Appendix

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Proposition 9. Let (U, C) be an FβCAS and B is a subset of C. Suppose that R_B^β is a fuzzy β-covering relation. For each $E, F \in \mathcal{F}(U)$, the upper $\overline{B_\beta}$ and lower $\underline{B_\beta}$ approximations satisfy the following properties:

(1) $\overline{B_\beta}(\phi) = \phi$, $\underline{B_\beta}(U) = U$.

(2) $\overline{B_\beta}(\mathcal{N}_S(E)) = \mathcal{N}_S(\underline{B_\beta}(E))$, $\underline{B_\beta}(\mathcal{N}_S(E)) = \mathcal{N}_S(\overline{B_\beta}(E))$, where $\mathcal{N}_S$ is a standard negator.

(3) $\overline{B_\beta}(\overline{B_\beta}(E)) = \underline{B_\beta}(\overline{B_\beta}(E)) = \overline{B_\beta}(E)$, $\underline{B_\beta}(\underline{B_\beta}(E)) = \overline{B_\beta}(\underline{B_\beta}(E)) = \underline{B_\beta}(E)$, when R_B^β is a fuzzy equivalence relation.

(4) $\overline{B_\beta}(E \cup F) = \overline{B_\beta}(E) \cup \overline{B_\beta}(F)$, $\underline{B_\beta}(E \cap F) = \underline{B_\beta}(E) \cap \underline{B_\beta}(F)$.

(5) $\overline{B_\beta}(E \cap F) \subseteq \overline{B_\beta}(E) \cap \overline{B_\beta}(F)$, $\underline{B_\beta}(E \cup F) \supseteq \underline{B_\beta}(E) \cup \underline{B_\beta}(F)$.

(6) $E \subseteq F \Rightarrow \overline{B_\beta}(E) \subseteq \overline{B_\beta}(F)$, $\underline{B_\beta}(E) \subseteq \underline{B_\beta}(F)$.

Proof. (1) For each $x \in U$, by Definition 9, we have

$$\overline{B_\beta}(\phi)(x) = \bigvee_{y \in U}\{R_B^\beta(x, y) \wedge \phi(y)\} = 0 = \phi(x),$$

$$\underline{B_\beta}(U)(x) = \bigwedge_{y \in U}\{(1 - R_B^\beta(x, y)) \vee U(y)\} = 1 = U(x).$$

Thus, $\overline{B_\beta}(\phi) = \phi$, $\underline{B_\beta}(U) = U$.

(2) For each $x \in U$, by Definition 9, we have

$$\begin{aligned}\overline{B_\beta}(\mathcal{N}_S(E))(x) &= \bigvee_{y \in U}\{R_B^\beta(x, y) \wedge \mathcal{N}_S(E(y))\} \\ &= 1 - \bigwedge_{y \in U}\{(1 - R_B^\beta(x, y)) \vee E(y)\} \\ &= 1 - \underline{B_\beta}(E)(x) = \mathcal{N}_S(\underline{B_\beta}(E)(x)).\end{aligned}$$

If E is replaced by $\mathcal{N}_S(E)$ in the proof, then we have $\underline{B_\beta}(\mathcal{N}_S(E))(x) = \mathcal{N}_S(\overline{B_\beta}(E)(x))$. Thus, $\overline{B_\beta}(\mathcal{N}_S(E)) = \mathcal{N}_S(\underline{B_\beta}(E))$, $\underline{B_\beta}(\mathcal{N}_S(E)) = \mathcal{N}_S(\overline{B_\beta}(E))$.

(3) For each $x, y \in U$, $R_B^\beta(x, y)$ is a fuzzy equivalence relation, which implies

$R_B^\beta(x,z)=R_B^\beta(y,z)$ for all $z\in U$, we have

$$
\begin{aligned}
&\overline{B_\beta}(\overline{B_\beta}(E)(x))\\
&=\bigvee_{y\in U}\{R_B^\beta(x,y)\wedge\overline{B_\beta}(E)(y)\}\\
&=\bigvee_{y\in U}\{R_B^\beta(x,y)\wedge\bigvee_{z\in U}\{R_B^\beta(y,z)\wedge E(z)\}\}\\
&=\bigvee_{y\in U}\{R_B^\beta(x,y)\wedge\bigvee_{z\in U}\{R_B^\beta(x,z)\wedge E(z)\}\}\\
&=\bigwedge_{y\in U}\{(1-R_B^\beta(x,y))\vee\bigvee_{z\in U}\{R_B^\beta(x,z)\wedge E(z)\}\}\\
&=\bigwedge_{y\in U}\{(1-R_B^\beta(x,y))\vee\bigvee_{z\in U}\{R_B^\beta(y,z)\wedge E(z)\}\}\\
&=\bigwedge_{y\in U}\{(1-R_B^\beta(x,y))\vee\overline{B_\beta}(E)(y)\}=\underline{B_\beta}(\overline{B_\beta}(E)(x))\\
&=\bigvee_{z\in U}\{R_B^\beta(x,z)\wedge E(z)\}=\overline{B_\beta}(E)(x).
\end{aligned}
$$

Similarly, there is $\underline{B_\beta}(\underline{B_\beta}(E)(x))=\overline{B_\beta}(\underline{B_\beta}(E)(x))=\underline{B_\beta}(E)(x)$ for each $x\in U$. Thus, when R_B^β is a fuzzy equivalence relation, $\overline{B_\beta}(\overline{B_\beta}(E))=\underline{B_\beta}(\overline{B_\beta}(E))=\overline{B_\beta}(E)$, $\underline{B_\beta}(\underline{B_\beta}(E))=\overline{B_\beta}(\underline{B_\beta}(E))=\underline{B_\beta}(E)$.

(4) For each $x\in U$, by Definition 9, we have

$$
\begin{aligned}
&\overline{B_\beta}(E\cup F)(x)\\
&=\bigvee_{y\in U}\{R_B^\beta(x,y)\wedge(E\cup F)(y)\}\\
&=\bigvee_{y\in U}\{R_B^\beta(x,y)\wedge(E(y)\vee F(y))\}\\
&=\bigvee_{y\in U}\{R_B^\beta(x,y)\wedge E(y)\}\vee\bigvee_{y\in U}\{R_B^\beta(x,y)\wedge F(y)\}\\
&=\overline{B_\beta}(E)(x)\vee\overline{B_\beta}(F)(x)=(\overline{B_\beta}(E)\cup\overline{B_\beta}(F))(x).
\end{aligned}
$$

Similarly, there is $\underline{B_\beta}(E\cap F)(x)=(\underline{B_\beta}(E)\cap\underline{B_\beta}(F))(x)$ for each $x\in U$. Thus, $\overline{B_\beta}(E\cup F)=\overline{B_\beta}(E)\cup\overline{B_\beta}(F)$, $\underline{B_\beta}(E\cap F)=\underline{B_\beta}(E)\cap\underline{B_\beta}(F)$.

(5) For each $x\in U$, by Definition 9, we have

$$
\begin{aligned}
&\overline{B_\beta}(E\cap F)(x)\\
&=\bigvee_{y\in U}\{R_B^\beta(x,y)\wedge(E\cap F)(y)\}\\
&=\bigvee_{y\in U}\{R_B^\beta(x,y)\wedge(E(y)\wedge F(y))\}\\
&\leqslant\bigvee_{y\in U}\{R_B^\beta(x,y)\wedge E(y)\}\wedge\bigvee_{y\in U}\{R_B^\beta(x,y)\wedge F(y)\}\\
&=\overline{B_\beta}(E)(x)\wedge\overline{B_\beta}(F)(x)=(\overline{B_\beta}(E)\cap\overline{B_\beta}(F))(x).
\end{aligned}
$$

Similarly, for each $x \in U$, there is

$$\begin{aligned}
&\underline{B_\beta}(E \cup F)(x) \\
=& \bigwedge_{y \in U} \{(1 - R_B^\beta(x,y)) \vee (E \cup F)(y)\} \\
=& \bigwedge_{y \in U} \{(1 - R_B^\beta(x,y)) \vee (E(y) \vee F(y))\} \\
\geqslant& \bigwedge_{y \in U} \{(1 - R_B^\beta(x,y)) \vee E(y)\} \vee \bigwedge_{y \in U} \{(1 - R_B^\beta(x,y)) \vee F(y)\} \\
=& \underline{B_\beta}(E)(x) \vee \underline{B_\beta}(F)(x) = (\underline{B_\beta}(E) \cup \underline{B_\beta}(F))(x).
\end{aligned}$$

Thus, $\overline{B_\beta}(E \cap F) \subseteq \overline{B_\beta}(E) \cap \overline{B_\beta}(F)$, $\underline{B_\beta}(E \cup F) \supseteq \underline{B_\beta}(E) \cup \underline{B_\beta}(F)$.

(6) For each $x \in U$, $E \subseteq F$, which implies $E(x) \leqslant F(x)$. By Definition 9, we have

$$\begin{aligned}
\overline{B_\beta}(E)(x) &= \bigvee_{y \in U} \{R_B^\beta(x,y) \wedge E(y)\} \\
&\leqslant \bigvee_{y \in U} \{R_B^\beta(x,y) \wedge F(y)\} = \overline{B_\beta}(F)(x).
\end{aligned}$$

Similarly, there is $\underline{B_\beta}(E)(x) \leqslant \underline{B_\beta}(F)(x)$ for each $x \in U$. Thus, $E \subseteq F \Rightarrow \overline{B_\beta}(E) \subseteq \overline{B_\beta}(F)$, $\underline{B_\beta}(E) \subseteq \underline{B_\beta}(F)$.

$\square$