SI 140A-02 Probability & Statistics for EECS, Fall 2024
Homework 8

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Read all the instructions below carefully before you start working on the assignment, and before you make a submission.

- You are required to write down all the major steps towards making your conclusions; otherwise you may obtain limited points of the problem.
- Write your homework in English; otherwise you will get no points of this homework.
- Any form of plagiarism will lead to 0 point of this homework.

Problem 1

Let X and Y be two continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx^2y & \text{if } 0 \le y \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of constant c.

(b) Find the conditional probability $P\left(Y \le \frac{X}{4} \,\middle|\, Y \le \frac{X}{2}\right)$.

Solution

(a):
According to the definition of joint PDF, we have

$$1 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = \int_0^1 \int_0^x cx^2y\,dy\,dx = \int_0^1 \frac{c}{2}x^4\,dx = \frac{c}{10}$$

Thus, we have $c = 10$.
(b):

$$P\left(Y \le \frac{X}{4} \,\middle|\, Y \le \frac{X}{2}\right) = \frac{P\left(Y \le \frac{X}{4}, Y \le \frac{X}{2}\right)}{P\left(Y \le \frac{X}{2}\right)} = \frac{\int_0^1 \int_0^{x/4} 10x^2y\,dy\,dx}{\int_0^1 \int_0^{x/2} 10x^2y\,dy\,dx} = \frac{1}{4}$$

Problem 2

Let X and Y be two integer random variables with joint PMF

$$P_{X,Y}(x,y) = \begin{cases} \frac{1}{6\cdot 2^{\min(x,y)}} & \text{if } x,y \geq 0, |x-y| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal distributions of X and Y.

(b) Are X and Y independent?

(c) Find $P(X = Y)$.

Solution

(a)
The marginal distributions of X is

$$P_X(X) = \sum_{y=0}^{\infty} P_{X,Y}(X,Y)$$

When $X = 0$, we have

$$P(X=0) = P(X=0, Y=0) + P(X=0, Y=1) = \frac{1}{3}.$$

When $X \neq 0$, we have

$$P(X=x) = P(X=x, Y=x-1) + P(X=x, Y=x) + P(X=x, Y=x+1) = \frac{1}{6\cdot 2^{x-2}}.$$

Thus, the marginal distribution of X is

$$P_X(X) = \begin{cases} \frac{1}{3}, & x=0 \\ \frac{1}{6\cdot 2^{x-2}}, & x>0 \\ 0, & \text{otherwise} \end{cases}$$

According to the symmetric, the marginal distribution of Y is

$$P_Y(Y) = \begin{cases} \frac{1}{3}, & y=0 \\ \frac{1}{6\cdot 2^{y-2}}, & y>0 \\ 0, & \text{otherwise} \end{cases}$$

(b)
Since that

$$P_{X,Y}(0,0) = \frac{1}{6},$$

and

$$P(X=0)P(Y=0) = \frac{1}{9}$$

X and Y are not independent.
(c)
According to symmetric, we have $P(X = Y) = P(X = Y-1) = P(X = Y+1)$ and $P(X = Y) + P(X = Y-1) + P(X = Y+1) = 1$. Thus, we have

$$P(X=Y) = \frac{1}{3}.$$

Problem 3

Let X and Y be i.i.d. $\mathcal{N}(0,1)$, and let S be a random sign 1 or -1 , with equal probabilities independent of (X, Y).

(a) Determine whether or not $(X, Y, X+Y)$ is Multivariate Normal.

(b) Determine whether or not $(X, Y, SX+SY)$ is Multivariate Normal.

(c) Determine whether or not (SX, SY) is Multivariate Normal.

Solution

(a)
Yes, $(X, Y, X+Y)$ is Multivariate Normal, because for any $a, b, c \in R$,

$$aX + bY + c(X+Y) = (a+c)X + (b+c)Y$$

and any linear combination of independent normally distributed variables are Normal.
(b)
Denote $Z = X + Y + SX + SY = (1+S)X + (1+S)Y$. $Z = 0$ is in fact $S = -1$, hence, we have that

$$P(Z=0) = P(S=-1) = \frac{1}{2}.$$

Hence, Z is not normally distributed.
(c)
Observe that random vector (X, Y) is identically distributed as (-X,-Y). So,

$$\begin{aligned}
P(SX+SY \leq k) &= P(SX+SY \leq k, S=1) + P(SX+SY \leq k, S=-1) \\
&= P(SX+SY \leq k \mid S=1)P(S=1) + P(SX+SY \leq k \mid S=-1)P(S=-1) \\
&= \frac{1}{2}P(X+Y \leq k) + \frac{1}{2}P(X+Y \geq -k) \\
&= \frac{1}{2}P(X+Y \leq k) + \frac{1}{2}P(X+Y \leq k) \\
&= P(X+Y \leq k).
\end{aligned}$$

So, (SX, SY) is equally distributed as (X, Y), and (X, Y) is Bivariate normal. Hence, (SX, SY) is Multivariate Normal.

Problem 4

Let Z_1, Z_2 be two i.i.d. random variables satisfying standard normal distributions, i.e., $Z_1, Z_2 \sim \mathcal{N}(0,1)$. Define

$$X = \sigma_X Z_1 + \mu_X$$
$$Y = \sigma_Y \left(\rho Z_1 + \sqrt{1-\rho^2} Z_2\right) + \mu_Y$$

where $\sigma_X > 0, \sigma_Y > 0, -1 < \rho < 1$.

(a) Show that X and Y are bivariate normal.

(b) Find the correlation coefficient between X and Y, i.e., $\operatorname{Corr}(X,Y)$.

(c) Find the joint PDF of X and Y.

Solution

(a)
For $a, b \in \mathbb{R}$, we have

$$aX + bY = \left(a\mathbf{\Sigma}_X + b\mathbf{\Sigma}_Y \rho\right) Z_1 + b\sqrt{1-\rho^2}\mathbf{\Sigma}_Y Z_2 + a\mu_X + b\mu_Y.$$

Since the linear combination of two Normal distribution follows Normal distribution, X and Y are bivariate normal.
(b)
Since $Z_1, Z_2 \sim \mathcal{N}(0,1)$. We have $\rho Z_1 + \sqrt{1-\rho^2} Z_2 \sim \mathcal{N}(0,1)$. So $X \sim \mathcal{N}\left(\mu_X, \mathbf{\Sigma}_X\right), Y \sim \mathcal{N}\left(\mu_Y, \mathbf{\Sigma}_Y\right)$. Thus, we have

$$\begin{aligned}
\operatorname{Cov}(X,Y) &= \operatorname{Cov}\left(\mathbf{\Sigma}_X Z_1 + \mu_X, \mathbf{\Sigma}_Y\left(\rho Z_1 + \sqrt{1-\rho^2} Z_2\right) + \mu_Y\right) \\
&= \mathbf{\Sigma}_X \mathbf{\Sigma}_Y \operatorname{Cov}\left(Z_1, \rho Z_1 + \sqrt{1-\rho^2} Z_2\right) \\
&= \mathbf{\Sigma}_X \mathbf{\Sigma}_Y \left(\rho \operatorname{Var}\left(Z_1\right) + \sqrt{1-\rho^2} \operatorname{Cov}\left(Z_1, Z_2\right)\right) \\
&= \mathbf{\Sigma}_X \mathbf{\Sigma}_Y \rho
\end{aligned}$$

Then correlation coefficient between X and y is

$$\operatorname{Corr}(X,Y) = \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}} = \frac{\mathbf{\Sigma}_X \mathbf{\Sigma}_Y \rho}{\mathbf{\Sigma}_X \mathbf{\Sigma}_Y}.$$

(c)
Since Z_1 and Z_2 are i.i.d., we have

$$f_{Z_1,Z_2}\left(z_1, z_2\right) = f_{Z_1}\left(z_1\right) f_{Z_2}\left(z_2\right) = \frac{1}{2\pi} e^{-\frac{z_1^2+z_2^2}{2}}.$$

Since $X = \mathbf{\Sigma}_X Z_1 + \mu_X, Y = \mathbf{\Sigma}_Y\left(\rho Z_1 + \sqrt{1-\rho^2} Z_2\right) + \mu_Y$, we have

$$Z_1 = \frac{X - \mu_X}{\mathbf{\Sigma}_X}$$

and

$$Z_2 = \frac{Y - \mu_Y}{\sqrt{1-\rho^2}\mathbf{\Sigma}_Y} - \rho\frac{X - \mu_X}{\sqrt{1-\rho^2}\mathbf{\Sigma}_X}.$$

Thus, we have

$$f_{X,Y}(x,y) = \left|\frac{\partial\left(Z_1, Z_2\right)}{\partial(X,Y)}\right| f_{Z_1,Z_2}\left(z_1, z_2\right)$$

Problem 4 continued on next page...

$$
\begin{aligned}
&= \frac{1}{\begin{vmatrix} \frac{\partial z_1}{\partial x} & \frac{\partial z_1}{\partial y} \\ \frac{\partial z_2}{\partial x} & \frac{\partial z_2}{\partial y} \end{vmatrix}} f_{Z_1,Z_2}(z_1, z_2) \\
&= \frac{1}{\begin{vmatrix} \frac{1}{\Sigma_X} & 0 \\ -\frac{\rho}{\sqrt{1-\rho^2}\Sigma_X} & \frac{1}{\sqrt{1-\rho^2}\Sigma_Y} \end{vmatrix}} f_{Z_1,Z_2}\left(\frac{x-\mu_X}{\Sigma_X}, \frac{y-\mu_Y}{\sqrt{1-\rho^2}\Sigma_Y} - \rho\frac{x-\mu_X}{\sqrt{1-\rho^2}\Sigma_X}\right) \\
&= \frac{1}{\Sigma_X\Sigma_Y\sqrt{1-\rho^2}} f_{Z_1,Z_2}\left(\frac{x-\mu_X}{\Sigma_X}, \frac{y-\mu_Y}{\sqrt{1-\rho^2}\Sigma_Y} - \rho\frac{x-\mu_X}{\sqrt{1-\rho^2}\Sigma_X}\right) \\
&= \frac{1}{2\pi\Sigma_X\Sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{\left(\frac{x-\mu_X}{\Sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sqrt{1-\rho^2}\Sigma_Y} - \rho\frac{x-\mu_X}{\sqrt{1-\rho^2}\Sigma_X}\right)^2}{2}\right) \\
&= \frac{1}{2\pi\Sigma_X\Sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{\left(\frac{x-\mu_X}{\Sigma_X}\right)^2 - 2\rho\frac{(x-\Sigma_X)(y-\Sigma_Y)}{\Sigma_X\Sigma_Y} + \left(\frac{y-\mu_Y}{\Sigma_Y}\right)^2}{2(1-\rho^2)}\right).
\end{aligned}
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Problem 5

(a) Let X and Y be i.i.d. $\mathcal{N}(0,1)$, and $Z = \frac{X}{Y}$. Find the PDF of Z.

(b) Let X and Y be i.i.d. $\text{Unif}(0,1), W = X \cdot Y$, and $Z = \frac{X}{Y}$. Find the joint PDF of (W, Z).

(c) A point (X, Y) is picked at random uniformly in the unit circle. Find the joint PDF of R and X, where $R = \sqrt{X^2 + Y^2}$.

(d) A point (X, Y, Z) is picked uniformly at random inside the unit ball of $\mathbb{R}^3$. Find the joint PDF of Z and R, where $R = \sqrt{X^2 + Y^2 + Z^2}$.

Solution

(a)
We're told that X and Y are independent random variables with a normal distribution with $\mu = 0$ and $\sigma = 1$. In other words, they both have a probability density function of:

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Let's now look at the case where $Z = X/Y$. This produces the ratio distribution, which is derived as follows: Note that we set $x = yz$, and apply the Jacobian determinant due to this transform, which is $|y|$

$$\begin{aligned} p_Z(z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |y| \exp\left(-\frac{(zy)^2}{2}\right) \exp\left(-\frac{y^2}{2}\right) dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |y| \exp\left(-\frac{y^2\left(z^2+1\right)}{2}\right) dy \end{aligned}$$

Using the known definite integral $\int_0^\infty x \exp\left(-cx^2\right) dx = \frac{1}{2c}$ we get

$$p_Z(z) = \frac{1}{\pi\left(z^2+1\right)}$$

which is the Cauchy distribution.

(b)
To find the joint PDF of (W, Z), we start with the joint PDF of (X, Y), which is $f_{X,Y}(x, y) = 1$ for $0 < x < 1$ and $0 < y < 1$. We then use the transformation $W = X \cdot Y$ and $Z = \frac{X}{Y}$. The Jacobian determinant for this transformation is $|J| = \frac{1}{2z}$. Thus, the joint PDF is:

$$f_{W,Z}(w, z) = f_{X,Y}(x, y)\,|J| = \tfrac{1}{2z} \text{ for } 0 < x < 1,\ 0 < y < 1,\ w = xy, \text{ and } z = \tfrac{x}{y}.$$

(c)
To find the joint PDF of R and X, where $R = \sqrt{X^2 + Y^2}$, we use the transformation $X = R\cos\Theta$ and $Y = R\sin\Theta$. Assuming that $f_{X,Y}(x, y) = \frac{1}{a}$ where a is a constant for $-1 \le x \le 1$ and $-1 \le y \le 1$, we have:

$$\textstyle\int_0^{2\pi} \int_0^1 f_{R,\Theta}(r, \theta)\, dr\, d\theta = 1 = \int_0^{2\pi} \int_0^1 f_{X,Y}(x, y) r\, dr\, d\theta$$

Solving for a, we find that $a = \pi$. Therefore, the joint PDF is:

$$f_{R,X}(r, x) = f(R = r, X = x) = f(R = r, \Theta = \arccos\tfrac{x}{r}) = \tfrac{r}{\pi} \text{ for } 0 \le r \le 1 \text{ and } -r \le x \le r.$$

(d)
To find the joint PDF of Z and R, where $R = \sqrt{X^2 + Y^2 + Z^2}$, we use the transformation $X = R\sin M\cos\Theta$, $Y = R\sin M\sin\Theta$, and $Z = R\cos M$. The joint PDF of (X, Y, Z) is $f_{X,Y,Z}(x, y, z) = \frac{1}{\frac{4\pi}{3}}$. Thus, we have:

$$f_{R,M}(r, m) = \textstyle\int_0^{2\pi} f_{R,M,\Theta}(r, m, \theta)\, d\theta = \frac{3r^2\sin(m)}{2}$$

Therefore, the joint PDF of R and Z is:

$$f_{R,Z}(r, z) = f(R = r, Z = z) = f(R = r, M = \arcsin\sqrt{1 - \tfrac{z^2}{r^2}}) = \frac{3r^2\sqrt{1-\frac{z^2}{r^2}}}{2}$$

for $0 \le r \le 1$ and $-r \le z \le r$.

Problem 6

(**Optional Challenging Problem**)
Let X and Y be i.i.d. Unif$(0, 1)$, and $Z = \frac{X}{Y}$. Find the probability that the integer close to Z is even.

Solution

If n is the integer nearest to Z, then

$$n - 0.5 \le Z \le n + 0.5$$
$$\frac{X}{n + 0.5} \le Y \le \frac{X}{n - 0.5}$$

Given $0 < X < 1$, the probability that Y yields a valid even number $n = 2k$ is

$$P(X, Y) = 2X \sum_{k=1}^{\infty} (\frac{1}{4k - 1} - \frac{1}{4k + 1}) = aX$$

And the probability that the integer close to Z is even is

$$P = \int_0^1 P(X, Y)dX = \int_0^1 aXdX = \frac{a}{2} = 2 \sum_{k=1}^{+\infty} \frac{1}{16k^2 - 1} = 1 - \frac{\pi}{4}$$

Now, we solve the probability to obtain 0 corresponding to the integer close to Z.

$$P(0) = \int_0^{1/2} (1 - 2X)dX = 1/4$$

Thus, the probability that the integer close to Z is even is $1 - \frac{\pi}{4} + 1/4 = \frac{5-\pi}{4}$.