Transformation of Inertia Tensor

Xiping Hu

XipingHu@hust.edu.cn
https://github.com/XipingHu

School of Physics
Huazhong University of Scienec and Technology

January 2, 2020

Outline

Introduction to the Inertia Tensor

The form of Inertia Tensor

We write the Inertia Tensor as:

$$I = \begin{bmatrix} \sum_\alpha m_\alpha \left(x_{\alpha,2}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha x_{\alpha,1} x_{\alpha,2} & \sum_\alpha m_\alpha x_{\alpha,1} x_{\alpha,3} \\ -\sum_\alpha m_\alpha x_{\alpha,2} x_{\alpha,3} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,3}^2\right) & \sum_\alpha m_\alpha x_{\alpha,2} x_{\alpha,3} \\ -\sum_\alpha m_\alpha x_{\alpha,3} x_{\alpha,1} & -\sum_\alpha m_\alpha x_{\alpha,3} x_{\alpha,2} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,2}^2\right) \end{bmatrix}$$

$$:= \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

Which is symmetric since

$$I_{ij} = I_{ji}$$

Introduction to the Inertia Tensor

The Rotational Kinetic Energy

$$T_{rot} = \sum_{i,j} \frac{1}{2}\omega_i I_{ij} \omega_j = \frac{1}{2}\boldsymbol{\omega}' \boldsymbol{I}\boldsymbol{\omega}$$

To make life much easier, we may find an axis in which the cross terms vanish.

Diagonalization of symmetric matrix

Some facts on Symmetric Matrices

Theorem: Any symmetric matrix

1. has only real eigenvalues
2. is always diagonalizable
3. has orthogonal eigenvectors

Diagonalization of symmetric matrix

Find eigenvalue and eigenvectors

Example (p113):

$$I = \begin{bmatrix} \frac{2}{3}\beta & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & \frac{2}{3}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta \end{bmatrix}$$

Our aim: Find I^*, which is diagonal and similar with I

Solve

$$|I - \lambda E = 0|$$

Whereas E represents the Elementary Matrix

Diagonalization of symmetric matrix

$$\begin{vmatrix} \frac{2}{3}\beta-\lambda & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & \frac{2}{3}\beta-\lambda & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta-\lambda \end{vmatrix} = 0 \tag{1}$$

Equation 1 can be simplified as:

$$\left(\frac{11}{12}\beta-\lambda\right)\left(\frac{11}{12}\beta-\lambda\right)\left(\frac{1}{6}\beta-\lambda\right)$$

The eigenvalues of $\boldsymbol{I}$ are

$$\lambda_1 = \lambda_2 = \frac{11}{12}\beta$$

$$\lambda_3 = \frac{1}{6}\beta$$

Diagonalization of symmetric matrix

To find eigenvectors, insert λ_i into $(\boldsymbol{I} - \lambda_i \boldsymbol{E})\,\boldsymbol{\omega} = 0$

for $\lambda_1 = \lambda_2 = \frac{11}{12}\beta$

$$\begin{bmatrix} \frac{2}{3}\beta - \frac{11}{12}\beta & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & \frac{2}{3}\beta - \frac{11}{12}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta - \frac{11}{12}\beta \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = 0$$

$$\omega_1 + \omega_2 + \omega_3 = 0$$

$$\omega_1 + \omega_2 + \omega_3 = 0$$

$$\omega_1 + \omega_2 + \omega_3 = 0$$

$$\boldsymbol{\omega} = [1, -1, 0]' \quad \text{or} \quad \boldsymbol{\omega} = [1, 0, -1]'$$

Diagonalization of symmetric matrix

for $\lambda_3 = \frac{1}{6}\beta$

$$\begin{bmatrix} \frac{2}{3}\beta - \frac{1}{6}\beta & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & \frac{2}{3}\beta - \frac{1}{6}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta - \frac{1}{6}\beta \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = 0$$

$$-2\omega_1 + \omega_2 + \omega_3 = 0$$

$$-2\omega_1 + \omega_2 + \omega_3 = 0$$

$$-2\omega_1 + \omega_2 + \omega_3 = 0$$

$$\boldsymbol{\omega} = [1, 1, 1]'$$

Diagonalization of symmetric matrix

Gram-Schmidt process

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u}$$

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, & \mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \\
\mathbf{u}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2), & \mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \\
\mathbf{u}_3 &= \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_3), & \mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \\
&\vdots & &\vdots \\
\mathbf{u}_k &= \mathbf{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}(\mathbf{v}_k), & \mathbf{e}_k &= \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}.
\end{aligned}$$

Diagonalization of symmetric matrix

for $\lambda_3 = \frac{1}{6}\beta$

$$\boldsymbol{\omega} = \frac{1}{\sqrt{3}}[1, 1, 1]'$$

for $\lambda_1 = \lambda_2 = \frac{11}{12}\beta$

$$\boldsymbol{\omega} = \frac{1}{\sqrt{3}}\left[-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 0\right]'$$

or

$$\boldsymbol{\omega} = \frac{1}{\sqrt{3}}\left[-\sqrt{\frac{3}{2}}, -\sqrt{\frac{1}{2}}, \sqrt{2}\right]'$$

Diagonalization of symmetric matrix

$$\boldsymbol{\Lambda} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -\sqrt{\frac{3}{2}} & -\sqrt{\frac{1}{2}} \\ 1 & \sqrt{\frac{3}{2}} & -\sqrt{\frac{1}{2}} \\ 1 & 0 & \sqrt{2} \end{bmatrix}$$

$$\boldsymbol{I}^* = \boldsymbol{\Lambda}' \boldsymbol{I} \boldsymbol{\Lambda} = \begin{bmatrix} \frac{1}{6}\beta & 0 & 0 \\ 0 & \frac{11}{12}\beta & 0 \\ 0 & 0 & \frac{11}{12}\beta \end{bmatrix}$$

Diagonalization of symmetric matrix

$$\boldsymbol{\omega}'\boldsymbol{I}\boldsymbol{\omega} = \boldsymbol{\omega}'\boldsymbol{\Lambda}\boldsymbol{\Lambda}'\boldsymbol{I}\boldsymbol{\Lambda}\boldsymbol{\Lambda}'\boldsymbol{\omega} = (\boldsymbol{\Lambda}'\boldsymbol{\omega})'(\boldsymbol{\Lambda}'\boldsymbol{I}\boldsymbol{\Lambda})(\boldsymbol{\Lambda}'\boldsymbol{\omega}) = (\boldsymbol{\omega}^*)'\boldsymbol{I}^*\boldsymbol{\omega}^*$$

$$\boldsymbol{\omega}^* = \boldsymbol{\Lambda}'\boldsymbol{\omega}$$

Why Λ' not Λ ?

$\boldsymbol{\omega}$ is the base vector of coordinates.

References

References

- Gao Deng Dai Shu. Weisheng Qiu
- Classical MEchanics (Third Edition), Herbert Goldstein

This Document is placed under GFDL 1.3

Source code of this document

https://github.com/XipingHu/Transformation-of-Inertia-Tensor.git