《电动力学》课后习题——第一章 电磁现象的基本规律

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1.1　根据算符 ∇ 的微分性与矢量性，推导下列公式：

$$\nabla\left(\vec{A}\cdot\vec{B}\right)=\vec{B}\times\left(\nabla\cdot\vec{A}\right)+\left(\vec{B}\cdot\nabla\right)\vec{A}+\vec{A}\times\left(\nabla\times\vec{B}\right) \tag{1}$$

$$\vec{A}\times\left(\nabla\times\vec{A}\right)=\frac{1}{2}\nabla A^2-\left(\vec{A}\cdot\nabla\right)\vec{A} \tag{2}$$

解

$$\nabla\left(\vec{A}\cdot\vec{B}\right)=\left(\partial_i\vec{e}_i\right)\left(A_jB_j\right)=\left(A_j\partial_iB_j+B_j\partial_iA_j\right)\vec{e}_i \tag{3}$$

$$\left(\vec{B}\cdot\nabla\right)\vec{A}=\left(B_i\vec{e}_i\cdot\partial_j\vec{e}_j\right)\vec{A}=\left(\delta_{ij}B_i\partial_j\right)\vec{A}=\left(B_i\partial_i\right)\left(A_j\vec{e}_j\right)=B_i\partial_iA_j\vec{e}_j \tag{4}$$

同理

$$\left(\vec{A}\cdot\nabla\right)\vec{B}=A_i\partial_iB_j\vec{e}_j \tag{5}$$

$$\begin{aligned}\vec{B}\times\left(\nabla\times\vec{A}\right)&=\vec{B}\times\left(\epsilon_{ijk}\partial_iA_j\vec{e}_k\right)=\epsilon_{mnl}B_m\left(\epsilon_{ijk}\partial_iA_j\vec{e}_k\right)_n\vec{e}_l=\epsilon_{mnl}B_m\epsilon_{ijn}\partial_iA_j\vec{e}_l=\epsilon_{lmn}\epsilon_{ijn}B_m\partial_iA_j\vec{e}_l\\&=\left(B_m\partial_lA_m-B_m\partial_mA_l\right)\vec{e}_l\end{aligned} \tag{6}$$

同理

$$\vec{A}\times\left(\nabla\times\vec{B}\right)=\left(A_m\partial_lB_m-A_m\partial_mB_l\right)\vec{e}_l \tag{7}$$

式（4）（5）（6）（7）相加，显然等于式（3），因此式（1）得证。

$$
\begin{aligned}
\vec{A}\times\left(\nabla\times\vec{A}\right) &= \vec{A}\times\left[\left(\partial_i\vec{e}_i\right)\times\left(A_j\vec{e}_j\right)\right] = \vec{A}\times\left(\epsilon_{ijk}\partial_i A_j\vec{e}_k\right) = \left(A_l\vec{e}_l\right)\times\left(\epsilon_{ijk}\partial_{ijk}\partial_i A_j\vec{e}_k\right) \\
&= \epsilon_{ijk}\epsilon_{lkn}A_l\partial_i A_j\vec{e}_n = \epsilon_{ijk}\epsilon_{nlk}A_l\partial_i A_j\vec{e}_n = \left(\delta_{in}\delta_{jl}-\delta_{il}\delta_{jn}\right)A_l\partial_i A_j\vec{e}_n \\
&= A_j\partial_i A_j\vec{e}_i
\end{aligned} \tag{8}
$$

$$
\left(\vec{A}\cdot\nabla\right)\vec{A} = \left(A_i\partial_i\right)\left(A_j\vec{e}_j\right) = A_j\partial_i A_j\vec{e}_j \tag{9}
$$

显然，式（8）和（9）是相等的，得证。

1.2 设 u 是空间坐标 x, y, z 的函数，证明：

$$
\nabla f\left(u\right) = \frac{\mathrm{d}f}{\mathrm{d}u}\nabla u
$$

$$
\nabla\cdot\vec{A}\left(u\right) = \nabla u\cdot\frac{\mathrm{d}\vec{A}}{\mathrm{d}u}
$$

$$
\nabla\times\vec{A}\left(u\right) = \nabla u\times\frac{\mathrm{d}\vec{A}}{\mathrm{d}u}
$$

解

$$
\nabla f = \partial_i f_i\vec{e}_i = \frac{\partial}{\partial x_i}f_i\vec{e}_i = \frac{\partial f_i}{\partial u}\frac{\partial u}{\partial x_i}\vec{e}_i = \frac{\mathrm{d}f}{\mathrm{d}u}\nabla u
$$

$$
\nabla\cdot\vec{A} = \partial_i A_i = \frac{\partial A_i}{\partial x_i} = \frac{\partial A_i}{\partial_u}\frac{\partial u}{\partial x_i} = \nabla u\cdot\frac{\mathrm{d}\vec{A}}{\mathrm{d}u}
$$

$$
\nabla\times\vec{A} = \epsilon_{ijk}\partial_i A_j\vec{e}_k = \epsilon_{ijk}\frac{\partial u}{\partial x_i}\frac{\partial A_j}{\partial u}\vec{e}_k = \nabla u\times\frac{\mathrm{d}\vec{A}}{\mathrm{d}u}
$$

1.3 设 $r=\sqrt{\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z-z'\right)^2}$ 为源点 x' 到 x 的距离，$\vec{r}$ 的方向规定为源点指向场点。

（1）证明以下结果，并体会对源变数求微商（$\nabla'=\vec{e}_i\dfrac{\partial}{\partial x_i'}$）和对场变数求微商（$\nabla=\vec{e}_i\dfrac{\partial}{\partial x_i}$）的关系。

$$
\nabla\vec{r} = -\nabla'\vec{r} = \frac{\vec{r}}{r}
$$

$$
\nabla\frac{1}{r} = -\nabla'\frac{1}{r} = -\frac{\vec{r}}{r^3}
$$

$$
\nabla\times\frac{\vec{r}}{r^3} = 0
$$

$$
\nabla\cdot\frac{\vec{r}}{r^3} = -\nabla'\cdot\frac{\vec{r}}{r^3} = 0
$$

解 因为

$$\nabla=\partial_i=\frac{\partial}{\partial x_i}=\frac{\partial}{\partial x_i'}\frac{\partial x_i'}{\partial x_i}=-\frac{\partial}{\partial x_i'}=-\nabla$$

所以

$$\nabla r=-\nabla' r$$

$$\nabla\frac{1}{r}=-\nabla'\frac{1}{r}$$

$$\nabla\times\frac{\vec{r}}{r^3}=0$$

$$\nabla\cdot\frac{\vec{r}}{r^3}=-\nabla'\cdot\frac{\vec{r}}{r^3}$$

只需要证明

$$\nabla\vec{r}=\frac{\vec{r}}{r} \tag{10}$$

$$\nabla\frac{1}{r}=-\frac{\vec{r}}{r^3} \tag{11}$$

$$\nabla\times\frac{\vec{r}}{r^3}=0 \tag{12}$$

$$\nabla\cdot\frac{\vec{r}}{r^3}=0 \tag{13}$$

对于（10）

$$\nabla r=\partial_i\sqrt{\Sigma\left(x_i-x_i'\right)^2}\vec{e}_i=\frac{x_i}{r}\vec{e}_i=\frac{\vec{r}}{r}$$

对于（11）

$$\nabla\frac{1}{r}=-\frac{1}{r^2}\nabla r=-\frac{\vec{r}}{r^3}$$

对于（12）

$$\nabla \times \frac{\vec{r}}{r^3} = \epsilon_{ijk}\partial_i \left(\frac{\vec{r}}{r^3}\right)_j \vec{e}_k = \epsilon_{ijk}\partial_i \left(\frac{x_j}{r^3}\right)\vec{e}_k = 0$$

对于（13）

$$\begin{aligned}\nabla \cdot \frac{\vec{r}}{r^3} = \partial_i\left(\frac{x_i}{r^3}\right) = \frac{r^3\partial_i x_i - x_i\partial_i r^3}{r^6} = \frac{r^3\partial_i x_i - 3r^2 x_i\partial_i r}{r^6} = \frac{r^3 - 3r^2 x_i \dfrac{x_i}{r}}{r^6} = \frac{r^3 - 3r x_i^2}{r^6} \\ = \frac{r^3 - 3r x_1^2}{r^6} + \frac{r^3 - 3r x_2^2}{r^6} + \frac{r^3 - 3r x_3^2}{r^6} = \frac{3r^3 - 3r\left(x_1^2 + x_2^2 + x_3^2\right)}{r^6} = 0\end{aligned}$$

由于分母上有 r，当 $r = 0$ 时，不一定成立。

（2）求 $\nabla\cdot\vec{r}$，$\nabla\times\vec{r}$，$(\vec{a}\cdot\nabla)\vec{r}$，$\nabla(\vec{a}\cdot\vec{r})$，$\nabla\cdot\left[\vec{E}_0\sin\left(\vec{k}\cdot\vec{r}\right)\right]$，$\nabla\times\left[\vec{E}_0\sin\left(\vec{k}\cdot\vec{r}\right)\right]$，$\vec{a}$、$\vec{k}$、$\vec{E}_0$ 是常矢量。

解

$$\nabla\cdot\vec{r} = \partial_i r_i = 3$$

$$\nabla\times\vec{r} = \epsilon_{ijk}\partial_i x_j \vec{e}_k = 0$$

$$(\vec{a}\cdot\nabla)\vec{r} = a_i\partial_i x_i \vec{e}_i = a_i\vec{e}_i = \vec{a}$$

$$\nabla(\vec{a}\cdot\vec{r}) = \partial_i(a_j r_j)\vec{e}_i = [a_j\partial_i x_j + x_j\partial_i a_j] = a_j\partial_i x_j = a_i\partial_i x_i\vec{e}_i = a_i\vec{e}_i = \vec{a}$$

$$\nabla\cdot\left[\vec{E}_0\sin\left(\vec{k}\cdot\vec{r}\right)\right] = \partial_i E_{0i}\sin(k_i x_i) = E_{0i}\cos(k_i x_i)k_i = \vec{k}\cdot\vec{E}_0\cos\left(\vec{k}\cdot\vec{r}\right)$$

$$\nabla\times\left[\vec{E}_0\sin\left(\vec{k}\cdot\vec{r}\right)\right] = \epsilon_{ijk}\partial_i E_{0j}\sin(k_j x_j)\vec{e}_k = \epsilon_{ijk}E_{0j}\cos(k_m x_m)k_i\vec{e}_k = \vec{k}\times\vec{E}_0\cos\left(\vec{k}\cdot\vec{r}\right)$$

1.4　利用高斯定理证明

$$\int_V \mathrm{d}V\nabla\times\vec{f} = \oint_S \mathrm{d}\vec{S}\times\vec{f}$$

利用斯托克斯定理证明

$$\int_S \mathrm{d}\vec{S}\times\nabla\varphi = \oint_L \varphi\,\mathrm{d}\vec{l}$$

解　引入常矢量 $\vec{c}$

$$\int_V \mathrm{d}V\nabla\times\vec{f}\cdot\vec{c}=\int_V\vec{c}\cdot\left(\nabla\times\vec{f}\right)\mathrm{d}V=\int_V\nabla\cdot\left(\vec{f}\times\vec{c}\right)\mathrm{d}V=\oint_S\left(\vec{f}\times\vec{c}\right)\cdot\mathrm{d}\vec{S}=\oint_S\mathrm{d}\vec{S}\times\vec{f}\cdot\vec{c}$$

$$\int_S\mathrm{d}\vec{S}\times\nabla\varphi\cdot\vec{c}=\int_S\nabla\varphi\times\vec{c}\cdot\mathrm{d}\vec{S}=\int_L\nabla\times(\varphi\vec{c})\cdot\mathrm{d}\vec{S}=\oint_L\varphi\vec{c}\cdot\mathrm{d}\vec{l}=\oint_L\varphi\,\mathrm{d}\vec{l}\cdot\vec{c}$$

因为 $\vec{c}$ 是任意的

$$\int_V\mathrm{d}V\nabla\times\vec{f}=\oint_S\mathrm{d}\vec{S}\times\vec{f}$$

$$\int_S\mathrm{d}\vec{S}\times\nabla\varphi=\oint_L\varphi\,\mathrm{d}\vec{l}$$

1.5　已知一个电荷系统的电偶极矩为

$$\vec{p}(t)=\int_V\rho\left(\vec{x}',t\right)\vec{x}'\,\mathrm{d}V'$$

利用电荷守恒定律 $\nabla\cdot\vec{J}+\dfrac{\partial\rho}{\partial t}=0$ 证明

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}=\int_V\vec{J}\left(\vec{x}',t\right)\mathrm{d}V'$$

解

$$\begin{aligned}\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}&=\int_V\frac{\mathrm{d}\rho\left(\vec{x}',t\right)}{\mathrm{d}t}\vec{x}'\,\mathrm{d}V'=-\int_V\left(\nabla'\cdot\vec{J}\right)\vec{x}'\,\mathrm{d}V'=-\int_V\left(\nabla'\cdot\vec{J}\vec{x}'\right)-\vec{J}\left(\nabla'\vec{x}'\right)\mathrm{d}V'\\&=-\oint_S\vec{J}\vec{x}'\cdot\mathrm{d}\vec{S}+\int_V\vec{J}\,\mathrm{d}V'=\int_V\vec{J}\,\mathrm{d}V'\end{aligned}$$

1.6　若 $\vec{m}$ 是常矢量，证明除 $R=0$ 以外，矢量 $\vec{A}=\dfrac{\vec{m}\times\vec{R}}{R^3}$ 的旋度等于标量 $\varphi=\dfrac{\vec{m}\cdot\vec{R}}{R^3}$ 的梯度的负值，即

$$\nabla\times\vec{A}=-\nabla\varphi$$

解

$$\begin{aligned}\nabla\times\vec{A}&=\nabla\times\left(\vec{m}\times\frac{\vec{R}}{R^3}\right)=\left(\frac{\vec{R}}{R^3}\cdot\nabla\right)\vec{m}+\left(\nabla\cdot\frac{\vec{R}}{R^3}\right)\vec{m}-\left(\vec{m}\cdot\nabla\right)\frac{\vec{R}}{R^3}-\left(\nabla\cdot\vec{m}\right)\frac{\vec{R}}{R^3}\\&=\left(\nabla\cdot\frac{\vec{R}}{R^3}\right)\vec{m}-\left(\vec{m}\cdot\nabla\right)\frac{\vec{R}}{R^3}=-\left(\vec{m}\cdot\nabla\right)\frac{\vec{R}}{R^3}\end{aligned}$$

$$
\begin{aligned}
-\nabla\varphi &= -\nabla\left(\frac{\vec{m}\cdot\vec{R}}{R^3}\right) = -\vec{m}\times\left(\nabla\times\frac{\vec{R}}{R^3}\right) - \left(\vec{m}\cdot\nabla\right)\frac{\vec{R}}{R^3} - \frac{\vec{R}}{R^3}\times\left(\nabla\times\vec{m}\right) - \left(\frac{\vec{R}}{R^3}\cdot\nabla\right)\vec{m} \\
&= -\vec{m}\times\left(\nabla\times\frac{\vec{R}}{R^3}\right) - \left(\vec{m}\cdot\nabla\right)\frac{\vec{R}}{R^3} = -\left(\vec{m}\cdot\nabla\right)\frac{\vec{R}}{R^3}
\end{aligned}
$$

得证