Stat 435 Lecture Notes 3

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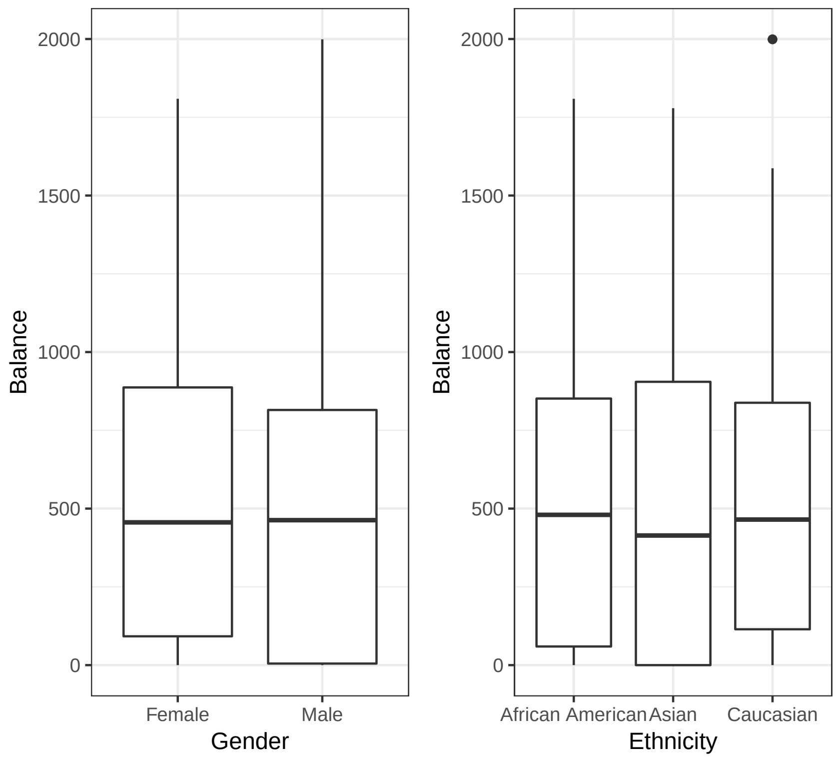
Contents

Linear regression with a qualitative predictor

Motivation

- How is `Balance` of a credit card related to a user's `Gender`?
- How is `Balance` of a credit card related to a user's `Ethnicity`?

Motivation



Model 1: 2 levels

- Coding: `Gender` has 2 levels, `Male` and `Female`
- *dummy variable*: $x_i = 0$ if ith person is `Female` and $x_i = 1$ if ith person is `Male`
- Model: $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, which induces 2 submodels:
- $y_i = \beta_0 + \beta_1 + \varepsilon_i$ if ith person is `Male`
- $y_i = \beta_0 + \varepsilon_i$ if ith person is `Female`

Note: dummy variable follows coding by R, for which the first level `Female` is the baseline

Model 1: 2 levels

Model: $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$

- $x_i = 0$ if ith person is `Female`, and $x_i = 1$ if ith person is `Male`
- β_0: average `Balance` for females
- β_1: average difference in balance between *males and females*

Remark: coding of a dummy variable is arbitrary and should be easily interpretable

Fitting the Model 1

```
Call:
lm(formula = Balance ~ Gender, data = creditData)

Coefficients:
(Intercept)   GenderMale
     529.54       -19.73
```

- `Female`s have an average `balance` of \$529.54; `Female` baseline
- `Male`s have an average `balance` of \$(529.54-19.73)= \$509.80

Note: in R, by default the first level `Female` is the baseline

Testing the Model 1

```
Call:
lm(formula = Balance ~ Gender, data = creditData)

Residuals:
    Min      1Q  Median      3Q     Max
-529.54 -455.35  -60.17  334.71 1489.20

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   529.54      31.99  16.554   <2e-16 ***
GenderMale    -19.73      46.05  -0.429    0.669
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 460.2 on 398 degrees of freedom
Multiple R-squared:  0.0004611, Adjusted R-squared:  -0.00205
F-statistic: 0.1836 on 1 and 398 DF,  p-value: 0.6685
```

- If model assumptions are met, `Gender` is not significant on affecting average `balance` at type I error level 0.05 based on F-statistic (or the p-value of `GenderMale`)

Model 2: 3 levels

`Ethnicity` has 3 levels `African American` (1st level and baseline in R), `Asian`, and `Caucasian`. *2 dummy variables* are needed:

- $x_{i1} = 0$ if ith person is *not* `Asian`, and $x_{i1} = 1$ if ith person is `Asian`
- $x_{i2} = 0$ if ith person is *not* `Caucasian`, and $x_{i2} = 1$ if ith person is `Caucasian`

- Model:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i$$

Model 2: 3 levels

Model:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i$$

- Codings on previous slide
- β_0: average `balance` for `African American`
- β_1: average difference in `balance` between `Asian` and `African American`
- β_2: average difference in `balance` between `Caucasian` and `African American`

Fitting the Model 2

```
Call:
lm(formula = Balance ~ Ethnicity, data = creditData)

Coefficients:
       (Intercept)      EthnicityAsian  EthnicityCaucasian
            531.00              -18.69              -12.50
```

- `African American`s have an average `balance` of \$531
- `Asian`s have an average `balance` of \$(531-18.69)= \$512.31
- `Caucasian`s have an average `balance` of \$(531-12.50)= \$518.5

Testing the Model 2

```
Call:
lm(formula = Balance ~ Ethnicity, data = creditData)

Residuals:
    Min      1Q  Median      3Q     Max
-531.00 -457.08  -63.25  339.25 1480.50

Coefficients:
                   Estimate Std. Error t value Pr(>|t|)
(Intercept)          531.00      46.32  11.464   <2e-16 ***
EthnicityAsian       -18.69      65.02  -0.287    0.774
EthnicityCaucasian   -12.50      56.68  -0.221    0.826
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 460.9 on 397 degrees of freedom
Multiple R-squared:  0.0002188, Adjusted R-squared:  -0.004818
F-statistic: 0.04344 on 2 and 397 DF,  p-value: 0.9575
```

If model assumptions are met, at type I error level 0.05, `Ethnicity` does not significantly affect average `balance` based on the F-statistic

Testing the Model 2

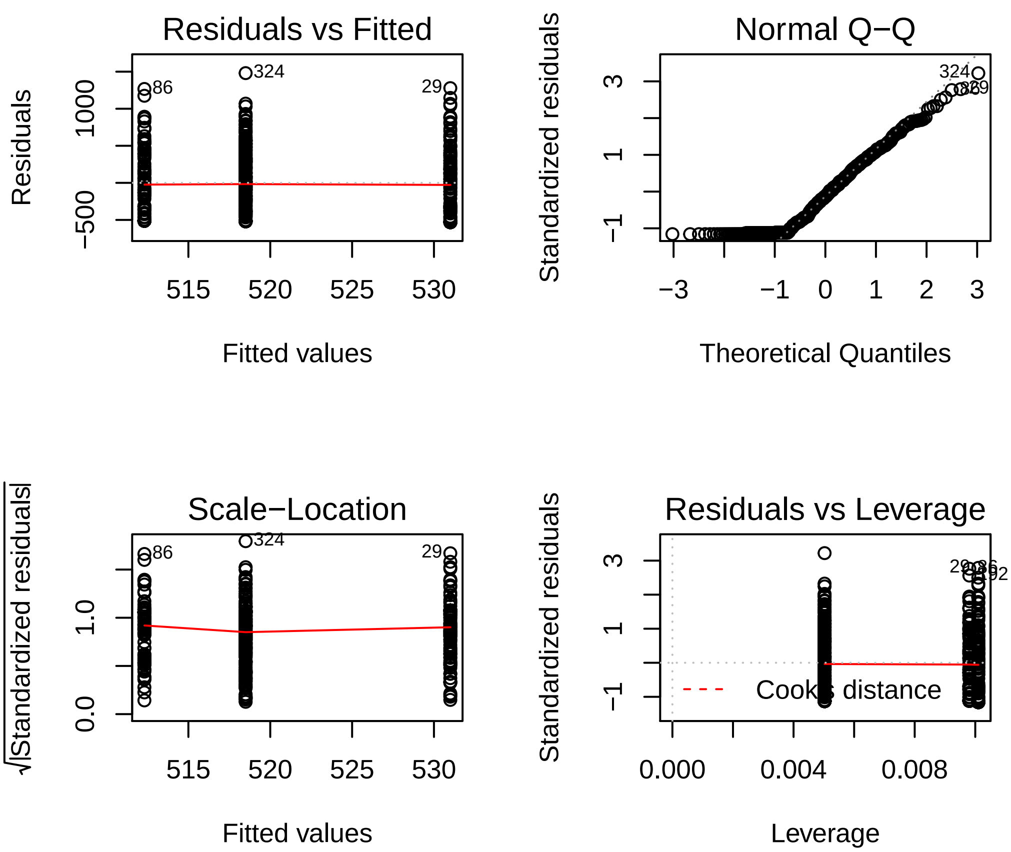
```
# A tibble: 3 x 5
  term                estimate std.error statistic  p.value
  <chr>                  <dbl>     <dbl>     <dbl>    <dbl>
1 (Intercept)            531.       46.3    11.5   1.77e-26
2 EthnicityAsian         -18.7      65.0    -0.287 7.74e- 1
3 EthnicityCaucasian     -12.5      56.7    -0.221 8.26e- 1
```

If model assumptions are met and `Ethnicity` does not significantly affect average `balance`, there is no need to check

- whether there is significant difference in average `balance` between `Asians` and `African Americans` or between `Caucasians` and `African Americans`

Diagnostics

Diagnostics are the same as those for simple linear regression with a quantitative predictor.



Multiple linear regression

Motivation

- How is `sales` (in thousands of units) for a particular product related to advertising budgets (in thousands of dollars) for `TV`, `radio` *and* `newspaper`?
- Model: `sales` $= \beta_0 + \beta_1 \times$ `TV` $+ \beta_2 \times$ `radio` $+ \beta_3 \times$ `newspaper` $+ \varepsilon$

We want to examine the relationship between `sales` and budgets for `TV`, `radio` and `newspaper` *jointly*, instead of *marginally*.

Model

Response Y and p predictors $X_1, X_2, \dots, X_p$, bound by model

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_p X_p + \varepsilon$$

- β_j: change in units in $E(Y)$ for a unit change in X_j while holding all other predictors fixed
- ε: random error term with $E(\varepsilon) = 0$ and $Var(\varepsilon) = \sigma^2$
- Estimate coefficient vector $\boldsymbol{\beta} = (\beta_0, \beta_1, \dots, \beta_p)$ by the *least squares method*; estimate $\hat{\boldsymbol{\beta}} = (\hat\beta_0, \hat\beta_1, \dots, \hat\beta_p)$ as *LSE (least squares estimate)*

Fitting model

Joint model vs marginal model:

```
# A tibble: 4 x 5
  term        estimate std.error statistic  p.value
  <chr>          <dbl>     <dbl>     <dbl>    <dbl>
1 (Intercept)  2.94      0.312       9.42  1.27e-17
2 TV           0.0458    0.00139    32.8   1.51e-81
3 radio        0.189     0.00861    21.9   1.51e-54
4 newspaper   -0.00104   0.00587    -0.177 8.60e- 1
```

```
# A tibble: 2 x 5
  term        estimate std.error statistic  p.value
  <chr>          <dbl>     <dbl>     <dbl>    <dbl>
1 (Intercept)  12.4       0.621      19.9  4.71e-49
2 newspaper     0.0547    0.0166      3.30 1.15e- 3
```

```
# A tibble: 2 x 5
  term        estimate std.error statistic  p.value
  <chr>          <dbl>     <dbl>     <dbl>    <dbl>
1 (Intercept)   7.03     0.458        15.4 1.41e-35
2 TV            0.0475   0.00269      17.7 1.47e-42
```

Testing on all coefficients

- Is there a relationship between the response and any of the predictors? Namely, is $H_0 : \beta_1 = \beta_2 = \cdots = \beta_p = 0$ true?

- Test statistic: F-statistic

$$F = \frac{(TSS - RSS)/p}{RSS/(n-p-1)},$$

  where $TSS = \sum_{i=1}^{n}(y_i - \bar{y})^2$ with $\bar{y} = n^{-1}\sum_{i=1}^{n} y_i$ and $RSS = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$
- If H_0 is true and the linear model assumptions are correct, F-statistic should be close to 1 on average; under suitable conditions, F-statistic approximately follows an F-distribution

Testing on all coefficients

Testing $H_0 : \beta_1 = \beta_2 = \cdots = \beta_p = 0$:

```
   value    numdf    dendf
570.2707   3.0000 196.0000
       value
1.575227e-96
```

F-statistic: 570.3 with numerator degrees of freedom 3 and denominator degrees of freedom 196; p-value: $<$ 2.2e-16

- Conclusion: reject H_0, meaning that *at least one of the predictors has a relationship with the response.*

Testing on some coefficients

Is there no relationship between the response and some predictors? Namely, for some $1 \leq q \leq p$, test

$$H_0 : \beta_{p-q+1} = \beta_{p-q+2} = \ldots = \beta_p = 0$$

- Fit $M_0 : Y = \beta_0 + \beta_1 X_1 + \ldots + \beta_{p-q} X_{p-q} + \varepsilon$, and obtain its residual sum of squares RSS_0
- Fit $M_1 : Y = \beta_0 + \beta_1 X_1 + \ldots + \beta_{p-q} X_{p-q} + \ldots + \beta_p X_p + \varepsilon$, and obtain its residual sum of squares RSS
- Use test statistic

$$F = \frac{(RSS_0 - RSS)/q}{RSS/(n-p-1)}$$

Testing on some coefficients

When H_0 and model assumptions are true, test statistic

$$F = \frac{(RSS_0 - RSS)/q}{RSS/(n-p-1)}$$

approximately follows an F-distribution with numerator degrees of freedom q and denominator degrees of freedom $n - p - 1$

Testing on model fit

- R^2 measures the proportion of variance that is explained by the postulated model
- With three predictors:

```
> FitL3c = lm(sales~TV+radio+newspaper,data=adData)
> summary(FitL3c)$r.squared
[1] 0.8972106
```

- With one predictor:

```
> FitL3d = lm(sales~newspaper,data=adData)
> summary(FitL3d)$r.squared
[1] 0.05212045
```

Interaction terms

Interaction terms: I

Consider predicting the average `sales` (in thousands of dollars) via budgets in advertisement through `TV` and `Radio`.

- Model 1: $E($`sales`$) = \beta_0 + \beta_1\times$ `TV` $+ \beta_2\times$ `Radio`
- Model 1: how is the change (in unit) in $E($`sales`$)$ relates to a unit change in `TV` and/or `Radio`?
- Is model 1 sensible when changes (in unit) in $E($`sales`$)$ are different for a unit change in `TV` when `Radio` takes different values?

Interaction terms: I

- If change (in unit) in $E($`sales`$)$ can be different for a unit change in `TV` at different values of `Radio` or for a unit change in `Radio` at different values of `TV`, then the model

$$E(\mathsf{sales}) = \beta_0 + \beta_1 \times \mathsf{TV} + \beta_2 \times \mathsf{Radio}$$

  is no longer suitable

- One way to account for this is to introduce an *interaction* term and use model

$$E(\mathsf{sales}) = \beta_0 + \beta_1 \times \mathsf{TV} + \beta_2 \times \mathsf{Radio} + \beta_3 \times \mathsf{TV} \times \mathsf{Radio}$$

- Does the following model do the job?

$$E(\mathsf{sales}) = \beta_0 + \beta_1 \times \mathsf{TV} + \beta_2 \times \mathsf{Radio} + \beta_3 \times \mathsf{TV}^2 + \beta_4 \times \mathsf{Radio}^2$$

Interaction terms: I

The model

$$E(\mathsf{sales}) = \beta_0 + \beta_1 \times \mathsf{TV} + \beta_2 \times \mathsf{Radio} + \beta_3 \times \mathsf{TV} \times \mathsf{Radio}$$

can be written as

$$E(\mathsf{sales}) = \beta_0 + \beta_1 \times \mathsf{TV} + (\beta_2 + \beta_3 \times \mathsf{TV}) \times \mathsf{Radio}$$

or as

$$E(\mathsf{sales}) = \beta_0 + (\beta_1 + \beta_3 \times \mathsf{Radio}) \times \mathsf{TV} + \beta_2 \times \mathsf{Radio}$$

Interaction terms: I

Fit the model with interaction:

```
Call:
lm(formula = sales ~ TV * radio, data = adData)

Residuals:
    Min      1Q  Median      3Q     Max 
-6.3366 -0.4028  0.1831  0.5948  1.5246 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) 6.750e+00  2.479e-01  27.233   <2e-16 ***
TV          1.910e-02  1.504e-03  12.699   <2e-16 ***
radio       2.886e-02  8.905e-03   3.241   0.0014 ** 
TV:radio    1.086e-03  5.242e-05  20.727   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.9435 on 196 degrees of freedom
Multiple R-squared:  0.9678,    Adjusted R-squared:  0.9673 
F-statistic:  1963 on 3 and 196 DF,  p-value: < 2.2e-16
```

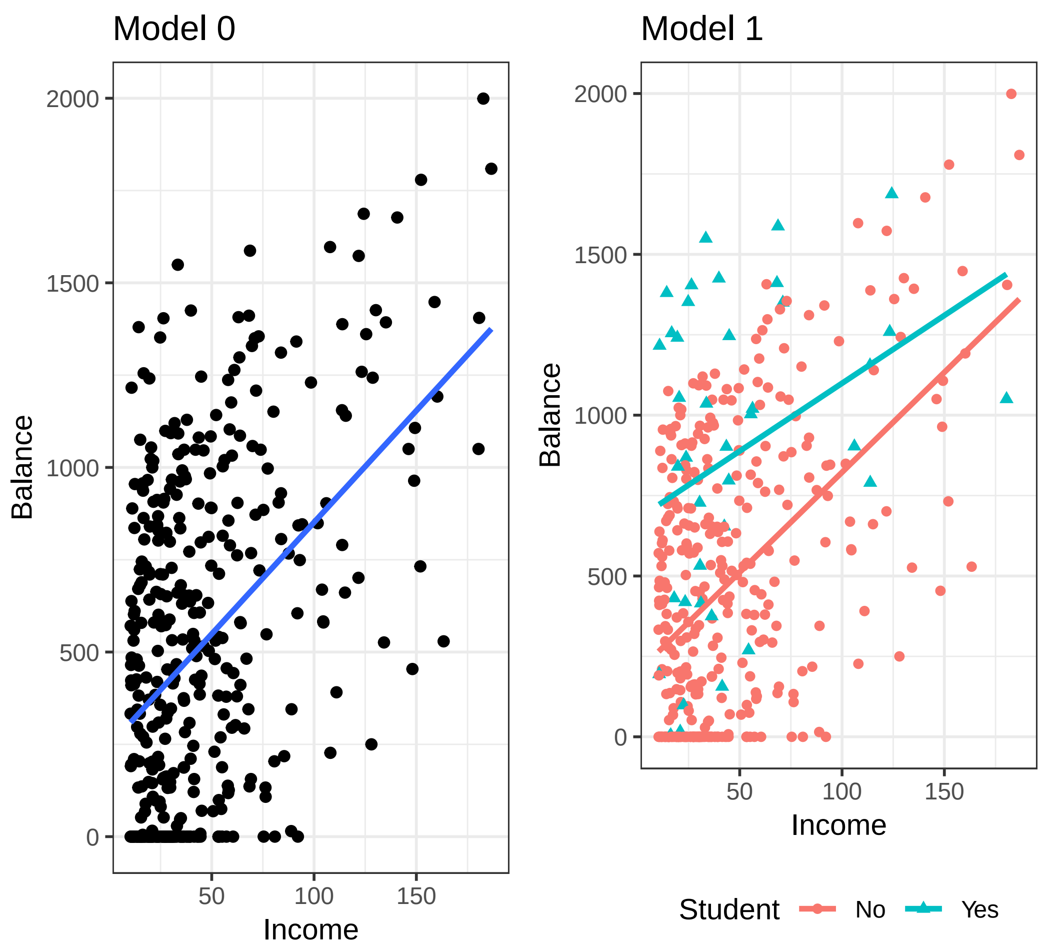
Interaction terms: II

Consider predicting the average `Balance` (of a credit card) using information on if a user is a `Student` ("Yes" or "No") and his/her `Income`

- Model 0: $E(\texttt{Balance}) = \beta_0 + \beta_1\times$ `Income`
- Model 1: $E(\texttt{Balance}) = \beta_0 + \beta_1\times$ `Student` $+ \beta_2\times$ `Income`
- Model 2: $E(\texttt{Balance}) = \beta_0 + \beta_1\times$ `Student` $+ \beta_2\times$ `Income` $+ \beta_3\times$ `Student` $\times$ `Income`

Coding in R: `Student`="No" is coded as 0 and the baseline, and `Student`="Yes" as 1

Interaction terms: II



Interaction terms: II

Fit the model with interaction:

```
Call:
lm(formula = Balance ~ Student * Income, data = creditData)

Residuals:
    Min      1Q  Median      3Q     Max 
-773.39 -325.70  -41.13  321.65  814.04 

Coefficients:
                  Estimate Std. Error t value Pr(>|t|)    
(Intercept)       200.6232    33.6984   5.953 5.79e-09 ***
StudentYes        476.6758   104.3512   4.568 6.59e-06 ***
Income              6.2182     0.5921  10.502  < 2e-16 ***
StudentYes:Income  -1.9992     1.7313  -1.155    0.249    
---
```

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 391.6 on 396 degrees of freedom
Multiple R-squared:  0.2799,	Adjusted R-squared:  0.2744 
F-statistic:  51.3 on 3 and 396 DF,  p-value: < 2.2e-16
```

Diagnostics

Diagnostics

- Diagnostics for multiple linear regression are very similar to those for simple linear regression with a quantitative predictor.
- Additional task: check on collinearity and *variance inflaction factor (VIF)*

Collinearity

Collinearity

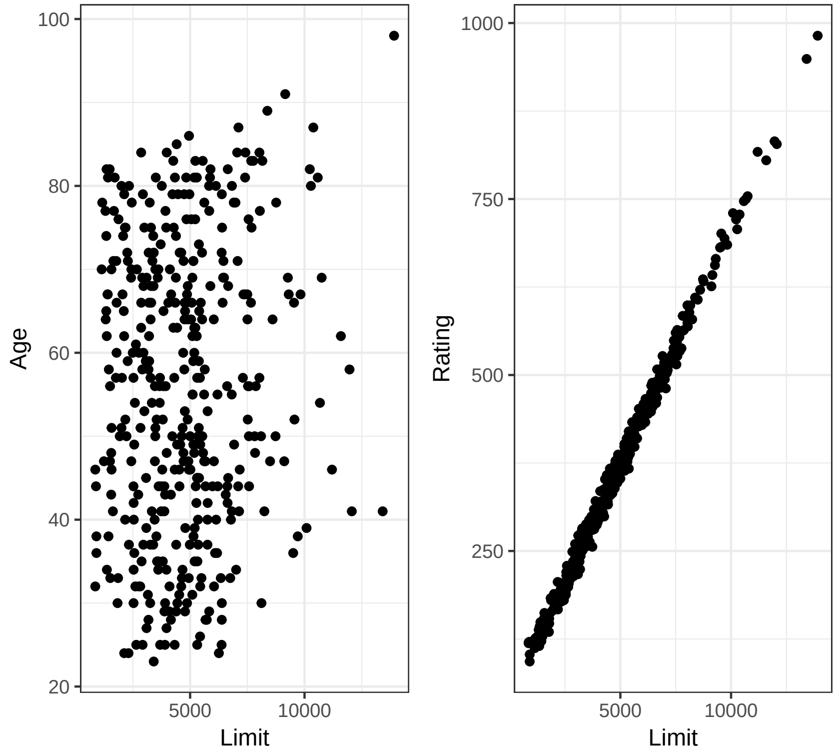
- refers to the situation in which two or more predictor variables are closely related to each other
- *often inflates the variances of estimated coefficients and makes the model unstable*
- can be measured by the *variance inflaction factor (VIF)*

A VIF value that exceeds 5 or 10 indicates a problematic amount of collinearity

Note: $\mathrm{VIF}(\hat{\beta}_j) = \frac{1}{1-R^2_{X_j|X_{-j}}}$; collinearity implies $R^2_{X_j|X_{-j}} \approx 1$

Collinearity

Collinearity among `Limit` and `Rating`:



Colllinearity

Model `Balance~Age+Limit`:

```
# A tibble: 3 x 5
  term        estimate std.error statistic   p.value
  <chr>          <dbl>     <dbl>     <dbl>     <dbl>
1 (Intercept) -173.      43.8        -3.96 9.01e-  5
2 Age           -2.29     0.672      -3.41 7.23e-  4
3 Limit          0.173    0.00503    34.5  1.63e-121
```

Model `Balance~Rating+Limit`:

```
# A tibble: 3 x 5
  term         estimate std.error statistic  p.value
  <chr>           <dbl>     <dbl>     <dbl>    <dbl>
1 (Intercept) -378.        45.3        -8.34  1.21e-15
2 Rating         2.20       0.952       2.31  2.13e- 2
3 Limit          0.0245     0.0638      0.384 7.01e- 1
```

Note: compare standard errors of $\hat{\beta}_{\text{Limit}}$ in both models

Colllinearity

```
> FitL3f = lm(Balance~Age+Rating+Limit,data=creditData)
> library(car)
> vif(FitL3f)
```

```
     Age     Rating      Limit
  1.011385 160.668301 160.592880
```

- VIFs indicate considerable collinearity in the data

In case of collinearity, either drop one of the problematic variables or combine some closely related variables

Non-linear models

Non-linear relationship

If there is evidence on a non-linear relationship between response and predictors, we can

- add high-order terms into the model (or employ more advanced non-linear methods); e.g.,

$$E(Y) = \beta_0 + \beta_1 X + \beta_2 X^2$$

- transform predictors (and/or response); e.g., e.g., $E(Y) = \beta_0 + \beta_1 \times f(X)$, where f can be $\log(X)$ or $\sqrt{X}$

License and session Information

License

```
> sessionInfo()
R version 3.5.0 (2018-04-23)
Platform: x86_64-w64-mingw32/x64 (64-bit)
Running under: Windows 10 x64 (build 19041)

Matrix products: default

locale:
[1] LC_COLLATE=English_United States.1252
[2] LC_CTYPE=English_United States.1252
[3] LC_MONETARY=English_United States.1252
[4] LC_NUMERIC=C
[5] LC_TIME=English_United States.1252

attached base packages:
[1] stats     graphics  grDevices utils     datasets  methods
[7] base

other attached packages:
[1] car_3.0-2      carData_3.0-2 broom_0.5.1   gridExtra_2.3
[5] ggplot2_3.1.0 knitr_1.21

loaded via a namespace (and not attached):
 [1] tidyselect_0.2.5 xfun_0.4          purrr_0.2.5
 [4] haven_2.0.0      lattice_0.20-35   colorspace_1.3-2
 [7] generics_0.0.2   htmltools_0.3.6   yaml_2.2.0
[10] utf8_1.1.4       rlang_0.4.4       pillar_1.3.1
[13] foreign_0.8-70   glue_1.3.0        withr_2.1.2
```

```
[16] readxl_1.2.0      plyr_1.8.4        stringr_1.3.1
[19] cellranger_1.1.0  munsell_0.5.0     gtable_0.2.0
[22] zip_1.0.0         evaluate_0.12     labeling_0.3
[25] rio_0.5.16        forcats_0.3.0     curl_3.2
[28] fansi_0.4.0       Rcpp_1.0.3        scales_1.0.0
[31] backports_1.1.3   abind_1.4-5       hms_0.4.2
[34] digest_0.6.18     openxlsx_4.1.0    stringi_1.2.4
[37] dplyr_0.8.4       grid_3.5.0        cli_1.0.1
[40] tools_3.5.0       magrittr_1.5      lazyeval_0.2.1
[43] tibble_2.1.3      crayon_1.3.4      tidyr_0.8.2
[46] pkgconfig_2.0.2   data.table_1.11.8 assertthat_0.2.0
[49] rmarkdown_1.11    rstudioapi_0.8    R6_2.3.0
[52] nlme_3.1-137      compiler_3.5.0
```