Optics

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Contents

Chapter 1

History of Optics

Chapter 2

Electromagnetic Theory and photons

2.1 Maxwell's Equation

Faraday's Induction Law

$$\oint_c \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint \vec{B} \cdot \mathrm{d}\vec{S}$$

Gauss's Law

$$\oiint_A \vec{E} \cdot \mathrm{d}\vec{S} = \frac{1}{\varepsilon_0} \iiint_v \rho \, \mathrm{d}V$$

$$\oiint_A \vec{B} \cdot \mathrm{d}\vec{S} = 0$$

Ampere's Circuital Law

$$\oint_C \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 \iint_A \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot \mathrm{d}\vec{S}$$

We can now take the derivatives of the 4 equations

$$\begin{cases} \nabla \times \vec{E} = -\dfrac{\partial B}{\partial t} \\ \nabla \times \vec{B} = \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} \\ \nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0} \\ \nabla \cdot \vec{B} = 0 \end{cases}$$

The Following Equation can be derived from Maxwell Equation above

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

Coincidentally

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Which indicates the speed of electromagnetic wave is the speed of light.

Furthermore, it can be seen that the electric field and magnetic field are transverse. They are perpendicular to each other. We assume the electric field is parallel to the y-axis.

$$E_y(x,t) = E_0 \cos\left[\omega\left(t - x/c\right) + \varepsilon\right]$$

According to Faraday's Law

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_Z}{\partial t}$$

We can calculate B_z

$$B_z = \frac{1}{c} E_0 \cos\left[\omega\left(t - x/c\right) + \varepsilon\right]$$

So that

$$E_y = cB_z$$

When not in vacuum, similarly

$$E_y = vB_z$$
$$v = {}^1\!/_{\varepsilon\mu}$$

2.2 Energy

$$
\begin{aligned}
u_E &= \frac{\varepsilon_0}{2} E^2 \\
u_B &= \frac{1}{2\mu_0} B^2 \\
u_E &= u_B \\
u &= u_E + u_B = \varepsilon_0 E^2 = \frac{1}{\mu_0} B^2 \\
S &= uc \\
\vec{S} &= \frac{1}{\mu} \vec{E} \times \vec{B} \quad \text{(PoyntingVector)} \\
I &= \frac{1}{2} \varepsilon v E_0^2
\end{aligned}
$$

2.3 Radiation Pressure

$$P(t) = \frac{S(t)}{c} = u = u_E + u_B$$

$$\langle P(t) \rangle_T = \frac{I}{c}$$

$$p_V = \frac{S}{c^2}$$

2.4 Light in Bulk Matter

2.4.1 Speed of light and Dielectric Constant

$$v = \frac{1}{\sqrt{\varepsilon\mu}}$$

$$n = \frac{c}{v} = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}} = \sqrt{\frac{\varepsilon}{\varepsilon_0}}$$

2.4.2 Dispersion

For gas and solid

$$m_e \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \gamma m_e \frac{\mathrm{d}x}{\mathrm{d}t} + m_e \omega_0^2 x = -eE(t)$$

$$E(t) = E_0 \exp(-i\omega t)$$

Assume

$$x = x_0 \exp(-i\omega t)$$

We got a solution

$$x_0 \left(\omega_0^2 - \omega^2 - i\gamma\omega\right) = -\frac{eE_0}{m_e}$$

$$x_0 = -\frac{eE_0}{m_e \left(\omega_0^2 - \omega^2 - i\gamma\omega\right)}$$

$$x(t) = -\frac{eE(t)}{m_e \left(\omega_0^2 - \omega^2 - i\gamma\omega\right)}$$

$$P(t) = -Nex(t) = \frac{Ne^2 E(t)}{m_e \left(\omega_0^2 - \omega^2 - i\gamma\omega\right)}$$

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = n^2 = 1 + \frac{P}{\varepsilon_0 E} = 1 + \frac{Ne^2}{\varepsilon_0 m_e \left(\omega_0^2 - \omega^2 - i\gamma\omega\right)}$$

$$\begin{cases} \mathrm{Re}(\varepsilon_r) = 1 + \dfrac{Ne^2\left(\omega_0^2 - \omega^2\right)}{\varepsilon_0 m_e \left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]} \\ \mathrm{Im}(\varepsilon_r) = \dfrac{Ne^2 \gamma\omega}{\varepsilon_0 m_e \left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]} \end{cases}$$

When $\gamma = 0$

$$\varepsilon_r = n^2 = 1 + \frac{Ne^2}{\varepsilon_0 m \left(\omega_0^2 - \omega^2\right)}$$

For metal

$$m_e \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \gamma m_e \frac{\mathrm{d}x}{\mathrm{d}t} = -eE(t)$$

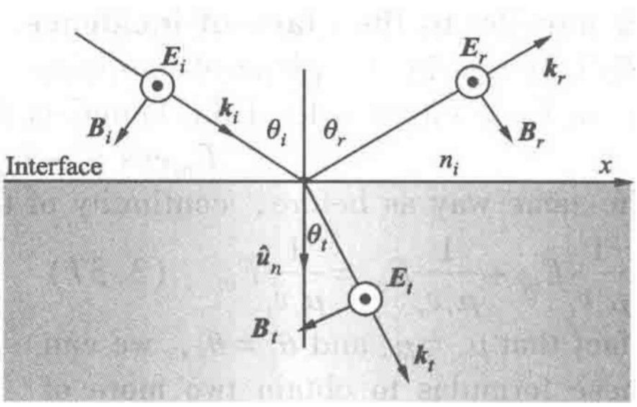
$$\varepsilon_r = 1 - \frac{Ne^2}{\varepsilon_0 m_e \left(\omega^2 + i\gamma\omega\right)} = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma)}$$

Chapter 3

The Propagation of Light

3.1 The Fresnel Equations

3.1.1 Electric Field Perpendicular to Plane of Incidence



$$\begin{cases} E_i + E_r = E_t \\ B_i \cos\theta_i = B_r \cos\theta_r + B_t \cos\theta_t \end{cases}$$

$$\begin{cases} E = vB \\ v = \dfrac{c}{n} \end{cases}$$

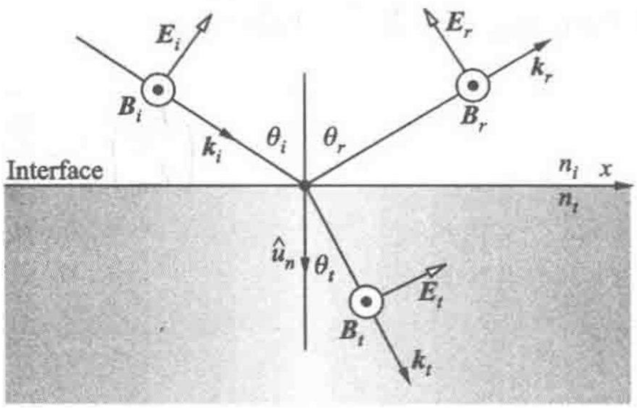
We define the amplitude reflection coefficient r, the amplitude transmission coefficient t

$$r = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}$$
$$t = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}$$

Considered that $n_i \sin\theta_i = n_t \sin\theta_t$

$$r_\perp = \frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$
$$t_\perp = \frac{2\sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)}$$

3.1.2 Electric Field Parallel to Plane of Incidence



$$\begin{cases} B_i + B_r = B_t \\ E_i \cos\theta_i = E_r \cos\theta_r + E_t \cos\theta_t \end{cases}$$

$$\begin{cases} E = vB \\ v = \dfrac{c}{n} \end{cases}$$

We define the amplitude reflection coefficient r, the amplitude transmission coefficient t

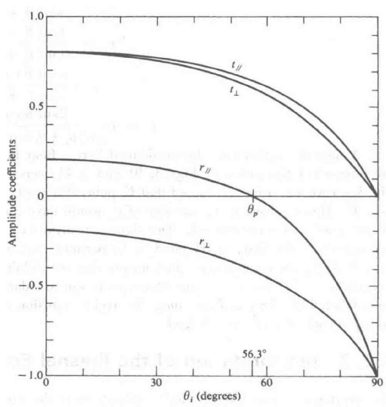
$$r = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t}$$
$$t = \frac{2n_i \cos\theta_i}{n_t \cos\theta_i + n_i \cos\theta_t}$$

Considered that $n_i \sin\theta_i = n_t \sin\theta_t$

$$r_\parallel = \frac{\sin(2\theta_i) - \sin(2\theta_t)}{\sin(2\theta_i) + \sin(2\theta_t)} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$
$$t_\parallel = \frac{2\sin\theta_t\theta_i}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)}$$

3.2 Polarization Angle

$$\theta_p = \arctan\left(\frac{\theta_p}{\theta_i}\right)$$



3.3 Critical Angle

$$\theta_c = \arcsin\left(\frac{n_t}{n_i}\right)$$

3.4 Phase Shift

When $\theta_i = 0$

$$\begin{aligned} r_\perp = -r_\parallel &= \frac{n_i - n_t}{n_i + n_t} \\ t_\parallel = \quad t_\perp &= \frac{2n_i}{n_i + n_t} \end{aligned}$$

While $n_i > n_t$ (Inner reflection)

$$\begin{aligned} r_\parallel &< 0 \\ r_\perp &> 0 \end{aligned}$$

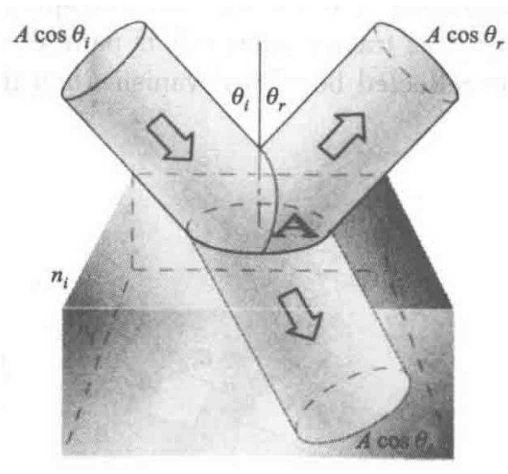
No phase shift.

While $n_i < n_t$ (Outer reflection)

$$\begin{aligned} r_\parallel &> 0 \\ r_\perp &< 0 \end{aligned}$$

Phase shifted by π.

3.5 Reflectance and Transmittance

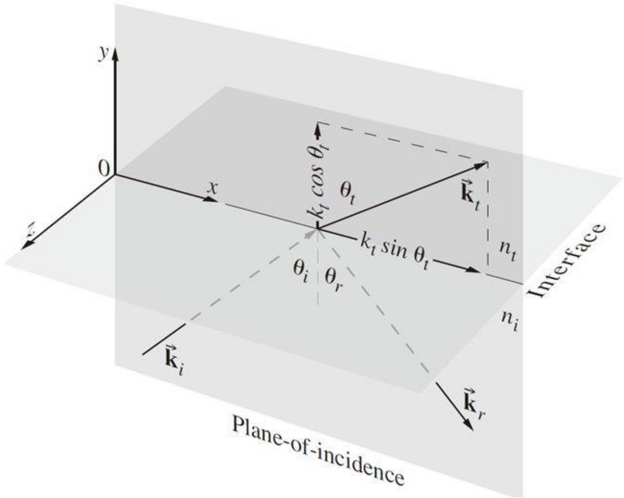


$$\begin{cases} R = \dfrac{I_t A \cos\theta_r}{I_i A \cos\theta_i} = \dfrac{I_t}{I_i} \\ T = \dfrac{I_t A \cos\theta_t}{I_i A \cos\theta_i} = \dfrac{I_t \cos\theta_t}{I_i \cos\theta_i} \end{cases}$$

$$I = \frac{1}{2}\varepsilon v E_0^2 = \frac{1}{2}\varepsilon_0 \varepsilon_r v E_0^2 = \frac{1}{2}\varepsilon_0 n^2 v E_0^2 = \frac{1}{2}\varepsilon_0 n c E_0^2$$

$$\begin{cases} R = \dfrac{I_t}{I_i} = \left(\dfrac{E_{0t}}{E_{0i}}\right)^2 = r^2 \\ T = \dfrac{I_t \cos\theta_t}{I_i \cos\theta_i} = \left(\dfrac{n_t \cos\theta_t}{n_i \cos\theta_i}\right)\left(\dfrac{E_{0t}}{E_{0i}}\right)^2 = \left(\dfrac{n_t \cos\theta_t}{n_i \cos\theta_i}\right) t^2 \end{cases}$$

3.6 The Evanescent Wave



$$\vec{E}_t = \vec{E}_{0t} \exp\left[i\left(\vec{k}_t \cdot \vec{r} - \omega t\right)\right]$$

$$\vec{k}_t \cdot \vec{r} = k_{tx}x + k_{ty}y$$

$$k_{tx} = k_t \sin\theta_t = \left(\frac{n_i}{n_t}\right) k_t \sin\theta_i = n_i k_0 \sin\theta_i$$
$$k_{ty} = k_t \cos\theta_t = i k_t \sqrt{\frac{n_i^2 \sin^2\theta_i}{n_t^2} - 1} = i\beta$$

$$\vec{E}_t = \vec{E}_{0t} \exp\left(-\beta y\right) \exp\left[i\left(n_i k_0 x \sin\theta_i - \omega t\right)\right]$$

3.7 Optical Properties of Metals

The index of refraction of metal is complex

$$\tilde{n} = n_R - i n_I$$

$$\nabla \times \vec{H} = \varepsilon_0 \varepsilon_r \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} = -i\omega\varepsilon_0\varepsilon_r \vec{E} + \sigma\vec{E} = -i\omega\varepsilon_0\tilde{\varepsilon}_r\vec{E}$$

Whereas

$$\tilde{\varepsilon}_r = \varepsilon_r + i\frac{\sigma}{\omega\varepsilon_0}$$

$$\tilde{n}^2 = \tilde{\varepsilon}_r = \varepsilon_r + i\frac{\sigma}{\omega\varepsilon_0} = (n_R + in_I)^2$$

Since $\frac{\sigma}{\omega\varepsilon_0\varepsilon_r} \gg 1$

$$n_I \approx n_R = \sqrt{\frac{\sigma}{2\omega\varepsilon_0}}$$

Skin depth

$$\delta = \sqrt{\frac{1}{2\omega\mu_0\sigma}}$$

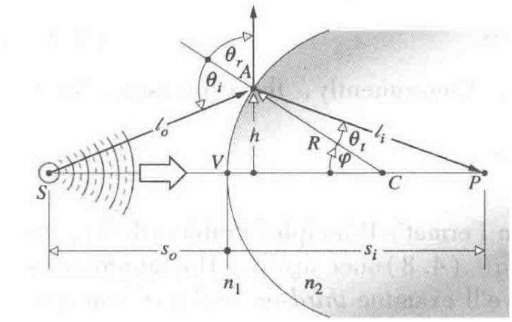
Reflectance

$$R = \left|\frac{n_i - n_t}{n_i + n_t}\right|^2 = \left(\frac{\tilde{n} - 1}{\tilde{n} + 1}\right)\left(\frac{\tilde{n} - 1}{\tilde{n} + 1}\right)^* = \frac{(n_R - 1)^2 + n_I^2}{(n_R + 1)^2 + n_I^2}$$

Chapter 4

Geometrical Optics

4.1 Refraction at a Spherical Interface



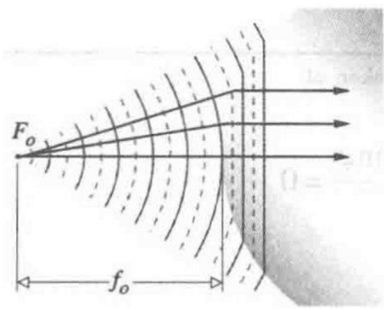
$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}$$

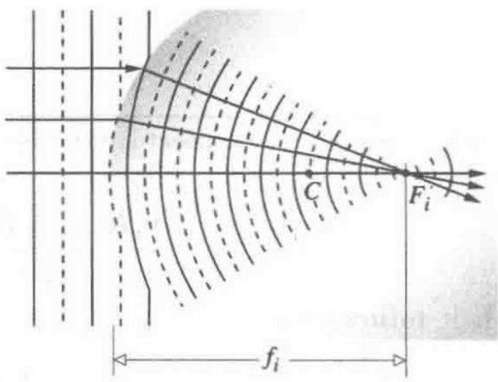
Let $s_i = \infty$, the object focus

$$f_0 = \frac{n_1}{n_2 - n_1} R$$

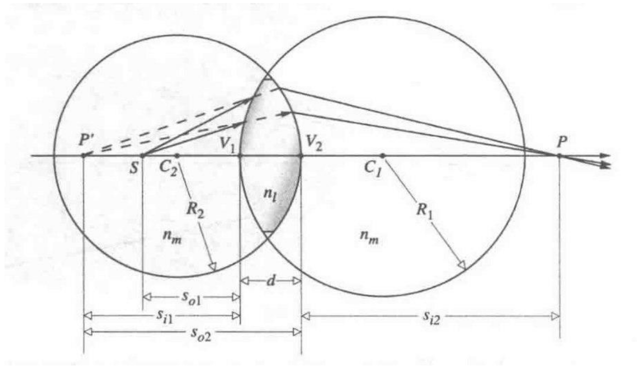
Let $s_o = \infty$, the image focus

$$f_i = \frac{n_2}{n_2 - n_1} R$$





4.2 Lenses



$$\frac{n_m}{s_{o1}}+\frac{n_m}{s_{i2}}=(n_l-n_m)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)+\frac{n_l d}{(s_{i1}-d)\,s_{i1}}$$

For lenses in the air, where $n_m=1$

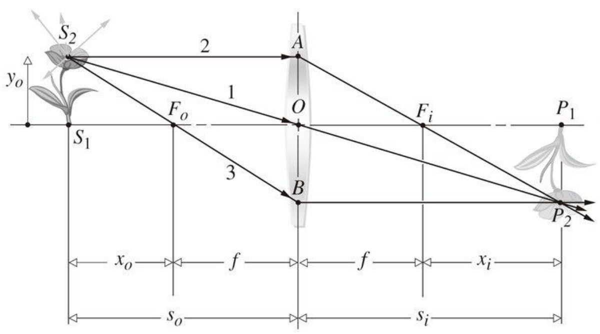
$$\frac{1}{s_{o1}}+\frac{1}{s_{i2}}=(n_l-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)+\frac{n_l d}{(s_{i1}-d)\,s_{i1}}$$

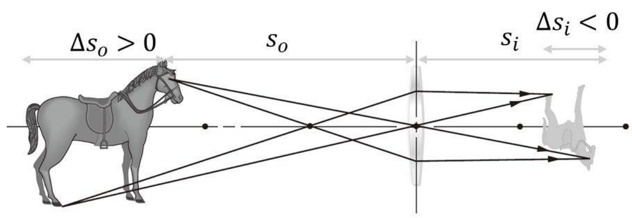
For thin lenses, $d\approx 0$

$$\frac{1}{s_{o1}}+\frac{1}{s_{i2}}=(n_l-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)=\frac{1}{f}$$

Which is the Gaussian Lens Formula.

4.3 Magnification





$$\begin{cases}\dfrac{y_o}{|y_i|}=\dfrac{f}{x_i}\\[2mm]\dfrac{|y_i|}{y_o}=\dfrac{f}{x_o}\end{cases}$$

Newton's formula

$$x_o x_i=f^2$$

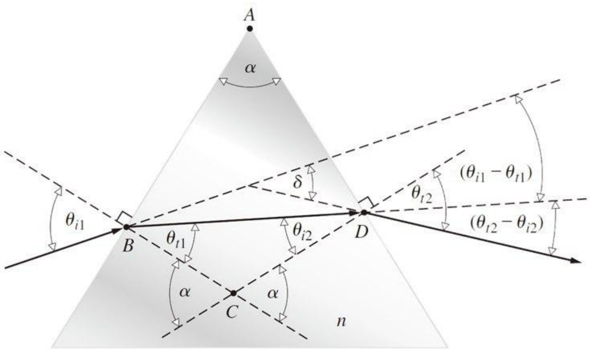
Transverse Magnification

$$M_T = \frac{y_i}{|y_o|} = -\frac{s_o}{s_i} = -\frac{f}{x_o} = -\frac{x_i}{f}$$

Longitudinal Magnification

$$M_L = \frac{\mathrm{d}x_i}{\mathrm{d}x_o} = \frac{\mathrm{d}}{\mathrm{d}x_o}\left(\frac{f^2}{x_o}\right) = -\frac{f^2}{x_o^2} = -M_T^2$$

4.4 Prism



$$\begin{cases} \delta = (\theta_{i1} - \theta_{t1}) + (\theta_{t2} - \theta_{i2}) \\ \alpha = \theta_{t1} + \theta_{i2} \end{cases}$$

$$\delta = \theta_{i1} + \theta_{t2} - \alpha$$

$$\begin{aligned} \theta_{t2} &= \arcsin\left(n \sin\theta_{i2}\right) = \arcsin\left[n \sin\left(\alpha - \theta_{t1}\right)\right] = \arcsin\left[n\left(\sin\alpha\cos\theta_{t1} - \cos\alpha\sin\theta_{t1}\right)\right] \\ &= \arcsin\left[n\left(\sin\alpha\sqrt{1-\sin^2\theta_{t1}} - \cos\alpha\sin\theta_{t1}\right)\right] \\ &= \arcsin\left[n\left(\sin\alpha\sqrt{1-n^2\sin^2\theta_{i1}} - \cos\alpha\sin\theta_{t1}\right)\right] \\ \delta &= \theta_{i1} + \arcsin\left[n\left(\sin\alpha\sqrt{1-n^2\sin^2\theta_{i1}} - \cos\alpha\sin\theta_{t1}\right)\right] - \alpha \end{aligned}$$

Chapter 5

The Superstition of Waves

5.1 The Addition of Waves

5.1.1 The Algebraic Method

$$E(x,t) = E_0 \sin[\omega t - (kx + \varepsilon)]$$

let

$$\alpha(x,\varepsilon) = -(kx + \varepsilon)$$

Then

$$E(x,t) = E_0 \sin[\omega t + \alpha(x,\varepsilon)]$$

Two waves of the same frequency

$$\begin{cases} E_1 = E_{01}\sin(\omega t + \alpha_1) \\ E_2 = E_{02}\sin(\omega t + \alpha_2) \end{cases}$$

$$\begin{aligned} E = E_1 + E_2 &= E_{01}(\sin\omega t\cos\alpha_1 + \cos\omega t\sin\alpha_1) + E_{02}(\sin\omega t\cos\alpha_2 + \cos\omega t\sin\alpha_2) \\ &= (E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2)\sin\omega t + (E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2)\cos\omega t \\ &= E_0\cos\alpha\sin\omega t + E_0\sin\alpha\cos\omega t \\ &= E_0\sin(\omega t + \alpha) \end{aligned}$$

$$\begin{cases} E_0\cos\alpha = E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2 \\ E_0\sin\alpha = E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2 \end{cases} \Rightarrow \begin{cases} E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos(\alpha_2 - \alpha_1) \\ \tan\alpha = \dfrac{E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2}{E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2} \end{cases}$$

The phase difference

$$\delta = (kx_1 + \varepsilon_1) - (kx_2 + \varepsilon_2) = \frac{2\pi}{\lambda}(x_1 - x_2) + (\varepsilon_1 - \varepsilon_2)$$

When $E_{01} = E_{02}$ and $\alpha_2 - \alpha_1 = \Delta x$

$$E_0^2 = 2E_{01}^2 + 2E_{01}^2\cos(k\Delta x) = 2E_{01}^2[1 + \cos(k\Delta x)]$$

$$\cos 2x = 2\cos^2 x - 1 \Rightarrow \cos(k\Delta x) = 2\cos^2\left(\frac{k\Delta x}{2}\right) - 1$$

$\Rightarrow$

$$E_0^2 = 2E_{01}^2 \cos^2\left(\frac{k\Delta x}{2}\right)$$

Period of the amplitude of addition

$$\frac{k\Delta x}{2} = \frac{\pi}{2} \Rightarrow k(\alpha_2 - \alpha_1) = \pi \Rightarrow \Delta x = \alpha_2 - \alpha_1 = \frac{\lambda}{2}$$

5.1.2 The Complex Method

$$E_1 = E_{01}\cos(kx \pm \omega t) \Rightarrow \tilde{E}_1 = E_{01}\exp[i(kx \pm \omega t)]$$

$$\begin{cases} E_1 = E_{01}\exp[i\alpha_1] \\ E_2 = E_{02}\exp[i\alpha_2] \\ E_0 = E_1 + E_2 \end{cases}$$

$$\begin{aligned} E_0^2 &= (E_{01}\exp[i\alpha_1] + E_{02}\exp[i\alpha_2]) \cdot (E_{01}\exp[-i\alpha_1] + E_{02}\exp[-i\alpha_2]) \\ &= E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos(\alpha_1 - \alpha_2) \end{aligned}$$

5.1.3 Phasor Addition Method

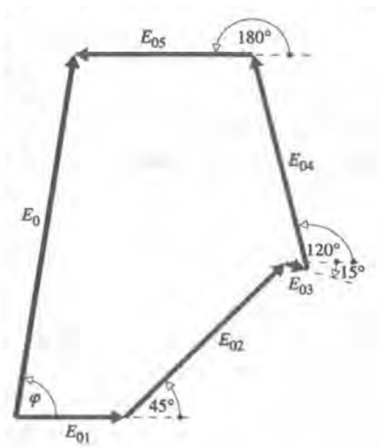


Figure 5.1: Phasor Addition Method

5.2 Standing Waves

$$\begin{aligned} &E_L = E_{0t}\sin(kx - \omega t) \\ &E_R = E_{0t}\sin(kx - \omega t) \quad E = E_L + E_R \end{aligned}$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$\Rightarrow$

$$E = 2E_{0t}\sin kx \cos\omega t$$

5.3 Addition of Waves of Different Frequency

$$\begin{cases} E_1 = E_{01}\cos\left(k_1 x - \omega_1 t\right) \\ E_2 = E_{02}\cos\left(k_2 x - \omega_2 t\right) \\ E = E_1 + E_2 \end{cases}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$\Rightarrow$

$$\begin{aligned} E &= E_{01}\left[\cos\left(k_1 x - \omega t\right) + \cos\left(k_2 x - \omega_2 t\right)\right] \\ &= 2E_{01}\cos\frac{1}{2}\left[\left(k_1 + k_2\right)x - \left(\omega_1 + \omega_2\right)t\right] \times \cos\frac{1}{2}\left[\left(k_1 - k_2\right)x - \left(\omega_1 - \omega_2\right)t\right] \end{aligned}$$

Define

$$\begin{aligned} \bar{\omega} &= \frac{1}{2}\left(\omega_1 + \omega_2\right) & \omega_m &= \frac{1}{2}\left(\omega_1 - \omega_2\right) \\ \bar{k} &= \frac{1}{2}\left(k_1 + k_2\right) & k_m &= \frac{1}{2}\left(k_1 - k_2\right) \end{aligned}$$

Then

$$E = 2E_{01}\cos\left(k_m x - \omega_m t\right)\cos\left(\bar{k}x - \bar{\omega}t\right) = E_0\left(x,t\right)\cos\left(\bar{k}x - \bar{\omega}t\right)$$

Noted that

$$\begin{aligned} \bar{\omega} &= \frac{1}{2}\left(\omega_1 + \omega_2\right) & & \omega_m = \frac{1}{2}\left(\omega_1 - \omega_2\right) \\ & & \gg & \\ \bar{k} &= \frac{1}{2}\left(k_1 + k_2\right) & & k_m = \frac{1}{2}\left(k_1 - k_2\right) \end{aligned}$$

$E_0 = 2E_{01}\cos\left(k_m x - \omega_m t\right)$ varies far less frequently than $\cos\left(\bar{k}x - \bar{\omega}t\right)$

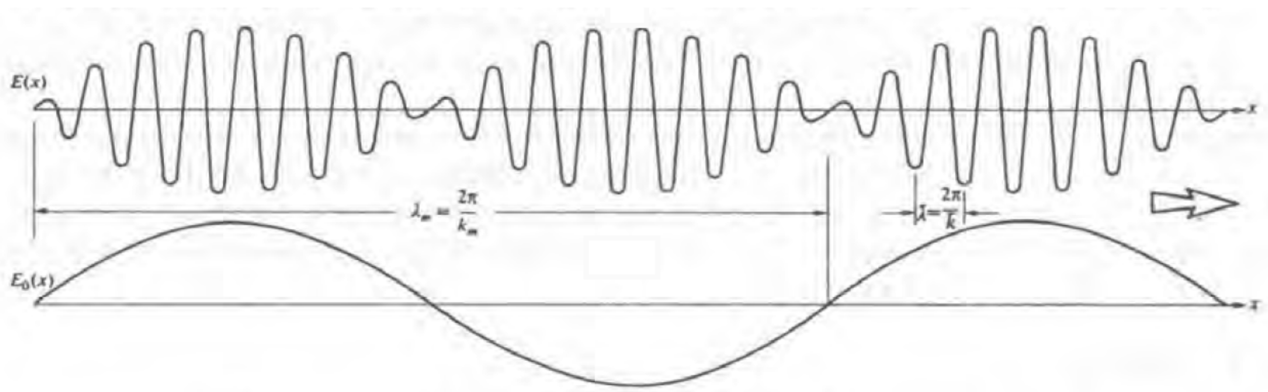


Figure 5.2: Standing Wave

Beat Frequency (Time)	$2\omega_m$
Beat Frequency (Space)	$2k_m$

Group Frequency	$v_g = \omega_m/k_m$
Phase Velocity	$v_p = \bar{\omega}/\bar{k}$

5.4 Light in Dispersible Media

Average Phase Velocity	$\bar{v}_p = \dfrac{c}{\bar{n}}$
Group Velocity	$v_g = \dfrac{c}{\bar{n}}\left(1 + \dfrac{\bar{\lambda}}{\bar{n}}\dfrac{\Delta n}{\Delta \lambda}\right)$

Normal Dispersion Media	$\bar{v}_p > v_g$
Anomalous Dispersion Media	$\bar{v}_p < v_g$