Homework for Chapter 3

Xiping Hu

https://hxp.plus/

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1. 波长为 1Å 的 X 光光子的动量和能量各为多少?

$$E = \frac{hc}{\lambda} = 1.98645 \times 10^{-15}\ \mathrm{J \cdot m}$$

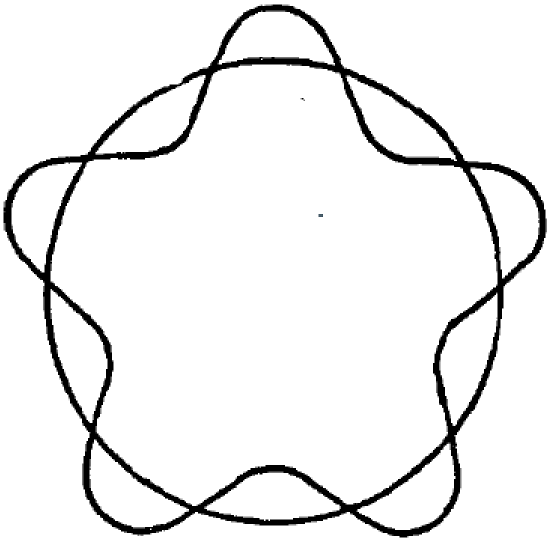
$$p = \frac{h}{\lambda} = 6.62607 \times 10^{-24}\ \mathrm{J \cdot s}$$

2. 经过 10000 伏特电势差加速的电子束的德布罗意波波长 $\lambda=$? 用上述电压加速的质子束,其德布罗意波波长是多少 Å?

$$v = \sqrt{\frac{2E_k}{m_0}} = 5.93097 \times 10^{7}\ \mathrm{m/s}$$

$$\lambda = \frac{h}{m_0 v} = 1.22643 \times 10^{-11}\ \mathrm{m}$$

4. 试证明氢原子稳定轨道上正好能容纳下整数个电子的德布罗意波波长(习题图 3.1)。上述结果不但适用于圆轨道，同样适用于椭圆轨道，试证明之。



习题图 3.1

In round orbits

$$\oint p_r \, \mathrm{d}q_r = n_r h$$
$$\oint p_\phi \, \mathrm{d}q_\phi = n_\phi h$$

Since $p_r = 0$

$$\oint mr^2\dot{\phi} \, \mathrm{d}\phi = n_\phi h$$
$$2\pi mvr = n_\phi h$$

$$2\pi r = n_\phi \frac{h}{mv}$$

The wave length of the electron is

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

So that

$$2\pi r = n_\phi \lambda$$

When the orbit is shaped in eclipse

$$\oint p_r \,\mathrm{d}q_r = n_r h$$
$$\oint p_\phi \,\mathrm{d}q_\phi = n_\phi h$$

So that

$$\oint (p_r \,\mathrm{d}r + p_\phi \,\mathrm{d}\phi) = nh$$

$$\oint \left(m\dot{r} \,\mathrm{d}r + mr^2\dot{\phi} \,\mathrm{d}\phi\right) = nh$$

$$\oint \left(m\dot{r}^2 \,\mathrm{d}t + mr^2\dot{\phi}^2 \,\mathrm{d}t\right) = nh$$

$$\oint \left[m\left(\dot{r}^2 + mr^2\dot{\phi}^2\right)\right] \mathrm{d}t = nh$$

$$\oint \left[mv^2\right] \mathrm{d}t = nh$$

$$\oint mv \,\mathrm{d}s = nh$$

$$\oint \frac{mv}{h} \,\mathrm{d}s = n$$

The wavelength of the electron is

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

insert it into

$$\oint \frac{\mathrm{d}s}{\lambda} = n$$

$$\oint \mathrm{d}s = n\lambda$$

5. 带电粒子在威耳孙云室（一种径迹探测器）中的轨迹是一串小雾滴，雾滴的线度约为 1 微米．当观察能量为 1000 电子伏的电子径迹时其动量与经典力学动量的相对偏差不小于多少？

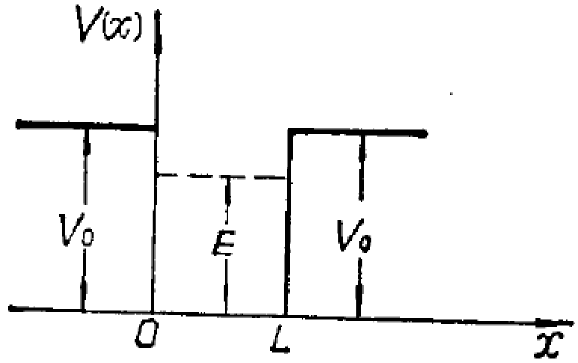
$$\Delta p \Delta x \geq \frac{\hbar}{2}$$

$$p = \sqrt{2mE_k}$$

$$\frac{\Delta p}{p} = \frac{\hbar}{2\Delta x\sqrt{2mE_k}} = 3.046 \times 10^{-5}$$

7. 粒子位于一维对称势场中，势场形式如下图，即

$$\begin{cases} 0<x<L, & V=0, \\ x<0, x>L, & V=V_0. \end{cases}$$



习题图 3.2

(1) 试推导粒子在 $E<V_0$ 情况下其总能量 E 满足的关系式.

(2) 试利用上述关系式，以图解法证明，粒子的能量只能是一些不连续的值.

Solution for problem (1)

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + \frac{2m(E-V)}{\hbar^2}\Psi = 0$$

For $x < 0$

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} - \frac{2m(V_0-E)}{\hbar^2}\Psi = 0$$

$$\Psi_1(x) = A\exp\left[\sqrt{\frac{2m(V_0-E)}{\hbar^2}}x\right]$$

For $0 < x < L$

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + \frac{2mE}{\hbar^2}\Psi = 0$$

$$\Psi_2(x) = B\sin\left[\sqrt{\frac{2mE}{\hbar^2}}x + \phi\right]$$

For $x > L$

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} - \frac{2m\left(V_0 - E\right)}{\hbar^2}\Psi = 0$$

$$\Psi_3\left(x\right) = C\exp\left[-\sqrt{\frac{2m\left(V_0 - E\right)}{\hbar^2}}x\right]$$

In conclusion

$$\begin{cases} \Psi_1\left(x\right) = A\exp\left[\sqrt{\frac{2m\left(V_0 - E\right)}{\hbar^2}}x\right] = A\exp\left[\alpha x\right] \\ \Psi_2\left(x\right) = B\sin\left[\sqrt{\frac{2mE}{\hbar^2}}x + \phi\right] = B\sin\left[\beta x + \phi\right] \\ \Psi_3\left(x\right) = C\exp\left[-\sqrt{\frac{2m\left(V_0 - E\right)}{\hbar^2}}x\right] = C\exp\left[-\alpha x\right] \end{cases}$$

By taking logarithms

$$\begin{cases} \log\left[\Psi_1\left(x\right)\right] = \log A + \alpha x \\ \log\left[\Psi_2\left(x\right)\right] = \log B + \log\left[\sin\left(\beta x + \phi\right)\right] \\ \log\left[\Psi_3\left(x\right)\right] = \log C - \alpha x \end{cases}$$

By taking derivatives

$$\begin{cases} \frac{1}{\Psi_1\left(x\right)}\frac{\mathrm{d}\Psi_1\left(x\right)}{\mathrm{d}x} = \alpha \\ \frac{1}{\Psi_2\left(x\right)}\frac{\mathrm{d}\Psi_2\left(x\right)}{\mathrm{d}x} = \beta\cot\left[\beta x + \phi\right] \\ \frac{1}{\Psi_3\left(x\right)}\frac{\mathrm{d}\Psi_3\left(x\right)}{\mathrm{d}x} = -\alpha \end{cases}$$

Due to the continuity of $\Psi\left(x\right)$

$$\begin{cases} \alpha = \beta\cot\phi \\ \beta\cot\left[\beta L + \phi\right] = -\alpha \end{cases}$$

So that

$$\begin{cases} \tan\phi = \frac{\beta}{\alpha} \\ \tan\left[\beta L + \phi\right] = -\frac{\beta}{\alpha} \end{cases}$$

Finally

$$\beta L = -2\phi + n\pi = -2\arctan\frac{\beta}{\alpha} + n\pi$$

Whereas

$$\alpha = \sqrt{\frac{2m\left(V_0 - E\right)}{\hbar^2}}$$
$$\beta = \sqrt{\frac{2mE}{\hbar^2}}$$

Solution for Problem (2)

$$\beta L = -2\arctan\frac{\beta}{\alpha} + n\pi$$

$$\frac{\beta}{\alpha} = \tan\left(\frac{n\pi}{2} - \frac{\beta L}{2}\right)$$

When $n = 1, 3, 5...$

$$\frac{\beta}{\alpha} = \cot\left(\frac{\beta L}{2}\right)$$

$$\sqrt{\frac{E}{V_0 - E}} = \cot\left(\frac{mEL^2}{2\hbar^2}\right)$$

When $n = 0, 2, 4...$

$$\frac{\beta}{\alpha} = -\tan\left(\frac{\beta L}{2}\right)$$

$$\sqrt{\frac{E}{V_0 - E}} = -\tan\left(\frac{mEL^2}{2\hbar^2}\right)$$

So that E is discontinued.

8. 有一粒子，其质量为 m，在一个三维势箱中运动．势箱的长、宽、高分别为 a, b, c。在势箱外，势能 $V = \infty$；在势箱内，$V = 0$。试算出粒子可能具有的能量。

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} + \frac{2m}{\hbar^2}\left[E - \left(V_x + V_y + V_Z\right)\right]\Psi = 0$$

$$V = \begin{cases} 0 & -a/2 < x < a/2, -b/2 < y < b/2, -c/2 < z < c/2 \\ \infty & \text{others} \end{cases}$$

Let

$$\Psi\left(x, y, z\right) = X\left(x\right)Y\left(y\right)Z\left(z\right)$$

Insert it into, we got

$$\left(\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} - \frac{2m}{\hbar^2}V_x\right) + \left(\frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} - \frac{2m}{\hbar^2}V_y\right) + \left(\frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} - \frac{2m}{\hbar^2}V_z\right) + \frac{2mE}{\hbar} = 0$$

So we can separate the 3 variables

$$\begin{cases} \left(\frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - \frac{2m}{\hbar^2}V_x\right) - \frac{2mE_x}{\hbar} = 0 \\ \left(\frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} - \frac{2m}{\hbar^2}V_y\right) - \frac{2mE_y}{\hbar} = 0 \\ \left(\frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} - \frac{2m}{\hbar^2}V_z\right) - \frac{2mE_z}{\hbar} = 0 \end{cases}$$

The solution is

$$\begin{cases} E_x = \frac{\pi^2\hbar^2}{2ma^2}n_x^2 \\ E_y = \frac{\pi^2\hbar^2}{2mb^2}n_y^2 \\ E_z = \frac{\pi^2\hbar^2}{2mc^2}n_z^2 \end{cases}$$

And the energy that the particle may have in total is

$$E = E_x + E_y + E_z = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}\right)$$