Homework for Chapter II

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Q1 Assume $\vec{A} = x\left(z-y\right)\hat{x} + y\left(x-z\right)\hat{y} + z\left(y-x\right)\hat{z}$, Solve the rotation of $\vec{A}$ at $M\left(1,0,1\right)$ and the circulation density along $\vec{n} = 2\hat{x} + 6\hat{y} + 3\hat{z}$

Solution The rotation of $\vec{A}$ is

$$\mathrm{rot}\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x\left(z-y\right) & y\left(x-z\right) & z\left(y-x\right) \end{vmatrix} = \left(y+z\right)\hat{x} + \left(x+z\right)\hat{y} + \left(x+y\right)\hat{z}$$

$$\mathrm{rot}\vec{A}\Big|_{(1,0,1)} = \hat{x} + 2\hat{y} + \hat{z}$$

The circulation density along $\vec{n} = 2\hat{x} + 6\hat{y} + 3\hat{z}$ is

$$\mathrm{rot}\vec{A} \cdot \hat{n} = \left(1,2,1\right) \cdot \frac{1}{7}\left(2,6,3\right) = \frac{17}{7}$$

Q2 The speed of light is 3×10^8 m/s. What is the wavelength of a red light, whose frequency is 5×10^4 Hz? Compare your result with a 60 Hz EM wave.

Solution The wavelength of the red light is

$$\lambda = \frac{3 \times 10^8}{5 \times 10^4} = 600 \text{ nm}$$

The wavelength of the 60 Hz EM wave is

$$\lambda = \frac{3 \times 10^8}{60} = 5 \times 10^6 \text{ m}$$

Q3 Two wave functions: $\psi_1 = 4\cos\left[2\pi\left(0.2x - 3t\right)\right]$, $\psi_2 = \cos\left(7x + 3.5t\right)/2.5$ Calculate the frequency, wavelength, period, amplitude and phase velocity for each function.

Solution For $\psi_1 = 4\cos\left[2\pi\left(0.2x - 3t\right)\right] = 4\cos\left[10\pi\left(x - 15t\right)\right]$

$$\begin{aligned} f &= \frac{v}{\lambda} = 75 \text{ Hz} \\ v &= 15 \text{ m/s} \\ \lambda &= \frac{2\pi}{k} = 0.2 \text{ m} \\ A &= 4 \text{ m} \\ T &= \frac{\lambda}{v} = 0.0133 \text{ s} \end{aligned}$$

For $\psi_2 = \cos(7x + 3.5t)/2.5 = \cos[2.8(x + 1.4t)]$

$$\begin{aligned}
f &= \frac{v}{\lambda} = 0.625 \text{ Hz} \\
v &= \frac{3}{0.2} = 1.4 \text{ m/s} \\
\lambda &= \frac{2\pi}{k} = 2.24 \text{ m} \\
A &= 1 \text{ m} \\
T &= \frac{\lambda}{v} = 1.60 \text{ s}
\end{aligned}$$

Q4 Verify that the following functions are solutions of wave function:

$$\begin{aligned}
\psi_1(x,t) &= A\exp[i(kx - \omega t)] \\
\psi_2(x,t) &= A\exp[i(-kx - \omega t)] \\
\psi_3(x,t) &= A\exp[i(kx - \omega t)] + B\exp[i(-kx - \omega t)]
\end{aligned}$$

Solution The form of one dimensional wave function is

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

We now insert ψ_1 into the wave function

$$-Ak^2 \exp[i(kx - \omega t)] = -A\frac{\omega^2}{v^2}\exp[i(kx - \omega t)]$$

So that ψ_1 is a wave function if $v = \omega/k$

Now we insert ψ_2 into the wave function, similarly,

$$-Ak^2 \exp[i(kx - \omega t)] = -A\frac{\omega^2}{v^2}\exp[i(kx - \omega t)]$$

So that ψ_2 is a wave function if $v = \omega/k$

Now we insert ψ_3 into the wave function, similarly,

$$-Ak^2 \exp[i(kx - \omega t)] - Ak^2 \exp[i(kx - \omega t)] = -A\frac{\omega^2}{v^2}\exp[i(kx - \omega t)] - A\frac{\omega^2}{v^2}\exp[i(kx - \omega t)]$$

ψ_3 is a wave function if $v = \omega/k$

Q4 Write the Maxwell Equation in 3 dimensions. Derive the wave function of in all components.

Solution The differentiation form of Maxwell's equation is

$$\begin{aligned}
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{B} &= \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \cdot \vec{E} &= 0
\end{aligned}$$

We expand the equation into 3 components along the 3 axis

$$\begin{aligned}
\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} =& -\frac{\partial B_x}{\partial t} \\
\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} =& -\frac{\partial B_y}{\partial t} \\
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} =& -\frac{\partial B_z}{\partial t} \\
\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} =& \mu_0\epsilon_0\frac{\partial E_x}{\partial t} \\
\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} =& \mu_0\epsilon_0\frac{\partial E_y}{\partial t} \\
\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} =& \mu_0\epsilon_0\frac{\partial E_z}{\partial t} \\
\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} =& 0 \\
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} =& 0
\end{aligned}$$

Assume $\vec{E}$ is along the $\hat{x}$ direction. Therefore $\vec{B}$ is along the $\hat{y}$ direction. If $\vec{E}$ and $\vec{B}$ is not in these 2 directions, we can rotate the axis.

So we have

$$\begin{aligned}
E_y =& 0 \\
E_z =& 0 \\
B_x =& 0 \\
B_z =& 0
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial E_x}{\partial z} =& -\frac{\partial B_y}{\partial t} \\
-\frac{\partial E_x}{\partial y} =& 0 \\
-\frac{\partial B_y}{\partial z} =& \mu_0\epsilon_0\frac{\partial E_x}{\partial t} \\
\frac{\partial B_y}{\partial x} =& 0 \\
\frac{\partial E_x}{\partial x} =& 0 \\
\frac{\partial B_y}{\partial y} =& 0
\end{aligned}$$

Finally

$$\begin{aligned}
\frac{\partial^2 B_y}{\partial z^2} =& -\mu_0\epsilon_0\frac{\partial^2 E_x}{\partial t\partial z} = \mu_0\epsilon_0\frac{\partial^2 B_y}{\partial t^2} \\
\frac{\partial^2 E_x}{\partial z^2} =& -\frac{\partial^2 B_y}{\partial t\partial z} = \mu_0\epsilon_0\frac{\partial^2 E_x}{\partial t^2}
\end{aligned}$$

Q6 A plane wave on y-z plane, traveling along $\vec{r}$. The angle between $\vec{r}$ and y-axis is θ. The initial phase is 0. Find the complex amplitude of this wave.

Solution

$$
\begin{aligned}
\vec{E}\left(\vec{r}, t\right) &= \vec{A}\exp\left(i\vec{k}\cdot\vec{r}\right)\exp\left(-i\omega t\right) \\
\vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} \\
\hat{k} &= k\left(\hat{y}\cos\theta + \hat{z}\sin\theta\right)
\end{aligned}
$$

Q7 A electromagnetic wave, $E_x = 0$, $E_y = 2\cos\left[2\pi \times 10^{14}\left(z/c - t\right)\right]$, $E_z = 0$. What is the amplitude, wavelength, frequency of the wave? How is $\vec{B}$ exists?

Solution

$$
E_y = 2\cos\left[2\pi \times 10^{14} c\left(z - ct\right)\right]
$$

$$
\begin{aligned}
\lambda &= 3.3 \times 10^{-21} \\
A &= 2 \\
f &= c/\lambda = 9.09 \times 10^{28} \\
B &= cE
\end{aligned}
$$