

# 理论力学复习

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## 1 达朗贝尔原理

对于稳定的系统，有

$$\sum_{\alpha} \mathbf{F}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} = 0 \quad (1)$$

将  $\mathbf{F}_{\alpha}$  分解为外力  $\mathbf{F}_{\alpha}^e$  和内力（约束力） $\mathbf{f}_{\alpha}$ ，其中

$$\sum_{\alpha} \mathbf{f}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} = 0 \quad (2)$$

因此有

$$\sum_{\alpha} \mathbf{F}_{\alpha}^e \cdot \delta \mathbf{r}_{\alpha} = 0 \quad (3)$$

当问题不是静力学问题的时候，我们添加一个惯性力使之化为静力学问题，由于  $\mathbf{F}_{\alpha} - \dot{\mathbf{p}}_{\alpha} = 0$

$$\sum_{\alpha} (\mathbf{F}_{\alpha}^e - \dot{\mathbf{p}}_{\alpha}) \cdot \delta \mathbf{r}_{\alpha} = 0 \quad (4)$$

这就是达朗贝尔准则

## 2 拉格朗日方程

### 2.1 广义坐标

位置  $\mathbf{r}_{\alpha}$  可以由坐标  $q_j$  表示

$$\mathbf{r}_{\alpha} = \mathbf{r}_{\alpha}(q_j, t) \quad (5)$$

对它求导

$$\dot{\mathbf{r}}_{\alpha} = \frac{d\mathbf{r}_{\alpha}}{dt} = \sum_j \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_{\alpha}}{\partial t} \quad (6)$$

进而得到

$$\delta \mathbf{r}_{\alpha} = \sum_j \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \delta q_j \quad (7)$$

## 2.2 拉格朗日方程的推导

又方程 4可以得到

$$\sum_{\alpha} \mathbf{F}_{\alpha}^e \cdot \delta \mathbf{r}_{\alpha} = \sum_{\alpha} \dot{\mathbf{p}}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} \quad (8)$$

对左边进行展开，代入 7

$$\sum_{\alpha} \mathbf{F}_{\alpha}^e \cdot \delta \mathbf{r}_{\alpha} = \sum_{\alpha} \mathbf{F}_{\alpha}^e \cdot \sum_j \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \delta q_j = \sum_j \sum_{\alpha} \mathbf{F}_{\alpha}^e \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \delta q_j \quad (9)$$

定义广义力  $Q_j$

$$\sum_{\alpha} \mathbf{F}_{\alpha}^e \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} = Q_j \quad (10)$$

则式 8左边可以化为

$$\sum_{\alpha} \mathbf{F}_{\alpha}^e \cdot \delta \mathbf{r}_{\alpha} = \sum_j Q_j \delta q_j \quad (11)$$

下面我们对右边进行展开，同样，代入 7

$$\sum_{\alpha} \dot{\mathbf{p}}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} = \sum_{\alpha} \dot{\mathbf{p}}_{\alpha} \cdot \sum_j \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \delta q_j = \sum_{\alpha} m_{\alpha} \ddot{\mathbf{r}}_{\alpha} \cdot \sum_j \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \delta q_j = \sum_{\alpha} \sum_j m_{\alpha} \ddot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \delta q_j \quad (12)$$

运用复合函数求导法则

$$\frac{d}{dt} \left( m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) = m_{\alpha} \ddot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} + m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{d}{dt} \left( \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) \quad (13)$$

即

$$m_{\alpha} \ddot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} = \frac{d}{dt} \left( m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) - m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{d}{dt} \left( \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) \quad (14)$$

把 14代入 12，得到

$$\sum_{\alpha} \dot{\mathbf{p}}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} = \sum_{\alpha} \sum_j \left[ \frac{d}{dt} \left( m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) - m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{d}{dt} \left( \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) \right] \delta q_j \quad (15)$$

因为

$$\frac{d}{dt} \left( \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) = \frac{\partial \dot{\mathbf{r}}_{\alpha}}{\partial q_j} \quad (16)$$

我们得到

$$\sum_{\alpha} \dot{\mathbf{p}}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} = \sum_{\alpha} \sum_j \left[ \frac{d}{dt} \left( m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) - m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \dot{\mathbf{r}}_{\alpha}}{\partial q_j} \right] \delta q_j \quad (17)$$

对 6求偏导

$$\frac{\partial \dot{\mathbf{r}}_{\alpha}}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \quad (18)$$

带入 17

$$\sum_{\alpha} \dot{\mathbf{p}}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} = \sum_{\alpha} \sum_j \left[ \frac{d}{dt} \left( m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) - m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \dot{\mathbf{r}}_{\alpha}}{\partial q_j} \right] \delta q_j \quad (19)$$

将 11和 19代入 8

$$\sum_{\alpha} \sum_j \left[ \frac{d}{dt} \left( m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) - m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \dot{\mathbf{r}}_{\alpha}}{\partial q_j} \right] \delta q_j = \sum_j Q_j \delta q_j \quad (20)$$

定义动能  $T = \sum_{\alpha} \frac{1}{2} m_{\alpha} \dot{\mathbf{r}}_{\alpha}^2$

$$\partial T = \partial \left( \sum_{\alpha} \frac{1}{2} m_{\alpha} \dot{\mathbf{r}}_{\alpha}^2 \right) = \sum_{\alpha} m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \partial \dot{\mathbf{r}}_{\alpha} \quad (21)$$

将 21 代入 20

$$\sum_j \left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j = \sum_j Q_j \delta q_j \quad (22)$$

因此

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j \quad (23)$$

即

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \left( \frac{\partial T}{\partial q_j} + Q_j \right) = 0 \quad (24)$$

当  $Q_j$  是保守力, 势能为  $U$  时

$$-\frac{\partial U}{\partial q_j} = Q_j \quad (25)$$

$$\frac{\partial U}{\partial \dot{q}_j} = 0 \quad (26)$$

方程 24 可以化为

$$\frac{d}{dt} \left( \frac{\partial (T - U)}{\partial \dot{q}_j} \right) - \frac{\partial (T - U)}{\partial q_j} = 0 \quad (27)$$

定义  $L = T - U$ , 则

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0 \quad (28)$$

### 3 哈密顿量的守恒

对  $L(q, \dot{q}, t)$  求导

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j} \dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial L}{\partial t} \quad (29)$$

式 28 告诉我们

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) = \frac{\partial L}{\partial q_j} \quad (30)$$

将 30 代入 29

$$\frac{dL}{dt} = \sum_j \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) \dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial L}{\partial t} \quad (31)$$

即

$$\frac{dL}{dt} = \sum_j \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j \right) + \frac{\partial L}{\partial t} \quad (32)$$

交换求和与积分顺序

$$\frac{dL}{dt} = \frac{d}{dt} \left( \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j \right) + \frac{\partial L}{\partial t} \quad (33)$$

移项

$$\frac{d}{dt} \left( \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L \right) + \frac{\partial L}{\partial t} = 0 \quad (34)$$

定义系统的哈密顿量

$$H = \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L \quad (35)$$

当  $\frac{\partial L}{\partial t} = 0$  时

$$\frac{dH}{dt} = 0 \quad (36)$$

哈密顿量守恒

## 4 Noether 定理与广义动量

由 33

$$\frac{dL}{dt} = \frac{d}{dt} \left( \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j \right) + \frac{\partial L}{\partial t} \quad (37)$$

可以得到

$$\delta L = \frac{d}{dt} \left( \sum_j \frac{\partial L}{\partial \dot{q}_j} \delta q_j \right) \quad (38)$$

当空间具有平移不变性时，即虚位移的产生对于拉格朗日量没有影响时

$$\delta L = 0 \quad (39)$$

因此

$$\frac{d}{dt} \left( \sum_j \frac{\partial L}{\partial \dot{q}_j} \delta q_j \right) = 0 \quad (40)$$

即

$$\sum_j \frac{\partial L}{\partial \dot{q}_j} \delta q_j = \text{const} \quad (41)$$

因此定义广义动量

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad (42)$$

则  $p_j = \text{const}$ ，广义动量守恒

## 5 欧拉变分原理

设

$$J = \int_{x_1}^{x_2} f \{y(x), y'(x); x\} dx \quad (43)$$

$J$  是函数  $y(x)$  的函数，求  $J$  的极值函数，方法是另

$$y(\alpha, x) = y(x) + \alpha \eta(x) \quad (44)$$

则

$$y'(\alpha, x) = y'(x) + \alpha \eta'(x) \quad (45)$$

$$\frac{\partial y}{\partial \alpha} = \eta(x) \quad (46)$$

$$\frac{\partial y'}{\partial \alpha} = \eta'(x) \quad (47)$$

求解

$$\frac{\partial J}{\partial \alpha} = 0 \quad (48)$$

解法为

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_{x_1}^{x_2} f\{y(x), y'(x); x\} dx = \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial \alpha} \right) dx \quad (49)$$

将 45和 46代入

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x) \right) dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \eta(x) dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \eta'(x) dx \quad (50)$$

利用分步积分法算式 50的最后一项

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \eta'(x) dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial y'} d\eta(x) = \frac{\partial f}{\partial y'} [\eta(x_1) - \eta(x_2)] - \int_{x_1}^{x_2} \eta(x) d\left(\frac{\partial f}{\partial y'}\right) \quad (51)$$

由于式 44中  $y(\alpha, x_1) = y(x_1)$  且  $y(\alpha, x_2) = y(x_2)$

$$\eta(x_1) = \eta(x_2) = 0 \quad (52)$$

代入 51

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \eta'(x) dx = - \int_{x_1}^{x_2} \eta(x) d\left(\frac{\partial f}{\partial y'}\right) = - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx} \left(\frac{\partial f}{\partial y'}\right) dx \quad (53)$$

代入 50

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \eta(x) dx - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx} \left(\frac{\partial f}{\partial y'}\right) dx = \int_{x_1}^{x_2} \left[ \frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'}\right) \right] \eta(x) dx = 0 \quad (54)$$

因此

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'}\right) = 0 \quad (55)$$

我们也可以使用  $\delta$  记号表示, 由式 54和 46

$$\frac{\partial J}{\partial \alpha} d\alpha = \int_{x_1}^{x_2} \left[ \frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'}\right) \right] \frac{\partial y}{\partial \alpha} d\alpha dx = 0 \quad (56)$$

即

$$\delta J = \int_{x_1}^{x_2} \left[ \frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'}\right) \right] \delta y dx = 0 \quad (57)$$

## 6 哈密顿原理

哈密顿原理的表示形式为

$$I = \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t) dt = 0 \quad (58)$$

或

$$\delta I = \delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t) dt = 0 \quad (59)$$

由式 55我们已经知道, 式 59的解是

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \quad (60)$$

即拉格朗日方程 28

## 7 拉格朗日乘子与约束条件

### 7.1 带约束的欧拉变分原理

有两个关于  $x$  的函数  $y(x)$ 、 $z(x)$ ，求

$$J = \int_{x_1}^{x_2} f \{y(x), y'(x), z(x), z'(x); x\} \quad (61)$$

的极值，有约束

$$g \{y(x), z(x); x\} = 0 \quad (62)$$

由 61

$$\delta J = \int_{x_1}^{x_2} \left[ \left( \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \delta y + \left( \frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} \right) \delta z \right] dx = 0 \quad (63)$$

$$\delta g = \frac{\partial g}{\partial y} \delta y + \frac{\partial g}{\partial z} \delta z = 0 \quad (64)$$

引入拉格朗日乘子  $\lambda(x)$

$$\lambda(x) \delta g = \lambda(x) \left( \frac{\partial g}{\partial y} \delta y + \frac{\partial g}{\partial z} \delta z \right) = 0 \quad (65)$$

将 65 代入 63

$$\int_{x_1}^{x_2} \left[ \left( \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \lambda(x) \frac{\partial g}{\partial y} \right) \delta y + \left( \frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} + \lambda(x) \frac{\partial g}{\partial z} \right) \delta z \right] dx = 0 \quad (66)$$

因此，我们有

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \lambda(x) \frac{\partial g}{\partial y} = 0 \quad (67)$$

和

$$\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} + \lambda(x) \frac{\partial g}{\partial z} = 0 \quad (68)$$

### 7.2 带不定乘子的拉格朗日方程

考察二维平面上的运动，拉格朗日量

$$L(x, \dot{x}, y, \dot{y}; t) \quad (69)$$

$x$  和  $y$  之间有约束

$$g(x, y; t) = 0 \quad (70)$$

由 67 和 68 可知

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \lambda(t) \frac{\partial g}{\partial x} = 0 \quad (71)$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} + \lambda(t) \frac{\partial g}{\partial y} = 0 \quad (72)$$

其中

$$Q_x = \lambda(t) \frac{\partial g}{\partial x} \quad (73)$$

$$Q_y = \lambda(t) \frac{\partial g}{\partial y} \quad (74)$$

为约束力

## 8 哈密顿动力学

### 8.1 广义动量

由广义动量的定义 42 和拉格朗日方程 28

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad (75)$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0 \quad (76)$$

得到

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) = \frac{\partial L}{\partial q_j} \quad (77)$$

进而得到

$$\dot{p}_j = \frac{\partial L}{\partial q_j} \quad (78)$$

### 8.2 哈密顿正则方程

由 35, 系统的哈密顿量

$$H(p_j, q_j, t) = \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L \quad (79)$$

将 75 代入

$$H = \sum_j p_j \dot{q}_j - L(q_j, \dot{q}_j, t) \quad (80)$$

对 80 求全微分

$$dH = \sum_j \left( \dot{q}_j dp_j + p_j d\dot{q}_j - \frac{\partial L}{\partial q_j} dq_j - \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j \right) - \frac{\partial L}{\partial t} dt \quad (81)$$

将 75 和 78 代入

$$dH = \sum_j (\dot{q}_j dp_j + p_j d\dot{q}_j - \dot{p}_j dq_j - p_j d\dot{q}_j) - \frac{\partial L}{\partial t} dt \quad (82)$$

整理得到

$$dH = \sum_j (\dot{q}_j dp_j - \dot{p}_j dq_j) - \frac{\partial L}{\partial t} dt \quad (83)$$

又因为

$$dH = \sum_j \left( \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j \right) + \frac{\partial H}{\partial t} dt \quad (84)$$

因此我们得到哈密顿正则方程

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \quad (85)$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} \quad (86)$$

$$\frac{dH}{dt} = -\frac{\partial H}{\partial t} \quad (87)$$

### 8.3 泊松括号

$$\{u, v\} = \sum_j \left( \frac{\partial u}{\partial q_j} \frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j} \frac{\partial v}{\partial q_j} \right) \quad (88)$$

用法

$$\frac{du}{dt} = \sum_j \left( \frac{\partial u}{\partial q_j} \dot{q}_j - \frac{\partial u}{\partial p_j} \dot{p}_j \right) + \frac{\partial u}{\partial t} = \sum_j \left( \frac{\partial u}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial u}{\partial p_j} \frac{\partial H}{\partial q_j} \right) + \frac{\partial u}{\partial t} \quad (89)$$

可以简写为

$$\frac{du}{dt} = \{u, H\} + \frac{\partial u}{\partial t} \quad (90)$$