Ecuaciones Diferenciales Lineales - Dennis G. Zill

1) $\frac{dy}{dx} = 5y \to \frac{dy}{dx} - 5y = 0 \to e^{-\int 5dx} \to e^{-5\int dx}$

$e^{-5x}\frac{dy}{dx} - e^{-5x}5y = 0 \to \frac{d}{dx}(ye^{-5x}) = \int 0dx + c$

$ye^{-5x} = 0 + C \to \boxed{y = Ce^{5x}}$

2) $\frac{dy}{dx} + 2y = 0 \to e^{2\int dx} \to e^{2x} \to F.I = e^{2x}$

$e^{2x}\frac{dy}{dx} + 2ye^{2x} = (e^{2x})0 \to \int\frac{d}{dx}(ye^{2x}) = \int 0 + C$

$ye^{2x} = C \to \boxed{y = Ce^{-2x}}$

3) $\frac{dy}{dx} + y = e^{3x} \to e^{\int dx} \to e^{x} \to F.I = e^{x}$

$e^{x}\frac{dy}{dx} + ye^{x} = e^{3x}e^{x} \to e^{x}\frac{dy}{dx} + ye^{x} = e^{4x}$

$\to \int\frac{d}{dx}(ye^{x}) = \int e^{4x} + c \to ye^{x} = \frac{1}{4}e^{4x} + C$

$y = \frac{1}{4}\frac{e^{4x}}{e^{x}} + \frac{c}{e^{x}} \to \boxed{y = \frac{1}{4}e^{3x} + ce^{-x}}$

4) $3\frac{dy}{dx} + 12y = 4 \to \frac{dy}{dx} + 4y = \frac{4}{3} \to e^{4\int dx} \to e^{4x}$

$e^{4x}\frac{dy}{dx} + 4ye^{4x} = \frac{4}{3}e^{4x} \to \int\frac{d}{dx}[ye^{4x}] = \int\frac{4}{3}e^{4x} + C$

$ye^{4x} = \frac{4}{3}\int e^{4x}dx + C \to ye^{4x} = \left(\frac{4}{3}\right)\left(\frac{1}{4}\right)e^{4x} + C$

$y = \frac{1}{3}\frac{e^{4x}}{e^{4x}} + Ce^{-4x} \to \boxed{y = \frac{1}{3} + Ce^{-4x}}$

5) $y' + 3x^2y = x^2 \to \frac{dy}{dx} + 3x^2y = x^2 \to e^{3\int x^2dx} \to e^{x^3}$

$e^{x^3}\frac{dy}{dx} + e^{x^3}(3x^2y) = e^{x^3}x^2 \to \int\frac{d}{dx}[e^{x^3}y] = \int e^{x^3}x^2 + C$

$ye^{x^3} = \int x^2e^{x^3}dx + C \qquad u = x^3 \quad du = 3x^2dx \quad \frac{du}{3} = x^2dx$

$ye^{x^3} = \frac{1}{3}\int e^{u}du + C \to ye^{x^3} = \frac{1}{3}e^{x^3} + C$

$y = \frac{1}{3}\frac{\cancel{e^{x^3}}}{\cancel{e^{x^3}}} + \frac{C}{e^{x^3}} \to \boxed{y = \frac{1}{3} + Ce^{-x^3}}$

6) $y' + 2xy = x^3 \to \frac{dy}{dx} + 2xy = x^3 \to e^{2\int xdx} \to e^{x^2}$

$e^{x^2}\frac{dy}{dx} + 2xye^{x^2} = x^3e^{x^2} \to \int\frac{d}{dx}[ye^{x^2}] = \int x^3e^{x^2} + C$

$ye^{x^2} = \int x^3e^{x^2} + C \to ye^{x^2} = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C$

$y = \frac{1}{2}\frac{x^2\cancel{e^{x^2}}}{\cancel{e^{x^2}}} - \frac{1}{2}\frac{e^{x^2}}{e^{x^2}} + Ce^{-x^2} \to \boxed{y = \frac{1}{2}x^2 - \frac{1}{2} + Ce^{-x^2}}$

7) $x^2y' + xy = 1 \to x^2\frac{dy}{dx} + xy = 1 \to \frac{dy}{dx} + \frac{xy}{x^2} = \frac{1}{x^2}$

$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2} \to e^{\int\frac{dx}{x}} \to e^{\ln x} \to F.I = X$

$x\frac{dy}{dx} + x\left(\frac{y}{x}\right) = x\left(\frac{1}{x^2}\right) \to x\frac{dy}{dx} + y = \frac{1}{x}$

$\int\frac{d}{dx}[xy] = \int\frac{1}{x} + C \to xy = \int\frac{dx}{x} + c \to xy = \ln|x| + C$

$y = \frac{\ln|x|}{x} + \frac{C}{x} \xrightarrow{\text{Para } x>0} \boxed{y = \frac{1}{x}\ln x + Cx^{-1}}$

8) $y' = 2y + x^2 + 5 \to \frac{dy}{dx} - 2y = x^2 + 5 \to e^{-2\int dx} \to e^{-2x} = F.I$

$e^{-2x}\frac{dy}{dx} - 2ye^{-2x} = e^{-2x}(x^2+5) \to e^{-2x}\frac{dy}{dx} - 2ye^{-2x} = x^2e^{-2x} + 5e^{-2x}$

$\frac{d}{dx}[ye^{-2x}] = \int x^2e^{-2x}dx + \int 5e^{-2x} + C \to ye^{-2x} =$ Continua abajo

Derivada Integral $ye^{-2x} = -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} - \frac{5}{2}e^{-2x} + C$

Derivada		Integral
x^2	+	e^{-2x}
$2x$	−	$-\frac{1}{2}e^{-2x}$
2	+	$\frac{1}{4}e^{-2x}$
0		$-\frac{1}{8}e^{-2x}$

$y = -\frac{1}{2}\frac{x^2\cancel{e^{-2x}}}{\cancel{e^{-2x}}} - \frac{1}{2}\frac{x\cancel{e^{-2x}}}{\cancel{e^{-2x}}} - \frac{1}{4}\frac{e^{-2x}}{e^{-2x}} - \frac{5}{2}\frac{\cancel{e^{-2x}}}{\cancel{e^{-2x}}} + \frac{C}{e^{-2x}}$

$y = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4} - \frac{5}{2} + \frac{C}{e^{-2x}}$

$\boxed{y = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{11}{4} + Ce^{2x}}$

9) $x\frac{dy}{dx} - y = x^2\operatorname{sen}x \to \frac{dy}{dx} - \frac{y}{x} = \frac{x^2\operatorname{sen}x}{x} \to \frac{dy}{dx} - \frac{y}{x} = x\operatorname{sen}x$

$e^{-\int\frac{dx}{x}} \to e^{-\ln x} \to e^{\ln x^{-1}} \to x^{-1} \to F.I = \frac{1}{x}$

$\frac{1}{x}\left(\frac{dy}{dx}\right) - \left(\frac{1}{x}\right)\left(\frac{y}{x}\right) = \frac{1}{x}(x\operatorname{sen}x) \to \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \operatorname{sen}x$

$\int\frac{d}{dx}\left[\frac{y}{x}\right] = \int\operatorname{sen}x\,dx + C \to \frac{y}{x} = -\cos x + C$

$y = -x\cos x + Cx \to \boxed{y = Cx - x\cos x}$

10) $x\frac{dy}{dx} + y = 3 \to \frac{dy}{dx} + \frac{y}{x} = \frac{3}{x} \to e^{\int\frac{dx}{x}} \to e^{\ln x} \to F.I = x$

$x\frac{dy}{dy} + x\left(\frac{y}{x}\right) = x\left(\frac{3}{x}\right) \to x\frac{dy}{dy} + y = 3 \to \int\frac{d}{dx}[xy] = \int 3dx + C$

$xy = 3\int dx + C \to xy = 3x + C \to \boxed{y = 3\frac{x}{y} + Cy^{-1}}$

11) $x\frac{dy}{dx} + 4y = x^3 - x \to \frac{dy}{dx} + 4\frac{y}{x} = \frac{1}{x}(x^3 - x)$

$\frac{dy}{dx} + 4\frac{y}{x} = x^2 - 1 \to e^{4\int\frac{dx}{x}} \to e^{4\ln x} \to e^{\ln x^4} \to x^4 = F.I$

$x^4\frac{dy}{dx} + x^4\left(4\frac{y}{x}\right) = x^4(x^2-1) \to x^4\frac{dy}{dx} + 4x^3y = x^6 - x^4$

$\int\frac{d}{dx}[x^4y] = \int x^6dx - \int x^4dx + C \to x^4y = \frac{x^7}{7} - \frac{x^5}{5} + C$

$y = \frac{1}{7}\frac{x^7}{x^4} - \frac{1}{5}\frac{x^5}{x^4} + Cx^{-4} \to \boxed{y = \frac{1}{7}x^3 - \frac{1}{5}x + Cx^{-4}}$

12) $(1+x)\frac{dy}{dx} - xy = x^2 + x \to \frac{dy}{dx} - \frac{xy}{x+1} = \frac{x\cancel{(x+1)}}{\cancel{(x+1)}}$

$\frac{dy}{dx} - \frac{xy}{x+1} = x \quad e^{\int\frac{x}{x+1}dx} \to \frac{A}{\cancel{x+1}} = \frac{x}{\cancel{x+1}} \to A = X$

$A = 1$

$e^{\int\frac{A}{x+1}dx} \to e^{\int\frac{1}{x+1}dx} \to e^{\ln|x+1|} \to F.I = (x+1)$

$(x+1)\frac{dy}{dx} - \cancel{(x+1)}\left(\frac{xy}{\cancel{x+1}}\right) = x(x+1) \to (x+1)\frac{dy}{dx} - xy = x^2 + x$

$\int\frac{d}{dx}[(x+1)y] = \int x^2 + x + C \to y(x+1) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$

$y = \frac{1}{3}\frac{x^3}{x+1} + \frac{1}{2}\frac{x^2}{x+1} + C(x-1)^{-1}$

* Verificar

13) $x^2y' + x(x+2)y = e^x \to \frac{dy}{dx} + \frac{x(x+2)y}{x^2} = \frac{e^x}{x^2}$

$\frac{dy}{dx} + \frac{(x+2)y}{x} = x^{-2}e^x \to \frac{dy}{dx} + \left(1+\frac{2}{x}\right)y = x^{-2}e^x$

$e^{\int(1+\frac{2}{x})dx} \to e^{\int dx + 2\int\frac{dx}{x}} \to e^{x+2\ln x} \to e^{x+\ln x^2} = F.I.$

$F.I. = x^2e^x \to x^2e^x\frac{dy}{dx} + (x^2e^x)\left(1+\frac{2}{y}\right)y = (\cancel{x^2}e^x)(\cancel{x^{-2}})e^x$

$\int\frac{d}{dx}[x^2ye^x] = \int e^{2x} + C \to x^2ye^x = \int e^{2x} + C$

$x^2ye^x = \frac12 e^{2x} + C \to y = \frac12\frac{\cancel{e^{2x}}}{x^2\cancel{e^x}} + \frac{C}{x^2e^{2x}}$

$y = \frac12\frac{e^{x}}{x^2} + \frac{C}{x^2e^{2x}} \to \boxed{y = \frac{e^x}{2}x^{-2} + Cx^{-2}e^{-2x}}$

14) $xy' + (1+x)y = e^{-x}\operatorname{sen}2x \to x\frac{dy}{dx} + (1+x)y = e^{-x}\operatorname{sen}2x$

$\frac{dy}{dx} + \frac{(x+1)}{x}y = \frac{e^{-x}\operatorname{sen}2x}{x} \to \frac{dy}{dx} + \left(1+\frac1x\right)y = \frac{e^{-x}\operatorname{sen}2x}{x}$

$e^{\int(1+\frac1x)dx} \to e^{\int dx + \int\frac{dx}{x}} \to e^{x+\ln x} \to F.I. = xe^x$

$xe^x\frac{dy}{dx} + xe^x\left(1+\frac1x\right)y = (x\cancel{e^x})\left(\frac{\cancel{e^{-x}}\operatorname{sen}2x}{\cancel{x}}\right)$

$xe^x\frac{dy}{dx} + xe^x\left(1+\frac1x\right)y = \operatorname{sen}2x \to \int\frac{d}{dx}[xye^x] = \int\operatorname{sen}2x + C$

$xye^x = -\frac12\cos2x + C \to \boxed{y = -\frac12\frac{\cos2x}{xe^x} + \frac{Ce^{-x}}{x}}$

15) $ydx - 4(x+y^6)dy = 0 \to ydx = 4(x+y^6)dy$

$y = 4(x+y^6)\frac{dy}{dx} \to \frac{y}{4} = (x+y^6)\frac{dy}{dx} \to (x+y^6)\frac{dy}{dx} - \frac14 y = 0$

*Verificar caso

17) $\cos x\frac{dy}{dx} + (\operatorname{sen}x)y = 1 \to \frac{dy}{dx} + \frac{\operatorname{sen}x}{\cos x}y = \frac{1}{\cos x}$

$\frac{dy}{dx} + y\tan y = \sec x \quad e^{\int\tan y\,dy} \to e^{\ln\sec x} \to \sec x = FI$

$\sec x\frac{dy}{dx} + (\sec x\tan y)y = \sec x\cdot\sec x \to \sec x$

$\sec x\frac{dy}{dx} + (\sec x+\tan y)y = \sec^2x \to \frac{d}{dx}[y\sec x] = \int\sec^2x + C$

$y\sec x = \tan x + C \to y = \frac{\tan x}{\sec x} + C\cos x$

$y = \frac{\frac{\operatorname{sen}x}{\cos x}}{\frac{1}{\cos x}} + C\cos x \to y = \left(\frac{\operatorname{sen}x}{\cos x}\right)\left(\frac{\cos x}{1}\right) + C\cos x$

$\boxed{y = \operatorname{sen}x + C\cos x}$

16) $\cos^2x\operatorname{sen}x\frac{dy}{dx} + (\cos^3x)y = 1 \to \frac{dy}{dx} + \frac{\cancel{\cos^3x}}{\cos^2x\operatorname{sen}x}y = \frac{1}{\cos^2x\operatorname{sen}x}$

$\frac{dy}{dx} + \frac{\cos x}{\operatorname{sen}x}y = \frac{1}{\cos^2x\operatorname{sen}x} \to \frac{dy}{dx} + y\cot x = \frac{1}{\cos^2x\operatorname{sen}x}$

$e^{\int\cot x\,dx} \to e^{\ln|\operatorname{sen}x|} \to \operatorname{sen}x$

$\operatorname{sen}x\frac{dy}{dx} + y\cancel{\operatorname{sen}x}\left(\frac{\cos x}{\cancel{\operatorname{sen}x}}\right) = \frac{\cancel{\operatorname{sen}x}}{\cos^2x\cancel{\operatorname{sen}x}}$

$\int\frac{d}{dx}[y\operatorname{sen}x] = \int\sec^2x + C \to y\operatorname{sen}x = \tan x + C$

$y = \frac{\tan x}{\operatorname{sen}x} + \frac{C}{\operatorname{sen}x} \to y = \frac{\frac{\operatorname{sen}x}{\cos x}}{\operatorname{sen}x} + C\csc x$

$y = \left(\frac{\operatorname{sen}x}{\cos x}\right)\left(\frac{1}{\operatorname{sen}x}\right) + C\csc x \to \boxed{y = \sec x + C\csc x}$

17) $(x+1)\frac{dy}{dx} + (x+2)y = 2xe^{-x} \to \frac{dy}{dx} + \frac{(x+2)}{(x+1)}y = \frac{2xe^{-x}}{x+1}$

$e^{\int\frac{x+2}{x+1}dx} \to \int\frac{x+2}{x+1}dx \to$ $x+2 \,|\, x+1$; $-x-1 \,|\, 1+\frac{1}{x+1}$; $\frac{x}{x} = 1(x+1)$; $= -x-1$

$e^{\int(1+\frac{1}{x+1})dx} \to e^{\int dx + \int\frac{dx}{x+1}} \to e^{x+\ln|x+1|} \to e^x(x+1) = F.I.$

$e^x(x+1)\frac{dy}{dx} + e^x(\cancel{x+1})\left(\frac{x+2}{\cancel{x+1}}\right)y = \cancel{e^x}(\cancel{x+1})\frac{2x\cancel{e^{-x}}}{\cancel{(x+1)}}$

$e^x(x+1)\frac{dy}{dx} + e^x(x+2)y = 2x \to \int\frac{d}{dx}[ye^x(x+1)] = \int 2x + C$

$ye^x(x+1) = x^2 + C \to \boxed{y = \frac{x^2e^{-x}}{x+1} + \frac{Ce^{-x}}{x+1}}$

18) $(x+2)^2\frac{dy}{dx} = 5 - 8y - 4xy \to (x+2)^2\frac{dy}{dx} + 8y + 4xy = 5$

$(x+2)^2\frac{dy}{dx} + 4y(x+2) = 5 \to \frac{dy}{dx} + \frac{4y\cancel{(x+2)}}{(x+2)^{\cancel{2}}} = \frac{5}{(x+2)^2}$

$\frac{dy}{dx} + \left(\frac{4}{x+2}\right)y = \frac{5}{(x-2)^2} \quad e^{4\int\frac{dx}{x+2}} \to e^{4\ln|x+2|} \to e^{\ln(x+2)^4}$

$F.I. = (x+2)^4 \to (x+2)^4\frac{dy}{dx} + (x+2)^{\cancel{4}3}\left(\frac{4}{\cancel{x+2}}\right)y = (x-2)^{\cancel{4}2}\frac{5}{\cancel{(x-2)^2}}$

$(x+2)^4\frac{dy}{dx} + 4(x+2)^3y = 5(x-2)^2 \to \int\frac{d}{dx}[(x+2)^4y] = \int5(x-2)^2 + C$

$\to y(x+2)^4 = 5\int(x-2)^2 + C \to y(x+2)^4 = 5\int u^2 + C$

$u = x-2 \quad du = dx$

$y(x+2)^4 = \frac53u^3 + C$

$y(x+2)^4 = \frac53(x-2)^3 + C \to y = \frac53\frac{(x-2)^{\cancel{3}}}{\cancel{(x+4)}^4} + \frac{C}{(x+4)^4}$

$y = \frac53\cdot\frac{1}{x+4} + \frac{C}{(x+4)^4} \to \boxed{y = \frac{5}{3x+12} + \frac{C}{(x+4)^4}}$

19) $\frac{dr}{d\theta} + r\sec\theta = \cos\theta \to e^{\int\sec\theta\,d\theta} \to e^{\ln|\sec x+\tan x|}$ $\boxed{\theta = x}$

$F.I. = \sec x + \tan x \to (\sec x+\tan x)\frac{dr}{d\theta} + r\sec\theta(\text{regla}) = \cos\theta(\text{regla})$

↳ regla

$\int\frac{d}{d\theta}[r(\sec x+\tan x)] = \int\frac{\cancel{\cos x}}{\cancel{\cos x}} + \int\cancel{\cos x}\left(\frac{\operatorname{sen}x}{\cancel{\cos x}}\right)$

$r(\sec x + \tan x) = \int 1 + \operatorname{sen}x + C$

$r(\sec x+\tan x) = x - \cos x + C$

$r = \frac{x}{\sec x+\tan x} - \frac{\cos x}{\sec x+\tan x} + \frac{C}{\sec x+\tan x}$

$\boxed{r = \frac{1}{\sec\theta+\tan\theta}\left(\theta - \cos\theta + C\right)}$

Ecuaciones Diferenciales Lineales – El factor de integración contraataca

20) $x\frac{dy}{dx}+(3x+1)y=e^{-3x} \rightarrow \frac{dy}{dx}+\frac{(3x+1)}{(x)}y=\frac{e^{-3x}}{x}$

$\frac{dy}{dx}+\left(3+\frac{1}{x}\right)y=\frac{e^{-3x}}{x} \rightarrow e^{\int\left(3+\frac{1}{x}\right)} \rightarrow e^{\int 3dx+\int\frac{dx}{x}}$

$e^{3x+\ln x}$

$\rightarrow F.I=xe^{3x} \rightarrow$ Tosonotachi

$xe^{3x}\frac{dy}{dx}+xe^{3x}\left(3+\frac{1}{x}\right)y=\cancel{xe^{3x}}\left(\frac{\cancel{e^{-3x}}}{\cancel{x}}\right)$

$xe^{3x}\frac{dy}{dx}+xe^{3x}\left(1+\frac{1}{x}\right)y=1 \rightarrow \int\frac{d}{dx}\left[xye^{3x}\right]=\int 1+C$

$xye^{3x}=x+C \rightarrow y=\frac{x}{xe^{3x}}+\frac{C}{xe^{3x}}$

$y=\frac{1}{e^{3x}}+\frac{C}{xe^{3x}} \rightarrow \boxed{y=e^{-3x}+\frac{Ce^{-3x}}{x}}$

21) $(x^2-1)\frac{dy}{dx}+2y=(x+1)^2 \rightarrow \frac{dy}{dy}+\frac{2y}{(x^2-1)}=\frac{(x+1)^2}{\cancel{(x+1)}(x-1)}$

$\frac{dy}{dx}+\frac{2y}{x^2-1}=\frac{x+1}{x-1} \rightarrow e^{\int\frac{dx}{x^2-1}} \rightarrow e^{\int\frac{dx}{(x+1)(x-1)}}$

$\frac{A}{x+1}+\frac{B}{x-1}=\frac{1}{\cancel{(x+1)(x-1)}}=A(x-1)+B(x+1)=1$

$Ax-A+Bx+B=1 \rightarrow x(A+B)-A+B=1$ $\begin{cases}A+B=0\\-A+B=1\end{cases}$

$A=-\frac{1}{2}$ $B=\frac{1}{2}$

$\int\frac{-1/2}{x+1}dx+\int\frac{1/2\,dx}{x-1} \rightarrow -\frac{1}{2}\int\frac{dx}{x+1}+\frac{1}{2}\int\frac{dx}{x-1}$

$-\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1| \rightarrow \frac{1}{2}\left[\ln(x-1)-\ln|x+1|\right]$

$\frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) \rightarrow \ln\left(\frac{x-1}{x+1}\right)^{1/2} \rightarrow e^{\ln\left(\frac{x-1}{x+1}\right)^{1/2}} \rightarrow F.I=\frac{x-1}{x+1}$

$\left(\frac{x-1}{x+1}\right)\frac{dy}{dx}+\left(\frac{\cancel{x-1}}{x+1}\right)\frac{2y}{(x+1)\cancel{(x-1)}}=\left(\frac{\cancel{x+1}}{\cancel{x-1}}\right)\left(\frac{\cancel{x-1}}{\cancel{x+1}}\right)$

$\left(\frac{x+1}{x-1}\right)\frac{dy}{dy}+\frac{2y}{(x+1)^2}=1 \rightarrow \frac{d}{dx}\left[y\left(\frac{x-1}{x+1}\right)\right]=\int dx+C$

$y\left(\frac{x-1}{x+1}\right)=x+C \rightarrow y=\frac{x}{\frac{x-1}{x+1}}+\frac{C}{\frac{x-1}{x+1}}$

$\boxed{y=x\left(\frac{x+1}{x-1}\right)+C\left(\frac{x+1}{x-1}\right)}$