SI 140A-02 Probability & Statistics for EECS, Fall 2024
Homework 13

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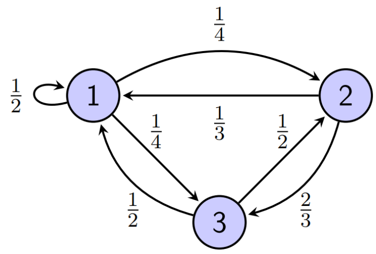
Due on Jan. 7, 2025, 11:59 UTC+8

Read all the instructions below carefully before you start working on the assignment, and before you make a submission.

- You are required to write down all the major steps towards making your conclusions; otherwise you may obtain limited points of the problem.
- Write your homework in English; otherwise you will get no points of this homework.
- Any form of plagiarism will lead to 0 point of this homework.

Problem 1

Given a Markov chain with state-transition diagram shown as follows:



(a) Is this chain irreducible?
(b) Is this chain aperiodic?
(c) Find the stationary distribution of this chain.
(d) Is this chain reversible?

Solution

For the Markov chain, we have the transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

(a) Yes, this chain is irreducible because all the elements apart from the diagonal elements are non-zero.

(b) Yes, we can see that 1 is a possible return time for state 1, thus $d(1) = 1$. Since both 2,3 are possible for state 2,3 $d(2) = d(3) = 1$. And because the chain is irreducible, so the chain is aperiodic.

(c) Denote the stationary distribution as $\pi = (\pi_1, \pi_2, \pi_3)$, we have

$$\pi P = \pi$$

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix}$$

We have the following equations:

$$\frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3 = \pi_1$$
$$\frac{1}{4}\pi_1 + \frac{2}{3}\pi_2 + \frac{1}{2}\pi_3 = \pi_2$$
$$\frac{1}{4}\pi_1 + \frac{2}{3}\pi_2 = \pi_3$$

Solving the above equations, we have

$$\pi_1 = \frac{16}{35}$$
$$\pi_2 = \frac{9}{35}$$
$$\pi_3 = \frac{2}{7}$$

Thus, the stationary distribution of this chain is $\pi = \left(\frac{16}{35}, \frac{9}{35}, \frac{2}{7}\right)$.

Problem 1 continued on next page...

(d) No, this chain is not reversible. If the chain is reversible, we should have

$$\pi_i p_{ij} = \pi_j p_{ji}$$

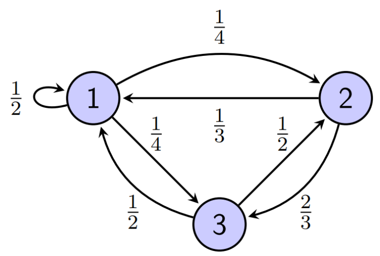
However, we can see that

$$\pi_1 p_{12} = \frac{16}{35} \cdot \frac{1}{4} = \frac{4}{35}$$
$$\pi_2 p_{21} = \frac{9}{35} \cdot \frac{1}{3} = \frac{3}{35}$$

Thus, the chain is not reversible.

Problem 2

Given a Markov chain with state-transition diagram shown as follows:



(a) Find $P(X_9 = 3 \mid X_8 = 1)$ and $P(X_8 = 2 \mid X_7 = 3)$.
(b) If $P(X_0 = 3) = \frac{1}{2}$, find $P(X_0 = 3, X_1 = 1, X_2 = 2, X_4 = 3)$.
(c) Find $E(X_8 \mid X_6 = 2)$.
(d) Find $\mathrm{Var}(X_7 \mid X_5 = 3)$.

Solution

For the Markov chain, we have the transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

(a)

$$P(X_9 = 3 \mid X_8 = 1) = p_{13} = \frac{1}{4}$$
$$P(X_8 = 2 \mid X_7 = 3) = p_{32} = \frac{1}{2}$$

(b)

$$\begin{aligned} P(X_0 = 3, X_1 = 1, X_2 = 2, X_4 = 3) &= P(X_0 = 3)P(X_1 = 1 \mid X_0 = 3)P(X_2 = 2 \mid X_1 = 1)P(X_4 = 3 \mid X_2 = 2) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{24} \end{aligned}$$

(c)

$$P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{11}{24} & \frac{1}{4} & \frac{7}{24} \\ \frac{1}{2} & \frac{5}{12} & \frac{1}{12} \\ \frac{5}{12} & \frac{1}{8} & \frac{11}{24} \end{bmatrix}$$

$$E(X_8 \mid X_6 = 2) = \sum_{i=1}^{3} i \cdot p_{2i} = 1 \cdot \frac{1}{2} + 2 \cdot \frac{5}{12} + 3 \cdot \frac{1}{12} = \frac{19}{12}$$

(d) To find $\mathrm{Var}(X_7 \mid X_5 = 3)$, we first need to find $P(X_7 \mid X_5 = 3)$. We have

$$\mathrm{Var}(X_7 \mid X_5 = 3) = E(X_7^2 \mid X_5 = 3) - E(X_7 \mid X_5 = 3)^2$$

We have

$$E(X_7^2 \mid X_5 = 3) = \sum_{i=1}^{3} i^2 \cdot p_{3i} = 1^2 \cdot \frac{5}{12} + 2^2 \cdot \frac{1}{8} + 3^2 \cdot \frac{1}{24} = \frac{31}{24}$$

$$E(X_7 \mid X_5 = 3) = \sum_{i=1}^{3} i \cdot p_{3i} = 1 \cdot \frac{5}{12} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{24} = \frac{19}{24}$$

Thus, we have

$$\mathrm{Var}(X_7 \mid X_5 = 3) = \frac{31}{24} - \left(\frac{19}{24}\right)^2 = \frac{31}{24} - \frac{361}{576} = \frac{383}{576}$$

Problem 3

There are two urns with a total of $2N$ distinguishable balls. Initially, the first urn has N white balls and the second urn has N black balls. At each stage, we pick a ball at random from each urn and interchange them. Let X_n be the number of black balls in the first urn at time n. This is a Markov chain on the state space $\{0, 1, \ldots, N\}$.

(a) Find the transition probabilities of the chain.

(b) Find the stationary distribution of the chain.

Solution

(a) Denote the transition probability as p_{ij} from i to j. The number of black balls change by at most 1 at each stage. Thus, we have $p_{ij} = 0$ if $|i - j| > 1$.
Also, we have

$$p_{i,i+1} = (\frac{N-i}{N})^2$$

$$p_{i,i-1} = (\frac{i}{N})^2$$

And for the number of black balls to stay the same, we have

$$p_{ii} = 1 - p_{i,i+1} - p_{i,i-1} = 1 - \frac{N-i}{N}^2 - \frac{i}{N}^2 = \frac{2i(N-i)}{N^2}$$

(b) Denote the stationary distribution as $s = (s_0, s_1, \ldots, s_N)$. According to the result of (a), we have $p_{i,i+1}$ and $p_{i+1,i}$ for each i,

$$p_{i,i+1} = (\frac{N-i}{N})^2 p_{i+1,i} = (\frac{i+1}{N})^2$$

$$p_{i,i-1} = (\frac{i}{N})^2 p_{i-1,i} = (\frac{N-i+1}{N})^2$$

We have an intuitive guess that the chain is reversible. Thus, to make it, we have

$$s_i p_{i,i+1} = s_{i+1} p_{i+1,i}$$

$$s_i p_{i,i-1} = s_{i-1} p_{i-1,i}$$

From that, we made a guess that the stationary distribution should be

$$s_i = \frac{\binom{N}{i}\binom{N}{N-i}}{\binom{2N}{N}}$$

To prove it, we need to show that

$$s_i p_{i,j} = s_j p_{j,i}$$

for all i, j.
However, if $|i - j| > 1$, we have $s_i p_{i,j} = 0 = s_j p_{j,i}$. If $i == j$, then both sides are equal to $s_i p_{i,i}$. Thus, we only need to discuss the case where $|i - j| = 1$.
If $j = i + 1$, we have

$$s_i p_{i,i+1} = \frac{\binom{N}{i}\binom{N}{N-i}}{\binom{2N}{N}} \cdot (\frac{N-i}{N})^2 = \frac{\binom{N}{i}^2 (N-i)^2}{\binom{2N}{N} N^2}$$

and

$$s_{i+1} p_{i+1,i} = \frac{\binom{N}{i+1}\binom{N}{N-i-1}}{\binom{2N}{N}} \cdot (\frac{i+1}{N})^2 = \frac{\binom{N}{i+1}^2 (i+1)^2}{\binom{2N}{N} N^2}$$

Problem 3 continued on next page...

We can see that the two sides are equal because

$$\binom{N}{i}(N-i) = \binom{N}{i+1}(i+1)$$

Similarly, we can prove that the two sides are equal when $j = i - 1$.

$$s_i p_{i,i-1} = \frac{\binom{N}{i}\binom{N}{N-i}}{\binom{2N}{N}} \cdot (\frac{i}{N})^2 = \frac{\binom{N}{i}^2 i^2}{\binom{2N}{N} N^2}$$

and

$$s_{i-1} p_{i-1,i} = \frac{\binom{N}{i-1}\binom{N}{N-i+1}}{\binom{2N}{N}} \cdot (\frac{N-i+1}{N})^2 = \frac{\binom{N}{i-1}^2 (N-i+1)^2}{\binom{2N}{N} N^2}$$

We can see that the two sides are equal because

$$\binom{N}{i} i = \binom{N}{i-1}(N-i+1)$$

Thus, the stationary distribution of the chain is

$$s_i = \frac{\binom{N}{i}\binom{N}{N-i}}{\binom{2N}{N}}$$