6,4),(7,5)}$ $\overline{(0,0),(1,1),(2,2),(3,3),(4,2),(5,3),(6,4),(7,5)}$
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$\overline{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,5)}$

2 Problem 2

To show that if A, B satisfy the T-condition, then there is a T-route from A to B, we can simply offer a possible T-Route.

Consider b-a steps, for the first $|\beta - \alpha|$ steps, we move in the direction of β-α. That is if β-α is positive, we move in the direction of upper right, otherwise we move in the direction of lower right.

After the first $|\beta - \alpha|$ steps, we are now at (a+$|\beta - \alpha|$, β).

Because A, B satisfy the T-condition, a+$|\beta - \alpha|$ is smaller than b. For the next b-(a+$|\beta - \alpha|$) steps, we take two steps as a unit: For each unit, we move in the direction of upper right for the first step, and move in the direction of lower right for the second step.

Because $2|(b - a + \beta - \alpha)$, we also have $2|(b - a - \beta + \alpha)$. So $2|b - (a + |\beta - \alpha|)$, and we can take the next b-(a+$|\beta - \alpha|$) steps as $\frac{b-(a+|\beta-\alpha|)}{2}$ units.

For each units, we are actually moving in the direction of right for two steps. So after b-(a+$|\beta - \alpha|$) steps, we are now at (b, β), which is B.

This is a T-route. So we have shown that if A, B satisfy the T-condition, then there is a T-route from A to B.

3 Problem 3

We are very clear that x_1 can only be 0. So we can take every x_i as x_{i-1} and throw away the stupid x_1.After that the system is reduced to:

$$\begin{cases} \mathrm{x}_1 + x_2 + x_3 + ... + x_{2n} = n \\ \mathrm{x}_1 + x_2 + x_3 + ... + x_{i-1} < \frac{i}{2} \\ \mathrm{x}_i \in \{0, 1\} \end{cases}$$

Further more, for the second case, $< \frac{i}{2}$ is just equal to $\leq \frac{i-1}{2}$, since the sum of x_i can only be an integer.

So we can rewrite the system as:

$$\begin{cases} \mathrm{x}_1 + x_2 + x_3 + ... + x_{2n} = n \\ \mathrm{x}_1 + x_2 + x_3 + ... + x_i \leq \frac{i}{2} \\ \mathrm{x}_i \in \{0, 1\} \end{cases}$$

That is just the typical case of Catalan number. So the number of solutions is C_{2n}, which is $\frac{(2n)!}{(n+1)!n!}$.

4 Problem 4

Consider $y_i = x_i - 1$, then we have $y_1 + y_2 + y_3 + ... + y_n = r - n$, where $y_i \geq 0$ and is an integer.

This corresponds to a choice of where to place n-1 addition signs in a row of r-n ones.

For example, let n=3 and r=6, then we have 111, $(1, 1, 1)$ is 1+1+1, and $(0, 2, 1)$ is +11+1

So totally we have r-1 signs(1 and +), and we need to choose n-1 places of signs to be +. This is just a set with r-1 elements $A = \{(n-1) \cdot +, (r-n) \cdot 1\}$, so the number of solutions is $\binom{r-1}{n-1}$

5 Problem 5

Let U $= \{u_1, u_2, u_3, ..., u_n\}, V = \{v_1, v_2, v_3, ..., v_n\}$be two sets of n elements each.

Consider the set X $= \{(A, B, C) : A \subseteq U, |A| = 1, B \subseteq U \cup V, |B| = n - 1, C \subseteq U \cup V, |C| = n.|A \cap B| = 0, |A \cap C| = 0, |B \cap C| = 0.\}$

If we choose A, then choose B, the rest are C, then: $|X| = n \cdot \binom{2n-1}{n-1}$

Or, we assume that $|A \cup B \cap U| = r$, where r can be any integer from 1 to n. Then of course $|C \cap V| = r$, that means there are r elements in U that are in A or B, and there are r elements in V that are in C. We first choose $A \cup B$ from U, then we choose A from $A \cup B$. Lastly, we choose C from V. Then we have $|X| = \sum_1^n r \binom{n}{r} \cdot \binom{n}{r}$

So in conclusion, $\sum_1^n r \cdot \binom{n}{r} \cdot \binom{n}{r} = n \cdot \binom{2n-1}{n-1}$

6 Problem 6

$a_n = \sum_{k=s}^{n}(-1)^{n-k}\binom{n}{k}b_k$

Then we have:$\sum_{k=s}^{n}\binom{n}{k}a_k = \sum_{k=s}^{n}\binom{n}{k}\sum_{i=s}^{k}(-1)^{k-i}\binom{k}{i}b_i = \sum_{i=s}^{n}\sum_{k=i}^{n}(-1)^{k-i}\binom{n}{k}\binom{k}{i}b_i$

Because $\sum_{k=i}^{n}(-1)^{k-i}\binom{n}{k}\binom{k}{i} = \sum_{k=i}^{n}(-1)^{k-i}\binom{n}{i}\binom{n-i}{k-i} = \binom{n}{i}\sum_{k=i}^{n}(-1)^{k-i}\binom{n-i}{k-i} = \binom{n}{i}\sum_{t=0}^{n-i}(-1)^{t}\binom{n-i}{t} = \begin{cases}1, & n=i\\ 0, & n>i\end{cases}$

So we have $\sum_{k=s}^{n}\binom{n}{k}a_k = \sum_{i=s}^{n}\sum_{k=i}^{n}(-1)^{k-i}\binom{n}{k}\binom{k}{i}b_i = b_n$