N-out-of-M Conditional Option Modeling in Convertible Bond

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Abstract

A soft-call Convertible Bond is a vanilla convertible bond with callable feature under certain constraint, pricing such financial asset is significative for investors and financial industries. In this paper, we are use four different methods to price a specific 20-out-of-30 soft-call convertible bond, explain the methodology, and show the result based on different methods.

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1 Executive Summary

In this report we investigate potential methods to better price convertible bonds with the n out of m soft-call feature, the most popular of which is the 20 out of 30 soft calls. Given the size of the convertible bond market a growing need exists for efficient yet accurate approximations of this category of convertible bond. We implemented 3 different methods of calculating a 20-out-of 30 Convertible bond, One-touch, Auxiliary Reverse Binomial (ARB) implemented in conjunction with the Ayache PDE with credit risk, and Auxiliary State Variable implemented using a Cox-Ross-Rubenstein (CRR) Binomial Tree.

One-touch methods are widely used in industry due to their ease of implementation and attractive performance with most methods requiring only single run through the grid (PDE) or tree. The accuracy of one-touch method though seems less than stellar, the most often referenced one-touch method, the Navin Algorithm (Navin 1999) itself admits that qualitatively that it is between a few pennies and a dime on the dollar off for valuing the provisional call protection and overall, about a bond point off in terms of evaluation. Our implementation confirmed these error rates, of particular concern is the fact that the error in One-touch is most significant when the bond is about to become provisionally callable or already callable and when the stock price is close (5-10%) to the trigger. This is problematic as these conditions are generally true for convertible bonds that we would want to price.

The ARB method presented by Liu gives us another method of approximating the price, this time using a backwards 30 step tree we generate all 30 step stock paths for each stock price. Counting the number of paths which satisfy the trigger condition, the probability of the trigger can be calculated. The value of the bond at any stock price can be approximated as the average of callable and non-callable bond based on the trigger probability. Thanks to the properties of the tree (CRR) the probabilities are expressed in terms of bounds for the trigger price given stock price S and parameter u. Therefore, we do not need to repeat the computationally intensive process of generating the paths and calculating the probabilities.

Our implementation of ARB found that though it provides an improvement of around 0.05 to 0.4 of a bond point (5 to 40 cent) depending on volatility and trigger price to the one-touch approximation in general, we have found cases where ARB provides a worse estimate than the one-touch method. Given the probability table, our implementation of ARB produced a run time comparable to one-touch though undoubtedly longer.

The Auxiliary State Variable method attempts to approximate the convertible by defining an additional variable L to count the number of days the stock price is above trigger H within the last 30 days. It therefore utilizes the 3 dimensions of i, j and l to expand upon a normal CRR Binomial tree. Utilizing L we can determine at every point within the tree whether the bond is callable or not-callable and we can price accordingly. This method is quite computationally taxing as we are adding an extra dimension to the tree and calculating the transitional probability for L throughout the tree is non-trivial. This method also had a much more complicated implementation with lots of potential pitfalls and issues regarding the transitional probabilities. We were unable to produce the results that Zhang showed in his paper, instead we found in our implementation that the approximations for this method was generally better than one-touch at stock prices close to the trigger while it had similar errors if not larger elsewhere. Surprisingly our implementation of the method had a computation time of around 7 minutes, which is quick and comparable to one-touch. Given the performance, further investigation of this method may prove fruitful for usage if Zhang's results of errors less than 50bp can be reproduced.

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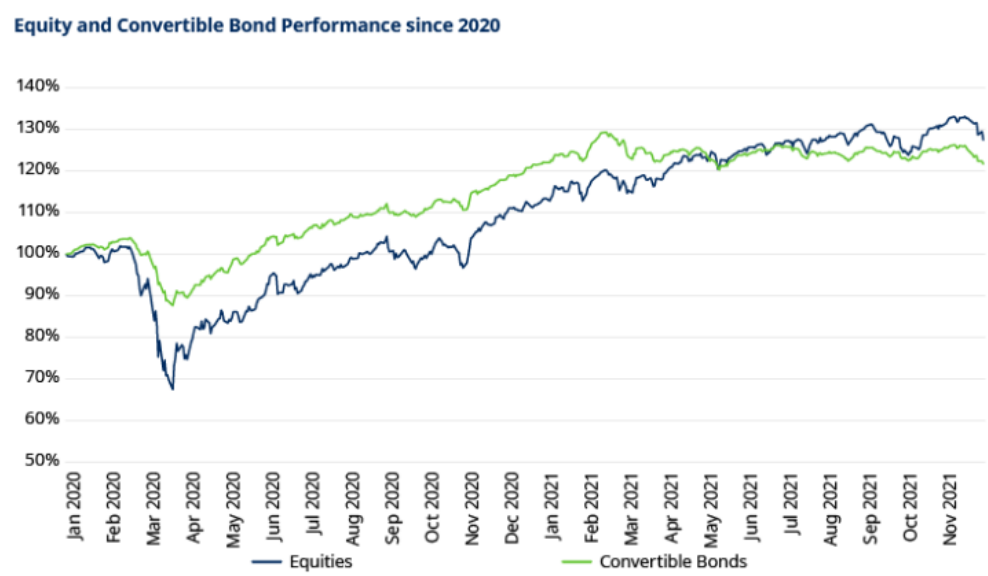
2 Introduction

2.1 Overview

Convertible bond is a fixed-income debt securities which allows the investors to convert the bond into common stocks in a predetermined number of shares, and the conversion can only happen in certain times before the bond matures. This feature makes the price of convertible bonds sensitive to both interest rate and underlying stock price, and also makes this type of hybrid security a very complicated financial instrument. Issuing convertible bonds can help the issuer to pay a lower interest rate and borrowing costs than they would have had they issued regular bonds. In addition, there would be less conflict of interest between shareholders and creditors. On the other hand, convertible bonds help investors to protect their principle if the market turns against them, while also helping them to enjoy the upside of the underlying stock price going up. Over the past decades, the convertible bonds market developed rapidly and it can be predicted that convertible bonds will be more popular in the next few years. It is of great significance to conduct research on convertible bonds at this stage.



Source: Bloomberg as of 30 November 2021.

In addition to the most basic conversion rights, modern convertible bonds usually have additional terms. These additional terms actually embed a variety of options for convertible bonds, making convertible bonds a very complicated financial derivative product.

The special terms and conditions of convertible bonds are designed to satisfy both issuers and investors. First of all, for issuers, convertible bonds can avoid the strong performance pressure brought by equity financing. If convertible bonds are successfully issued and all of them are converted into shares, it is actually equivalent to the issuer issuing new shares of stocks at the conversion price. The difference is that, the issuance of new stock shares will include the newly issued shares in the shareholders' equity immediately, after the new stock shares are issued, the earnings per share will be diluted immediately, and after the raised capital are invested in new projects, it often takes a while to get the benefits. During this period, the company will face great performance pressure. On the other hand, the issuance of convertible bonds is within a period of time, and as convertible bond holders start to convert the bonds into shares, the raised capital will be gradually included in the shareholders' equity.

Secondly, when the issuer thinks that the stock price of the company is underperforming the market but still wants to raise capital, they can use convertible bonds to finance. To be more specific, when the issuer believes that the company will achieve a better performance in the future and the company's stock price will rise sharply over the next period of time, they can issue convertible bonds at the expected future stock price, and wait until the stock price goes up before investors start to convert shares, so as to reduce the cost of financing. Moreover, the coupon rate of convertible bonds is relatively low, which will prevent the company from financial crisis.

Thirdly, the investment from convertible bond investors can reduce the conflict of interests with the company's shareholders. The value of the company is equal to the market value of the company's equity value plus the company's debt value, therefore the company's equity investors tend to carry out high-risk business activities and increase the company's risk in order to reduce the market value of the company's debt and increase the market value of equity, which is harmful to the bond investors. Convertible bonds can share in the benefits of rising equity values, which reduces the incentive for company shareholders to take risks.

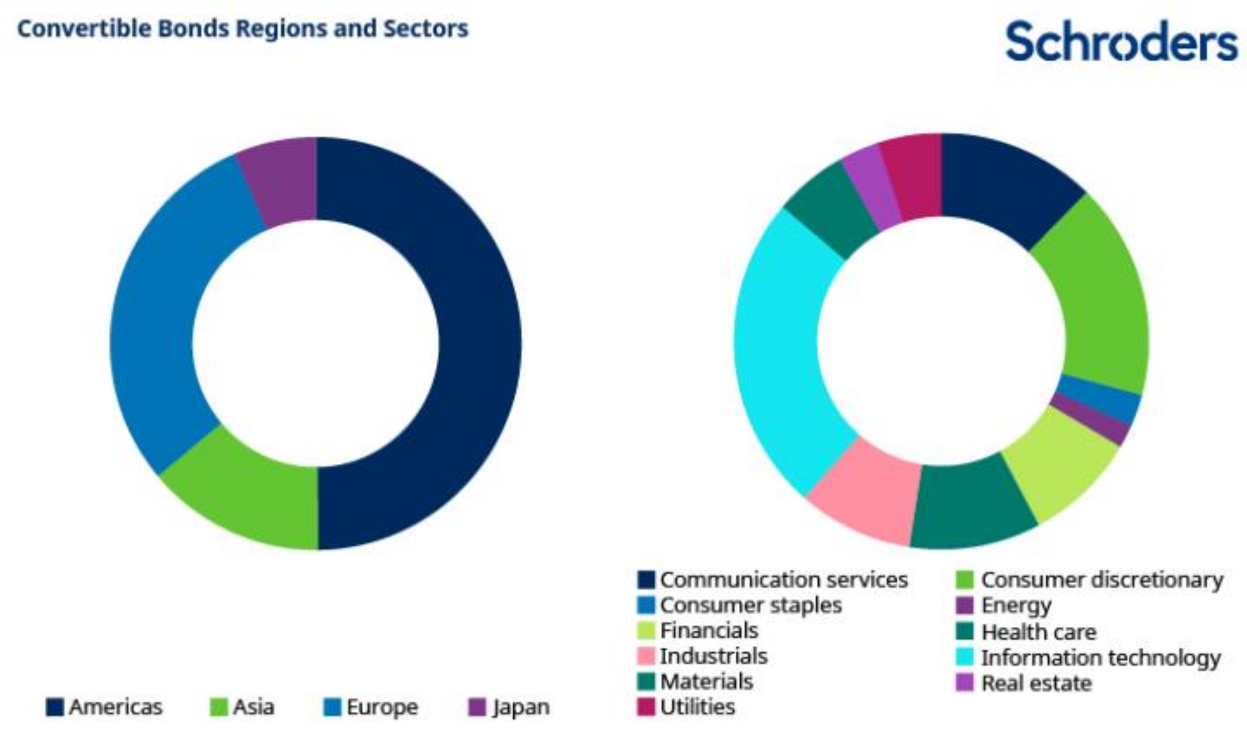
2.2 Background

In 1843, New York Erie in the United States issued the world's first convertible bond. Due to the lack of support by financial theories, the convertible bond market had been developed slowly. After the 1970s, with the development of option theory, the convertible bond market began to develop rapidly. By the end of 2021, the total market value of global convertible bonds had reached $400 billion US dollars.

The United States has now the largest convertible bond market in the world, around $200 billion outstanding. Moreover, the United States has the most complete types of convertible bonds, and many innovative terms of convertible bonds first occurred in the United States. Europe is the second largest convertible bond market. The current market size is around $130 billion US dollars. The issuers are mainly in developed countries such as France, Germany, Britain, and Italy. Before 1999, Japan was the largest convertible bond market in the world. After the burst of the Japanese economic bubble, the scale of the Japanese market continued to decrease as the old bonds matured and the scale of new bond issuance decreased. The current market stock is around

$50 billion US dollars. The convertible bond market in Asia is dominated by Taiwan, Hong Kong, and South Korea. By the end of 2005, the market size was about $32 billion US dollars. From the industry perspective, the US convertible bond issuers are mainly in the industries of technology, health protection, finance and consumer goods, the European convertible bond issuers are mainly in the industries of finance, industry, telecommunications, consumer goods, Japanese convertible bond issuances are mainly in consumer discretionary, utilities and technology. Asian convertible bonds are mainly in the financial, telecommunications and public utilities industries, and together account for about 70% of the entire Asian convertible bond market. Among them, Taiwan's convertible bonds mainly focus on finance and telecommunications, and South Korea's convertible bonds mainly focus on telecommunications and public utilities.

From the perspective of the credit rating of convertible bonds, it is relatively low in the US, and the credit rating of convertible bonds in Japan and Europe is relatively high, and there is a large proportion of unrated convertible bonds in Asia. This is because the corporate bond market in the United States is relatively developed, and convertible bond is playing a supplement role in the market, convertible bond is a financing method for companies with lower credit rating. The issuers in the Japanese convertible bond market are mainly large enterprises like Sony and Panasonic, so the credit rating of convertible bonds is very high, and 97% of convertible bonds are rated higher than BBB.



Source: Schroders, based on constituents of the Refinitiv Global Focus convertible bond index, credit ratings based on official and implied ratings, as of 30 November 2021.

2.3 Terms & Explanation

A large range of terms have come to become associated with convertibles and have made some to become very complicated financial instruments.

Conversion ratio: The number of shares the bond holder gets when converting one bond into shares of stocks.

Conversion price: The implied purchase price of the stock that the bondholders pay.

Callable: Callable gives the issuer the right to call the bond back before it matures.

Puttable: puttable gives the bondholder the right to sell the bond back if the stock price is trading below a certain level of the stock price.

Call type: Hard or soft. Hard call gives the issuer the right to call after prespecified time, while soft call allows the issuer to call only if the stock is trading above a certain level.

Call trigger: Only when the stock price is trading at a certain specified level higher than the trigger price, the bond is able to be called.

One of the crucial issues in studying convertible bonds is the pricing problem. From the perspective of the issuer, pricing is related to whether convertible bonds can be issued and converted smoothly. From the perspective of the investor, pricing is related to whether they can formulate investment strategies and obtain benefits from investing convertible bonds. Therefore, this paper is focusing on different pricing models of convertible bonds, and comparisons between each model.

3 Literature review

Convertible Bonds is a type of debt security that allows investors to exchange their bonds for a predetermined number of shares. Convertibles have grown to prominence in recent years due to its attractiveness to both investors and issuers. Investors benefit from appreciation of the stock value while not assuming the risk of loss in equity value and also enjoying a stable rate of return. While issuers benefit from having to pay a lower interest rate than they would have had they issued regular bonds. A large range of terms have come to become associated with convertibles and have made some to become very complicated financial instruments.

Most convertible bonds have a call feature, this allows the issuer to call a bond back at an earlier time than the maturity date. This introduces reinvestment risk to the investor of the bond, in response some constraints are placed on the call feature to limit when the issuer can call back the bond. One type is a hard-call restriction which means the bond is restricted to be called for a period of time, usually 2-3 years, after issuance. The other type is the soft-call which limits the bond from being called conditional on certain restrictions. Of particular interest to us and this paper are the so called n-out-of-m conditional soft call, which places a restriction on the bond that it can only be called if the stock price closes above a trigger price for n out of the last m days. Once the issuer exercises the call option, the holder must redeem the bond at the call price or convert the bond into stocks, choosing the better value.

This condition presents challenges for pricing as this condition introduces path-dependency, and remains one of the most difficult path-dependent problems in derivative pricing. Currently around 20% of total notional value of convertible bonds have the soft-call feature. We will look at methods that will be useful at pricing this particular feature.

We started from getting known of the Basic Concepts of Soft-call Convertible bond, and then stepped into various Valuing Methods, including PDE, ARB, and Auxiliary State Variable. In this section, all of the methods' methodology are going to be explained in detail below.

3.1 One-touch

The most widely used approximation method for this conditional callable feature is the one-touch barrier, in which instead of considering the complicated 20-out-of 30 path dependent problem, we simplify the problem by instead only looking at 1 out of 1 in which the bond is callable any day the stock closes above the trigger, various strategies exist that utilize this style of approximation with different adjustments to trigger. A particularly interesting modification is the Navin Algorithm (Navin 1999) which is a perturbation method with the order of perturbation based on how many times the stock price crosses the barrier. The leading order is the one-touch barrier option with a daily indicator for the option to exercise for the entier valuation period. This essentially boils down to making the bond 1 touch callable on some days and not on others by using a daily indicator. Navin provides a example with 3-out-of-5 provisional trigger with a 7 days window for exercise but goes short from explicitly describing how to generate the daily indicator. Its primary advantage is that it is a method that only requires a single running of the grid. The algorithm assumes that we know the relevant risk-neutral distributions Green's function and therefore "evolution operator", which is the repeated application of the one-day "grid algorithm" operator.

3.2 Partial Differential Equation

PDE (Partial Differential Equation) is widely used in industry to estimate the value of the convertible bond based on its path-dependent feature. And PDE is also easy to construct and save time. Ayache(2003) propose that PDE can be applied into pricing the convertible bond with various characteristics.

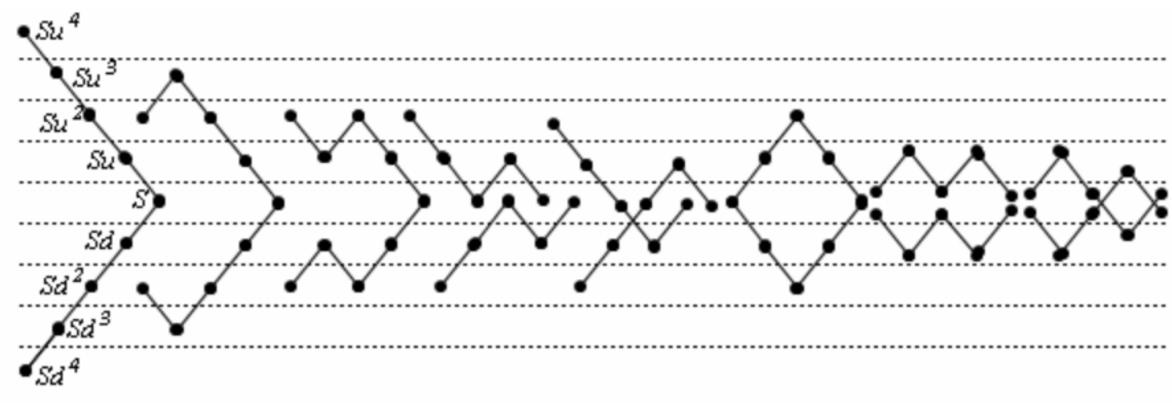
According to Ayache's valuation methods of convertible bonds, the hedge model assumes that there's credit risk along with convertible bonds, and the credit risk is been described as hazard rate, which refers to the percentage of chance that the asset could be in the event of default, Ayache(2003) gives out the PDE:

$$V_t + (r(t) + p\eta)SV_S + \frac{\sigma^2 S^2}{2} V_{SS} + p[max(\mathcal{K}S(1-\eta), RX)] = (r(t) + p)V$$

3.3 Auxiliary Reversed Binomial Tree

The Auxiliary Reversed Binomial (ARB) tree approximation (Liu 2006) is a method that seeks to provide a reasonable yet computationally efficient approximation for estimating the contribution of the embedded soft-call option to the value of the convertible bond. This method is useful in a backwards induction setting, either finite difference or tree method and can be used to approximate the price of a convertible bond when used in conjunction with those methods. The method involves at every grid point looking back and connecting earlier stock prices with the current stock price. We can then enumerate all m length paths that end at each particular end point, using this information we can determine a probability for which the bond is callable and the contribution of the soft call can be calculated using a probability weighted average of the callable price and non-callable price of the bond.

For our purposes a binomial tree is used here but alternatives are valid and may yield better results, though the initial probability tree calculation will be very costly and time-consuming. To demonstrate how this probability can be calculated let's first look at the simple 3-out of 5 case. In this case only 16 possible paths exist that lead to S_j.



Possible trigger positions are indicated by dotted lines. Using the Cox-Ross-Rubenstein (CRR) risk neutral formulation

$$u = e^{\sigma\sqrt{\Delta t}}, ud = 1$$

Given these parameters we can determine the number of paths that exist that are callable and similarly the call probability which are displayed in the table below. It is assumed from the parameters that each path is equally likely to occur, which is reasonable as the

$$Prob_{up} \approx \frac{1}{2} + \frac{1}{2\sigma}\left(r - \frac{1}{2}\sigma^2\right)\sqrt{\Delta t} \ \rightarrow \frac{1}{2} \ when \ \Delta t \ \rightarrow 0$$

$S_{Tr} \in$	Number of Callable Paths	Call Probability
(Su^4, ∞)	0	0
$(Su^3, Su^4]$	0	0
$(Su^2, Su^3]$	0	0
$(Su, Su^2]$	2	2/16
$(S, Su]$	5	5/16
$(Sd, S]$	11	11/16
$(Sd^2, Sd]$	14	14/16
$(Sd^3, Sd^2]$	16	16/16
$(Sd^4, Sd^3]$	16	16/16
$[0, Sd^4]$	16	16/16

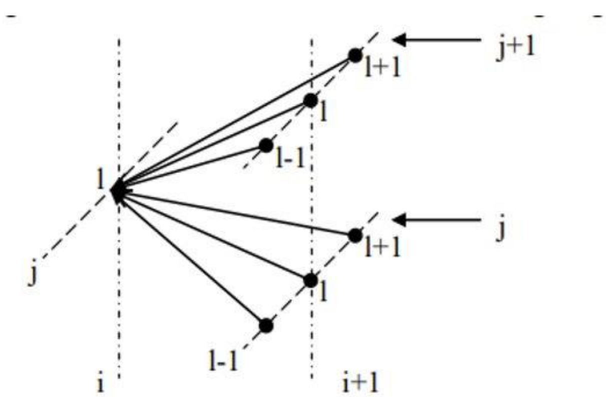
This process can be applied to the 20-out-of-30 condition, in this case, there are 536870912 (2^{29}) paths. This is quite computationally intensive, thankfully, it is unnecessary to recalculate the table as this is calculated irrespective of specific prices for S and trigger price but in terms of their relation to each other. The probabilities for 20-out-of-30 conditions are provided below from Liu (2006). The probabilities can then be used to approximate the price of the CB by taking the probability weighted average of the price of the non-callable bond and callable bond. We used these probabilities in our implementation of this method.

Table 2. Number of callable paths and call probability in *20* out of *30* for all various trigger levels calculated with *S=100* and *u=1.1*.

$S_{T'} \in$	Number of Callable Paths	Probability of Call (%)
(Su^{10}, ∞)	0	0.00
$(Su^{9}, Su^{10}]$	92378	0.02
$(Su^{8}, Su^{9}]$	277134	0.05
$(Su^{7}, Su^{8}]$	1368874	0.25
$(Su^{6}, Su^{7}]$	3367598	0.63
$(Su^{5}, Su^{6}]$	9330178	1.74
$(Su^{4}, Su^{5}]$	19256614	3.59
$(Su^{3}, Su^{4}]$	39237394	7.31
$(Su^{2}, Su^{3}]$	69272518	12.90
$(Su, Su^{2}]$	115309708	21.48
$(S, Su]$	177348964	33.03
$(Sd, S]$	254907724	47.48
$(Sd^{2}, Sd]$	324706732	60.48
$(Sd^{3}, Sd^{2}]$	381413581	71.04
$(Sd^{4}, Sd^{3}]$	429260863	79.96
$(Sd^{5}, Sd^{4}]$	463636309	86.36
$(Sd^{6}, Sd^{5}]$	490888807	91.44
$(Sd^{7}, Sd^{6}]$	507955766	94.61
$(Sd^{8}, Sd^{7}]$	520697698	96.99
$(Sd^{9}, Sd^{8}]$	527518414	98.26
$(Sd^{10}, Sd^{9}]$	532324922	99.15
$(Sd^{11}, Sd^{10}]$	534464675	99.55
$(Sd^{12}, Sd^{11}]$	535891177	99.82
$(Sd^{13}, Sd^{12}]$	536398475	99.91
$(Sd^{14}, Sd^{13}]$	536719133	99.97
$(Sd^{15}, Sd^{14}]$	536804570	99.99
$(Sd^{16}, Sd^{15}]$	536855882	100.00
$(Sd^{17}, Sd^{16}]$	536864990	100.00
$(Sd^{18}, Sd^{17}]$	536870198	100.00
$(Sd^{19}, Sd^{18}]$	536870660	100.00
$[0, Sd^{19}]$	536870912	100.00

3.4 Auxiliary State Variable

This model is using a kind of improved method based on ARB. It is also pricing on a binomial tree. What is different is that, instead of using the direct probability, we use the conditional probability in the Auxiliary State Variable model. What's more, it introduces another parameter, l, into the model as well, representing the number of days the stock price is above trigger H within the last m days. And then, at each evaluation point in the tree, we can calculate the M-table for the point, which gives the number of paths from the evaluation point to m-1 days back that would match level l. This M table is pre-calculable. Finally, we calculate the price at each node by using the weighted expectation with conditional probability of callability of the node in next steps.



For each node $S_{i,j}$, it now consists one more index, l , so it becomes $S_{i,j,l}$. Then The option value can be approximated with

$$O_{i,j,l} = \max\left(I_{i,j,l}, \frac{p\pi_{i+1,j+1,l} + q\pi_{i+1,j,l}}{R}\right)$$

$$\pi_{i+1,j+1,l} = h_{i+1,j+1,l+1}O_{i+1,j+1,l+1} + h_{i+1,j+1,l}O_{i+1,j+1,l} + h_{i+1,j+1,l-1}O_{i+1,j+1,l-1}$$

$$\pi_{i+1,j,l} = h_{i+1,j,l+1}O_{i+1,j,l+1} + h_{i+1,j,l}O_{i+1,j,l} + h_{i+1,j,l-1}O_{i+1,j,l-1}$$

Where

$$S_{i,j,l} = S_{i,j}, I_{i,j,l} = \max\left(0, S_{i,j} - k\right) \parallel (l \geq n)$$

Here h is defined by the general conditional transition probability which is path dependent.

$$h_{i+1,j+1,l+1} = Prob\{L_{i+1,j+1} = L_{i,j} + 1 | S_{i+1,j+1} = uS_{i,j}\}$$

Is the probability of level increased by 1 conditioned on that the stock $S_{i,j}$ moving up to notes $S_{i+1,j+1}$.

1) if $S_{i,j} > H * u^{m-1}$, $L_{i,j}$ has only one state $L_{i,j} = m$ and $h_{i+1,j+1,m} = 1$, all other probabilities are 0.

2) if $S_{i,j} < H * d^{m-1}$, $L_{i,j}$ has only one state $L_{i,j} = 0$ and $h_{i+1,j+1,0} = 1$, all other probabilities are 0.

So we only need to calculate the conditional probability on the whole tree within nodes with $H * u^{m-1} >= S_{i,j} >= H * d^{m-1}$, outside of this range the level is fixed.

What's more, when the stock is above the trigger H in the next step, then the total level cannot decrease. Same idea, when the stock price is not above the trigger H in the next step, the total level cannot increase. So we only need to solve two probabilities instead of there at any node. Starting from $S_{i,j}$, on the binomial tree, we can look backward m-1 day to check the history. So the probability

$$h_{i+1,(*),l+1} = Prob\{L_{i+1} = L_i + 1 \middle| S_{i+1,(*)} > H\}$$

is equivalent to get the probability

$$\rho(S_{i-m+1} \leq H | S_{i,j}, S_{0,0}, l)$$

which is the conditional probability of the stock price S_{i-m+1} is below the barrier H with process starting from time 0 with$S_{i,j}$, end at $S_{i,j}$ with fixed number of level l. With same idea we get

$$h_{i+1,(*),l} = Prob\{L_{i+1} = L_i \middle| S_{i+1,(*)} > H\} = \rho(S_{i-m+1} > H | S_{i,j}, S_{0,0}, l)$$

$$h_{i+1,(*),l} = Prob\{L_{i+1} = L_i \middle| S_{i+1,(*)} \leq H\} = \rho(S_{i-m+1} \leq H | S_{i,j}, S_{0,0}, l)$$

$$h_{i+1,(*),l} = Prob\{L_{i+1} = L_i - 1 \middle| S_{i+1,(*)} > H\} = \rho(S_{i-m+1} > H | S_{i,j}, S_{0,0}, l)$$

The table below describes how the level l changes due to stock price:

	The stock price being deleted	The stock price being added	New Level Status
Stock price above or below the trigger	×	√	$l-1$
	×	×	l
	√	√	l
	√	×	$l+1$

This can be done by just counting the path with fixed level l within the 2^{m-1} total paths within the m - 1 steps, the number of paths above the trigger, and also the number of paths below the trigger separately, weighted by the probability that the stock is coming from time 0 to that state.

$$\rho\left(S_{i-m+1,(*)} \le H \middle| \mathrm{S_{i,j}}, S_{(0,0)}, l\right) = \rho\left(\xi_{m-1,(*)} \le H \middle| \xi_{0,0} = \mathrm{S_{i,j}}, S_{(0,0)}, l\right)$$

$$= \begin{cases} \dfrac{\sum_{k=0}^{b} M(k,b,l;m)\binom{j+k-m+1}{i-m+1}}{\sum_{k=0}^{m-1} M(k,b,l;m)\binom{j+k-m+1}{i-m+1}} & if \sum_{k=0}^{m-1} M(k,b,l;m)\binom{j+k-m+1}{i-m+1} > 0 \\ 0 & if \sum_{k=0}^{m-1} M(k,b,l;m)\binom{j+k-m+1}{i-m+1} = 0 \end{cases}$$

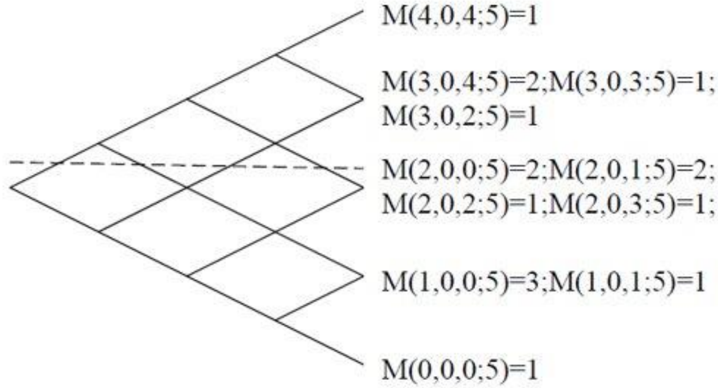
With the same idea we get:

$$\rho\left(S_{i-m+1,(*)} > H \middle| \mathrm{S_{i,j}}, S_{(0,0)}, l\right) = \rho\left(\xi_{m-1,(*)} > H \middle| \xi_{0,0} = \mathrm{S_{i,j}}, S_{(0,0)}, l\right)$$

$$= \begin{cases} 1 - \rho\left(S_{i-m+1,(*)} \le H \middle| \mathrm{S_{i,j}}, S_{(0,0)}, l\right) & if \sum_{k=0}^{m-1} M(k,b,l;m)\binom{j+k-m+1}{i-m+1} > 0 \\ 0 & if \sum_{k=0}^{m-1} M(k,b,l;m)\binom{j+k-m+1}{i-m+1} = 0 \end{cases}$$

$M(k,b,l,m)$ is a pre-calculate a table, which gives the number of path for an m-1 step tree, starting from $R_{i,j} = S_{0,0}$ end at notes $R_{m-1,k}$ with trigger at $R_{0,0}\, u^b d^{m-1-b} < H < R_{0,0}\, u^{b+1} d^{m-2-b}$ and the level equal l. It gives the weight from $S_{i,j}$ to $S_{0,0}$ with fixed level l.

The table below show the $M(k, b, l, m)$ table on a 5-step tree as an example.



When the time step with $i <= m-1$, since we know the historical path of the stock price S_{i-m+1}, so

$$\rho\left(S_{i-m+1,(*)} \leq H \middle| \mathrm{S}_{\mathrm{i,j}}, S_{(0,0)}, l\right) = \begin{cases} 0, S_{i-m+1} > H \\ 1, S_{i-m+1} \leq H \end{cases}$$

Since we solved the conditional transition probability, then we can apply into the rollback scheme to get the option price rollback equations

$$O_{i,j,l} = \max\,(I_{i,j,l}, \frac{p\pi_{i+1,j+1,l} + q\pi_{i+1,j,l}}{R})$$

$\pi_{i+1,j+1,l}$

$$= \begin{cases} \left[\rho\left(S_{i-m+1,(*)} \leq H \middle| S_{i.j}, S_{0,0}, l\right) O_{i+1,j+1,l+1} + \rho\left(S_{i-m+1,(*)} > H \middle| S_{i.j}, S_{0,0}, l\right) O_{i+1,j+1,l}\right] \mathbb{1}\left(S_{i+1,j+1} > H\right) + \\ \left[\rho\left(S_{i-m+1,(*)} \leq H \middle| S_{i.j}, S_{0,0}, l\right) O_{i+1,j+1,l} + \rho\left(S_{i-m+1,(*)} > H \middle| S_{i.j}, S_{0,0}, l\right) O_{i+1,j+1,l-1}\right] \mathbb{1}\left(S_{i+1,j+1} \leq H\right) \end{cases}$$

$\pi_{i+1,j,l}$

$$= \begin{cases} \left[\rho\left(S_{i-m+1,(*)} \leq H \middle| S_{i.j}, S_{0,0}, l\right) O_{i+1,j,l+1} + \rho\left(S_{i-m+1,(*)} > H \middle| S_{i.j}, S_{0,0}, l\right) O_{i+1,j,l}\right] \mathbb{1}\left(S_{i+1,j} > H\right) + \\ \left[\rho\left(S_{i-m+1,(*)} \leq H \middle| S_{i.j}, S_{0,0}, l\right) O_{i+1,j,l} + \rho\left(S_{i-m+1,(*)} > H \middle| S_{i.j}, S_{0,0}, l\right) O_{i+1,j,l-1}\right] \mathbb{1}\left(S_{i+1,j} \leq H\right) \end{cases}$$

4 Mathematical Inference

4.1 Explicit Finite Difference method

According to Ayache's valuation methods of convertible bonds, the hedge model assumes that there's credit risk along with convertible bonds, and the credit risk is been described as hazard rate, which refers to the percentage of chance that the asset could be in the event of default, Ayache (2003) gives out the PDE:

$$V_t + (r(t) + p\eta)SV_S + \frac{\sigma^2 S^2}{2} V_{SS} + p[max(\mathcal{K}S(1-\eta), RX)] = (r(t) + p)V$$

Where p is hazard rate, R is the bond recovery, $\mathcal{K}$ is the conversion ratio and η is the index factor, where we set η=1 and it implies the underlying stock price jumps to zero once it defaults as the "total default" case. Rearrange the equation:

$$V_t + (r + p)SV_S + \frac{\sigma^2 S^2}{2} + pRV = (r + p)V$$

Then for a simple model, one can assume that there's no recovery after the bond defaults, then the PDE solving right now turns into the desired form:

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{ss} + (r + p)SV_s - (r + p)V = 0$$

The equation right now doesn't satisfy the requirements for Finite Difference Method, because coefficient $\sigma^2 S^2$ and $(r + p)S$ don't have borders, so we need to transform:

$$z = \text{In}\, S\,, V(S, t) = V(e^z, t)$$

Then:

$$V_t = V_t\ (cause\ no\ stock\ price\ involved)$$

$$V_s = \frac{\partial V}{\partial S} = \frac{\partial V}{\partial Z} * \frac{\partial Z}{\partial S} = V_z * \frac{1}{S}$$

$$V_{ss} = \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial}{\partial S}\left(V_z * \frac{1}{S}\right) = \frac{1}{S^2}V_{zz} - \frac{1}{S^2}V_z$$

Then the PDE will change to:

$$V_t + \frac{1}{2}\sigma^2 V_{zz} + \left(r + p - \frac{1}{2}\sigma^2\right)V_z - (r+p)V = 0$$

By using explicit method (backward), one can derive that:

$$V_t = \frac{v_j^t - v_j^{t-1}}{\Delta t}, V_{zz} = \frac{v_{j+1}^t + v_{j-1}^t - 2v_j^t}{(\Delta z)^2}, V_z = \frac{v_j^t - v_{j-1}^t}{\Delta z}$$

Placing them into the PDE and Reorganizing the PDE, the iterative equation will be:

$$v_j^{t-1} = v_j^t\left(1 - \sigma^2\frac{\Delta t}{(\Delta z)^2}\right) + v_{j+1}^t\left(\frac{1}{2}\sigma^2\frac{\Delta t}{(\Delta z)^2} + \frac{1}{2}\left(r + p - \frac{1}{2}\sigma^2\right)\frac{\Delta t}{\Delta z}\right)$$

$$+ v_{j-1}^t\left(\frac{1}{2}\sigma^2\frac{\Delta t}{(\Delta z)^2} - \frac{1}{2}\left(r + p - \frac{1}{2}\sigma^2\right)\frac{\Delta t}{\Delta z}\right) - (r+p)\Delta t v_j^t$$

For each grid point in PDE, each stock price has a predetermined call probabilities, calculated from the ARB method, then we got a call $Prob_j$ vector:

$$Prob_j = F(S_j, Trigger, u)$$

Applying the Probabilities into the PDE, we get the soft-call convertible bond value by the average of callable and noncallable bond based on the callable probability:

$$V(t,S) = P_{sc} * V_{sc}(t,S) + [1 - P_{sc}] * V_{nc}(t,S)$$

4.2 Auxiliary State Variable construction

4.2.1 Parameters setup

In order to keep consistency in all results, we choose the same parameter for every method, the feature of the specific convertible bond and the binomial tree is shown as below:

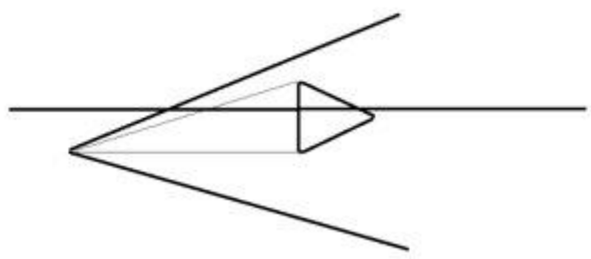
Convertible Bond	
Soft-call Period	2
Conversion Ratio	1
Call(strike) price	110
Coupon rate	8%
Coupon frequency	semiannually
Hazard rate	0.02
Risk-free rate	0.05
Volatility	0.2/0.4/0.9
Trigger Price	varies

Tree Setup	
u	$e^{\sigma\sqrt{\Delta t}}$
d	$1/u$
p_u	$\frac{1}{2}+\frac{1}{2\sigma}(r - \frac{1}{2}\sigma^2)\sqrt{\Delta t}$
p_d	$1-\mathrm{p_u}$

4.2.2 Looking backward algorithm

The project use Looking backward algorithm in Auxiliary State Variable model, the overall methodology is fully described in 3.4, the key part for this algorithm is to calculate the condition probability, where we construct a reversed tree for every point in the binomial tree, and calculate the conditional probabilities accordingly.

Sketch map for the Looking backward algorithm



When we are implementing M table into probability calculation, we need to take care of the structure of M table and h matrix:

(1) $M(k, b, l; m)$ is essentially a 3-dim pre-calculated table since m is fixed, and for each node at $S_{i,j}$, we only need a 2-dim M table since b is fixed at each node.

(2) The actual domain for each parameter:

$$k: [0, m-1] \qquad b: [-m+1, m-2]$$

$$l: [0, m] \qquad m: 30$$

(3) ρ matrix gives us the conditional transition probabilities for each node at $S_{i,j}$, function $H(i, j, l)$ gives the 3-dim matrix of pho that indicates the conditional transition probability that m days ago from node $S_{i,j}$, with the level l of the stock path for the passed m day including today.

The boundary effect:

(1) when $i \leq m-1$, all the nodes is now deterministic, which means they only rely on the historical path, and we just need to stimulate the m-days historical path and to keep track of their

prices to see if they exceed the trigger price or not. We can use random work or use assumption in the article, which it assumes all historical prices are below the barrier.

(2) When the nodes are very close to the $S_{i,j}$ tree boundary, the previous calculation of the conditional probability cannot hold anymore, because now the M table is in complete.

So if the nodes are very close to the upper boundary, to calculate the probability we only need to count the table $M(k, b, l; m)$ with all the k that

$$0 \leq k \leq w$$

if the nodes are very close to the lower boundary, to calculate the probability we only need to count the table $M(k, b, l; m)$ with all the k that

$$w \leq k \leq m - 1$$

The boundary effect, in our practical cases, is very small even neglectable, since our stock price binomial tree is relatively huge and these nodes that have to calculate the conditional probabilities (not 0 or 1) are all lies between a narrow range.

We constructed this model in a programming language C++. We started from constructing the M table of a 30-step tree by recursively counting the number of paths of each level. Then, we built a function to calculate the conditional probabilities of each node. Finally, we calculated the price of the bond from the end and then rollback at each layer. The final results are shown below.

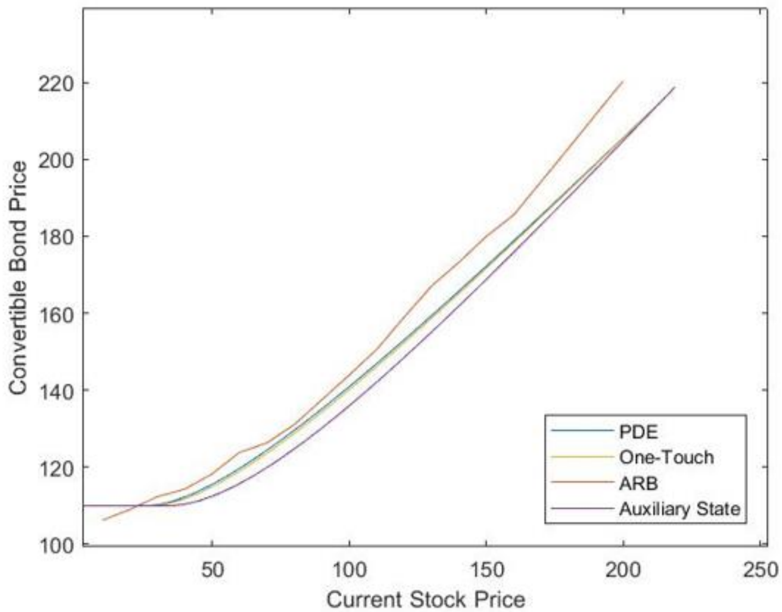
5 Implementation Results

The final result that achieved was quite satisfactory. In brief, the Auxiliary State Variable algorithm gives out results that are most close to the reference values. But the ARB tree and the One-Touch neither give deviated results.

5.1 Overall comparison

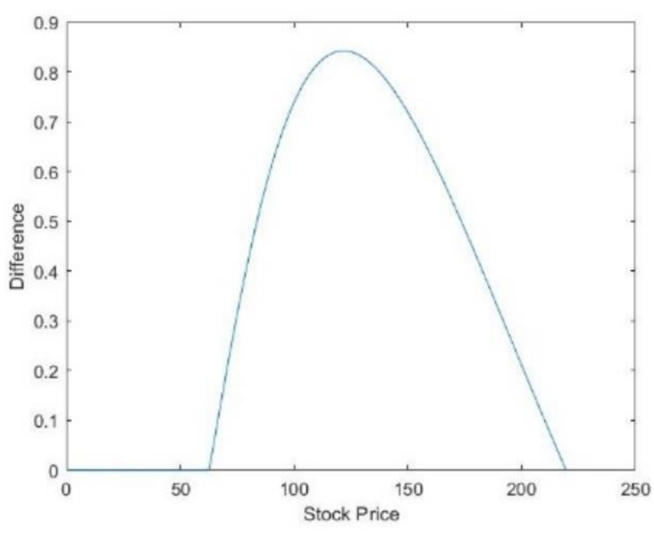
By using the same data as mentioned in 4.2, a summary graph shows different method's pricing result regarding 20-out-of-30 soft-call convertible bond, in order to make the difference clearer, here we set up the volatility to 90%. As we can see in the graph, all the convertible bond prices converges to a certain price when the current stock price is low, and then steadily rise up when stock price increases. The ARB model we constructed gives the highest estimate and Auxiliary State model gives the lowest estimate, One-touch and PDE method gives approximately the same result, and One-touch result is also calculated by PDE in this project.

Convertible Bond price with different methods



To make it clearer, the figure below draws the Numerical difference between One-touch and PDE method, it approach to its maximum around the trigger price, approximately 117. Noticeably ARB results tends to be more volatile and unstable compared to other three results, which may caused by algorithm's complexity, means we need to set up larger timesteps to cut down computation time, therefore the result fluctuates. So in order to keep accuracy, we decide to exclude ARB from further comparison.

Numerical difference between OT and PDE

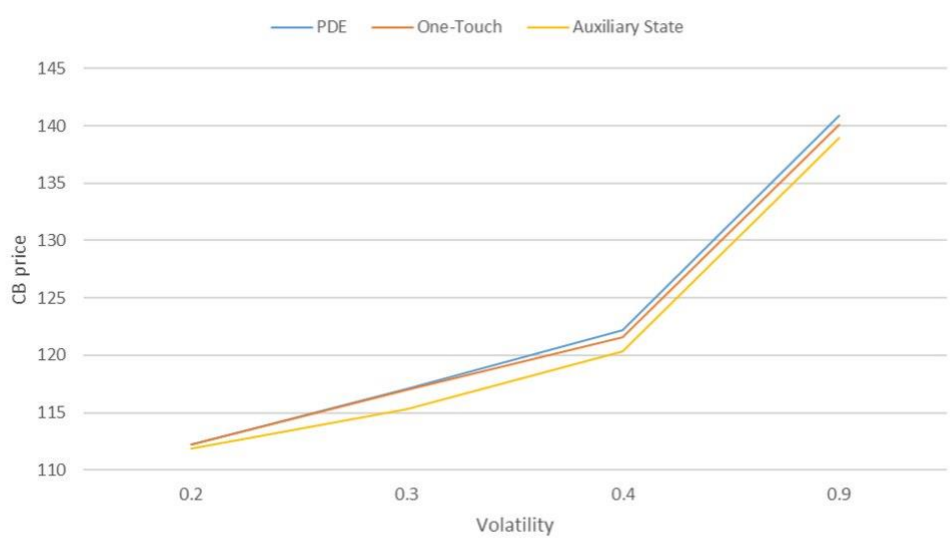


5.2 Convertible Bond prices for different volatilities

In order to discover how volatilities effect the result in different methods, the following table shows the convertible bond prices for different volatilities, the trigger price is set up to 110. As we can see in the graph, the CB price increases as the volatility rises, which is consistent with intuition, and can also be seen as a cross-validation for these three different methods, which eliminates the doubt for internal validity.

Convertible Bond price with different volatilities

Volatilities (Trigger=110)	PDE(ARB)	One-touch	Auxiliary State
0.2	112.25	112.20	111.87
0.3	117.08	117.02	115.27
0.4	119.84	119.27	120.32
0.9	140.91	140.102	138.96

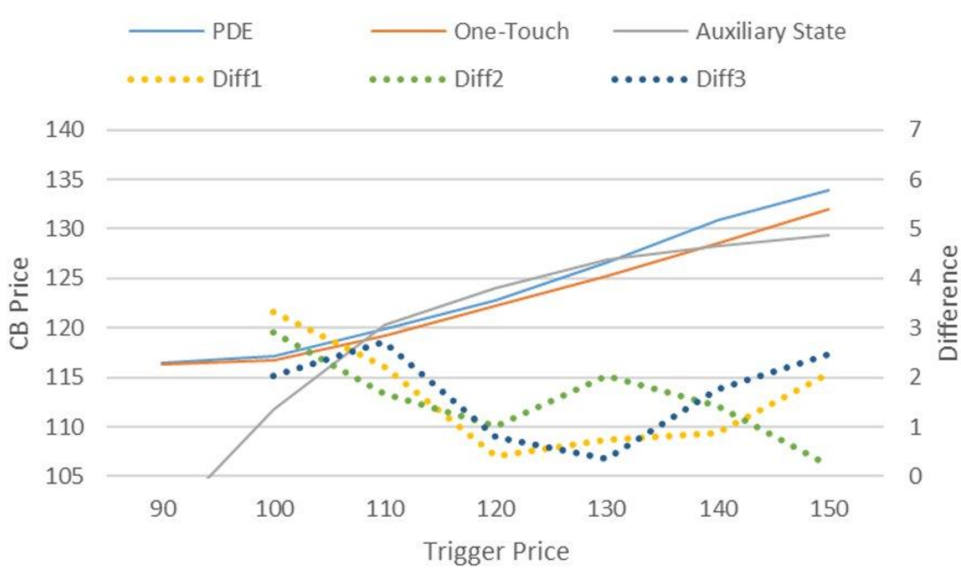


5.3 Convertible Bond prices for different trigger price

We also try compare each model with different trigger price while all other parameters are the same. The results are shown as below.

Convertible Bond price with different trigger

Trigger (Vol=0.4)	PDE (ARB)	One-Touch	Auxiliary State	Exact
100	117.16	116.75	111.79	113.84
110	119.84	119.27	120.32	117.60
120	122.78	122.18	124.00	123.20
130	126.56	125.27	126.95	127.29
140	130.90	128.59	128.26	130.01
150	133.88	132.00	129.32	131.79
Computation Time	>20 mins	10 mins if PDE-based	7mins for each	/



It is obvious that the Auxiliary State gives the most accurate result compared to the other two methods, while ARB and One-Touch also work quite well, when the trigger price is higher than S0. But when the trigger price is lower than S0, the Auxiliary State cannot give accurate result.

6 Conclusions

This practicum project compares three different ways to compute the convertible bond value with embedded soft-call option and compare the differences based on various parameters; our implementation results show that while ARB and Auxiliary State Variable methods produces better approximations while the trigger is between 20-40% above S. They do not produce better approximations in all cases, though it could be argued that the conditions where they outperform one-touch are more realistic and more generally true for CBs that exist in the real world. A comparison of the performance time of the methods shows that both methods are comparable to the one-touch method with the Auxiliary State Variable method having a computation time of around 7 minutes for our example bond, this is promising in fulfilling the requirements of a fast approximation technique. This combined in their better performance under realistic convertible bond conditions makes these methods a realistic alternative for usage in approximating bond prices, in particular Auxiliary State Variable method deserves additional investigation as it has a reasonable computational time as well as potential accuracy improvement if Zhang(2011) results can be attained.

Thank you for reading! We would like to say thank you also to the sponsors Yunsong Huang and Charlie Che from J.P. Morgan for their tutelage.

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