理论力学复习

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1 达朗贝尔原理

对于稳定的系统，有

$$\sum_\alpha \mathbf{F}_\alpha \cdot \delta \mathbf{r}_\alpha = 0 \tag{1}$$

将 $\mathbf{F}_\alpha$ 分解为外力 $\mathbf{F}_\alpha^e$ 和内力（约束力）$\mathbf{f}_\alpha$，其中

$$\sum_\alpha \mathbf{f}_\alpha \cdot \delta \mathbf{r}_\alpha = 0 \tag{2}$$

因此有

$$\sum_\alpha \mathbf{F}_\alpha^e \cdot \delta \mathbf{r}_\alpha = 0 \tag{3}$$

当问题不是静力学问题的时候，我们添加一个惯性力使之化为静力学问题，由于 $\mathbf{F}_\alpha - \dot{\mathbf{p}}_\alpha = 0$

$$\sum_\alpha \left(\mathbf{F}_\alpha^e - \dot{\mathbf{p}}_\alpha\right) \cdot \delta \mathbf{r}_\alpha = 0 \tag{4}$$

这就是达朗贝尔准则

2 拉格朗日方程

2.1 广义坐标

位置 $\mathbf{r}_\alpha$ 可以由坐标 q_j 表示

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha\left(q_j, t\right) \tag{5}$$

对它求导

$$\dot{\mathbf{r}}_\alpha = \frac{\mathrm{d}\mathbf{r}_\alpha}{\mathrm{d}t} = \sum_j \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_\alpha}{\partial t} \tag{6}$$

进而得到

$$\delta \mathbf{r}_\alpha = \sum_j \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\delta q_j \tag{7}$$

2.2 拉格朗日方程的推导

又方程 4可以得到

$$\sum_{\alpha}\mathbf{F}^{e}_{\alpha}\cdot\delta\mathbf{r}_{\alpha}=\sum_{\alpha}\dot{\mathbf{p}}_{\alpha}\cdot\delta\mathbf{r}_{\alpha} \tag{8}$$

对左边进行展开，代入 7

$$\sum_{\alpha}\mathbf{F}^{e}_{\alpha}\cdot\delta\mathbf{r}_{\alpha}=\sum_{\alpha}\mathbf{F}^{e}_{\alpha}\cdot\sum_{j}\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\delta q_j=\sum_{j}\sum_{\alpha}\mathbf{F}^{e}_{\alpha}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\delta q_j \tag{9}$$

定义广义力 Q_j

$$\sum_{\alpha}\mathbf{F}^{e}_{\alpha}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}=Q_j \tag{10}$$

则式 8左边可以化为

$$\sum_{\alpha}\mathbf{F}^{e}_{\alpha}\cdot\delta\mathbf{r}_{\alpha}=\sum_{j}Q_j\delta q_j \tag{11}$$

下面我们对右边进行展开，同样，代入 7

$$\sum_{\alpha}\dot{\mathbf{p}}_{\alpha}\cdot\delta\mathbf{r}_{\alpha}=\sum_{\alpha}\dot{\mathbf{p}}_{\alpha}\cdot\sum_{j}\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\delta q_j=\sum_{\alpha}m_{\alpha}\ddot{\mathbf{r}}_{\alpha}\cdot\sum_{j}\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\delta q_j=\sum_{\alpha}\sum_{j}m_{\alpha}\ddot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\delta q_j \tag{12}$$

运用复合函数求导法则

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\right)=m_{\alpha}\ddot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}+m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\right) \tag{13}$$

即

$$m_{\alpha}\ddot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}=\frac{\mathrm{d}}{\mathrm{d}t}\left(m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\right)-m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\right) \tag{14}$$

把 14代入 12，得到

$$\sum_{\alpha}\dot{\mathbf{p}}_{\alpha}\cdot\delta\mathbf{r}_{\alpha}=\sum_{\alpha}\sum_{j}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\right)-m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\right)\right]\delta q_j \tag{15}$$

因为

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\right)=\frac{\partial\dot{\mathbf{r}}_{\alpha}}{\partial q_j} \tag{16}$$

我们得到

$$\sum_{\alpha}\dot{\mathbf{p}}_{\alpha}\cdot\delta\mathbf{r}_{\alpha}=\sum_{\alpha}\sum_{j}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\right)-m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\dot{\mathbf{r}}_{\alpha}}{\partial q_j}\right]\delta q_j \tag{17}$$

对 6求偏导

$$\frac{\partial\dot{\mathbf{r}}_{\alpha}}{\partial\dot{q}_j}=\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j} \tag{18}$$

带入 17

$$\sum_{\alpha}\dot{\mathbf{p}}_{\alpha}\cdot\delta\mathbf{r}_{\alpha}=\sum_{\alpha}\sum_{j}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\dot{\mathbf{r}}_{\alpha}}{\partial\dot{q}_j}\right)-m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\dot{\mathbf{r}}_{\alpha}}{\partial q_j}\right]\delta q_j \tag{19}$$

将 11和 19代入 8

$$\sum_{\alpha}\sum_{j}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\dot{\mathbf{r}}_{\alpha}}{\partial\dot{q}_j}\right)-m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\dot{\mathbf{r}}_{\alpha}}{\partial q_j}\right]\delta q_j=\sum_{j}Q_j\delta q_j \tag{20}$$

定义动能 $T=\sum\limits_{\alpha}\frac{1}{2}m_\alpha\dot{\mathbf{r}}_\alpha^2$

$$\partial T=\partial\left(\sum_{\alpha}\frac{1}{2}m_\alpha\dot{\mathbf{r}}_\alpha^2\right)=\sum_{\alpha}m_\alpha\dot{\mathbf{r}}_\alpha\cdot\partial\dot{\mathbf{r}}_\alpha \tag{21}$$

将 21代入 20

$$\sum_j\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_j}\right)-\frac{\partial T}{\partial q_j}\right]\delta q_j=\sum_j Q_j\delta q_j \tag{22}$$

因此

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=Q_j \tag{23}$$

即

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_j}\right)-\left(\frac{\partial T}{\partial q_j}+Q_j\right)=0 \tag{24}$$

当 Q_j 是保守力，势能为 U 时

$$-\frac{\partial U}{\partial q_j}=Q_j \tag{25}$$

$$\frac{\partial U}{\partial\dot{q}_j}=0 \tag{26}$$

方程 24可以化为

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(T-U)}{\partial\dot{q}_j}\right)-\frac{\partial(T-U)}{\partial q_j}=0 \tag{27}$$

定义 $L=T-U$，则

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0 \tag{28}$$

3 哈密顿量的守恒

对 $L(q,\dot{q},t)$ 求导

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\sum_j\frac{\partial L}{\partial q_j}\dot{q}_j+\sum_j\frac{\partial L}{\partial\dot{q}_j}\ddot{q}_j+\frac{\partial L}{\partial t} \tag{29}$$

式 28告诉我们

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_j}\right)=\frac{\partial L}{\partial q_j} \tag{30}$$

将 30代入 29

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\sum_j\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_j}\right)\dot{q}_j+\sum_j\frac{\partial L}{\partial\dot{q}_j}\ddot{q}_j+\frac{\partial L}{\partial t} \tag{31}$$

即

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\sum_j\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_j}\dot{q}_j\right)+\frac{\partial L}{\partial t} \tag{32}$$

交换求和与积分顺序

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j\frac{\partial L}{\partial\dot{q}_j}\dot{q}_j\right)+\frac{\partial L}{\partial t} \tag{33}$$

移项

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j\frac{\partial L}{\partial\dot{q}_j}\dot{q}_j-L\right)+\frac{\partial L}{\partial t}=0 \tag{34}$$

定义系统的哈密顿量

$$H = \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L \tag{35}$$

当 $\frac{\partial L}{\partial t} = 0$ 时

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \tag{36}$$

哈密顿量守恒

4 Noether 定理与广义动量

由 33

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j\right) + \frac{\partial L}{\partial t} \tag{37}$$

可以得到

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j \frac{\partial L}{\partial \dot{q}_j} \delta q_j\right) \tag{38}$$

当空间具有平移不变性时，即虚位移的产生对于拉格朗日量没有影响时

$$\delta L = 0 \tag{39}$$

因此

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j \frac{\partial L}{\partial \dot{q}_j} \delta q_j\right) = 0 \tag{40}$$

即

$$\sum_j \frac{\partial L}{\partial \dot{q}_j} \delta q_j = \mathrm{const} \tag{41}$$

因此定义广义动量

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \tag{42}$$

则 $p_j = \mathrm{const}$，广义动量守恒

5 欧拉变分原理

设

$$J = \int_{x_1}^{x_2} f\left\{y(x), y'(x); x\right\} \mathrm{d}x \tag{43}$$

J 是函数 $y(x)$ 的函数，求 J 的极值函数，方法是另

$$y(\alpha, x) = y(x) + \alpha \eta(x) \tag{44}$$

则

$$y'(\alpha, x) = y'(x) + \alpha \eta'(x) \tag{45}$$

$$\frac{\partial y}{\partial \alpha} = \eta(x) \tag{46}$$

$$\frac{\partial y'}{\partial \alpha} = \eta'(x) \tag{47}$$

求解

$$\frac{\partial J}{\partial \alpha} = 0 \tag{48}$$

解法为

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_{x_1}^{x_2} f\left\{y(x), y'(x); x\right\} \mathrm{d}x = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right)\mathrm{d}x \tag{49}$$

将 45和 46代入

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\eta'(x)\right)\mathrm{d}x = \int_{x_1}^{x_2}\frac{\partial f}{\partial y}\eta(x)\,\mathrm{d}x + \int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\eta'(x)\,\mathrm{d}x \tag{50}$$

利用分步积分法算式 50的最后一项

$$\int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\eta'(x)\,\mathrm{d}x = \int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\,\mathrm{d}\eta(x) = \frac{\partial f}{\partial y'}\left[\eta(x_1) - \eta(x_2)\right] - \int_{x_1}^{x_2}\eta(x)\,\mathrm{d}\left(\frac{\partial f}{\partial y'}\right) \tag{51}$$

由于式 44中 $y(\alpha, x_1) = y(x_1)$ 且 $y(\alpha, x_2) = y(x_2)$

$$\eta(x_1) = \eta(x_2) = 0 \tag{52}$$

代入 51

$$\int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\eta'(x)\,\mathrm{d}x = -\int_{x_1}^{x_2}\eta(x)\,\mathrm{d}\left(\frac{\partial f}{\partial y'}\right) = -\int_{x_1}^{x_2}\eta(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\mathrm{d}x \tag{53}$$

代入 50

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\frac{\partial f}{\partial y}\eta(x)\,\mathrm{d}x - \int_{x_1}^{x_2}\eta(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\mathrm{d}x = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)\,\mathrm{d}x = 0 \tag{54}$$

因此

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0 \tag{55}$$

我们也可以用 δ 记号表示，由式 54和 46

$$\frac{\partial J}{\partial \alpha}\mathrm{d}\alpha = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right]\frac{\partial y}{\partial \alpha}\,\mathrm{d}\alpha\,\mathrm{d}x = 0 \tag{56}$$

即

$$\delta J = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right]\delta y\,\mathrm{d}x = 0 \tag{57}$$

6 哈密顿原理

哈密顿原理的表示形式为

$$I = \int_{t_1}^{t_2} L\left(q_j, \dot{q}_j, t\right)\mathrm{d}t = 0 \tag{58}$$

或

$$\delta I = \delta\int_{t_1}^{t_2} L\left(q_j, \dot{q}_j, t\right)\mathrm{d}t = 0 \tag{59}$$

由式 55我们已经知道，式 59的解是

$$\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \tag{60}$$

即拉格朗日方程 28

7　拉格朗日乘子与约束条件

7.1　带约束的欧拉变分原理

有两个关于 x 的函数 $y(x)$、$z(x)$，求

$$J=\int_{x_1}^{x_2} f\left\{y(x), y'(x), z(x), z'(x); x\right\} \tag{61}$$

的极值，有约束

$$g\left\{y(x), z(x); x\right\}=0 \tag{62}$$

由 61

$$\delta J=\int_{x_1}^{x_2}\left[\left(\frac{\partial f}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\delta y+\left(\frac{\partial f}{\partial z}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'}\right)\delta z\right]\mathrm{d}x=0 \tag{63}$$

$$\delta g=\frac{\partial g}{\partial y}\delta y+\frac{\partial g}{\partial z}\delta z=0 \tag{64}$$

引入拉格朗日乘子 $\lambda(x)$

$$\lambda(x)\delta g=\lambda(x)\left(\frac{\partial g}{\partial y}\delta y+\frac{\partial g}{\partial z}\delta z\right)=0 \tag{65}$$

将 65代入 63

$$\int_{x_1}^{x_2}\left[\left(\frac{\partial f}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}+\lambda(x)\frac{\partial g}{\partial y}\right)\delta y+\left(\frac{\partial f}{\partial z}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'}+\lambda(x)\frac{\partial g}{\partial z}\right)\delta z\right]\mathrm{d}x=0 \tag{66}$$

因此，我们有

$$\frac{\partial f}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}+\lambda(x)\frac{\partial g}{\partial y}=0 \tag{67}$$

和

$$\frac{\partial f}{\partial z}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'}+\lambda(x)\frac{\partial g}{\partial z}=0 \tag{68}$$

7.2　带不定乘子的拉格朗日方程

考察二维平面上的运动，拉格朗日量

$$L\left(x, \dot{x}, y, \dot{y}; t\right) \tag{69}$$

x 和 y 之间有约束

$$g\left(x, y; t\right)=0 \tag{70}$$

由 67和 68可知

$$\frac{\partial L}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}+\lambda(t)\frac{\partial g}{\partial x}=0 \tag{71}$$

$$\frac{\partial L}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}}+\lambda(t)\frac{\partial g}{\partial y}=0 \tag{72}$$

其中

$$Q_x=\lambda(t)\frac{\partial g}{\partial x} \tag{73}$$

$$Q_y=\lambda(t)\frac{\partial g}{\partial y} \tag{74}$$

为约束力