2

Laplace Transforms

Exercises 2.2.6

▪ **1(a)** $\mathcal{L}\{\cosh 2t\} = \mathcal{L}\{\frac{1}{2}(e^{2t} + e^{-2t}) = \frac{1}{2}\big[\frac{1}{s-2} + \frac{1}{s+2}\big] = \frac{s}{s^2-4},\ Re(s) > 2$

1(b) $\mathcal{L}\{t^2\} = \frac{2}{s^3},\ Re(s) > 0$

1(c) $\mathcal{L}\{3+t\} = \frac{3}{s} + \frac{1}{s^2} = \frac{3s+1}{s^2},\ Re(s) > 0$

1(d) $\mathcal{L}\{te^{-t}\} = \frac{1}{(s+1)^2},\ Re(s) > -1$

▪ **2(a)** 5 **(b)** -3 **(c)** 0 **(d)** 3 **(e)** 2 **(f)** 0
(g) 0 **(h)** 0 **(i)** 2 **(j)** 3

▪ **3(a)** $\mathcal{L}\{5-3t\} = \frac{5}{s} - \frac{3}{s^2} = \frac{5s-3}{s^2},\ Re(s) > 0$

3(b) $\mathcal{L}\{7t^3 - 2\sin 3t\} = 7.\frac{6}{s^4} - 2.\frac{3}{s^2+9} = \frac{42}{s^4} - \frac{6}{s^2+9},\ Re(s) > 0$

3(c) $\mathcal{L}\{3 - 2t + 4\cos 2t\} = \frac{3}{s} - \frac{2}{s^2} + 4.\frac{s}{s^2+4} = \frac{3s-2}{s^2} + \frac{4s}{s^2+4},\ Re(s) > 0$

3(d) $\mathcal{L}\{\cosh 3t\} = \frac{s}{s^2-9},\ Re(s) > 3$

3(e) $\mathcal{L}\{\sinh 2t\} = \frac{2}{s^2-4},\ Re(s) > 2$

3(f) $\mathcal{L}\{5e^{-2t} + 3 - 2\cos 2t\} = \dfrac{5}{s+2} + \dfrac{3}{s} - 2.\dfrac{s}{s^2+4},\ \text{Re}(s) > 0$

3(g) $\mathcal{L}\{4te^{-2t}\} = \dfrac{4}{(s+2)^2},\ Re(s) > -2$

3(h) $\mathcal{L}\{2e^{-3t}\sin 2t\} = \dfrac{4}{(s+3)^2+4} = \dfrac{4}{s^2+6s+13},\ Re(s) > -3$

3(i) $\mathcal{L}\{t^2e^{-4t}\} = \dfrac{2}{(s+4)^3},\ Re(s) > -4$

3(j)

$$\begin{aligned}\mathcal{L}\{6t^3 - 3t^2 + 4t - 2\} &= \frac{36}{s^4} - \frac{6}{s^3} + \frac{4}{s^2} - \frac{2}{s}\\ &= \frac{36 - 6s + 4s^2 - 2s^3}{s^4},\ Re(s) > 0\end{aligned}$$

3(k) $\mathcal{L}\{2\cos 3t + 5\sin 3t\} = 2.\dfrac{s}{s^2+9} + 5.\dfrac{3}{s^2+9} = \dfrac{2s+15}{s^2+9},\ Re(s) > 0$

3(l)

$$\begin{aligned}\mathcal{L}\{\cos 2t\} &= \frac{s}{s^2+4}\\ \mathcal{L}\{t\cos 2t\} &= -\frac{d}{ds}\Big[\frac{s}{s^2+4}\Big] = \frac{s^2-4}{(s^2+4)^2}\end{aligned}$$

3(m)

$$\begin{aligned}\mathcal{L}\{t\sin 3t\} &= -\frac{d}{ds}\Big[\frac{3}{s^2+9}\Big] = \frac{6s}{(s^2+9)^2}\\ \mathcal{L}\{t^2\sin 3t\} &= -\frac{d}{ds}\Big[\frac{6s}{(s^2+9)^2}\Big] = -\Big[\frac{(s^2+9)^2 6 - 6s(s^2+9)^2 4s}{(s^2+9)^4}\Big]\\ &= \frac{18s^2-54}{(s^2+9)^3},\ Re(s) > 0\end{aligned}$$

3(n) $\mathcal{L}\{t^2 - 3\cos 4t\} = \dfrac{2}{s^3} - \dfrac{3s}{s^2+16},\ Re(s) > 0$

3(o)

$$\begin{aligned}\mathcal{L}\{t^2e^{-2t} - e^{-t}\cos 2t + 3\} &= \frac{2}{(s+2)^3} + \frac{(s+1)}{(s+1)^2+4} + \frac{3}{s} \\ &= \frac{2}{(s+2)^3} + \frac{s+1}{s^2+2s+5} + \frac{3}{s},\ Re(s) > 0\end{aligned}$$

Exercises 2.2.10

■ **4(a)** $\mathcal{L}^{-1}\{\frac{1}{(s+3)(s+7)}\} = \mathcal{L}^{-1}\{\frac{\frac{1}{4}}{s+3} - \frac{\frac{1}{4}}{s+7}\} = \frac{1}{4}[e^{-3t} - e^{-7t}]$

4(b) $\mathcal{L}^{-1}\{\frac{s+5}{(s+1)(s-3)}\} = \mathcal{L}^{-1}\{\frac{-1}{s+1} + \frac{2}{s-3}\} = -e^{-t} + 2e^{3t}$

4(c) $\mathcal{L}^{-1}\{\frac{s-1}{s^2(s+3)}\} = \mathcal{L}^{-1}\{\frac{\frac{4}{9}}{s} - \frac{\frac{1}{3}}{s^2} - \frac{\frac{4}{9}}{s+3}\} = \frac{4}{9} - \frac{1}{3}t - \frac{4}{9}e^{-3t}$

4(d) $\mathcal{L}^{-1}\{\frac{2s+6}{s^2+4}\} = \mathcal{L}^{-1}\{2.\frac{s}{s^2+2^2} + 3.\frac{2}{s^2+2^2}\} = 2\cos 2t + 3\sin 2t$

4(e)

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{1}{s^2(s^2+16)}\} &= \mathcal{L}^{-1}\{\frac{0}{s} + \frac{\frac{1}{16}}{s^2} - \frac{\frac{1}{16}}{s^2+16}\} \\ &= \frac{1}{16}t - \frac{1}{64}\sin 4t = \frac{1}{64}[4t - \sin 4t]\end{aligned}$$

4(f) $\mathcal{L}^{-1}\{\frac{s+8}{s^2+4s+5}\} = \mathcal{L}^{-1}\{\frac{(s+2)+6}{(s+2)^2+1}\} = e^{-2t}[\cos t + 6\sin t]$

4(g)

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{s+1}{s^2(s^2+4s+8)}\} &= \mathcal{L}^{-1}\{\frac{\frac{1}{8}}{s} + \frac{-\frac{1}{8}s + \frac{1}{2}}{(s+2)^2+2^2}\} \\ &= \mathcal{L}^{-1}\{\frac{1}{8}.\frac{1}{s} - \frac{1}{8}\frac{(s+2)-3(2)}{(s+2)^2+2^2}\} \\ &= \frac{1}{8}[1 - e^{-2t}\cos 2t + 3e^{-2t}\sin 2t]\end{aligned}$$

4(h)

$$\mathcal{L}^{-1}\Big\{\frac{4s}{(s-1)(s+1)^2}\Big\} = \mathcal{L}^{-1}\Big\{\frac{1}{s-1} - \frac{1}{(s+1)} + \frac{2}{(s+1)^2}\Big\}$$
$$= e^t - e^{-t} + 2te^t$$

4(i) $$\mathcal{L}^{-1}\Big\{\frac{s+7}{s^2+2s+5}\Big\} = \mathcal{L}^{-1}\Big\{\frac{(s+1)+3(2)}{(s+1)^2+2^2}\Big\} = e^{-t}[\cos 2t + 3\sin 2t]$$

4(j)

$$\mathcal{L}^{-1}\Big\{\frac{s^2-7s+5}{(s-1)(s-2)(s-3)}\Big\} = \mathcal{L}^{-1}\Big\{\frac{\frac{1}{2}}{s-1} - \frac{3}{s-2} + \frac{\frac{1}{2}}{s-3}\Big\}$$
$$= \frac{1}{2}e^t - 3e^{2t} + \frac{11}{2}e^{3t}$$

4(k)

$$\mathcal{L}^{-1}\Big\{\frac{5s-7}{(s+3)(s^2+2)}\Big\} = \mathcal{L}^{-1}\Big\{\frac{-2}{s+3} + \frac{2s-1}{s^2+2}\Big\}$$
$$= -2e^{-3t} + 2\cos\sqrt{2}t - \frac{1}{\sqrt{2}}\sin\sqrt{2}t$$

4(l)

$$\mathcal{L}^{-1}\Big\{\frac{s}{(s-1)(s^2+2s+2)}\Big\} = \mathcal{L}^{-1}\Big\{\frac{\frac{1}{5}}{s-1} - \frac{1}{5}\frac{s-2}{s^2+2s+2}\Big\}$$
$$= \mathcal{L}^{-1}\Big\{\frac{\frac{1}{5}}{s-1} - \frac{1}{5}\frac{(s+1)-3}{(s+1)^2+1}\Big\}$$
$$= \frac{1}{5}e^t - \frac{1}{5}e^{-t}(\cos t - 3\sin t)$$

4(m) $$\mathcal{L}^{-1}\Big\{\frac{s-1}{s^2+2s+5}\Big\} = \mathcal{L}^{-1}\Big\{\frac{(s+1)-2}{(s+1)^2+2^2}\Big\} = e^{-t}(\cos 2t - \sin 2t)$$

4(n)

$$\mathcal{L}^{-1}\Big\{\frac{s-1}{(s-2)(s-3)(s-4)}\Big\} = \mathcal{L}^{-1}\Big\{\frac{\frac{1}{2}}{s-2} - \frac{2}{s-3} + \frac{\frac{3}{2}}{s-4}\Big\}$$
$$= \frac{1}{2}e^{2t} - 2e^{3t} + \frac{3}{2}e^{-4t}$$

4(o)

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{3s}{(s-1)(s^2-4)}\} &= \mathcal{L}^{-1}\{\frac{3s}{(s-1)(s-2)(s+2)}\}\\ &= \mathcal{L}^{-1}\{\frac{-1}{s-1}+\frac{\frac{3}{2}}{s-2}-\frac{\frac{1}{2}}{s+2}\}\\ &= -e^t+\frac{3}{2}e^{2t}-\frac{1}{2}e^{-2t}\end{aligned}$$

4(p)

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{36}{s(s^2+1)(s^2+9)}\} &= \mathcal{L}^{-1}\{\frac{4}{s}-\frac{\frac{9}{2}s}{s^2+1}+\frac{\frac{1}{2}s}{s^2+9}\}\\ &= 4-\frac{9}{2}\cos t+\frac{1}{2}\cos 3t\end{aligned}$$

4(q)

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{2s^2+4s+9}{(s+2)(s^2+3s+3)}\} &= \mathcal{L}^{-1}\{\frac{9}{s+2}-\frac{7s+9}{(s+\frac{3}{2})^2+3/4}\}\\ &= \mathcal{L}^{-1}\{\frac{9}{s+2}-\frac{7(s+\frac{3}{2})-\sqrt{3}.\sqrt{3}/2}{(s+\frac{3}{2})^2+(\sqrt{3}/2)^2}\}\\ &= 9e^{-2t}-e^{-\frac{3}{2}t}[7\cos\frac{\sqrt{3}}{2}e^{2t}-\sqrt{3}\sin\frac{\sqrt{3}}{2}t]\end{aligned}$$

4(r)

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{1}{(s+1)(s+2)(s^2+2s+10)}\} &= \mathcal{L}^{-1}\{\frac{\frac{1}{9}}{s+1}-\frac{\frac{1}{10}}{s+2}-\frac{\frac{1}{90}s+\frac{1}{9}}{s^2+2s+10}\}\\ &= \mathcal{L}^{-1}\left\{\frac{\frac{1}{9}}{s+1}-\frac{\frac{1}{10}}{s+2}-\frac{1}{90}[\frac{s+10}{(s+1)^2+3^2}]\right\}\\ &= \mathcal{L}^{-1}\left\{\frac{\frac{1}{9}}{s+1}-\frac{\frac{1}{10}}{s+2}-\frac{1}{90}[\frac{(s+1)+3(3)}{(s+1)^2+3^2}]\right\}\\ &= \frac{1}{9}e^{-t}-\frac{1}{10}e^{-2t}-\frac{1}{90}e^{-t}(\cos 3t+3\sin 3t)\end{aligned}$$

Exercises 2.3.5

5(a)

$$(s+3)X(s) = 2 + \frac{1}{s+2} = \frac{2s+5}{s+2}$$
$$X(s) = \frac{2s+5}{(s+2)(s+3)} = \frac{1}{s+2} + \frac{1}{s+3}$$
$$x(t) = \mathcal{L}^{-1}\{X(s)\} = e^{-2t} + e^{-3t}$$

5(b)

$$(3s-4)X(s) = 1 + \frac{2}{s^2+4} = \frac{s^2+6}{s^2+4}$$
$$X(s) = \frac{s^2+6}{(3s-4)(s^2+4)} = \frac{\frac{35}{26}}{3s-4} - \frac{\frac{3}{26}s + \frac{4}{26}}{s^2+4}$$
$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{35}{78}e^{\frac{4}{3}t} - \frac{3}{26}(\cos 2t + \frac{2}{3}\sin 2t)$$

5(c)

$$(s^2+2s+5)X(s) = \frac{1}{s}$$
$$X(s) = \frac{1}{s(s^2+2s+5)} = \frac{\frac{1}{5}}{s} - \frac{1}{5}\cdot\frac{s+2}{s^2+2s+5}$$
$$= \frac{\frac{1}{5}}{s} - \frac{1}{5}\frac{(s+1)+\frac{1}{2}(2)}{(s+1)^2+2^2}$$
$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{5}(1 - e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t)$$

5(d)

$$(s^2+2s+1)X(s) = 2 + \frac{4s}{s^2+4} = \frac{2s^2+4s+8}{s^2+4}$$
$$X(s) = \frac{2s^2+4s+8}{(s+1)^2(s^2+4)}$$
$$= \frac{\frac{12}{25}}{(s+1)} + \frac{\frac{6}{5}}{(s+1)^2} - \frac{1}{25}\left[\frac{12s-32}{s^2+4}\right]$$
$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{12}{25}e^{-t} + \frac{6}{5}te^{-t} - \frac{12}{25}\cos 2t + \frac{16}{25}\sin 2t$$

5(e)

$$(s^2-3s+2)X(s)=1+\frac{2}{s+4}=\frac{s+6}{s+4}$$

$$X(s)=\frac{s+6}{(s+4)(s-1)(s-2)}=\frac{\frac{1}{15}}{s+4}-\frac{\frac{7}{5}}{s-1}+\frac{\frac{4}{3}}{s-2}$$

$$x(t)=\mathcal{L}^{-1}\{X(s)\}=\frac{1}{15}e^{-4t}-\frac{7}{5}e^{t}+\frac{4}{3}e^{2t}$$

5(f)

$$(s^2+4s+5)X(s)=(4s-7)+16+\frac{3}{s+2}$$

$$X(s)=\frac{4s^2+17s+21}{(s+2)(s^2+4s+5)}=\frac{3}{s+2}+\frac{(s+2)+1}{(s+2)^2+1}$$

$$x(t)=\mathcal{L}^{-1}\{X(s)\}=3e^{-2t}+e^{-2t}\cos t+e^{-2t}\sin t$$

5(g)

$$(s^2+s-2)X(s)=s+1+\frac{5(2)}{(s+1)^2+4}$$

$$X(s)=\frac{s^3+3s^2+7s+15}{(s+2)(s-1)(s^2+2s+5)}$$

$$=\frac{-\frac{1}{3}}{s+2}+\frac{\frac{13}{12}}{s-1}+\frac{\frac{1}{4}s-\frac{5}{4}}{s^2+2s+5}$$

$$=\frac{-\frac{1}{3}}{s+2}+\frac{\frac{13}{12}}{s-1}+\frac{1}{4}\left[\frac{(s+1)-3(2)}{(s+1)^2+2^2}\right]$$

$$x(t)=\mathcal{L}^{-1}\{X(s)\}=-\frac{1}{3}e^{-2t}+\frac{13}{12}e^{t}+\frac{1}{4}e^{-t}\cos 2t-\frac{3}{4}e^{-t}\sin 2t$$

5(h)

$$(s^2+2s+3)Y(s) = 1 + \frac{3}{s^2}$$

$$Y(s) = \frac{s^2+3}{s^2(s^2+2s+3)} = \frac{-\frac{2}{3}}{s} + \frac{1}{s^2} + \frac{\frac{2}{3}s+\frac{4}{3}}{s^2+2s+3}$$

$$= \frac{-\frac{2}{3}}{s} + \frac{1}{s^2} + \frac{2}{3}\Big[\frac{(s+1) - \frac{1}{\sqrt{2}}(\sqrt{2})}{(s+1)^2+(\sqrt{2})^2}\Big]$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{2}{3} + t + \frac{2}{3}e^{-t}(\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t)$$

5(i)

$$(s^2+4s+4)X(s) = \frac{1}{2}s + 2 + \frac{2}{s^3} + \frac{1}{s+2}$$

$$X(s) = \frac{s^5+6s^4+10s^3+4s+8}{2s^3(s+2)^3}$$

$$= \frac{\frac{3}{8}}{s} - \frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{2}}{s^3} + \frac{\frac{1}{8}}{s+2} + \frac{\frac{3}{4}}{(s+2)^2} + \frac{1}{(s+2)^3}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{3}{8} - \frac{1}{2}t + \frac{1}{4}t^2 + \frac{1}{8}e^{-2t} + \frac{3}{4}te^{-2t} + \frac{1}{2}t^2e^{-2t}$$

5(j)

$$(9s^2+12s+5)X(s) = \frac{1}{s}$$

$$X(s) = \frac{1}{9s(s^2+\frac{4}{3}s+\frac{5}{9})} = \frac{\frac{1}{5}}{s} - \frac{\frac{1}{5}s+\frac{4}{15}}{(s+\frac{2}{3})^2+\frac{1}{9}}$$

$$= \frac{\frac{1}{5}}{s} - \frac{1}{5}\frac{[(s+\frac{2}{3})+\frac{2}{3}]}{(s+\frac{2}{3})^2+(\frac{1}{3})^2}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{5} - \frac{1}{5}e^{-\frac{2}{3}t}(\cos\frac{1}{3}t + 2\sin\frac{1}{3}t)$$

5(k)

$$(s^2+8s+16)X(s) = -\frac{1}{2}s+1-4+16\cdot\frac{4}{s^2+16} = \frac{-s^3-6s^2-16s+32}{2(s^2+16)}$$

$$X(s) = \frac{-s^3-6s^2-16s+32}{2(s+4)^2(s^2+16)}$$

$$= \frac{0}{s+4}+\frac{1}{(s+4)^2}-\frac{\frac{1}{2}s}{s^2+16}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = te^{-4t}-\frac{1}{2}\cos 4t$$

5(l)

$$(9s^2+12s+4)Y(s) = 9(s+1)+12+\frac{1}{s+1}$$

$$Y(s) = \frac{9s^2+30s+22}{(3s+2)^2(s+1)}$$

$$= \frac{1}{s+1}+\frac{0}{3s+2}+\frac{18}{(3s+2)^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{-t}+2te^{-\frac{2}{3}t}$$

5(m)

$$(s^3-2s^2-s+2)X(s) = s-2+\frac{2}{s}+\frac{1}{s^2}$$

$$X(s) = \frac{s^3-2s^2+2s+1}{s^2(s-1)(s-2)(s+1)}$$

$$= \frac{\frac{5}{4}}{s}+\frac{\frac{1}{2}}{s^2}-\frac{1}{s-1}+\frac{\frac{5}{12}}{s-2}-\frac{\frac{2}{3}}{s+1}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{5}{4}+\frac{1}{2}t-e^t+\frac{5}{12}e^{2t}-\frac{2}{3}e^{-t}$$

5(n)

$$(s^3+s^2+s+1) = (s+1)+1+\frac{s}{s^2+9}$$

$$X(s) = \frac{s^3+2s^2+10s+18}{(s^2+9)(s+1)(s^2+1)} = \frac{\frac{9}{20}}{s+1}-\frac{1}{16}\frac{7s-25}{s^2+1}-\frac{1}{80}\frac{s+9}{s^2+9}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{9}{20}e^{-t}-\frac{7}{16}\cos t+\frac{25}{16}\sin t-\frac{1}{80}\cos 3t-\frac{3}{80}\sin 3t$$

6(a)

$$2sX(s) - (2s+9)Y(s) = -\frac{1}{2} + \frac{1}{s+2}$$
$$(2s+4)X(s) + (4s-37)Y(s) = 1$$

Eliminating $X(s)$

$$[-(2s+9)(2s+4) - 2s(4s-37)]Y(s) = (-\frac{1}{2} + \frac{1}{s+2})(2s+4) - 2s = -3s$$
$$Y(s) = \frac{3s}{12s^2 - 48s + 36} = \frac{1}{4}\cdot\frac{s}{(s-3)(s-1)}$$
$$= \frac{1}{4}[\frac{\frac{3}{2}}{s-3} - \frac{\frac{1}{2}}{s-1}]$$
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{4}[\frac{3}{2}e^{3t} - \frac{1}{2}e^{t}] = \frac{3}{8}e^{3t} - \frac{1}{8}e^{t}$$

Eliminating $\frac{dx}{dt}$ from the two equations

$$6\frac{dy}{dt} + 4x - 28y = -e^{-2t}$$
$$x(t) = \frac{1}{4}[-e^{-2t} + 28y - 6\frac{dy}{dt}] = \frac{1}{4}[-e^{-2t} + \frac{21}{4}e^{3t} - \frac{7}{2}e^{t} - \frac{27}{3}e^{3t} + \frac{3}{4}e^{t}]$$
$$\text{i.e. } x(t) = \frac{1}{4}(\frac{15}{4}e^{3t} - \frac{11}{4}e^{t} - e^{-2t}) \ , y(t) = \frac{1}{8}(3e^{3t} - e^{t})$$

6(b)

$$(s+1)X(s) + (2s-1)Y(s) = \frac{5}{s^2+1}$$
$$(2s+1)X(s) + (3s-1)Y(s) = \frac{1}{s-1}$$

Eliminating $X(s)$

$$[(2s-1)(2s+1) - (3s-1)(s+1)]Y(s) = \frac{5}{s^2+1}(2s+1) - \frac{s+1}{s-1}$$
$$Y(s) = \frac{10s+5}{s(s^2+1)(s-2)} - \frac{s+1}{s(s-1)(s-2)}$$
$$= [\frac{-\frac{5}{2}}{s} + \frac{\frac{5}{2}}{s-2} - \frac{5}{s^2+1}] - [\frac{\frac{1}{2}}{s} - \frac{2}{s-1} + \frac{\frac{3}{2}}{s-2}]$$
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{5}{2} + \frac{5}{2}e^{2t} - 5\sin t - \frac{1}{2} + 2e^{t} - \frac{3}{2}e^{2t}$$
$$= -3 + e^{2t} + 2e^{t} - 5\sin t$$

Eliminating $\frac{dx}{dt}$ from the original equations

$$\frac{dy}{dt} + x - y = 10\sin t - e^t$$

$$x(t) = 10\sin t - e^t - 3 + e^{2t} + 2e^t - 5\sin t - 2e^{2t} - 2e^t + 5\cos t$$

$$= 5\sin t + 5\cos t - 3 - e^t - e^{2t}$$

6(c)

$$(s+2)X(s) + (s+1)Y(s) = 3 + \frac{1}{s+3} = \frac{3s+10}{s+3}$$

$$5X(s) + (s+3)Y(s) = 4 + \frac{5}{s+2} = \frac{4s+13}{s+2}$$

Eliminating $X(s)$

$$[5(s+1) - (s+2)(s+3)]Y(s) = \frac{15s+50}{s+3} - (4s+13) = \frac{-4s^2-10s+11}{s+3}$$

$$Y(s) = \frac{4s^2+10s-11}{(s+3)(s^2+1)} = \frac{-\frac{1}{2}}{s+3} + \frac{\frac{9}{2}s - \frac{7}{2}}{s^2+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{2}e^{-3t} + \frac{9}{2}\cos t - \frac{7}{2}\sin t$$

From the second differential equation

$$5x = 5e^{-2t} + \frac{3}{2}e^{-3t} - \frac{27}{2}\cos t + \frac{21}{2}\sin t - \frac{3}{2}e^{-3t}$$

$$+ \frac{9}{2}\sin t + \frac{7}{2}\cos t$$

$$x(t) = 3\sin t - 2\cos t + e^{-2t}$$

6(d)

$$(3s-2)X(s) + 3sY(s) = 6 + \frac{1}{s-1} = \frac{6s-5}{s-1}$$

$$sX(s) + (2s-1)Y(s) = 3 + \frac{1}{s} = \frac{3s+1}{s}$$

Eliminating $X(s)$

$$[3s^2-(3s-2)(2s-1)]Y(s)=\frac{s(6s-5)}{s-1}-\frac{(3s-2)(3s+1)}{s}$$

$$Y(s)=\frac{9s^2-3s-2}{s(3s-1)(s-2)}-\frac{6s^2-5s}{(s-1)(3s-1)(s-2)}$$

$$=[-\frac{1}{s}+\frac{\frac{18}{5}}{3s-1}+\frac{\frac{14}{5}}{s-2}]$$

$$-[\frac{-\frac{1}{2}}{s-1}-\frac{\frac{9}{10}}{3s-1}+\frac{\frac{14}{5}}{s-2}]$$

$$=-\frac{1}{s}+\frac{\frac{1}{2}}{s-1}+\frac{\frac{9}{2}}{3s-1}$$

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=-1+\frac{1}{2}e^t+\frac{3}{2}e^{\frac{t}{3}}$$

Eliminating $\dfrac{dx}{dt}$ from the original equations

$$x(t)=\frac{1}{2}[3-e^t-3+\frac{3}{2}e^t+\frac{9}{2}e^{\frac{t}{3}}-\frac{3}{2}e^t-\frac{3}{2}e^{\frac{t}{3}}]$$

$$=\frac{3}{2}e^{\frac{t}{3}}-\frac{1}{2}e^t$$

6(e)

$$(3s-2)X(s)+sY(s)=-1+\frac{3}{s^2+1}+\frac{5s}{s^2+1}=\frac{-s^2+5s+2}{s^2+1}$$

$$2sX(s)+(s+1)Y(s)=-1+\frac{1}{s^2+1}+\frac{s}{s^2+1}=\frac{-s^2+s}{s^2+1}$$

Eliminating $Y(s)$

$$[(3s-2)(s+1)-2s^2]X(s)=\frac{1}{s^2+1}[(-s^2+5s+2)(s+1)-(-s^2+s)s]$$

$$X(s)=\frac{3s^2+7s+2}{(s+2)(s-1)(s^2+1)}=\frac{3s+1}{(s-1)(s^2+1)}$$

$$=\frac{2}{s-1}-\frac{2s-1}{s^2+1}$$

$$x(t)=\mathcal{L}^{-1}\{X(s)\}=2e^t-2\cos t+\sin t$$

Eliminating $\dfrac{dy}{dt}$ from the original equation

$$\begin{aligned}
y(t) &= -2\sin t - 4\cos t - 2x + \frac{dx}{dt}\\
&= -2\sin t - 4\cos t - 4e^t + 4\cos t - 2\sin t + 2e^t + 2\sin t + \cos t\\
\text{i.e. } y(t) &= -2e^t - 2\sin t + \cos t,\ x(t) = 2e^t - 2\cos t + \sin t
\end{aligned}$$

6(f)

$$\begin{aligned}
sX(s) + (s+1)Y(s) &= 1 + \frac{1}{s^2} = \frac{s^2+1}{s^2}\\
(s+1)X(s) + 4sY(s) &= 1 + \frac{1}{s} = \frac{s+1}{s}
\end{aligned}$$

Eliminating $Y(s)$

$$\begin{aligned}
[4s^2 - (s+1)^2]X(s) &= 4s\Big(\frac{s^2+1}{s^2}\Big) - \frac{(s+1)^2}{s} = \frac{3s^2 - 2s + 3}{s}\\
X(s) &= \frac{3s^2 - 2s + 3}{s(s-1)(3s+1)} = \frac{-3}{s} - \frac{1}{s-1} + \frac{9}{3s+1}\\
x(t) = \mathcal{L}^{-1}\{X(s)\} &= -3 + e^t + 3e^{-\frac{t}{3}}
\end{aligned}$$

Eliminating $\dfrac{dy}{dt}$ from the original equation

$$\begin{aligned}
y &= \frac{1}{4}\Big[4t - 1 + x + 3\frac{dx}{dt}\Big]\\
&= \frac{1}{4}\Big[4t - 1 - 3 + e^t + 3e^{-\frac{t}{3}} - 3e^t + 3e^{-\frac{t}{3}}\Big]\\
\text{i.e. } y(t) &= t - 1 - \frac{1}{2}e^t + \frac{3}{2}e^{-\frac{t}{3}},\ x(t) = -3 + e^t + 3e^{-\frac{t}{3}}
\end{aligned}$$

6(g)

$$\begin{aligned}
(2s+7)X(s) + 3sY(s) &= \frac{12}{s^2} + \frac{7}{s} = \frac{14+7s}{s^2}\\
(5s+4)X(s) - (3s-6)Y(s) &= \frac{14}{s^2} - \frac{14}{s} = \frac{14-14s}{s^2}
\end{aligned}$$

Eliminating $Y(s)$

$$
\begin{aligned}
[(2s+7)(3s-6)+(5s+4)(3s)]X(s) &= \frac{1}{s^2}[(3s-6)(14+7s)+3s(14-14s)] \\
21(s^2+s-2)X(s) = 21(s+2)(s-1)X(s) &= \frac{21}{s^2}(-s^2+2s-4) \\
X(s) &= \frac{-s^2+2s-4}{s^2(s+2)(s-1)} \\
&= \frac{-1}{s-1}+\frac{1}{s+2}+\frac{0}{s}+\frac{2}{s^2} \\
x(t) = \mathcal{L}^{-1}\{X(s)\} &= -e^t+e^{-2t}+2t
\end{aligned}
$$

Eliminating $\dfrac{dy}{dt}$ from the original equations

$$
\begin{aligned}
6y &= 28t-7-11x-7\frac{dx}{dt} \\
&= 28t-7+7e^t+14e^{-2t}-14+11e^t-11e^{-2t}-22t
\end{aligned}
$$

$$
\text{giving } y(t) = t-\frac{7}{2}+3e^t+\frac{1}{2}e^{-2t},\ x(t) = -e^t+e^{-2t}+2t.
$$

6(h)

$$
\begin{aligned}
(s^2+2)X(s)-Y(s) &= 4s \\
-X(s)+(s^2+2)Y(s) &= 2s
\end{aligned}
$$

Eliminating $Y(s)$

$$
\begin{aligned}
[(s^2+2)^2-1]X(s) &= 4s(s^2+2)+2s \\
(s^4+4s^2+3)X(s) &= 4s^3+10s \\
X(s) &= \frac{4s^3+10s}{(s^2+1)(s^2+3)} = \frac{3s}{s^2+1}+\frac{s}{s^2+3} \\
x(t) = \mathcal{L}^{-1}\{X(s)\} &= 3\cos t+\cos\sqrt{3}t
\end{aligned}
$$

From the first of the given equations

$$
\begin{aligned}
y(t) &= 2x+\frac{d^2x}{dt^2} = 6\cos t+2\cos\sqrt{3}t-3\cos t-3\cos\sqrt{3}t \\
\text{i.e. } y(t) &= 3\cos t-\cos\sqrt{3}t,\ x(t) = 3\cos t-\cos\sqrt{3}t
\end{aligned}
$$

6(i)

$$(5s^2+6)X(s)+12s^2Y = s\big[\frac{35}{4}+12\big] = \frac{83}{4}s$$
$$5s^2X(s)+(16s^2+6)Y(s) = s\big[\frac{35}{4}+16\big] = \frac{99}{4}s$$

Eliminating $X(s)$

$$[60s^4-(5s^2+6)(16s^2+6)]Y(s) = \frac{s}{4}[83(5s^2)-99(5s^2+6)]$$
$$[-20s^4-126s^2-36]Y(s) = \frac{s}{4}[-80s^2-594]$$

$$Y(s) = \frac{s(40s^2+297)}{4(s^2+6)(10s^2+3)}$$
$$= \frac{-\frac{1}{4}s}{s^2+6}+\frac{\frac{25}{2}s}{10s^2+3}$$
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{4}\cos\sqrt{6}t+\frac{5}{4}\cos\sqrt{\frac{3}{10}}t$$

Eliminating $\dfrac{d^2x}{dt^2}$ from the original equations

$$3x = 3y+3\frac{d^2y}{dt^2} = \big(\frac{15}{4}-\frac{3}{4}\big)\cos\sqrt{\frac{3}{10}}t+\big(-\frac{3}{4}+3\big)\cos\sqrt{6}t$$
$$\text{i.e. } x(t) = \cos\sqrt{\frac{3}{10}}t+\frac{3}{4}\cos\sqrt{6}t,\ x(t) = \frac{5}{4}\cos\sqrt{\frac{3}{10}}t-\frac{1}{4}\cos\sqrt{6}t.$$

6(j)

$$(2s^2-s+9)X(s)-(s^2+s+3)Y(s) = 2(s+1)-1 = 2s+1$$
$$(2s^2+s+7)X(s)-(s^2-s+5)Y(s) = 2(s+1)+1 = 2s+3$$

Subtract

$$(-2s+2)X(s)-(2s-2)Y(s) = -2 \ \Rightarrow\ X(s)+Y(s) = \frac{1}{s-1}$$
$$\Rightarrow\ x(t)+y(t) = e^t \qquad \text{(i)}$$

Add

$$(4s^2+16)X(s)-(2s+8)Y(s) = 4(s+1)$$
$$2X(s)-Y(s) = \frac{2(s+1)}{s^2+4} \Rightarrow 2x(t)-y(t)$$
$$= 2\cos 2t+\sin 2t \qquad \text{(ii)}$$

Then from (i) and (ii)

$$x(t) = \frac{1}{3}e^t + \frac{2}{3}\cos 2t + \frac{1}{3}\sin 2t,\ y(t) = \frac{2}{3}e^t - \frac{2}{3}\cos 2t - \frac{1}{3}\sin 2t$$

Exercises 2.4.3

- **7** $1\mu\text{F} = 10^{-6}\text{F}$ so $50\mu = 5.10^5\text{F}$

Applying Kirchhoff's second law to the left hand loop

$$\frac{1}{5.10^5}\int i_1 dt + 2\Big(\frac{di_1}{dt} - \frac{di_2}{dt}\Big) = E.\sin 100t$$

Taking Laplace transforms

$$\frac{2.10^4}{s}I_1(s) + 2s[I_1(s) - I_2(s)] = E.\frac{100}{s^2+10^4}$$

$$(10^4 + s^2)I_1(s) - s^2 I_2(s) = E.\frac{50s}{s^2+10^4} \qquad \text{(i)}$$

Applying Kirchhoff's law to the right hand loop

$$100i_2(t) - 2\Big(\frac{di_1}{dt} - \frac{di_2}{dt}\Big) = 0$$

which on taking Laplace transforms gives

$$sI_1(s) = (50+s)I_2(s) \qquad \text{(ii)}$$

Substituting in (i)

$$(10^4+s^2)(50+s)I_2(s) - s^2 I_2(s) = E.\frac{50s^2}{s^2+10^4}$$

$$(s^2 + 200s + 10^4)I_2(s) = \frac{Es^2}{s^2+10^4}$$

$$I_2(s) = E\Big[\frac{s^2}{(s^2+10^4)(s+100)^2}\Big]$$

$$\text{then from (ii) } I_1(s) = E\Big[\frac{s(50+s)}{(s^2+10^4)(s+100)^2}\Big]$$

Expanding in partial functions

$$I_2(s) = E\Big[\frac{-\frac{1}{200}}{s+100} + \frac{\frac{1}{2}}{(s+100)^2} + \frac{\frac{1}{200}s}{s^2+10^4}\Big]$$

$$i_2(t) = \mathcal{L}^{-1}\{I_2(s)\} = E\Big[-\frac{1}{200}e^{-100t} + \frac{1}{2}te^{-100t} + \frac{1}{200}\cos 100t\Big]$$

▪ **8** Applying Kirchhoff's second law to the primary and secondary circuits respectively gives

$$2i_1 + \frac{di_1}{dt} + 1\frac{di_2}{dt} = 10\sin t$$

$$2i_2 + 2\frac{di_2}{dt} + \frac{di_1}{dt} = 0$$

Taking Laplace transforms

$$(s+2)I_1(s) + sI_2(s) = \frac{10}{s^2+1}$$

$$sI_1(s) + 2(s+1)I_2(s) = 0$$

Eliminating $I_1(s)$

$$[s^2 - 2(s+1)(s+2)]I_2(s) = \frac{10s}{s^2+1}$$

$$I_2(s) = -\frac{10s}{(s^2+1)(s^2+7s+6)} = -\frac{10s}{(s^2+1)(s+6)(s+1)}$$

$$= -\Big[\frac{-1}{s+1} + \frac{\frac{12}{37}}{s+6} + \frac{\frac{25}{37}s + \frac{35}{37}}{s^2+1}\Big]$$

$$i_2(t) = \mathcal{L}^{-1}\{I_2(s)\} = e^{-t} - \frac{12}{37}e^{-6t} - \frac{25}{37}\cos t - \frac{35}{37}\sin t$$

▪ **9** Applying Kirchhoff's law to the left and right hand loops gives

$$(i_1 + i_2) + \frac{d}{dt}(i_1 + i_2) + 1\int i_1 dt = E_0 = 10$$

$$i_2 + \frac{di_2}{dt} - 1\int i_1 dt = 0$$

Applying Laplace transforms

$$(s+1)I_1(s) + (s+1)I_2(s) + \frac{1}{s}I_1(s) = \frac{10}{s}$$
$$(s+1)I_2(s) - \frac{1}{s}I_1(s) = 0 \ \Rightarrow \ I_1(s) = s(s+1)I_2(s) \qquad \text{(i)}$$

Substituting back in the first equation

$$s(s+1)^2 I_2(s) + (s+1)I_2(s) + (s+1)I_2(s) = \frac{10}{s}$$
$$(s^2+s+2)I_2(s) = \frac{10}{s(s+1)}$$
$$I_2(s) = \frac{10}{s(s+1)(s^2+s+2)}$$

Then from (i)

$$I_1(s) = \frac{10}{s^2+s+2} = \frac{10}{(s+\frac{1}{2})^2+\frac{7}{4}}$$
$$i_1(t) = \mathcal{L}^{-1}\{I_1(s)\} = \frac{20}{\sqrt{7}}e^{-\frac{1}{2}t}\sin\frac{\sqrt{7}}{2}t$$

■ **10** Applying Newton's law to the motion of each mass

$$\ddot{x}_1 = 3(x_2 - x_1) - x_1 = 3x_2 - 4x_1$$
$$\ddot{x}_2 = -9x_2 - 3(x_2 - x_1) = -12x_2 + 3x_1$$

giving

$$\ddot{x}_1 + 4x_1 - 3x_2 = 0, \ x_1(0) = -1, \ x_2(0) = 2$$
$$\ddot{x}_2 + 12x_2 - 3x_1 = 0$$

Taking Laplace transforms

$$(s^2+4)X_1(s) - 3X_2(s) = -s$$
$$-3X_1(s) + (s^2+12)X_2(s) = 2s$$

Eliminating $X_2(s)$

$$\begin{aligned}
[(s^2+4)(s^2+12)-9]X_1(s) &= -s(s^2+12)+6s \\
(s^2+13)(s^2+3)X_1(s) &= -s^3-6s \\
X_1(s) &= \frac{-s^3-6s}{(s^2+13)(s^2+3)} = \frac{-\frac{3}{10}s}{s^2+3} - \frac{\frac{7}{10}s}{s^2+13} \\
x_1(t) = \mathcal{L}^{-1}\{X_1(s)\} &= -\frac{3}{10}\cos\sqrt{3}t - \frac{7}{10}\cos\sqrt{13}t
\end{aligned}$$

From the first differential equation

$$\begin{aligned}
3x_2 &= 4x_1 + \ddot{x}_1 \\
&= -\frac{6}{5}\cos\sqrt{3}t - \frac{14}{5}\cos\sqrt{13}t + \frac{9}{10}\cos\sqrt{3}t + \frac{91}{10}\cos\sqrt{13}t \\
x_2(t) &= \frac{1}{10}[21\cos\sqrt{13}t - \cos\sqrt{3}t]
\end{aligned}$$

Thus $x_1(t) = -\frac{1}{10}(3\cos\sqrt{3}t + 7\cos\sqrt{13}t)$, $x_2(t) = \frac{1}{10}[21\cos\sqrt{13}t - \cos\sqrt{3}t]$
Natural frequencies are $\sqrt{13}$ and $\sqrt{3}$.

- **11** The equation of motion is

$$M\ddot{x} + b\dot{x} + Kx = Mg \ ; \ x(0) = 0 \ , \ \dot{x}(0) = \sqrt{2gh}$$

The problem is then an investigative one where students are required to investigate for different h values either analytically or by simulation.

- **12** By Newton's second law of motion

$$\begin{aligned}
M_2\ddot{x}_2 &= -K_2x_2 - B_1(\dot{x}_2 - \dot{x}_1) + u_2 \\
M_1\ddot{x}_1 &= B_1(\dot{x}_2 - \dot{x}_1) - K_1x_1 + u_1
\end{aligned}$$

Taking Laplace transforms and assuming quiescent initial state

$$\begin{aligned}
(M_2s^2 + B_1s + K_2)X_2(s) - B_1sX_1(s) &= U_2(s) \\
-B_1sX_2(s) + (M_1s^2 + B_1s + K_1)X_1(s) &= U_1(s)
\end{aligned}$$

Eliminating $X_1(s)$

$$\begin{aligned}[(M_1s^2+B_1s+K_1)(M_2s^2+B_1s+K_2)-B_1^2s^2]X_2(s)&\\ =(M_1s^2+B_1s+K_1)U_2(s)+B_1sU_1(s)&\\ \text{i.e. } X_2(s)=\frac{B_1s}{\Delta}U_1(s)+\frac{(M_1s^2+B_1s+K_1)}{\Delta}U_2(s)&\\ \text{and } x_2(t)=\mathcal{L}^{-1}\{X_2(s)\}=\mathcal{L}^{-1}\Big\{\frac{B_1s}{\Delta}U_1(s)+\frac{(M_1s^2+B_1s+K_1)}{\Delta}U_2(s)\Big\}&\end{aligned}$$

Likewise eliminating $X_2(t)$ from the original equation gives

$$x_1(t)=\mathcal{L}^{-1}\{X_1(s)\}=\mathcal{L}^{-1}\Big\{\frac{(M_1s+B_1s+K_2)}{\Delta}U_1(s)+\frac{B_1s}{\Delta}U_2(s)\Big\}$$

Exercises 2.5.7

- **13**

$$\begin{aligned}f(t)&=tH(t)-tH(t-1)\\ &=tH(t)-(t-1)H(t-1)-1H(t-1)\end{aligned}$$

Thus, using theorem 2.4

$$\mathcal{L}\{f(t)\}=\frac{1}{s^2}-e^{-s}\frac{1}{s^2}-e^{-s}=\frac{1}{s^2}(1-e^{-s})-\frac{1}{s}e^{-s}$$

- **14(a)**

$$\begin{aligned}f(t)&=3t^2H(t)-(3t^2-2t+3)H(t-4)-(2t-8)H(t-6)\\ &=3t^2H(t)-[3(t-4)^2+22(t-4)+43]H(t-4)-[2(t-6)+4]H(t-6)\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{L}\{f(t)\}&=\frac{6}{s^3}-e^{-4s}\mathcal{L}[3t^2+22t+43]-e^{-6s}\mathcal{L}[2t+4]\\ &=\frac{6}{s^3}-[\frac{6}{s^3}+\frac{22}{s^2}+\frac{43}{5}]e^{-4s}-[\frac{2}{s^2}+\frac{4}{s}]e^{-6s}\end{aligned}$$

14(b)

$$\begin{aligned} f(t) &= tH(t) + (2-2t)H(t-1) - (2-t)H(t-2) \\ &= tH(t) - 2(t-1)H(t-1) - (t-2)H(t-2) \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \frac{1}{s^2} - 2e^{-s}\mathcal{L}\{t\} + e^{-2s}\mathcal{L}\{t\} \\ &= \frac{1}{s^2}[1 - 2e^{-s} + e^{-2s}] \end{aligned}$$

▪ **15(a)** $\mathcal{L}^{-1}\{\dfrac{e^{-5s}}{(s-2)^4}\} = \mathcal{L}^{-1}\{e^{-5s}F(s)\}$ where $F(s) = \dfrac{1}{(s-2)^4}$ and by the first shift theorem $f(t) = \mathcal{L}^{-1}\{F(s)\} = \dfrac{1}{6}t^3e^{2t}$.
Thus by the second shift theorem

$$\begin{aligned} \mathcal{L}^{-1}\{\frac{e^{-5s}}{(s-2)^4}\} &= f(t-5)H(t-5) \\ &= \frac{1}{6}(t-5)^3e^{2(t-5)}H(t-5) \end{aligned}$$

15(b) $\mathcal{L}^{-1}\{\dfrac{3e^{-2s}}{(s+3)(s+1)}\} = \mathcal{L}^{-1}\{e^{-2s}F(s)\}$ where

$$\begin{aligned} F(s) &= \frac{3}{(s+3)(s+1)} = \frac{-\frac{3}{2}}{s+3} + \frac{\frac{3}{2}}{s+1} \\ f(t) = \mathcal{L}^{-1}\{F(s)\} &= \frac{3}{2}e^{-t} - \frac{3}{2}e^{-3t} \\ \text{so} \quad \mathcal{L}^{-1}\{\frac{3e^{-2s}}{(s+3)(s+1)}\} &= f(t-2)H(t-2) \\ &= \frac{3}{2}[e^{-(t-2)} - e^{-3(t-2)}]H(t-2) \end{aligned}$$

15(c) $\mathcal{L}^{-1}\{\dfrac{s+1}{s^2(s^2+1)}e^{-s}\} = \mathcal{L}^{-1}\{e^{-s}F(s)\}$ where

$$\begin{aligned} F(s) &= \frac{s+1}{s^2(s^2+1)} = \frac{1}{s} + \frac{1}{s^2} - \frac{s+1}{s^2+1} \\ f(t) = \mathcal{L}^{-1}\{F(s)\} &= 1 + t - \cos t - \sin t \end{aligned}$$

so

$$\begin{aligned}\mathcal{L}^{-1}\Big\{\frac{s+1}{s^2(s^2+1)}e^{-s}\Big\} &= f(t-1)H(t-1)\\ &= [1+(t-1)-\cos(t-1)-\sin(t-1)]H(t-1)\\ &= [t-\cos(t-1)-\sin(t-1)]H(t-1)\end{aligned}$$

15(d) $\mathcal{L}^{-1}\Big\{\dfrac{s+1}{s^2+s+1}e^{-\pi s}\Big\} = \mathcal{L}^{-1}\{e^{-\pi s}F(s)\}$ where

$$F(s) = \frac{s+1}{(s^2+s+1)} = \frac{(s+\frac{1}{2})+\frac{1}{\sqrt{3}}(\frac{\sqrt{3}}{2})}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$f(t) = e^{-\frac{1}{2}t}\Big\{\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\Big\}$$

so

$$\mathcal{L}^{-1}\Big\{\frac{s+1}{s^2+s+1}e^{-\pi s}\Big\} = \frac{1}{\sqrt{3}}e^{-\frac{1}{2}(t-\pi)}\Big[\sqrt{3}\cos\frac{\sqrt{3}}{2}(t-\pi)+\sin\frac{\sqrt{3}}{2}(t-\pi)\Big].H(t-\pi)$$

15(e) $\mathcal{L}^{-1}\Big\{\dfrac{s}{s^2+25}e^{-4\pi s/5}\Big\} = \mathcal{L}^{-1}\{e^{-4\pi s/5}F(s)\}$ where

$$F(s) = \frac{s}{s^2+25} \;\Rightarrow\; f(t) = \mathcal{L}^{-1}\{F(s)\} = \cos 5t$$

so

$$\begin{aligned}\mathcal{L}^{-1}\Big\{\frac{s}{s^2+25}e^{-4\pi s/5}\Big\} &= f\Big(t-\frac{4\pi}{5}\Big)H\Big(t-\frac{4\pi}{5}\Big)\\ &= \cos(5t-4\pi)H\Big(t-\frac{4\pi}{5}\Big)\\ &= \cos 5t\ H\Big(t-\frac{4\pi}{5}\Big)\end{aligned}$$

15(f) $\mathcal{L}^{-1}\Big\{\dfrac{e^{-s}(1-e^{-s})}{s^2(s^2+1)}\Big\} = \mathcal{L}^{-1}\{(e^{-s}-e^{-2s})F(s)\}$ where

$$F(s) = \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t - \sin t$$

so

$$\begin{aligned}\mathcal{L}^{-1}\{(e^{-s}-e^{-2s})F(s)\} =& f(t-1)H(t-1)-f(t-2)H(t-2)\\ =& [(t-1)-\sin(t-1)]H(t-1)\\ & -[(t-2)-\sin(t-2)]H(t-2)\end{aligned}$$

■ **16** $\dfrac{dx}{dt}+x=f(t),\ \mathcal{L}\{f(t)\}=\dfrac{1}{s^2}(1-e^{-s}-se^{-s})$

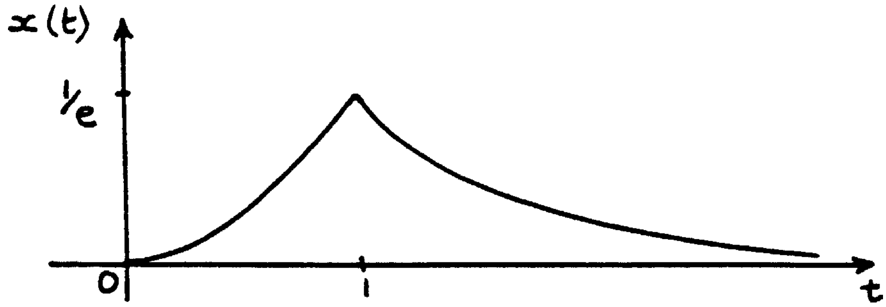
Taking Laplace transforms with $x(0)=0$

$$\begin{aligned}(s+1)X(s) &= \frac{1}{s^2}-e^{-s}\frac{(1+s)}{s^2}\\ X(s) &= \frac{1}{s^2(s+1)}-e^{-s}\frac{1}{s^2}\\ &= -\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s+1}-e^{-s}\mathcal{L}\{t\}\end{aligned}$$

Taking inverse transforms

$$\begin{aligned}x(t) &= -1+e^{-t}+t-(t-1)H(t-1)\\ &= e^{-t}+(t-1)[1-H(t-1)]\\ \text{or } x(t) &= e^{-t}+(t-1) \text{ for } t\le 1\\ x(t) &= e^{-t} \text{ for } t\ge 1\end{aligned}$$

Sketch of response is



■ **17** $\dfrac{d^2x}{dt^2}+\dfrac{dx}{dt}+x=g(t),\ x(0)=1,\ \dot{x}(0)=0$

with $\mathcal{L}\{g(t)\}=\dfrac{1}{s^2}(1-2e^{-s}+e^{-2s})$

Taking Laplace transforms

$$(s^2+s+1)X(s)=s+1+\frac{1}{s^2}(1-2e^{-s}+e^{-2s})$$

$$\begin{aligned}X(s)&=\frac{s+1}{(s^2+s+1)}+\frac{1}{s^2(s^2+s+1)}(1-2e^{-s}+e^{-2s})\\&=\frac{(s+1)}{(s^2+s+1)}+[-\frac{1}{s}+\frac{1}{s^2}+\frac{s}{s^2+s+1}][1-2e^{-s}+e^{-2s}]\\&=\frac{(s+\frac{1}{2})+\frac{1}{\sqrt{3}}(\frac{\sqrt{3}}{2})}{(s+\frac{1}{2})^2+(\dfrac{\sqrt{3}}{2})^2}+\left[-\frac{1}{s}+\frac{1}{s^2}+\frac{(s+\frac{1}{2})-\frac{1}{\sqrt{3}}(\frac{\sqrt{3}}{2})}{(s+\frac{1}{2})^2+(\dfrac{\sqrt{3}}{2})^2}\right][1-2e^{-s}+e^{-2s}]\end{aligned}$$

$$\begin{aligned}x(t)=\mathcal{L}^{-1}\{X(s)\}=e^{-\frac{1}{2}t}\big(&\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\big)\\&+t-1+e^{-\frac{1}{2}t}\big(\cos\frac{\sqrt{3}}{2}t-\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\big)\\&-2H(t-1)\Big[t-2+e^{-\frac{1}{2}(t-1)}\big\{\cos\frac{\sqrt{3}}{2}(t-1)\\&-\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(t-1)\big\}\Big]\\&+H(t-2)\Big[t-3+e^{-\frac{1}{2}(t-2)}\big\{\cos\frac{\sqrt{3}}{2}(t-2)\\&-\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(t-2)\big\}\Big]\end{aligned}$$

i.e.

$$\begin{aligned}x(t)=2e^{-\frac{1}{2}t}&\cos\tfrac{\sqrt{3}}{2}t+t-1\\&-2H(t-1)\Big[t-2+e^{-\frac{1}{2}(t-1)}\big\{\cos\frac{\sqrt{3}}{2}(t-1)-\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(t-1)\big\}\Big]\\&+H(t-2)\Big[t-3+e^{-\frac{1}{2}(t-2)}\big\{\cos\frac{\sqrt{3}}{2}(t-2)-\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(t-2)\big\}\Big]\end{aligned}$$

■ **18**

$$f(t) = \sin t H\big(t - \frac{\pi}{2}\big) = \cos\big(t - \frac{\pi}{2}\big) H\big(t - \frac{\pi}{2}\big)$$
$$\text{since } \cos\big(t - \frac{\pi}{2}\big) = \sin t.$$

Taking Laplace transforms with $x(0) = 1, \dot{x}(0) = -1$

$$\begin{aligned}
(s^2 + 3s + 2)X(s) &= s + 2 + \mathcal{L}\big\{\cos\big(t - \frac{\pi}{2}\big) H\big(t - \frac{\pi}{2}\big)\big\} \\
&= s + 2 + e^{-\frac{\pi}{2}s}\mathcal{L}\{\cos t\} \\
&= s + 2 + e^{-\frac{\pi}{2}s} \cdot \frac{s}{s^2 + 1} \\
X(s) &= \frac{1}{s+1} + e^{-\frac{\pi}{2}s}\big[\frac{s}{(s+1)(s+2)(s^2+1)}\big] \\
&= \frac{1}{s+1} + e^{-\frac{\pi}{2}s}\big[\frac{-\frac{1}{2}}{s+1} + \frac{\frac{2}{5}}{s+2} + \frac{1}{10} \cdot \frac{s+3}{s^2+1}\big] \\
&= \frac{1}{s+1} + e^{-\frac{\pi}{2}s}\mathcal{L}\big\{-\frac{1}{2}e^{-t} + \frac{2}{5}e^{-2t} + \frac{1}{10}(\cos t + 3\sin t)\big\} \\
\text{so } x(t) = \mathcal{L}^{-1}\{X(s)\} &= e^{-t} + \bigg[-\frac{1}{2}e^{-(t-\frac{\pi}{2})} + \frac{2}{5}e^{-2(t-\frac{\pi}{2})} + \frac{1}{10}(\cos\big(t - \frac{\pi}{2}\big) \\
&\qquad + 3\sin\big(t - \frac{\pi}{2}\big))\bigg] H\big(t - \frac{\pi}{2}\big) \\
&= e^{-t} + \frac{1}{10}\big[\sin t - 3\cos t + 4e^{\pi}e^{-2t} - 5e^{\frac{\pi}{2}}e^{-t}\big] H\big(t - \frac{\pi}{2}\big)
\end{aligned}$$

■ **19**

$$\begin{aligned}
f(t) &= 3H(t) - (8 - 2t)H(t - 4) \\
&= 3H(t) + 2(t - 4)H(t - 4) \\
\mathcal{L}\{f(t)\} &= \frac{3}{s} + 2e^{-4s}\mathcal{L}\{t\} = \frac{3}{s} + \frac{2}{s^2}e^{-4s}
\end{aligned}$$

Taking Laplace transforms with $x(0) = 1,\ \dot{x}(0) = 0$

$$\begin{aligned}
(s^2 + 1)X(s) &= s + \frac{3}{s} + \frac{2}{s^2}e^{-4s} \\
X(s) &= \frac{s}{s^2+1} + \frac{3}{s(s^2+1)} + \frac{2}{s^2(s^2+1)}e^{-4s} \\
&= \frac{s}{s^2+1} + \frac{3}{5} - \frac{3}{s^2+1} + 2\big[\frac{1}{s^2} - \frac{1}{s^2+1}\big]e^{-4s} \\
&= \frac{3}{5} - \frac{2}{s^2+1} + 2e^{-4s}\mathcal{L}\{t - \sin t\}
\end{aligned}$$

Thus taking inverse transforms

$$x(t) = 3 - 2\cos t + 2(t - 4 - \sin(t-4))H(t-4)$$

■ **20**

$$\ddot{\theta}_0 + 6\dot{\theta}_0 + 10\theta_0 = \theta_i \tag{1}$$

$$\theta_i(t) = 3H(t) - 3H(t-a)$$

$$\text{so } \mathcal{L}\{\theta_i\} = \frac{3}{s} - \frac{3}{s}e^{-as} = \frac{3}{s}(1 - e^{-as})$$

Taking Laplace transforms in (1) with $\theta_0 = \dot{\theta}_0 = 0$ at $t = 0$

$$\begin{aligned}(s^2 + 6s + 10)\Phi_0(s) &= \frac{3}{s}(1 - e^{-as})\\ \Phi_0(s) &= 3(1 - e^{-as})\Big[\frac{1}{s(s^2+6s+10)}\Big]\\ &= \frac{3}{10}(1 - e^{-as})\Big[\frac{1}{s} - \frac{(s+3)+3}{(s+3)^2+1}\Big]\\ &= \frac{3}{10}(1 - e^{-as})\mathcal{L}\big[1 - e^{-3t}\cos t - 3e^{-3t}\sin t\big]\end{aligned}$$

Thus taking inverse transforms

$$\begin{aligned}\theta_0(t) = &\frac{3}{10}[1 - e^{-3t}\cos t - 3e^{-3t}\sin t]H(t)\\ &- \frac{3}{10}[1 - e^{-3(t-a)}\cos(t-a) - 3e^{-3(t-a)}\sin(t-a)]H(t-a)\end{aligned}$$

If $T > a$ then $H(T) = 1,\ H(T-a) = 1$ giving

$$\begin{aligned}\theta_0(T) = &-\frac{3}{10}[e^{-3T}\cos T - e^{-3(T-a)}\cos(T-a)]\\ &- \frac{3}{10}[3e^{-3T}\sin T - 3e^{-3(T-a)}\sin(T-a)]\\ = &-\frac{3}{10}e^{-3T}\{\cos T + 3\sin T - e^{3a}[\cos(T-a) + 3\sin(T-a)]\}\end{aligned}$$

■ **21**

$$\begin{aligned}\theta_i(t) = f(t) &= (1-t)H(t) - (1-t)H(t-1)\\ &= (1-t)H(t) + (t-1)H(t-1)\end{aligned}$$

so

$$\mathcal{L}\{\theta_i(t)\} = \frac{1}{s} - \frac{1}{s^2} + e^{-s}\mathcal{L}\{t\}$$
$$= \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-s} = \frac{s-1}{s^2} + \frac{1}{s^2}e^{-s}$$

Then taking Laplace transforms, using $\theta_0(0) = \dot{\theta}_0(0) = 0$

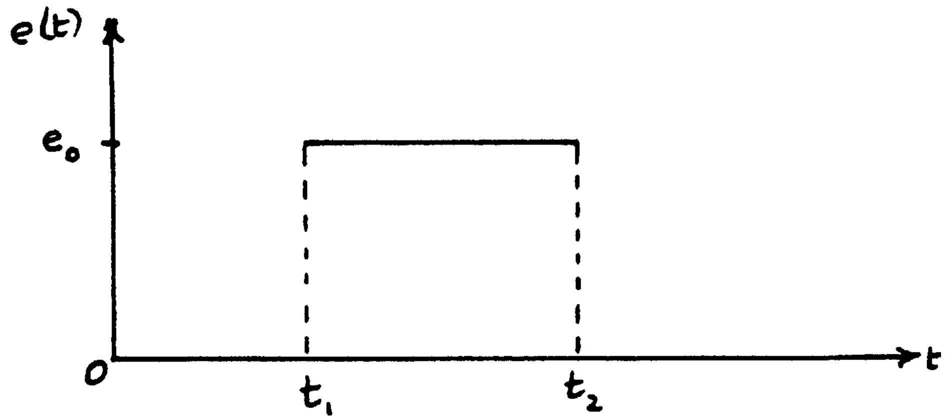
$$(s^2 + 8s + 16)\Phi_0(s) = \frac{s-1}{s^2} + \frac{1}{s^2}e^{-s}$$

$$\Phi_0(s) = \frac{s-1}{s^2(s+4)^2} + e^{-s}\Big[\frac{1}{s^2(s+4)^2}\Big]$$
$$= \frac{1}{s^2}\Big[\frac{3}{s} - \frac{2}{s^2} - \frac{3}{s+4} - \frac{10}{(s+4)^2}\Big] + \frac{e^{-s}}{32}\Big[\frac{3}{s} + \frac{2}{s^2} + \frac{1}{s+4} + \frac{2}{(s+4)^2}\Big]$$

which on taking inverse transforms gives

$$\theta_0(t) = \mathcal{L}^{-1}\{\Phi_0(s)\} = \frac{1}{32}[3 - 2t - 3e^{-4t} - 10te^{-4t}]$$
$$+ \frac{1}{32}[-1 + 2(t-1) + e^{-4(t-1)} + 2(t-1)e^{-4(t-1)}]H(t-1)$$
$$= \frac{1}{32}[3 - 2t - 3e^{-4t} - 10te^{-4t}]$$
$$+ \frac{1}{32}[2t - 3 + (2t-1)e^{-4(t-1)}]H(t-1)$$

▪ **22**



$$e(t) = e_0H(t-t_1) - e_0H(t-t_2)$$
$$\mathcal{L}\{e(t)\} = \frac{e_0}{s}(e^{-st_1} - e^{-st_2})$$

By Kirchhoff's second law current in the circuit is given by

$$Ri + \frac{1}{C}\int i dt = e$$

which on taking Laplace transforms

$$RI(s) + \frac{1}{Cs}I(s) = \frac{e_0}{s}(e^{-st_1} - e^{-st_2})$$

$$I(s) = \frac{e_0 C}{RCs+1}(e^{-st_1} - e^{-st_2})$$

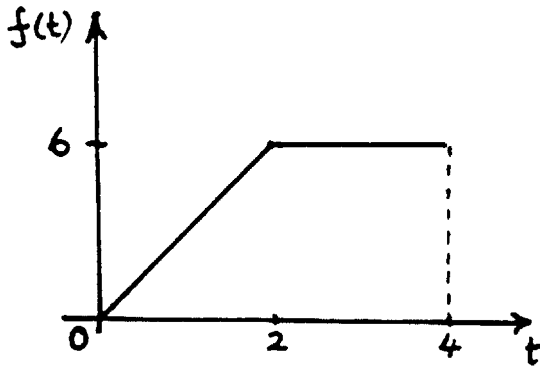
$$= \frac{e_0/R}{s+\frac{1}{RC}}(e^{-st_1} - e^{-st_2})$$

$$= \frac{e_0/R}{s+\frac{1}{RC}}e^{-st_1} - \frac{e_0/R}{s+\frac{1}{RC}}e^{-st_2}$$

then

$$\begin{aligned} i(t) &= \mathcal{L}^{-1}\{I(s)\} \\ &= \frac{e_0}{R}\left[e^{-(t-t_1)/RC}H(t-t_1) - e^{-(t-t_2)/RC}H(t-t_2)\right] \end{aligned}$$

▪ **23**



Sketch over one period as shown and readily extended to $0 \le t < 12$.

$$\begin{aligned} f_1(t) &= 3tH(t) - (3t-6)H(t-2) - 6H(t-4) \\ &= 3tH(t) - 3(t-2)H(t-2) - 6H(t-4) \end{aligned}$$

$$\mathcal{L}\{f_1(t)\} = F_1(s) = \frac{3}{s^2} - \frac{3}{s^2}e^{-2s} - \frac{6}{s}e^{-4s}$$

Then by theorem 2.5

$$\begin{aligned} \mathcal{L}\{f(t)\} = F(s) &= \frac{1}{1-e^{-4s}}F_1(s) \\ &= \frac{1}{s^2(1-e^{-4s})}(3 - 3e^{-2s} - 6se^{-4s}) \end{aligned}$$

- **24** Take

$$\begin{aligned} f_1(t) &= \frac{K}{T}t, \quad 0 < t < T \\ &= 0, \quad t > T \end{aligned}$$

then $f_1(t) = \frac{K}{T}tH(t) - \frac{Kt}{T}H(t-T) = \frac{K}{T}tH(t) - \frac{K}{T}(t-T)H(t-T) - KH(t-T)$

$$\mathcal{L}\{f_1(t)\} = F_1(s) = \frac{K}{Ts^2} - e^{-sT}\frac{K}{Ts^2} - e^{-sT}\frac{K}{s} = \frac{K}{Ts^2}(1-e^{-sT}) - \frac{K}{s}e^{-sT}$$

Then by theorem 2.5

$$\mathcal{L}\{f(t)\} = F(s) = \frac{1}{1-e^{-sT}}F_1(s) = \frac{K}{Ts^2} - \frac{K}{s}\frac{e^{-sT}}{1-e^{-sT}}$$

Exercises 2.5.12

- **25(a)**

$$\frac{2s^2+1}{(s+2)(s+3)} = 2 - \frac{10s+11}{(s+2)(s+3)} = 2 + \frac{9}{s+2} - \frac{19}{s+3}$$

$$\mathcal{L}^{-1}\left\{\frac{2s^2+1}{(s+2)(s+3)}\right\} = 2\delta(t) + 9e^{-2t} - 19e^{-3t}$$

25(b)

$$\frac{s^2-1}{s^2+4} = 1 - \frac{5}{s^2+4}$$
$$\mathcal{L}^{-1}\{\frac{s^2-1}{s^2+4}\} = \delta(t) - \frac{5}{2}\sin 2t$$

25(c)

$$\begin{aligned}\frac{s^2+2}{s^2+2s+5} &= 1 - \frac{2s+3}{s^2+2s+5} \\ &= 1 - [\frac{2(s+1)+\frac{1}{2}(2)}{(s+1)^2+s^2}] \\ \mathcal{L}^{-1}\{\frac{s^2+2}{s^2+2s+5}\} &= \delta(t) - e^{-t}\big(2\cos 2t + \frac{1}{2}\sin 2t\big)\end{aligned}$$

- **26(a)** $(s^2+7s+12)X(s) = \dfrac{2}{s} + e^{-2s}$

$$\begin{aligned}X(s) &= \frac{2}{s(s+4)(s+3)} + [\frac{1}{(s+4)(s+3)}]e^{-2s} \\ &= \frac{\frac{1}{6}}{s} - \frac{\frac{2}{3}}{s+3} + \frac{\frac{1}{2}}{s+4} + [\frac{1}{s+3} - \frac{1}{s+4}]e^{-2s} \\ x(t) = \mathcal{L}^{-1}\{X(s)\} &= \big(\frac{1}{6} - \frac{2}{3}e^{-3t} + \frac{1}{2}e^{-4t}\big) + \big(e^{-3(t-2)} - e^{-4(t-2)}\big)H(t-2)\end{aligned}$$

26(b)

$$\begin{aligned}(s^2+6s+13)X(s) &= e^{-2\pi s} \\ X(s) &= \frac{1}{(s+3)^2+2^2}e^{-2\pi s} \\ &= e^{-2\pi s}\mathcal{L}\{\frac{1}{2}e^{-3t}\sin 2t\} \\ \text{so } x(t) = \mathcal{L}^{-1}\{X(s)\} &= \frac{1}{2}e^{-3(t-2\pi)}\sin 2(t-2\pi).H(t-2\pi) \\ &= \frac{1}{2}e^{6\pi}e^{-3t}\sin 2t.H(t-2\pi)\end{aligned}$$

26(c)

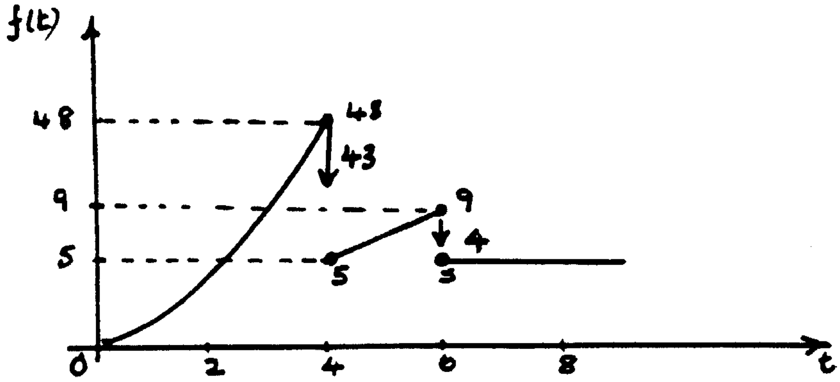
$$(s^2 + 7s + 12)X(s) = s + 8 + e^{-3s}$$

$$X(s) = \frac{s+8}{(s+4)(s+3)} + [\frac{1}{(s+4)(s+3)}]e^{-3s}$$

$$= [\frac{5}{s+3} - \frac{4}{s+4}] + [\frac{1}{s+3} - \frac{1}{s+4}]e^{-3s}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = 5e^{-3t} - 4e^{-4t} + [e^{-3(t-3)} - e^{-4(t-3)}]H(t-3)$$

▪ **27(a)**



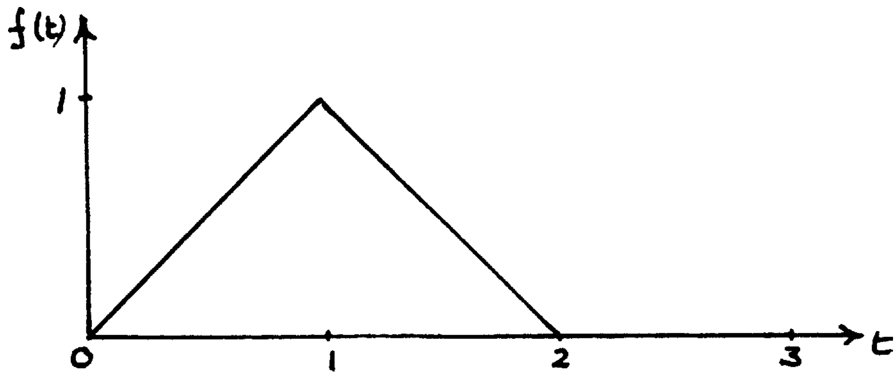
Generalised derivative is

$$f'(t) = g'(t) - 43\delta(t-4) - 4\delta(t-6)$$

where

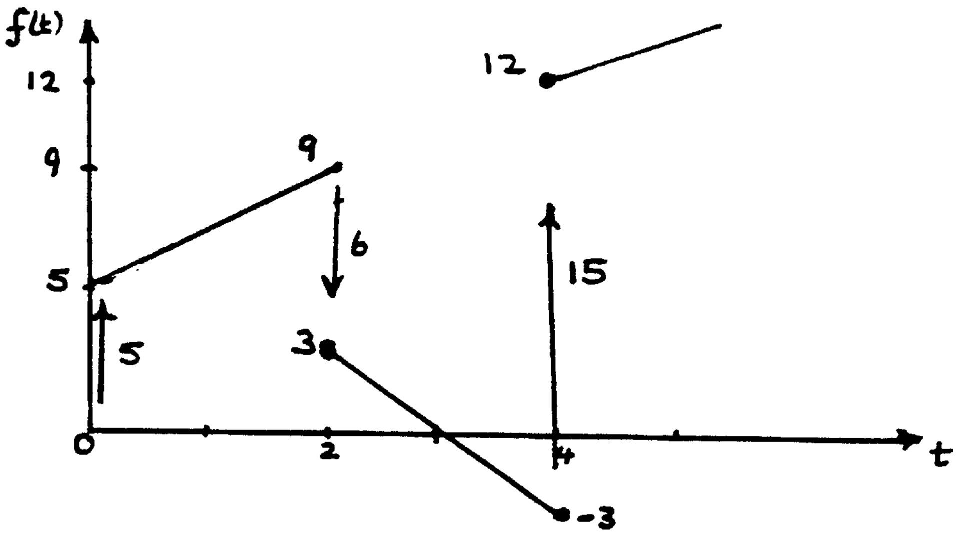
$$g'(t) = \begin{cases} 6t, & 0 \le t < 4 \\ 2, & 4 \le t < 6 \\ 0, & t \ge 6 \end{cases}$$

27(b)



$$f'(t) = g'(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$$

27(c)



$$f'(t) = g'(t) + 5\delta(t) - 6\delta(t-2) + 15\delta(t-4)$$

where

$$g'(t) = \begin{cases} 2, & 0 \le t < 2 \\ -3, & 2 \le t < 4 \\ 2t-1, & t \ge 4 \end{cases}$$

■ **28**

$$(s^2 + 7s + 10)X(s) = 2 + (3s + 2)U(s)$$

$$= 2 + (3s + 2)\frac{1}{s+2} = \frac{5s+6}{s+2}$$

$$X(s) = \frac{5s+6}{(s+2)^2(s+5)}$$

$$= \frac{\frac{19}{9}}{s+2} - \frac{\frac{4}{3}}{(s+2)^2} - \frac{\frac{19}{9}}{(s+5)}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{19}{9}e^{-2t} - \frac{4}{3}te^{-2t} - \frac{19}{9}e^{-5t}$$

■ **29** $f(t) = \sum\limits_{n=0}^{\infty} \delta(t - nT)$

Thus

$$F(s) = \mathcal{L}\{f(t)\} = \sum_{n=0}^{\infty} \mathcal{L}\{\delta(t - nT)\} = \sum_{n=0}^{\infty} e^{-snT}$$

This is an infinite GP with first term 1 and common ratio e^{-sT} and therefore having sum $(1 - e^{-sT})^{-1}$. Hence

$$F(s) = \frac{1}{1 - e^{-sT}}$$

Assuming zero initial conditions and taking Laplace transforms the response of the harmonic oscillator is given by

$$(s^2 + w^2)X(s) = F(s) = \frac{1}{1 - e^{-sT}}$$

$$X(s) = \Big(\sum_{n=0}^{\infty} e^{-snT}\Big)\Big(\frac{1}{s^2 + w^2}\Big)$$

$$= [1 + e^{-sT} + e^{-2sT} + \ldots]\mathcal{L}\{\frac{1}{w}\sin wt\}$$

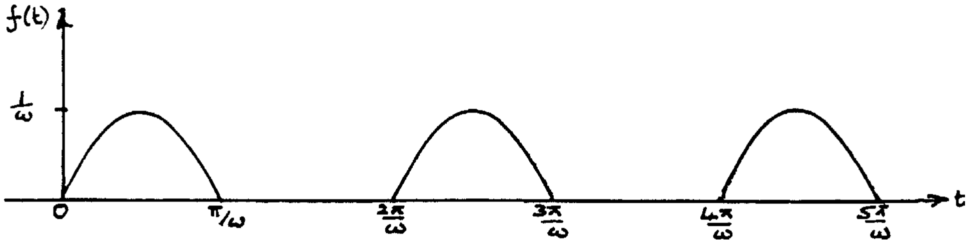
giving $x(t) = \mathcal{L}^{-1}\{X(s)\} = \dfrac{1}{w}[\sin wt + H(t - T).\sin w(t - T) + H(t - 2T).$
$\sin w(t - 2T) + \ldots]$
or $x(t) = \dfrac{1}{w}\sum\limits_{n=0}^{\infty} H(t - nT)\sin w(t - nT)$.

29(a)

$$T = \frac{\pi}{w} \; ; \quad x(t) = \frac{1}{w}\sum_{n=0}^{\infty} H\big(t - \frac{n\pi}{w}\big)\sin(wt - n\pi)$$

$$= \frac{1}{w}\Big[\sin wt - \sin wt.\, H\big(t - \frac{\pi}{w}\big) + \sin wt.\, H\big(t - \frac{2\pi}{w}\big) + \ldots\Big]$$

and a sketch of the response is as follows

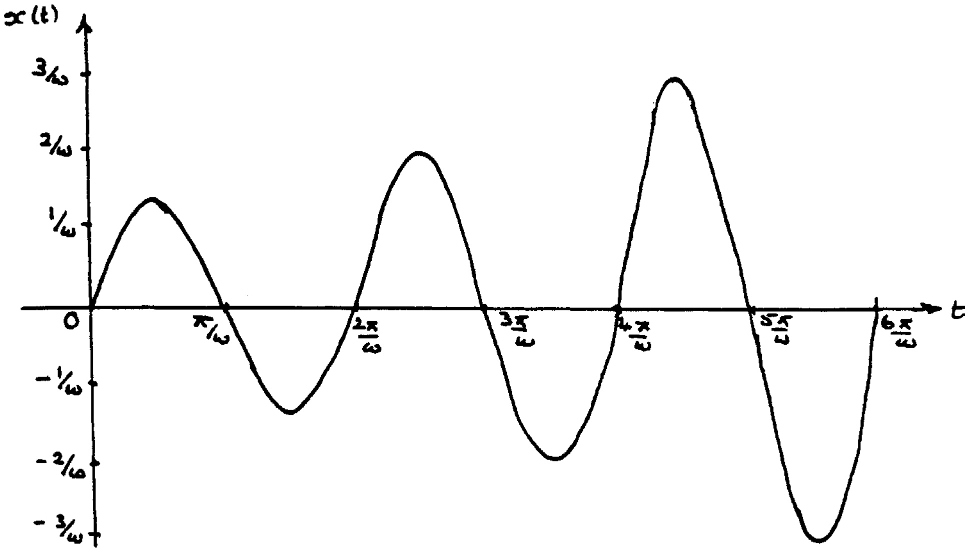


29(b)

$$T = \frac{2\pi}{w} \; ; \quad x(t) = \frac{1}{w}\sum_{n=0}^{\infty} H\big(t - \frac{2\pi n}{w}\big)\sin(wt - 2\pi n)$$

$$= \frac{1}{w}\Big[\sin wt + \sin wt.H\big(t - \frac{2\pi}{w}\big) + \sin wt.H\big(t - \frac{4\pi}{w}\big) + \ldots\Big]$$

and the sketch of the response is as follows



▪ **30** The charge q on the LCR circuit is determined by

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e(t)$$

where $e(t) = E\delta(t), q(0) = \dot{q}(0) = 0$.
Taking Laplace transforms

$$\left(Ls^2 + Rs + \frac{1}{C}\right)Q(s) = \mathcal{L}\{E\delta(t)\} = E$$

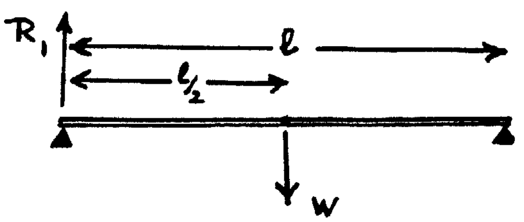
$$\begin{aligned} Q(s) &= \frac{E/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{E/L}{(s + \frac{R}{2L})^2 + (\frac{1}{LC} - \frac{R^2}{4L^2})} \\ &= \frac{E/L}{(s+\mu)^2 + \eta^2}, \mu = \frac{R}{2L}, \eta = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \end{aligned}$$

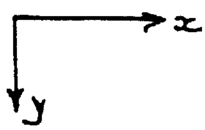
Thus $q(t) = \dfrac{E}{L\eta}e^{-\mu t}\sin\eta t$

and current $i(t) = \dot{q}(t) = \dfrac{E}{L\eta}e^{-\mu t}(\eta\cos\eta t - \mu\sin\eta t)$

Exercises 2.5.14

▪ **31**





Load $W(x) = \dfrac{M}{\ell}H(x) + W\delta\left(x - \dfrac{\ell}{2}\right) - R_1\delta(x)$, where $R_1 = \dfrac{1}{2}(M+W)$
so the force function is

$$W(x) = \frac{M}{\ell}H(x) + W\delta\left(x - \frac{\ell}{2}\right) - \left(\frac{M+W}{2}\right)\delta(x)$$

having Laplace transform

$$W(s) = \frac{M}{\ell s} + We^{-\ell s/2} - \frac{(M+W)}{2}$$

Since the beam is freely supported at both ends

$$y(0) = y_2(0) = y(\ell) = y_2(\ell) = 0$$

and the transformed equation (2.64) of the text becomes

$$Y(s) = \frac{1}{EI}\left[\frac{M}{\ell s^5} + \frac{W}{s^4}e^{-\ell s/2} - (\frac{M+W}{2})\frac{1}{s^4}\right] + \frac{y_1(0)}{s^2} + \frac{y_3(0)}{s^4}$$

Taking inverse transforms gives

$$y(x) = \frac{1}{EI}\left[\frac{1}{24}\frac{M}{\ell}x^4 + \frac{1}{6}W(x - \frac{\ell}{2})^3 \cdot H(x - \frac{\ell}{2}) - \frac{1}{12}(M+W)x^3\right]$$

$$+ y_1(0)x + \frac{1}{6}y_3(0)x^3$$

for $x > \frac{\ell}{2}$

$$y(x) = \frac{1}{EI}\left[\frac{1}{24}\frac{M}{\ell}x^4 + \frac{1}{6}W\left(x - \frac{\ell}{2}\right)^3 - \frac{1}{12}(M+W)x^3\right] + y_1(0)x + \frac{1}{6}y_3(0)x^3$$

$$y_2(x) = \frac{1}{EI}\left[\frac{1}{2}\frac{M}{\ell}x^2 + W\left(x - \frac{\ell}{2}\right) - \frac{1}{2}(M+W)x\right] + y_3(0)x$$

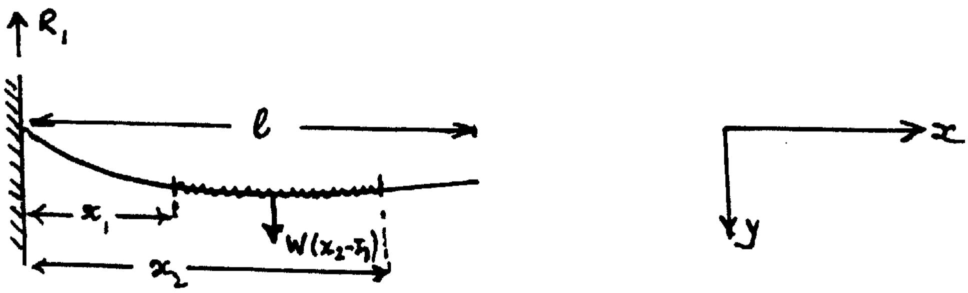
$y_2(\ell) = 0$ then gives $y_3(0) = 0$ and $y(\ell) = 0$ gives

$$0 = \frac{1}{EI}\left[\frac{M\ell^3}{24} + \frac{W\ell^3}{24} - \frac{1}{12}M\ell^3 - \frac{1}{2}W\ell^3\right] + y_1(0)\ell$$

$$y_1(0) = \frac{1}{EI}\left[\frac{1}{24}M\ell^2 + \frac{1}{16}W\ell^2\right]$$

so $$y(x) = \frac{1}{48EI}\left[\frac{2}{\ell}Mx^4 + 8W(x - \frac{\ell}{2})^3 H\left(x - \frac{\ell}{2}\right) - 4(M+W)x^3 + (2M+3W)\ell^2 x\right]$$

■ **32**



Load $W(x) = w(H(x-x_1) - H(x-x_2)) - R_1\delta(x),\ R_1 = w(x_2 - x_1)$
so the force function is

$$W(x) = w(H(x-x_1) - H(x-x_2)) - w(x_2 - x_1)\delta(x)$$

having Laplace transform

$$W(s) = w\big(\frac{1}{s}e^{-x_1 s} - \frac{1}{s}e^{-x_2 s}\big) - w(x_2 - x_1)$$

with corresponding boundary conditions

$$y(0) = y_1(0) = 0,\ y_2(\ell) = y_3(\ell) = 0$$

The transformed equation (2.64) of the text becomes

$$Y(s) = \frac{w}{EI}\left[\frac{1}{s^5}e^{-x_1 s} - \frac{1}{s^5}e^{-x_2 s} - \frac{(x_2 - x_1)}{s^4}\right] + \frac{y_2(0)}{s^3} + \frac{y_3(0)}{s^4}$$

which on taking inverse transforms gives

$$y(x) = \frac{w}{EI}\big[\frac{1}{24}(x-x_1)^4 H(x-x_1) - \frac{1}{24}(x-x_2)^4 H(x-x_2) \\ - \frac{1}{6}(x_2 - x_1)x^3\big] + y_2(0)\frac{x^2}{2} + y_3(0)\frac{x^3}{6}$$

For $x > x_2$

$$y(x) = \frac{w}{EI}\big[\frac{1}{24}(x-x_1)^4 - \frac{1}{24}(x-x_2)^4 - \frac{1}{6}(x_2 - x_1)x^3\big] + y_2(0)\frac{x^2}{2} + y_3(0)\frac{x^3}{6}$$
$$y_2(x) = \frac{w}{EI}\big[\frac{1}{24}(x-x_1)^2 - \frac{1}{2}(x-x_2)^2 - (x_2 - x_1)x\big] + y_2(0) + y_3(0)x$$
$$y_3(x) = \frac{w}{EI}\big[(x-x_1) - (x-x_2) - (x_2 - x_1)\big] + y_3(0)\ \Rightarrow\ y_3(0) = 0$$

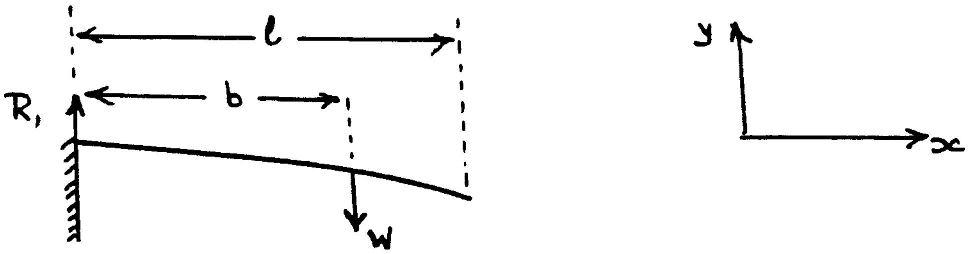
The boundary condition $y_2(\ell) = 0$ then gives

$$\begin{aligned}
0 &= \frac{w}{EI}\Big[\frac{1}{2}(\ell^2 - 2\ell x_1 + x_1^2) - \frac{1}{2}(\ell^2 - 2\ell x_2 + x_2^2) - x_2\ell + x_1\ell\Big] + y_2(0) \\
&\Rightarrow\ y_2(0) = \frac{w}{2EI}(x_2^2 - x_1^2) \\
y(x) &= \frac{w}{24EI}\big[(x - x_1)^4 H(x - x_1) - (x - x_2)^4 H(x - x_2) - 4(x_2 - x_1)x^3 \\
&\quad + 6(x_2^2 - x_1^2)x^2\big]
\end{aligned}$$

When $x_1 = 0, x_2 = \ell$, max deflection at $x = \ell$

$$y_{\max} = \frac{w}{24EI}\{\ell^4 - 4\ell^4 + 6\ell^4\} = \frac{w\ell^4}{8EI}$$

33



Load $W(x) = W\delta(x - b) - R_1\delta(x)$, $R_1 = W$ so the force function is

$$W(x) = W\delta(x - b) - W\delta(x)$$

having Laplace transform

$$W(s) = We^{-bs} - W$$

with corresponding boundary conditions

$$y(0) = y_1(0) = 0,\ y_2(\ell) = y_3(\ell) = 0$$

The transformed equation (2.64) of the text becomes

$$Y(s) = -\frac{1}{EI}\Big[\frac{W}{s^4}e^{-bs} - \frac{W}{s^4}\Big] + \frac{y_2(0)}{s^3} + \frac{y_3(0)}{s^4}$$

which on taking inverse transforms gives

$$y(x) = -\frac{W}{EI}\Big[\frac{1}{6}(x - b)^3 H(x - b) - \frac{1}{6}x^3\Big] + y_2(0)\frac{x^2}{2} + y_3(0)\frac{x^3}{6}$$

For $x > b$

$$y(x) = -\frac{W}{EI}\left[\frac{1}{6}(x-b)^3 - \frac{1}{6}x^3\right] + y_2(0)\frac{x^2}{2} + y_3(0)\frac{x^3}{6}$$

$$y_2(x) = -\frac{W}{EI}\left[(x-b) - x\right] + y_2(0) + y_3(0)x$$

$$y_3(x) = -\frac{W}{EI}\left[1-1\right] + y_3(0) \;\Rightarrow\; y_3(0) = 0$$

Using the boundary condition $y_2(\ell) = 0$

$$0 = -\frac{W}{EI}(-h) + y_2(0) \;\Rightarrow\; y_2(0) = -\frac{Wb}{EI}$$

giving

$$y(x) = \frac{W}{EI}\left[\frac{x^3}{6} - \frac{(x-b)^3}{6}H(x-b) - \frac{bx^2}{2}\right]$$

$$= \begin{cases} -\dfrac{Wx^2}{6EI}(3b-x), & 0 < x \le b \\ -\dfrac{Wb^2}{6EI}(3x-b), & b < x \le \ell \end{cases}$$

Exercises 2.6.5

- **34(a)** Assuming all the initial conditions are zero taking Laplace transforms gives

$$(s^2 + 2s + 5)X(s) = (3s+2)U(s)$$

so that the system transfer function is given by

$$G(s) = \frac{X(s)}{U(s)} = \frac{3s+2}{s^2+2s+5}$$

34(b) The characteristic equation of the system is

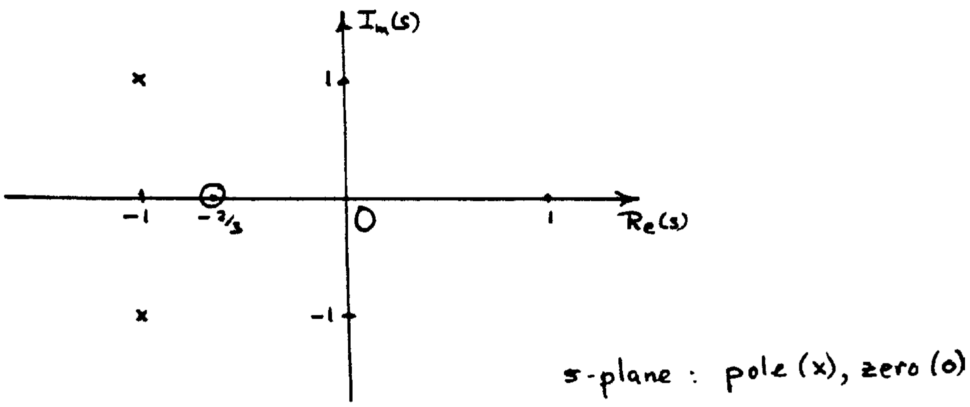
$$s^2 + 2s + 5 = 0$$

and the system is of order 2.

34(c) The transfer function poles are the roots of the characteristic equation

$$s^2 + 2s + 5 = 0$$

which are $s = -1 \pm j$. That is, the transfer function has single poles at $s = -1 + j$ and $s = -1 - j$.
The transfer function zeros are determined by equating the numerator polynomial to zero; that is, a single zero at $s = -\dfrac{2}{3}$.



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35 Following the same procedure as for Exercise 34

35(a) The transfer function characterising the system is

$$G(s) = \frac{s^3 + 5s + 6}{s^3 + 5s^2 + 17s + 13}$$

35(b) The characteristic equation of the system is

$$s^3 + 5s^2 + 17s + 13 = 0$$

and the system is of order 3.

35(c) The transfer function poles are given by

$$\begin{aligned} s^3 + 5s^2 + 17s + 13 &= 0 \\ \text{i.e. } (s+1)(s^2 + 4s + 13) &= 0 \end{aligned}$$

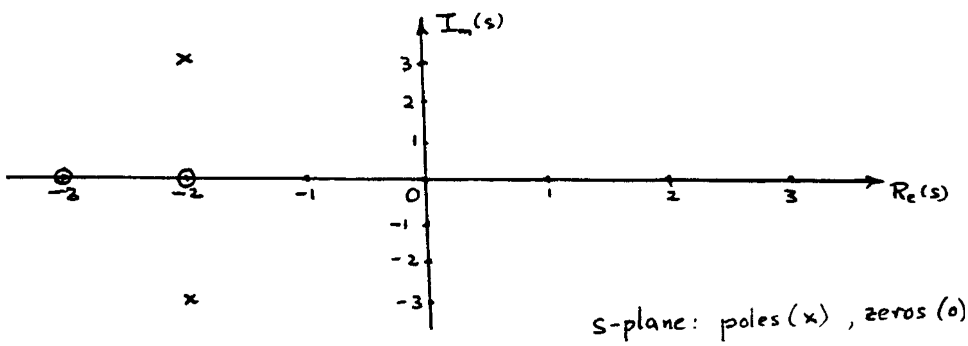
That is, the transfer function has simple poles at

$$s = -1, \; s = -2 + j3, \; s = -2 - j3$$

The transfer function zeros are given by

$$\begin{aligned} s^2 + 5s + 6 &= 0 \\ (s+3)(s+2) &= 0 \end{aligned}$$

i.e. zeros at $s = -3$ and $s = -2$.



- **36(a)** Poles at $(s+2)(s^2+4)=0$; i.e. $s=-2,\ s=+2j,\ s=-j$.

Since we have poles on the imaginary axis in the s-plane, system is marginally stable.

36(b) Poles at $(s+1)(s-1)(s+4)=0$; i.e. $s=-1,\ s=1,\ s=-4$.

Since we have the pole $s=1$ in the right hand half of the s-plane, the system is unstable.

36(c) Poles at $(s+2)(s+4)=0$; i.e. $s=-2,\ s=-4$.

Both the poles are in the left hand half of the plane so the system is stable.

36(d) Poles at $(s^2+s+1)(s+1)^2=0$; i.e. $s=-1$ (repeated), $s=-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}$.

Since all the poles are in the left hand half of the s-plane the system is stable.

36(e) Poles at $(s+5)(s^2-s+10)=0$; i.e. $s=-5,\ s=\frac{1}{2}\pm j\frac{\sqrt{39}}{2}$.

Since both the complex poles are in the right hand half of the s-plane the system is unstable.

- **37(a)** $s^2-4s+13=0 \ \Rightarrow\ s=2\pm j3$.

Thus the poles are in the right hand half s-plane and the system is unstable.

37(b)

$$\begin{array}{cccc} 5s^3 & +\,13s^2 & +\,31s & +\,15=0 \\ a_3 & a_2 & a_1 & a_0 \end{array}$$

Routh–Hurwitz (R-H) determinants are:

$$\Delta_1=13>0,\ \Delta_2=\begin{vmatrix} 13 & 5 \\ 15 & 31 \end{vmatrix}>0,\ \Delta_3=15\Delta_2>0$$

so the system is stable.

37(c) $s^3 + s^2 + s + 1 = 0$
R-H determinants are

$$\Delta_1 = 1 > 0,\ \Delta_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0,\ \Delta_3 = 1\Delta_2 = 0$$

Thus system is marginally stable. This is readily confirmed since the poles are at $s = -1, s = \pm j$

37(d) $24s^4 + 11s^3 + 26s^2 + 45s + 36 = 0$
R-H determinants are

$$\Delta_1 = 11 > 0,\ \Delta_2 = \begin{vmatrix} 11 & 24 \\ 45 & 26 \end{vmatrix} < 0$$

so the system is unstable.

37(e) $s^3 + 2s^2 + 2s + 1 = 0$
R-H determinants are

$$\Delta_1 = 2 > 0,\ \Delta_2 = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1 > 0, \Delta_3 = 1\Delta_2 > 0$$

and the system is stable. The poles are at $s = -1,\ s = -\dfrac{1}{2} \pm j\dfrac{\sqrt{3}}{2}$ confirming the result.

■ **38** $m\dfrac{d^3x}{dt^3} + c\dfrac{d^2x}{dt^2} + K\dfrac{dx}{dt} + Krx = 0;\ m, K, r, c > 0$
R-H determinants are

$$\begin{aligned}
\Delta_1 &= c > 0 \\
\Delta_2 &= \begin{vmatrix} c & m \\ Kr & K \end{vmatrix} = cK - mKr > 0 \text{ provided } r < \frac{c}{m} \\
\Delta_3 &= Kr\Delta_2 > 0 \text{ provided } \Delta_2 > 0
\end{aligned}$$

Thus system stable provided $r < \dfrac{c}{m}$

39

$$\begin{array}{c} s^4 + 2s^2 + (K+2)s^2 + 7s + K = 0 \\ \quad a_3 \qquad\quad a_2 \qquad\quad a_1 \quad a_0 \end{array}$$

R-H determinants are

$$\Delta_1 = |\, a_3 \,| = 9 > 0$$

$$\Delta_2 = \begin{vmatrix} a_3 & a_4 \\ a_1 & a_2 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 7 & K+2 \end{vmatrix} = 2K - 3 > 0 \text{ provided } K > \frac{3}{2}$$

$$\Delta_3 = \begin{vmatrix} a_3 & a_4 & 0 \\ a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 7 & K+2 & 2 \\ 0 & K & 7 \end{vmatrix} = 10K - 21 > 0 \text{ provided } K > 2$$

$$\Delta_4 = K\Delta_3 > 0 \text{ provided } \Delta_3 > 0$$

Thus the system is stable provided $K > 2.1$.

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40 $s^2 + 15Ks^2 + (2K-1)s + 5K = 0,\ K > 0$

R-H determinants are

$$\Delta_1 = 15K > 0$$

$$\Delta_2 = \begin{vmatrix} 15K & 1 \\ 5K & (2K-1) \end{vmatrix} = 30K^2 - 20K$$

$$\Delta_3 = 5K\Delta_2 > 0 \text{ provided } \Delta_2 > 0$$

Thus system stable provided $K(3K-2) > 0$ that is $K > \frac{2}{3}$, since $K > 0$.

41(a) Impulse response $h(t)$ is given by the solution of

$$\frac{d^2h}{dt^2} + 15\frac{dh}{dt} + 56h = 3\delta(t)$$

with zero initial conditions. Taking Laplace transforms

$$(s^2 + 15s + 56)H(s) = 3$$

$$H(s) = \frac{3}{(s+7)(s+8)} = \frac{3}{s+7} - \frac{3}{s+8}$$

$$\text{so } h(t) = \mathcal{L}^{-1}\{H(s)\} = 3e^{-7t} - 3e^{-8t}$$

Since $h(t) \to 0$ as $t \to \infty$ the system is stable.

41(b) Following (a) impulse response is given by

$$(s^2 + 8s + 25)H(s) = 1$$

$$H(s) = \frac{1}{(s+4)^2 + 3^2}$$

$$\text{so } h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{3}e^{-4t}\sin 3t$$

Since $h(t) \to 0$ as $t \to \infty$ the system is stable.

41(c) Following (a) impulse response is given by

$$(s^2 - 2s - 8)H(s) = 4$$

$$H(s) = \frac{4}{(s-4)(s+2)} = \frac{2}{3}\frac{1}{s-4} - \frac{2}{3}\frac{1}{s+2}$$

$$\text{so } h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{2}{3}(e^{4t} - e^{-2t})$$

Since $h(t) \to \infty$ as $t \to \infty$ system is unstable.

41(d) Following (a) impulse response is given by

$$(s^2 - 4s + 13)H(s) = 1$$

$$H(s) = \frac{1}{s^2 - 4s + 13} = \frac{1}{(s-2)^2 + 3^2}$$

$$\text{so } h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{3}e^{2t}\sin 3t$$

Since $h(t) \to \infty$ as $t \to \infty$ system is unstable.

■ **42** Impulse response $h(t) = \dfrac{dx}{dt} = \dfrac{7}{3}e^{-t} - 3e^{-2t} + \dfrac{2}{3}e^{-4t}$
System transfer function $G(s) = \mathcal{L}\{h(t)\}$; that is

$$G(s) = \frac{7}{3(s+1)} - \frac{3}{s+2} + \frac{2}{3(s+4)}$$

$$= \frac{s+8}{(s+1)(s+2)(s+4)}$$

Note The original unit step response can be reconstructed by evaluating $\mathcal{L}^{-1}\{G(s)\frac{1}{s}\}$.

■ **43(a)** $f(t) = 2 - 3\cos t \ , \ F(s) = \dfrac{2}{s} - \dfrac{3s}{s^2+1}$

$$sF(s) = 2 - \frac{3s^2}{s^2+1} = 2 - \frac{3}{1+\frac{1}{s^2}}$$

Thus $\lim\limits_{t\to 0+}(2 - 3\cos t) = 2 - 3 = -1$

and $\lim\limits_{s\to\infty} sF(s) = 2 - \dfrac{3}{1} = -1$ so confirming the i.v. theorem.

43(b)

$$f(t) = (3t-1)^2 = 9t^2 - 6t + 1, \ \lim_{t\to 0+} f(t) = 1$$

$$F(s) = \frac{18}{s^3} - \frac{6}{s^2} + \frac{1}{s} \text{ so } \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{18}{s^2} - \frac{6}{s} + 1\right] = 1$$

thus confirming the i.v. theorem.

43(c)

$$f(t) = t + 3\sin 2t \ , \ \lim_{t\to 0+} = 0$$

$$F(s) = \frac{1}{s^2} - \frac{6}{s^2+4} \text{ so } \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{1}{s} + \frac{6}{s+\frac{4}{s}}\right] = 0$$

thus confirming the i.v. theorem.

■ **44(a)**

$$f(t) = 1 + 3e^{-t}\sin 2t \ , \ \lim_{t\to\infty} f(t) = 1$$

$$F(s) = \frac{1}{s} + \frac{6}{(s+1)^2+4} \text{ and } \lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[1 + \frac{6s}{(s+1)^2+4}\right] = 1$$

thus confirming the f.v. theorem. Note that $sF(s)$ has its poles in the left half of the s-plane so the theorem is applicable.

44(b)

$$f(t) = t^2 3e^{-2t}\ ,\ \lim_{t\to\infty} f(t) = 0$$

$$F(s) = \frac{2}{(s+2)^3} \text{ and } \lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{2s}{(s+2)^3}\right] = 0$$

thus confirming the f.v. theorem. Again note that $sF(s)$ has its poles in the left half of the s-plane.

44(c)

$$f(t) = 3 - 2e^{-3t} + e^{-t}\cos 2t\ ,\ \lim_{t\to\infty} f(t) = 3$$

$$F(s) = \frac{3}{s} - \frac{2}{s+3} + \frac{(s+1)}{(s+1)^2+4}$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[3 - \frac{2s}{s+3} + \frac{s(s+1)}{(s+1)^2+4}\right] = 3$$

confirming the f.v. theorem. Again $sF(s)$ has its poles in the left half of the s-plane.

▪ **45** For the circuit of Example 2.28

$$I_2(s) = \frac{3.64}{s} + \frac{1.22}{s+59.1} - \frac{4.86}{s+14.9}$$

Then by the f.v. theorem

$$\lim_{t\to\infty} i_2(t) = \lim_{s\to 0} sI_2(s) = \lim_{s\to 0}\left[3.64 + \frac{1.22s}{s+59.1} - \frac{4.86s}{s+14.9}\right]$$
$$= 3.64$$

which confirms the answer obtained in Example 2.28. Note that $sI_2(s)$ has all its poles in the left half of the s-plane.

▪ **46** For the circuit of Example 2.29

$$sI_2(s) = \frac{28s^2}{(3s+10)(s+1)(s^2+4)}$$

and since it has poles at $s = \pm j2$ not in the left hand half of the s-plane the f.v. theorem is not applicable.

■ **47** Assuming quiescent initial state taking Laplace transforms gives

$$\begin{aligned}(7s+5)Y(s) &= \frac{4}{s}+\frac{1}{s+3}+2\\ Y(s) &= \frac{4}{s(7s+5)}+\frac{1}{(s+3)(7s+5)}+\frac{2}{7s+5}\\ sY(s) &= \frac{4}{7s+5}+\frac{s}{(s+3)(7s+5)}+\frac{2s}{7s+5}\end{aligned}$$

By the f.v. theorem

$$\begin{aligned}\lim_{t\to\infty} y(t) = \lim_{s\to 0} sF(s) &= \lim_{s\to 0}\Big[\frac{4}{7s+5}+\frac{s}{(s+3)(7s+5)}+\frac{2s}{7s+5}\Big]\\ &= \frac{4}{5}\end{aligned}$$

By the i.v. theorem

$$\begin{aligned}\lim_{t\to 0^+} y(t) = y(0+) = \lim_{s\to\infty} sF(s) &= \lim_{s\to\infty}\Big[\frac{4}{7s+5}+\frac{s}{(1+\frac{3}{s})(7s+5)}+\frac{2}{7+\frac{5}{s}}\Big]\\ &= \frac{2}{7}\end{aligned}$$

Thus jump at $t=0=y(0+)-y(0-)=1\dfrac{2}{7}$.

Exercises 2.6.8

■ **48(a)**

$$\begin{aligned}f*g(t) &= \int_0^t \tau\cos(3t-3\tau)d\tau\\ &= \Big[-\frac{1}{3}\tau\sin(3t-3\tau)+\frac{1}{9}\cos(3t-3\tau)\Big]_0^t\\ &= \frac{1}{9}(1-\cos 3t)\\ g*f(t) &= \int_0^t (t-\tau)\cos 3\tau d\tau\\ &= \Big[\frac{t}{3}\sin 3\tau-\frac{\tau}{3}\sin 3\tau-\frac{1}{9}\cos 3\tau\Big]_0^t = \frac{1}{9}(1-\cos 3t)\end{aligned}$$

48(b)

$$\begin{aligned}
f * g(t) &= \int_0^t (\tau+1)e^{-2(t-\tau)}d\tau \\
&= \Big[\frac{1}{2}(\tau+1)e^{-2(t-\tau)} - \frac{1}{4}e^{-2(t-\tau)}\Big]_0^t \\
&= \frac{1}{2}t + \frac{1}{4} - \frac{1}{4}e^{-2t} \\
g * f(t) &= \int_0^t (t-\tau+1)e^{-2\tau}d\tau \\
&= \Big[-\frac{1}{2}(t-\tau+1)e^{-2\tau} + \frac{1}{4}e^{-2\tau}\Big]_0^t \\
&= \frac{1}{2}t + \frac{1}{4} - \frac{1}{4}e^{-2t}
\end{aligned}$$

48(c) Integration by parts gives

$$\begin{aligned}
\int_0^t \tau^2 \sin 2(t-\tau)d\tau &= \int_0^t (t-\tau)^2 \sin 2\tau d\tau \\
&= \frac{1}{4}\cos 2t + \frac{1}{2}t^2 - \frac{1}{4}
\end{aligned}$$

48(d) Integration by parts gives

$$\begin{aligned}
\int_0^t e^{-\tau}\sin(t-\tau)d\tau &= \int_0^t e^{-(t-\tau)}\sin\tau d\tau \\
&= \frac{1}{2}(\sin t - \cos t + e^{-t})
\end{aligned}$$

▪ **49(a)** Since $\mathcal{L}^{-1}\{\frac{1}{s}\} = 1 = f(t)$ and $\mathcal{L}^{-1}\{\frac{1}{(s+3)^3}\} = \frac{1}{2}t^2e^{-3t}$

$$\begin{aligned}
\mathcal{L}^{-1}\{\frac{1}{s}\cdot\frac{1}{(s+3)^3}\} &= \int_0^t f(t-\tau)g(\tau)d\tau \\
&= \int_0^t 1.\frac{1}{2}\tau^2e^{-3\tau}d\tau \\
&= \frac{1}{4}\Big[-\tau^2e^{-3\tau} - \frac{2}{3}\tau e^{-3\tau} - \frac{2}{9}e^{-3\tau}\Big]_0^t \\
&= \frac{1}{54}[2 - e^{-3t}(9t^2+6t+2)]
\end{aligned}$$

Directly

$$\mathcal{L}^{-1}\Big\{\frac{1}{s}\cdot\frac{1}{(s+3)^3}\Big\} = \mathcal{L}^{-1}\frac{1}{54}\Big\{\frac{2}{s} - \frac{18}{(s+3)^3} - \frac{6}{(s+3)^2} - \frac{2}{(s+3)}\Big\}$$
$$= \frac{1}{54}[2 - e^{-3t}(9t^2 + 6t + 2)]$$

49(b) $\mathcal{L}^{-1}\Big\{\dfrac{1}{(s-2)^2}\Big\} = te^{2t} = f(t),\ \mathcal{L}^{-1}\Big\{\dfrac{1}{(s+3)^2}\Big\} = te^{-3t} = g(t)$

$$\mathcal{L}^{-1}\Big\{\frac{1}{(s-2)^2}\cdot\frac{1}{(s+3)^2}\Big\} = \int_0^t (t-\tau)e^{2(t-\tau)}.\tau e^{-3\tau}d\tau$$
$$= e^{-2t}\int_0^t (t\tau - \tau^2)e^{-5\tau}d\tau$$
$$= e^{2t}\Big[-\frac{1}{5}(t\tau - \tau^2)e^{-5\tau} - \frac{1}{25}(t-2\tau)e^{-5\tau} + \frac{2}{125}e^{-5\tau}\Big]_0^t$$
$$= e^{2t}\Big[\frac{t}{25}e^{-5t} + \frac{2}{125}e^{-5t} + \frac{t}{25} - \frac{2}{125}\Big]$$
$$= \frac{1}{125}\big[e^{2t}(5t-2) + e^{-3t}(5t+2)\big]$$

Directly

$$\frac{1}{(s-2)^2(s+3)^2} = \frac{\frac{-2}{125}}{s-2} + \frac{\frac{1}{25}}{(s-2)^2} + \frac{\frac{2}{125}}{(s+3)} + \frac{\frac{1}{25}}{(s+3)^2}$$
$$\therefore \mathcal{L}^{-1}\Big\{\frac{1}{(s-2)^2(s+3)^2}\Big\} = \frac{-2}{125}e^{2t} + \frac{1}{25}te^{2t} + \frac{2}{125}e^{-3t} + \frac{1}{25}te^{-3t}$$
$$= \frac{1}{125}[e^{2t}(5t-2) + e^{-3t}(5t+2)]$$

49(c) $\mathcal{L}^{-1}\Big\{\dfrac{1}{s^2}\Big\} = t = f(t),\ \mathcal{L}^{-1}\Big\{\dfrac{1}{(s+4)}\Big\} = e^{-4t} = g(t)$

$$\mathcal{L}^{-1}\Big\{\frac{1}{s^2}\cdot\frac{1}{s+4}\Big\} = \int_0^t (t-\tau)e^{-4t}d\tau$$
$$= \Big[-\frac{1}{4}(t-\tau)e^{-4\tau} + \frac{1}{16}e^{-4\tau}\Big]_0^t$$
$$= \frac{1}{16}e^{-4t} + \frac{1}{4}t - \frac{1}{16}$$

Directly

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{1}{s^2(s+4)}\} &= \mathcal{L}^{-1}\{\frac{1}{16}\cdot\frac{1}{s+4} - \frac{1}{16}\cdot\frac{1}{s} + \frac{1}{4}\cdot\frac{1}{s^2}\} \\ &= \frac{1}{16}e^{-4t} - \frac{1}{16} + \frac{1}{4}t\end{aligned}$$

▪ **50** Let $f(\lambda) = \lambda$ and $g(\lambda) = e^{-\lambda}$ so

$$F(s) = \frac{1}{s^2} \text{ and } G(s) = \frac{1}{s+1}$$

Considering the integral equation

$$y(t) = \int_0^t \lambda e^{-(t-\lambda)} d\lambda$$

By (2.80) in the text

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)G(s)\} &= \int_0^t f(\lambda)g(t-\lambda)d\lambda \\ &= \int_0^t \lambda e^{-(t-\lambda)} d\lambda = y(t)\end{aligned}$$

so

$$\begin{aligned}y(t) = \mathcal{L}^{-1}\{F(s)G(s)\} &= \mathcal{L}^{-1}\{\frac{1}{s^2(s+1)}\} \\ &= \mathcal{L}^{-1}\{-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\} \\ &= (t-1) + e^{-t}\end{aligned}$$

▪ **51** Impulse response $h(t)$ is given by the solution of

$$\frac{d^2h}{dt^2} + \frac{7dh}{dt} + 12h = \delta(t)$$

subject to zero initial conditions. Taking Laplace transforms

$$(s^2 + 7s + 12)H(s) = 1$$

$$H(s) = \frac{1}{(s+3)(s+4)} = \frac{1}{s+3} - \frac{1}{s+4}$$

giving $h(t) = \mathcal{L}^{-1}\{H(s)\} = e^{-3t} - e^{-4t}$

Response to pulse input is

$$\begin{aligned}
x(t) &= A\{\int_0^t [e^{-3(t-\tau)} - e^{-4(t-\tau)}]d\tau\}H(t) \\
&\quad - A\{\int_T^t [e^{-3(t-\tau)} - e^{-4(t-\tau)}]d\tau\}H(t-T) \\
&= A\Big\{[\frac{1}{3} - \frac{1}{4} - \frac{1}{3}e^{-3t} + \frac{1}{4}e^{-4t}]H(t) \\
&\quad -[\frac{1}{3} - \frac{1}{4} - \frac{1}{3}e^{-3(t-T)} - \frac{1}{4}e^{-4(t-T)}]H(t-T)\Big\} \\
&= \frac{1}{12}A\big[1 - 4e^{-3t} + 3e^{-4t} - (1 - 4e^{-3(t-T)} + 3e^{-4(t-T)})H(t-T)\big]
\end{aligned}$$

or directly

$$u(t) = A[H(t) - H(t-T)] \text{ so } U(s) = \mathcal{L}\{u(t)\} = \frac{A}{s}[1 - e^{-sT}]$$

Thus taking Laplace transforms with initial quiescent state

$$\begin{aligned}
(s^2 + 7s + 12)X(s) &= \frac{A}{s}[1 - e^{-sT}] \\
X(s) &= A[\frac{1}{12}\cdot\frac{1}{s} - \frac{1}{3}\cdot\frac{1}{s+3} + \frac{1}{4}\cdot\frac{1}{s+4}](1 - e^{-sT}) \\
x(t) = \mathcal{L}^{-1}\{X(s)\} &= \frac{A}{12}[1 - 4e^{-3t} + 3e^{-4t} - (1 - 4e^{-3(t-T)} + 3e^{-4(t-T)})H(t-T)]
\end{aligned}$$

■ **52** Impulse response $h(t)$ is the solution of

$$\frac{d^2h}{dt^2} + 4\frac{dh}{dt} + 5h = \delta(t),\ h(0) = \dot{h}(0) = 0$$

Taking Laplace transforms

$$\begin{aligned}
(s^2 + 4s + 5)H(s) &= 1 \\
H(s) &= \frac{1}{s^2 + 4s + 5} = \frac{1}{(s+2)^2 + 1}
\end{aligned}$$

so $h(t) = \mathcal{L}^{-1}\{H(s)\} = e^{-2t}\sin t.$

By the convolution integral response to unit step is

$$\theta_0(t) = \int_0^t e^{-2(t-\tau)}\sin(t-\tau).1d\tau$$
$$= e^{-2t}\int_0^t e^{2\tau}\sin(t-\tau)d\tau$$

which using integration by parts gives

$$\theta_0(t) = \frac{e^{-2t}}{5}\left[e^{2\tau}[2\sin(t-\tau)+\cos(t-\tau)]\right]_0^t$$
$$= \frac{1}{5} - \frac{1}{5}e^{-2t}(2\sin t + \cos t)$$

<u>Check</u>

Solving

$$\frac{d^2\theta_0}{dt^2} + 4\frac{d\theta_0}{dt} + 5\theta_0 = 1\ ,\ \dot{\theta}_0(0) = \theta_0(0) = 0$$

gives

$$(s^2+4s+5)\Phi_0(s) = \frac{1}{s}$$
$$\Phi_0(s) = \frac{1}{s(s^2+4s+5)} = \frac{1}{5s} - \frac{1}{5}\cdot\frac{s+4}{(s+2)^2+1}$$
$$\text{so } \theta_0(t) = \mathcal{L}^{-1}\{\Phi_0(s)\} = \frac{1}{5} - \frac{1}{5}[\cos t + 2\sin t]e^{-2t}.$$

Review Exercises 2.8

- **1(a)** $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 5x = 8\cos t,\ x(0) = \dot{x}(0) = 0$ Taking Laplace transforms

$$(s^2+4s+5)X(s) = \frac{8s}{s^2+1}$$
$$X(s) = \frac{8s}{(s^2+1)(s^2+4s+5)}$$
$$= \frac{s+1}{s^2+1} - \frac{s+5}{s^2+4s+5}$$
$$= \frac{s}{s^2+1} + \frac{1}{s^2+1} - \frac{(s+2)+3}{(s+2)^2+1}$$

giving $x(t) = \mathcal{L}^{-1}\{X(s)\} = \cos t + \sin t - e^{-2t}[\cos t + 3\sin t]$

1(b) $5\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} - 2x = 6,\ x(0) = \dot{x}(0) = 1$

Taking Laplace transforms

$$
\begin{aligned}
(5s^2 - 3s - 2)X(s) &= 5(s+1) - 3(1) + \frac{6}{s} = \frac{5s^2 + 2s + 6}{s} \\
X(s) &= \frac{5s^2 + 2s + 6}{5s(s+\frac{2}{5})(s+1)} \\
&= -\frac{3}{5} + \frac{\frac{13}{7}}{s-1} + \frac{\frac{15}{7}}{s+\frac{2}{5}}
\end{aligned}
$$

giving $x(t) = \mathcal{L}^{-1}\{X(s)\} = -3 + \dfrac{13}{7}e^{t} + \dfrac{15}{7}e^{-\frac{2}{5}t}$

▪ **2(a)**

$$
\begin{aligned}
\frac{1}{(s+1)(s+2)(s^2+2s+2)} &= \frac{1}{s+1} - \frac{1}{2}\cdot\frac{1}{s+2} - \frac{1}{2}\cdot\frac{s+2}{s^2+2s+2} \\
&= \frac{1}{s+1} - \frac{1}{2}\cdot\frac{1}{s+2} - \frac{1}{2}\cdot\frac{(s+1)+1}{(s+1)^2+1}
\end{aligned}
$$

Thus $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)(s+2)(s^2+2s+2)}\right\} = e^{-t} - \dfrac{1}{2}e^{-2t} - \dfrac{1}{2}e^{-t}(\cos t + \sin t)$

2(b) From equation (2.26) in the text the equation is readily deduced.
Taking Laplace transforms

$$
(s^2 + 3s + 2)I(s) = s + 2 + 3 + V.\frac{1}{(s+1)^2+1}
$$

$$
\begin{aligned}
I(s) &= \frac{s+5}{(s+2)(s+1)} + V\Big[\frac{1}{(s+2)(s+1)(s^2+2s+2)}\Big] \\
&= \frac{4}{s+1} - \frac{3}{s+2} + V\big[\text{extended as in (a)}\big]
\end{aligned}
$$

Thus using the result of (a) above

$$i(t) = \mathcal{L}^{-1}\{I(s)\} = 4e^{-t} - 3e^{-2t} + V\left[e^{-t} - \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-t}(\cos t + \sin t)\right]$$

---

3 Taking Laplace transforms

$$\begin{aligned}(s^2-1)X(s) + 5sY(s) &= \frac{1}{s^2}\\ -2sX(s) + (s^2-4)Y(s) &= -\frac{2}{s}\end{aligned}$$

Eliminating $Y(s)$

$$[(s^2-1)(s^2-4) + 2s(5s)]X(s) = \frac{s^2-4}{s^2} + 10 = \frac{11s^2-4}{s^2}$$

$$\begin{aligned}X(s) &= \frac{11s^2-4}{s^2(s^2+1)(s^2+4)}\\ &= -\frac{1}{s^2} + \frac{5}{s^2+1} - \frac{4}{s^2+4}\end{aligned}$$

giving $x(t) = \mathcal{L}^{-1}\{X(s)\} = -t + 5\sin t - 2\sin 2t$
From the first differential equation

$$\begin{aligned}\frac{dy}{dt} &= \frac{1}{5}\left[t + x - \frac{d^2x}{dt^2}\right]\\ &= \frac{1}{5}[t - t + 5\sin t - 2\sin 2t + 5\sin t - 8\sin 2t]\\ &= (2\sin t - 2\sin 2t)\end{aligned}$$

then $y = -2\cos t + \cos 2t + \text{const.}$

and since $y(0) = 0$, const. $= 1$ giving

$$\begin{aligned}y(t) &= 1 - 2\cos t + \cos 2t\\ x(t) &= -t + 5\sin t - 2\sin 2t\end{aligned}$$

---

4 Taking Laplace transforms

$$(s^2 + 2s + 2)X(s) = sx_0 + x_1 + 2x_0 + \frac{s}{s^2+1}$$

$$
\begin{aligned}
X(s) &= \frac{sx_0 + x_1 + 2x_0}{s^2+2s+2} + \frac{s}{(s^2+1)(s^2+2s+2)} \\
&= \frac{x_0(s+1) + (x_1+x_0)}{(s+1)^2+1} + \frac{1}{5}\cdot\frac{s+2}{s^2+1} - \frac{1}{5}\cdot\frac{s+4}{(s+1)^2+1}
\end{aligned}
$$

giving

$$
\begin{aligned}
x(t) &= \mathcal{L}^{-1}\{X(s)\} \\
&= e^{-t}(x_0\cos t + (x_1+x_0)\sin t) + \frac{1}{5}(\cos t + 2\sin t) \\
&\quad - \frac{1}{5}e^{-t}(\cos t + 3\sin t)
\end{aligned}
$$

i.e.

$$
x(t) = \underbrace{\frac{1}{5}(\cos t + 2\sin t)}_{\text{steady state}} + \underbrace{e^{-t}\big[(x_0 - \frac{1}{5})\cos t + (x_1 + x_0 - \frac{3}{5})\sin t\big]}_{\text{transient}}
$$

Steady state solution is $x_s(t) = \frac{1}{5}\cos t + \frac{2}{5}\sin t \equiv A\cos(t-\alpha)$

having amplitude $A = \sqrt{(\frac{1}{5})^2 + (\frac{2}{5})^2} = \frac{1}{\sqrt{5}}$

and phase lag $\alpha = \tan^{-1} 2 = 63.4°$.

■ **5** Denoting the currents in the primary and secondary circuits by $i_1(t)$ and $i_2(t)$ respectively Kirchoff's second law gives

$$
\begin{aligned}
5i_1 + 2\frac{di_1}{dt} + \frac{di_2}{dt} &= 100 \\
20i_2 + 3\frac{di_2}{dt} + \frac{di_1}{dt} &= 0
\end{aligned}
$$

Taking Laplace transforms

$$
\begin{aligned}
(5+2s)I_1(s) + sI_2(s) &= \frac{100}{s} \\
sI_1(s) + (3s+20)I_2(s) &= 0
\end{aligned}
$$

Eliminating $I_1(s)$

$$
[s^2 - (3s+20)(2s+5)]I_2(s) = 100
$$

$$
\begin{aligned}
I_2(s) &= \frac{-100}{5s^2+55s+100} = -\frac{20}{s^2+11s+20} \\
&= -\frac{20}{(s+\frac{11}{2})^2 - \frac{41}{4}} = -\frac{20}{\sqrt{41}}\Big[\frac{1}{(s+\frac{11}{2}-\frac{\sqrt{41}}{2})} - \frac{1}{(s+\frac{11}{2}+\frac{\sqrt{41}}{2})}\Big]
\end{aligned}
$$

giving the current $i_2(t)$ in the secondary loop as

$$i_2(t) = \mathcal{L}^{-1}\{I_2(s)\} = \frac{20}{\sqrt{41}}\left[e^{-(11+\sqrt{41})t/2} - e^{-(11-\sqrt{41})t/2}\right]$$

■ **6(a)**

(i)

$$\begin{aligned}\mathcal{L}\{\cos(wt+\phi)\} &= \mathcal{L}\{\cos\phi\cos wt - \sin\phi\sin wt\}\\ &= \cos\phi\frac{s}{s^2+w^2} - \sin\phi\frac{w}{s^2+w^2}\\ &= (s\cos\phi - w\sin\phi)/(s^2+w^2)\end{aligned}$$

(ii)

$$\begin{aligned}\mathcal{L}\{e^{-wt}\sin(wt+\phi)\} &= \mathcal{L}\{e^{-wt}\sin wt\cos\phi + e^{-wt}\cos wt\sin\phi\}\\ &= \cos\phi\frac{w}{(s+w)^2+w^2} + \sin\phi\frac{s+w}{(s+w)^2+w^2}\\ &= [\sin\phi + w(\cos\phi+\sin\phi)]/(s^2+2sw+2w^2)\end{aligned}$$

6(b) Taking Laplace transforms

$$\begin{aligned}(s^2+4s+8)X(s) &= (2s+1)+8+\frac{s}{s^2+4}\\ &= \frac{2s^3+9s^2+9s+36}{(s^2+4)(s^2+4s+8)}\\ &= \frac{1}{20}\cdot\frac{s+4}{s^2+4} + \frac{1}{20}\cdot\frac{39s+172}{s^2+4s+8}\\ &= \frac{1}{20}\cdot\frac{s+4}{s^2+4} + \frac{1}{20}\cdot\frac{39(s+2)+47(2)}{(s+2)^2+(2)^2}\end{aligned}$$

giving $x(t) = \mathcal{L}^{-1}\{X(s)\} = \dfrac{1}{20}(\cos 2t + 2\sin 2t) + \dfrac{1}{20}e^{-2t}(39\cos 2t + 47\sin 2t)$.

■ **7(a)**

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s-4}{s^2+4s+13}\right] &= \mathcal{L}^{-1}\left[\frac{(s+2)-2(3)}{(s+2)^2+3^2}\right]\\ &= e^{-2t}[\cos 3t - 2\sin 3t]\end{aligned}$$

7(b) Taking Laplace transforms

$$(s+2)Y(s) = -3 + \frac{4}{s} + \frac{2s}{s^2+1} + \frac{4}{s^2+1}$$

$$Y(s) = \frac{-3s^3+6s^2+s+4}{s(s+2)(s^2+1)}$$

$$= \frac{2}{s} - \frac{5}{s+2} + \frac{2}{s^2+1}$$

$$\therefore y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2 - 5e^{-2t} + 2\sin t$$

■ **8** Taking Laplace transforms

$$(s+5)X(s) + 3Y(s) = 1 + \frac{5}{s^2+1} - \frac{2s}{s^2+1} = \frac{s^2-2s+6}{s^2+1}$$

$$5X(s) + (s+3)Y(s) = \frac{6}{s^2+1} - \frac{3s}{s^2+1} = \frac{6-3s}{s^2+1}$$

Eliminating $Y(s)$

$$[(s+5)(s+3)-15]X(s) = \frac{(s+3)(s^2-2s+6)}{s^2+1} - \frac{3(6-3s)}{s^2+1}$$

$$(s^2+8s)X(s) = \frac{s^3+s^2+9s}{s^2+1}$$

$$X(s) = \frac{s^2+s+9}{(s+8)(s^2+1)} = \frac{1}{s+8} + \frac{1}{s^2+1}$$

$$\text{so } x(t) = \mathcal{L}^{-1}\{X(s)\} = e^{-8t} + \sin t$$

From the first differential equation

$$3y = 5\sin t - 2\cos t - 5x - \frac{dx}{dt} = 3e^{-8t} - 3\cos t$$

Thus $x(t) = e^{-8t} + \sin t,\ y(t) = e^{-8t} - \cos t$.

■ **9** Taking Laplace transforms

$$(s^2+300s+2\times10^4)Q(s)=200\cdot\frac{100}{s^2+10^4}$$
$$(s+100)(s+200)Q(s)=10^4\cdot\frac{2}{s^2+10^4}$$
$$Q(s)=\frac{2.10^4}{(s+100)(s+200)(s^2+10^4)}$$
$$=\frac{1}{100}\cdot\frac{1}{s+100}-\frac{2}{500}\cdot\frac{1}{s+200}-\frac{1}{500}\cdot\frac{3s-100}{s^2+10^4}$$

giving $q(t)=\mathcal{L}^{-1}\{Q(s)\}=\frac{1}{100}e^{-100t}-\frac{2}{500}e^{-200t}-\frac{1}{500}(3\cos100t-\sin100t)$
i.e.

$$q(t)=\underbrace{\frac{1}{500}[5e^{-100t}-2e^{-200t}]}_{\text{transient}}-\underbrace{\frac{1}{500}[3\cos100t-\sin100t]}_{\text{steady state}}$$

Steady state current $=\frac{3}{5}\sin100t+\frac{1}{5}\cos100t\equiv A\sin(100t+\alpha)$
where $\alpha=\tan^{-1}\frac{1}{5}\simeq18\frac{1}{2}^{\circ}$.
Hence the current leads the applied emf by about $18\frac{1}{2}^{\circ}$.

■ **10**

$$4\frac{dx}{dt}+6x+y=2\sin2t \qquad \text{(i)}$$
$$\frac{d^2x}{dt^2}+x-\frac{dy}{dt}=3e^{-2t} \qquad \text{(ii)}$$

Given $x=2$ and $\frac{dx}{dt}=-2$ when $t=0$ so from (i) $y=-4$ when $t=0$.
Taking Laplace transforms

$$(4s+6)X(s)+Y(s)=8+\frac{4}{s^2+4}=\frac{8s^2+36}{s^2+4}$$
$$(s^2+1)X(s)-sY(s)=2s-2+4+\frac{3}{s+2}=\frac{2s^2+6s+7}{s+2}$$

Eliminating $Y(s)$

$$[s(4s+6)+(s^2+1)]X(s)=\frac{8s^2+36}{s^2+4}+\frac{2s^2+6s+7}{s+2}$$

$$
\begin{aligned}
X(s) &= \frac{8s^2+36}{s(s^2+4)(s+1)(s+\frac{1}{5})} + \frac{2s^2+6s+7}{5(s+2)(s+1)(s+\frac{1}{5})} \\
&= \frac{\frac{11}{5}}{s+1} - \frac{\frac{227}{505}}{s+\frac{1}{5}} - \frac{1}{505}\cdot\frac{76s-96}{s^2+4} + \frac{\frac{1}{3}}{s+2} - \frac{\frac{3}{4}}{s+1} + \frac{\frac{49}{60}}{s+\frac{1}{5}} \\
&= \frac{\frac{29}{20}}{s+1} + \frac{\frac{1}{3}}{s+2} + \frac{\frac{445}{1212}}{s+\frac{1}{5}} - \frac{1}{505}\Big[\frac{76s-96}{s^2+4}\Big]
\end{aligned}
$$

giving

$$
x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{29}{20}e^{-t} + \frac{1}{3}e^{-2t} + \frac{445}{1212}e^{-\frac{1}{5}t} - \frac{1}{505}(76\cos 2t - 48\sin 2t)
$$

▪ **11(a)** Taking Laplace transforms

$$
\begin{aligned}
(s^2+8s+16)\Phi(s) &= \frac{2}{s^2+4} \\
\Phi(s) &= \frac{2}{(s+4)^2(s^2+4)} \\
&= \frac{1}{25}\cdot\frac{1}{s+4} + \frac{1}{10}\cdot\frac{1}{(s+4)^2} - \frac{1}{50}\cdot\frac{2s-3}{s^2+4}
\end{aligned}
$$

so $\theta(t) = \mathcal{L}^{-1}\{\Phi(s)\} = \frac{1}{25}e^{-4t} + \frac{1}{10}\cdot te^{-4t} - \frac{1}{100}(4\cos 2t - 3\sin 2t)$

i.e. $\theta(t) = \frac{1}{100}(4e^{-4t} + 10te^{-4t} - 4\cos 2t + 3\sin 2t)$

11(b) Taking Laplace transforms

$$
\begin{aligned}
(s+2)I_1(s) + 6I_2(s) &= 1 \\
I_1(s) + (s-3)I_2(s) &= 0
\end{aligned}
$$

Eliminating $I_2(s)$

$$
\begin{aligned}
[(s+2)(s-3)-6]I_1(s) &= s-3 \\
I_1(s) &= \frac{s-3}{(s-4)(s+3)} = \frac{\frac{1}{7}}{s-4} + \frac{\frac{6}{7}}{s+3}
\end{aligned}
$$

giving $i_1(t) = \mathcal{L}^{-1}\{I_1(s)\} = \frac{1}{7}(e^{4t} + 6e^{-3t})$

Then from the first differential equation

$$6i_2 = -2i_1 - \frac{di_1}{dt} = -\frac{6}{7}e^{4t} + \frac{6}{7}e^{-3t}$$
$$\text{giving } i_2(t) = \frac{1}{7}(e^{-3t} - e^{4t}),\ i_1(t) = \frac{1}{7}(e^{4t} + 6e^{-3t}).$$

▪ **12** The differential equation

$$LCR\,\frac{d^2i}{dt^2} + L\frac{di}{dt} + Ri = V$$

follows using Kirchhoff's second law.
Substituting $V = E$ and $L = 2R^2C$ gives

$$2R^3C^2\frac{d^2i}{dt^2} + 2R^2C\frac{di}{dt} + Ri = E$$

which on substituting $CR = \dfrac{1}{2n}$ leads to

$$\frac{1}{2n^2}\frac{d^2i}{dt^2} + \frac{1}{n}\frac{di}{dt} + i = \frac{E}{R}$$

and it follows that

$$\frac{d^2i}{dt^2} + 2n\frac{di}{dt} + 2n^2i = 2n^2\frac{E}{R}$$

Taking Laplace transforms

$$(s^2 + 2ns + 2n^2)I(s) = \frac{2n^2E}{R}\cdot\frac{1}{s}$$

$$\begin{aligned} I(s) &= \frac{E}{R}\Big[\frac{2n^2}{s(s^2 + 2ns + 2n^2)}\Big] \\ &= \frac{E}{R}\Big[\frac{1}{s} - \frac{s + 2n}{(s + n)^2 + n^2}\Big] \end{aligned}$$

so that

$$i(t) = \frac{E}{R}[1 - e^{-nt}(\cos nt + n\sin nt)]$$

---

13 The equations are readily deduced by applying Kirchhoff's second law to the left and right hand circuits.

Note that from the given initial conditions we deduce that $i_2(0) = 0$.

Taking Laplace transforms then gives

$$\begin{aligned}(sL+2R)I_1(s) - RI_2(s) &= \frac{E}{s}\\ -RI_1(s) + (sL+2R)I_2(s) &= 0\end{aligned}$$

Eliminating $I_2(s)$

$$\begin{aligned}[(sL+2R)^2 - R^2]I_1(s) &= \frac{E}{s}(sL+2R)\\ (sL+3R)(sL+R)I_1(s) &= \frac{E}{s}(sL+2R)\end{aligned}$$

$$\begin{aligned}I_1(s) &= \frac{E}{L}\Big[\frac{s+\frac{2R}{L}}{s(s+\frac{R}{L})(s+\frac{3R}{L})}\Big]\\ &= \frac{E}{R}\Big[\frac{\frac{2}{3}}{s} - \frac{\frac{1}{2}}{s+\frac{R}{L}} - \frac{\frac{1}{6}}{s+\frac{3R}{L}}\Big]\\ \text{giving } i_1(t) = \mathcal{L}^{-1}\{I_1(s)\} &= \frac{1}{6}\frac{E}{R}\big[4 - 3e^{-\frac{R}{L}t} - e^{-\frac{3R}{L}t}\big]\end{aligned}$$

For large t the exponential terms are approximately zero and

$$i_1(t) \simeq \frac{2}{3}\frac{E}{R}$$

From the first differential equation

$$Ri_2 = 2Ri_1 + L\frac{di_1}{dt} - E$$

Ignoring the exponential terms we have that for large t

$$i_2 \simeq \frac{4}{3}\frac{E}{R} - \frac{E}{R} = \frac{1}{3}\frac{E}{R}$$

■ **14** Taking Laplace transforms

$$\begin{aligned}(s^2+2)X_1(s)-X_2(s)&=\frac{2}{s^2+4}\\ -X_1(s)+(s^2+2)X_2(s)&=0\end{aligned}$$

Eliminating $X_1(s)$

$$[(s^2+2)^2-1]X_2(s)=\frac{2}{s^2+4}$$

$$\begin{aligned}X_2(s)&=\frac{2}{(s^2+4)(s^2+1)(s^2+3)}\\ &=\frac{\frac{2}{3}}{s^2+4}+\frac{\frac{1}{3}}{s^2+1}-\frac{1}{s^2+3}\\ \text{so } x_2(t)=\mathcal{L}^{-1}\{X_2(s)\}&=\frac{1}{3}\sin 2t+\frac{1}{3}\sin t-\frac{1}{\sqrt{3}}\sin\sqrt{3}t\end{aligned}$$

Then from the second differential equation

$$\begin{aligned}x_1(t)=2x_2+\frac{d^2x_2}{dt^2}&=\frac{2}{3}\sin 2t+\frac{2}{3}\sin t-\frac{2}{\sqrt{3}}\sin\sqrt{3}t-\frac{4}{3}\sin 2t-\frac{1}{3}\sin t+\sqrt{3}\sin\sqrt{3}t\\ \text{or } x_1(t)&=-\frac{2}{3}\sin 2t+\frac{1}{3}\sin t+\frac{1}{\sqrt{3}}\sin\sqrt{3}t\end{aligned}$$

■ **15(a)**

(i)

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{s+4}{s^2+2s+10}\}&=\mathcal{L}^{-1}\{\frac{(s+1)+3}{(s+1)^2+3^2}\}\\ &=e^{-t}(\cos 3t+\sin 3t)\end{aligned}$$

(ii)

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{s-3}{(s-1)^2(s-2)}\}&=\mathcal{L}^{-1}\{\frac{1}{(s-1)}+\frac{2}{(s-1)^2}-\frac{1}{s-2}\}\\ &=e^t+2te^t-e^{2t}\end{aligned}$$

15(b) Taking Laplace transforms

$$\begin{aligned}
(s^2+2s+1)Y(s) &= 4s+2+8+\mathcal{L}\{3te^{-t}\} \\
(s+1)^2Y(s) &= 4s+10+\frac{3}{(s+1)^2} \\
Y(s) &= \frac{4s+10}{(s+1)^2}+\frac{3}{(s+1)^4} \\
&= \frac{4}{s+1}+\frac{6}{(s+1)^2}+\frac{3}{(s+1)^4} \\
\text{giving } y(t)=\mathcal{L}^{-1}\{Y(s)\} &= 4e^{-t}+6te^{-t}+\frac{1}{2}t^3e^{-t} \\
\text{i.e. } y(t) &= \frac{1}{2}e^{-t}(8+12t+t^3)
\end{aligned}$$

- **16(a)**

$$\begin{aligned}
F(s) &= \frac{5}{s^2-14s+53}=\frac{5}{2}\cdot\frac{2}{(s-7)^2+2^2} \\
\therefore f(t)=\mathcal{L}^{-1}\{F(s)\} &= \frac{5}{2}e^{7t}\sin 2t
\end{aligned}$$

16(b) $\dfrac{d^2\theta}{dt^2}+2K\dfrac{d\theta}{dt}+n^2\theta=\dfrac{n^2i}{K}$, $\theta(0)=\dot{\theta}(0)=0,\ i$ const.

Taking Laplace transforms

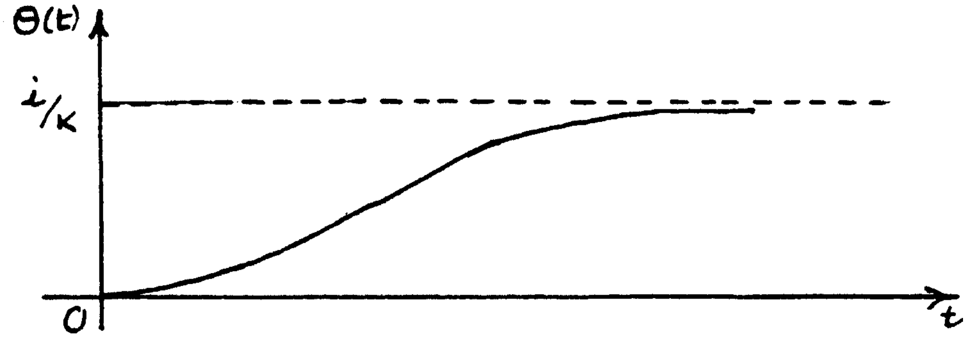
$$\begin{aligned}
(s^2+2Ks+n^2)\Phi(s) &= \frac{n^2}{K}\frac{i}{s} \\
\therefore \Phi(s) &= \frac{n^2i}{Ks(s^2+2Ks+n^2)}
\end{aligned}$$

For the case of critical damping $n=K$ giving

$$\Phi(s)=\frac{Ki}{s(s+K)^2}=Ki\Big[\frac{\frac{1}{K^2}}{s}-\frac{\frac{1}{K^2}}{s+K}-\frac{\frac{1}{K}}{(s+K)^2}\Big]$$

Thus

$$\theta(t) = \mathcal{L}^{-1}\{\Phi(s)\} = \frac{i}{K}[1 - e^{-Kt} - Kte^{-Kt}]$$



■ **17(a)**

(i)

$$\begin{aligned}\mathcal{L}\{\sin tH(t-\alpha)\} &= \mathcal{L}\{\sin[(t-\alpha)+\alpha]H(t-\alpha)\}\\ &= \mathcal{L}\{[\sin(t-\alpha)\cos\alpha + \cos(t-\alpha)\sin\alpha]H(t-\alpha)\}\\ &= \frac{\cos\alpha + s\sin\alpha}{s^2+1}.\,e^{-\alpha s}\end{aligned}$$

(ii)

$$\begin{aligned}\mathcal{L}^{-1}\frac{se^{-\alpha s}}{s^2+2s+5} &= \mathcal{L}^{-1}\Big\{e^{-\alpha s}\frac{(s+1)-1}{(s+1)^2+4}\Big\}\\ &= \mathcal{L}^{-1}\Big\{e^{\alpha s}\mathcal{L}[e^{-t}(\cos 2t - \frac{1}{2}\sin 2t)]\Big\}\\ &= e^{-(t-\alpha)}[\cos 2(t-\alpha) - \frac{1}{2}\sin 2(t-\alpha)]H(t-\alpha)\end{aligned}$$

17(b) Taking Laplace transforms

$$\begin{aligned}(s^2+2s+5)Y(s) &= \frac{1}{s^2+1} - \Big[\frac{-e^{-s\pi}}{s^2+1}\Big] \text{ by (i) above in part (a)}\\ &= \frac{1+e^{-\pi s}}{s^2+1}\end{aligned}$$

$$Y(s) = \frac{1+e^{-\pi s}}{(s^2+1)(s^2+2s+5)} = \Big[-\frac{1}{10}\frac{s-2}{s^2+1} + \frac{1}{10}\frac{s}{s^2+2s+5}\Big](1+e^{-\pi s})$$

giving

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=\frac{1}{10}[2\sin t-\cos t+e^{-(t-\pi)}[2\sin(t-\pi)-\cos(t-\pi)]H(t-\pi)]$$
$$+e^{-t}(\cos 2t-\frac{1}{2}\sin 2t)+e^{-(t-\pi)}[\cos 2(t-z)$$
$$-\frac{1}{2}\sin 2(t-\pi)]H(t-\pi)]$$
$$=\frac{1}{10}\big[e^{-t}(\cos 2t-\frac{1}{2}\sin 2t)+2\sin t-\cos t$$
$$+[e^{-(t-\pi)}(\cos 2t-\frac{1}{2}\sin 2t)+\cos t-2\sin t]H(t-\pi)\big]$$

■ **18** By theorem 2.5

$$\mathcal{L}\{v(t)\}=V(s)=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}v(t)dt$$
$$=\frac{1}{1-e^{-sT}}\left[\int_0^{T/2}e^{-st}dt-\int_{T/2}^{T}e^{-st}dt\right]$$
$$=\frac{1}{1-e^{-sT}}\left\{\left[-\frac{1}{s}e^{-st}\right]_0^{T/2}-\left[-\frac{1}{s}e^{-st}\right]_{T/2}^{T}\right\}$$
$$=\frac{1}{s}\cdot\frac{1}{1-e^{-sT}}(e^{-sT}-e^{-sT/2}-e^{-sT/2}+1)$$
$$=\frac{1}{s}\frac{(1-e^{-sT/2})^2}{(1-e^{-sT/2})(1+e^{-sT/2})}=\frac{1}{s}\left[\frac{1-e^{-sT/2}}{1+e^{-sT/2}}\right]$$

Equation for current flowing is

$$250i+\frac{1}{C}(q_0+\int_0^t i(\tau)d\tau)=v(t),\qquad q_0=0$$

Taking Laplace transforms

$$250I(s)+\frac{1}{10^{-4}}\cdot\frac{1}{s}\cdot I(s)=V(s)=\frac{1}{s}\left[\frac{1-e^{-sT/2}}{1+e^{-sT/2}}\right]$$
$$(s+40)I(s)=\frac{1}{250}\left[\frac{1-e^{-sT/2}}{1+e^{-sT/2}}\right]$$
$$\text{or }I(s)=\frac{1}{250(s+40)}\cdot\frac{1-e^{-sT/2}}{1+e^{-sT/2}}$$

$$I(s) = \frac{1}{250(s+40)}(1 - e^{-sT/2})(1 - e^{-sT/2} + e^{-sT} - e^{-\frac{3}{2}sT} + e^{-2sT} \ldots)$$
$$= \frac{1}{250(s+40)}[1 - 2e^{-sT/2} + 2e^{-sT} - 2e^{-\frac{3}{2}sT} + 2e^{-2sT} \ldots]$$

Since $\mathcal{L}^{-1}\{\frac{1}{250(s+40)}\} = \frac{1}{250}e^{-40t}$ using the second shift theorem gives

$$i(t) = \frac{1}{250}\Big[e^{-40t} - 2H\big(t - \frac{T}{2}\big)e^{-40(t-T/2)} + 2H(t-T)e^{-40(t-T)}$$
$$-2H\big(t - \frac{3T}{2}\big)e^{-40(t-3T/2)} + \ldots\Big]$$

If $T = 10^{-3}$s then the first few terms give a good representation of the steady state since the time constant $\frac{1}{4}$ of the circuit is large compared to the period T.

■ **19** The impulse response $h(t)$ is the solution of

$$\frac{d^2h}{dt^2} + \frac{2dh}{dt} + 2h = \delta(t)$$

subject to the initial conditions $h(0) = \dot{h}(0) = 0$. Taking Laplace transforms

$$(s^2 + 2s + s)H(s) = \mathcal{L}\{\delta(t)\} = 1$$
$$H(s) = \frac{1}{(s+1)^2 + 1}$$
$$\text{i.e. } h(t) = \mathcal{L}^{-1}\{H(s)\} = e^{-t}\sin t.$$

Using the convolution integral the step response $x_s(t)$ is given by

$$x_s(t) = \int_0^t h(\tau)u(t-\tau)d\tau$$

with $u(t) = 1H(t)$; that is

$$x_s(t) = \int_0^t 1.e^{-\tau}\sin\tau d\tau$$
$$= -\frac{1}{2}[e^{-\tau}\cos\tau + e^{-\tau}\sin\tau]_0^t$$
$$\text{i.e. } x_s(t) = \frac{1}{2}[1 - e^{-t}(\cos t + \sin t)].$$

Solving $\dfrac{d^2x_s}{dt^2} + \dfrac{2dx_s}{dt} + 2x_s = 1$ directly we have taking Laplace transforms

$$(s^2 + 2s + 2)X_s(s) = \frac{1}{s}$$

$$\begin{aligned} X_s(s) &= \frac{1}{s(s^2 + 2s + 2)} \\ &= \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\Big[\frac{s+2}{(s+1)^2 + 1}\Big] \end{aligned}$$

giving as before

$$x_s(t) = \frac{1}{2} - \frac{1}{2}e^{-t}(\cos t + \sin t)$$

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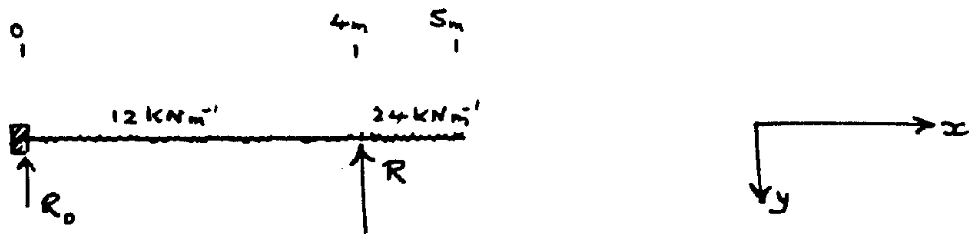


$$\begin{aligned} EI\frac{d^4y}{dx^4} &= 12 + 12H(x-4) - R\delta(x-4) \\ y(0) = y'(0) = 0,\ &y(4) = 0,\ y''(5) = y'''(5) = 0 \end{aligned}$$

With $y''(0) = A, y'''(0) = B$ taking Laplace transforms

$$\begin{aligned} EIs^4Y(s) &= EI(sA + B) + \frac{12}{s} + \frac{12}{s}e^{-4s} - Re^{-4s} \\ Y(s) &= \frac{A}{s^3} + \frac{B}{s^4} + \frac{12}{EI}\cdot\frac{1}{s^5} + \frac{12}{EI}\cdot\frac{1}{s^5}e^{-4s} - \frac{R}{EI}\cdot\frac{1}{s^4}e^{-4s} \end{aligned}$$

giving

$$\begin{aligned} y(x) = \mathcal{L}^{-1}\{Y(s)\} =& \frac{A}{2}x^2 + \frac{B}{6}x^3 + \frac{1}{2EI}x^4 + \frac{1}{2EI}(x-4)^4H(x-4) \\ &- \frac{R}{6EI}(x-4)^3H(x-4) \end{aligned}$$

or

$$EIy(x) = \frac{1}{2}A_1x^2 + \frac{1}{6}B_1x^3 + \frac{1}{2}x^4 + \frac{1}{2}(x-4)^4H(x-4) - \frac{R}{6}(x-4)^3H(x-4)$$

$$y(4) = 0 \;\Rightarrow\; 0 = 8A_1 + \frac{32}{3}B_1 + 128 \;\Rightarrow 3A_1 + 4B_1 = -48$$
$$y''(5) = 0 \;\Rightarrow\; 0 = A_1 + 5B_1 + 6(25) + 6 - R \Rightarrow A_1 + 5B_1 - R = -156$$
$$y'''(5) = 0 \;\Rightarrow\; 0 = B_1 + 12(5) + 12 - R \;\Rightarrow B_1 - R = -72$$

which solve to give $A_1 = 18,\ B_1 = -25.5,\ R = 46.5$
Thus

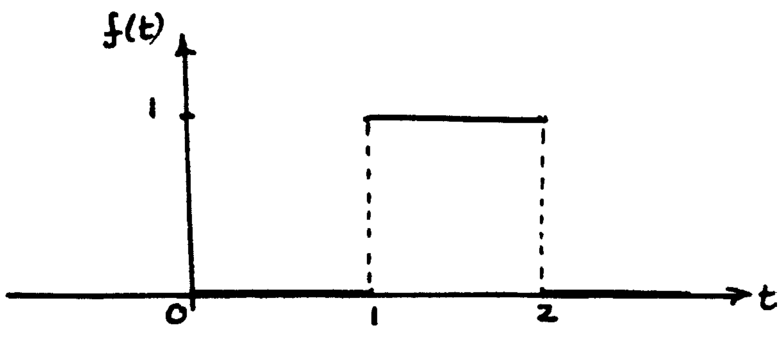
$$y(x) = \begin{cases} \frac{1}{2}x^4 - 4.25x^3 + 9x^2,\ 0 \le x \le 4 \\ \frac{1}{2}x^4 - 4.25x^3 + 9x^2 + \frac{1}{2}(x-4)^4 - 7.75(x-4)^3,\ 4 \le x \le 5 \end{cases}$$

$$R_0 = -EIy'''(0) = 25.5\text{kN},\ M_0 = EIy''(0) = 18\text{kN.m}$$

Check $\quad R_0 + R = 72$kN, Total load $= 12 \times 4 + 24 = 72$kN $\quad\surd$
Moment about $x = 0$ is

$$12 \times 4 \times 2 + 24 \times 4.5 - 4R = 18 = M_0 \quad \surd$$

- **21(a)**



$f(t) = H(t-1) - H(t-2)$
and $\mathcal{L}\{f(t)\} = F(s) = \dfrac{e^{-s}}{s} - \dfrac{e^{-2s}}{s}$

Taking Laplace transforms throughout the differential equation

$$(s+1)X(s) = \frac{1}{s}(e^{-s} - e^{-2s})$$
$$X(s) = \frac{1}{s(s+1)}(e^{-s} - e^{-2s})$$
$$= [\frac{1}{s} - \frac{1}{s+1}]e^{-s} - [\frac{1}{s} - \frac{1}{s+1}]e^{-2s}$$

giving $x(t) = \mathcal{L}^{-1}\{X(s)\} = [1 - e^{-(t-1)}]H(t-1) - [1 - e^{-(t-2)}]H(t-2)$

21(b) $I(s) = \dfrac{E}{s[Ls + R/(1+Cs)]}$

(i) By the initial value theorem

$$\lim_{t\to 0} i(t) = \lim_{s\to\infty} sI(s) = \lim_{s\to\infty} \frac{E}{Ls + R/(1+Cs)} = 0$$

(ii) Since $sI(s)$ has all its poles in the left half of the s-plane the conditions of the final value theorem hold so

$$\lim_{t\to\infty} i(t) = \lim_{s\to 0} sI(s) = \frac{E}{R}$$

▪ **22** We have that for a periodic function $f(t)$ of period T

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-sT} f(t)dt$$

Thus the Laplace transform of the half-rectified sine wave is

$$\mathcal{L}\{v(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^\pi e^{-sT} \sin t dt$$
$$= I_m\{\frac{1}{1-e^{-2\pi s}} \int_0^\pi e^{(j-s)t} dt\}$$
$$= I_m\{\frac{1}{1-e^{-2\pi s}} [\frac{e^{(j-s)t}}{j-s}]_0^\pi\}$$
$$= I_m\{\frac{1}{1-e^{-2\pi s}} [\frac{(-e^{-\pi s}-1)(-j-s)}{(j-s)(-j-s)}]\} = \frac{1+e^{-\pi s}}{(1-e^{-2\pi s})(1+s^2)}$$

i.e. $\mathcal{L}\{v(t)\} = \dfrac{1}{(1+s^2)(1-e^{-\pi s})}$

Applying Kirchoff's law to the circuit the current is determined by

$$\frac{di}{dt} + i = v(t)$$

which on taking Laplace transforms gives

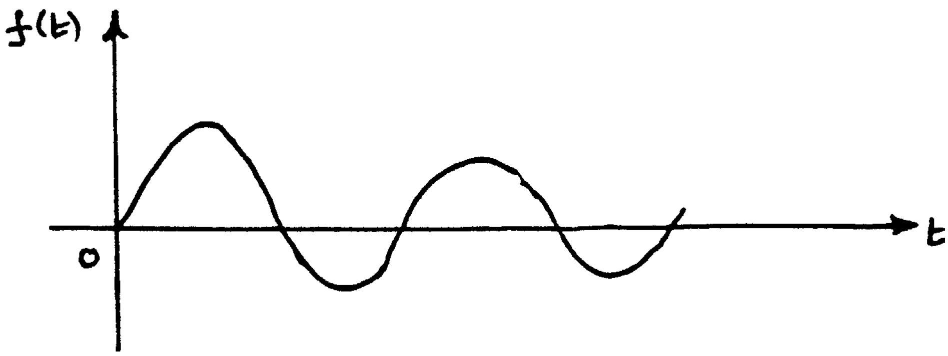
$$\begin{aligned}
(s+1)I(s) &= \frac{1}{(1+s^2)(1-e^{-\pi s})} \\
I(s) &= \frac{1}{1-e^{-\pi s}}\Big[\frac{1}{s+1} - \frac{s+1}{s^2+1}\Big]\cdot\frac{1}{2} \\
&= \frac{1}{2}\Big[\frac{1}{s+1} - \frac{s+1}{s^2+1}\Big]\big[1 + e^{-\pi s} + e^{-2\pi s} + \ldots\big]
\end{aligned}$$

Since $\mathcal{L}^{-1}\{\frac{1}{2}[\frac{1}{s+1} - \frac{s+1}{s^2+1}]\} = \frac{1}{2}(\sin t - \cos t + e^{-t})H(t) = f(t)$

we have by the second shift theorem that

$$i(t) = f(t) + f(t-\pi) + f(t-2\pi) + \ldots = \sum_{n=0}^{\infty} f(t-n\pi)$$

The graph may be plotted by computer and should take the form



- **23(a)** Since $\mathcal{L}\{t\} = \dfrac{1}{s^2}$, $\mathcal{L}\{te^{-t}\} = \dfrac{1}{(s+1)^2}$

taking $f(t) = t$ and $g(t) = te^{-t}$ in the convolution theorem

$$\mathcal{L}^{-1}[F(s)G(s)] = f * g(t)$$

gives

$$\begin{aligned}
\mathcal{L}^{-1}\Big[\frac{1}{s^2}\cdot\frac{1}{(s+1)^2}\Big] &= \int_0^t f(t-\tau)g(\tau)d\tau \\
&= \int_0^t (t-\tau)\tau e^{-\tau} \\
&= \big[-(t-\tau)\tau e^{-\tau} - (t-2\tau)e^{-\tau} + 2e^{-\tau}\big]_0^t \\
\text{i.e. } \mathcal{L}^{-1}\Big[\frac{1}{s^2}\cdot\frac{1}{(s+2)^2}\Big] &= t - 2 + 2e^{-t} + te^{-t}.
\end{aligned}$$

23(b) $y(t) = t + 2\int_0^t y(u)\cos(t-u)du$

Taking $f(t) = y(t), g(t) = \cos t \Rightarrow F(s) = Y(s), G(s) = \dfrac{s}{s^2+1}$ giving on taking transforms

$$\begin{aligned}
Y(s) &= \frac{1}{s^2} + 2Y(s)\frac{s}{s^2+1} \\
(s^2+1-2s)Y(s) &= \frac{s^2+1}{s^2} \\
\text{or } Y(s) &= \frac{s^2+1}{s^2(s-1)^2} = \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{2}{(s-1)^2} \\
\text{and } y(t) &= \mathcal{L}^{-1}\{Y(s)\} = 2 + t - 2e^t + 2te^t.
\end{aligned}$$

Taking transforms

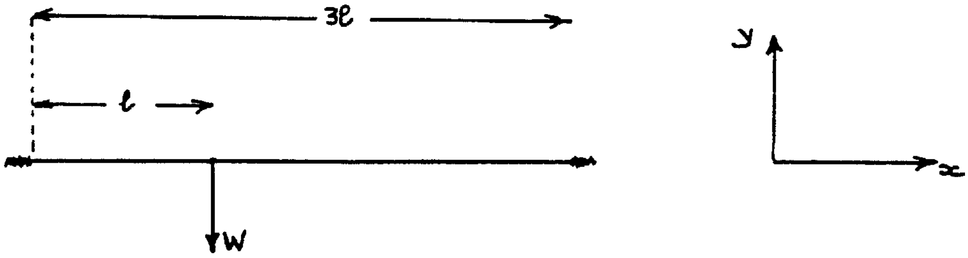
$$\begin{aligned}
(s^2Y(s) - sy(0) - y'(0))(sY(s) - y(0)) &= Y(s) \\
\text{or } (s^2Y(s) - y_1)(sY(s)) &= Y(s) \\
\text{giving } Y(s) = 0 \text{ or } Y(s) &= \frac{y_1}{s^2} + \frac{1}{s^3}
\end{aligned}$$

which on inversion gives

$$y(t) = 0 \text{ or } y(t) = \frac{1}{2}t^2 + ty_1$$

In the second of these solutions the condition on $y'(0)$ is arbitrary.

■ **24**



Equation for displacement is

$$EI\frac{d^4y}{dx^4} = -W\delta(x-\ell)$$

with $y(0)=0,\ y(3\ell)=0,\ y'(0)=y'(3\ell)=0$
with $y''(0)=A,\ y'''(0)=B$ then taking Laplace transforms gives

$$\begin{aligned}
EIs^4Y(s) &= EI(sA+B) - We^{-\ell s}\\
Y(s) &= \frac{-W}{EIs^4}e^{-\ell s} + \frac{A}{s^3} + \frac{B}{s^4}\\
\text{giving } y(x) &= \frac{-W}{6EI}(x-\ell)^3.H(x-\ell) + \frac{A}{2}x^2 + \frac{B}{6}x^3
\end{aligned}$$

For $x>\ell, y'(x) = \dfrac{-3W}{6EI}(x-\ell)^2 + Ax + \dfrac{B}{2}x^2$
so $y'(3\ell)=0$ and $y(3\ell)=0$ gives

$$\begin{aligned}
0 &= -\frac{2W\ell^2}{EI} + 3A\ell + 9B\frac{\ell^2}{2}\\
0 &= -\frac{4W\ell^3}{3EI} + \frac{9}{2}A\ell^2 + \frac{9}{2}B\ell^3\\
\text{giving } A &= -\frac{4W\ell}{9EI} \text{ and } B = \frac{20}{27}\frac{W}{EI}
\end{aligned}$$

Thus deflection $y(x)$ is

$$y(x) = -\frac{W}{6EI}(x-\ell)^3H(x-\ell) - \frac{2}{9}\frac{W\ell}{EI}x^2 + \frac{10}{81}\frac{W}{EI}x^3$$

With the added uniform load the differential equation governing the deflection is

$$EI\frac{d^4y}{dx^4} = -W\delta(x-\ell) - w[H(x) - H(x-\ell)]$$

25(a) Taking Laplace transforms

$$(s^2 - 3s + 3)X(s) = \frac{1}{s}e^{-as}$$

$$X(s) = \frac{1}{s(s^2-3s+3)}\cdot e^{-as} = \Big[\frac{\frac{1}{6}}{s} - \frac{\frac{1}{6}s - \frac{1}{2}}{s^2-3s+3}\Big]\cdot e^{-as}$$

$$= \frac{1}{6}\Big[\frac{1}{s} - \frac{(s-\frac{3}{2}) - \sqrt{3}(\frac{\sqrt{3}}{2})}{(s-\frac{3}{2})^2 + (\frac{\sqrt{3}}{2})^2}\Big]e^{-as}$$

$$= \frac{e^{-as}}{6}\mathcal{L}\Big\{1 - e^{-\frac{3}{2}t}\big(\cos\frac{\sqrt{3}}{2}t - \sqrt{3}\sin\frac{\sqrt{3}}{2}t\big)\Big\}$$

giving

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{6}\Big[1 - e^{-\frac{3}{2}(t-a)}\big(\cos\frac{\sqrt{3}}{2}(t-a) - \sqrt{3}\sin\frac{\sqrt{3}}{2}(t-a)\big)\Big]H(t-a)$$

25(b)

$$X(s) = G(s)\mathcal{L}\{\sin wt\} = G(s)\frac{w}{s^2+w^2}$$

$$= \frac{w}{(s+jw)(s-jw)}G(s)$$

Since the system is stable all the poles of $G(s)$ have negative real part. Expanding in partial fractions and inverting gives

$$x(t) = 2R_e\Big[\frac{F(jw)w}{2jw}\cdot e^{jwt}\Big] + \text{ terms from } G(s) \text{ with negative exponentials}$$

Thus as $t \to \infty$ the added terms tend to zero and $x(t) \to x_s(t)$ with

$$x_s(t) = R_e\Big[\frac{e^{jwt}F(jw)}{j}\Big]$$

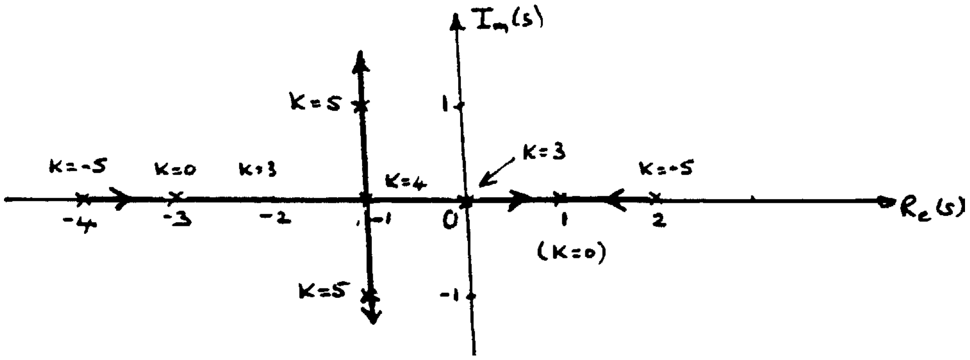
■ **26(a)** In the absence of feedback the system has poles at

$$s = -3 \text{ and } s = 1$$

and is therefore unstable.

26(b) $G_1(s) = \dfrac{G(s)}{1 + KG(s)} = \dfrac{1}{(s-1)(s+3)+K} = \dfrac{1}{s^2 + 2s + (K-3)}$

26(c) Poles $G_1(s)$ given by $s = -1 \pm \sqrt{4-K}$.
These may be plotted in the s-plane for different values of K. Plot should be as in the figure



26(d) Clearly from the plot in (c) all the poles are in the left half plane when $K > 3$. Thus system stable for $K > 3$.

26(e)
$$\overset{a_2}{1s^2} + \overset{a_1}{2s} + \overset{a_0}{(K-3)} = 0$$
Routh–Hurwitz determinants are

$$\Delta_1 = 2 > 0$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_2 \\ 0 & a_0 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & K-3 \end{vmatrix} = 2(K-3) > 0 \text{ if } K > 3$$

thus confirming the result in (d).

■ **27(a)** Closed loop transfer function is

$$G_1(s) = \frac{G(s)}{1+G(s)} = \frac{2}{s^2 + \alpha s + 5}$$

Thus $\mathcal{L}^{-1}\{\frac{2}{s^2+\alpha s+5}\} = h(t) = 2e^{-2t}\sin t$
i.e. $\mathcal{L}^{-1}\{\frac{2}{(s+\frac{\alpha}{2})^2+(5-\frac{\alpha^2}{4})}\} = 2e^{-\frac{\alpha}{2}t}\sin\sqrt{(5-\frac{\alpha^2}{4})}t = 2e^{-2t}\sin t$
giving $\alpha = 4$

27(b) Closed loop transfer function is

$$G(s) = \frac{\frac{10}{s(s-1)}}{1-\frac{(1+Ks)10}{s(s-1)}} = \frac{10}{s^2+(10K-1)s+10}$$

Poles of the system are given by

$$s^2+(10K-1)s+10 = 0$$

which are both in the negative half plane of the s-plane provided $(10K-1) > 0$; that is, $K > \frac{1}{10}$. Thus the critical value of K for stability of the closed loop system is $K = \frac{1}{10}$.

▪ **28(a)** Overall closed loop transfer function is

$$G(s) = \frac{\frac{K}{s(s+1)}}{1+\frac{K}{s(s+1)}(1+K_1 s)} = \frac{K}{s^2+s(1+KK_1)+K}$$

28(b) Assuming zero initial conditions step response $x(t)$ is given by

$$\begin{aligned}
X(s) = G(s)\mathcal{L}\{1.H(t)\} &= \frac{K}{s[s^2+s(1+KK_1)+K]} \\
&= \frac{w_n}{s[s^2+2\xi w_n s+w_n^2]} \\
&= \frac{1}{s} - \frac{s+2\xi w_n}{s^2+2\xi w_n s+w_n^2} \\
&= \frac{1}{s} - \left[\frac{(s+\xi w_n)+\xi w_n}{(s+\xi w_n)^2+[w_n^2(1-\xi^2)]}\right] \\
&= \frac{1}{s} - \left[\frac{(s+\xi w_n)+\xi w_n}{(s+\xi w_n)^2+w_d^2}\right]
\end{aligned}$$

giving $x(t) = \mathcal{L}^{-1}\{X(s)\} = 1 - e^{-\xi w_n t}\big[\cos w_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin w_d t\big], t \geq 0.$

■ **28(c)** The peak time t_p is given by the solution of $\frac{dx}{dt}\big|_{t=t_p} = 0$

$$\frac{dx}{dt} = e^{-\xi w_n t}\left[(\xi w_n - \frac{\xi w_d}{\sqrt{1-\xi^2}})\cos w_d t(\frac{\xi^2 w_n}{\sqrt{1-\xi^2}} + w_d)\sin w_d t\right]$$

$$= e^{-\xi w_n t}\frac{w_n}{\sqrt{1-\xi^2}}\sin w_d t$$

Thus t_p given by the solution of

$$e^{-\xi w_n t_p}\frac{w_n}{\sqrt{1-\xi^2}}\sin w_d t_p = 0$$

$$\text{i.e. } \sin w_d t_p = 0$$

Since the peak time corresponds to the first peak overshoot

$$w_d t_p = \pi \text{ or } t_p = \frac{\pi}{w_d}$$

The maximum overshoot M_p occurs at the peak time t_p. Thus

$$M_p = x(t_p) - 1 = e^{-\frac{\xi w_n \pi}{w_d}}\left[\cos\pi + \frac{\xi}{\sqrt{1-\xi^2}}\sin\pi\right]$$

$$= e^{-\frac{\xi w_n \pi}{w_d}} = e^{-\xi\pi/\sqrt{1-\xi^2}}\pi$$

We wish M_p to be 0.2 and t_p to be 1s, thus

$$e^{-\xi\pi/\sqrt{1-\xi^2}} = 0.2 \text{ giving } \xi = 0.456$$

and

$$t_p = \frac{\pi}{w_d} = 1 \text{ giving } w_d = 3.14$$

Then it follows that $w_n = \frac{w_d}{\sqrt{1-\xi^2}} = 3.53$ from which we deduce that

$$K = w_n^2 = 12.5$$

$$\text{and } K_1 = \frac{2w_n\xi - 1}{K} = 0.178.$$

28(d) The rise time t_r is given by the solution of

$$x(t_r) = 1 = 1 - e^{-\xi w_n t_r}\left[\cos w_d t_r + \frac{\xi}{\sqrt{1-\xi^2}}\sin w_d t_r\right]$$

Since $e^{-\xi w_n t_r} \neq 0$

$$\cos w_d t_r + \frac{\xi}{\sqrt{1-\xi^2}}\sin w_d t_r = 0$$

giving $\tan w_d t_r = -\dfrac{\sqrt{1-\xi^2}}{\xi}$

$$\text{or} \qquad t_r = \frac{1}{w_d}\tan^{-1}\left(-\frac{\sqrt{1-\xi^2}}{\xi}\right) = \frac{\pi - 1.10}{w_d} = 0.65\text{s}.$$

The response $x(t)$ in (b) may be written as

$$x(t) = 1 - \frac{e^{-\xi w_n t}}{\sqrt{1-\xi^2}}\sin\left[w_\alpha t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right]$$

so the curves $1 \pm \dfrac{e^{-\xi w_n t}}{\sqrt{1-\xi^2}}$ are the envelope curves of the transient response to a unit step input and have a time constant $T = \dfrac{1}{\xi w_n}$. The settling time t_s may be measured in terms of T. Using the 2% criterion t_s is approximately 4 times the time constant and for the 5% criterion it is approximately 3 times the time constant. Thus

$$2\% \text{ criterion} \;:\; t_s = 4T = \frac{4}{\xi w_n} = 2.48\text{s}$$
$$5\% \text{ criterion} \;:\; t_s = 3T = \frac{3}{\xi w_n} = 1.86\text{s}$$

Footnote This is intended to be an extended exercise with students being encouraged to carry out simulation studies in order to develop a better understanding of how the transient response characteristics can be used in system design.

▪ **29** As for Exercise 28 this is intended to be an extended problem supported by simulation studies. The following is simply an outline of a possible solution.

Figure 2.63(a) is simply a mass-spring damper system represented by the differential equation

$$M_1\frac{d^2x}{dt^2}+B\frac{dx}{dt}+K_1x=\sin wt$$

Assuming that it is initially in a quiescent state taking Laplace transforms

$$X(s)=\frac{1}{M_1s^2+Bs+K_1}\cdot\frac{w}{s^2+w^2}$$

The steady state response will be due to the forcing term and determined by the $\dfrac{\alpha s+\beta}{s^2+w^2}$ term in the partial fractions expansion of $X(s)$. Thus, the steady state response will be of the form $A\sin(wt+\delta)$; that is, a sinusoid having the same frequency as the forcing term but with a phase shift δ and amplitude scaling A.
In the situation of Figure 2.63(b) the equations of motion are

$$\begin{aligned}M_1\frac{d^2x}{dt^2}&=-K_1x-B\frac{dx}{dt}+K_2(y-x)+\sin wt\\ M_2\frac{d^2y}{dt^2}&=-K_2(y-x)\end{aligned}$$

Assuming an initial quiescent state taking Laplace transforms gives

$$[M_1s^2+Bs+(K_1+K_2)]X(s)-K_2Y(s)=w/(s^2+w^2)$$
$$-K_2X(s)+(s^2M_2+K_2)Y(s)=0$$

Eliminating $Y(s)$ gives

$$X(s)=\frac{w(s^2M_2+K_2)}{(s^2+w^2)p(s)}$$

where $p(s)=(M_1s^2+Bs+K_1+K_2)(s^2M_2+K_2)$.
Because of the term (s^2+w^2) in the denominator $x(t)$ will contain terms in $\sin wt$ and $\cos wt$. However, if $(s^2M_2+K_2)$ exactly cancels (s^2+w^2) this will be avoided. Thus choose $K_2=M_2w^2$. This does make practical sense for if the natural frequency of the secondary system is equal to the frequency of the applied force then it may resonate and therefore damp out the steady state vibration of M_1.
It is also required to show that the polynomial $p(s)$ does not give rise to any undamped oscillations. That is, it is necessary to show that $p(s)$ does not possess purely imaginary roots of the form $j\theta,\theta$ real, and that it has no roots with a positive real part. This can be checked using the Routh–Hurwitz criterion.

To examine the motion of the secondary mass M_2 solve for $Y(s)$ giving

$$Y(s) = \frac{K_2 w}{(s^2 + w^2)p(s)}$$

Clearly due to the term $(s^2 + w^2)$ in the denominator the mass M_2 possesses an undamped oscillation. Thus, in some sense the secondary system has absorbed the energy produced by the applied sinusoidal force $\sin wt$.

▪ **30** Again this is intended to be an extended problem requiring wider exploration by the students. The following is an outline of the solution.

30(a) Students should be encouraged to plot the Bode plots using the steps used in example 2.62 of the text and using a software package. Sketches of the magnitude and phase Bode plots are given in the figures below.

30(b) With unity feedback the amplifier is unstable. Since the $-180°$ crossover gain is greater than 0dB (from the plot it is +92dB).

30(c) Due to the assumption that the amplifier is ideal it follows that for marginal stability the value of $\frac{1}{\beta}$ must be 92dB (that is, the plot is effectively lowered by 92dB). Thus

$$\begin{aligned} 20\log\frac{1}{\beta} &= 92 \\ \frac{1}{\beta} &= \text{ antilog } \left(\frac{92}{20}\right) \Rightarrow \beta \simeq 2.5 \times 10^{-5} \end{aligned}$$

30(d) From the amplitude plot the effective 0dB axis is now drawn through the 100dB point. Comparing this to the line drawn through the 92dB point, corresponding to marginal stability, it follows that

$$\begin{aligned} \text{Gain margin } &= -8\text{dB} \\ \text{and Phase margin } &= 24°. \end{aligned}$$

30(e)

$$G(s) = \frac{K}{(1 + s\tau_1)(1 - s\tau_2)(1 + s\tau_3)}$$

Given low frequency gain $K = 120\text{dB}$ so

$$20 \log K = 120 \;\Rightarrow\; K = 10^6$$

$T_i = \dfrac{1}{f_i}$ where f_i is the oscillating frequency in cycles per second of the pole. Since $1\text{MHz} = 10$ cycles per second

$$\begin{aligned}
\tau_1 &= \frac{1}{f_1} = \frac{1}{10^6} \text{ since } f_1 = 1\text{MHz} \\
\tau_2 &= \frac{1}{f_2} = \frac{1}{10.10^6} \text{ since } f_2 = 10\text{MHz} \\
\tau_3 &= \frac{1}{f_3} = \frac{1}{25.10^6} \text{ since } f_3 = 25\text{MHz}
\end{aligned}$$

Thus

$$\begin{aligned}
G(s) &= \frac{10^6}{(1 + \frac{s}{10^6})(1 + \frac{s}{10.10^6})(1 + \frac{s}{25.10^6})} \\
&= \frac{250.10^{24}}{(s + 10^6)(s + 10^7)(s + \frac{5}{2}.10^7)}
\end{aligned}$$

The closed loop transfer function $G_1(s)$ is

$$G(s) = \frac{G(s)}{1 + \beta G(s)}$$

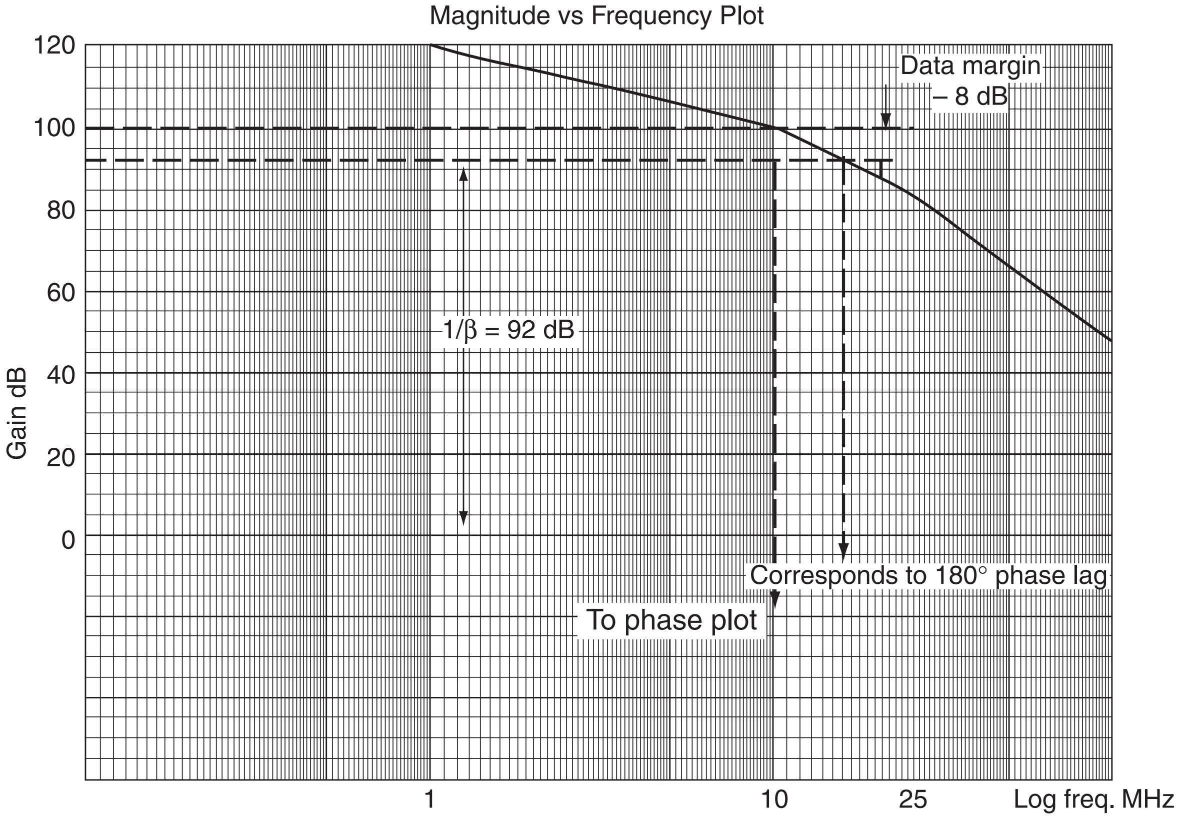
30(f) The characteristic equation for the closed loop system is

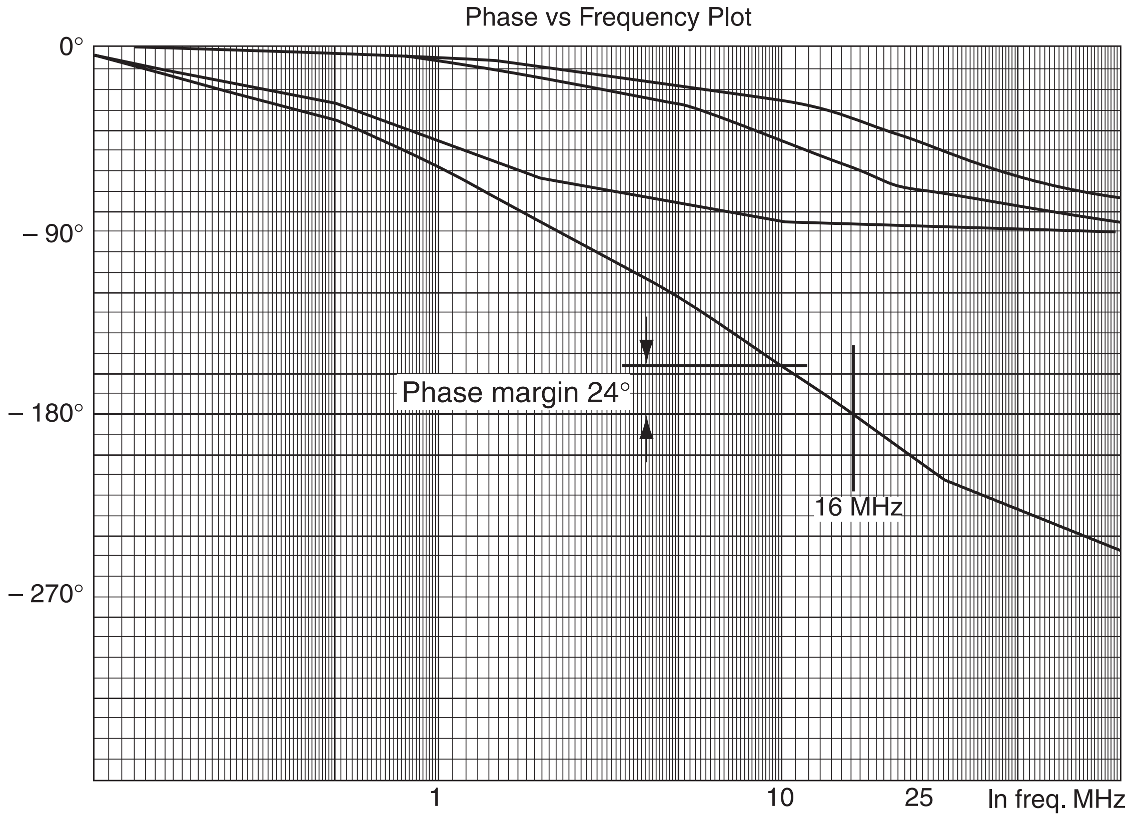
$$(s + 10^6)(s + 10^7)(s + \tfrac{5}{2}.10^7) + \beta 25.10^{25} = 0$$

or

$$\underset{\substack{\downarrow \\ A_1}}{s^3 + 36(10^6)}s^2 + \underset{\substack{\downarrow \\ A_2}}{(285)10^{12}}s + \underset{\substack{\downarrow \\ A_3}}{10^{19}(25 + 25\beta 10^6)} = 0$$

By Routh–Hurwitz criterion system stable provided $A_1 > 0$ and $A_1A_2 > A_3$. If $\beta = 1$ then $A_1A_2 < A_3$ and the system is unstable as determined in (b). For marginal stability $A_1A_2 = A_3$ giving $\beta = 1.40^{-5}$ (compared with $\beta = 2.5.10^{-5}$ using the Bode plot).


Magnitude vs Frequency Plot
120
100
80
60
40
20
0
Gain dB
Data margin
– 8 dB
1/β = 92 dB
Corresponds to 180° phase lag
To phase plot
1
10
25
Log freq. MHz


Phase vs Frequency Plot
0°
– 90°
– 180°
– 270°
Phase margin 24°
16 MHz
1
10
25
In freq. MHz