SI114H-Computational Science and Engineering, 2025 Spring

Homework Set #3

Requirements:

1) Deadline: **11pm, 17 May 2025**.
2) About your codes:
   a) Make sure that your codes can run and are consistent with your results.
   b) Attach a Readme.txt file to clearly identify the function of each file.
3) You need to compress three files **code** , **readme** (Add supplementary explanations to the code), and **PDF** (Show your results) into one file, name this file as student ID + your name and send it to the blackboard system.

Problem 1. (50 points)

Find the Fourier series on $-\pi \leq x \leq \pi$ for

(a) $f(x) = \sin^3 x$, an odd function

(b) $f(x) = |\sin x|$, an even function

(c) $f(x) = x$

(d) $f(x) = e^x$, using the complex form of the series.

A. *Solution*

(a) $f(x) = \sin^3 x$, an odd function

Since $f(x)$ is odd, all cosine coefficients $a_n = 0$. For the sine coefficients:

$\sin^3 x = \sin x \cdot \sin^2 x = \sin x \cdot \frac{1-\cos 2x}{2} = \frac{\sin x - \sin x \cos 2x}{2}$

Using the identity $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$: $\sin x \cos 2x = \frac{1}{2}[\sin(x+2x) + \sin(x-2x)] = \frac{1}{2}[\sin 3x - \sin x]$

Therefore: $\sin^3 x = \frac{\sin x - \frac{1}{2}[\sin 3x - \sin x]}{2} = \frac{3\sin x - \sin 3x}{4}$

So the Fourier series is: $f(x) = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

(b) $f(x) = |\sin x|$, an even function

Since $f(x)$ is even, all sine coefficients $b_n = 0$. For the constant term and cosine coefficients:

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| dx = \frac{2}{\pi}\int_0^{\pi} |\sin x| dx = \frac{2}{\pi}\int_0^{\pi} \sin x dx = \frac{2}{\pi}[2] = \frac{4}{\pi}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| \cos(nx) dx$

For odd n, these integrals are zero due to symmetry. For even n $(n = 2k)$, $a_{2k} = \frac{-4}{\pi}\frac{1}{4k^2-1}$

The Fourier series is: $f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2kx)}{4k^2-1}$

(c) $f(x) = x$

This is neither even nor odd, so we need both sine and cosine terms:

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x dx = 0$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)dx = 0$ for all n (can be shown by integration by parts)

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx = \frac{2(-1)^{n+1}}{n}$

The Fourier series is: $f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx) = 2[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - ...]$

(d) $f(x) = e^x$ using the complex form

For the complex Fourier series $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$:

$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x-inx}dx$

$c_n = \frac{1}{2\pi}\left[\frac{e^{x-inx}}{1-in}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\frac{e^{\pi-in\pi}-e^{-\pi+in\pi}}{1-in} = \frac{e^{\pi}(-1)^n - e^{-\pi}(-1)^n}{2\pi(1-in)}$

$c_n = \frac{(-1)^n(e^{\pi}-e^{-\pi})}{2\pi(1-in)} = \frac{(-1)^n \cdot 2\sinh(\pi)}{2\pi(1-in)} = \frac{(-1)^n\sinh(\pi)}{\pi(1-in)}$

Therefore, the complex Fourier series is: $f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n\sinh(\pi)}{\pi(1-in)}e^{inx}$

Problem 2. (50 points)

Show the Gibbs phenomenon using a numerical example.

Gibbs Phenomenon Demonstration:

The plot shows the Fourier series approximation of a square wave.

Notice the overshoot and undershoot near the discontinuities (t=0, t=pi, t=-pi).

As the number of terms (N) increases, the approximation gets better overall, but the height of the overshoot remains significant (approx. 9% of the jump), though it gets squeezed into a narrower region around the discontinuity.

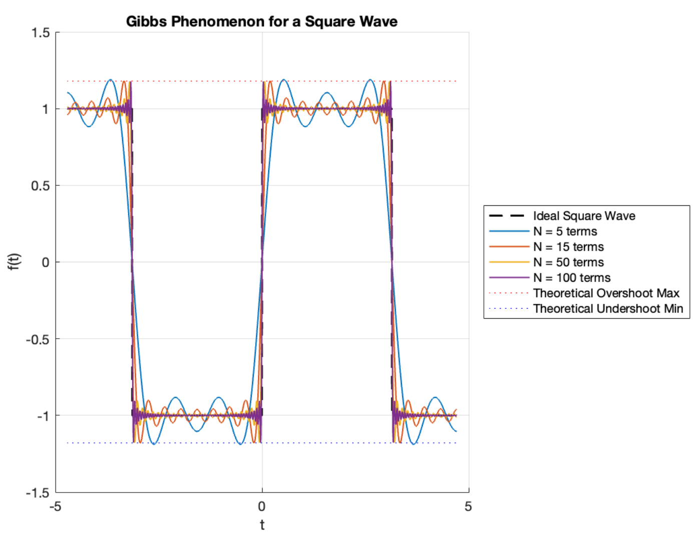

Figure 1. Gibbs Phenomenon