《电动力学》课后习题——第一章 电磁现象的基本规律

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1.1　根据算符 ∇ 的微分性与矢量性，推导下列公式：

$$\nabla\left(\vec{A}\cdot\vec{B}\right)=\vec{B}\times\left(\nabla\cdot\vec{A}\right)+\left(\vec{B}\cdot\nabla\right)\vec{A}+\vec{A}\times\left(\nabla\times\vec{B}\right) \tag{1}$$

$$\vec{A}\times\left(\nabla\times\vec{A}\right)=\frac{1}{2}\nabla A^2-\left(\vec{A}\cdot\nabla\right)\vec{A} \tag{2}$$

解

$$\nabla\left(\vec{A}\cdot\vec{B}\right)=\left(\partial_i\vec{e}_i\right)\left(A_jB_j\right)=\left(A_j\partial_iB_j+B_j\partial_iA_j\right)\vec{e}_i \tag{3}$$

$$\left(\vec{B}\cdot\nabla\right)\vec{A}=\left(B_i\vec{e}_i\cdot\partial_j\vec{e}_j\right)\vec{A}=\left(\delta_{ij}B_i\partial_j\right)\vec{A}=\left(B_i\partial_i\right)\left(A_j\vec{e}_j\right)=B_i\partial_iA_j\vec{e}_j \tag{4}$$

同理

$$\left(\vec{A}\cdot\nabla\right)\vec{B}=A_i\partial_iB_j\vec{e}_j \tag{5}$$

$$\begin{aligned}\vec{B}\times\left(\nabla\times\vec{A}\right)&=\vec{B}\times\left(\epsilon_{ijk}\partial_iA_j\vec{e}_k\right)=\epsilon_{mnl}B_m\left(\epsilon_{ijk}\partial_iA_j\vec{e}_k\right)_n\vec{e}_l=\epsilon_{mnl}B_m\epsilon_{ijn}\partial_iA_j\vec{e}_l=\epsilon_{lmn}\epsilon_{ijn}B_m\partial_iA_j\vec{e}_l\\&=\left(B_m\partial_lA_m-B_m\partial_mA_l\right)\vec{e}_l\end{aligned} \tag{6}$$

同理

$$\vec{A}\times\left(\nabla\times\vec{B}\right)=\left(A_m\partial_lB_m-A_m\partial_mB_l\right)\vec{e}_l \tag{7}$$

式（4）（5）（6）（7）相加，显然等于式（3），因此式（1）得证。

$$\begin{aligned}\vec{A}\times\left(\nabla\times\vec{A}\right)&=\vec{A}\times\left[\left(\partial_i\vec{e}_i\right)\times\left(A_j\vec{e}_j\right)\right]=\vec{A}\times\left(\epsilon_{ijk}\partial_iA_j\vec{e}_k\right)=\left(A_l\vec{e}_l\right)\times\left(\epsilon_{ijk}\partial_{ijk}\partial_iA_j\vec{e}_k\right)\\&=\epsilon_{ijk}\epsilon_{lkn}A_l\partial_iA_j\vec{e}_n=\epsilon_{ijk}\epsilon_{nlk}A_l\partial_iA_j\vec{e}_n=\left(\delta_{in}\delta_{jl}-\delta_{il}\delta_{jn}\right)A_l\partial_iA_j\vec{e}_n\\&=A_j\partial_iA_j\vec{e}_i\end{aligned}\tag{8}$$

$$\left(\vec{A}\cdot\nabla\right)\vec{A}=\left(A_i\partial_i\right)\left(A_j\vec{e}_j\right)=A_j\partial_iA_j\vec{e}_j\tag{9}$$

显然，式（8）和（9）是相等的，得证。

1.2 设 u 是空间坐标 x,y,z 的函数，证明：

$$\nabla f\left(u\right)=\frac{\mathrm{d}f}{\mathrm{d}u}\nabla u$$

$$\nabla\cdot\vec{A}\left(u\right)=\nabla u\cdot\frac{\mathrm{d}\vec{A}}{\mathrm{d}u}$$

$$\nabla\times\vec{A}\left(u\right)=\nabla u\times\frac{\mathrm{d}\vec{A}}{\mathrm{d}u}$$

解

$$\nabla f=\partial_if_i\vec{e}_i=\frac{\partial}{\partial x_i}f_i\vec{e}_i=\frac{\partial f_i}{\partial u}\frac{\partial u}{\partial x_i}\vec{e}_i=\frac{\mathrm{d}f}{\mathrm{d}u}\nabla u$$

$$\nabla\cdot\vec{A}=\partial_iA_i=\frac{\partial A_i}{\partial x_i}=\frac{\partial A_i}{\partial_u}\frac{\partial u}{\partial x_i}=\nabla u\cdot\frac{\mathrm{d}\vec{A}}{\mathrm{d}u}$$

$$\nabla\times\vec{A}=\epsilon_{ijk}\partial_iA_j\vec{e}_k=\epsilon_{ijk}\frac{\partial u}{\partial x_i}\frac{\partial A_j}{\partial u}\vec{e}_k=\nabla u\times\frac{\mathrm{d}\vec{A}}{\mathrm{d}u}$$

1.3 设 $r=\sqrt{\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z-z'\right)^2}$ 为源点 x' 到 x 的距离，$\vec{r}$ 的方向规定为源点指向场点。

（1）证明以下结果，并体会对源变数求微商（$\nabla'=\vec{e}_i\dfrac{\partial}{\partial x_i'}$）和对场变数求微商（$\nabla=\vec{e}_i\dfrac{\partial}{\partial x_i}$）的关系。

$$\nabla\vec{r}=-\nabla'\vec{r}=\frac{\vec{r}}{r}$$

$$\nabla\frac{1}{r}=-\nabla'\frac{1}{r}=-\frac{\vec{r}}{r^3}$$

$$\nabla\times\frac{\vec{r}}{r^3}=0$$

$$\nabla\cdot\frac{\vec{r}}{r^3}=-\nabla'\cdot\frac{\vec{r}}{r^3}=0$$

解　因为

$$\nabla = \partial_i = \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x'_i}\frac{\partial x'_i}{\partial x_i} = -\frac{\partial}{\partial x'_i} = -\nabla$$

所以

$$\nabla r = -\nabla' r$$

$$\nabla \frac{1}{r} = -\nabla' \frac{1}{r}$$

$$\nabla \times \frac{\vec{r}}{r^3} = 0$$

$$\nabla \cdot \frac{\vec{r}}{r^3} = -\nabla' \cdot \frac{\vec{r}}{r^3}$$

只需要证明

$$\nabla \vec{r} = \frac{\vec{r}}{r} \tag{10}$$

$$\nabla \frac{1}{r} = -\frac{\vec{r}}{r^3} \tag{11}$$

$$\nabla \times \frac{\vec{r}}{r^3} = 0 \tag{12}$$

$$\nabla \cdot \frac{\vec{r}}{r^3} = 0 \tag{13}$$

对于（10）

$$\nabla r = \partial_i \sqrt{\Sigma \left(x_i - x'_i\right)^2}\vec{e}_i = \frac{x_i}{r}\vec{e}_i = \frac{\vec{r}}{r}$$

对于（11）

$$\nabla \frac{1}{r} = -\frac{1}{r^2}\nabla r = -\frac{\vec{r}}{r^3}$$

对于（12）

$$\nabla \times \frac{\vec{r}}{r^3} = \epsilon_{ijk}\partial_i \left(\frac{\vec{r}}{r^3}\right)_j \vec{e}_k = \epsilon_{ijk}\partial_i \left(\frac{x_j}{r^3}\right)\vec{e}_k = 0$$

对于（13）

$$\begin{aligned}\nabla \cdot \frac{\vec{r}}{r^3} = \partial_i \left(\frac{x_i}{r^3}\right) = \frac{r^3\partial_i x_i - x_i\partial_i r^3}{r^6} = \frac{r^3\partial_i x_i - 3r^2 x_i \partial_i r}{r^6} = \frac{r^3 - 3r^2 x_i \dfrac{x_i}{r}}{r^6} = \frac{r^3 - 3rx_i^2}{r^6} \\ = \frac{r^3 - 3rx_1^2}{r^6} + \frac{r^3 - 3rx_2^2}{r^6} + \frac{r^3 - 3rx_3^2}{r^6} = \frac{3r^3 - 3r\left(x_1^2 + x_2^2 + x_3^2\right)}{r^6} = 0\end{aligned}$$

由于分母上有 r，当 $r = 0$ 时，不一定成立。

（2）求 $\nabla \cdot \vec{r}$，$\nabla \times \vec{r}$，$(\vec{a}\cdot\nabla)\,\vec{r}$，$\nabla(\vec{a}\cdot\vec{r})$，$\nabla\cdot\left[\vec{E}_0 \sin\left(\vec{k}\cdot\vec{r}\right)\right]$，$\nabla\times\left[\vec{E}_0 \sin\left(\vec{k}\cdot\vec{r}\right)\right]$，$\vec{a}$、$\vec{k}$、$\vec{E}_0$ 是常矢量。

解

$$\nabla \cdot \vec{r} = \partial_i r_i = 3$$

$$\nabla \times \vec{r} = \epsilon_{ijk}\partial_i x_j \vec{e}_k = 0$$

$$(\vec{a}\cdot\nabla)\,\vec{r} = a_i \partial_i x_i \vec{e}_i = a_i \vec{e}_i = \vec{a}$$

$$\nabla(\vec{a}\cdot\vec{r}) = \partial_i (a_j r_j)\,\vec{e}_i = [a_j\partial_i x_j + x_j \partial_i a_j] = a_j \partial_i x_j = a_i\partial_i x_i \vec{e}_i = a_i\vec{e}_i = \vec{a}$$

$$\nabla\cdot\left[\vec{E}_0 \sin\left(\vec{k}\cdot\vec{r}\right)\right] = \partial_i E_{0i}\sin(k_i x_i) = E_{0i}\cos(k_i x_i)\,k_i = \vec{k}\cdot\vec{E}_0\cos\left(\vec{k}\cdot\vec{r}\right)$$

$$\nabla\times\left[\vec{E}_0 \sin\left(\vec{k}\cdot\vec{r}\right)\right] = \epsilon_{ijk}\partial_i E_{0j}\sin(k_j x_j)\,\vec{e}_k = \epsilon_{ijk}E_{0j}\cos(k_m x_m)\,k_i\vec{e}_k = \vec{k}\times\vec{E}_0\cos\left(\vec{k}\cdot\vec{r}\right)$$

1.4 利用高斯定理证明

$$\int_V \mathrm{d}V \nabla\times\vec{f} = \oint_S \mathrm{d}\vec{S}\times\vec{f}$$

利用斯托克斯定理证明

$$\int_S \mathrm{d}\vec{S}\times\nabla\varphi = \oint_L \varphi\,\mathrm{d}\vec{l}$$

解 引入常矢量 $\vec{c}$

$$\int_V \mathrm{d}V\nabla\times\vec{f}\cdot\vec{c}=\int_V \vec{c}\cdot\left(\nabla\times\vec{f}\right)\mathrm{d}V=\int_V \nabla\cdot\left(\vec{f}\times\vec{c}\right)\mathrm{d}V=\oint_S\left(\vec{f}\times\vec{c}\right)\cdot\mathrm{d}\vec{S}=\oint_S \mathrm{d}\vec{S}\times\vec{f}\cdot\vec{c}$$

$$\int_S \mathrm{d}\vec{S}\times\nabla\varphi\cdot\vec{c}=\int_S \nabla\varphi\times\vec{c}\cdot\mathrm{d}\vec{S}=\int_L \nabla\times(\varphi\vec{c})\cdot\mathrm{d}\vec{S}=\oint_L \varphi\vec{c}\cdot\mathrm{d}\vec{l}=\oint_L \varphi\,\mathrm{d}\vec{l}\cdot\vec{c}$$

因为 $\vec{c}$ 是任意的

$$\int_V \mathrm{d}V\nabla\times\vec{f}=\oint_S \mathrm{d}\vec{S}\times\vec{f}$$

$$\int_S \mathrm{d}\vec{S}\times\nabla\varphi=\oint_L \varphi\,\mathrm{d}\vec{l}$$

1.5 已知一个电荷系统的电偶极矩为

$$\vec{p}(t)=\int_V \rho\left(\vec{x}',t\right)\vec{x}'\,\mathrm{d}V'$$

利用电荷守恒定律 $\nabla\cdot\vec{J}+\dfrac{\partial\rho}{\partial t}=0$ 证明

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}=\int_V \vec{J}\left(\vec{x}',t\right)\mathrm{d}V'$$

解

$$\begin{aligned}\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}&=\int_V \frac{\mathrm{d}\rho\left(\vec{x}',t\right)}{\mathrm{d}t}\vec{x}'\,\mathrm{d}V'=-\int_V\left(\nabla'\cdot\vec{J}\right)\vec{x}'\,\mathrm{d}V'=-\int_V\left(\nabla'\cdot\vec{J}\vec{x}'\right)-\vec{J}\left(\nabla'\vec{x}'\right)\mathrm{d}V'\\&=-\oint_S \vec{J}\vec{x}'\cdot\mathrm{d}\vec{S}+\int_V\vec{J}\,\mathrm{d}V'=\int_V\vec{J}\,\mathrm{d}V'\end{aligned}$$

1.6 若 $\vec{m}$ 是常矢量，证明除 $R=0$ 以外，矢量 $\vec{A}=\dfrac{\vec{m}\times\vec{R}}{R^3}$ 的旋度等于标量 $\varphi=\dfrac{\vec{m}\cdot\vec{R}}{R^3}$ 的梯度的负值，即

$$\nabla\times\vec{A}=-\nabla\varphi$$

解

$$\begin{aligned}\nabla\times\vec{A}&=\nabla\times\left(\vec{m}\times\frac{\vec{R}}{R^3}\right)=\left(\frac{\vec{R}}{R^3}\cdot\nabla\right)\vec{m}+\left(\nabla\cdot\frac{\vec{R}}{R^3}\right)\vec{m}-\left(\vec{m}\cdot\nabla\right)\frac{\vec{R}}{R^3}-\left(\nabla\cdot\vec{m}\right)\frac{\vec{R}}{R^3}\\&=\left(\nabla\cdot\frac{\vec{R}}{R^3}\right)\vec{m}-\left(\vec{m}\cdot\nabla\right)\frac{\vec{R}}{R^3}=-\left(\vec{m}\cdot\nabla\right)\frac{\vec{R}}{R^3}\end{aligned}$$

$$
\begin{aligned}
-\nabla\varphi &= -\nabla\left(\frac{\vec{m}\cdot\vec{R}}{R^3}\right) = -\vec{m}\times\left(\nabla\times\frac{\vec{R}}{R^3}\right) - (\vec{m}\cdot\nabla)\frac{\vec{R}}{R^3} - \frac{\vec{R}}{R^3}\times(\nabla\times\vec{m}) - \left(\frac{\vec{R}}{R^3}\cdot\nabla\right)\vec{m} \\
&= -\vec{m}\times\left(\nabla\times\frac{\vec{R}}{R^3}\right) - (\vec{m}\cdot\nabla)\frac{\vec{R}}{R^3} = -(\vec{m}\cdot\nabla)\frac{\vec{R}}{R^3}
\end{aligned}
$$

得证

1.7 有一内外半径为 r_1 和 r_2 的空心介质球，介质的电容率为 ε，是介质内均匀带静自由电荷密度 ρ_f，求：

（1）空间各点的电场

（2）极化体电荷和极化面电荷分布

解 在 $r<r_1$ 时

$$
\oint_S \vec{D}\cdot\mathrm{d}\vec{S} = \int_V \rho_f\,\mathrm{d}V \Rightarrow 4\pi r^2 D = 0 \Rightarrow \vec{D}=0 \Rightarrow \vec{E}=0
$$

在 $r_1<r<r_2$ 时

$$
\oint_S \vec{D}\cdot\mathrm{d}\vec{S} = \int_V \rho_f\,\mathrm{d}V \Rightarrow 4\pi r^2 D = \frac{4}{3}\pi\left(r^3-r_1^3\right)\rho_f \Rightarrow \vec{D} = \frac{r^3-r_1^3}{3r^3}\rho_f\vec{r} \Rightarrow \vec{E} = \frac{r^3-r_1^3}{3\varepsilon r^3}\rho_f\vec{r}
$$

在 $r>r_2$ 时

$$
\oint_S \vec{D}\cdot\mathrm{d}\vec{S} = \int_V \rho_f\,\mathrm{d}V \Rightarrow 4\pi r^2 D = \frac{4}{3}\pi\left(r_2^3-r_1^3\right)\rho_f \Rightarrow \vec{D} = \frac{r_2^3-r_1^3}{3r^3}\rho_f\vec{r} \Rightarrow \vec{D} = \frac{r_2^3-r_1^3}{3\varepsilon_0 r^3}\rho_f\vec{r}
$$

在 $r_1<r<r_2$ 时

$$
\begin{aligned}
&D = \varepsilon_0 E + P \Rightarrow P = D - \varepsilon_0 E = D\left(1-\frac{\varepsilon_0}{\varepsilon}\right) \Rightarrow \vec{P} = \left(1-\frac{\varepsilon_0}{\varepsilon}\right)\frac{r^3-r_1^3}{3r^3}\rho_f\vec{r} \\
&\quad\Rightarrow \rho_P = -\nabla\cdot\vec{P} = -\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\rho_f
\end{aligned}
$$

在 $r=r_2$ 时

$$
\sigma_P = -\vec{e}_n\cdot\left(\vec{P}_3-\vec{P}_2\right) = \left(1-\frac{\varepsilon_0}{\varepsilon}\right)\frac{r_2^3-r_1^3}{3r_2^2}\rho_f
$$

在 $r=r_1$ 时

$$
\sigma_P = -\vec{e}_n\cdot\left(\vec{P}_2-\vec{P}_1\right) = 0
$$

1.8 内外半径分别为 r_1 和 r_2 的无穷长中空导体圆柱，沿轴向流有恒定均匀自由电流 $\vec{J}_f$，导体磁导率为 μ，求磁感应强度和磁化电流

解 在 $r < r_1$ 时

$$\oint \vec{H} \cdot \mathrm{d}\vec{l} = I_f + \frac{\mathrm{d}}{\mathrm{d}t} \int_S \vec{D} \cdot \mathrm{d}\vec{S} \Rightarrow \vec{H} = 0 \Rightarrow \vec{B} = 0$$

在 $r_1 < r < r_2$ 时

$$\oint \vec{H} \cdot \mathrm{d}\vec{l} = I_f + \frac{\mathrm{d}}{\mathrm{d}t} \int_S \vec{D} \cdot \mathrm{d}\vec{S} \Rightarrow 2\pi r H = \pi \left(r^2 - r_1^2\right) J_f \Rightarrow \vec{H} = \frac{r^2 - r_1^2}{2r^2} \vec{J}_f \times \vec{r} \Rightarrow \vec{B} = \mu \frac{r^2 - r_1^2}{2r^2} \vec{J}_f \times \vec{r}$$

在 $r > r_2$ 时

$$\oint \vec{H} \cdot \mathrm{d}\vec{l} = I_f + \frac{\mathrm{d}}{\mathrm{d}t} \int_S \vec{D} \cdot \mathrm{d}\vec{S} \Rightarrow 2\pi r H = \pi \left(r_2^2 - r_1^2\right) J_f \Rightarrow \vec{H} = \frac{r_2^2 - r_1^2}{2r^2} \vec{J}_f \times \vec{r} \Rightarrow \vec{B} = \mu_0 \frac{r_2^2 - r_1^2}{2r^2} \vec{J}_f \times \vec{r}$$

当 $r_1 < r < r_2$ 时

$$\vec{J}_m = \nabla \times \vec{M} = \left(\frac{\mu}{\mu_0} - 1\right) \nabla \times \vec{H} = \left(\frac{\mu}{\mu_0} - 1\right) \vec{J}_f$$

当 $r = r_2$ 时

$$\vec{\alpha}_M = \vec{e}_r \times \left(\vec{M}_3 - \vec{M}_2\right) = -\left(\frac{\mu}{\mu_0} - 1\right) \vec{e}_r \times \vec{H}_3 = -\left(\frac{\mu}{\mu_0} - 1\right) \frac{r_2^2 - r_1^2}{2r_2^2} \vec{J}_f$$

当 $r = r_1$ 时

$$\vec{\alpha}_M = \vec{e}_r \times \left(\vec{M}_2 - \vec{M}_1\right) = 0$$

1.9 证明均匀介质内部的极化电荷体密度 ρ_P 总是等于自由电荷体密度 ρ_f 的 $-\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)$ 倍

解

$$\varepsilon E = D \Rightarrow \nabla \cdot \varepsilon E = \nabla \cdot D \Rightarrow \varepsilon\varepsilon_0 \nabla \cdot E = \varepsilon_0 \nabla \cdot D \Rightarrow \varepsilon \left(\rho_P + \rho_f\right) = \varepsilon_0 \rho_f \Rightarrow \rho_P = -\left(1 - \frac{\varepsilon_0}{\varepsilon}\right) \rho_f$$

1.10 证明两个闭合的恒定电流圈之间的相互作用力大小相等，方向相反（但两个电流元之间的相互作用力一般不服从牛顿第三定律）

解 设两个电流圈的电流为 I_1 和 I_2

$$
\begin{aligned}
\vec{F}_{12} &= \oint_{L2} I_2 \,\mathrm{d}\vec{l}_2 \times \frac{\mu_0}{4\pi} \oint_{L1} \frac{I_1 \,\mathrm{d}\vec{l}_1 \times \vec{r}_{12}}{r_{12}^3} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L2} \oint_{L1} \frac{\mathrm{d}\vec{l}_2 \times \mathrm{d}\vec{l}_1 \times \vec{r}_{12}}{r_{12}^3} \\
&= \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L2} \oint_{L1} \frac{\left(\mathrm{d}\vec{l}_2 \cdot \vec{r}_{12}\right) \mathrm{d}\vec{l}_1 - \left(\mathrm{d}\vec{l}_2 \cdot \mathrm{d}\vec{l}_1\right) \vec{r}_{12}}{r_{12}^3}
\end{aligned}
$$

因为

$$
\oint_{L1} \oint_{L2} \frac{\left(\mathrm{d}\vec{l}_2 \cdot \vec{r}_{12}\right) \mathrm{d}\vec{l}_1}{r_{12}^3} = \oint_{L1} \left[\oint_{L2} \frac{\vec{r}_{12}}{r_{12}^3} \cdot \mathrm{d}\vec{l}_2\right] \mathrm{d}\vec{l}_1 = \oint_{L1} \left[\oint_{S2} \nabla \times \frac{\vec{r}_{12}}{r_{12}^3} \cdot \mathrm{d}\vec{S}_2\right] \mathrm{d}\vec{l}_1 = 0
$$

所以

$$
\vec{F}_{12} = -\frac{\mu_0 I_1 I_2}{4\pi} \oint_{L2} \oint_{L1} \frac{\mathrm{d}\vec{l}_2 \cdot \mathrm{d}\vec{l}_1}{r^3} \vec{r}_{12}
$$

同理

$$
\vec{F}_{21} = -\frac{\mu_0 I_2 I_1}{4\pi} \oint_{L1} \oint_{L2} \frac{\mathrm{d}\vec{l}_1 \cdot \mathrm{d}\vec{l}_2}{r^3} \vec{r}_{21}
$$

其中 $r_{12} = -r_{21}$，因此

$$
\vec{F}_{21} = -\vec{F}_{12}
$$