

SI 140A-02 Probability & Statistics for EECS, Fall 2024

Homework 13

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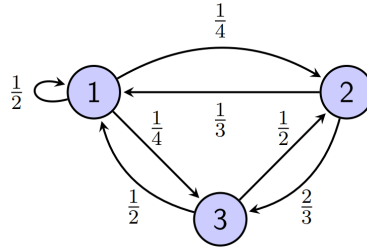
Due on Jan. 7, 2025, 11:59 UTC+8

Read all the instructions below carefully before you start working on the assignment, and before you make a submission.

- You are required to write down all the major steps towards making your conclusions; otherwise you may obtain limited points of the problem.
- Write your homework in English; otherwise you will get no points of this homework.
- Any form of plagiarism will lead to 0 point of this homework.

Problem 1

Given a Markov chain with state-transition diagram shown as follows:



- Is this chain irreducible?
- Is this chain aperiodic?
- Find the stationary distribution of this chain.
- Is this chain reversible?

Solution

For the Markov chain, we have the transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

- Yes, this chain is irreducible because all the elements apart from the diagonal elements are non-zero.
- Yes, we can see that 1 is a possible return time for state 1, thus $d(1) = 1$. Since both 2,3 are possible for state 2,3 $d(2) = d(3) = 1$. And because the chain is irreducible, so the chain is aperiodic.
- Denote the stationary distribution as $\pi = (\pi_1, \pi_2, \pi_3)$, we have

$$\pi P = \pi$$

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix}$$

We have the following equations:

$$\begin{aligned} \frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3 &= \pi_1 \\ \frac{1}{4}\pi_1 + \frac{2}{3}\pi_2 + \frac{1}{2}\pi_3 &= \pi_2 \\ \frac{1}{4}\pi_1 + \frac{2}{3}\pi_2 &= \pi_3 \end{aligned}$$

Solving the above equations, we have

$$\begin{aligned} \pi_1 &= \frac{16}{35} \\ \pi_2 &= \frac{9}{35} \\ \pi_3 &= \frac{2}{7} \end{aligned}$$

Thus, the stationary distribution of this chain is $\pi = (\frac{16}{35}, \frac{9}{35}, \frac{2}{7})$.

(d) No, this chain is not reversible. If the chain is reversible, we should have

$$\pi_i p_{ij} = \pi_j p_{ji}$$

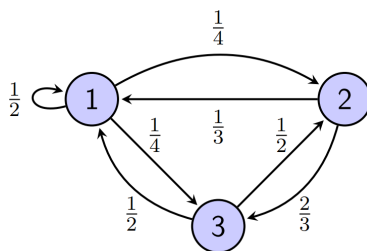
However, we can see that

$$\begin{aligned}\pi_1 p_{12} &= \frac{16}{35} \cdot \frac{1}{4} = \frac{4}{35} \\ \pi_2 p_{21} &= \frac{9}{35} \cdot \frac{1}{3} = \frac{3}{35}\end{aligned}$$

Thus, the chain is not reversible.

Problem 2

Given a Markov chain with state-transition diagram shown as follows:



- Find $P(X_9 = 3 \mid X_8 = 1)$ and $P(X_8 = 2 \mid X_7 = 3)$.
- If $P(X_0 = 3) = \frac{1}{2}$, find $P(X_0 = 3, X_1 = 1, X_2 = 2, X_4 = 3)$.
- Find $E(X_8 \mid X_6 = 2)$.
- Find $\text{Var}(X_7 \mid X_5 = 3)$.

Solution

For the Markov chain, we have the transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

(a)

$$P(X_9 = 3 \mid X_8 = 1) = p_{13} = \frac{1}{4}$$

$$P(X_8 = 2 \mid X_7 = 3) = p_{32} = \frac{1}{2}$$

(b)

$$P(X_0 = 3, X_1 = 1, X_2 = 2, X_4 = 3) = P(X_0 = 3)P(X_1 = 1 \mid X_0 = 3)P(X_2 = 2 \mid X_1 = 1)P(X_4 = 3 \mid X_2 = 2)$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{12} = \frac{1}{192}$$

(c)

$$P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{11}{24} & \frac{1}{4} & \frac{7}{24} \\ \frac{1}{2} & \frac{5}{12} & \frac{1}{12} \\ \frac{5}{12} & \frac{1}{8} & \frac{11}{24} \end{bmatrix}$$

$$E(X_8 \mid X_6 = 2) = \sum_{i=1}^3 i \cdot p_{2i} = 1 \cdot \frac{1}{2} + 2 \cdot \frac{5}{12} + 3 \cdot \frac{1}{12} = \frac{19}{12}$$

(d) To find $\text{Var}(X_7 \mid X_5 = 3)$, we first need to find $P(X_7 \mid X_5 = 3)$. We have

$$\text{Var}(X_7 \mid X_5 = 3) = E(X_7^2 \mid X_5 = 3) - E(X_7 \mid X_5 = 3)^2$$

We have

$$E(X_7^2 \mid X_5 = 3) = \sum_{i=1}^3 i^2 \cdot p_{3i} = 1^2 \cdot \frac{5}{12} + 2^2 \cdot \frac{1}{8} + 3^2 \cdot \frac{11}{24} = \frac{121}{24}$$

$$E(X_7 \mid X_5 = 3) = \sum_{i=1}^3 i \cdot p_{3i} = 1 \cdot \frac{5}{12} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{11}{24} = \frac{49}{24}$$

Thus, we have

$$\text{Var}(X_7 \mid X_5 = 3) = \frac{121}{24} - \left(\frac{49}{24}\right)^2 = \frac{503}{576}$$

Problem 3

There are two urns with a total of $2N$ distinguishable balls. Initially, the first urn has N white balls and the second urn has N black balls. At each stage, we pick a ball at random from each urn and interchange them. Let X_n be the number of black balls in the first urn at time n . This is a Markov chain on the state space $\{0, 1, \dots, N\}$.

- Find the transition probabilities of the chain.
- Find the stationary distribution of the chain.

Solution

- Denote the transition probability as p_{ij} from i to j . The number of black balls change by at most 1 at each stage. Thus, we have $p_{ij} = 0$ if $|i - j| > 1$.

Also, we have

$$p_{i,i+1} = \left(\frac{N-i}{N}\right)^2$$

$$p_{i,i-1} = \left(\frac{i}{N}\right)^2$$

And for the number of black balls to stay the same, we have

$$p_{ii} = 1 - p_{i,i+1} - p_{i,i-1} = 1 - \frac{N-i}{N} - \frac{i}{N} = \frac{2i(N-i)}{N^2}$$

- Denote the stationary distribution as $s = (s_0, s_1, \dots, s_N)$. According to the result of (a), we have $p_{i,i+1}$ and $p_{i+1,i}$ for each i ,

$$p_{i,i+1} = \left(\frac{N-i}{N}\right)^2 p_{i+1,i} = \left(\frac{i+1}{N}\right)^2$$

$$p_{i,i-1} = \left(\frac{i}{N}\right)^2 p_{i-1,i} = \left(\frac{N-i+1}{N}\right)^2$$

We have an intuitive guess that the chain is reversible. Thus, to make it, we have

$$s_i p_{i,i+1} = s_{i+1} p_{i+1,i}$$

$$s_i p_{i,i-1} = s_{i-1} p_{i-1,i}$$

From that, we made a guess that the stationary distribution should be

$$s_i = \frac{\binom{N}{i} \binom{N}{N-i}}{\binom{2N}{N}}$$

To prove it, we need to show that

$$s_i p_{i,j} = s_j p_{j,i}$$

for all i, j .

However, if $|i - j| > 1$, we have $s_i p_{i,j} = 0 = s_j p_{j,i}$. If $i = j$, then both sides are equal to $s_i p_{i,i}$. Thus, we only need to discuss the case where $|i - j| = 1$.

If $j = i + 1$, we have

$$s_i p_{i,i+1} = \frac{\binom{N}{i} \binom{N}{N-i}}{\binom{2N}{N}} \cdot \left(\frac{N-i}{N}\right)^2 = \frac{\binom{N}{i}^2 (N-i)^2}{\binom{2N}{N} N^2}$$

and

$$s_{i+1} p_{i+1,i} = \frac{\binom{N}{i+1} \binom{N}{N-i-1}}{\binom{2N}{N}} \cdot \left(\frac{i+1}{N}\right)^2 = \frac{\binom{N}{i+1}^2 (i+1)^2}{\binom{2N}{N} N^2}$$

We can see that the two sides are equal because

$$\binom{N}{i}(N-i) = \binom{N}{i+1}(i+1)$$

Similarly, we can prove that the two sides are equal when $j = i - 1$.

$$s_i p_{i,i-1} = \frac{\binom{N}{i} \binom{N}{N-i}}{\binom{2N}{N}} \cdot \left(\frac{i}{N}\right)^2 = \frac{\binom{N}{i}^2 i^2}{\binom{2N}{N} N^2}$$

and

$$s_{i-1} p_{i-1,i} = \frac{\binom{N}{i-1} \binom{N}{N-i+1}}{\binom{2N}{N}} \cdot \left(\frac{N-i+1}{N}\right)^2 = \frac{\binom{N}{i-1}^2 (N-i+1)^2}{\binom{2N}{N} N^2}$$

We can see that the two sides are equal because

$$\binom{N}{i} i = \binom{N}{i-1} (N-i+1)$$

Thus, the stationary distribution of the chain is

$$s_i = \frac{\binom{N}{i} \binom{N}{N-i}}{\binom{2N}{N}}$$