理论力学公式推导

胡喜平

2020 年 1 月 2 日

1 达朗贝尔原理

对于稳定的系统，有

$$\sum_{\alpha} \mathbf{F}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} = 0 \tag{1}$$

将 $\mathbf{F}_{\alpha}$ 分解为外力 $\mathbf{F}_{\alpha}^{e}$ 和内力（约束力）$\mathbf{f}_{\alpha}$，其中

$$\sum_{\alpha} \mathbf{f}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} = 0 \tag{2}$$

因此有

$$\sum_{\alpha} \mathbf{F}_{\alpha}^{e} \cdot \delta \mathbf{r}_{\alpha} = 0 \tag{3}$$

当问题不是静力学问题的时候，我们添加一个惯性力使之化为静力学问题，由于 $\mathbf{F}_{\alpha} - \dot{\mathbf{p}}_{\alpha} = 0$

$$\sum_{\alpha} \left(\mathbf{F}_{\alpha}^{e} - \dot{\mathbf{p}}_{\alpha}\right) \cdot \delta \mathbf{r}_{\alpha} = 0 \tag{4}$$

这就是达朗贝尔准则

2 拉格朗日方程

2.1 广义坐标

位置 $\mathbf{r}_{\alpha}$ 可以由坐标 q_j 表示

$$\mathbf{r}_{\alpha} = \mathbf{r}_{\alpha}\left(q_j, t\right) \tag{5}$$

对它求导

$$\dot{\mathbf{r}}_{\alpha} = \frac{\mathrm{d}\mathbf{r}_{\alpha}}{\mathrm{d}t} = \sum_{j} \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_{\alpha}}{\partial t} \tag{6}$$

进而得到

$$\delta \mathbf{r}_{\alpha} = \sum_{j} \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \delta q_j \tag{7}$$

2.2 拉格朗日方程的推导

又方程 4可以得到

$$\sum_\alpha \mathbf{F}_\alpha^e \cdot \delta\mathbf{r}_\alpha = \sum_\alpha \dot{\mathbf{p}}_\alpha \cdot \delta\mathbf{r}_\alpha \tag{8}$$

对左边进行展开，代入 7

$$\sum_\alpha \mathbf{F}_\alpha^e \cdot \delta\mathbf{r}_\alpha = \sum_\alpha \mathbf{F}_\alpha^e \cdot \sum_j \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\delta q_j = \sum_j \sum_\alpha \mathbf{F}_\alpha^e \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\delta q_j \tag{9}$$

定义广义力 Q_j

$$\sum_\alpha \mathbf{F}_\alpha^e \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j} = Q_j \tag{10}$$

则式 8左边可以化为

$$\sum_\alpha \mathbf{F}_\alpha^e \cdot \delta\mathbf{r}_\alpha = \sum_j Q_j \delta q_j \tag{11}$$

下面我们对右边进行展开，同样，代入 7

$$\sum_\alpha \dot{\mathbf{p}}_\alpha \cdot \delta\mathbf{r}_\alpha = \sum_\alpha \dot{\mathbf{p}}_\alpha \cdot \sum_j \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\delta q_j = \sum_\alpha m_\alpha \ddot{\mathbf{r}}_\alpha \cdot \sum_j \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\delta q_j = \sum_\alpha \sum_j m_\alpha \ddot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\delta q_j \tag{12}$$

运用复合函数求导法则

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\right) = m_\alpha \ddot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j} + m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_j}\right) \tag{13}$$

即

$$m_\alpha \ddot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\right) - m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_j}\right) \tag{14}$$

把 14代入 12，得到

$$\sum_\alpha \dot{\mathbf{p}}_\alpha \cdot \delta\mathbf{r}_\alpha = \sum_\alpha \sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\right) - m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_j}\right)\right]\delta q_j \tag{15}$$

因为

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_j}\right) = \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_j} \tag{16}$$

我们得到

$$\sum_\alpha \dot{\mathbf{p}}_\alpha \cdot \delta\mathbf{r}_\alpha = \sum_\alpha \sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\right) - m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_j}\right]\delta q_j \tag{17}$$

对 6求偏导

$$\frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \tag{18}$$

带入 17

$$\sum_\alpha \dot{\mathbf{p}}_\alpha \cdot \delta\mathbf{r}_\alpha = \sum_\alpha \sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_j}\right) - m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_j}\right]\delta q_j \tag{19}$$

将 11和 19代入 8

$$\sum_\alpha \sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_j}\right) - m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_j}\right]\delta q_j = \sum_j Q_j \delta q_j \tag{20}$$

定义动能 $T=\sum\limits_{\alpha}\frac{1}{2}m_\alpha\dot{\mathbf{r}}_\alpha^2$

$$\partial T=\partial\left(\sum_{\alpha}\frac{1}{2}m_\alpha\dot{\mathbf{r}}_\alpha^2\right)=\sum_{\alpha}m_\alpha\dot{\mathbf{r}}_\alpha\cdot\partial\dot{\mathbf{r}}_\alpha \tag{21}$$

将 21代入 20

$$\sum_j\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_j}\right)-\frac{\partial T}{\partial q_j}\right]\delta q_j=\sum_j Q_j\delta q_j \tag{22}$$

因此

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=Q_j \tag{23}$$

即

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_j}\right)-\left(\frac{\partial T}{\partial q_j}+Q_j\right)=0 \tag{24}$$

当 Q_j 是保守力，势能为 U 时

$$-\frac{\partial U}{\partial q_j}=Q_j \tag{25}$$

$$\frac{\partial U}{\partial\dot{q}_j}=0 \tag{26}$$

方程 24可以化为

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(T-U)}{\partial\dot{q}_j}\right)-\frac{\partial(T-U)}{\partial q_j}=0 \tag{27}$$

定义 $L=T-U$，则

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0 \tag{28}$$

3 哈密顿量的守恒

对 $L(q,\dot{q},t)$ 求导

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\sum_j\frac{\partial L}{\partial q_j}\dot{q}_j+\sum_j\frac{\partial L}{\partial\dot{q}_j}\ddot{q}_j+\frac{\partial L}{\partial t} \tag{29}$$

式 28告诉我们

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_j}\right)=\frac{\partial L}{\partial q_j} \tag{30}$$

将 30代入 29

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\sum_j\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_j}\right)\dot{q}_j+\sum_j\frac{\partial L}{\partial\dot{q}_j}\ddot{q}_j+\frac{\partial L}{\partial t} \tag{31}$$

即

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\sum_j\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_j}\dot{q}_j\right)+\frac{\partial L}{\partial t} \tag{32}$$

交换求和与微分顺序

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j\frac{\partial L}{\partial\dot{q}_j}\dot{q}_j\right)+\frac{\partial L}{\partial t} \tag{33}$$

移项

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j\frac{\partial L}{\partial\dot{q}_j}\dot{q}_j-L\right)+\frac{\partial L}{\partial t}=0 \tag{34}$$

定义系统的哈密顿量

$$H=\sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j-L \tag{35}$$

当 $\frac{\partial L}{\partial t}=0$ 时

$$\frac{\mathrm{d}H}{\mathrm{d}t}=0 \tag{36}$$

哈密顿量守恒

4 Noether 定理与广义动量

由 33

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right)+\frac{\partial L}{\partial t} \tag{37}$$

可以得到

$$\delta L=\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j \frac{\partial L}{\partial \dot{q}_j}\delta q_j\right) \tag{38}$$

当空间具有平移不变性时，即虚位移的产生对于拉格朗日量没有影响时

$$\delta L=0 \tag{39}$$

因此

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j \frac{\partial L}{\partial \dot{q}_j}\delta q_j\right)=0 \tag{40}$$

即

$$\sum_j \frac{\partial L}{\partial \dot{q}_j}\delta q_j=\mathrm{const} \tag{41}$$

因此定义广义动量

$$p_j=\frac{\partial L}{\partial \dot{q}_j} \tag{42}$$

则 $p_j=\mathrm{const}$，广义动量守恒

5 欧拉变分原理

设

$$J=\int_{x_1}^{x_2} f\left\{y(x),y'(x);x\right\}\mathrm{d}x \tag{43}$$

J 是函数 $y(x)$ 的函数，求 J 的极值函数，方法是另

$$y\left(\alpha,x\right)=y(x)+\alpha\eta(x) \tag{44}$$

则

$$y'\left(\alpha,x\right)=y'(x)+\alpha\eta'(x) \tag{45}$$

$$\frac{\partial y}{\partial \alpha}=\eta(x) \tag{46}$$

$$\frac{\partial y'}{\partial \alpha} = \eta'(x) \tag{47}$$

求解

$$\frac{\partial J}{\partial \alpha} = 0 \tag{48}$$

解法为

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_{x_1}^{x_2} f\left\{y(x), y'(x); x\right\} \mathrm{d}x = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial \alpha} \right) \mathrm{d}x \tag{49}$$

将 45和 46代入

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x) \right) \mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \eta(x) \, \mathrm{d}x + \int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \eta'(x) \, \mathrm{d}x \tag{50}$$

利用分步积分法算式 50的最后一项

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \eta'(x) \, \mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \, \mathrm{d}\eta(x) = \frac{\partial f}{\partial y'} \left[\eta(x_1) - \eta(x_2) \right] - \int_{x_1}^{x_2} \eta(x) \, \mathrm{d} \left(\frac{\partial f}{\partial y'} \right) \tag{51}$$

由于式 44中 $y(\alpha, x_1) = y(x_1)$ 且 $y(\alpha, x_2) = y(x_2)$

$$\eta(x_1) = \eta(x_2) = 0 \tag{52}$$

代入 51

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \eta'(x) \, \mathrm{d}x = - \int_{x_1}^{x_2} \eta(x) \, \mathrm{d} \left(\frac{\partial f}{\partial y'} \right) = - \int_{x_1}^{x_2} \eta(x) \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) \mathrm{d}x \tag{53}$$

代入 50

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \eta(x) \, \mathrm{d}x - \int_{x_1}^{x_2} \eta(x) \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) \mathrm{d}x = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) \right] \eta(x) \, \mathrm{d}x = 0 \tag{54}$$

因此

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) = 0 \tag{55}$$

我们也可以用 δ 记号表示，由式 54和 46

$$\frac{\partial J}{\partial \alpha} \mathrm{d}\alpha = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) \right] \frac{\partial y}{\partial \alpha} \mathrm{d}\alpha \, \mathrm{d}x = 0 \tag{56}$$

即

$$\delta J = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) \right] \delta y \, \mathrm{d}x = 0 \tag{57}$$

6 哈密顿原理

哈密顿原理的表示形式为

$$I = \int_{t_1}^{t_2} L\left(q_j, \dot{q}_j, t\right) \mathrm{d}t = 0 \tag{58}$$

或

$$\delta I = \delta \int_{t_1}^{t_2} L\left(q_j, \dot{q}_j, t\right) \mathrm{d}t = 0 \tag{59}$$

由式 55我们已经知道，式 59的解是

$$\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_j} \right) = 0 \tag{60}$$

即拉格朗日方程 28

7 拉格朗日乘子与约束条件

7.1 带约束的欧拉变分原理

有两个关于 x 的函数 $y(x)$、$z(x)$，求

$$J=\int_{x_1}^{x_2} f\left\{y(x),y'(x),z(x),z'(x);x\right\} \tag{61}$$

的极值，有约束

$$g\left\{y(x),z(x);x\right\}=0 \tag{62}$$

由 61

$$\delta J=\int_{x_1}^{x_2}\left[\left(\frac{\partial f}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\delta y+\left(\frac{\partial f}{\partial z}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'}\right)\delta z\right]\mathrm{d}x=0 \tag{63}$$

$$\delta g=\frac{\partial g}{\partial y}\delta y+\frac{\partial g}{\partial z}\delta z=0 \tag{64}$$

引入拉格朗日乘子 $\lambda(x)$

$$\lambda(x)\delta g=\lambda(x)\left(\frac{\partial g}{\partial y}\delta y+\frac{\partial g}{\partial z}\delta z\right)=0 \tag{65}$$

将 65代入 63

$$\int_{x_1}^{x_2}\left[\left(\frac{\partial f}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}+\lambda(x)\frac{\partial g}{\partial y}\right)\delta y+\left(\frac{\partial f}{\partial z}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'}+\lambda(x)\frac{\partial g}{\partial z}\right)\delta z\right]\mathrm{d}x=0 \tag{66}$$

因此，我们有

$$\frac{\partial f}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}+\lambda(x)\frac{\partial g}{\partial y}=0 \tag{67}$$

和

$$\frac{\partial f}{\partial z}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'}+\lambda(x)\frac{\partial g}{\partial z}=0 \tag{68}$$

7.2 带不定乘子的拉格朗日方程

考察二维平面上的运动，拉格朗日量

$$L\left(x,\dot{x},y,\dot{y};t\right) \tag{69}$$

x 和 y 之间有约束

$$g\left(x,y;t\right)=0 \tag{70}$$

由 67和 68可知

$$\frac{\partial L}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}+\lambda(t)\frac{\partial g}{\partial x}=0 \tag{71}$$

$$\frac{\partial L}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}}+\lambda(t)\frac{\partial g}{\partial y}=0 \tag{72}$$

其中

$$Q_x=\lambda(t)\frac{\partial g}{\partial x} \tag{73}$$

$$Q_y=\lambda(t)\frac{\partial g}{\partial y} \tag{74}$$

为约束力

8 哈密顿动力学

8.1 广义动量

由广义动量的定义 42和拉格朗日方程 28

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \tag{75}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \tag{76}$$

得到

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \tag{77}$$

进而得到

$$\dot{p}_j = \frac{\partial L}{\partial q_j} \tag{78}$$

8.2 哈密顿正则方程

由 35，系统的哈密顿量

$$H\left(p_j, q_j, t\right) = \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L \tag{79}$$

将 75代入

$$H = \sum_j p_j \dot{q}_j - L\left(q_j, \dot{q}_j, t\right) \tag{80}$$

对 80求全微分

$$\mathrm{d}H = \sum_j \left(\dot{q}_j \, \mathrm{d}p_j + p_j \, \mathrm{d}\dot{q}_j - \frac{\partial L}{\partial q_j}\, \mathrm{d}q_j - \frac{\partial L}{\partial \dot{q}_j}\, \mathrm{d}\dot{q}_j\right) - \frac{\partial L}{\partial t}\, \mathrm{d}t \tag{81}$$

将 75和 78带入

$$\mathrm{d}H = \sum_j \left(\dot{q}_j \, \mathrm{d}p_j + p_j \, \mathrm{d}\dot{q}_j - \dot{p}_j \, \mathrm{d}q_j - p_j \, \mathrm{d}\dot{q}_j\right) - \frac{\partial L}{\partial t}\, \mathrm{d}t \tag{82}$$

整理得到

$$\mathrm{d}H = \sum_j \left(\dot{q}_j \, \mathrm{d}p_j - \dot{p}_j \, \mathrm{d}q_j\right) - \frac{\partial L}{\partial t}\, \mathrm{d}t \tag{83}$$

又因为

$$\mathrm{d}H = \sum_j \left(\frac{\partial H}{\partial p_j}\, \mathrm{d}p_j + \frac{\partial H}{\partial q_j}\, \mathrm{d}q_j\right) + \frac{\partial H}{\partial t}\, \mathrm{d}t \tag{84}$$

因此我们得到哈密顿正则方程

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \tag{85}$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} \tag{86}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{87}$$

8.3 泊松括号

$$\{u,v\} = \sum_j \left(\frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_j} \right) \tag{88}$$

用法

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \sum_j \left(\frac{\partial u}{\partial q_j}\dot{q}_j + \frac{\partial u}{\partial p_j}\dot{p}_j \right) + \frac{\partial u}{\partial t} = \sum_j \left(\frac{\partial u}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial H}{\partial q_j} \right) + \frac{\partial u}{\partial t} \tag{89}$$

可以简写为

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \{u, H\} + \frac{\partial u}{\partial t} \tag{90}$$

9 有心运动

9.1 基本概念

考虑两个质点的系统，系统的质心是 R，定义等效质量 μ

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{91}$$

则系统的动能为

$$T = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 \tag{92}$$

系统的势能为 $U(r)$，则

$$L = T - U = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r) \tag{93}$$

因为有

$$(m_1 + m_2)\ddot{\mathbf{R}} = 0 \tag{94}$$

式 93的第一项可以丢弃

$$L = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r) \tag{95}$$

用极坐标表示

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r) \tag{96}$$

广义角动量为

$$l = p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{const} \tag{97}$$

面积速度为

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\,\mathrm{d}\theta = \frac{l}{2\mu} = \text{const} \tag{98}$$

总能量为

$$E = T + U = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r) = \sqrt{\frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)} \tag{99}$$

9.2 运动方程的求解

9.2.1 第一种办法

可以用式 97和式 99解出 $\dot{\theta}$ 和 $\dot{r}$

$$\dot{\theta} = \frac{l}{\mu r^2} \tag{100}$$

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}\left[E - U(r)\right] - \frac{l^2}{\mu^2 r^2}} \tag{101}$$

并用

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\dot{\theta}}{\dot{r}} \tag{102}$$

并积分解出

$$\theta(r) = \pm\int \frac{\dot{\theta}}{\dot{r}}\,\mathrm{d}r + \mathrm{const} = \pm\int \frac{\frac{l}{\mu r^2}}{\sqrt{\frac{2}{\mu}\left[E - U(r)\right] - \frac{l^2}{\mu^2 r^2}}}\,\mathrm{d}r + \mathrm{const} \tag{103}$$

9.2.2 第二种解法

拉格朗日方程

$$\frac{\partial L}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} = 0 \tag{104}$$

将式 96代入

$$\mu r\dot{\theta}^2 - \frac{\partial U}{\partial r} - \mu\ddot{r} = 0 \tag{105}$$

整理得到

$$-\mu r\dot{\theta}^2 + \mu\ddot{r} = -\frac{\partial U}{\partial r} \tag{106}$$

$$\mu\left(\ddot{r} - r\dot{\theta}^2\right) = F(r) \tag{107}$$

换元，另

$$u = \frac{1}{r} \tag{108}$$

则

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} = -\frac{\mu^2}{l^2}r^2\ddot{r} \tag{109}$$

$$\ddot{r} = -\frac{1}{r^2}\frac{l^2}{\mu^2}\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} = -u^2\frac{l^2}{\mu^2}\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} \tag{110}$$

结合

$$l = \mu r^2\dot{\theta} \tag{111}$$

即

$$\dot{\theta}^2 = \frac{l^2}{\mu^2 r^4} \tag{112}$$

将 112与 110代入 107

$$\mu\left(-u^2\frac{l^2}{\mu^2}\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} - r\frac{l^2}{\mu^2 r^4}\right) = F(r) \tag{113}$$

$$\left(-u^2\frac{l^2}{\mu}\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} - r\frac{l^2}{\mu r^4}\right) = F(r) \tag{114}$$

$$\left(-u^2\frac{l^2}{\mu}\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}-\frac{l^2u^3}{\mu}\right)=F(r) \tag{115}$$

$$\left(\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+u\right)=-\frac{\mu}{u^2l^2}F(r) \tag{116}$$

即

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{\mu r^2}{l^2}F(r) \tag{117}$$

9.3 有效势能

由式 103分母，

$$\sqrt{\frac{2}{\mu}\left[E-U(r)\right]-\frac{l^2}{\mu^2r^2}}=\sqrt{\frac{2}{\mu}\left[E-\left(U(r)+\frac{l^2}{2\mu r^2}\right)\right]} \tag{118}$$

定义有效势能 $V(r)$

$$V(r)=U(r)+\frac{l^2}{2\mu r^2} \tag{119}$$

离心势能

$$U_C(r)=\frac{l^2}{2\mu r^2} \tag{120}$$

10 刚体的运动

转动惯量的张量定义为

$$\mathbf{I}=\begin{bmatrix}\sum\limits_\alpha m_\alpha\left(x_{\alpha,2}x_{\alpha,3}\right) & -\sum\limits_\alpha m_\alpha x_{\alpha,1}x_{\alpha,2} & -\sum\limits_\alpha m_\alpha x_{\alpha,1}x_{\alpha,3}\\ -\sum\limits_\alpha m_\alpha x_{\alpha,2}x_{\alpha,1} & \sum\limits_\alpha m_\alpha\left(x_{\alpha,1}x_{\alpha,3}\right) & -\sum\limits_\alpha m_\alpha x_{\alpha,2}x_{\alpha,3}\\ -\sum\limits_\alpha m_\alpha x_{\alpha,3}x_{\alpha,1} & -\sum\limits_\alpha m_\alpha x_{\alpha,3}x_{\alpha,2} & \sum\limits_\alpha m_\alpha\left(x_{\alpha,1}x_{\alpha,2}\right)\end{bmatrix} \tag{121}$$

定义

$$\mathbf{I}=\begin{bmatrix}I_{11} & I_{12} & I_{13}\\ I_{21} & I_{22} & I_{23}\\ I_{31} & I_{32} & I_{33}\end{bmatrix} \tag{122}$$

刚体的转动动能

$$T_{rot}=\frac{1}{2}\omega'\mathbf{I}\omega \tag{123}$$

这里面的 ω 是列向量，刚体的角动量

$$\mathbf{L}=\mathbf{I}\omega \tag{124}$$

$\mathbf{I}$ 是对称的，可以相似对角化相似对角化后的转动惯量

$$\mathbf{I}'=\lambda'\mathbf{I}\lambda \tag{125}$$

$$\omega'=\lambda'\omega \tag{126}$$