UNCERTAINTY IN SAMPLED SYSTEMS

by

Charles E. Rohrs*, Gunter Stein**, and Karl J. Astrom***

ABSTRACT

The recently obtained evidence of the need for a positive real element in an adaptive system leaves us with a disturbing gap in adaptive control theory. It is a fact that in some applications adaptive controllers are performing well in practice. How can these systems behave well in practical situations which must contain modeling error? This paper introduces a preliminary result which indicates that it may be possible to maintain the needed positive real system in the presence of modeling error. The result shows that if a continuous-time system with large high frequency uncertainty is treated appropriately with anti-aliasing filters and sampled slowly enough, the resulting discrete-time system may contain very little uncertainty. With small enough uncertainty in the plant, a positive real system in the adaptive loop is possible.

INTRODUCTION

In [1], it was shown that a passivity condition appears necessary for stability in adaptive algorithms. This is a severe restriction since, in practice, the phase of the plant will be completely uncertain at high frequencies. Thus, there is an apparent contradiction between what adaptive control requires for stability and what is achievable in practice. Yet, adaptive controllers are functioning in practice. How can this be?

In this paper, a theorem which states that, while a continuous-time plant may contain much high frequency uncertainty, that uncertainty may appear small in the discrete-time equivalent after sampling. Thus, passivity or a similar condition is achievable for some practical sampled data systems. This reduction of uncertainty is achieved at the expense of bandwidth reduction. An example of this effect in adaptive systems has appeared in [2] and [3].

THE THEOREM

A typical sample data system can be modeled as an impulse reconstructor, a zero-order hold, the plant, an anti-aliasing filter, and an impulse sampler in series. Let $g(j\omega)$ represent the continuous frequency response function of the series combination of the zero-order hold, the plant, and the anti-aliasing filter. Let the plant uncertainty be represented by a multiplicative perturbation, $1 + L(j\omega)$, to the nominal system, $g_0(j\omega)$.

$$g(j\omega) = g_0(j\omega)(1 + L(j\omega)). \tag{1}$$

Typically, all that is known about $L(j\omega)$ is a bound on its magnitude, [4].

$$|L(j\omega)| < \ell(j\omega). \tag{2}$$

Typically, the magnitude of $\ell(j\omega)$ increases as frequency increases. A typical $\ell(j\omega)$ may be given by

$$\ell(j\omega) = \frac{\omega^2 + a^2}{b^2} \tag{3}$$

with $b > a$. When $\ell(j\omega) > 1$, there is complete uncertainty about the phase of $g(j\omega)$.

Let $g_d(j\omega)$ be the equivalent discrete-time frequency response attained by preceding $g(j\omega)$ with an impulse reconstructer and following it with an impulse sampler, synchronized and using sampling period T. Then

$$g_d(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} g(j\omega + jk\frac{2\pi}{T}). \tag{4}$$

Likewise, let

$$g_{d_0}(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} g_0(j\omega + jk\frac{2\pi}{T}). \tag{5}$$

Finally, let $L_d(j\omega)$ be a discrete-time multiplicative perturbation, i.e. let $L_d(j\omega)$ be such that

$$g_d(j\omega) = g_{d_0}(j\omega)\,(1 + L_d(j\omega)). \tag{6}$$

---

*Tellabs Research Laboratory, South Bend, IN and Department of Electrical Engineering, University of Notre Dame, Notre Dame, Indiana 46556.
This research has been supported in part by the National Science Foundation under Grant NSF/ECS-8307479.

**Systems and Research Center, Honeywell, Inc., Minneapolis, Minnesota, and Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA.

***Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.

Theorem: Given the above definitions and assuming that all infinite sums converge, then

$$|L_d(j\omega)| < \ell_d(j\omega) \tag{7}$$

with

$$\ell_d(j\omega) = \frac{1}{T} \frac{\sum_{k=-\infty}^{\infty} \ell(j\omega + j\frac{2\pi}{T})|g_0(j\omega + j\frac{2\pi}{T})|}{|g_{d_0}(j\omega)|} \tag{8}$$

Proof:
$$G_d(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} g(j\omega + jk\frac{2\pi}{T})$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} g_0(j\omega + jk\frac{2\pi}{T})(1 + L(j\omega + jk\frac{2\pi}{T}))$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} g_0(j\omega + jk\frac{2\pi}{T}) +$$

$$\frac{1}{T}\sum_{k=-\infty}^{\infty} g_0(j\omega + jk\frac{2\pi}{T})\, L(j\omega + jk\frac{2\pi}{T})$$

$$= g_{d_0}(j\omega) + \frac{1}{T}\sum_{k=-\infty}^{\infty} g_0(j\omega + jk\frac{2\pi}{T})\, L(j\omega + jk\frac{2\pi}{T})$$

$$= g_{d_0}(j\omega)\left(1 + \frac{1}{T}\frac{\sum_{k=-\infty}^{\infty} g_0(j\omega + jk\frac{2\pi}{T})\, L(j\omega + jk\frac{2\pi}{T})}{g_{d_0}(j\omega)}\right)$$

Making the identification with equation (6) gives

$$L_d(j\omega) = \frac{1}{T}\frac{\sum_{k=-\infty}^{\infty} g_0(j\omega + jk\frac{2\pi}{T})\, L(j\omega + jk\frac{2\pi}{T})}{g_{d_0}(j\omega)}$$

The result follows from a simple bounding argument.

Q.E.D.

The theorem provides some information about the proper choice of anti-aliasing filters and sampling interval. First of all, the anti-aliasing filters which have been considered to be a part of $g_0(j\omega)$ must be chosen so that the rolloff provided by these filters when added to the rolloff naturally provided by the plant will overcome the increase in $\ell(j\omega)$ and allow the infinite sum in the numerator of $\ell_d(j\omega)$ (equation (8)) to converge.

Assume that the prefilters are chosen based upon the sampling period so that for $-\pi/T < \omega < \pi/T$ the only significant term in each of the sums of equation (5) and equation (8) is the k=0 term. In practice, prefilters are chosen so this assumption holds. This assumption implies that

$$|g_{d_0}(j\omega)| \approx \frac{1}{T}|g_0(j\omega)| \qquad -\frac{\pi}{T} < \omega < \frac{\pi}{T} \tag{9}$$

and also that

$$\ell_d(j\omega) \approx \ell(j\omega) \qquad -\frac{\pi}{T} < \omega < \frac{\pi}{T} \tag{10}$$

However, while $\ell(j\omega)$ increases with increasing ω, $\ell_d(j\omega)$ is periodic with period $2\pi/T$. Determining $\ell_d(j\omega)$ for $-\pi/T < \omega < \pi/T$ determines it for all ω.

If we consider $\ell(j\omega)$ to be a monotone increasing function of ω then

$$\ell_{dsup} = \sup_{\omega} \ell_d(j\omega) \approx \ell(j\frac{\pi}{T}). \tag{11}$$

If a system is sampled rapidly, T will be small and ℓ_{dsup} will be quite large. If, however, the system is sampled slowly enough, ℓ_{dsup} will be approximately equal to the small value that $\ell(j\omega)$ has for low frequencies.

The quantity $\ell_d(j\omega)$ measures how closely the actual discrete-time system $g_d(j\omega)$ is approximated by the nominal discrete-time system, $g_d\,(j\omega)$. In particular if

$$\ell_d(j\omega) < \cos\phi_{d_0}(j\omega) \tag{12}$$

where $\phi_{d_0}(j\omega)$ is the phase angle associated with $g_{d_0}(j\omega)$, then a sampled data system which is nominally positive real will remain positive real in the face of all admissible continuous-time multiplicative perturbations satisfying the magnitude bound of equation (13). Notice also that, from the periodicity of $g_{d_0}(j\omega)$, the discrete-time equivalent of $g_0(j\omega)$, it can be seen that slow sampling also helps promote a positive real nominal sampled data system and a large right hand side of equation (12).

Further explanations of the meaning of this theorem and its effect on adaptive control theory are given in [5].

CONCLUSIONS

A a new result is shown which proves that, by proper sampling of a continuous-time system, a discrete-time system can result which contains very little uncertainty even though the original continuous-time system contains a great deal of high frequency uncertainty. Thus, this paper shows that the passivity requirement needed in adaptive systems may be attainable in at least some discrete-time implementations of adaptive systems.

REFERENCES

1. C. Rohrs, "Stability Mechanisms and Adaptive Control", Control Systems Technical Note #26, Dept. of Electrical Engineering, University of Notre Dame, September 1984.

2. C.E. Rohrs, Adaptive Control in the Presence of Unmodeled Dynamics, Ph.D. Thesis, Massachusetts Institute of Technology, Cambridge, MA, LIDS Report #TH-1254, November 1982.

3. C.E. Rohrs, M. Athans, L. Valavani, and G. Stein, "Some Design Guidelines for Discrete-Time Adaptive Controllers", Proceeding of the 9th IFAC World Congress, Budapest, Hungary, July 1984; to appear, Automatica.

4. J.C. Doyle and G. Stein, "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis", IEEE Trans. Automatic Control, Vol. AC-26, No. 1, pp. 4-16, February 1981.

5. C.E. Rohrs, G. Stein, and K.J. Åstrom, "Sampling and Robustness in Adaptive Control", in preparation.