

# SI 140A-02 Probability & Statistics for EECS, Fall 2024

## Homework 9

Name:

Student ID:

Due on Dec. 10, 2024, 11:59 UTC+8

Read all the instructions below carefully before you start working on the assignment, and before you make a submission.

- You are required to write down all the major steps towards making your conclusions; otherwise you may obtain limited points of the problem.
- Write your homework in English; otherwise you will get no points of this homework.
- Any form of plagiarism will lead to 0 point of this homework.

Note: for the programming assignment, you are required to submit the whole solution in the format of Jupyter Notebook (Formerly known as the IPython Notebook), including source codes, theory, algorithms, simulation result and analysis. Do NOT print the file or the source codes.

## Problem 1

Use the methods of inverse transform sampling (or called the method of inverse CDF) to obtain samples from each of the following continuous distributions:

- (a) Logistic distribution with CDF  $F(x) = e^x / (1 + e^x), x \in \mathbb{R}$ .
- (b) Rayleigh distribution with CDF  $F(x) = 1 - e^{-x^2/2}, x > 0$ .
- (c) Exponential distribution with CDF  $F(x) = 1 - e^{-x}, x > 0$ .

After obtaining enough samples, please plot the corresponding histogram and corresponding theoretical PDF.

## Problem 2

Acceptance-Rejection Method

- (a) Use the Acceptance-Rejection Method to obtain samples from Beta distribution  $\text{Beta}(4, 6)$ .
- (b) Use both the Box-Muller method and the Acceptance-Rejection Method to obtain samples from the standard Normal distribution  $\mathcal{N}(0, 1)$ , then discuss the pros and cons of each method.

After obtaining enough samples, please plot the corresponding histogram and corresponding theoretical PDF.

**Problem 3**

Monte Carlo Integration

- (a) Evaluate the integration

$$\int_0^1 \frac{4}{1+x^2} dx.$$

- (b) Evaluate the integration

$$\int_0^4 \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x}}}} dx.$$

- (c) Evaluate the probability of rare event
- $c = \mathbb{P}(Y > 8)$
- , where
- $Y \sim \mathcal{N}(0, 1)$
- .

## Problem 4

Use your own words to describe the geometric perspective of Jacobian Matrix and Jacobian Determinant.

### Solution:

The Jacobian matrix of a vector-valued function represents the best linear approximation to the function at a given point. Geometrically, it describes how the function stretches, rotates, and shears the space around that point. Each entry in the Jacobian matrix corresponds to the partial derivative of one component of the function with respect to one of the input variables, indicating how sensitive that component is to changes in that variable.

The Jacobian determinant provides a measure of how the function transforms volumes near a given point. Geometrically, it represents the factor by which the function scales volumes in the input space when mapping them to the output space. If the determinant is positive, the orientation of the space is preserved; if negative, the orientation is reversed. A determinant of zero indicates that the function compresses the space into a lower dimension at that point.

In summary, the Jacobian matrix describes the local linear transformation properties of a function, while the Jacobian determinant quantifies the local volume scaling effect of that transformation.

## Problem 5

The PDF of Gamma distribution  $\text{Gamma}(a, \lambda)$  is:

$$f(x) = \frac{1}{\Gamma(a)} (\lambda x)^a e^{-\lambda x} \frac{1}{x}, x > 0,$$

where  $a > 0, \lambda > 0$ , and  $\Gamma(a)$  is the Gamma Function:

$$\Gamma(a) = \int_0^\infty z^{a-1} e^{-z} dz.$$

- (a) If  $X \sim \text{Gamma}(a, \lambda)$ , find  $E(X)$  and  $\text{Var}(X)$ .
- (b) If  $Y \sim \mathcal{N}(0, 1)$ , show that  $Y^2 \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$ .
- (c) If  $V = Y_1^2 + \dots + Y_n^2$ , where  $Y_i, i = 1, \dots, n$  are i.i.d. random variables and  $Y_i \sim \mathcal{N}(0, 1)$ , then  $V$  satisfies chi-square distribution, i.e.  $V \sim \chi_n^2$ . Show that  $V \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$  and find the PDF of  $V$ .
- (d) If  $Y$  and  $V$  are independent, define random variable  $Z$  as follows

$$Z = \frac{Y}{\sqrt{\frac{V}{n}}}.$$

then  $Z$  satisfies Student's t-distribution, i.e.  $Z \sim t_n$ . Please adopt the change of variable method to find the PDF of  $Z$ .

- (e) Given two independent random variables  $V_1$  and  $V_2$ , where  $V_1 \sim \chi_m^2$  and  $V_2 \sim \chi_n^2$ . Define random variable  $W$  as follows

$$W = \frac{\frac{V_1}{m}}{\frac{V_2}{n}}.$$

then  $W$  satisfies F-distribution, i.e.  $W \sim F(m, n)$ . Please adopt the change of variable method to find the PDF of  $W$ .

## Solution:

(a):

To find the expected value  $E(X)$  and variance  $\text{Var}(X)$  of a Gamma distribution  $X \sim \text{Gamma}(a, \lambda)$ , we use the properties of the Gamma function.

The expected value is given by:

$$E(X) = \int_0^\infty x \frac{1}{\Gamma(a)} (\lambda x)^a e^{-\lambda x} \frac{1}{x} dx = \frac{\Gamma(a+1)}{\lambda \Gamma(a)} = \frac{a}{\lambda}$$

The second moment is:

$$E(X^2) = \int_0^\infty x^2 \frac{1}{\Gamma(a)} (\lambda x)^a e^{-\lambda x} \frac{1}{x} dx = \frac{\Gamma(a+2)}{\lambda^2 \Gamma(a)} = \frac{a(a+1)}{\lambda^2}$$

Thus, the variance is:

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{a(a+1)}{\lambda^2} - \left(\frac{a}{\lambda}\right)^2 = \frac{a}{\lambda^2}$$

(b):

To show that  $Y^2 \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$  for  $Y \sim \mathcal{N}(0, 1)$ , we start with the PDF of the standard normal distribution:

$$\int_{-\infty}^{\infty} f(x) dx = 1 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

Using the change of variable  $u = \frac{x^2}{2}$ , we get:

$$\frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{1}{\sqrt{u}} e^{-u} du = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right)$$

Since  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ , we have:

$$f(x) = \frac{1}{\sqrt{2\pi x}} e^{-\frac{x}{2}}$$

This matches the PDF of the Gamma distribution  $\text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$ :

$$f_G(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \left(\frac{1}{2}x\right)^{\frac{1}{2}-1} e^{-\frac{x}{2}} = \frac{1}{\sqrt{2\pi x}} e^{-\frac{x}{2}}$$

Thus,  $Y^2 \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$ .

(c):

To show that  $V = Y_1^2 + \dots + Y_n^2 \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$ , we use the property that the sum of independent Gamma random variables is also Gamma distributed.

Assume  $X \sim \text{Gamma}(a, \lambda)$  and  $Y \sim \text{Gamma}(b, \lambda)$ . Then  $Z = X + Y$  has the PDF:

$$f_Z(z) = \int_0^z f_X(x) f_Y(z-x) dx = \frac{e^{-z\lambda} \lambda^{a+b}}{\Gamma(a)\Gamma(b)} \int_0^z x^{a-1} (z-x)^{b-1} dx$$

Using the Beta function identity, we get:

$$f_Z(z) = \frac{\lambda^{a+b}}{\Gamma(a+b)} z^{a+b-1} e^{-z\lambda}$$

Thus,  $V = Y_1^2 + \dots + Y_n^2 \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$ .

(d):

To find the PDF of  $Z = \frac{Y}{\sqrt{\frac{V}{n}}}$  where  $Y \sim \mathcal{N}(0, 1)$  and  $V \sim \chi_n^2$ , we use the joint PDF of  $Y$  and  $V$ :

$$f_{Y,V}(y, v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \frac{1}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{v}{2}\right)^{\frac{n}{2}-1} e^{-\frac{v}{2}}$$

Using the change of variables and the Jacobian determinant  $|J| = \frac{v}{n}$ , we get:

$$f_Z(z) = \int_0^{\infty} f_{Z,V}(z, v) dv = 2 \frac{1}{\sqrt{2\pi n}} \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^{\infty} e^{-\frac{z^2 v}{n^2}} \left(\frac{v}{2}\right)^{\frac{n-1}{2}} e^{-\frac{v}{2}} d\left(\frac{v}{2}\right)$$

This simplifies to:

$$f_Z(z) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{\left(1 + \frac{z^2}{n}\right)^{-\frac{n+1}{2}}}{\sqrt{\pi n}}$$

(e):

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To find the PDF of  $W = \frac{V_1}{\frac{V_2}{n}}$  where  $V_1 \sim \chi_m^2$  and  $V_2 \sim \chi_n^2$ , we use the joint PDF and the Jacobian determinant  $|J| = \frac{m}{n}z$ :

$$f_W(w) = \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1} \left(1 + \frac{nw}{m}\right)^{-\frac{m+n}{2}}$$



## Problem 6

(Optional Challenging Problem). Given  $n$  i.i.d. continuous random variables  $X_1, \dots, X_n$ , where  $X_i \sim F_X$ . Define

$$D_n = \sup_{x \in \mathbb{R}} \left| \hat{F}_n(x) - F_X(x) \right|.$$

where

$$\hat{F}_n(x) = \frac{1}{n} \sum_{j=1}^n I(X_j \leq x)$$

- (a) Find the CDF of  $D_n$ .
- (b) Find the limiting CDF of  $\sqrt{n}D_n$  when  $n \rightarrow \infty$ .