Weeks 5-6 Mixed and Continuous Strategies

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Outline

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Infinite Action Set

The games we have looked at have so far been simple in at least one respect: the number of moves in the action set A_i has been finite.

In this chapter we allow a *continuum* of moves, such as a player chooses a price between 10 and 20 or a purchase probability between 0 and 1.

Pure and Mixed Strategies

A **pure strategy** maps each of a player's possible information sets to one action. $s_i : \omega_i \to a_i$.

A **mixed strategy** maps each of a player's possible information sets to a probability distribution over actions.

$$s_i : \omega_i \to m(a_i), \text{ where } m \in [0,1], \text{ and } \int_{A_i} m(a_i)\, d\, a_i = 1$$

where m represents a density function of a probability distribution.

A **completely mixed** strategy puts positive probability on every action, so $m > 0$.

Outline

The Welfare Game

Example

A government wishes to aid a pauper if he searchers for work but not otherwise, and a pauper who searches for work only if he cannot depend on government aid.

		Pauper	
		Work (γ_w)	*Loaf* ($1-\gamma_w$)
Government	*Aid* (θ_a)	3, 2	-1, 3
	No Aid (1-θ_a)	-1, 1	0, 0

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How Government Responds?

Government's expected payoff

$$\begin{aligned}\pi_G &= \theta_a[3\gamma_w + (-1)(1-\gamma_w)] + [1-\theta_a][-\gamma_w + 0(1-\gamma_w)] \\ &= \theta_a[3\gamma_w - 1 + \gamma_w] - \gamma_w + \theta_a\gamma_w \\ &= \theta_a[5\gamma_w - 1] - \gamma_w\end{aligned}$$

Government chooses θ_a

- If $\gamma_w > 0.2$, $\theta_a = 1$.
- If $\gamma_w = 0.2$, $\theta_a \in [0,1]$.
- If $\gamma_w < 0.2$, $\theta_a = 0$.

How the Pauper Responds?

Pauper's expected payoff

$$\begin{aligned}\pi_P &= \theta_a[2\gamma_w + 3(1-\gamma_w)] + (1-\theta_a)[\gamma_w + 0(1-\gamma_w)] \\ &= 2\theta_a\gamma_w + 3\theta_a - 3\theta_a\gamma_w + \gamma_w - \theta_a\gamma_w \\ &= -\gamma_w(2\theta_a - 1) + 3\theta_a\end{aligned}$$

Pauper chooses γ_a

- If $\theta_a > 0.5$, $\gamma_a = 0$.
- If $\theta_a = 0.5$, $\gamma_a \in [0,1]$.
- If $\theta_a < 0.5$, $\gamma_a = 1$.

The Unique Mixed Strategy Equilibrium

For Government to randomize, $\gamma_w = 0.2$.
For Pauper to randomize, $\theta_a = 0.5$.
So that $(\theta_a = 0.5,\ \gamma_w = 0.2)$ is the Nash equilibrium.

The Payoff-Equating Method

The idea behind this approach

In the mixed strategy equilibrium, Government must be indifferent between *Aid* and *No Aid* and Pauper must be indifferent between *Work* and *Loaf*.

Solving for θ_a

$$\pi_P(Work) = 2\theta_a + (1-\theta_a) = 3\theta_a + 0(1-\theta_a) = \pi_P(Loaf)$$

The solution is $\theta_a = 0.5$

Solving for γ_w

$$\pi_G(Aid) = 3\gamma_w - (1-\gamma_w) = -\gamma_w + 0(1-\gamma_w) = \pi_G(No\ Aid)$$

The solution is $\gamma_w = 0.2$

Interpreting Mixed Strategies

Mixed strategies are not as intuitive as pure strategies.

Objections

1. People in the real world do not take random actions.
2. Mixing requires indifference. A small deviation in probability destroys the equilibrium completely.

Defend Mixed Strategies

1. It only requires actions appear random to observers.
2. There is a population of players divided into different fractions that take pure strategies (e.g. 20% work and 80% loaf). A player is randomly drawn from a population.

Serving the Balls at Wimbleton

Tennis Play

Each tennis player can choose to serve the ball to the *left* or *right*, which will affect the *win rate* of that point.

Research finds that for almost all matches, players' win rates by serving to both directions are very close. This is evidence for the payoff-equating feature.

Reference: Walker and Wooders (2001) "Minimax Play at Wimbleton."

Unpredictability is helpful in competitive environments.
"兵者，诡道也。故能而示之不能，用而示之不用，远而示之近，近而示之远。" — 孙子兵法

Existence of Nash Equilibrium

Pure strategy NE may not exist. Think of "Matching Pennies."

Mixed strategy NE always exists in finite games.

By a "finite game" I mean a game with finite players, actions, payoffs, and stages.

Outline

The Game of Chicken

Smith drives south down the middle of Route 1, and Jones drives north. Each side decides whether *Continue* in the middle or *Swerve* to the side. If a player is the only one to *Swerve*, he loses face, but if neither players picks *Swerve* they are both killed. If a player is the only one to *Continue*, he is covered with glory, and if both *Swerve* they are both embarassed.

		Jones	
		Continue (θ)	*Swerve* ($1-\theta$)
Smith	*Continue* (θ)	-3, -3	2, 0
	Swerve ($1-\theta$)	0, 2	1, 1

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The Game of Chicken

Advantage of Mixed strategy equilibrium over pure strategy equilibria

Symmetric.

Problems with Asymmetry

- How do players know which equilibrium is the one that will be played out?
- Even if they talk before the game started, it is not clear how they could arrive at an asymmetric result.
- We encountered this dilemma in the battle of the sexes as well.
- The best description is the mixed strategy equilibrium.

The War of Attrition

- The War of Attrition is like *Chicken* stretched out over time.
- Smith and Jones control two firms in an industry which is natural monopoly, with demand strong enough for one firm to operate profitably, but not two.
- There could be infinite periods.
- In each period, the possible actions are to *Exit* or to *Continue*.
  - Both continue, each earns -1. The game continues.
  - Continue and the opponent exits, earns 3. The game ends.
  - Exit, earns 0. The game ends.
- Discount rate r.

The War of Attrition

Solve for a symmetric equilibrium

Each player exits with probability θ in any period.

V_{stay}: the expected discounted value of Smith's payoffs if he stays.
V_{exit}: the expected discounted value of Smith's payoffs if he exits.

$$V_{stay} = 3\theta + (1-\theta)\left(-1 + \frac{V_{stay}}{1+r}\right) \tag{1}$$

$$V_{stay} = V_{exit} = 0 \tag{2}$$

Combining Eq. (1)(2), we have $\theta = \frac{1}{4}$.

Patent Game

- Three companies A, B, C simultaneously choose their $R\&D$ funds $x_i \geq 0$ $(i = a, b, c)$.
- The innovation occurs at the point $T(x_i)$, where $T' < 0$ (first-order derivative).
- The company that achieves innovation first can apply for the patent, whose value is V.
- If more than one company obtains the patent at the same time, they will share the value equally.

Patent Game

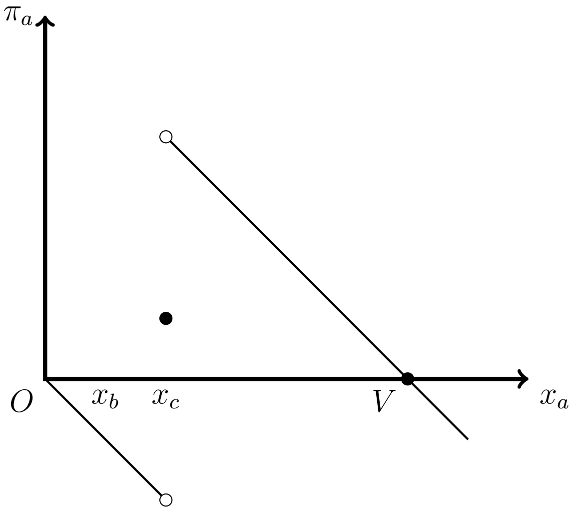
Each company maximizes its expected payoff

$$\pi_i = \begin{cases} V - x_i, & \text{if } T(x_i) < Min\{T(x_j), T(x_k)\} \\ \frac{V}{2} - x_i, & \text{if } T(x_i) = T(x_j) \neq T(x_k) \\ \frac{V}{3} - x_i, & \text{if } T(x_i) = T(x_j) = T(x_k) \\ -x_i, & \text{if } T(x_i) > Min\{T(x_j), T(x_k)\} \end{cases}$$

There is no pure strategy equilibrium

- Payoff function discontinuous.
- Every company wants to outperform the other two by a little.

Patent Game



Mixed Strategy Equilibrium

Mixed strategy

- Each company randomizes over $[0, V]$.
- The cumulative distribution function is $M_i(x_i)$.
- Strategies are symmetric, i.e., $M_i = M$ $(i = 1, 2, 3)$.

Payoff under $\{M_i\}_{i=a,b,c}$

$$\pi_a(x_a) = V \cdot M_b(x_a) M_c(x_a) - x_a$$

Payoff-equating

$$\pi_a(x_a) = \pi_a(0) = 0 \quad \Rightarrow \quad M(x) = \left(\frac{x}{V}\right)^{1/2}$$

The Civic Duty Game

Smith and Jones observe a burglary taking place. Each would like someone to call the police and stop the burglary, but neither wishes to make the call himself.

		Jones	
		Ignore (θ)	*Call* ($1-\theta$)
Smith	*Ignore* (θ)	0, 0	10. 7
	Call ($1-\theta$)	7, 10	7, 7

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N observers

Suppose there are N players. From Smith's perspective, the probability that at least one of the other $N-1$ players calls the police is equal to $1-\theta^{N-1}$. Then we can use the payoff equating method.

$$\pi_S(Call) = 7 = 0 \cdot \theta^{N-1} + 10 \cdot (1-\theta^{N-1}) = \pi_S(Ignore)$$

Solving the Equation we have $\boxed{\theta^{N-1} = 0.3}$ and

$$\theta^* = 0.3^{\frac{1}{N-1}} \tag{3}$$

Comparative Statics

From Eq.(3), as N goes up, the equilibrium value of θ^* goes up.
With more observers, each individual has lower probability of calling the police.

The probability that nobody calls equals θ^{*N}.
Because $\boxed{\theta^{N-1} = 0.3}$, $\theta^{*N} = 0.3\theta^*$, which increases with N.
The more people that watch a crime, the less likely it is to be reported.

Implications of Mixed Equilibrium

The mixed-strategy outcome is clearly *bad*, which suggests a role for active policy.

The solution is the division of responsibility. One person must be made responsible for calling the police.

- By tradition (e.g., the oldest person on the block always calls the police)
- By direction (e.g., Smith shouts to Jones: “Call the police!”)

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The Auditing Game I

The Internal Revenue Service (IRS) must decide whether to audit a certain class of suspect tax returns to discover whether they are accurate or not. The goal of IRS is to either prevent or catch cheating at minimum cost.

Suspect

		Cheat (θ)		Obey ($1-\theta$)
IRS	Audit (γ)	4-C, -F	$\longrightarrow$	4-C, -1
	Trust ($1-\gamma$)	0, 0	$\longleftarrow$	4, -1

;

where $C < 4$, $F > 1$.

The Auditing Game I

Payoff Equating

$$
\begin{aligned}
\theta^* &= \frac{C}{4} \\
\gamma^* &= \frac{1}{F}
\end{aligned}
$$

Expected payoffs

$$
\begin{aligned}
\pi_{IRS} &= 4 - C \\
\pi_{Suspect} &= -1
\end{aligned}
$$

The Auditing Game II

A sequential game with randomizing policy

- Government announces a policy that it will audit a random sample with size $\alpha \in [0, 1]$.
- Suspect sees α, and then decides to cheat or not.

Simultaneous vs. Sequential

It has different effects to have $\theta = \alpha$ in the simultaneous game.

The Auditing Game II

Optimal α

- IRS is willing to deter the suspects from cheating.
- It wants lower α to save costs.
- Look for the smallest possible α.

Suspect wants to obey the rule

$$\begin{aligned}
\pi_{Suspect}(Obey) &\geq \pi_{Suspect}(Cheat) \\
-1 &\geq \alpha(-F) + (1-\alpha)(0) \\
\alpha &= 1/F
\end{aligned}$$

Expected payoff

$\pi_{IRS} = 4 - \alpha C$. It is larger than in Audit Game I.

Outline

The Cournot Game (Oligopoly)

Players

Firms Apex and Brydox

The Order of Play

Apex and Brydox simultaneously choose quantities q_a and q_b from the set $[0, \infty)$.

Payoffs

Production costs are zero. Demand is a function of the total quantity sold, $Q = q_a + q_b$.

$$P(Q) = 120 - q_a - q_b$$

Payoffs are profits, which are given by a firm's price times its quantity, i.e.,

$$\begin{aligned} \pi_A &= 120 q_a - q_a^2 - q_a q_b \\ \pi_B &= 120 q_b - q_a q_b - q_b^2 \end{aligned}$$

The Monopoly Output

If the game is cooperative and two companies collude, they would choose the total output Q to maximize:

$$(120 - Q)Q$$

$$Q^* = 60$$

Best Response Function

The best response of Company A to q_b

$q_b = 0$, $q_a = 60$.
Generally, given q_b, the problem for Company A is

$$\max_{q_a} 120q_a - q_a^2 - q_a q_b$$

$$F.O.C. \quad q_a = 60 - q_b/2 \tag{4}$$

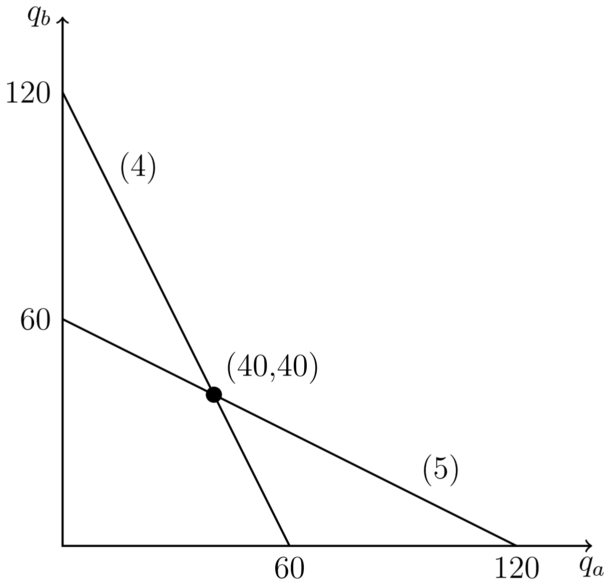
Eq.(4) is the **best response function for Company A**.

The best response function for Company B

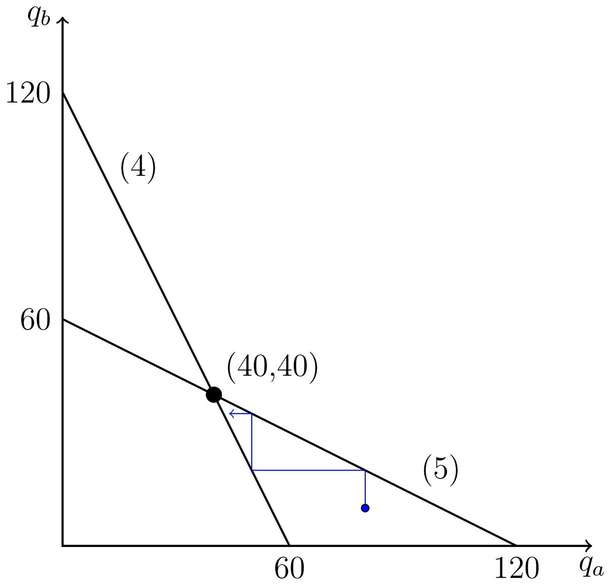
Similarly, we have

$$q_b = 60 - q_a/2 \tag{5}$$

Nash Equilibrium



Stability



Stackelberg Equilibrium

Sequential Setting

Stackelberg Equilibrium differs from Cournot in that Company A moves first.

Company A knows how B will react to its choice, so it picks the point on B's best response function that maximizes A's profit.

Solution

Company A is the **Stackelberg leader** and Company B is the **Stackelberg follower**.

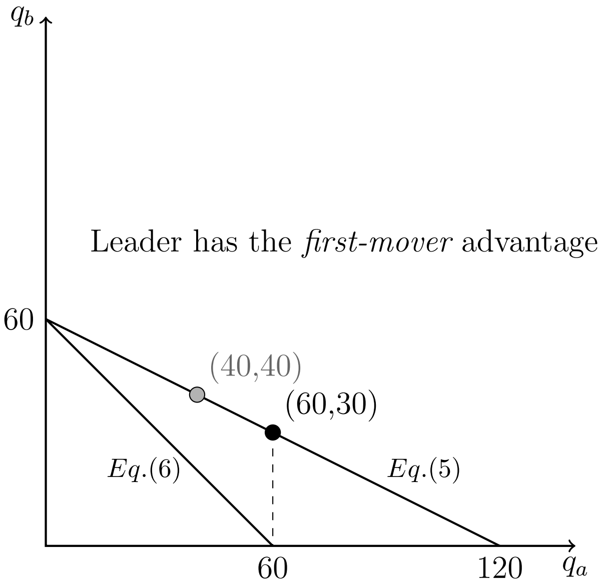
Plug Eq.(5) into Eq.(4):

$$\pi_A = 120q_a - q_a^2 - q_a(60 - q_a/2)$$

Then, maximize π_A with respect to q_a and the derivative equals

$$\partial \pi_A / \partial q_a = 60 - 3q_a \tag{6}$$

Stackelberg Equilibrium



Leader has the *first-mover* advantage.

Vocabulary

pure strategy	纯策略	mixed strategy	混合策略
completely mixed strategy	完全混合策略	welfare game	福利博弈
payoff-equating method	支付均等化方法	game of chicken	斗鸡博弈
war of attrition	消耗战	civic duty game	市民责任博弈
comparative statics	静态比较	auditing game	审计博弈
Cournot game	古诺博弈	Stackelberg equilibrium	斯皮尔伯格均衡
best response function	最优反应函数		