理论力学复习

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1 达朗贝尔原理

对于稳定的系统，有

$$\sum_\alpha \mathbf{F}_\alpha \cdot \delta \mathbf{r}_\alpha = 0 \tag{1}$$

将 $\mathbf{F}_\alpha$ 分解为外力 $\mathbf{F}_\alpha^e$ 和内力（约束力）$\mathbf{f}_\alpha$，其中

$$\sum_\alpha \mathbf{f}_\alpha \cdot \delta \mathbf{r}_\alpha = 0 \tag{2}$$

因此有

$$\sum_\alpha \mathbf{F}_\alpha^e \cdot \delta \mathbf{r}_\alpha = 0 \tag{3}$$

当问题不是静力学问题的时候，我们添加一个惯性力使之化为静力学问题，由于 $\mathbf{F}_\alpha - \dot{\mathbf{p}}_\alpha = 0$

$$\sum_\alpha (\mathbf{F}_\alpha^e - \dot{\mathbf{p}}_\alpha) \cdot \delta \mathbf{r}_\alpha = 0 \tag{4}$$

这就是达朗贝尔准则

2 拉格朗日方程

2.1 广义坐标

位置 $\mathbf{r}_\alpha$ 可以由坐标 q_j 表示

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_j, t) \tag{5}$$

对它求导

$$\dot{\mathbf{r}}_\alpha = \frac{\mathrm{d}\mathbf{r}_\alpha}{\mathrm{d}t} = \sum_j \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_\alpha}{\partial t} \tag{6}$$

进而得到

$$\delta \mathbf{r}_\alpha = \sum_j \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \delta q_j \tag{7}$$

2.2 拉格朗日方程的推导

又方程 4可以得到

$$\sum_\alpha \mathbf{F}_\alpha^e \cdot \delta\mathbf{r}_\alpha = \sum_\alpha \dot{\mathbf{p}}_\alpha \cdot \delta\mathbf{r}_\alpha \tag{8}$$

对左边进行展开，代入 7

$$\sum_\alpha \mathbf{F}_\alpha^e \cdot \delta\mathbf{r}_\alpha = \sum_\alpha \mathbf{F}_\alpha^e \cdot \sum_j \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \delta q_j = \sum_j \sum_\alpha \mathbf{F}_\alpha^e \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \delta q_j \tag{9}$$

定义广义力 Q_j

$$\sum_\alpha \mathbf{F}_\alpha^e \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j} = Q_j \tag{10}$$

则式 8左边可以化为

$$\sum_\alpha \mathbf{F}_\alpha^e \cdot \delta\mathbf{r}_\alpha = \sum_j Q_j \delta q_j \tag{11}$$

下面我们对右边进行展开，同样，代入 7

$$\sum_\alpha \dot{\mathbf{p}}_\alpha \cdot \delta\mathbf{r}_\alpha = \sum_\alpha \dot{\mathbf{p}}_\alpha \cdot \sum_j \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \delta q_j = \sum_\alpha m_\alpha \ddot{\mathbf{r}}_\alpha \cdot \sum_j \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \delta q_j = \sum_\alpha \sum_j m_\alpha \ddot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \delta q_j \tag{12}$$

运用复合函数求导法则

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\right) = m_\alpha \ddot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j} + m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_j}\right) \tag{13}$$

即

$$m_\alpha \ddot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\right) - m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_j}\right) \tag{14}$$

把 14代入 12，得到

$$\sum_\alpha \dot{\mathbf{p}}_\alpha \cdot \delta\mathbf{r}_\alpha = \sum_\alpha \sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\right) - m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_j}\right)\right] \delta q_j \tag{15}$$

因为

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_j}\right) = \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_j} \tag{16}$$

我们得到

$$\sum_\alpha \dot{\mathbf{p}}_\alpha \cdot \delta\mathbf{r}_\alpha = \sum_\alpha \sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\right) - m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_j}\right] \delta q_j \tag{17}$$

对 6求偏导

$$\frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q_j}} = \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \tag{18}$$

带入 17

$$\sum_\alpha \dot{\mathbf{p}}_\alpha \cdot \delta\mathbf{r}_\alpha = \sum_\alpha \sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q_j}}\right) - m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_j}\right] \delta q_j \tag{19}$$

将 11和 19代入 8

$$\sum_\alpha \sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q_j}}\right) - m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_j}\right] \delta q_j = \sum_j Q_j \delta q_j \tag{20}$$

定义动能 $T = \sum\limits_{\alpha} \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha^2$

$$\partial T = \partial \left(\sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha^2 \right) = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \partial \dot{\mathbf{r}}_\alpha \tag{21}$$

将 21代入 20

$$\sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j = \sum_j Q_j \delta q_j \tag{22}$$

因此

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j \tag{23}$$

即

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \left(\frac{\partial T}{\partial q_j} + Q_j \right) = 0 \tag{24}$$

当 Q_j 是保守力, 势能为 U 时

$$-\frac{\partial U}{\partial q_j} = Q_j \tag{25}$$

$$\frac{\partial U}{\partial \dot{q}_j} = 0 \tag{26}$$

方程 24可以化为

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial (T - U)}{\partial \dot{q}_j} \right) - \frac{\partial (T - U)}{\partial q_j} = 0 \tag{27}$$

定义 $L = T - U$, 则

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0 \tag{28}$$

3　哈密顿量的守恒

对 $L(q, \dot{q}, t)$ 求导

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_j \frac{\partial L}{\partial q_j} \dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial L}{\partial t} \tag{29}$$

式 28告诉我们

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_j} \right) = \frac{\partial L}{\partial q_j} \tag{30}$$

将 30代入 29

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_j \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_j} \right) \dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial L}{\partial t} \tag{31}$$

即

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_j \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_j} \dot{q}_j \right) + \frac{\partial L}{\partial t} \tag{32}$$

交换求和与积分顺序

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j \right) + \frac{\partial L}{\partial t} \tag{33}$$

移项

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L \right) + \frac{\partial L}{\partial t} = 0 \tag{34}$$

定义系统的哈密顿量

$$H = \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L \tag{35}$$

当 $\frac{\partial L}{\partial t} = 0$ 时

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \tag{36}$$

哈密顿量守恒

4　Noether 定理与广义动量

由 33

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j \right) + \frac{\partial L}{\partial t} \tag{37}$$

可以得到

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_j \frac{\partial L}{\partial \dot{q}_j} \delta q_j \right) \tag{38}$$

当空间具有平移不变性时，即虚位移的产生对于拉格朗日量没有影响时

$$\delta L = 0 \tag{39}$$

因此

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_j \frac{\partial L}{\partial \dot{q}_j} \delta q_j \right) = 0 \tag{40}$$

即

$$\sum_j \frac{\partial L}{\partial \dot{q}_j} \delta q_j = \mathrm{const} \tag{41}$$

因此定义广义动量

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \tag{42}$$

则 $p_j = \mathrm{const}$，广义动量守恒

5　欧拉变分原理

设

$$J = \int_{x_1}^{x_2} f\left\{ y(x), y'(x); x \right\} \mathrm{d}x \tag{43}$$

J 是函数 $y(x)$ 的函数，求 J 的极值函数，方法是另

$$y(\alpha, x) = y(x) + \alpha \eta(x) \tag{44}$$

则

$$y'(\alpha, x) = y'(x) + \alpha \eta'(x) \tag{45}$$

$$\frac{\partial y}{\partial \alpha} = \eta(x) \tag{46}$$

$$\frac{\partial y'}{\partial \alpha} = \eta'(x) \tag{47}$$

求解

$$\frac{\partial J}{\partial \alpha} = 0 \tag{48}$$

解法为

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_{x_1}^{x_2} f\{y(x), y'(x); x\} \, \mathrm{d}x = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial \alpha} \right) \mathrm{d}x \tag{49}$$

将 45和 46代入

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x) \right) \mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \eta(x) \, \mathrm{d}x + \int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \eta'(x) \, \mathrm{d}x \tag{50}$$

利用分步积分法算式 50的最后一项

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \eta'(x) \, \mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \, \mathrm{d}\eta(x) = \frac{\partial f}{\partial y'} [\eta(x_1) - \eta(x_2)] - \int_{x_1}^{x_2} \eta(x) \, \mathrm{d} \left(\frac{\partial f}{\partial y'} \right) \tag{51}$$

由于式 44中 $y(\alpha, x_1) = y(x_1)$ 且 $y(\alpha, x_2) = y(x_2)$

$$\eta(x_1) = \eta(x_2) = 0 \tag{52}$$

代入 51

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \eta'(x) \, \mathrm{d}x = - \int_{x_1}^{x_2} \eta(x) \, \mathrm{d} \left(\frac{\partial f}{\partial y'} \right) = - \int_{x_1}^{x_2} \eta(x) \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) \mathrm{d}x \tag{53}$$

代入 50

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \eta(x) \, \mathrm{d}x - \int_{x_1}^{x_2} \eta(x) \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) \mathrm{d}x = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) \right] \eta(x) \, \mathrm{d}x = 0 \tag{54}$$

因此

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) = 0 \tag{55}$$

我们也可以用 δ 记号表示，由式 54和 46

$$\frac{\partial J}{\partial \alpha} \, \mathrm{d}\alpha = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) \right] \frac{\partial y}{\partial \alpha} \, \mathrm{d}\alpha \, \mathrm{d}x = 0 \tag{56}$$

即

$$\delta J = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) \right] \delta y \, \mathrm{d}x = 0 \tag{57}$$

6 哈密顿原理

哈密顿原理的表示形式为

$$I = \int_{t_1}^{t_2} L(q, \dot{q}, t) \, \mathrm{d}t = 0 \tag{58}$$

或

$$\delta I = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t) \, \mathrm{d}t = 0 \tag{59}$$

由式 55我们已经知道，式 59的解是

$$\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_j} \right) = 0 \tag{60}$$

即拉格朗日方程