理论力学复习

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1　达朗贝尔原理

对于稳定的系统，有

$$\sum_{\alpha} \mathbf{F}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} = 0 \tag{1}$$

将 $\mathbf{F}_{\alpha}$ 分解为外力 $\mathbf{F}_{\alpha}^{e}$ 和内力（约束力）$\mathbf{f}_{\alpha}$，其中

$$\sum_{\alpha} \mathbf{f}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} = 0 \tag{2}$$

因此有

$$\sum_{\alpha} \mathbf{F}_{\alpha}^{e} \cdot \delta \mathbf{r}_{\alpha} = 0 \tag{3}$$

当问题不是静力学问题的时候，我们添加一个惯性力使之化为静力学问题，由于 $\mathbf{F}_{\alpha} - \dot{\mathbf{p}}_{\alpha} = 0$

$$\sum_{\alpha} \left(\mathbf{F}_{\alpha}^{e} - \dot{\mathbf{p}}_{\alpha} \right) \cdot \delta \mathbf{r}_{\alpha} = 0 \tag{4}$$

这就是达朗贝尔准则

2　拉格朗日方程

2.1　广义坐标

位置 $\mathbf{r}_{\alpha}$ 可以由坐标 q_j 表示

$$\mathbf{r}_{\alpha} = \mathbf{r}_{\alpha} \left(q_j, t \right) \tag{5}$$

对它求导

$$\dot{\mathbf{r}}_{\alpha} = \frac{\mathrm{d}\mathbf{r}_{\alpha}}{\mathrm{d}t} = \sum_{j} \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_{\alpha}}{\partial t} \tag{6}$$

进而得到

$$\delta \mathbf{r}_{\alpha} = \sum_{j} \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \delta q_j \tag{7}$$

2.2 拉格朗日方程的推导

又方程 4可以得到

$$\sum_{\alpha} \mathbf{F}_{\alpha}^{e} \cdot \delta \mathbf{r}_{\alpha} = \sum_{\alpha} \dot{\mathbf{p}}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} \tag{8}$$

对左边进行展开，代入 7

$$\sum_{\alpha} \mathbf{F}_{\alpha}^{e} \cdot \delta \mathbf{r}_{\alpha} = \sum_{\alpha} \mathbf{F}_{\alpha}^{e} \cdot \sum_{j} \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \delta q_j = \sum_{j} \sum_{\alpha} \mathbf{F}_{\alpha}^{e} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \delta q_j \tag{9}$$

定义广义力 Q_j

$$\sum_{\alpha} \mathbf{F}_{\alpha}^{e} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} = Q_j \tag{10}$$

则式 8左边可以化为

$$\sum_{\alpha} \mathbf{F}_{\alpha}^{e} \cdot \delta \mathbf{r}_{\alpha} = \sum_{j} Q_j \delta q_j \tag{11}$$

下面我们对右边进行展开，同样，代入 7

$$\sum_{\alpha} \dot{\mathbf{p}}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} = \sum_{\alpha} \dot{\mathbf{p}}_{\alpha} \cdot \sum_{j} \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \delta q_j = \sum_{\alpha} m_{\alpha} \ddot{\mathbf{r}}_{\alpha} \cdot \sum_{j} \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \delta q_j = \sum_{\alpha} \sum_{j} m_{\alpha} \ddot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \delta q_j \tag{12}$$

运用复合函数求导法则

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) = m_{\alpha} \ddot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} + m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) \tag{13}$$

即

$$m_{\alpha} \ddot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} = \frac{\mathrm{d}}{\mathrm{d}t} \left(m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) - m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) \tag{14}$$

把 14代入 12，得到

$$\sum_{\alpha} \dot{\mathbf{p}}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} = \sum_{\alpha} \sum_{j} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) - m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) \right] \delta q_j \tag{15}$$

因为

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) = \frac{\partial \dot{\mathbf{r}}_{\alpha}}{\partial q_j} \tag{16}$$

我们得到

$$\sum_{\alpha} \dot{\mathbf{p}}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} = \sum_{\alpha} \sum_{j} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \right) - m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \dot{\mathbf{r}}_{\alpha}}{\partial q_j} \right] \delta q_j \tag{17}$$

对 6求偏导

$$\frac{\partial \dot{\mathbf{r}}_{\alpha}}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_{\alpha}}{\partial q_j} \tag{18}$$

带入 17

$$\sum_{\alpha} \dot{\mathbf{p}}_{\alpha} \cdot \delta \mathbf{r}_{\alpha} = \sum_{\alpha} \sum_{j} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \dot{\mathbf{r}}_{\alpha}}{\partial \dot{q}_j} \right) - m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \dot{\mathbf{r}}_{\alpha}}{\partial q_j} \right] \delta q_j \tag{19}$$

将 11和 19代入 8

$$\sum_{\alpha} \sum_{j} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \dot{\mathbf{r}}_{\alpha}}{\partial \dot{q}_j} \right) - m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \dot{\mathbf{r}}_{\alpha}}{\partial q_j} \right] \delta q_j = \sum_{j} Q_j \delta q_j \tag{20}$$

定义动能 $T=\sum\limits_{\alpha}\frac{1}{2}m_\alpha\dot{\mathbf{r}}_\alpha^2$

$$\partial T=\partial\left(\sum_{\alpha}\frac{1}{2}m_\alpha\dot{\mathbf{r}}_\alpha^2\right)=\sum_{\alpha}m_\alpha\dot{\mathbf{r}}_\alpha\cdot\partial\dot{\mathbf{r}}_\alpha \tag{21}$$

将 21代入 20

$$\sum_j\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_j}\right)-\frac{\partial T}{\partial q_j}\right]\delta q_j=\sum_j Q_j\delta q_j \tag{22}$$

因此

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=Q_j \tag{23}$$

即

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_j}\right)-\left(\frac{\partial T}{\partial q_j}+Q_j\right)=0 \tag{24}$$

当 Q_j 是保守力，势能为 U 时

$$-\frac{\partial U}{\partial q_j}=Q_j \tag{25}$$

$$\frac{\partial U}{\partial\dot{q}_j}=0 \tag{26}$$

方程 24可以化为

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(T-U)}{\partial\dot{q}_j}\right)-\frac{\partial(T-U)}{\partial q_j}=0 \tag{27}$$

定义 $L=T-U$，则

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0 \tag{28}$$

3 哈密顿量的守恒

对 $L(q,\dot{q},t)$ 求导

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\sum_j\frac{\partial L}{\partial q_j}\dot{q}_j+\sum_j\frac{\partial L}{\partial\dot{q}_j}\ddot{q}_j+\frac{\partial L}{\partial t} \tag{29}$$

式 28告诉我们

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_j}\right)=\frac{\partial L}{\partial q_j} \tag{30}$$

将 30代入 29

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\sum_j\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_j}\right)\dot{q}_j+\sum_j\frac{\partial L}{\partial\dot{q}_j}\ddot{q}_j+\frac{\partial L}{\partial t} \tag{31}$$

即

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\sum_j\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_j}\dot{q}_j\right)+\frac{\partial L}{\partial t} \tag{32}$$

交换求和与积分顺序

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j\frac{\partial L}{\partial\dot{q}_j}\dot{q}_j\right)+\frac{\partial L}{\partial t} \tag{33}$$

移项

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j\frac{\partial L}{\partial\dot{q}_j}\dot{q}_j-L\right)+\frac{\partial L}{\partial t}=0 \tag{34}$$

定义系统的哈密顿量

$$H = \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L \tag{35}$$

当 $\frac{\partial L}{\partial t} = 0$ 时

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \tag{36}$$

哈密顿量守恒