Stat 435 Lecture Notes 2a

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Inference on coefficients

Inference on estimates

Since the observations $(x_i, y_i), i = 1, \ldots, n$ for (X, Y) are random, so is the LSE $\left(\hat{\beta}_0, \hat{\beta}_1\right)$.

- $\left(\hat{\beta}_0, \hat{\beta}_1\right)$ is unbiased, i.e., $E(\hat{\beta}_0) = \beta_0$ and $E(\hat{\beta}_1) = \beta_1$, when ε_i's are uncorrelated
- How accurate is $\left(\hat{\beta}_0, \hat{\beta}_1\right)$ with respect to (β_0, β_1)?

Variability of estimates

Recall

$$\hat{\beta}_1 = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}$$

and $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$.

When ε_i's are uncorrelated,

- $[SE(\hat{\beta}_0)]^2 = \sigma^2 \left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2}\right]$
- $[SE(\hat{\beta}_1)]^2 = \frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{x})^2}$

How does the sample size n and the sample variance of x_i's affect these variances?

Variability of estimates

- Variances of $\hat{\beta}_0$ and $\hat{\beta}_1$ contain information on the accuracy of $\hat{\beta}_0$ and $\hat{\beta}_1$
- Without information on σ, variances of $\hat{\beta}_0$ and $\hat{\beta}_1$ cannot be accurately assessed
- An estimate of σ is

$$RSE = \sqrt{RSS/(n-2)} = \sqrt{\frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{n-2}}$$

CI for estimates

The approximate 95% *confidence interval (CI)*

- for $\hat{\beta}_0$ is $\hat{\beta}_0 \pm 2 \cdot SE(\hat{\beta}_0)$
- for $\hat{\beta}_1$ is $\hat{\beta}_1 \pm 2 \cdot SE(\hat{\beta}_1)$

Note: the above follows a general principle for constructing a CI when the distribution of “estimate minus parameter” is symmetric around 0

Testing the slope

- Recall the model
$$Y = \beta_0 + \beta_1 X + \varepsilon$$
with $E(\varepsilon) = 0$ and $Var(\varepsilon) = \sigma^2$
- If $\beta_1 = 0$, then $Y = \beta_0 + \varepsilon$, and Y is independent of X (when ε is independent X).
- Testing "$H_0 : \beta_1 = 0$" often is equivalent to checking if "H_0: there is no relationship between X and Y".

Caution: The random error ε may not be independent of X and Y due to latent dependence, which is common in genetics studies.

Testing the slope

- A test statistic for this purpose is the *t-statistic*

$$t = \frac{\hat{\beta}_1 - 0}{SE(\hat{\beta}_1)}$$

- If $SE(\hat{\beta}_1)$ is small, then relatively small $|\hat{\beta}_1|$ provides strong evidence that $\beta_1 \neq 0$

Note: When $H_0 : \beta_1 = 0$ holds, t approximately has a t-distribution with $n - 2$ degrees of freedom when ε_i's are not much dependent on each other; when n is large, a t-distribution will be close to a Gaussian distribution.

Testing the slope

Model: $E($`sales`$) = \beta_0 + \beta_1$ `TV`:

```
# A tibble: 2 x 5
  term        estimate std.error statistic  p.value
  <chr>          <dbl>     <dbl>     <dbl>    <dbl>
1 (Intercept)   7.03     0.458        15.4 1.41e-35
2 TV            0.0475   0.00269      17.7 1.47e-42
```

- Is $H_0 : \beta_1 = 0$ retained or rejected? In either case, at which Type I error level?
- How trustable are our conclusions?

Testing the slope

Model $E($`mpg`$) = \beta_0 + \beta_1$ `horsepower`:

```
# A tibble: 2 x 5
  term        estimate std.error statistic   p.value
  <chr>          <dbl>     <dbl>     <dbl>     <dbl>
1 (Intercept)   39.9     0.717        55.7 1.22e-187
2 horsepower    -0.158   0.00645     -24.5 7.03e- 81
```

- Is $H_0 : \beta_1 = 0$ retained or rejected? In either case, at which Type I error level?
- How trustable are our conclusions?

Model diagnostics

Things to check

- Nonlinearity of relationship between response and predictor
- Correlation of error terms
- Non-constant variance of error terms
- Outliers
- High-leverage points

Nonlinearity

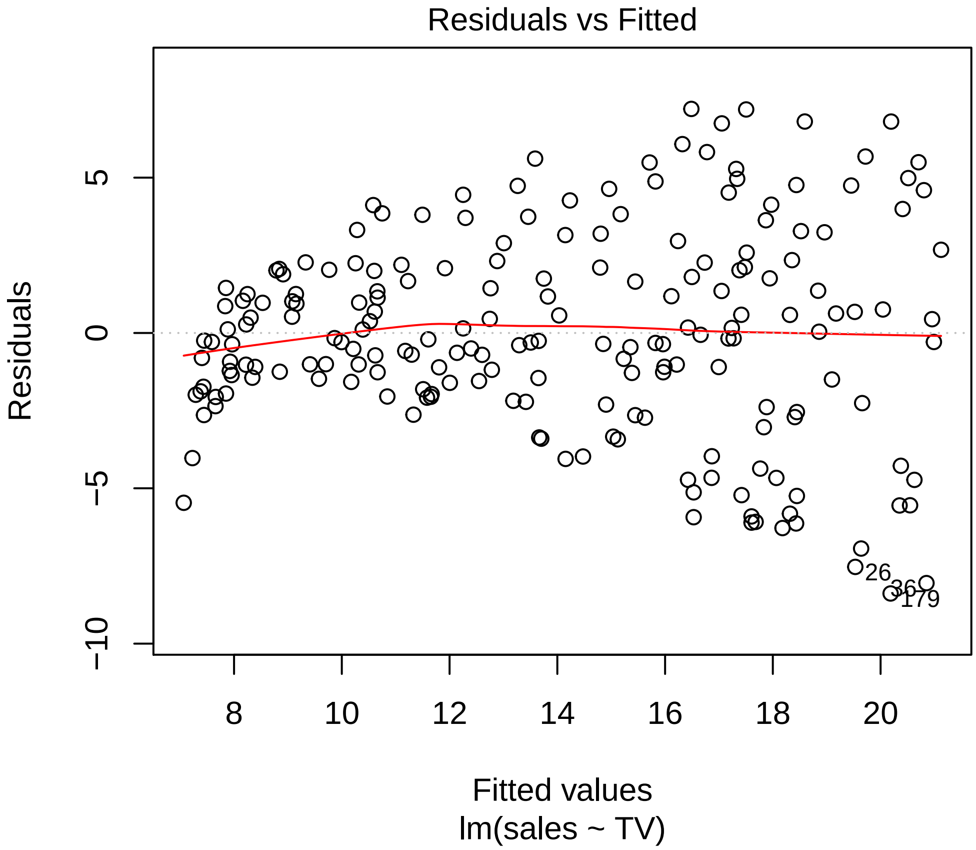
If the linear model
$$Y = \beta_0 + \beta_1 X + \varepsilon$$
is plausible, then

- $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, where ε_i is a realization of ε for each pair (x_i, y_i)
- the residuals e_i's*, where*
$$e_i = y_i - \hat{y}_i, \quad \hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i,$$
are random errors (as estimated realizations from ε), and should have no specific relationship with x_i's

So, the residuals e_i's should contain no specific pattern on the fitted values $\hat{y}_i$'s or x_i's if the model is plausible.

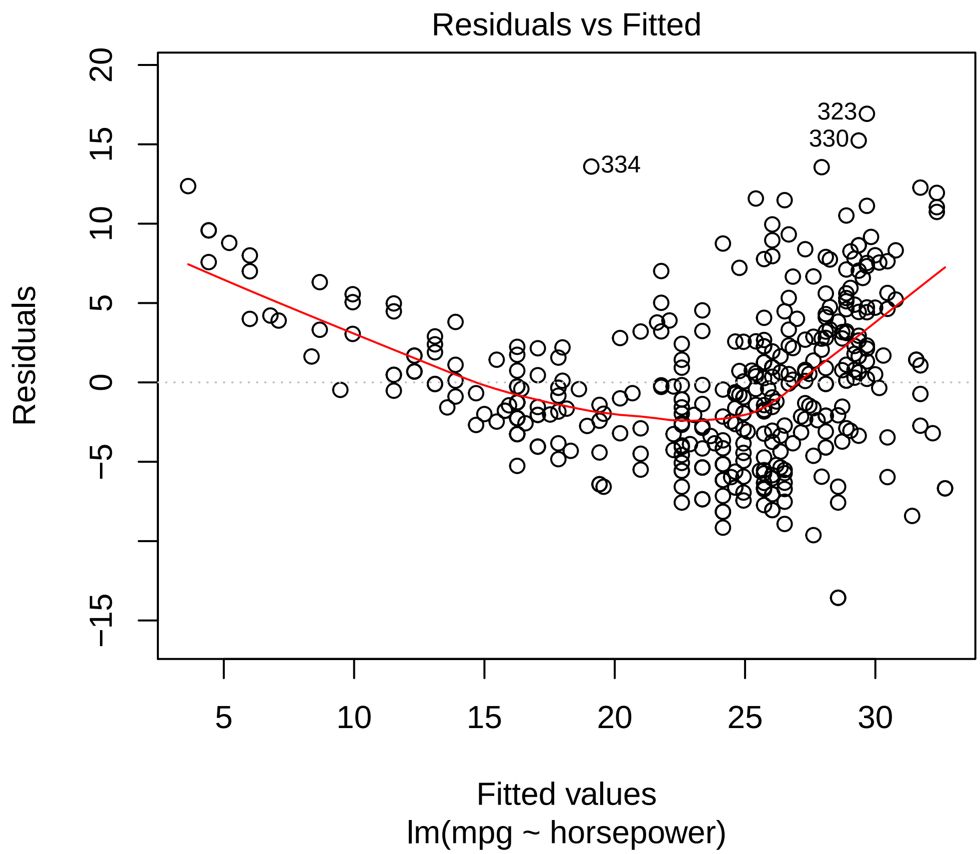
Nonlinearity

Model: $E(\texttt{sales}) = \beta_0 + \beta_1\ \texttt{TV}$:



Nonlinearity

Model $E(\texttt{mpg}) = \beta_0 + \beta_1$ `horsepower`:



Check on error terms

- Inference on the LSE $(\hat{\beta}_0, \hat{\beta}_1)$ depends critically on $SE(\hat{\beta}_0)$ and $SE(\hat{\beta}_1)$, which depend on the unknown $\sigma = \sqrt{Var(\varepsilon)}$.

- Information on ε, hence on σ, is contained in the residuals e_i (as estimates of ε_i)

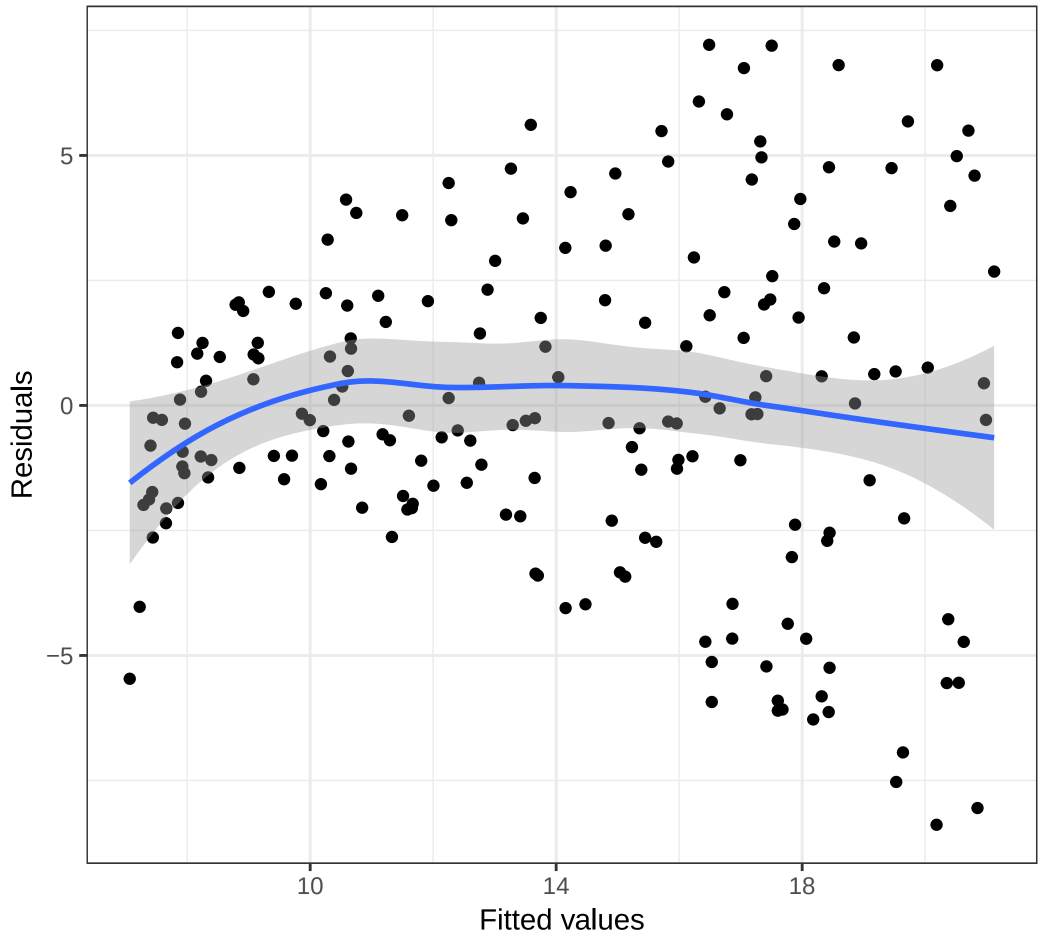
Relatively accurate inference requires e_i's to

- be *uncorrelated*
- have *identical variance*

Otherwise, the formulae for $SE(\hat{\beta}_0)$ and $SE(\hat{\beta}_1)$ are (usually) invalid, leading to invalid inference.

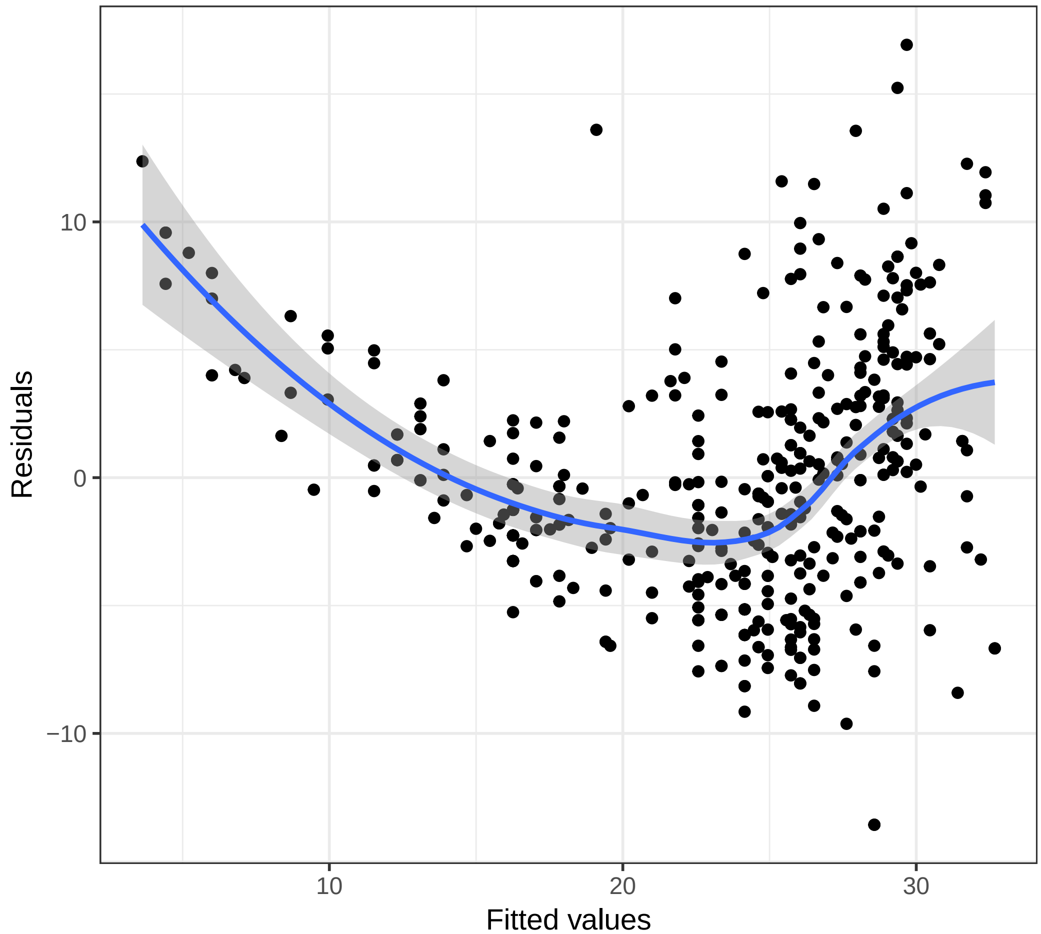
Heterogeneous error variances

Model: $E(\texttt{sales}) = \beta_0 + \beta_1\ \texttt{TV}$:



Heterogeneous error variances

Model $E(\texttt{mpg}) = \beta_0 + \beta_1\ \texttt{horsepower}$:

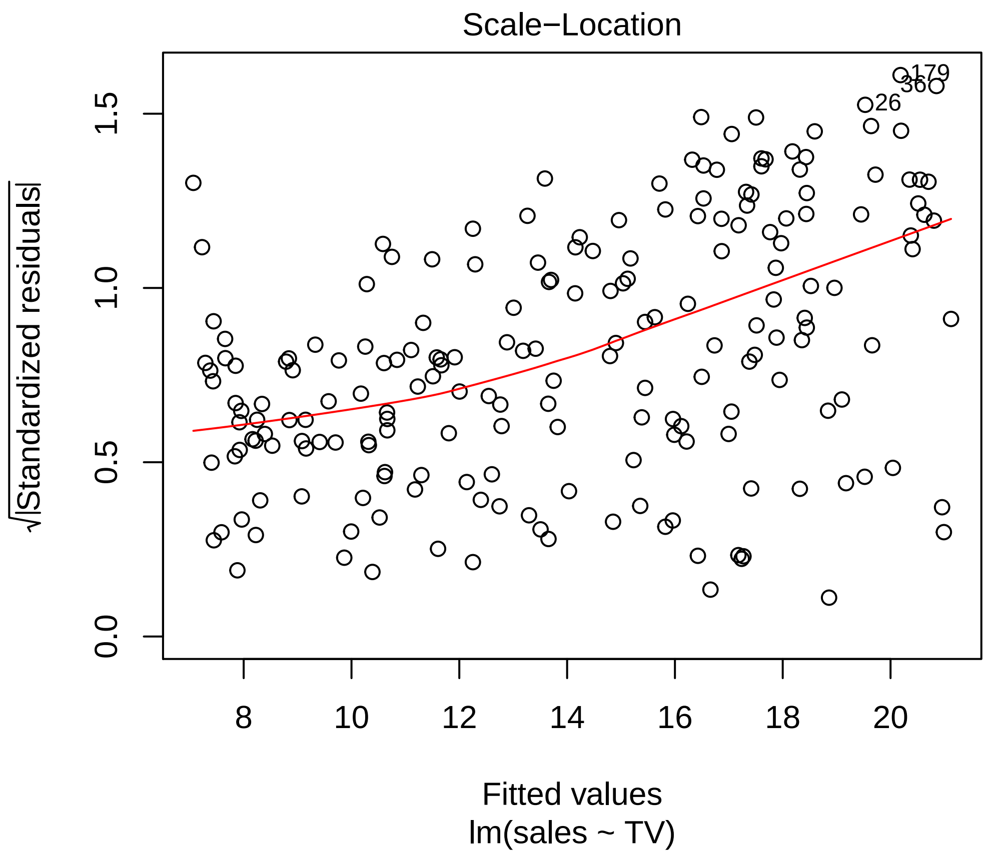


Outliers

- *An outlier* is an observation (x_i, y_i) for which y_i is far from the value predicted by the model
- Observations whose studentized residuals are greater than 3 in absolute value are possible outliers
- An outlier may significantly affect RSE (i.e., residual standard error) and R^2
- Removing an outlier may or may not significantly affect the subsequent estimated regression line, and this is related to *high-leverage points*

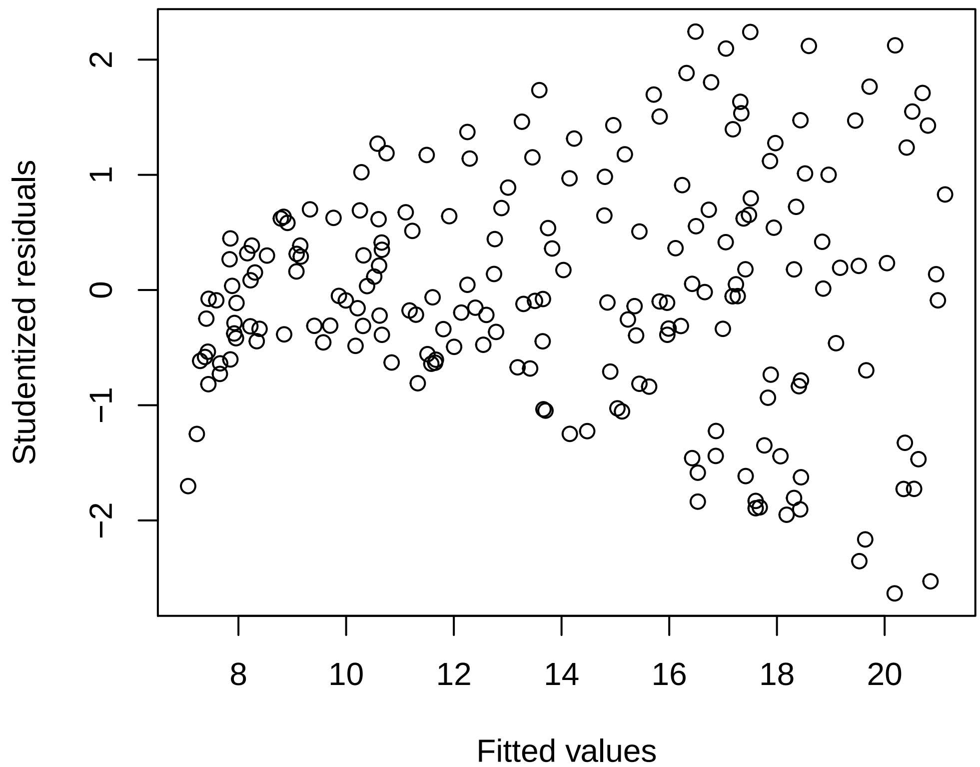
Outliers

Model: $E(\texttt{sales}) = \beta_0 + \beta_1 \, \texttt{TV}$:



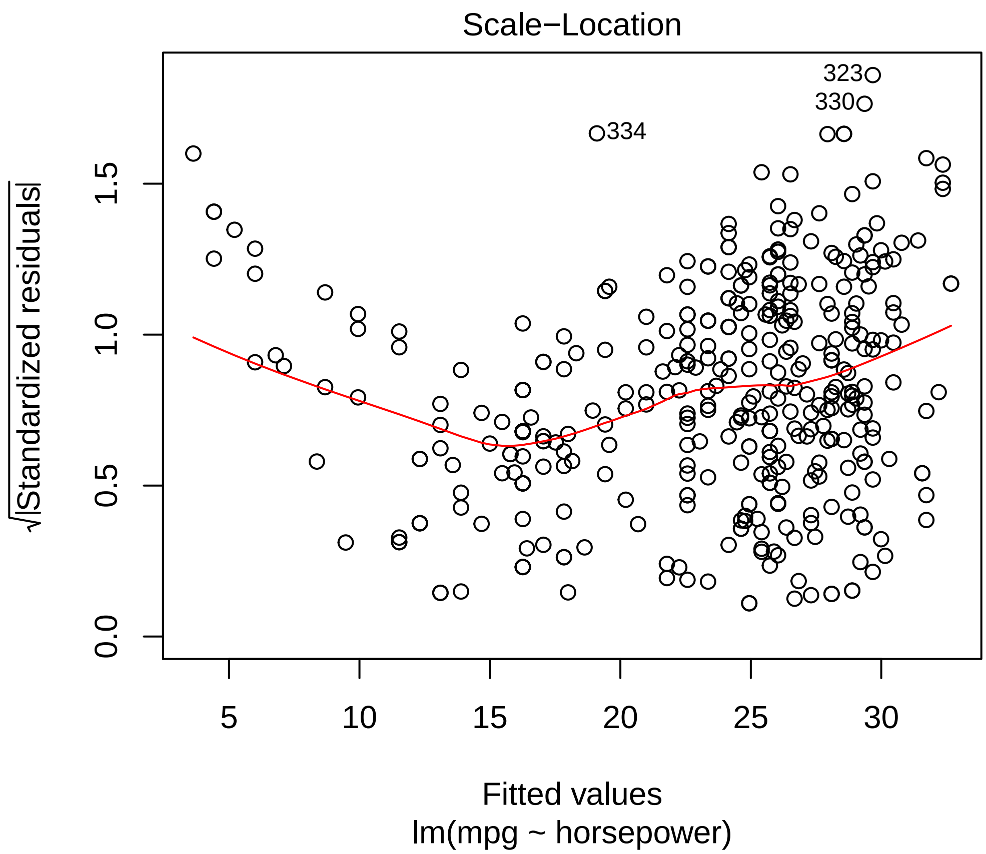
Outliers

Model: $E(\texttt{sales}) = \beta_0 + \beta_1 \ \texttt{TV}$:



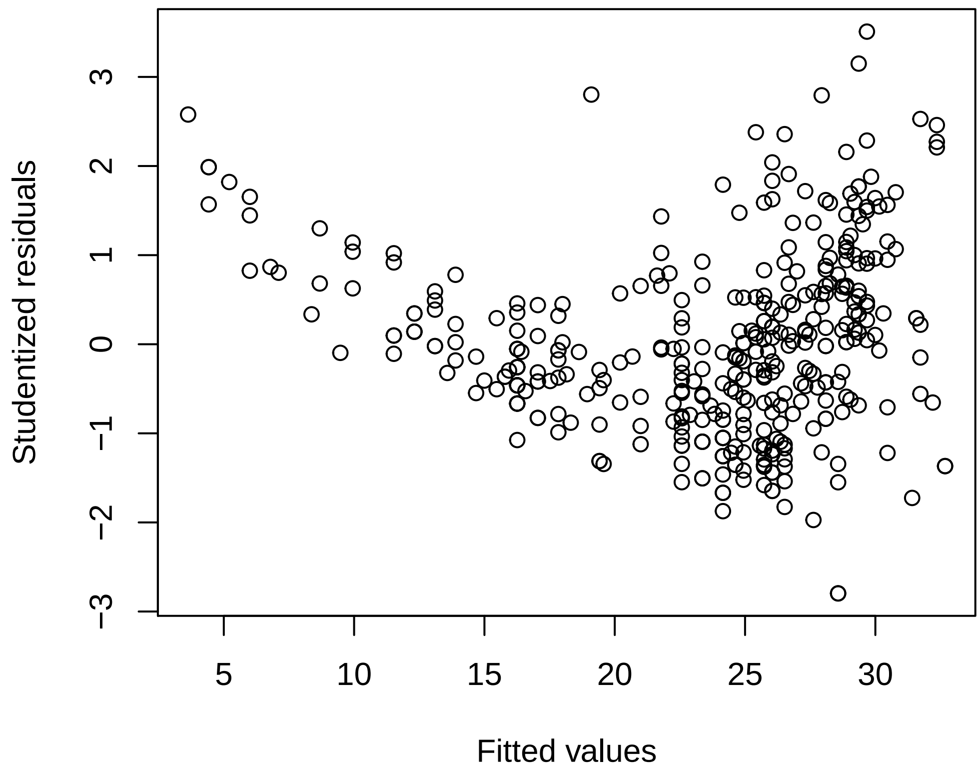
Outliers

Model $E(\texttt{mpg}) = \beta_0 + \beta_1$ `horsepower`:



Outliers

Model $E(\texttt{mpg}) = \beta_0 + \beta_1\ \texttt{horsepower}$:

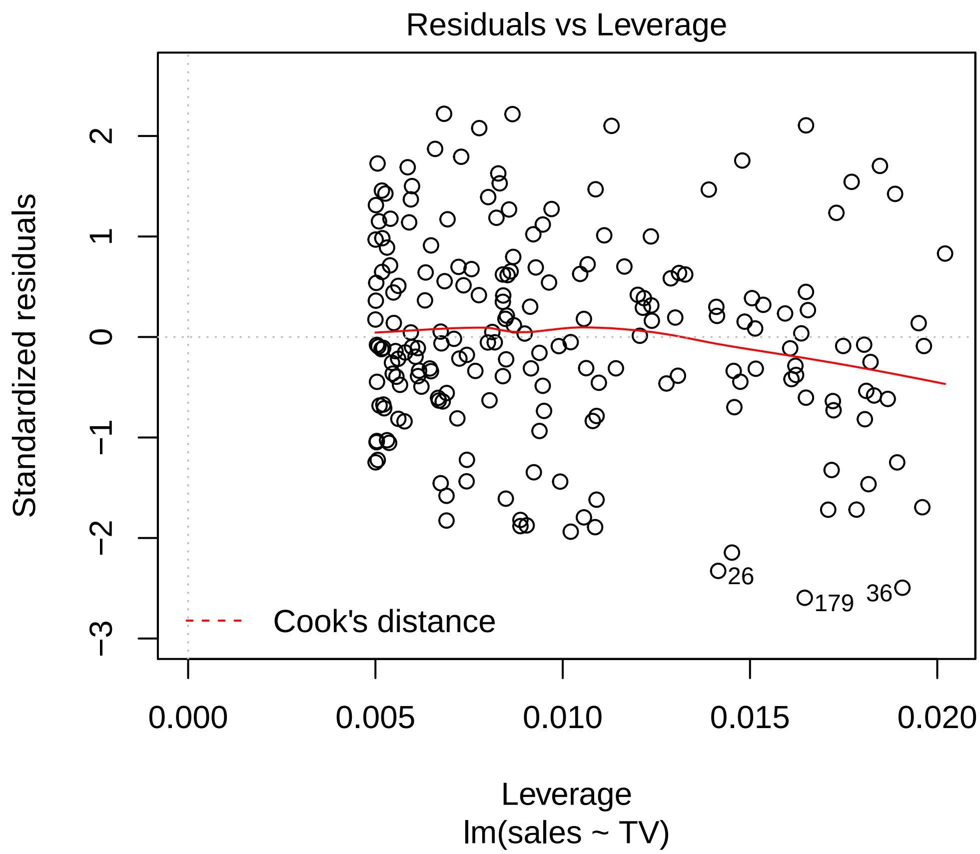


High-leverage points

- *A high-leverage point* is an observation (x_i, y_i) for which x_i is unusual among all observations for X
- Removing a high-leverage point often significantly affects the subsequent estimated regression line
- Let p be the number of predictors in the model and n the sample size, if the leverage statistic for an observation greatly exceeds $(p + 1)/n$, then it can be considered a high-leverage point

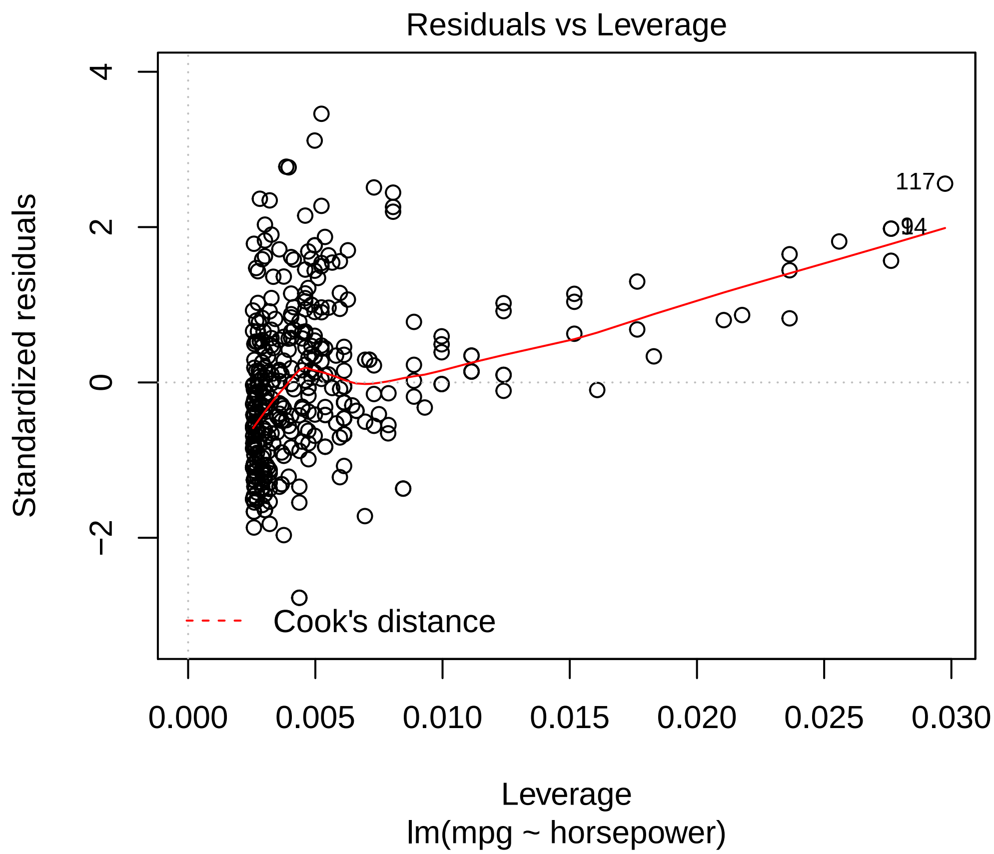
High-leverage points

Model: $E(\texttt{sales}) = \beta_0 + \beta_1\ \texttt{TV}$ ($p = 1$,$n = 397$, $\tilde{h} = (p + 1)/n \approx 0.005$):



High-leverage points

Model $E(\texttt{mpg}) = \beta_0 + \beta_1$ `horsepower` $(p = 1, n = 200, \tilde{h} = (p+1)/n = 0.01)$:



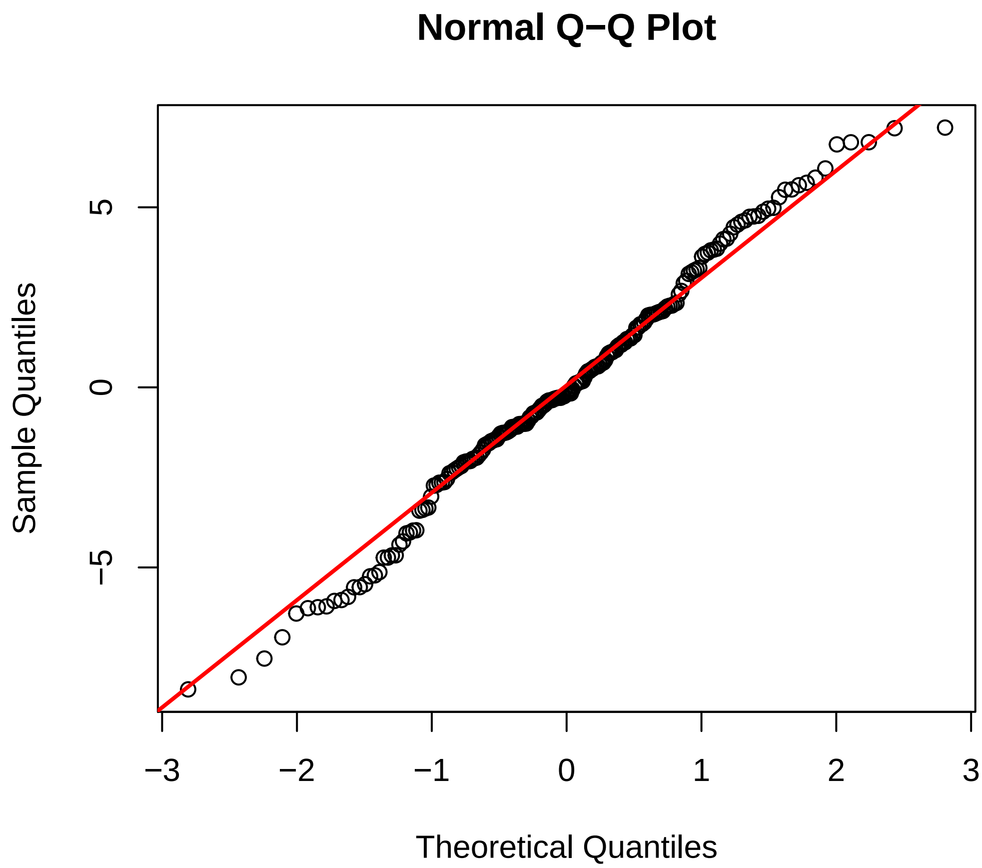
Q-Q Plot

A *Q-Q (quantile-quantile) plot* plots observed quantiles against the quantiles of a theoretical distribution, and hence provides information on whether observations under investigation have a distribution that matches this theoretical distribution

- A normal Q-Q plot does this with the standard Normal distribution as the theoretical distribution
- In a normal Q-Q plot, x-axis plots the theoretical quantiles from the standard Normal distribution with mean 0 and standard deviation 1

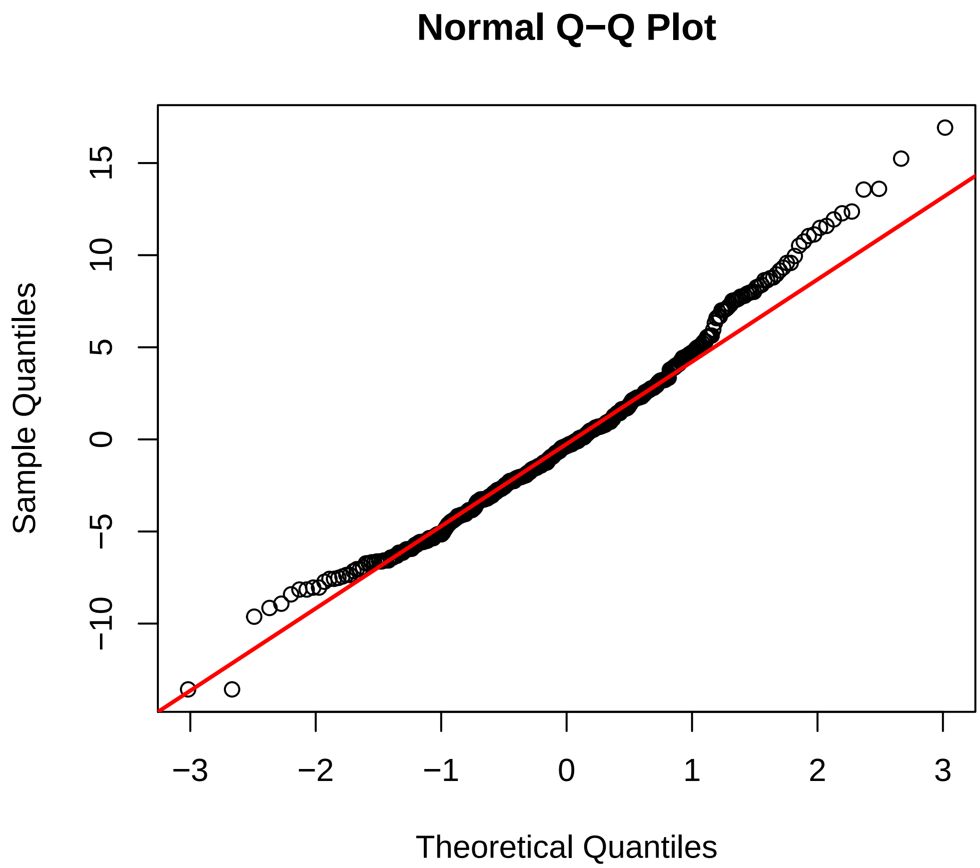
Test on Normality

- Model: $E(\texttt{sales}) = \beta_0 + \beta_1\ \texttt{TV}$
- Test Normality of random error



Test on Normality

- Model $E(\texttt{mpg}) = \beta_0 + \beta_1$ `horsepower`
- Test Normality of random error



Kolmogorov-Smirnov Test

- Model: $E(\texttt{sales}) = \beta_0 + \beta_1$ `TV`
- `Fit1` is the object obtained from fitting the model
- Test Normality of random error

```
    One-sample Kolmogorov-Smirnov test

data:  Fit1$residuals
D = 0.041533, p-value = 0.8806
alternative hypothesis: two-sided
```

Kolmogorov-Smirnov Test

- Model $E(\texttt{mpg}) = \beta_0 + \beta_1$ `horsepower`
- `Fit2` is the object obtained from fitting the model
- Test Normality of random error

```
    One-sample Kolmogorov-Smirnov test
```

```
data:  Fit2$residuals
D = 0.060525, p-value = 0.1131
alternative hypothesis: two-sided
```

Correlation of error terms

It is extremely important that the error terms are uncorrelated. Correlated error terms often present in time series data and in data with latent variables. Such correlation affects

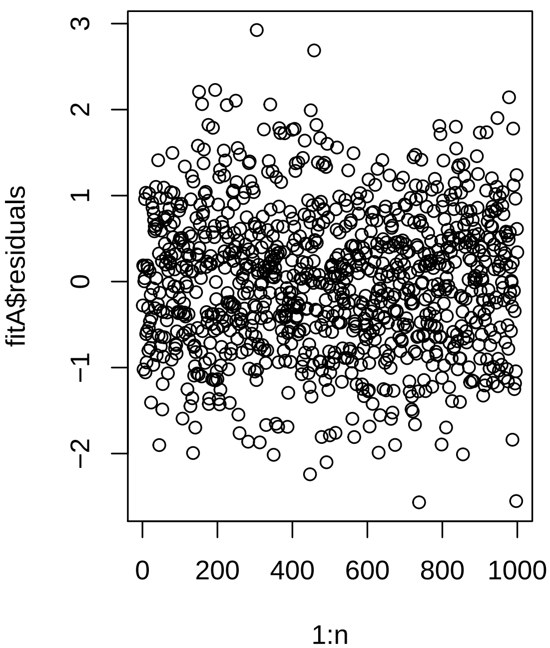
- testing if random errors are Normally distributed
- variances of estimated coefficients and variance of random error term

- Testing independence is a *highly nontrival* issue in statistical learning

Correlation of error terms

- The true model is
$$Y = 1 + 2X + \varepsilon$$
with $n = 1000$ observations
$$y_i = 1 + 2x_i + \varepsilon_i$$
- The random errors are *equally correlated*, such that
$$\varepsilon_i = \sqrt{1 - 0.3}X_i + \sqrt{0.3}X_0$$
and $X_0, X_1, \ldots, X_{1000}$ are i.i.d. standard Normal.
- Fit a simple linear model and obtain fitted values and residuals

Correlation of error terms



Correlation of error terms

For this example, when trying to check if the random errors are independent or uncorrelated by a visual check, we see the following:

- no pattern in the left plot where each residuals is plotted against its index
- the random errors are dependent, since if they were independent, the fitted values should be independent of the residuals
- the random errors do not seem to be correlated with the fitted values

Appendix

True model

- For two quantitative random variables Y and X, a simple linear model is

$$E(Y) = \beta_0 + \beta_1 X,$$

  where β_0 (*intercept*) and β_1 (*slope*) are *unknown, true model* parameters (or *coefficients*), and β_1 is called the *regression coefficient*.

- The above model is equivalent to

$$Y = \beta_0 + \beta_1 X + \varepsilon \quad \text{with} \quad E(\varepsilon) = 0, Var(\varepsilon) = \sigma^2$$

  which is called the *population regression line*.

The least squares estimate

With observations $(x_i, y_i), i = 1, \ldots, n$ for (X, Y), the LS method gives the *least squares estimate (LSE)*:

- $\hat{\beta}_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$ with $\bar{y} = n^{-1} \sum_{i=1}^n y_i$
- $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$ with $\bar{x} = n^{-1} \sum_{i=1}^n x_i$

Namely, the fitted model is

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x,$$

also called the *least squares line*.

License and session Information

License

```
> sessionInfo()
R version 3.5.0 (2018-04-23)
Platform: x86_64-w64-mingw32/x64 (64-bit)
Running under: Windows 10 x64 (build 19041)

Matrix products: default

locale:
[1] LC_COLLATE=English_United States.1252
[2] LC_CTYPE=English_United States.1252
[3] LC_MONETARY=English_United States.1252
[4] LC_NUMERIC=C
[5] LC_TIME=English_United States.1252

attached base packages:
[1] stats     graphics  grDevices utils     datasets  methods
[7] base

other attached packages:
[1] ggplot2_3.1.0 broom_0.5.1   knitr_1.21

loaded via a namespace (and not attached):
 [1] Rcpp_1.0.3       plyr_1.8.4       pillar_1.3.1
 [4] compiler_3.5.0   tools_3.5.0      digest_0.6.18
 [7] evaluate_0.12    tibble_2.1.3     nlme_3.1-137
[10] gtable_0.2.0     lattice_0.20-35  pkgconfig_2.0.2
[13] rlang_0.4.4      cli_1.0.1        rstudioapi_0.8
[16] yaml_2.2.0       xfun_0.4         withr_2.1.2
[19] dplyr_0.8.4      stringr_1.3.1    generics_0.0.2
[22] grid_3.5.0       tidyselect_0.2.5 glue_1.3.0
```

```
[25] R6_2.3.0         fansi_0.4.0      rmarkdown_1.11
[28] purrr_0.2.5      tidyr_0.8.2      magrittr_1.5
[31] backports_1.1.3  scales_1.0.0     htmltools_0.3.6
[34] assertthat_0.2.0 colorspace_1.3-2 labeling_0.3
[37] utf8_1.1.4       stringi_1.2.4    lazyeval_0.2.1
[40] munsell_0.5.0    crayon_1.3.4
```