数学物理方法要背的东西汇总

胡喜平

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1 解析函数

函数 $f(x) = u + v\mathrm{i}$ 解析的充分必要条件

$$\begin{cases} \dfrac{\partial u}{\partial x} = +\dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

2 复变函数的积分

2.1 柯西定理

对于单连通区域

$$\oint_c f(z)\,\mathrm{d}z = 0$$

2.2 柯西公式

单连通区域内点 z

$$f(z) = \frac{1}{2\pi\mathrm{i}} \oint_l \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

3 留数定理

b_i 为所有 l 内奇点，则

$$\oint_l f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum \mathrm{Res} f(b_i)$$

包括无穷远点在内的所有的奇点的留数和为零

4 留数的运算

n 阶极点留数运算

$$\mathrm{Res}\left[f(z), \eta\right] = \frac{1}{(n-1)!} \lim_{z\to\eta} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[(z-\eta)^n f(z)\right]$$

当 $f(z)=\dfrac{P(z)}{Q(z)}$ 时，一阶级点的计算可以简化

$$\mathrm{Res}f(z_0)=\lim_{z\to z_0}(z-z_0)f(z)=\lim_{z\to z_0}(z-z_0)\frac{P(z)}{Q(z)}=\frac{P(z_0)}{Q'(z_0)}$$

5 利用留数定理计算实变函数的积分

5.1 几个类型的积分

非常重要，请仔细阅读梁昆淼《数学物理方法》P56-P63，就那几种类型

5.2 大圆弧定理

当 $z\to\infty$ 时，如果 $zf(z)\to k$，积分路径是半径为 R 的圆弧 C_R，则

$$\lim_{R\to\infty}\int_{C_R}f(z)\,\mathrm{d}z=\mathrm{i}k(\beta-\alpha)$$

5.3 小圆弧定理

当 $z\to a$ 时，如果 $(z-a)f(z)\to k$，积分路径是半径为 R 的圆弧 C_R，则

$$\lim_{\rho\to 0}\int_{C_\rho}f(z)\,\mathrm{d}z=\mathrm{i}k\,(\beta-\alpha)$$

6 傅立叶级数展开

6.1 函数，周期 $2l$

$$\begin{aligned}
f(x)&=\frac{a_0}{2}+\sum_{i=1}^{n}a_k\cos\frac{k\pi x}{l}+\sum_{i=1}^{n}b_k\sin\frac{k\pi x}{l}\\
a_k&=\frac{1}{l}\int_{-l}^{l}f(\xi)\cos\frac{k\pi\xi}{l}\,\mathrm{d}\xi\\
b_k&=\frac{1}{l}\int_{-l}^{l}f(\xi)\sin\frac{k\pi\xi}{l}\,\mathrm{d}\xi
\end{aligned}$$

积分前面系数的确定方法：利用正交性

$$\begin{aligned}
\int_{-l}^{l}\sin\frac{k\pi\xi}{l}\sin\frac{k\pi\xi}{l}\,\mathrm{d}\xi&=\int_{-l}^{l}\sin^2\frac{k\pi\xi}{l}\,\mathrm{d}\xi\\
&=\frac{1}{2}\int_{-l}^{l}1-\cos\frac{2k\pi\xi}{l}\,\mathrm{d}\xi\\
&=\frac{1}{2}\frac{l}{2k\pi}\int_{-l}^{l}1-\cos\frac{2k\pi\xi}{l}\,\mathrm{d}\frac{2k\pi\xi}{l}\\
&=\frac{l}{2k\pi}\frac{2k\pi}{l}\cdot 2l\\
&=l
\end{aligned}$$

6.2 奇函数, 周期 $2l$

$$f(x)=\sum_{i=1}^{n} b_k \sin\frac{k\pi x}{l}$$
$$b_k=\frac{2}{l}\int_0^l f(\xi)\sin\frac{k\pi\xi}{l}\,\mathrm{d}\xi$$

6.3 偶函数, 周期 $2l$

$$f(x)=\frac{a_0}{2}+\sum_{i=1}^{n} a_k \cos\frac{k\pi x}{l}$$
$$a_k=\frac{2}{l}\int_0^l f(\xi)\cos\frac{k\pi\xi}{l}\,\mathrm{d}\xi$$

7 数学物理方程的求解

7.1 波动方程的行波解

达朗贝尔解

$$u(x,t)=\frac{1}{2}\left[\phi(x+vt)+\phi(x-vt)\right]+\frac{1}{2v}\int_{x-vt}^{x+vt}\psi(\xi)\,\mathrm{d}\xi$$

7.2 齐次边界条件，齐次方程

7.2.1 波动方程

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}-a^2\dfrac{\partial^2 u}{\partial x^2}=0\\ u|_{x=0}=0\\ u|_{x=l}=0\\ u|_{t=0}=\phi(x)\\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=\psi(x)\end{cases}$$

将分离变量形式的解

$$u(x,t)=X(x)T(t)$$

带入方程，求解本征值

$$\frac{1}{v^2}\frac{T''(t)}{T(t)}=\frac{X''(x)}{X(x)}=-\lambda$$
$$T''(t)+\lambda a^2 T(t)=0$$
$$X''(x)+\lambda X(x)=0$$

设 $X(x)$ 的解 $X(x)=A\sin\sqrt{\lambda}x+B\cos\sqrt{\lambda}x$，代入边界条件 $u|_{x=0}=0$ 和 $u|_{x=l}=0$

$$B=0$$
$$A\sin\sqrt{\lambda}l=0$$

本征值与本征函数

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2$$

$$X_n(x) = \sin\frac{n\pi}{l}x$$

用本征函数解 $T(t)$

$$T_n(t) = C_n \sin\frac{n\pi a}{l}t + D_n \cos\frac{n\pi a}{l}t$$

叠加出 $u(x,t)$

$$u(x,t) = \sum_{n=1}^{\infty}\left(C_n \sin\frac{n\pi a}{l}t + D_n \cos\frac{n\pi a}{l}t\right)\sin\frac{n\pi}{l}x$$

代入

$$\begin{cases} u|_{t=0} = \phi(x) \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = \psi(x) \end{cases}$$

利用傅立叶积分计算 C_n 和 D_n

7.2.2　矩形区域的稳定问题

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \\ u|_{x=0} = 0 \\ u|_{y=0} = f(x) \\ \left.\dfrac{\partial u}{\partial x}\right|_{x=a} = 0 \\ \left.\dfrac{\partial u}{\partial y}\right|_{y=b} = 0 \end{cases}$$

设

$$u\left(x,y\right) = X(x)Y(y)$$

则

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

求解本征值问题

$$X''(x) + \lambda X(x) = 0$$

$$Y''(y) - \lambda Y(y) = 0$$

边界条件给出

$$X(0) = 0$$

$$X'(a) = 0$$

设

$$X(x) = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x$$

则

$$B = 0$$

$$\cos\sqrt{\lambda}a = 0$$

本征值

$$\sqrt{\lambda_n}a = \frac{2n+1}{2}\pi$$

$$\lambda_n = \left(\frac{2n+1}{2a}\pi\right)^2$$

对应本正函数

$$X(x) = \sin\frac{2n+1}{2a}\pi x$$

$Y(y)$ 的解为

$$Y(y) = C\cosh\sqrt{\lambda}x + D\sinh\sqrt{\lambda}x$$

代入本征值

$$Y_n(y) = C_n\cosh\frac{2n+1}{2a}\pi x + D_n\sinh\frac{2n+1}{2a}\pi x$$

分离变量形式的解

$$u(x,y) = \sum_{n=0}^{\infty}\left(C_n\cosh\frac{2n+1}{2a}\pi x + D_n\sinh\frac{2n+1}{2a}\pi x\right)\sin\frac{2n+1}{2a}\pi x$$

代入边界条件，使用傅立叶积分计算常数

7.3 非其次方程 (特解法)

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2\dfrac{\partial^2 u}{\partial x^2} = f(x,t) \\ u|_{x=0} = 0 \\ u|_{x=l} = 0 \\ u|_{t=0} = 0 \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0 \end{cases}$$

设

$$u(x,t) = v(x,t) + w(x,t)$$

其中 $v(x,t)$ 可以不满足初始条件

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} - a^2\dfrac{\partial^2 v}{\partial x^2} = f(x,t) \\ v|_{x=0} = 0 \\ v|_{x=l} = 0 \end{cases}$$

$w(x,t)$ 满足

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} - a^2\dfrac{\partial^2 w}{\partial x^2} = 0 \\ w|_{x=0} = 0 \\ w|_{x=l} = 0 \\ w|_{t=0} = -v|_{t=0} \\ \left.\dfrac{\partial w}{\partial t}\right|_{t=0} = -\left.\dfrac{\partial v}{\partial t}\right|_{t=0} \end{cases}$$

7.4 非齐次方程 (展开法)

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = f(x,t) \\ u|_{x=0} = 0 \\ u|_{x=l} = 0 \\ u|_{t=0} = 0 \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0 \end{cases}$$

将 $f(x,t)$ 用相应齐次方程的本征函数 $X_n(x)$ 展开

$$f(x,t) = \sum_{n=1}^{\infty} g_n(t) X_n(x)$$

于是有

$$\sum_{n=1}^{\infty} X_n(x) T_n''(t) - a^2 X_n''(x) T_n(t) = \sum_{n=1}^{\infty} g_n(t) X_n(x)$$

由于

$$X_n''(x) + \lambda_n X(x) = 0$$

有

$$\sum_{n=1}^{\infty} X_n(x) T_n''(t) + a^2 \lambda_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} g_n(t) X_n(x)$$

即

$$T_n''(t) + a^2 \lambda_n T_n(t) = g_n(t)$$

初始条件给出

$$\sum_{n=1}^{\infty} X_n(0) T_n(0) = 0$$

$$\sum_{n=1}^{\infty} X_n(0) T_n'(0) = 0$$

即

$$T_n(0) = 0$$

$$T_n'(0) = 0$$

解为

$$T_n(t) = \frac{l}{n\pi a} \int_0^t g_n(\tau) \sin \frac{n\pi a}{l} (t - \tau) \,\mathrm{d}\tau$$

其中 $g_n(\tau)$ 需要利用本征函数 $X_n(x)$ 的正交性积分解出

7.5 非齐次边界条件的处理

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = 0 \\
u|_{x=0} = \mu(t) \\
u|_{x=l} = \nu(t) \\
u|_{t=0} = 0 \\
\left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0
\end{cases}
$$

令

$$u(x,t) = v(x,t) + w(x,t)$$

$v(x,t)$ 满足

$$
\begin{cases}
v|_{x=0} = \mu(t) \\
v|_{x=l} = \nu(t)
\end{cases}
$$

可以设

$$v(x,t) = A(t)x + B(t)$$

$w(x,t)$ 满足

$$
\begin{cases}
\dfrac{\partial^2 w}{\partial t^2} - a^2 \dfrac{\partial^2 w}{\partial x^2} = -\left(\dfrac{\partial^2 v}{\partial t^2} - a^2 \dfrac{\partial^2 v}{\partial x^2}\right) \\
w|_{x=0} = 0 \\
w|_{x=l} = 0 \\
w|_{t=0} = -v|_{t=0} \\
\left.\dfrac{\partial w}{\partial t}\right|_{t=0} = -\left.\dfrac{\partial v}{\partial t}\right|_{t=0}
\end{cases}
$$

7.6 极坐标系定解问题

7.6.1 拉普拉斯算符

极坐标系下拉普拉斯算符

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

柱坐标系下拉普拉斯算符

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

球坐标系下拉普拉斯算符

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

7.6.2 极坐标系下自然边界条件

极坐标系定解问题

$$\begin{cases}\Delta u=0\\ u(r,\theta)|_{r=a}=f(\theta)\\ u(r,\theta)|_{\theta=0}=u(r,\theta)|_{\theta=2\pi}\\ \left.\dfrac{\partial u}{\partial\theta}\right|_{\theta=0}=\left.\dfrac{\partial u}{\partial\theta}\right|_{\theta=2\pi}\\ u(r,\theta)|_{r=0}\neq\infty\end{cases}$$

最后三个补充的条件称为自然边界条件

8 勒让德方程

8.1 在 $x=0$ 的邻域求解勒让德方程

l 阶勒让德方程

$$\left(1-x^2\right)y''-2xy'+l\left(l+1\right)y=0$$

的解法为:
设

$$y\left(x\right)=\sum_{k=0}^{\infty}a_kz^k$$

则

$$\begin{aligned}y\left(x\right)&=a_0+a_1x+a_2x^2+a_3x^3+\cdots\\ y'\left(x\right)&=a_1x+2a_2x+3a_3x^2+4a_4x^3\cdots\\ y''\left(x\right)&=2a_2+3\cdot2a_3x+4\cdot3a_4x^2+5\cdot4a_5x^3\cdots\end{aligned}$$

代入方程，比较系数，得到奇数项和偶数项

8.2 勒让德多项式

$$P_l\left(x\right)=\frac{1}{2^ll!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}\left(x^2-1\right)^l$$

8.3 广义傅立叶级数的展开

$$\begin{cases}f(x)=\displaystyle\sum_0^{\infty}f_lP_l(x)\\ \quad f_l=\dfrac{2l+1}{2}\displaystyle\int_{-1}^{+1}f(x)P_l(x)\,\mathrm{d}x\end{cases}$$