Trayectorias Ortogonales

1) Encuentre las trayectorias ortogonales de la familia $x^2+y^2=cx$

$\frac{x^2+y^2}{x}=C \rightarrow \frac{x\frac{d}{dx}(x^2+y^2)-(x^2+y^2)\frac{d}{dx}(x)}{x^2}=0$

$\rightarrow \frac{x(2x+2yy')-(x^2+y^2)}{x^2}=0 \rightarrow 2x^2+2xyy'-x^2-y^2=0$

$2xyy'=x^2+y^2-2x^2 \rightarrow y'=\frac{y^2-x^2}{2xy} \rightarrow \frac{dy}{dx}=\frac{y^2-x^2}{2xy}$

$2xy\,dy=(y^2-x^2)dx \quad y=ux \quad dy=u\,dx+x\,du \quad u=\frac{y}{x}$

$2x(ux)(u\,dx+x\,du)=(u^2x^2-x^2)dx$

$[2ux^2(u\,dx+x\,du)]=(u^2x^2-x^2)dx \rightarrow [2u^2x^2dx+2ux^3du]$

$=(u^2x^2-x^2)dx \rightarrow (2u^2x^2-u^2x^2+x^2)dx+2ux^3du=0$

$(u^2x^2+x^2)dx+2ux^3du=0 \rightarrow x^2(u^2+1)dx+2ux^3du=0$

$x^2(u^2+1)dx=-2ux^3du \rightarrow \frac{x^2dx}{x^3}=\frac{-2u\,du}{u^2+1}$

$\int\frac{dx}{x}=\int\frac{-2u\,du}{u^2+1} \rightarrow \ln|x|=-\ln|u^2+1|+C$ Verificar

$e^{\ln|x|}+e^{\ln|u^2+1|}=e^{\ln C} \rightarrow x+u^2+1=C$

$x+\frac{y^2}{x^2}+1=C \rightarrow \frac{x^2+y^2+x^2}{x^2}=C \rightarrow \boxed{\frac{2x^2+y^2}{x^2}=C}$

2) Encuentre las trayectorias ortogonales de la familia $x^2-y^2=C$

$2x-2yy'=0 \rightarrow 2yy'=2x \rightarrow y'=\frac{2x}{2y} \rightarrow y'=\frac{x}{y}$

$y'=\frac{-1}{\frac{x}{y}} \rightarrow y'=(-1)\left(\frac{y}{x}\right) \rightarrow y'=\frac{-y}{x} \rightarrow \frac{dy}{dx}=-\frac{y}{x}$

$\int\frac{dy}{y}=\int\frac{dx}{x} \rightarrow \ln y+\ln x=\ln C \rightarrow \boxed{xy=C}$

3) Encuentre las trayectorias ortogonales de la familia $x^2+y^2-2Kx=0$

$x^2+y^2=2Kx \rightarrow \frac{x^2+y^2}{x}=2K \rightarrow \frac{x\frac{d}{dx}(x^2+y^2)+(x^2+y^2)}{x^2}=0$

$x(2x+2yy')-x^2-y^2=0 \rightarrow 2x^2+2xyy'=x^2-y^2$

$2xyy'=x^2+y^2-2x^2 \rightarrow y'=\frac{y^2-x^2}{2xy} \rightarrow \frac{dy}{dx}=\frac{y^2-x^2}{2xy}$

$\frac{dy}{dx}=\frac{-1}{\frac{y^2-x^2}{2xy}} \rightarrow \frac{dy}{dx}=\frac{2xy}{x^2-y^2} \rightarrow (x^2-y^2)dy=2xy\,dx$

$x=vy \quad dx=v\,dy+y\,dv \quad (v^2y^2-y^2)dy=2(vy)y[v\,dy+y\,dv]$

$(v^2y^2-y^2)dy=2vy^2[v\,dy+y\,dv] \rightarrow (v^2y^2-y^2)dy=(2v^2y^2dy+2vy^3dv)$

$(v^2y^2-y^2-2v^2y^2)dy=2vy^3dv \rightarrow (-v^2y^2-y^2)dy=2vy^3dv$

$-y^2(v^2+1)dy=2vy^3dv \rightarrow -\int\frac{y^2dy}{y^3}=\int\frac{2v\,dv}{v^2+1}$

$\int\frac{dy}{y}=\int\frac{2v\,dv}{v^2+1} \rightarrow e^{\ln|y|}-e^{\ln|v^2+1|}=e^{\ln C}$

$y-\frac{x^2}{y^2}+1=C \rightarrow \frac{y^2-x^2+y^2}{y^2}=C \rightarrow \boxed{\frac{2y^2-x^2}{y^2}=C}$

4) Encuentre las trayectorias ortogonales de la familia $x^2-4xy-y^2+1=$

$x^2-4xy-y^2=C-1 \rightarrow 2x-[4xy'+y\frac{d}{dx}(4x)]-2yy'=0$

$2x-4xy'+4y-2yy'=0 \rightarrow 2x+4y=2yy'+4xy'$

$y'(2y+4x)=2x+4y \rightarrow y'=\frac{2x+4y}{2y+4x} \rightarrow \frac{dy}{dx}=\frac{2x+4y}{2y+4x}$

$\frac{dy}{dx}=\frac{-1}{\frac{2x+4y}{2y+4x}} \rightarrow \frac{dy}{dx}=-\frac{2y+4x}{2x+4y}$

$(2x+4y)dy=-(2y+4x)dx \rightarrow \overset{M}{(2y+4x)}dx+\overset{N}{(2x+4y)}dy=0$

$\frac{\partial M}{\partial y}=2 \quad \frac{\partial N}{\partial x}=2 \quad \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ Sí es exacta

$f(x,y)=\int 2y\,dx+\int 4x\,dx+c(y) \rightarrow f(x,y)=2xy+4\frac{x^2}{2}+c(y)$

$f(x,y)=2xy+2x^2+c(y) \rightarrow \frac{\partial f(x,y)}{\partial y}=2y+c'(y)$

$2y+c'(y)=2x+4y \rightarrow c'(y)=4y-2y+2x$

$c'(y)=\int 2y\,dy+\int 2x\,dy \rightarrow c(y)=2\frac{y^2}{2}+2xy$

$c(y)=y^2+2xy \rightarrow f(x,y)=2xy+2x^2+y^2+2xy$

$\boxed{f(x,y)=2x^2+y^2+4xy}$

5) Encuentre las trayectorias ortogonales de la familia $y=cx^{-1}$

$y=\frac{C}{x} \rightarrow xy=C \rightarrow xy'+y=0 \rightarrow xy'=-y$

$y'=-\frac{y}{x} \rightarrow \frac{dy}{dx}=\frac{-1}{-\frac{y}{x}} \rightarrow \frac{dy}{dx}=(-1)\left(-\frac{x}{y}\right) \rightarrow \frac{dy}{dx}=\frac{x}{y}$

$\int y\,dy=\int x\,dx \rightarrow \frac{1}{2}y^2-\frac{1}{2}x^2=C \rightarrow y^2=\frac{2x^2}{2}+2C$

$y^2=x^2+2C \rightarrow \boxed{y=\sqrt{x^2+C}}$

6) Encuentre las trayectorias ortogonales de la familia $y=\frac{C}{1+x^2}$

$(y)(1+x^2)=C \rightarrow (1+x^2)\frac{d}{dx}(y)+(y)\frac{d}{dx}(1+x^2)=0$

$(1+x^2)(y')+(y)(2x)=0 \rightarrow y'+x^2y'+2xy=0$

$y'(1+x^2)=-2xy \rightarrow y'=\frac{-2xy}{1+x^2} \rightarrow \frac{dy}{dx}=\frac{-2xy}{1+x^2}$

$\frac{dy}{dx}=\frac{-1}{\frac{-2xy}{1+x^2}} \rightarrow \frac{dy}{dx}=\frac{1+x^2}{2xy} \rightarrow (2xy)dy=(1+x^2)dx$

$\int(2y)\,dy=\int\frac{(1+x^2)dx}{x} \rightarrow 2\int y\,dy=\int\frac{dx}{x}+\int\frac{x^2}{x}dx$

$2\int y\,dy=\int\frac{dx}{x}+\int x\,dx \rightarrow 2\frac{y^2}{2}=\ln|x|+\frac{x^2}{2}+C$

$y^2=\ln|x|+\frac{x^2}{2}+C \rightarrow \boxed{y=\sqrt{\ln|x|+\frac{x^2}{2}+C}}$

7) Encuentre las trayectorias ortogonales de la familia $y = \dfrac{cx}{1+x}$

$(y)(1+x) = cx \rightarrow \dfrac{y+xy}{x} = c \rightarrow \dfrac{x\frac{d}{dx}(y+xy) - (y+xy)\frac{d}{dx}(x)}{x^2} = 0$

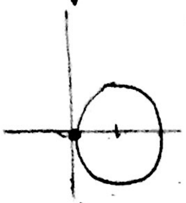
$x(y'+y) - (y+xy) = 0 \rightarrow xy' + \cancel{xy} - y - \cancel{xy} = 0$

$xy' - y = 0 \rightarrow xy' = y \rightarrow y' = \dfrac{y}{x} \rightarrow \dfrac{dy}{dx} = \dfrac{-1}{\frac{y}{x}}$

$\dfrac{dy}{dx} = \dfrac{-x}{y} \rightarrow \int y\,dy = \int x\,dx \rightarrow \dfrac{y^2}{2} = -\dfrac{x^2}{2} + c$

$\frac{1}{2}y^2 = -\dfrac{x^2}{2} + c \rightarrow y^2 = -2\dfrac{x^2}{2} + 2c \rightarrow \boxed{y^2 = 2c - x^2}$

8) Determine las trayectorias ortogonales de la familia de circunferencias que son tangentes al eje y en el origen



$r = h$

$(x-h)^2 + (y-0)^2 = r^2$

$(x-h)^2 + y^2 = r^2$

$(x-r)^2 + y^2 = r^2$

$x^2 - 2xr + r^2 + y^2 = r^2$

$\rightarrow x^2 - 2xr + y^2 = r^2 - r^2$

$x^2 - 2xr + y^2 = 0$

$2x - 2r + 2yy' = 0$

$x^2 + y^2 = 2xr \rightarrow \dfrac{x^2+y^2}{2x} = r$

$2x - \cancel{2}\left(\dfrac{x^2+y^2}{\cancel{2}x}\right) + 2yy' = 0 \rightarrow 2x - \dfrac{x^2+y^2}{x} + 2yy' = 0$

$\dfrac{2x^2 - x^2 + y^2}{x} + 2yy' = 0 \rightarrow \dfrac{x^2+y^2}{x} + 2yy' = 0 \rightarrow 2yy' = -\dfrac{x^2+y^2}{x}$

$y' = \dfrac{y^2 - x^2}{2xy} \rightarrow \dfrac{dy}{dx} = \dfrac{-1}{\frac{y^2-x^2}{2xy}} \rightarrow \dfrac{dy}{dx} = -\left(\dfrac{2xy}{y^2-x^2}\right)$

$v = \dfrac{x}{y}$

$\dfrac{dy}{dx} = \dfrac{2xy}{x^2-y^2} \rightarrow (x^2-y^2)\,dy = (2xy)\,dx$

$x = vy \quad dx = v\,dy + y\,dv$

$(v^2y^2 - y^2)\,dy = (2)(vy)(y)[v\,dy + y\,dv] \rightarrow (v^2y^2 - y^2)\,dy = 2vy^2[v\,dy + y\,dv]$

$(v^2y^2 - y^2)\,dy = [2v^2y^2\,dy + 2vy^3\,dv] \rightarrow (v^2y^2 - 2v^2y^2 - y^2)\,dy = 2vy^3\,dv$

$-y^2(v^2+1)\,dy = 2vy^3\,dv \rightarrow \int -\dfrac{y^2}{y^3}\,dy = \int \dfrac{2v\,dv}{v^2+1} \rightarrow -\int \dfrac{dy}{y} = \int \dfrac{2v\,dv}{v^2+1}$

$-\ln|y| = \ln|v^2+1| + \ln c \rightarrow e^{\ln|v^2+1|} + e^{\ln|y|} + e^{\ln|c|} = 0$

$v^2 + 1 + y + c = 0 \rightarrow \dfrac{x^2}{y^2} + y + 1 + c = 0 \rightarrow \dfrac{x^2 + y^2 + y^2}{y^2} = c$

$\dfrac{x^2 + 2y^2}{y^2} = c \rightarrow \boxed{2y^2 + x^2 = cy^2}$