理论力学复习

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2019 年 12 月 31 日

1 达朗贝尔原理

对于稳定的系统，有

$$\sum_{\alpha} \mathbf{F}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} = 0 \tag{1}$$

将 $\mathbf{F}_{\alpha}$ 分解为外力 $\mathbf{F}_{\alpha}^{e}$ 和内力（约束力）$\mathbf{f}_{\alpha}$，其中

$$\sum_{\alpha} \mathbf{f}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} = 0 \tag{2}$$

因此有

$$\sum_{\alpha} \mathbf{F}_{\alpha}^{e} \cdot \delta \mathbf{r}_{\alpha} = 0 \tag{3}$$

当问题不是静力学问题的时候，我们添加一个惯性力使之化为静力学问题，由于 $\mathbf{F}_{\alpha} - \dot{\mathbf{p}}_{\alpha} = 0$

$$\sum_{\alpha} \left(\mathbf{F}_{\alpha}^{e} - \dot{\mathbf{p}}_{\alpha} \right) \cdot \delta \mathbf{r}_{\alpha} = 0 \tag{4}$$

这就是达朗贝尔准则

2 拉格朗日方程

2.1 广义坐标

位置 $\mathbf{r}_{\alpha}$ 可以由坐标 q_j 表示

$$\mathbf{r}_{\alpha} = \mathbf{r}_{\alpha} \left(q_j, t \right) \tag{5}$$

对它求导

$$\dot{\mathbf{r}}_{\alpha} = \frac{\mathrm{d}\mathbf{r}_{\alpha}}{\mathrm{d}t} = \sum_{j} \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_{\alpha}}{\partial t} \tag{6}$$

进而得到

$$\delta \mathbf{r}_{\alpha} = \sum_{j} \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \delta q_j \tag{7}$$

2.2 拉格朗日方程的推导

又方程 4可以得到

$$\sum_{\alpha}\mathbf{F}_{\alpha}^{e}\cdot\delta\mathbf{r}_{\alpha}=\sum_{\alpha}\dot{\mathbf{p}}_{\alpha}\cdot\delta\mathbf{r}_{\alpha} \tag{8}$$

对左边进行展开，代入 7

$$\sum_{\alpha}\mathbf{F}_{\alpha}^{e}\cdot\delta\mathbf{r}_{\alpha}=\sum_{\alpha}\mathbf{F}_{\alpha}^{e}\cdot\sum_{j}\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\delta q_j=\sum_{j}\sum_{\alpha}\mathbf{F}_{\alpha}^{e}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\delta q_j \tag{9}$$

定义广义力 Q_j

$$\sum_{\alpha}\mathbf{F}_{\alpha}^{e}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}=Q_j \tag{10}$$

则式 8左边可以化为

$$\sum_{\alpha}\mathbf{F}_{\alpha}^{e}\cdot\delta\mathbf{r}_{\alpha}=\sum_{j}Q_j\delta q_j \tag{11}$$

下面我们对右边进行展开，同样，代入 7

$$\sum_{\alpha}\dot{\mathbf{p}}_{\alpha}\cdot\delta\mathbf{r}_{\alpha}=\sum_{\alpha}\dot{\mathbf{p}}_{\alpha}\cdot\sum_{j}\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\delta q_j=\sum_{\alpha}m_{\alpha}\ddot{\mathbf{r}}_{\alpha}\cdot\sum_{j}\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\delta q_j=\sum_{\alpha}\sum_{j}m_{\alpha}\ddot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\delta q_j \tag{12}$$

运用复合函数求导法则

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\right)=m_{\alpha}\ddot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}+m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\right) \tag{13}$$

即

$$m_{\alpha}\ddot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}=\frac{\mathrm{d}}{\mathrm{d}t}\left(m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\right)-m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\right) \tag{14}$$

把 14代入 12，得到

$$\sum_{\alpha}\dot{\mathbf{p}}_{\alpha}\cdot\delta\mathbf{r}_{\alpha}=\sum_{\alpha}\sum_{j}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\right)-m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\right)\right]\delta q_j \tag{15}$$

因为

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\right)=\frac{\partial\dot{\mathbf{r}}_{\alpha}}{\partial q_j} \tag{16}$$

我们得到

$$\sum_{\alpha}\dot{\mathbf{p}}_{\alpha}\cdot\delta\mathbf{r}_{\alpha}=\sum_{\alpha}\sum_{j}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j}\right)-m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\dot{\mathbf{r}}_{\alpha}}{\partial q_j}\right]\delta q_j \tag{17}$$

对 6求偏导

$$\frac{\partial\dot{\mathbf{r}}_{\alpha}}{\partial\dot{q}_j}=\frac{\partial\mathbf{r}_{\alpha}}{\partial q_j} \tag{18}$$

带入 17

$$\sum_{\alpha}\dot{\mathbf{p}}_{\alpha}\cdot\delta\mathbf{r}_{\alpha}=\sum_{\alpha}\sum_{j}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\dot{\mathbf{r}}_{\alpha}}{\partial\dot{q}_j}\right)-m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\dot{\mathbf{r}}_{\alpha}}{\partial q_j}\right]\delta q_j \tag{19}$$

将 11和 19代入 8

$$\sum_{\alpha}\sum_{j}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\dot{\mathbf{r}}_{\alpha}}{\partial\dot{q}_j}\right)-m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\frac{\partial\dot{\mathbf{r}}_{\alpha}}{\partial q_j}\right]\delta q_j=\sum_{j}Q_j\delta q_j \tag{20}$$

定义动能 $T=\sum\limits_{\alpha}\frac{1}{2}m_{\alpha}\dot{\mathbf{r}}_{\alpha}^{2}$

$$\partial T=\partial\left(\sum_{\alpha}\frac{1}{2}m_{\alpha}\dot{\mathbf{r}}_{\alpha}^{2}\right)=\sum_{\alpha}m_{\alpha}\dot{\mathbf{r}}_{\alpha}\cdot\partial\dot{\mathbf{r}}_{\alpha} \tag{21}$$

将 21代入 20

$$\sum_{j}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_{j}}\right)-\frac{\partial T}{\partial q_{j}}\right]\delta q_{j}=\sum_{j}Q_{j}\delta q_{j} \tag{22}$$

因此

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_{j}}\right)-\frac{\partial T}{\partial q_{j}}=Q_{j} \tag{23}$$

即

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_{j}}\right)-\left(\frac{\partial T}{\partial q_{j}}+Q_{j}\right)=0 \tag{24}$$

当 Q_j 是保守力，势能为 U 时

$$\frac{\partial U}{\partial q_{j}}=Q_{j} \tag{25}$$

$$\frac{\partial U}{\partial\dot{q}_{j}}=0 \tag{26}$$

方程 24可以化为

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(T+U)}{\partial\dot{q}_{j}}\right)-\frac{\partial(T+U)}{\partial q_{j}}=0 \tag{27}$$

定义 $L=T+U$，则

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_{j}}\right)-\frac{\partial L}{\partial q_{j}}=0 \tag{28}$$