电动力学学习笔记

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1　数学基础

1.1　三位旋转坐标系

下图中 x' 和 x、y、z 的夹角为 α_1、β_1、γ_1，y' 和 x、y、z 的夹角为 α_2、β_2、γ_2，z' 和 x、y、z 的夹角为 α_3、β_3、γ_3

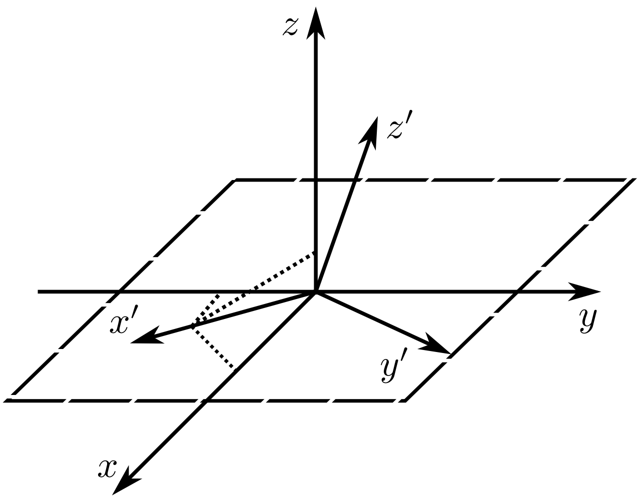

图 1: 三维旋转坐标系

以 x 方向为例

$$x' = x\cos\alpha_1 + y\cos\beta_1 + z\cos\gamma_1$$

y、z 同理

$$y' = y\cos\alpha_2 + y\cos\beta_2 + z\cos\gamma_2$$
$$z' = x\cos\alpha_3 + y\cos\beta_3 + z\cos\gamma_3$$

因此旋转矩阵为

$$R=\begin{bmatrix}\cos\alpha_1 & \cos\beta_1 & \cos\gamma_1\\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2\\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3\end{bmatrix}$$

则

$$\begin{bmatrix}x'\\ y'\\ z'\end{bmatrix}=R\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}\cos\alpha_1 & \cos\beta_1 & \cos\gamma_1\\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2\\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

1.2 矢量分析

Kronecker 符号定义为:

$$\delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$$

基矢之间的点乘可以表示为

$$\vec{e}_i\cdot\vec{e}_j=\delta_{ij}$$

Levi-Civita 张量定义为

$$\epsilon_{ijk}=\begin{cases}1 & i,j,k=(1,2,3),(2,3,1),(3,1,2)\\ -1 & i,j,k=(3,2,1),(2,1,3),(1,3,2)\\ 0 & i=j \text{ or } j=k \text{ or } i=k\end{cases}$$

基矢之间的点乘可以表示为

$$\vec{e}_i\times\vec{e}_j=\epsilon_{ijk}\vec{e}_k$$

其中运用了爱因斯坦求和转换，要对重复的字母 k 进行递归求和。

Levi-Civita 张量还可以表示为

$$\epsilon_{ijk}=\begin{bmatrix}\delta_{i1} & \delta_{i2} & \delta_{i3}\\ \delta_{j1} & \delta_{j2} & \delta_{j3}\\ \delta_{k1} & \delta_{k2} & \delta_{k3}\end{bmatrix}$$

矢量的点乘和叉乘可以表示为

$$
\begin{aligned}
\vec{A}\cdot\vec{B} &= \delta_{ij}A_iB_j = A_iB_i \\
\left(\vec{A}\times\vec{B}\right)_i &= \epsilon_{ijk}A_jB_k
\end{aligned}
$$

常用的一些公式有

$$
\begin{aligned}
&\delta_{ij}\delta_{mj} = \delta_{i1}\delta_{m1} + \delta_{i2}\delta_{m2} + \delta_{i3}\delta_{m3} = \delta_{im} \\
&\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3 \\
&\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}
\end{aligned}
$$

推论有

$$
\begin{aligned}
&\epsilon_{ijk}\epsilon_{mjk} = 2\delta_{im} \\
&\epsilon_{ijk}\epsilon_{ijk} = 6
\end{aligned}
$$

从此矢量的运算可以表示为

$$
\begin{aligned}
&\vec{A}+\vec{B} = (A_i + B_i)\,\vec{e}_i \\
&\vec{A}\cdot\vec{B} = (A_i\vec{e}_i)\cdot(B_j\vec{e}_j) = (A_iB_j)\,(\vec{e}_i\cdot\vec{e}_j) = \delta_{ij}A_iB_j = A_iB_i \\
&\vec{A}\times\vec{B} = (A_i\vec{e}_i)\times(B_j\vec{e}_j) = A_iB_j\,(\vec{e}_i\times\vec{e}_j) = \epsilon_{ijk}A_iB_j\vec{e}_k
\end{aligned}
$$

显然

$$
\begin{aligned}
\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) &= \vec{A}\cdot(\epsilon_{ijk}B_iC_j)\,\vec{e}_k = (A_l\vec{e}_l)\,(\epsilon_{ijk}B_iC_j\vec{e}_k) = \delta_{lk}\epsilon_{ijk}A_lB_iC_j \\
&= \epsilon_{ijk}A_kB_iC_j
\end{aligned}
$$

$$
\begin{aligned}
\vec{A}\times\left(\vec{B}\times\vec{C}\right) &= \vec{A}\times(\epsilon_{ijk}B_iC_j\vec{e}_k) = \epsilon_{mnl}A_m\,(\epsilon_{ijk}B_iC_j\vec{e}_k)_n\,\vec{e}_l = \epsilon_{mnl}A_m\,(\epsilon_{ijn}B_iC_j)\,\vec{e}_l \\
&= \epsilon_{mnl}\epsilon_{ijn}A_mB_iC_j\vec{e}_l = (\epsilon_{lmn}\epsilon_{ijn})\,A_mB_iC_j\vec{e}_l \\
&= (\delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi})\,A_mB_iC_j\vec{e}_l \\
&= (A_mB_lC_m - A_mB_mC_l)\,\vec{e}_i = (A_mC_m)\,B_l\vec{e}_l - (A_mB_m)\,C_l\vec{e}_l \\
&= \left(\vec{A}\cdot\vec{C}\right)\vec{B} - \left(\vec{A}\cdot\vec{B}\right)\vec{C}
\end{aligned}
$$

梯度算符可以简化为

$$
\nabla = \vec{e}_1\frac{\partial}{\partial x_1} + \vec{e}_2\frac{\partial}{\partial x_2} + \vec{e}_3\frac{\partial}{\partial x_3} = \vec{e}_i\frac{\partial}{\partial x_i} = \vec{e}_i\partial_i
$$

在直角坐标系中

$$\partial_i x_j = \frac{\partial x_j}{\partial x_i} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

矢量的散度为

$$\begin{aligned}
\nabla \cdot \vec{A} &= \vec{e}_i \partial_i \cdot \left(A_j \vec{e}_j\right) = \left(\vec{e}_i \cdot \vec{e}_j\right) \partial_i A_j + A_j \vec{e}_i \left(\partial_i \vec{e}_j\right) = \delta_{ij} \partial_i A_j = \partial_i A_i \\
&= \partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 \\
&= \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}
\end{aligned}$$

矢量场的旋度为

$$\begin{aligned}
\nabla \times \vec{A} &= \left(\vec{e}_i \partial_i\right) \times \left(A_j \vec{e}_j\right) = \epsilon_{lmn} \left(\vec{e}_i \partial_i\right)_l \left(A_j \vec{e}_j\right)_m \vec{e}_n = \epsilon_{lmn} \partial_l A_m \vec{e}_n \\
&= \left(\partial_2 A_3 - \partial_3 A_2\right) \vec{e}_1 + \left(\partial_3 A_1 - \partial_1 A_3\right) \vec{e}_2 + \left(\partial_1 A_2 - \partial_2 A_1\right) \\
&= \left[\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right] \vec{e}_1 + \left[\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right] \vec{e}_2 + \left[\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right] \vec{e}_3
\end{aligned}$$