

《电动力学》课后习题——第二章 静电场

物理 (4+4) 1801 胡喜平 学号 U201811966

网站 <https://hxp.plus/> 邮件 hxp201406@gmail.com

2020 年 10 月 28 日

2.1 半径为 R 的电介质球, 极化强度为 $\vec{P} = K \frac{\vec{r}}{r^2}$, 电容率为 ϵ

- (1) 计算束缚电荷的体密度和面密度
- (2) 计算自由电荷体密度
- (3) 计算球外和球内的电势
- (4) 求该带电介质球产生的静电场总能量

解 计算束缚电荷密度, 只需要对极化强度求梯度

$$\rho_p = -\nabla \cdot \vec{P} = \frac{K}{r^2}$$

面电荷密度用电场的边值关系求解

$$\sigma_p = -\vec{e}_n \cdot (0 - \vec{P}) = \frac{K}{r}$$

在自由电荷密度为

$$\begin{aligned}\rho_f &= -\nabla \cdot \vec{D} = -\nabla \cdot (\epsilon_0 \vec{E} + \vec{P}) = -\nabla \cdot \left(\frac{\epsilon_0}{\epsilon_0 \chi_e} + 1 \right) \vec{P} = -\nabla \cdot \left(\frac{\epsilon_0}{\epsilon_0 (\epsilon_r - 1)} + 1 \right) \vec{P} \\ &= -\nabla \cdot \left(\frac{\epsilon_0}{\epsilon - \epsilon_0} + 1 \right) \vec{P} = -\nabla \cdot \frac{\epsilon}{\epsilon - \epsilon_0} \vec{P} = \frac{\epsilon}{\epsilon - \epsilon_0} (-\nabla \cdot \vec{P}) = \frac{\epsilon}{\epsilon - \epsilon_0} \frac{K}{r^2}\end{aligned}$$

球内的电场为

$$\vec{E}_i = \frac{\vec{P}}{\epsilon_0 \chi_e} = \frac{\vec{P}}{\epsilon_0 (\epsilon_r - 1)} = \frac{\vec{P}}{\epsilon - \epsilon_0} = \frac{K}{\epsilon - \epsilon_0} \cdot \frac{\vec{r}}{r^2}$$

球外电场为

$$\vec{E}_o = \frac{\int_0^R 4\pi r^2 \rho_f dr}{4\pi \varepsilon_0 r^3} \vec{r} = \frac{\int_0^R r^2 \frac{\varepsilon}{\varepsilon - \varepsilon_0} \frac{K}{r^2} dr}{\varepsilon_0 r^3} \vec{r} = \int_0^R \frac{\varepsilon K}{\varepsilon_0 (\varepsilon - \varepsilon_0)} dr \cdot \frac{\vec{r}}{r^3} = \frac{\varepsilon K R}{\varepsilon_0 (\varepsilon - \varepsilon_0)} \cdot \frac{\vec{r}}{r^3}$$

球外电势为

$$\varphi_o = \int_r^\infty \vec{E}_o d\vec{r} = \frac{\varepsilon K R}{\varepsilon_0 (\varepsilon - \varepsilon_0)} \frac{1}{r}$$

球内电势为

$$\varphi_i = \int_R^\infty \vec{E}_o d\vec{r} + \int_r^R \vec{E}_i d\vec{r} = \frac{K}{\varepsilon - \varepsilon_0} \left[\ln \frac{R}{r} + \frac{\varepsilon}{\varepsilon_0} \right]$$

总能量为

$$W = \frac{1}{2} \int_0^R 4\pi r^2 \varepsilon \vec{E}_i^2 dr + \frac{1}{2} \int_R^\infty 4\pi r^2 \varepsilon_0 \vec{E}_o^2 dr = 2\pi \varepsilon R \left(\frac{K}{\varepsilon - \varepsilon_0} \right)^2 \left(1 + \frac{\varepsilon}{\varepsilon_0} \right)$$

2.2 在均匀外电场中置入半径为 R_0 的导体球, 试用分离变量法求下列两种情况的电势:

(1) 导体球上接有电池, 使球与地保持电势差 Φ_0

(2) 导体球上带总电荷 Q

解 导体边界电势为 Φ_0 时, 设解为

$$\varphi = \sum_n \left(a_n R^n + \frac{b_n}{R^{n+1}} \right) P_n(\cos \theta)$$

边界条件为

$$\varphi|_{r=\infty} = -E_0 R \cos \theta$$

$$\varphi|_{r=R_0} = \Phi_0$$

代入边界条件得出

$$a_1 = -E_0$$

$$a_n = 0 \quad n \neq 1$$

$$b_0 = \Phi_0 R_0$$

$$b_1 = E_0 R_0^3$$

$$b_n = 0 \quad n \neq 0, 1$$

所以

$$\varphi = \Phi_0 R_0 - E_0 \left(R - \frac{R_0^3}{R^2} \right) \cos \theta$$

当没有外接电池而是在导体球上放置电荷 Q 时, 球面电势为

$$\Phi'_0 = \frac{Q}{4\pi\epsilon_0}$$

方程和边界条件相同, 因此

$$\varphi = \Phi'_0 R_0 - E_0 \left(R - \frac{R_0^3}{R^2} \right) \cos \theta = \frac{Q}{4\pi\epsilon_0} R_0 - E_0 \left(R - \frac{R_0^3}{R^2} \right) \cos \theta$$

2.4 均匀介质球 (电容率为 ϵ_1) 的中心置一自由电偶极子 \vec{p}_f , 球外充满了另一种介质 (电容率为 ϵ_2), 求空间各点电势和极化电荷分布。

提示: $\varphi = \frac{\vec{p}_f \cdot \vec{R}}{4\pi\epsilon_1 R^3} + \varphi'$, 而 φ' 满足拉普拉斯方程

解 电偶极子产生的电势为

$$\varphi = \frac{\vec{p}_f \cdot \vec{R}}{4\pi\epsilon_0 R^3}$$

设

$$\begin{aligned} \varphi_i &= \frac{p_f \cos \theta}{4\pi\epsilon_1 R^2} + \sum_n a_n R^n P_n(\cos \theta) \quad R < R_0 \\ \varphi_o &= \frac{p_f \cos \theta}{4\pi\epsilon_2 R^2} + \sum_n \frac{b_n}{R^{n+1}} P_n(\cos \theta) \quad R > R_0 \end{aligned}$$

因为边界上电势连续

$$\begin{aligned} \frac{p_f \cos \theta}{4\pi\epsilon_1 R_0^2} + \sum_n a_n R_0^n P_n(\cos \theta) &= \frac{p_f \cos \theta}{4\pi\epsilon_2 R_0^2} + \sum_n \frac{b_n}{R_0^{n+1}} P_n(\cos \theta) \\ \epsilon_1 \left\{ -\frac{2p_f \cos \theta}{4\pi\epsilon_1 R_0^3} + \sum_n n a_n R_0^{n-1} P_n(\cos \theta) \right\} &= \epsilon_2 \left\{ -\frac{2p_f \cos \theta}{4\pi\epsilon_2 R_0^3} - \sum_n (n+1) \frac{b_n}{R_0^{n+2}} P_n(\cos \theta) \right\} \end{aligned}$$

当 $n=1$ 时

$$\frac{p_f}{4\pi\epsilon_1 R_0^2} + a_1 R_0 = \frac{p_f}{4\pi\epsilon_2 R_0^2} + \frac{b_1}{R_0^2}$$

$$\varepsilon_1 \left\{ -\frac{2p_f}{4\pi\varepsilon_1 R_0^3} + a_1 \right\} = \varepsilon_2 \left\{ -\frac{2p_f}{4\pi\varepsilon_2 R_0^3} - 2\frac{b_1}{R_0^3} \right\} \Rightarrow \varepsilon_1 a_1 = -\frac{2\varepsilon_2 b_1}{R_0^3}$$

解得

$$a_1 = \frac{(\varepsilon_1 - \varepsilon_2) p_f}{2\pi\varepsilon_1 (\varepsilon_1 + 2\varepsilon_2) R_0^3} \quad b_1 = \frac{(\varepsilon_2 - \varepsilon_1) p_f}{4\pi\varepsilon_2 (\varepsilon_1 + 2\varepsilon_2)}$$

当 $n \neq 1$ 时, $a_n = b_n = 0$, 因此方程的解为

$$\begin{aligned} \varphi_i &= \frac{p_f \cos \theta}{4\pi\varepsilon_1 R^2} + \frac{(\varepsilon_1 - \varepsilon_2) p_f \cos \theta}{2\pi\varepsilon_1 (\varepsilon_1 + 2\varepsilon_2) R_0^3} R \quad R < R_0 \\ \varphi_o &= \frac{p_f \cos \theta}{4\pi\varepsilon_2 R^2} + \frac{(\varepsilon_2 - \varepsilon_1) p_f \cos \theta}{4\pi\varepsilon_2 (\varepsilon_1 + 2\varepsilon_2)} \frac{1}{R^2} \quad R > R_0 \end{aligned}$$

2.8 半径为 R_0 的导体球外充满均匀绝缘介质 ε , 导体球接地, 离球心 a 处 ($a > R_0$) 置一点电荷 Q_f , 试用分离变量法求空间各点电势, 证明所得结果与镜像法结果相同

解 球内电势为零, 设球外电势为

$$\varphi = \frac{Q_f}{4\pi\varepsilon_0 r} + \sum_n \frac{b_n}{R^{n+1}} P_n(\cos \theta)$$

当 $R < a$ 时, 展开

$$\frac{1}{r} = \frac{1}{\sqrt{R^2 + a^2 - 2Ra \cos \theta}} = \frac{1}{a} \sum_n \left(\frac{R}{a} \right)^n P_n(\cos \theta)$$

因为 $R_0 < a$, 代入得到

$$\varphi = \frac{Q_f}{4\pi\varepsilon_0} \frac{1}{a} \sum_n \left(\frac{R}{a} \right)^n P_n(\cos \theta) + \sum_n \frac{b_n}{R^{n+1}} P_n(\cos \theta) = \left[\sum_n \frac{Q_f}{4\pi\varepsilon_0} \frac{R^n}{a^{n+1}} + \frac{b_n}{R^{n+1}} \right] P_n(\cos \theta)$$

当 $R = R_0$ 时, $\varphi = 0$

$$\frac{Q_f}{4\pi\varepsilon_0} \frac{R_0^n}{a^{n+1}} + \frac{b_n}{R_0^{n+1}} = 0 \Rightarrow b_n = -\frac{Q_f}{4\pi\varepsilon_0} \frac{R_0^{2n+1}}{a^{n+1}}$$

因此方程的解为

$$\varphi = \frac{Q_f}{4\pi\varepsilon_0} \left[\frac{1}{r} - \sum_n \frac{R_0^{2n+1}}{a^{n+1}} \frac{1}{R^{n+1}} P_n(\cos \theta) \right] \quad (1)$$

用电像法时只需要在 $b = \frac{R_0^2}{a}$ 处放置 $q_f = -\frac{R_0}{a}Q_f$ 的电荷, 空间各点电势为

$$\varphi = \frac{Q_f}{4\pi\epsilon_0} \left[\frac{1}{r} - \frac{R_0}{a} \frac{1}{r'} \right] \quad (2)$$

其中

$$\frac{1}{r'} = \frac{1}{\sqrt{R^2 + b^2 - 2Rb \cos \theta}}$$

因为 $b < R$, 展开式为

$$\frac{1}{\sqrt{R^2 + b^2 - 2Rb \cos \theta}} = \frac{1}{R} \sum_n \left(\frac{b}{R} \right)^n P_n(\cos \theta) = \frac{1}{R} \sum_n \left(\frac{R_0^2}{aR} \right)^n P_n(\cos \theta) = \sum_n \frac{R_0^{2n}}{a^n} \frac{1}{R^{n+1}} P_n(\cos \theta)$$

代入式 (2) 得到

$$\varphi = \frac{Q_f}{4\pi\epsilon_0} \left[\frac{1}{r} - \frac{R_0}{a} \sum_n \frac{R_0^{2n}}{a^n} \frac{1}{R^{n+1}} P_n(\cos \theta) \right]$$

和分离变量法的结果式 (1) 相同

2.11 在接地的导体平面上有一半径为 a 的半球凸部, 半球的球心在导体平面上, 点电荷 Q 位于系统的对称轴上, 并与平面相距为 b ($b > a$), 试用镜像法求空间电势

解 在原点的下方距离为 a 的地方放置一个电量为 $-Q$ 的电荷, 距离为 $\frac{a^2}{b}$ 的地方放置一个 $+qa/b$ 的电荷。之后在原点上方 $\frac{a^2}{b}$ 处放置一个 $-qa/b$ 的电荷, 电势为

$$\varphi = \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{x^2 + y^2 + (z-b)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+b)^2}} - \frac{a/b}{\sqrt{x^2 + y^2 + (z-a^2/b)^2}} + \frac{a/b}{\sqrt{x^2 + y^2 + (z+a^2/b)^2}} \right]$$

2.12 有一点电荷 Q 位于两个相互垂直的接地导体平面所围成的直角空间内, 它到两个平面的距离为 a 和 b , 求空间电势

解 在 $(0, -a, b)$ 、 $(0, a, -b)$ 、 $(0, -a, -b)$ 各放置三个虚拟电荷 $-Q$ 、 $-Q$ 、 $+Q$, 电势为

$$\varphi = \frac{Q}{4\pi\epsilon_0} \left\{ \left[(x-a)^2 + (y-b)^2 + z^2 \right]^{-1/2} + \left[(x+a)^2 + (y+b)^2 + z^2 \right]^{-1/2} - \left[(x+a)^2 + (y-b)^2 + z^2 \right]^{-1/2} - \left[(x-a)^2 + (y+b)^2 + z^2 \right]^{-1/2} \right\}$$

2.18 一半径为 R_0 的球面, 在球坐标 $0 < \theta < \frac{\pi}{2}$ 的半球面上电势为 φ_0 , 在 $\frac{\pi}{2} < \theta < \pi$ 的半球面上电势为 $-\varphi_0$, 球空间各点电势

提示:

$$\int_0^1 P_n(x) dx = \left. \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1} \right|_0^1$$

$$P_n(1) = 1$$

$$P_n(0) = \begin{cases} 0 & n = 2k+1 \\ (-1)^{n/2} \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} & n = 2k \end{cases}$$

解 设电势为

$$\begin{aligned} \varphi_i &= \sum_n a_n R^n P(\cos \theta) \quad R < R_0 \\ \varphi_o &= \sum_n \frac{b_n}{R^{n+1}} P_n(\cos \theta) \quad R > R_0 \end{aligned}$$

下面求解球内电势分布, 将 φ_i 展开

$$\varphi_i(x) = \sum_n \frac{2n+1}{2} \int_{-1}^{+1} \varphi_i(x) P_n(x) dx P(x)$$

比较系数得

$$\frac{2n+1}{2} \int_{-1}^{+1} \varphi_i(x) P_n(x) dx = a_n R^n$$

在 $R = R_0$ 处

$$\begin{aligned} a_n R_0^n &= \frac{2n+1}{2} \left[- \int_{-1}^0 \varphi_i(x) P_n(x) dx + \int_0^{+1} \varphi_i(x) P_n(x) dx \right] = (2n+1) \varphi_i(x) \int_0^1 P_n(x) dx \\ &= (2n+1) \varphi_i(x) \left[\frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1} \right] \Big|_0^1 \\ &= \varphi_i(x) [P_{n+1}(x) - P_{n-1}(x)] \Big|_0^1 \end{aligned}$$

其中

$$[P_{n+1}(x) - P_{n-1}(x)] \Big|_0^1 = P_{n+1}(1) - P_{n-1}(1) - P_{n+1}(0) + P_{n-1}(0) = P_{n-1}(0) - P_{n+1}(0)$$

当 n 为偶数时, $P_{n-1} = P_{n+1} = 0$, $a_n = 0$, 当 n 为奇数时

$$\begin{aligned} P_{n-1}(0) &= (-1)^{(n-1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (n-2)}{2 \cdot 4 \cdot 6 \cdots (n-1)} \\ P_{n+1}(0) &= (-1)^{(n+1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (n-2)}{2 \cdot 4 \cdot 6 \cdots (n-1)} \cdot \frac{n}{n+1} \end{aligned}$$

因此

$$\begin{aligned} a_n R_0^n &= P_{n-1}(0) - P_{n+1}(0) \\ &= (-1)^{(n-1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (n-2)}{2 \cdot 4 \cdot 6 \cdots (n-1)} \cdot \left[1 + \frac{n}{n+1} \right] \\ &= (-1)^{(n-1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (n-2)}{2 \cdot 4 \cdot 6 \cdots (n-1)} \cdot \frac{2n+1}{n+1} \end{aligned}$$

球内电势为

$$\varphi_i = \sum_n \left[(-1)^{(n-1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (n-2)}{2 \cdot 4 \cdot 6 \cdots (n-1)} \cdot \frac{2n+1}{n+1} \right] \frac{R^n}{R_0^n} P(\cos \theta) \quad R < R_0, n = 2k+1$$

下面求解球外电势分布, 将 φ_o 展开

$$\varphi_o(x) = \sum_n \frac{2n+1}{2} \int_{-1}^{+1} \varphi_o(x) P_n(x) dx P(x)$$

比较系数得

$$\frac{2n+1}{2} \int_{-1}^{+1} \varphi_o(x) P_n(x) dx = \frac{b_n}{R^{n+1}}$$

在 $R = R_0$ 处

$$\begin{aligned} \frac{b_n}{R_0^{n+1}} &= \frac{2n+1}{2} \left[- \int_{-1}^0 \varphi_o(x) P_n(x) dx + \int_0^{+1} \varphi_o(x) P_n(x) dx \right] = (2n+1) \varphi_o(x) \int_0^1 P_n(x) dx \\ &= (2n+1) \varphi_o(x) \left[\frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1} \right] \Big|_0^1 \\ &= \varphi_o(x) [P_{n+1}(x) - P_{n-1}(x)] \Big|_0^1 \end{aligned}$$

其中

$$[P_{n+1}(x) - P_{n-1}(x)] \Big|_0^1 = P_{n+1}(1) - P_{n-1}(1) - P_{n+1}(x) + P_{n-1}(0) = P_{n-1}(0) - P_{n+1}(0)$$

当 n 为偶数时, $P_{n-1} = P_{n+1} = 0$, $a_n = 0$, 当 n 为奇数时

$$P_{n-1}(0) = (-1)^{(n-1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \dots (n-2)}{2 \cdot 4 \cdot 6 \dots (n-1)}$$

$$P_{n+1}(0) = (-1)^{(n+1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \dots (n-2)}{2 \cdot 4 \cdot 6 \dots (n-1)} \cdot \frac{n}{n+1}$$

因此

$$\begin{aligned} \frac{b_n}{R_0^{n+1}} &= P_{n-1}(0) - P_{n+1}(0) \\ &= (-1)^{(n-1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \dots (n-2)}{2 \cdot 4 \cdot 6 \dots (n-1)} \cdot \left[1 + \frac{n}{n+1} \right] \\ &= (-1)^{(n-1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \dots (n-2)}{2 \cdot 4 \cdot 6 \dots (n-1)} \cdot \frac{2n+1}{n+1} \end{aligned}$$

球外电势为

$$\varphi_o = \sum_n \left[(-1)^{(n-1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \dots (n-2)}{2 \cdot 4 \cdot 6 \dots (n-1)} \cdot \frac{2n+1}{n+1} \right] \frac{R_0^{n+1}}{R^{n+1}} P(\cos \theta) \quad R > R_0, n = 2k+1$$

综上

$$\varphi = \begin{cases} \sum_n \left[(-1)^{(n-1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \dots (n-2)}{2 \cdot 4 \cdot 6 \dots (n-1)} \cdot \frac{2n+1}{n+1} \right] \frac{R^n}{R_0^n} P(\cos \theta) & R < R_0, n = 2k+1 \\ \sum_n \left[(-1)^{(n-1)/2} \cdot \frac{1 \cdot 3 \cdot 5 \dots (n-2)}{2 \cdot 4 \cdot 6 \dots (n-1)} \cdot \frac{2n+1}{n+1} \right] \frac{R_0^{n+1}}{R^{n+1}} P(\cos \theta) & R > R_0, n = 2k+1 \end{cases}$$

2.19 上题能用格林函数解吗? 结果如何?

解