《电动力学》课后习题——第二章 静电场

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2.1 半径为 R 的电介质球，极化强度为 $\vec{P}=K\dfrac{\vec{r}}{r^2}$，电容率为 ϵ

（1）计算束缚电荷的体密度和面密度

（2）计算自由电荷体密度

（3）计算球外和球内的电势

（4）求该带电介质球产生的静电场总能量

解 计算束缚电荷密度，只需要对极化强度求梯度

$$\rho_p=-\nabla\cdot\vec{P}=\frac{K}{r^2}$$

面电荷密度用电场的边值关系求解

$$\sigma_p=-\vec{e}_n\cdot\left(0-\vec{P}\right)=\frac{k}{r}$$

在自由电荷密度为

$$\begin{aligned}\rho_f&=-\nabla\cdot\vec{D}=-\nabla\cdot\left(\varepsilon_0\vec{E}+\vec{P}\right)=-\nabla\cdot\left(\frac{\varepsilon_0}{\varepsilon_0\chi_e}+1\right)\vec{P}=-\nabla\cdot\left(\frac{\varepsilon_0}{\varepsilon_0\left(\varepsilon_r-1\right)}+1\right)\vec{P}\\&=-\nabla\cdot\left(\frac{\varepsilon_0}{\varepsilon-\varepsilon_0}+1\right)\vec{P}=-\nabla\cdot\frac{\varepsilon}{\varepsilon-\varepsilon_0}\vec{P}=\frac{\varepsilon}{\varepsilon-\varepsilon_0}\left(-\nabla\vec{P}\right)=\frac{\varepsilon}{\varepsilon-\varepsilon_0}\frac{K}{r^2}\end{aligned}$$

球内的电场为

$$\vec{E}_i=\frac{\vec{P}}{\varepsilon_0\chi_e}=\frac{\vec{P}}{\varepsilon_0\left(\varepsilon_r-1\right)}=\frac{\vec{P}}{\varepsilon-\varepsilon_0}=\frac{K}{\varepsilon-\varepsilon_0}\cdot\frac{\vec{r}}{r^2}$$

球外电场为

$$\vec{E}_o=\frac{\int_0^R 4\pi r^2\rho_f\,\mathrm{d}r}{4\pi\varepsilon_0 r^3}\vec{r}=\frac{\int_0^R r^2\dfrac{\varepsilon}{\varepsilon-\varepsilon_0}\dfrac{K}{r^2}\,\mathrm{d}r}{\varepsilon_0 r^3}\vec{r}=\int_0^R\frac{\varepsilon K}{\varepsilon_0\left(\varepsilon-\varepsilon_0\right)}\,\mathrm{d}r\cdot\frac{\vec{r}}{r^3}=\frac{\varepsilon KR}{\varepsilon_0\left(\varepsilon-\varepsilon_0\right)}\cdot\frac{\vec{r}}{r^3}$$

球外电势为

$$\varphi_o=\int_r^\infty\vec{E}_o\,\mathrm{d}\vec{r}=\frac{\varepsilon KR}{\varepsilon_0\left(\varepsilon-\varepsilon_0\right)}\frac{1}{r}$$

球内电势为

$$\varphi_i=\int_R^\infty\vec{E}_o\,\mathrm{d}\vec{r}+\int_r^R\vec{E}_i\,\mathrm{d}\vec{r}=\frac{K}{\varepsilon-\varepsilon_0}\left[\ln\frac{R}{r}+\frac{\varepsilon}{\varepsilon_0}\right]$$

总能量为

$$W=\frac{1}{2}\int_0^R 4\pi r^2\varepsilon\vec{E}_i^2\,\mathrm{d}r+\frac{1}{2}\int_R^\infty 4\pi r^2\varepsilon_0\vec{E}_o^2\,\mathrm{d}r=2\pi\varepsilon R\left(\frac{K}{\varepsilon-\varepsilon_0}\right)^2\left(1+\frac{\varepsilon}{\varepsilon_0}\right)$$

2.2 在均匀外电场中置入半径为 R_0 的导体球，试用分离变量法求下列两种情况的电势：

（1）导体球上接有电池，使球与地保持电势差 Φ_0

（2）导体球上带总电荷 Q

解 导体边界电势为 Φ_0 时，设解为

$$\varphi=\sum_n\left(a_nR^n+\frac{b_n}{R_n^{n+1}}\right)P_n\left(\cos\theta\right)$$

边界条件为

$$\begin{aligned}\varphi|_{r=\infty}&=-E_0R\cos\theta\\ \varphi|_{r=R_0}&=\Phi_0\end{aligned}$$

代入边界条件得出

$$\begin{aligned}&a_1=-E_0\\ &a_n=0 \qquad n\neq 1\\ &b_0=\Phi_0R_0\\ &b_1=E_0R_0^3\\ &b_n=0 \qquad n\neq 0,1\end{aligned}$$

所以

$$\varphi = \Phi_0 R_0 - E_0 \left(R - \frac{R_0^3}{R^2} \right) \cos\theta$$

当没有外接电池而是在导体球上放置电荷 Q 时，球面电势为

$$\Phi_0' = \frac{Q}{4\pi\varepsilon_0}$$

方程和边界条件相同，因此

$$\varphi = \Phi_0' R_0 - E_0 \left(R - \frac{R_0^3}{R^2} \right) \cos\theta = \frac{Q}{4\pi\varepsilon_0} R_0 - E_0 \left(R - \frac{R_0^3}{R^2} \right) \cos\theta$$

2.4 均匀介质球（电容率为 ε_1）的中心置一自由电偶极子 $\vec{p}_f$，球外充满了另一种介质（电容率为 ε_2），求空间各点电势和极化电荷分布。

提示：$\varphi = \dfrac{\vec{p}_f \cdot \vec{R}}{4\pi\varepsilon_1 R^3} + \varphi'$，而 φ' 满足拉普拉斯方程

解 电偶极子产生的电势为

$$\varphi = \frac{\vec{p}_f \cdot \vec{R}}{4\pi\varepsilon_0 R^3}$$

设

$$\begin{aligned}
\varphi_i &= \frac{p_f \cos\theta}{4\pi\varepsilon_1 R^2} + \sum_n a_n R^n P_n(\cos\theta) \quad R < R_0 \\
\varphi_o &= \frac{p_f \cos\theta}{4\pi\varepsilon_2 R^2} + \sum_n \frac{b_n}{R^{n+1}} P_n(\cos\theta) \quad R > R_0
\end{aligned}$$

因为边界上电势连续

$$\frac{p_f \cos\theta}{4\pi\varepsilon_1 R_0^2} + \sum_n a_n R_0^n P_n(\cos\theta) = \frac{p_f \cos\theta}{4\pi\varepsilon_2 R_0^2} + \sum_n \frac{b_n}{R_0^{n+1}} P_n(\cos\theta)$$

$$\varepsilon_1 \left\{ -\frac{2p_f \cos\theta}{4\pi\varepsilon_1 R_0^3} + \sum_n n a_n R_0^{n-1} P_n(\cos\theta) \right\} = \varepsilon_2 \left\{ -\frac{2p_f \cos\theta}{4\pi\varepsilon_2 R_0^3} - \sum_n (n+1) \frac{b_n}{R_0^{n+2}} P_n(\cos\theta) \right\}$$

当 $n = 1$ 时

$$\frac{p_f}{4\pi\varepsilon_1 R_0^2} + a_1 R_0 = \frac{p_f}{4\pi\varepsilon_2 R_0^2} + \frac{b_1}{R_0^2}$$

$$\varepsilon_1 \left\{-\frac{2p_f}{4\pi\varepsilon_1 R_0^3} + a_1\right\} = \varepsilon_2 \left\{-\frac{2p_f}{4\pi\varepsilon_2 R_0^3} - 2\frac{b_1}{R_0^3}\right\} \Rightarrow \varepsilon_1 a_1 = -\frac{2\varepsilon_2 b_1}{R_0^3}$$

解得

$$a_1 = \frac{(\varepsilon_1 - \varepsilon_2)\, p_f}{2\pi\varepsilon_1 (\varepsilon_1 + 2\varepsilon_2)\, R_0^3} \qquad b_1 = \frac{(\varepsilon_2 - \varepsilon_1)\, p_f}{4\pi\varepsilon_2 (\varepsilon_1 + 2\varepsilon_2)}$$

当 $n \neq 1$ 时，$a_n = b_n = 0$，因此方程的解为

$$\varphi_i = \frac{p_f \cos\theta}{4\pi\varepsilon_1 R^2} + \frac{(\varepsilon_1 - \varepsilon_2)\, p_f \cos\theta}{2\pi\varepsilon_1 (\varepsilon_1 + 2\varepsilon_2)\, R_0^3} R \quad R < R_0$$
$$\varphi_o = \frac{p_f \cos\theta}{4\pi\varepsilon_2 R^2} + \frac{(\varepsilon_2 - \varepsilon_1)\, p_f \cos\theta}{4\pi\varepsilon_2 (\varepsilon_1 + 2\varepsilon_2)} \frac{1}{R^2} \quad R > R_0$$

2.8 半径为 R_0 的导体球外充满均匀绝缘介质 ε，导体球接地，离球心 a 处（$a > R_0$）置一点电荷 Q_f，试用分离变量法求空间各点电势，证明所得结果与镜像法结果相同

解 球内电势为零，设球外电势为

$$\varphi = \frac{Q_f}{4\pi\varepsilon_0 r} + \sum_n \frac{b_n}{R^{n+1}} P_n(\cos\theta)$$

当 $R < a$ 时，展开

$$\frac{1}{r} = \frac{1}{\sqrt{R^2 + a^2 - 2Ra\cos\theta}} = \frac{1}{a}\sum_n \left(\frac{R}{a}\right)^n P_n(\cos\theta)$$

因为 $R_0 < a$，代入得到

$$\varphi = \frac{Q_f}{4\pi\varepsilon_0}\frac{1}{a}\sum_n \left(\frac{R}{a}\right)^n P_n(\cos\theta) + \sum_n \frac{b_n}{R^{n+1}} P_n(\cos\theta) = \left[\sum_n \frac{Q_f}{4\pi\varepsilon_0}\frac{R^n}{a^{n+1}} + \frac{b_n}{R^{n+1}}\right] P_n(\cos\theta)$$

当 $R = R_0$ 时，$\varphi = 0$

$$\frac{Q_f}{4\pi\varepsilon_0}\frac{R_0^n}{a^{n+1}} + \frac{b_n}{R_0^{n+1}} = 0 \Rightarrow b_n = -\frac{Q_f}{4\pi\varepsilon_0}\frac{R_0^{2n+1}}{a^{n+1}}$$

因此方程的解为

$$\varphi = \frac{Q_f}{4\pi\varepsilon_0}\left[\frac{1}{r} - \sum_n \frac{R_0^{2n+1}}{a^{n+1}}\frac{1}{R^{n+1}} P_n(\cos\theta)\right] \tag{1}$$

用电像法时只需要在 $b=\dfrac{R_0^2}{a}$ 处放置 $q_f=-\dfrac{R_0}{a}Q_f$ 的电荷，空间各点电势为

$$\varphi=\frac{Q_f}{4\pi\varepsilon_0}\left[\frac{1}{r}-\frac{R_0}{a}\frac{1}{r'}\right] \tag{2}$$

其中

$$\frac{1}{r'}=\frac{1}{\sqrt{R^2+b^2-2Rb\cos\theta}}$$

因为 $b<R$，展开式为

$$\frac{1}{\sqrt{R^2+b^2-2Rb\cos\theta}}=\frac{1}{R}\sum_n\left(\frac{b}{R}\right)^n P_n\left(\cos\theta\right)=\frac{1}{R}\sum_n\left(\frac{R_0^2}{aR}\right)^n P_n\left(\cos\theta\right)=\sum_n\frac{R_0^{2n}}{a^n}\frac{1}{R^{n+1}}P_n\left(\cos\theta\right)$$

代入式（2）得到

$$\varphi=\frac{Q_f}{4\pi\varepsilon_0}\left[\frac{1}{r}-\frac{R_0}{a}\sum_n\frac{R_0^{2n}}{a^n}\frac{1}{R^{n+1}}P_n\left(\cos\theta\right)\right]$$

和分离变量法的结果式（1）相同

2.11　在接地的导体平面上有一半径为 a 的半球凸部，半球的球心在导体平面上，点电荷 Q 位于系统的对称轴上，并与平面相距为 b（$b>a$），试用镜像法求空间电势

解　在原点的下方距离为 a 的地方放置一个电量为 $-Q$ 的电荷，距离为 $\dfrac{a^2}{b}$ 的地方放置一个 $+qa/b$ 的电荷。之后在原点上方 $\dfrac{a^2}{b}$ 处放置一个 $-qa/b$ 的电荷，电势为

$$\begin{aligned}\varphi=\frac{Q}{4\pi\varepsilon_0}\Bigg[&\frac{1}{\sqrt{x^2+y^2+(z-b)^2}}-\frac{1}{\sqrt{x^2+y^2+(z+b)^2}}\\&-\frac{a/b}{\sqrt{x^2+y^2+(z-a^2/b)^2}}+\frac{a/b}{\sqrt{x^2+y^2+(z+a^2/b)^2}}\Bigg]\end{aligned}$$

2.12　有一点电荷 Q 位于两个相互垂直的接地导体平面所围成的直角空间内，它到两个平面的距离为 a 和 b，求空间电势

解　在 $(0,-a,b)$、$(0,a,-b)$、$(0,-a,-b)$ 各放置三个虚拟电荷 $-Q$、$-Q$、$+Q$，电势为

$$\begin{aligned}\varphi=\frac{Q}{4\pi\varepsilon_0}\Bigg\{&\left[(x-a)^2+(y-b)^2+z^2\right]^{-1/2}+\left[(x+a)^2+(y+b)^2+z^2\right]^{-1/2}\\&-\left[(x+a)^2+(y-b)^2+z^2\right]^{-1/2}-\left[(x-a)^2+(y+b)^2+z^2\right]^{-1/2}\Bigg\}\end{aligned}$$

2.18 一半径为 R_0 的球面，在球坐标 $0<\theta<\dfrac{\pi}{2}$ 的半球面上电势为 φ_0，在 $\dfrac{\pi}{2}<\theta<\pi$ 的半球面上电势为 $-\varphi_0$，球空间各点电势

提示：

$$\int_0^1 P_n\left(x\right)\mathrm{d}x=\left.\frac{P_{n+1}\left(x\right)-P_{n-1}\left(x\right)}{2n+1}\right|_0^1$$

$$P_n\left(1\right)=1$$

$$P_n\left(0\right)=\begin{cases}0 & n=2k+1\\ \left(-1\right)^{n/2}\dfrac{1\cdot3\cdot5\cdots\left(n-1\right)}{2\cdot4\cdot6\cdots n} & n=2k\end{cases}$$

解 设电势为

$$\begin{aligned}\varphi_i&=\sum_n a_nR^nP_n\left(\cos\theta\right)\quad R<R_0\\ \varphi_o&=\sum_n\frac{b_n}{R^{n+1}}P_n\left(\cos\theta\right)\quad R>R_0\end{aligned}$$

下面求解球内电势分布，将 φ_i 展开

$$\varphi_i\left(x\right)=\sum_n\frac{2n+1}{2}\int_{-1}^{+1}\varphi_i\left(x\right)P_n\left(x\right)\mathrm{d}xP_n\left(x\right)$$

比较系数得

$$\frac{2n+1}{2}\int_{-1}^{+1}\varphi_i\left(x\right)P_n\left(x\right)\mathrm{d}x=a_nR^n$$

在 $R=R_0$ 处

$$\begin{aligned}a_nR_0^n&=\frac{2n+1}{2}\left[-\int_{-1}^{0}\varphi_0\left(x\right)P_n\left(x\right)\mathrm{d}x+\int_0^{+1}\varphi_0\left(x\right)P_n\left(x\right)\mathrm{d}x\right]=\left(2n+1\right)\varphi_0\left(x\right)\int_0^1P_n\left(x\right)\mathrm{d}x\\ &=\left(2n+1\right)\varphi_0\left(x\right)\left.\left[\frac{P_{n+1}\left(x\right)-P_{n-1}\left(x\right)}{2n+1}\right]\right|_0^1\\ &=\varphi_0\left(x\right)\left.\left[P_{n+1}\left(x\right)-P_{n-1}\left(x\right)\right]\right|_0^1\end{aligned}$$

其中

$$\left.\left[P_{n+1}\left(x\right)-P_{n-1}\left(x\right)\right]\right|_0^1=P_{n+1}\left(1\right)-P_{n-1}\left(1\right)-P_{n+1}\left(x\right)+P_{n-1}\left(0\right)=P_{n-1}\left(0\right)-P_{n+1}\left(0\right)$$

当 n 为偶数时，$P_{n-1}=P_{n+1}=0$，$a_n=0$，当 n 为奇数时

$$
\begin{aligned}
P_{n-1}\left(0\right)&=\left(-1\right)^{\left(n-1\right)/2}\cdot\frac{1\cdot3\cdot5\ldots\left(n-2\right)}{2\cdot4\cdot6\ldots\left(n-1\right)}\\
P_{n+1}\left(0\right)&=\left(-1\right)^{\left(n+1\right)/2}\cdot\frac{1\cdot3\cdot5\ldots\left(n-2\right)}{2\cdot4\cdot6\ldots\left(n-1\right)}\cdot\frac{n}{n+1}
\end{aligned}
$$

因此

$$
\begin{aligned}
\frac{a_nR_0^n}{\varphi_0}&=P_{n-1}\left(0\right)-P_{n+1}\left(0\right)\\
&=\left(-1\right)^{\left(n-1\right)/2}\cdot\frac{1\cdot3\cdot5\ldots\left(n-2\right)}{2\cdot4\cdot6\ldots\left(n-1\right)}\cdot\left[1+\frac{n}{n+1}\right]\\
&=\left(-1\right)^{\left(n-1\right)/2}\cdot\frac{1\cdot3\cdot5\ldots\left(n-2\right)}{2\cdot4\cdot6\ldots\left(n-1\right)}\cdot\frac{2n+1}{n+1}
\end{aligned}
$$

球内电势为

$$
\varphi_i=\varphi_0\sum_n\left[\left(-1\right)^{\left(n-1\right)/2}\cdot\frac{1\cdot3\cdot5\ldots\left(n-2\right)}{2\cdot4\cdot6\ldots\left(n-1\right)}\cdot\frac{2n+1}{n+1}\right]\frac{R^n}{R_0^n}P_n\left(\cos\theta\right)\quad R<R_0,n=2k+1
$$

下面求解球外电势分布，将 φ_o 展开

$$
\varphi_o\left(x\right)=\sum_n\frac{2n+1}{2}\int_{-1}^{+1}\varphi_o\left(x\right)P_n\left(x\right)\mathrm{d}xP_n\left(x\right)
$$

比较系数得

$$
\frac{2n+1}{2}\int_{-1}^{+1}\varphi_o\left(x\right)P_n\left(x\right)\mathrm{d}x=\frac{b_n}{R^{n+1}}
$$

在 $R=R_0$ 处

$$
\begin{aligned}
\frac{b_n}{R_0^{n+1}}&=\frac{2n+1}{2}\left[-\int_{-1}^{0}\varphi_0\left(x\right)P_n\left(x\right)\mathrm{d}x+\int_{0}^{+1}\varphi_0\left(x\right)P_n\left(x\right)\mathrm{d}x\right]=\left(2n+1\right)\varphi_0\left(x\right)\int_0^1P_n\left(x\right)\mathrm{d}x\\
&=\left(2n+1\right)\varphi_0\left(x\right)\left[\frac{P_{n+1}\left(x\right)-P_{n-1}\left(x\right)}{2n+1}\right]\Bigg|_0^1\\
&=\varphi_0\left(x\right)\left[P_{n+1}\left(x\right)-P_{n-1}\left(x\right)\right]\Big|_0^1
\end{aligned}
$$

其中

$$
\left[P_{n+1}\left(x\right)-P_{n-1}\left(x\right)\right]\Big|_0^1=P_{n+1}\left(1\right)-P_{n-1}\left(1\right)-P_{n+1}\left(x\right)+P_{n-1}\left(0\right)=P_{n-1}\left(0\right)-P_{n+1}\left(0\right)
$$

当 n 为偶数时，$P_{n-1}=P_{n+1}=0$，$a_n=0$，当 n 为奇数时

$$
\begin{aligned}
P_{n-1}(0) &= (-1)^{(n-1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \ldots (n-2)}{2 \cdot 4 \cdot 6 \ldots (n-1)} \\
P_{n+1}(0) &= (-1)^{(n+1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \ldots (n-2)}{2 \cdot 4 \cdot 6 \ldots (n-1)} \cdot \frac{n}{n+1}
\end{aligned}
$$

因此

$$
\begin{aligned}
\frac{b_n}{R_0^{n+1}\varphi_0} &= P_{n-1}(0) - P_{n+1}(0) \\
&= (-1)^{(n-1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \ldots (n-2)}{2 \cdot 4 \cdot 6 \ldots (n-1)} \cdot \left[1 + \frac{n}{n+1}\right] \\
&= (-1)^{(n-1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \ldots (n-2)}{2 \cdot 4 \cdot 6 \ldots (n-1)} \cdot \frac{2n+1}{n+1}
\end{aligned}
$$

球外电势为

$$
\varphi_o = \varphi_0 \sum_n \left[(-1)^{(n-1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \ldots (n-2)}{2 \cdot 4 \cdot 6 \ldots (n-1)} \cdot \frac{2n+1}{n+1}\right] \frac{R_0^{n+1}}{R^{n+1}} P_n(\cos\theta) \quad R > R_0, n = 2k+1
$$

综上

$$
\varphi = \begin{cases}
\varphi_0 \sum_n \left[(-1)^{(n-1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \ldots (n-2)}{2 \cdot 4 \cdot 6 \ldots (n-1)} \cdot \frac{2n+1}{n+1}\right] \frac{R^n}{R_0^n} P_n(\cos\theta) & R < R_0, n = 2k+1 \\
\varphi_0 \sum_n \left[(-1)^{(n-1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \ldots (n-2)}{2 \cdot 4 \cdot 6 \ldots (n-1)} \cdot \frac{2n+1}{n+1}\right] \frac{R_0^{n+1}}{R^{n+1}} P_n(\cos\theta) & R > R_0, n = 2k+1
\end{cases}
$$

将 n 换成 k

$$
\varphi = \begin{cases}
\varphi_0 \sum_k \left[(-1)^k \cdot \frac{1 \cdot 3 \cdot 5 \ldots (2k-1)}{2 \cdot 4 \cdot 6 \ldots (2k+2)} \cdot (4k+3)\right] \frac{R^{2k+1}}{R_0^{2k+1}} P_{2k+1}(\cos\theta) & R < R_0 \\
\varphi_0 \sum_k \left[(-1)^k \cdot \frac{1 \cdot 3 \cdot 5 \ldots (2k-1)}{2 \cdot 4 \cdot 6 \ldots (2k+2)} \cdot (4k+3)\right] \frac{R_0^{2k+2}}{R^{2k+2}} P_{2k+1}(\cos\theta) & R > R_0
\end{cases}
$$

2.19 上题能用格林函数解吗？结果如何？

解 格林函数为

$$
G(x', x) = \frac{1}{4\pi\varepsilon_0}\left\{\left[\left(R^2 + R'^2 - 2RR'\cos\alpha\right)\right]^{-0.5} - \left[\left(\frac{RR'}{R_0}\right)^2 + R_0^2 - 2RR'\cos\alpha\right]^{-0.5}\right\}
$$

在球外，格林函数展开为

$$
\begin{aligned}
G\left(x', x\right) &= \frac{1}{4\pi\varepsilon_0}\left\{\left[\left(R^2 + R'^2 - 2RR'\cos\alpha\right)\right]^{-0.5} - \left[\left(\frac{RR'}{R_0}\right)^2 + R_0^2 - 2RR'\cos\alpha\right]^{-0.5}\right\} \\
&= \frac{1}{4\pi\varepsilon_0}\sum_n\left[\frac{R'^n}{R^{n+1}} - \frac{R_0^n}{\left(RR'/R_0\right)^{n+1}}\right] P_n\left(x\right) P_n\left(x'\right)
\end{aligned}
$$

$$
\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial R'} = \frac{1}{4\pi\varepsilon_0}\sum_n\left[\frac{nR'^{n-1}}{R^{n+1}} + \frac{\left(n+1\right)R_0^{2n+1}}{R^{n+1}R'^{n+2}}\right] P_n\left(x\right) P_n\left(x'\right)
$$

$$
\begin{aligned}
\left.\frac{\partial G}{\partial R'}\right|_{R'=R_0} &= \frac{1}{4\pi\varepsilon_0}\sum_n\left[\frac{nR_0^{n-1}}{R^{n+1}} + \frac{\left(n+1\right)R_0^{n-1}}{R^{n+1}}\right] P_n\left(x\right) P_n\left(x'\right) \\
&= \frac{1}{4\pi\varepsilon_0}\sum_n\left[\left(2n+1\right)\frac{R_0^{n-1}}{R^{n+1}}\right] P_n\left(x\right) P_n\left(x'\right)
\end{aligned}
$$

空间中电势的解为

$$
\varphi\left(x\right) = -\varepsilon_0\oint_S \varphi\left(x'\right)\frac{\partial G}{\partial n'}\,\mathrm{d}S'
$$

其中

$$
\mathrm{d}S' = R_0^2\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\phi' = -R_0^2\,\mathrm{d}x'\,\mathrm{d}\phi'
$$

所以

$$
\begin{aligned}
\varphi\left(x\right) &= -\varepsilon_0\oint_S \varphi\left(x'\right)\frac{\partial G}{\partial R'}R_0^2\,\mathrm{d}x'\,\mathrm{d}\phi' = -2\pi\varepsilon_0 R_0^2\int_{+1}^{-1}\varphi\left(x'\right)\frac{\partial G}{\partial R'}\,\mathrm{d}x' \\
&= -2\pi\varepsilon_0 R_0^2\int_{+1}^{-1}\varphi\left(x'\right)\left[\frac{1}{4\pi\varepsilon_0}\sum_n\left[\left(2n+1\right)\frac{R_0^{n-1}}{R^{n+1}}\right]\right] P_n\left(x\right) P_n\left(x'\right)\mathrm{d}x' \\
&= -4\pi\varepsilon_0 R_0^2\int_{+1}^{0}\varphi\left(x'\right)\left[\frac{1}{4\pi\varepsilon_0}\sum_n\left[\left(2n+1\right)\frac{R_0^{n-1}}{R^{n+1}}\right]\right] P_n\left(x\right) P_n\left(x'\right)\mathrm{d}x' \\
&= 4\pi\varepsilon_0 R_0^2\int_{0}^{1}\varphi\left(x'\right)\left[\frac{1}{4\pi\varepsilon_0}\sum_n\left[\left(2n+1\right)\frac{R_0^{n-1}}{R^{n+1}}\right]\right] P_n\left(x\right) P_n\left(x'\right)\mathrm{d}x' \\
&= \int_0^1\varphi_0\left[\sum_n\left[\left(2n+1\right)\frac{R_0^{n+1}}{R^{n+1}}\right] P_n\left(x\right) P_n\left(x'\right)\right]\mathrm{d}x' \\
&= \varphi_0\sum_n\left[\left(2n+1\right)\frac{R_0^{n+1}}{R^{n+1}}\right] P_n\left(x\right)\int_0^1 P_n\left(x'\right)\mathrm{d}x' \\
&= \varphi_0\sum_n\left[\frac{R_0^{n+1}}{R^{n+1}}\right] P_n\left(x\right)\left(2n+1\right)\int_0^1 P_n\left(x'\right)\mathrm{d}x' \\
&= \varphi_0\sum_n\left[\left(-1\right)^{\left(n-1\right)/2}\cdot\frac{1\cdot3\cdot5\ldots\left(n-2\right)}{2\cdot4\cdot6\ldots\left(n-1\right)}\cdot\frac{2n+1}{n+1}\right]\frac{R_0^{n+1}}{R^{n+1}}P_n\left(x\right)
\end{aligned}
$$

其中 n 为奇数。在球外，电势为

$$\varphi_o=\varphi_0\sum_n\left[(-1)^{(n-1)/2}\cdot\frac{1\cdot3\cdot5\ldots(n-2)}{2\cdot4\cdot6\ldots(n-1)}\cdot\frac{2n+1}{n+1}\right]\frac{R_0^{n+1}}{R^{n+1}}P_n\left(x\right)\quad R>R_0,n=2k+1$$

在球内，格林函数可以展开为

$$\begin{aligned}G\left(x',x\right)&=\frac{1}{4\pi\varepsilon_0}\left\{\left[\left(R^2+R'^2-2RR'\cos\alpha\right)\right]^{-0.5}-\left[\left(\frac{RR'}{R_0}\right)^2+R_0^2-2RR'\cos\alpha\right]^{-0.5}\right\}\\&=\frac{1}{4\pi\varepsilon_0}\sum_n\left[\frac{R^n}{R'^{n+1}}-\frac{\left(RR'/R_0\right)^n}{R_0^{n+1}}\right]P_n\left(x\right)P_n\left(x'\right)\end{aligned}$$

$$\frac{\partial G}{\partial n'}=\frac{\partial G}{\partial R'}=-\frac{1}{4\pi\varepsilon_0}\sum_n\left[\frac{n+1}{R'^{n+2}}R^n+\frac{nR^nR'^{n-1}}{R_0^{2n+1}}\right]P_n\left(x\right)P_n\left(x'\right)$$

$$\begin{aligned}\left.\frac{\partial G}{\partial R'}\right|_{R'=R_0}&=-\frac{1}{4\pi\varepsilon_0}\sum_n\left[\frac{n+1}{R_0^{n+2}}R^n+\frac{nR^nR_0^{n-1}}{R_0^{2n+1}}\right]P_n\left(x\right)P_n\left(x'\right)\\&=-\frac{1}{4\pi\varepsilon_0}\sum_n\left[\frac{n+1}{R_0^{n+2}}R^n+\frac{nR^n}{R_0^{n+2}}\right]P_n\left(x\right)P_n\left(x'\right)\\&=-\frac{1}{4\pi\varepsilon_0}\sum_n\left[\left(2n+1\right)\frac{R^n}{R_0^{n+2}}\right]P_n\left(x\right)P_n\left(x'\right)\end{aligned}$$

同理

$$\begin{aligned}\varphi\left(x\right)&=\varepsilon_0\oint_S\varphi\left(x'\right)\frac{\partial G}{\partial R'}R_0^2\,\mathrm{d}x'\,\mathrm{d}\phi'=2\pi\varepsilon_0R_0^2\int_{+1}^{-1}\varphi\left(x'\right)\frac{\partial G}{\partial R'}\,\mathrm{d}x'\\&=2\pi\varepsilon_0R_0^2\int_{+1}^{-1}\varphi\left(x'\right)\left[-\frac{1}{4\pi\varepsilon_0}\sum_n\left[\left(2n+1\right)\frac{R^n}{R_0^{n+2}}\right]P_n\left(x\right)P_n\left(x'\right)\right]\mathrm{d}x'\\&=4\pi\varepsilon_0R_0^2\int_{+1}^{0}\varphi_0\left[-\frac{1}{4\pi\varepsilon_0}\sum_n\left[\left(2n+1\right)\frac{R^n}{R_0^{n+2}}\right]P_n\left(x\right)P_n\left(x'\right)\right]\mathrm{d}x'\\&=\int_0^1\varphi_0\left[\sum_n\left[\left(2n+1\right)\frac{R^n}{R_0^n}\right]P_n\left(x\right)P_n\left(x'\right)\right]\mathrm{d}x'\\&=\varphi_0\sum_n\left[\left(2n+1\right)\frac{R^n}{R_0^n}\right]P_n\left(x\right)\int_0^1P_n\left(x'\right)\mathrm{d}x'\\&=\varphi_0\sum_n\left[\frac{R^n}{R_0^n}\right]P_n\left(x\right)\left(2n+1\right)\int_0^1P_n\left(x'\right)\mathrm{d}x'\\&=\varphi_0\sum_n\left[\frac{R^n}{R_0^n}\right]P_n\left(x\right)\left[(-1)^{(n-1)/2}\cdot\frac{1\cdot3\cdot5\ldots(n-2)}{2\cdot4\cdot6\ldots(n-1)}\cdot\frac{2n+1}{n+1}\right]\\&=\varphi_0\sum_n\left[(-1)^{(n-1)/2}\cdot\frac{1\cdot3\cdot5\ldots(n-2)}{2\cdot4\cdot6\ldots(n-1)}\cdot\frac{2n+1}{n+1}\right]\frac{R^n}{R_0^n}P_n\left(x\right)\end{aligned}$$

其中 n 为奇数。在球内，电势为

$$\varphi_i = \varphi_0 \sum_n \left[(-1)^{(n-1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \ldots (n-2)}{2 \cdot 4 \cdot 6 \ldots (n-1)} \cdot \frac{2n+1}{n+1} \right] \frac{R^n}{R_0^n} P_n\left(x\right) \quad R < R_0, n = 2k+1$$

因此

$$\varphi = \begin{cases} \varphi_0 \sum\limits_n \left[(-1)^{(n-1)/2} \cdot \dfrac{1 \cdot 3 \cdot 5 \ldots (n-2)}{2 \cdot 4 \cdot 6 \ldots (n-1)} \cdot \dfrac{2n+1}{n+1} \right] \dfrac{R^n}{R_0^n} P_n\left(x\right) & R < R_0, n = 2k+1 \\ \varphi_0 \sum\limits_n \left[(-1)^{(n-1)/2} \cdot \dfrac{1 \cdot 3 \cdot 5 \ldots (n-2)}{2 \cdot 4 \cdot 6 \ldots (n-1)} \cdot \dfrac{2n+1}{n+1} \right] \dfrac{R_0^{n+1}}{R^{n+1}} P_n\left(x\right) & R > R_0, n = 2k+1 \end{cases}$$

将 n 换成 k

$$\varphi = \begin{cases} \varphi_0 \sum\limits_k \left[(-1)^k \cdot \dfrac{1 \cdot 3 \cdot 5 \ldots (2k-1)}{2 \cdot 4 \cdot 6 \ldots (2k+2)} \cdot (4k+3) \right] \dfrac{R^{2k+1}}{R_0^{2k+1}} P_{2k+1}\left(\cos\theta\right) & R < R_0 \\ \varphi_0 \sum\limits_k \left[(-1)^k \cdot \dfrac{1 \cdot 3 \cdot 5 \ldots (2k-1)}{2 \cdot 4 \cdot 6 \ldots (2k+2)} \cdot (4k+3) \right] \dfrac{R_0^{2k+2}}{R^{2k+2}} P_{2k+1}\left(\cos\theta\right) & R > R_0 \end{cases}$$