Week 2 The Rules of the Game

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Outline

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Outline

Who are Players in a "Game"?

Game theory is concerned with the actions of *decision makers* who are conscious that their actions affect each other.

Two publishers in a city are "players."

- Their sales jointly determine the price.

Readers are not players.

- They are price takers.

Game theory is not useful when decisions are made that

- ignore the *reactions* of others
- treat others as *impersonal* market forces

Which of the Following are Games?

1. OPEC members choose their annual output;
2. General Motors purchases steel from U.S. Steel;
3. Two manufactures, one of nuts and one of bolts, decide whether to use metric or American standards;
4. A board of directors sets up a stock option plan for the Chief Executive Officer;
5. The US Air Force hires jet fighter pilots;
6. An electric company decides whether to order a new power plant given its estimate of demand for electricity in ten years.

Which of the Following are Games?

1. ✓
2. ✓
3. ✓
4. ✓
5. ×
6. Lies in the field of "decision theory"

Decision theory is NOT game theory

- One single decision maker faces uncertainty.
- He does not interact *strategically* with other decision makers.

Describing a Game

The **rules of the game** include four essential elements, for short PAPI:

- Players
- Actions
- Payoffs
- Information

Objective

Describe a situation in terms of the rules of a game so as to explain *what will happen* in that situation.

Dry Cleaners Game

		OldCleaner: Low Price	OldCleaner: High Price
NewCleaner	Enter	-100, -50	100, 100
	Stay Out	0, 50	0, 300

Table: Payoffs to (New, Old) in thousands of dollars (normal economy)

		OldCleaner: Low Price	OldCleaner: High Price
NewCleaner	Enter	-160, -110	40, 40
	Stay Out	0, -10	0, 240

Table: Payoffs to (New, Old) in thousands of dollars (recession)

Players

Players are the individuals who make decisions. Each player's goal is to maximize his utility by choice of actions.

In the example, players are $\{OldCleaner, NewCleaner\}$.

Nature is a pseudo-player who takes *random* actions at specified points in the game with *specified* probabilities.

In the example, Nature determines the state of economy: 70% normal, 30% recession. The **realizations** of a game depend on the results of random moves.

Actions

Suppose there are n players, $n \in \mathbb{Z}$.

An **action** or **move** by player i, denoted a_i, is a choice he can make. Player i's **action set**, $A_i = \{a_i\}$, is the entire set of actions available to him. An **action combination** is an ordered set $a = (a_1, \ldots, a_n)$ of one action for each of the n players in the game.

In the example, we simplify the action sets for players to be:

- OldCleaner: $\{Low, High\}$
- NewCleaner: $\{Enter, Stay\ Out\}$

Order of Play

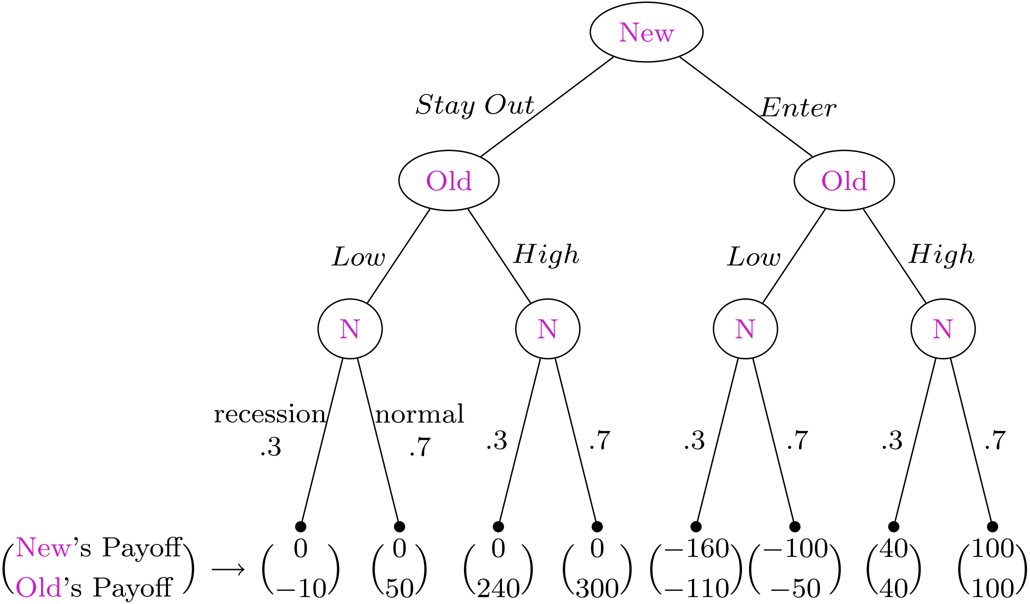
The **order of play** specifies who moves at which point.

For example,

1. NewCleaner chooses from $\{Enter, Stay\ Out\}$.
2. OldCleaner chooses from $\{Low, High\}$.
3. Nature picks demand, D, to be $Recession$ with probability .3 or $Normal$ with probability .7.

We can depict the order of play as a **game tree**.

Game Tree of the Cleaners Game



Strategies

Player i's **strategy** s_i is a rule that tells him which action to choose at *each* instant of the game, given his information.
Player i's **strategy set** or **strategy space** $S_i = \{s_i\}$ is the set of strategies available to him.
A **strategy profile** is an ordered set $s = (s_1, \ldots, s_n)$ consisting of one strategy for each of the n players in the game.

In the example, the *strategy set* for NewCleaner is $\{Enter, Stay\ Out\}$, but that for OldCleaner is

$$\left\{\begin{array}{l}\text{High if NewCleaner Enters, Low if NewCleaner Stays Out}\\ \text{Low if NewCleaner Enters, High if NewCleaner Stays Out}\\ \text{High No Matter What}\\ \text{Low No Matter What}\end{array}\right\}$$

Outcome

The **outcome** of the game is a set of interesting elements that we pick from the values of actions, payoffs, and other variables *after* the game is played out.

In the example, if

- NewCleaner chooses *Enter*
- OldCleaner chooses *Low if NewCleaner Enters, High if NewCleaner Stays Out*
- Nature chooses *Normal*

Then, one outcome of the game is that

- NewCleaner's payoff= -100
- OldCleaner's payoff= -50.

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Payoffs

By player i's **payoff** $\pi_i(s_1, \ldots, s_n)$, we mean either:
(1) The utility player i receives after all players and Nature have picked their strategies and the game has been played out; or
(2) The *expected* utility he receives as a function of the strategies chosen by himself and the other players.

Note: Definitions (1) and (2) are different: (1) actual payoff; (2) expected payoff.
We use π_i for both, and the context will make clear which is meant.

Payoff = Utility ≠ Money

Payoffs or *payoff functions* reflect players' preferences over outcomes induced by strategy profiles.

$\pi_i(s) > \pi_i(s^*)$ if and only if player i prefers the outcome under s to the outcome under s^*.

Caveat: payoffs are not necessarily equal to monetary payoffs.

Rationality of a player lies in the consistency of his decisions, not in the nature of his likes and dislikes.

- We can allow him to be *altruistic*.

Equilibrium

It is the interaction of the different players' strategies that determines what happens.

An **equilibrium** $s^* = (s_1^*, \ldots, s_n^*)$ is a strategy profile consisting of a *best* strategy for *each* of n players in the game.
The **equilibrium strategies** are the strategies players pick in trying to maximize their individual payoffs.
The outcome of a game generated by equilibrium strategies is called **equilibrium outcome**.

Caveat: Pay attention to the difference between "equilibrium" and "equilibrium outcome," and "action" and "strategy."

Solution Concept

To find the equilibrium, we need to decide what "best strategy" means.

An **equilibrium concept** or **solution concept** $F : \{S_1, \ldots, S_n, \pi_1, \ldots, \pi_n\} \to s^*$ is a rule that defines an equilibrium based on the possible strategy combinations and the payoff functions.

Solution Concept

To find the equilibrium, we need to decide what "best strategy" means.

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Are the assumptions valid?

Are the outcomes interesting?

Uniqueness

There could be:

- One equilibrium
- No equilibrium
- Multiple equilibria (plural of "equilibrium")

A *unique* equilibrium provides precise predictions.

No and *multiple* equilibria are the downsides of a model.

No Rewrite the model

Multiple
- admits incompleteness
- change the assumptions or solution concepts

Outline

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Notation s_{-i}

Players must try to figure out each others' actions to choose their own. So we need a shorthand for "all the other players' strategies."

For any vector $y = (y_1, \ldots, y_n)$, denote by y_{-i} the vector $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$, which is the portion of y not associated with player i.

For example, s_{-Smith} is the combination of strategies of every player except player $Smith$.

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Best Response

Player i's **best response** or **best reply to the strategies** s_{-i} chosen by the other players is the strategy s_i^* that yields him the greatest payoff; that is,

$$\pi_i(s_i^*, s_{-i}) \geq \pi_i(s_i', s_{-i}) \quad \forall s_i' \neq s_i^*.$$

Dominated Strategy

The strategy s_i^d is a **dominated strategy** if it is inferior as a response to any strategies the other players might pick; in the sense that whatever strategies other players pick, his payoff is relatively low with s_i^d.

Mathematically, a strategy s_i^d is a dominated strategy for player i, if there exists another strategy s_i' such that

$$\pi_i(s_i^d, s_{-i}) < \pi_i(s_i', s_{-i}) \quad \forall s_{-i}.$$

Dominant Strategy

The strategy s_i^* is a **dominant strategy** if it is a player's strictly best response to any strategies the other players might pick; in the sense that whatever strategies they pick, his payoff is highest with s_i^*. Mathematically,

$$\pi_i(s_i^*, s_{-i}) > \pi_i(s_i', s_{-i}) \quad \forall s_{-i}, \forall s_i' \neq s_i^*$$

A **dominant strategy equilibrium** is a strategy profile consisting of each player's dominant strategy.

The Prisoner's Dilemma

The Prisoner's Dilemma is a **2-by-2 game** with <u>two</u> players, *Row* and *Column*, and <u>two</u> actions, *Deny* and *Confess*, for each player.

The unique dominant strategy equilibrium: (*Confess*, *Confess*).

		Column	
		Deny	*Confess*
Row	*Deny*	-1, -1 $\longrightarrow$	-10, 0
		$\downarrow$	$\downarrow$
	Confess	0, -10 $\longrightarrow$	-8, -8

Payoffs to (Row, Column)

Outline

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The Battle of the Bismarck Sea

Example

In 1943 in the South Pacific,

- **General Imanura** ("木村") has been ordered to transport Japanese troops across the Bismarck Sea ("俾斯麦海") to New Guinea ("新几内亚")
  - Imamura must choose between a shorter *northern* route or longer *southern* route to New Guinea.
- **General Kenny** wants to bomb the troop transports.
  - Kenny must decide where to send his planes to look for Japanese.
  - If Kenney sends his planes to the wrong route he can recall them, but the number of days of bombing is reduced.

If Kenny chooses North

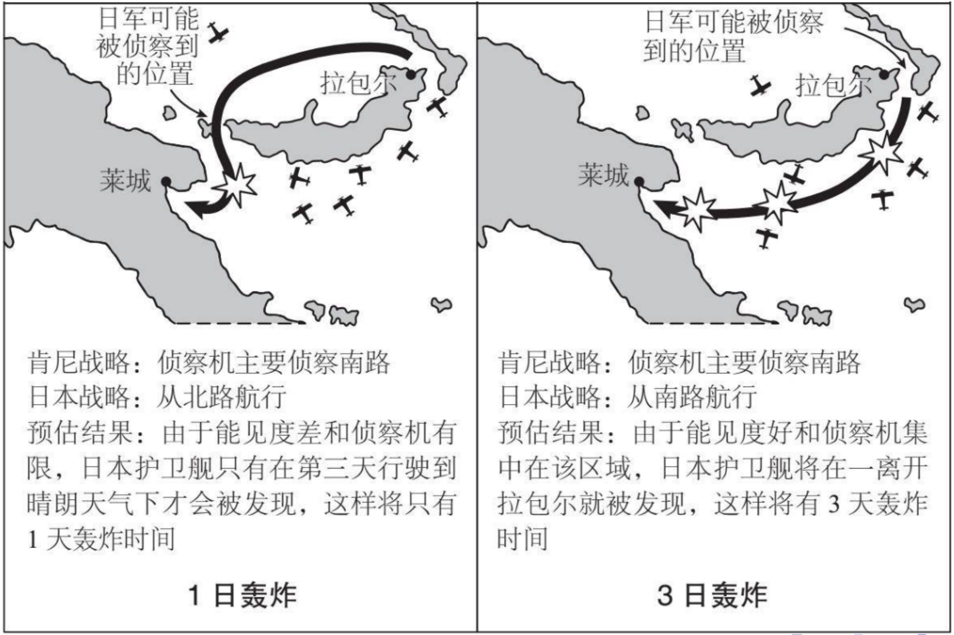
肯尼战略：侦察机主要侦察北路
日本战略：从北路航行
预估结果：虽然能见度低会影响侦察机视野，但第二天盟军将会找到日本护卫舰，这样将有 2 天轰炸时间

2 日轰炸

肯尼战略：侦察机主要侦察北路
日本战略：从南路航行
预估结果：日本护卫舰将在晴朗天气下航行，然而该区域盟军侦察机数量有限，护卫舰会躲过第一天，第二天会被发现，将有 2 天轰炸时间

2 日轰炸

If Kenny chooses South



2×2 Matrix Game

		Imanura	
		North	*South*
Kenny	*North*	2, -2	2, -2
	South	1, -1	3, -3

2×2 Matrix Game

$$
\begin{array}{ccc}
 & & \text{Imanura} \\
 & & \begin{array}{ccc} \textit{North} & & \textit{South} \end{array} \\
\text{Kenny} & \begin{array}{c} \textit{North} \\ \\ \textit{South} \end{array} & \begin{array}{ccc} 2,\ \text{-}2 & \longleftrightarrow & 2,\ \text{-}2 \\ \uparrow & & \downarrow \\ 1,\ \text{-}1 & \longleftarrow & 3,\ \text{-}3 \end{array}
\end{array}
$$

Weak Dominance

Neither player has a dominant strategy.

For player i, strategy s_i' is **weakly dominated** if there exists some other strategy s_i'' which is *possibly better and never worse*, yielding a higher payoff in some strategy profile and never yielding a lower payoff.

Mathematically, s_i' is **weakly dominated** if there exists s_i'' such that

$$\pi_i(s_i'', s_{-i}) \geq \pi_i(s_i', s_{-i}) \quad \forall s_{-i},$$

and for some s_{-i}',

$$\pi_i(s_i'', s_{-i}') > \pi_i(s_i', s_{-i}').$$

Iterated Dominance

- Imanura has a weakly dominated strategy *South*.
- Kenny does not have a weakly dominated strategy.
    - Eliminate Imanura's *South*, Kenny will have a weakly dominated strategy *South*.

An **iterated dominance equilibrium** is a strategy profile found by deleting a weakly dominated strategy from the strategy set of one of the players, recalculating to find which remaining strategies are weakly dominated, deleting one of them, and continuing the process until only one strategy remains for each player.

Iterated Dominance Equilibrium of the Example

(*North*, *North*) is an iterated dominance equilibrium.

It is achieved by deleting *South* from Samura's strategies first and then *South* from Kenny's strategies.

		Imanura	
		North	*South*
Kenny	*North*	**2, -2**	2, -2
	South	1, -1	3, -3

Zero-sum game

A **zero-sum game** is a game in which the sum of the payoffs of all the players is zero whatever strategies they choose.

The Battle of the Bismarck Sea is a zero-sum game, but the Prisoner's Dilemma and the Dry Cleaners Game are not.
Zero-sum games are uncommon in economics.

Multiple Iterated Dominance Equilibria

There could be multiple equilibria depending on the order of deletion.
In the following example:

- $(r_3, c_3, c_2, r_2) \longrightarrow (r_1, c_1)$
- $(r_2, c_2, c_1, r_3) \longrightarrow (r_1, c_3)$

Column

Row		c_1	c_2	c_3
	r_1	**2, 12**	1, 10	**1, 12**
	r_2	0, 12	0, 10	0, 11
	r_3	0, 12	1, 10	0, 13

Multiple Iterated Dominance Equilibria

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In the following example:

- $(r_3, c_3, c_2, r_2) \longrightarrow (r_1, c_1)$
- $(r_2, c_2, c_1, r_3) \longrightarrow (r_1, c_3)$

Column

		c_1	c_2	c_3
Row	r_1	**2, 12**	1, 10	**1, 12**
	r_2	0, 12	0, 10	0, 11
	r_3	0, 12	1, 10	0, 13

What are the Nash Equilibrium of This Game?

Outline

Boxed Pigs

Example

Two pigs, one big and one small, are put in a box.

Structure of box One end: a button. One end: food dispenser.

Press the button Cost of 2 incurs. 10 units of food dispensed.

Wait No cost. Get the food first.

Note: There is neither dominant strategy equilibrium nor iterated dominance equilibrium.

Boxed Pigs

		Small Pig	
		Press	*Wait*
Big Pig	*Press*	5, 1	4, 4
	Wait	9, -1	0, 0

Boxed Pigs

		Small Pig	
		Press	*Wait*
Big Pig	*Press*	5, 1	4, 4
	Wait	9, -1	0, 0

Boxed Pigs

		Small Pig	
		Press	*Wait*
Big Pig	*Press*	5, 1	$\boxed{4}$, $\boxed{4}$
	Wait	$\boxed{9}$, -1	0, $\boxed{0}$

Nash Equilibrium

Nash equilibrium is the most important and widespread equilibrium concept.

The strategy profile s^* is a **Nash equilibrium** if no player has incentive to deviate from his strategy given that the other players do not deviate. Formally,

$$\forall i, \quad \pi_i(s_i^*, s_{-i}^*) \geq \pi_i(s_i', s_{-i}^*) \quad \forall s_i'$$

The way to approach Nash equilibrium:

1. Propose a strategy profile
2. Test whether each player's strategy is a best response to others' strategies.

The Battle of the Sexes

Example

A man and a woman are in love.

- They want to stay together.
- Man wants to go to a prize fight. Woman wants to go to a ballet.

		Woman	
		Prize Fight	*Ballet*
Man	*Prize Fight*	2, 1	0, 0
	Ballet	0, 0	1, 2

The Battle of the Sexes

Example

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- They want to stay together.
- Man wants to go to a prize fight. Woman wants to go to a ballet.

		Woman	
		Prize Fight	*Ballet*
Man	*Prize Fight*	2, 1	0, 0
	Ballet	0, 0	1, 2

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The Stag Hunt

Example

Two hunters go after a stag. If both of them remain attentive, they catch the stag and share it equally. If one devotes his energy to catching a hare, the stag escapes, and the hare belongs to the defecting hunter alone.

		Hunter 2	
		Stag	*Hare*
Hunter 1	*Stag*	2, 2	0, 1
	Hare	1, 0	1, 1

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$$
\begin{array}{cccccc}
 & & & \text{Hunter 2} & & \\
 & & \textit{Stag} & & \textit{Hare} \\
\text{Hunter 1} & \textit{Stag} & 2,\ 2 & \leftarrow & 0,\ 1 \\
 & & \uparrow & & \downarrow \\
 & \textit{Hare} & 1,\ 0 & \rightarrow & 1,\ 1
\end{array}
$$

Matching Pennies

Example

Two players, Row and Column, simultaneously choose whether to show the Head or the Tail of a coin.

- Match: Row gets 1, Column loses 1
- Different: Row loses 1, Column gets 1

		Column	
		Head	*Tail*
Row	*Head*	1, -1	-1, 1
	Tail	-1, 1	1, -1

Matching Pennies

Example

Two players, Row and Column, simultaneously choose whether to show the Head or the Tail of a coin.

- Match: Row gets 1, Column loses 1
- Different: Row loses 1, Column gets 1

$$
\begin{array}{ccccc}
 & & & \text{Column} & \\
 & & \textit{Head} & & \textit{Tail} \\
\text{Row} & \textit{Head} & 1, -1 & \rightarrow & -1, 1 \\
 & & \uparrow & & \downarrow \\
 & \textit{Tail} & -1, 1 & \leftarrow & 1, -1
\end{array}
$$

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Vocabulary

player	参与人	action	行动
payoff	支付	rules of the game	博弈规则
strategy	策略	outcome	结果
order of play	博弈顺序	game tree	博弈树
strategy profile	策略组合	equilibrium	均衡
equilibrium strategy	均衡策略	equilibrium outcome	均衡结果
equilibrium concept	均衡概念	dominated strategy	劣策略
dominant strategy	优势策略	weakly dominated	弱劣于
iterated dominance equilibrium		重复剔除优势均衡	
zero-sum game	零和博弈	boxed pigs	智猪博弈
battle of the sexes	性别战	stag hunt	猎鹿博弈