

# 数学物理方法要背的东西汇总

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## 1 解析函数

函数  $f(x) = u + vi$  解析的充分必要条件

$$\begin{cases} \frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

## 2 复变函数的积分

### 2.1 柯西定理

对于单连通区域

$$\oint_c f(z) dz = 0$$

### 2.2 柯西公式

单连通区域内点  $z$

$$f(z) = \frac{1}{2\pi i} \oint_l \frac{f(\zeta)}{\zeta - z} d\zeta$$

## 3 留数定理

$b_i$  为所有  $l$  内奇点, 则

$$\oint_l f(z) dz = 2\pi i \sum \text{Res} f(b_i)$$

包括无穷远点在内的所有的奇点的留数和为零

## 4 留数的运算

$n$  阶极点留数运算

$$\text{Res}[f(z), \eta] = \frac{1}{(n-1)!} \lim_{z \rightarrow \eta} \frac{d^{n-1}}{dz^{n-1}} [(z - \eta)^n f(z)]$$

当  $f(z) = \frac{P(z)}{Q(z)}$  时, 一阶级点的计算可以简化

$$\operatorname{Res} f(z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z) = \lim_{z \rightarrow z_0} (z - z_0) \frac{P(z)}{Q(z)} = \frac{P(z_0)}{Q'(z_0)}$$

## 5 利用留数定理计算实变函数的积分

### 5.1 几个类型的积分

非常重要, 请仔细阅读梁昆森《数学物理方法》P56-P63, 就那几种类型

#### 5.1.1 类型一

$$\int_0^{2\pi} R(\cos(x), \sin(x)) dx$$

做变换  $z = e^{ix}$

$$I = \oint_{|z|=1} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \frac{dz}{iz}$$

#### 5.1.2 类型二

$$\int_{-\infty}^{+\infty} f(x) dx$$

其中

$$zf(z) \rightarrow 0$$

则

$$I = 2\pi i \{\text{上半平面奇点留数和}\}$$

**大圆弧定理** 当  $z \rightarrow \infty$  时, 如果  $zf(z) \rightarrow k$ , 积分路径是半径为  $R$  的圆弧  $C_R$ , 则

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = ik(\beta - \alpha)$$

**小圆弧定理** 当  $z \rightarrow a$  时, 如果  $(z-a)f(z) \rightarrow k$ , 积分路径是半径为  $R$  的圆弧  $C_R$ , 则

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} f(z) dz = ik(\beta - \alpha)$$

#### 5.1.3 类型三

$$\int_0^\infty F(x) \cos mx dx$$

$$\int_0^\infty G(x) \sin mx dx$$

其中  $F(x) \cos mx$  和  $G(x) \sin mx$  为偶函数

$$\int_0^\infty F(x) \cos mx dx = \pi i \{F(z) e^{imz} \text{ 上半平面奇点留数和} \}$$

$$\int_0^\infty G(x) \sin mx dx = \pi \{G(z) e^{imz} \text{ 上半平面奇点留数和} \}$$

#### 5.1.4 当实轴上有奇点时

每一个奇点导致最后积分加  $\pi i \operatorname{Res} f(z)$

## 6 傅立叶级数展开

### 6.1 函数, 周期 $2l$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^n a_k \cos \frac{k\pi x}{l} + \sum_{k=1}^n b_k \sin \frac{k\pi x}{l}$$

$$a_k = \frac{1}{l} \int_{-l}^l f(\xi) \cos \frac{k\pi \xi}{l} d\xi$$

$$b_k = \frac{1}{l} \int_{-l}^l f(\xi) \sin \frac{k\pi \xi}{l} d\xi$$

积分前面系数的确定方法: 利用正交性

$$\begin{aligned} \int_{-l}^l \sin \frac{k\pi \xi}{l} \sin \frac{k\pi \xi}{l} d\xi &= \int_{-l}^l \sin^2 \frac{k\pi \xi}{l} d\xi \\ &= \frac{1}{2} \int_{-l}^l 1 - \cos \frac{2k\pi \xi}{l} d\xi \\ &= \frac{1}{2} \frac{l}{2k\pi} \int_{-l}^l 1 - \cos \frac{2k\pi \xi}{l} d \frac{2k\pi \xi}{l} \\ &= \frac{l}{2k\pi} \frac{2k\pi}{l} \cdot 2l \\ &= l \end{aligned}$$

### 6.2 奇函数, 周期 $2l$

$$f(x) = \sum_{k=1}^n b_k \sin \frac{k\pi x}{l}$$

$$b_k = \frac{2}{l} \int_0^l f(\xi) \sin \frac{k\pi \xi}{l} d\xi$$

### 6.3 偶函数, 周期 $2l$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^n a_k \cos \frac{k\pi x}{l}$$

$$a_k = \frac{2}{l} \int_0^l f(\xi) \cos \frac{k\pi \xi}{l} d\xi$$

## 7 数学物理方程的求解

### 7.1 波动方程的行波解

达朗贝尔解

$$u(x, t) = \frac{1}{2} [\phi(x + vt) + \phi(x - vt)] + \frac{1}{2v} \int_{x-vt}^{x+vt} \psi(\xi) d\xi$$

### 7.2 齐次边界条件, 齐次方程

#### 7.2.1 波动方程

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \\ u|_{x=0} = 0 \\ u|_{x=l} = 0 \\ u|_{t=0} = \phi(x) \\ \left. \frac{\partial u}{\partial t} \right|_{t=0} = \psi(x) \end{cases}$$

将分离变量形式的解

$$u(x, t) = X(x)T(t)$$

带入方程, 求解本征值

$$\frac{1}{v^2} \frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$$T''(t) + \lambda a^2 T(t) = 0$$

$$X''(x) + \lambda X(x) = 0$$

设  $X(x)$  的解  $X(x) = A \sin \sqrt{\lambda}x + B \cos \sqrt{\lambda}x$ , 代入边界条件  $u|_{x=0} = 0$  和  $u|_{x=l} = 0$

$$B = 0$$

$$A \sin \sqrt{\lambda}l = 0$$

本征值与本征函数

$$\lambda_n = \left( \frac{n\pi}{l} \right)^2$$

$$X_n(x) = \sin \frac{n\pi}{l}x$$

用本征函数解  $T(t)$

$$T_n(t) = C_n \sin \frac{n\pi a}{l}t + D_n \cos \frac{n\pi a}{l}t$$

叠加出  $u(x, t)$

$$u(x, t) = \sum_{n=1}^{\infty} \left( C_n \sin \frac{n\pi a}{l}t + D_n \cos \frac{n\pi a}{l}t \right) \sin \frac{n\pi}{l}x$$

代入

$$\begin{cases} u|_{t=0} = \phi(x) \\ \left. \frac{\partial u}{\partial t} \right|_{t=0} = \psi(x) \end{cases}$$

利用傅立叶积分计算  $C_n$  和  $D_n$

### 7.2.2 矩形区域的稳定问题

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ u|_{x=0} = 0 \\ u|_{y=0} = f(x) \\ \frac{\partial u}{\partial x}\bigg|_{x=a} = 0 \\ \frac{\partial u}{\partial y}\bigg|_{y=b} = 0 \end{cases}$$

设

$$u(x, y) = X(x)Y(y)$$

则

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

求解本征值问题

$$X''(x) + \lambda X(x) = 0$$

$$Y''(y) - \lambda Y(y) = 0$$

边界条件给出

$$X(0) = 0$$

$$X'(a) = 0$$

设

$$X(x) = A \sin \sqrt{\lambda}x + B \cos \sqrt{\lambda}x$$

则

$$B = 0$$

$$\cos \sqrt{\lambda}a = 0$$

本征值

$$\sqrt{\lambda_n}a = \frac{2n+1}{2}\pi$$

$$\lambda_n = \left(\frac{2n+1}{2a}\pi\right)^2$$

对应本正函数

$$X(x) = \sin \frac{2n+1}{2a}\pi x$$

$Y(y)$  的解为

$$Y(y) = C \cosh \sqrt{\lambda}x + D \sinh \sqrt{\lambda}x$$

代入本征值

$$Y_n(y) = C_n \cosh \frac{2n+1}{2a}\pi x + D_n \sinh \frac{2n+1}{2a}\pi x$$

分离变量形式的解

$$u(x, y) = \sum_{n=0}^{\infty} \left( C_n \cosh \frac{2n+1}{2a}\pi x + D_n \sinh \frac{2n+1}{2a}\pi x \right) \sin \frac{2n+1}{2a}\pi x$$

代入边界条件, 使用傅立叶积分计算常数

### 7.3 非其次方程 (特解法)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x, t) \\ u|_{x=0} = 0 \\ u|_{x=l} = 0 \\ u|_{t=0} = 0 \\ \left. \frac{\partial u}{\partial t} \right|_{t=0} = 0 \end{cases}$$

设

$$u(x, t) = v(x, t) + w(x, t)$$

其中  $v(x, t)$  可以不满足初始条件

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} - a^2 \frac{\partial^2 v}{\partial x^2} = f(x, t) \\ v|_{x=0} = 0 \\ v|_{x=l} = 0 \end{cases}$$

$w(x, t)$  满足

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} - a^2 \frac{\partial^2 w}{\partial x^2} = 0 \\ w|_{x=0} = 0 \\ w|_{x=l} = 0 \\ w|_{t=0} = -v|_{t=0} \\ \left. \frac{\partial w}{\partial t} \right|_{t=0} = - \left. \frac{\partial v}{\partial t} \right|_{t=0} \end{cases}$$

### 7.4 非齐次方程 (展开法)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x, t) \\ u|_{x=0} = 0 \\ u|_{x=l} = 0 \\ u|_{t=0} = 0 \\ \left. \frac{\partial u}{\partial t} \right|_{t=0} = 0 \end{cases}$$

将  $f(x, t)$  用相应齐次方程的本征函数  $X_n(x)$  展开

$$f(x, t) = \sum_{n=1}^{\infty} g_n(t) X_n(x)$$

于是有

$$\sum_{n=1}^{\infty} X_n(x) T_n''(t) - a^2 X_n''(x) T_n(t) = \sum_{n=1}^{\infty} g_n(t) X_n(x)$$

由于

$$X_n''(x) + \lambda_n X(x) = 0$$

有

$$\sum_{n=1}^{\infty} X_n(x) T_n''(t) + a^2 \lambda_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} g_n(t) X_n(x)$$

即

$$T_n''(t) + a^2 \lambda_n T_n(t) = g_n(t)$$

初始条件给出

$$\begin{aligned} \sum_{n=1}^{\infty} X_n(0) T_n(0) &= 0 \\ \sum_{n=1}^{\infty} X_n(0) T_n'(0) &= 0 \end{aligned}$$

即

$$\begin{aligned} T_n(0) &= 0 \\ T_n'(0) &= 0 \end{aligned}$$

解为

$$T_n(t) = \frac{l}{n\pi a} \int_0^t g_n(\tau) \sin \frac{n\pi a}{l}(t - \tau) d\tau$$

其中  $g_n(\tau)$  需要利用本征函数  $X_n(x)$  的正交性积分解出

## 7.5 非齐次边界条件的处理

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \\ u|_{x=0} = \mu(t) \\ u|_{x=l} = \nu(t) \\ u|_{t=0} = 0 \\ \left. \frac{\partial u}{\partial t} \right|_{t=0} = 0 \end{cases}$$

令

$$u(x, t) = v(x, t) + w(x, t)$$

$v(x, t)$  满足

$$\begin{cases} v|_{x=0} = \mu(t) \\ v|_{x=l} = \nu(t) \end{cases}$$

可以设

$$v(x, t) = A(t)x + B(t)$$

$w(x, t)$  满足

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} - a^2 \frac{\partial^2 w}{\partial x^2} = - \left( \frac{\partial^2 v}{\partial t^2} - a^2 \frac{\partial^2 v}{\partial x^2} \right) \\ w|_{x=0} = 0 \\ w|_{x=l} = 0 \\ w|_{t=0} = -v|_{t=0} \\ \left. \frac{\partial w}{\partial t} \right|_{t=0} = - \left. \frac{\partial v}{\partial t} \right|_{t=0} \end{cases}$$

## 7.6 极坐标系定解问题

### 7.6.1 拉普拉斯算符

极坐标系下拉普拉斯算符

$$\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

柱坐标系下拉普拉斯算符

$$\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

球坐标系下拉普拉斯算符

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

### 7.6.2 极坐标系下自然边界条件

极坐标系定解问题

$$\begin{cases} \Delta u = 0 \\ u(r, \theta)|_{r=a} = f(\theta) \\ u(r, \theta)|_{\theta=0} = u(r, \theta)|_{\theta=2\pi} \\ \left. \frac{\partial u}{\partial \theta} \right|_{\theta=0} = \left. \frac{\partial u}{\partial \theta} \right|_{\theta=2\pi} \\ u(r, \theta)|_{r=0} \neq \infty \end{cases}$$

最后三个补充的条件称为自然边界条件

## 8 勒让德方程

### 8.1 在 $x = 0$ 的邻域求解勒让德方程

$l$  阶勒让德方程

$$(1 - x^2) y'' - 2xy' + l(l+1)y = 0$$

的解法为：

设

$$y(x) = \sum_{k=0}^{\infty} a_k z^k$$



则

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

$$y'(x) = a_1x + 2a_2x + 3a_3x^2 + 4a_4x^3 \cdots$$

$$y''(x) = 2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + 5 \cdot 4a_5x^3 \cdots$$

代入方程，比较系数，得到奇数项和偶数项

## 8.2 勒让德多项式

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l$$

## 8.3 广义傅立叶级数的展开

$$\begin{cases} f(x) = \sum_0^{\infty} f_l P_l(x) \\ f_l = \frac{2l+1}{2} \int_{-1}^{+1} f(x) P_l(x) dx \end{cases}$$