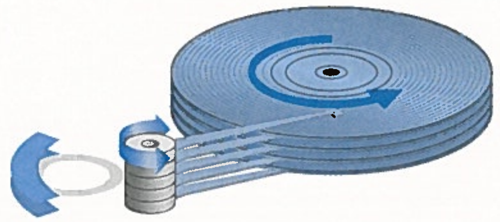
2 Dynamic Models



A Perspective on Dynamic Models

The overall goal of feedback control is to use the principle of feedback to cause the output variable of a dynamic process to follow a desired reference variable accurately, regardless of the reference variable's path and regardless of any external disturbances or any changes in the dynamics of the process. This complex goal is met as the result of a number of simple, distinct steps. The first of these is to develop a mathematical description (called a **dynamic model**) of the process to be controlled. The term **model**, as it is used and understood by control engineers, means a set of differential equations that describe the dynamic behavior of the process. A model can be obtained using principles of the underlying physics or by testing a prototype of the device, measuring its response to inputs, and using the data to construct an analytical model. We will focus only on using physics in this chapter. There are entire books written on experimentally determining models, sometimes called System Identification, and these techniques are described very briefly in Chapter 3. A careful control system designer will typically rely on at least some experiments to verify the accuracy of the model when it is derived from physical principles.

In many cases the modeling of complex processes is difficult and expensive, especially when the important steps of building and testing prototypes are included. However, in this introductory text, we will focus on the most basic principles of modeling for the most common physical systems. More comprehensive sources and specialized texts will be referenced throughout the text where appropriate for those wishing more detail.

In later chapters we will explore a variety of analysis methods for dealing with the equations of motion and their solution for purposes of designing feedback control systems.

Chapter Overview

The fundamental step in building a dynamic model is writing the equations of motion for the system. Through discussion and a variety of examples, Section 2.1 demonstrates how to write the equations of motion for a variety of mechanical systems. In addition, the section demonstrates the use of MATLAB® to find the time response of a simple system to a step input. Furthermore, the ideas of transfer functions and block diagrams are introduced, along with the idea that problems can also be solved via SIMULINK®.

Electric circuits and electromechanical systems are modeled in Sections 2.2 and 2.3, respectively.

For those wanting modeling examples for more diverse dynamic systems, Section 2.4, which is optional, extends the discussion to heat and fluid-flow systems.

The chapter concludes with Section 2.5, a discussion of the history behind the discoveries that led to the knowledge that we take for granted today.

The differential equations developed in modeling are often nonlinear. Because nonlinear systems are significantly more challenging to solve than linear ones and because linear models are usually adequate, the emphasis in the early chapters is primarily on linear systems. However, we do show how to linearize simple nonlinearities here in Chapter 2 and show how to use SIMULINK to numerically solve for the motion of a nonlinear system. A much more extensive discussion of linearization and analysis of nonlinear systems is contained in Chapter 9.

In order to focus on the important first step of developing mathematical models, we will defer explanation of the computational methods used to solve the equations of motion developed in this chapter until Chapter 3.

2.1 Dynamics of Mechanical Systems

2.1.1 Translational Motion

Newton's law for translational motion

The cornerstone for obtaining a mathematical model, or the **equations of motion**, for any mechanical system is Newton's law,

$$\mathbf{F} = m\mathbf{a}, \tag{2.1}$$

where

$\mathbf{F}$ = the vector sum of all forces applied to each body in a system, newtons (N) or pounds (lb),

$\mathbf{a}$ = the vector acceleration of each body with respect to an inertial reference frame (i.e., one that is neither accelerating nor rotating with respect to the stars); often called **inertial acceleration**, m/sec^2 or ft/sec^2,

m = mass of the body, kg or slug.

Note that here in Eq. (2.1), as throughout the text, we use the convention of boldfacing the type to indicate that the quantity is a matrix or vector, possibly a vector function.

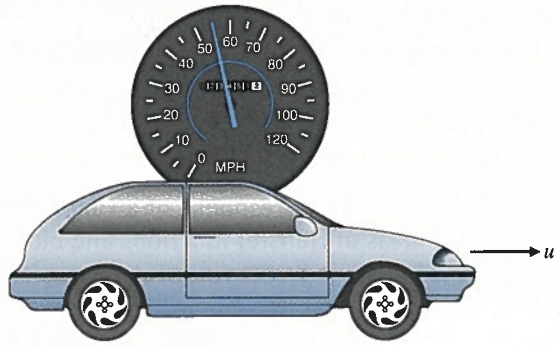


Figure 2.1
Cruise control model

In SI units a force of 1 N will impart an acceleration of 1 m/sec^2 to a mass of 1 kg. In English units a force of 1 lb will impart an acceleration of 1 ft/sec^2 to a mass of 1 slug. The "weight" of an object is mg, where g is the acceleration of gravity ($= 9.81$ m/sec^2 $= 32.2$ ft/sec^2). In English units it is common usage to refer to the mass of an object in terms of its weight in pounds, which is the quantity measured on scales. To obtain the mass in slugs for use in Newton's law, divide the weight by g. Therefore, an object weighing 1 lb has a mass of $^1/_{32.2}$ slugs. A slug has units lb$\cdot$sec^2/ft. In metric units, scales are typically calibrated in kilograms, which is a direct measure of mass.

Use of free-body diagram in applying Newton's law

Application of this law typically involves defining convenient coordinates to account for the body's motion (position, velocity, and acceleration), determining the forces on the body using a free-body diagram, and then writing the equations of motion from Eq. (2.1). The procedure is simplest when the coordinates chosen express the position with respect to an inertial frame because, in this case, the accelerations needed for Newton's law are simply the second derivatives of the position coordinates.

EXAMPLE 2.1 *A Simple System; Cruise Control Model*

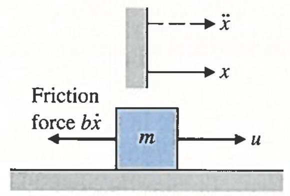
1. Write the equations of motion for the speed and forward motion of the car shown in Fig. 2.1 assuming that the engine imparts a force u as shown. Take the Laplace transform of the resulting differential equation and find the transfer function between the input u and the output v.
2. Use MATLAB to find the response of the velocity of the car for the case in which the input jumps from being $u = 0$ at time $t = 0$ to a constant $u = 500$ N thereafter. Assume that the car mass m is 1000 kg and viscous drag coefficient, $b = 50$ N$\cdot$sec/m.

Solution

1. **Equations of motion:** For simplicity we assume that the rotational inertia of the wheels is negligible and that there is friction retarding the motion of the car that is proportional to the car's speed with a proportionality constant, b.[1] The car can

[1] If the speed is v, the aerodynamic friction force is proportional to v^2. In this simple model we have taken a linear approximation.

Figure 2.2
Free-body diagram for cruise control



then be approximated for modeling purposes using the free-body diagram seen in Fig. 2.2, which defines coordinates, shows all forces acting on the body (heavy lines), and indicates the acceleration (dashed line). The coordinate of the car's position x is the distance from the reference line shown and is chosen so that positive is to the right. Note that in this case the inertial acceleration is simply the second derivative of x (i.e., $\mathbf{a} = \ddot{x}$) because the car position is measured with respect to an inertial reference. The equation of motion is found using Eq. (2.1). The friction force acts opposite to the direction of motion; therefore it is drawn opposite the direction of positive motion and entered as a negative force in Eq. (2.1). The result is

$$u - b\dot{x} = m\ddot{x}, \tag{2.2}$$

or

$$\ddot{x} + \frac{b}{m}\dot{x} = \frac{u}{m}. \tag{2.3}$$

For the case of the automotive cruise control where the variable of interest is the speed, v ($=\dot{x}$), the equation of motion becomes

$$\dot{v} + \frac{b}{m}v = \frac{u}{m}. \tag{2.4}$$

The solution of such an equation will be covered in detail in Chapter 3; however, the essence is that you assume a solution of the form $v = V_o e^{st}$ given an input of the form $u = U_o e^{st}$. Then, since $\dot{v} = sV_o e^{st}$, the differential equation can be written as

$$\left(s + \frac{b}{m}\right) V_o e^{st} = \frac{1}{m} U_o e^{st}. \tag{2.5}$$

The e^{st} term cancels out, and we find that

$$\frac{V_o}{U_o} = \frac{\frac{1}{m}}{s + \frac{b}{m}}. \tag{2.6}$$

For reasons that will become clear in Chapter 3, this is usually written as

$$\frac{V(s)}{U(s)} = \frac{\frac{1}{m}}{s + \frac{b}{m}}. \tag{2.7}$$

Transfer function

This expression of the differential equation (2.4) is called the **transfer function** and will be used extensively in later chapters. Note that, in essence, we have substituted s for d/dt in Eq. (2.4).[2]

[2] The use of an operator for differentiation was developed by Cauchy about 1820 based on the Laplace transform, which was developed about 1780. In Chapter 3 we will show how to derive transfer functions using the Laplace transform. Reference: Gardner and Barnes, 1942.

2. **Time response:** The dynamics of a system can be prescribed to MATLAB in terms of row vectors containing the coefficients of the polynomials describing the numerator and denominator of its transfer function. The transfer function for this problem is that given in part (a). In this case, the numerator (called num) is simply one number since there are no powers of s, so that num $= 1/m = 1/1000$. The denominator (called den) contains the coefficients of the polynomial $s + {}^{b}/_{m}$, which are

$$\text{den} = \begin{bmatrix} 1 & \frac{b}{m} \end{bmatrix} = \begin{bmatrix} 1 & \frac{50}{1000} \end{bmatrix}.$$

The step function in MATLAB calculates the time response of a linear system to a unit step input. Because the system is linear, the output for this case can be multiplied by the magnitude of the input step to derive a step response of any amplitude. Equivalently, num can be multiplied by the magnitude of the input step.

Step response with MATLAB

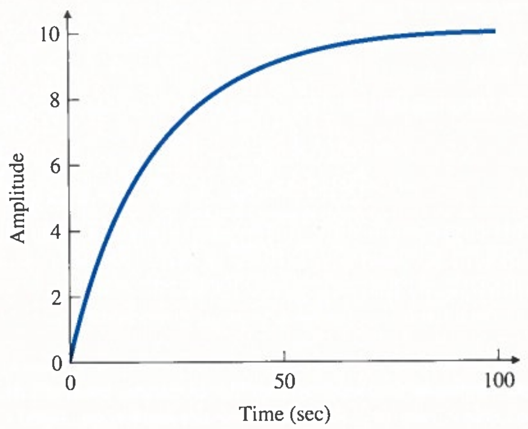
The statements

```
num = 1/1000;            % 1/m
den = [1  50/1000];      % s + b/m
sys = tf(num*500, den);  % step gives unit step response, so num*500
                           gives u = 500.
step(sys);               % plots the step response
```

calculate and plot the time response for an input step with a 500-N magnitude. The step response is shown in Fig. 2.3.

Newton's law also can be applied to systems with more than one mass. In this case it is particularly important to draw the free-body diagram of each mass, showing the applied external forces as well as the equal and opposite internal forces that act from each mass on the other.

Figure 2.3
Response of the car velocity to a step in u



EXAMPLE 2.2 *A Two-Mass System: Suspension Model*

Figure 2.4 shows an automobile suspension system. Write the equations of motion for the automobile and wheel motion assuming one-dimensional vertical motion of one quarter of the car mass above one wheel. A system consisting of one of the four wheel suspensions is usually referred to as a quarter-car model. Assume that the model is for a car with a mass of 1580 kg, including the four wheels, which have a mass of 20 kg each. By placing a known weight (an author) directly over a wheel and measuring the car's deflection, we find that $k_s = 130{,}000$ N/m. Measuring the wheel's deflection for the same applied weight, we find that $k_w \simeq 1{,}000{,}000$ N/m. By using the results in Section 3.3, Fig. 3.18(b), and qualitatively observing that the car's response as the author jumps off matches the $\zeta = 0.7$ curve, we conclude that $b = 9800$ N·sec/m.

Solution. The system can be approximated by the simplified system shown in Fig. 2.5. The coordinates of the two masses, x and y, with the reference directions as shown, are the displacements of the masses from their equilibrium conditions. The equilibrium positions are offset from the springs' unstretched positions because of the force of gravity. The shock absorber is represented in the schematic diagram by a dashpot symbol with friction constant b. The magnitude of the force from the shock

Figure 2.4
Automobile suspension

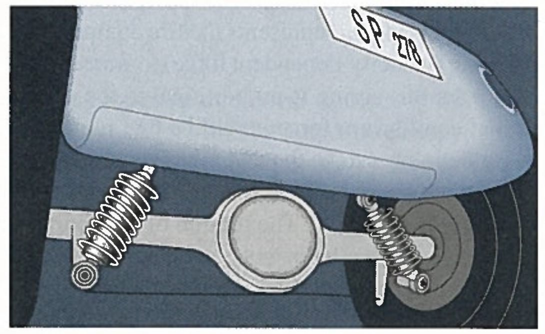


Figure 2.5
The quarter-car model

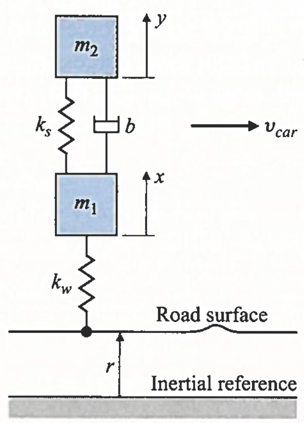
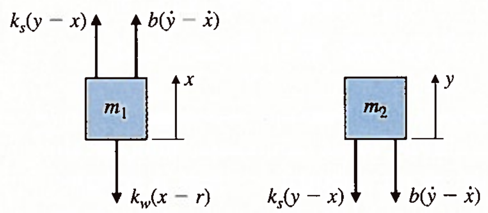


Figure 2.6
Free-body diagrams for suspension system



absorber is assumed to be proportional to the rate of change of the relative displacement of the two masses—that is, the force $= b(\dot{y}-\dot{x})$. The force of gravity could be included in the free-body diagram; however, its effect is to produce a constant offset of x and y. By defining x and y to be the distance from the equilibrium position, the need to include the gravity forces is eliminated.

The force from the car suspension acts on both masses in proportion to their relative displacement with spring constant k_s. Figure 2.6 shows the free-body diagram of each mass. Note that the forces from the spring on the two masses are equal in magnitude but act in opposite directions, which is also the case for the damper. A positive displacement y of mass m_2 will result in a force from the spring on m_2 in the direction shown and a force from the spring on m_1 in the direction shown. However, a positive displacement x of mass m_1 will result in a force from the spring k_s on m_1 in the opposite direction to that drawn in Fig. 2.6, as indicated by the *minus* x term for the spring force.

The lower spring k_w represents the tire compressibility, for which there is insufficient damping (velocity-dependent force) to warrant including a dashpot in the model. The force from this spring is proportional to the distance the tire is compressed and the nominal equilibrium force would be that required to support m_1 and m_2 against gravity. By defining x to be the distance from equilibrium, a force will result if either the road surface has a bump (r changes from its equilibrium value of zero) or the wheel bounces (x changes). The motion of the simplified car over a bumpy road will result in a value of $r(t)$ that is not constant.

As previously noted, there is a constant force of gravity acting on each mass; however, this force has been omitted, as have the equal and opposite forces from the springs. Gravitational forces can always be omitted from vertical-spring mass systems (1) if the position coordinates are defined from the equilibrium position that results when gravity is acting, and (2) if the spring forces used in the analysis are actually the perturbation in spring forces from those forces acting at equilibrium.

Applying Eq. (2.1) to each mass and noting that some forces on each mass are in the negative (down) direction yields the system of equations

$$b(\dot{y}-\dot{x}) + k_s(y-x) - k_w(x-r) = m_1\ddot{x}, \tag{2.8}$$

$$-k_s(y-x) - b(\dot{y}-\dot{x}) = m_2\ddot{y}. \tag{2.9}$$

Some rearranging results in

$$\ddot{x} + \frac{b}{m_1}(\dot{x}-\dot{y}) + \frac{k_s}{m_1}(x-y) + \frac{k_w}{m_1}x = \frac{k_w}{m_1}r, \tag{2.10}$$

$$\ddot{y} + \frac{b}{m_2}(\dot{y}-\dot{x}) + \frac{k_s}{m_2}(y-x) = 0. \tag{2.11}$$

Check for sign errors

The most common source of error in writing equations for systems like these are sign errors. The method for keeping the signs straight in the preceding development entailed mentally picturing the displacement of the masses and drawing the resulting force in the direction that the displacement would produce. Once you have obtained the equations for a system, a check on the signs for systems that are obviously stable from physical reasoning can be quickly carried out. As we will see when we study stability in Section 3.6, a stable system always has the same signs on similar variables. For this system, Eq. (2.10) shows that the signs on the $\ddot{x}$, $\dot{x}$, and x terms are all positive, as they must be for stability. Likewise, the signs on the $\ddot{y}$, $\dot{y}$, and y terms are all positive in Eq. (2.11).

The transfer function is obtained in a similar manner as before. Substituting s for d/dt in the differential equations yields

$$s^2X(s) + s\frac{b}{m_1}(X(s)-Y(s)) + \frac{k_s}{m_1}(X(s)-Y(s)) + \frac{k_w}{m_1}X(s) = \frac{k_w}{m_1}R(s),$$

$$s^2Y(s) + s\frac{b}{m_2}(Y(s)-X(s)) + \frac{k_s}{m_2}(Y(s)-X(s)) = 0,$$

which, after some algebra and rearranging, yields the transfer function

$$\frac{Y(s)}{R(s)} = \frac{\frac{k_w b}{m_1 m_2}\left(s+\frac{k_s}{b}\right)}{s^4 + \left(\frac{b}{m_1}+\frac{b}{m_2}\right)s^3 + \left(\frac{k_s}{m_1}+\frac{k_s}{m_2}+\frac{k_w}{m_1}\right)s^2 + \left(\frac{k_w b}{m_1 m_2}\right)s + \frac{k_w k_s}{m_1 m_2}}. \tag{2.12}$$

To determine numerical values, we subtract the mass of the four wheels from the total car mass of 1580 kg and divide by 4 to find that $m_2 = 375$ kg. The wheel mass was measured directly to be $m_1 = 20$ kg. Therefore, the transfer function with the numerical values is

$$\frac{Y(s)}{R(s)} = \frac{1.31e06(s+13.3)}{s^4 + (516.1)s^3 + (5.685e04)s^2 + (1.307e06)s + 1.733e07}. \tag{2.13}$$

2.1.2 Rotational Motion

Newton's law for rotational motion

Application of Newton's law to one-dimensional rotational systems requires that Eq. (2.1) be modified to

$$M = I\alpha, \tag{2.14}$$

where

M = the sum of all external moments about the center of mass of a body, N·m or lb·ft,

I = the body's mass moment of inertia about its center of mass, kg·m^2 or slug·ft^2,

α = the angular acceleration of the body, rad/sec^2.

EXAMPLE 2.3 *Rotational Motion: Satellite Attitude Control Model*

Satellites, as shown in Fig. 2.7, usually require attitude control so that antennas, sensors, and solar panels are properly oriented. Antennas are usually pointed toward a particular location on earth, while solar panels need to be oriented toward the sun for maximum power generation. To gain insight into the full three-axis attitude control system, it is helpful to consider one axis at a time. Write the equations of motion for one axis of this system and show how they would be depicted in a block diagram. In addition, determine the transfer function of this system and construct the system as if it were to be evaluated via MATLAB's SIMULINK.

Solution. Figure 2.8 depicts this case, where motion is allowed only about the axis perpendicular to the page. The angle θ that describes the satellite orientation must be measured with respect to an inertial reference—that is, a reference that has no angular acceleration. The control force comes from reaction jets that produce a moment of $F_c d$ about the mass center. There may also be small disturbance moments M_D on the

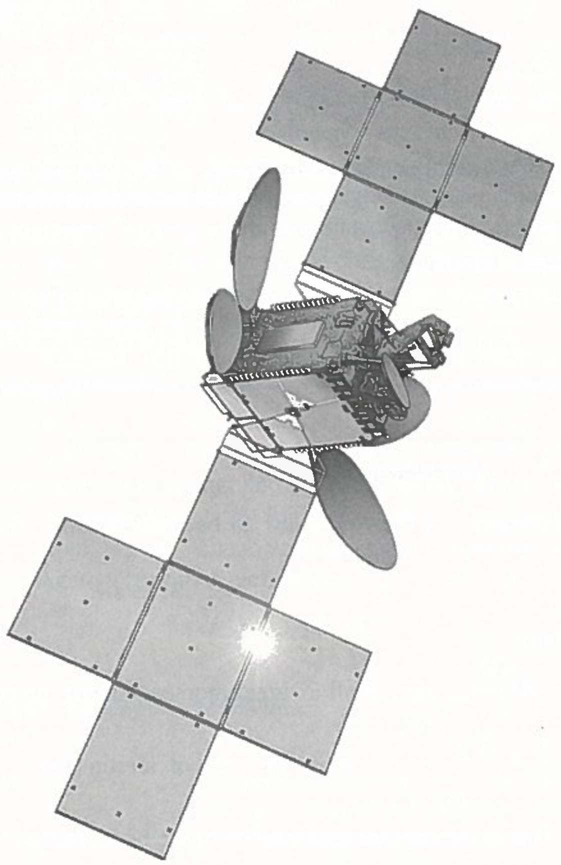


Figure 2.7
Communications satellite
Source: Courtesy Space Systems/Loral

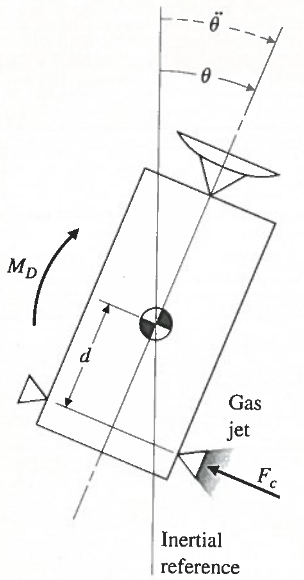


Figure 2.8
Satellite control schematic

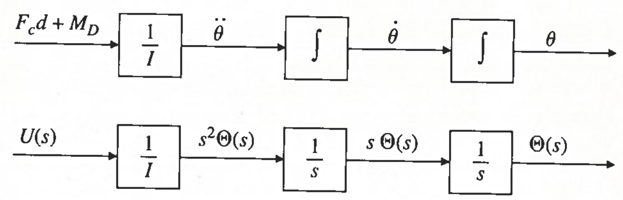


Figure 2.9
Block diagrams representing Eq. (2.15) in the upper half and Eq. (2.16) in the lower half

satellite, which arise primarily from solar pressure acting on any asymmetry in the solar panels. Applying Eq. (2.14) yields the equation of motion

$$F_c d + M_D = I\ddot{\theta}. \tag{2.15}$$

Double-integrator plant

The output of this system, θ, results from integrating the sum of the input torques twice; hence this type of system is often referred to as the **double-integrator plant**. The transfer function can be obtained as described for Eq. (2.7) and is

$$\frac{\Theta(s)}{U(s)} = \frac{1}{I}\frac{1}{s^2}, \tag{2.16}$$

$1/s^2$ plant

where $U = F_c d + M_D$. In this form, the system is often referred to as the **$1/s^2$ plant**.

Figure 2.9 shows a block diagram representing Eq. (2.15) in the upper half and a block diagram representing Eq. (2.16) in the lower half. This simple system can be analyzed using the linear analysis techniques that are described in later chapters, or via MATLAB as we saw in Example 2.1. It can also be numerically evaluated for an arbitrary input time history using SIMULINK. SIMULINK is a sister software package to MATLAB for interactive, nonlinear simulation and has a graphical user interface with drag and drop properties. Figure 2.10 shows a block diagram of the system as depicted by SIMULINK.

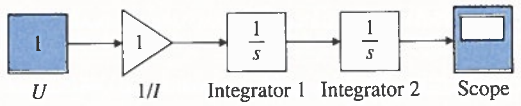


Figure 2.10
SIMULINK block diagram of the double-integrator plant

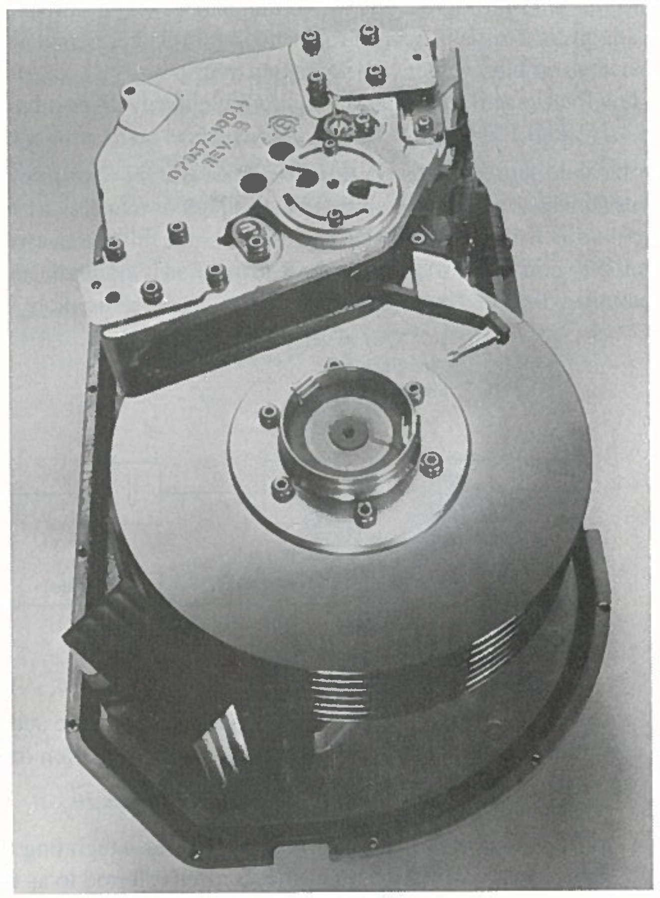


Figure 2.11
Disk read/write mechanism
Source: Courtesy of Hewlett-Packard Company

In many cases a system, such as the disk-drive read/write head shown in Fig. 2.11, in reality has some flexibility, which can cause problems in the design of a control system. Particular difficulty arises when there is flexibility, as in this case, between the sensor and actuator locations. Therefore, it is often important to include this flexibility in the model even when the system seems to be quite rigid.

EXAMPLE 2.4 *Flexibility: Flexible Read/Write for a Disk Drive*

Assume that there is some flexibility between the read head and the drive motor in Fig. 2.11. Find the equations of motion relating the motion of the read head to a torque applied to the base.

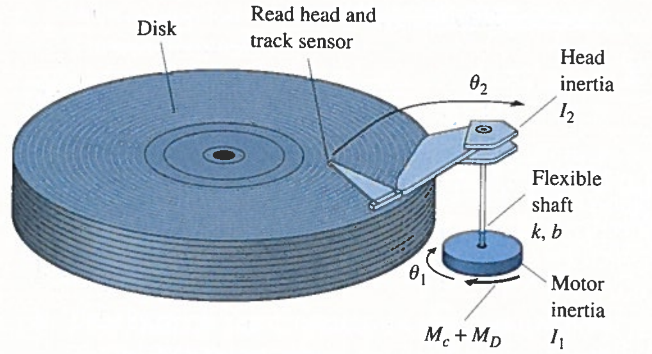


Figure 2.12
Disk read/write head schematic for modeling

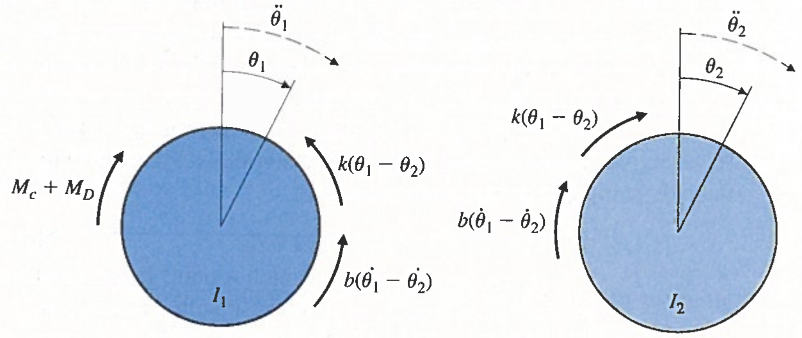


Figure 2.13
Free-body diagrams of the disk read/write head

Solution. The dynamic model for this situation is shown schematically in Fig. 2.12. This model is dynamically similar to the resonant system shown in Fig. 2.5 and results in equations of motion that are similar in form to Eqs. (2.10) and (2.11). The moments on each body are shown in the free-body diagrams in Fig. 2.13. The discussion of the moments on each body is essentially the same as the discussion for Example 2.2, except that the springs and damper in that case produced forces, instead of moments that act on each inertia, as in this case. When the moments are summed, equated to the accelerations according to Eq. (2.14), and rearranged, the result is

$$I_1\ddot{\theta}_1 + b(\dot{\theta}_1 - \dot{\theta}_2) + k(\theta_1 - \theta_2) = M_c + M_D, \tag{2.17}$$

$$I_2\ddot{\theta}_2 + b(\dot{\theta}_2 - \dot{\theta}_1) + k(\theta_2 - \theta_1) = 0. \tag{2.18}$$

Ignoring the disturbance torque M_D and the damping b for simplicity, we find the transfer function from the applied torque M_c to the read head motion to be

$$\frac{\Theta_2(s)}{M_c(s)} = \frac{k}{I_1 I_2 s^2\left(s^2 + \frac{k}{I_1} + \frac{k}{I_2}\right)}. \tag{2.19}$$

It might also be possible to sense the motion of the inertia where the torque is applied, θ_1, in which case the transfer function with the same simplifications would be

$$\frac{\Theta_1(s)}{M_c(s)} = \frac{I_2 s^2 + k}{I_1 I_2 s^2\left(s^2 + \frac{k}{I_1} + \frac{k}{I_2}\right)}. \tag{2.20}$$

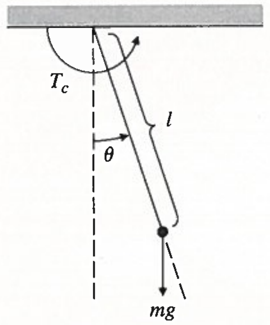


Figure 2.14
Pendulum

Collocated sensor and actuator

These two cases are typical of many situations in which the sensor and actuator may or may not be placed in the same location in a flexible body. We refer to the situation between sensor and actuator in Eq. (2.19) as the "noncollocated" case, whereas Eq. (2.20) describes the "collocated" case. You will see in Chapter 5 that it is far more difficult to control a system when there is flexibility between the sensor and actuator (noncollocated case) than when the sensor and actuator are rigidly attached to one another (the collocated case).

In the special case in which a point in a rotating body is fixed with respect to an inertial reference, as is the case with a pendulum, Eq. (2.14) can be applied such that M is the sum of all moments about the *fixed* point and I is the moment of inertia about the fixed point.

EXAMPLE 2.5 *Rotational Motion: Pendulum*

1. Write the equations of motion for the simple pendulum shown in Fig. 2.14, where all the mass is concentrated at the end point and there is a torque, T_c, applied at the pivot.
2. Use MATLAB to determine the time history of θ to a step input in T_c of 1 N·m. Assume $l = 1$ m, $m = 1$ kg, and $g = 9.81$ m/sec^2.

Solution

1. **Equations of motion:** The moment of inertia about the pivot point is $I = ml^2$. The sum of moments about the pivot point contains a term from gravity as well as the applied torque T_c. The equation of motion, obtained from Eq. (2.14), is
$$T_c - mgl\sin\theta = I\ddot{\theta}, \tag{2.21}$$
which is usually written in the form
$$\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{T_c}{ml^2}. \tag{2.22}$$
This equation is nonlinear due to the $\sin\theta$ term. A general discussion of nonlinear equations is contained in Chapter 9; however, we can proceed with a linearization of this case by assuming the motion is small enough that $\sin\theta \cong \theta$. Then Eq. (2.22) becomes the linear equation
$$\ddot{\theta} + \frac{g}{l}\theta = \frac{T_c}{ml^2}. \tag{2.23}$$

With no applied torque, the natural motion is that of a harmonic oscillator with a natural frequency of[3]
$$\omega_n = \sqrt{\frac{g}{l}}. \tag{2.24}$$
The transfer function can be obtained as described for Eq. (2.7), yielding
$$\frac{\Theta(s)}{T_c(s)} = \frac{\frac{1}{ml^2}}{s^2 + \frac{g}{l}}. \tag{2.25}$$

2. **Time history:** The dynamics of a system can be prescribed to MATLAB in terms of row vectors containing the coefficients of the polynomials describing the numerator and denominator of its transfer function. In this case, the numerator (called num) is simply one number, since there are no powers of s, so that
$$\text{num} = \frac{1}{ml^2} = \frac{1}{(1)(1)^2} = [1],$$
and the denominator (called den) contains the coefficients of the descending powers of s in $(s^2 + g/l)$ and is a row vector with three elements:
$$\text{den} = \begin{bmatrix} 1 & 0 & \frac{g}{l} \end{bmatrix} = [1 \quad 0 \quad 9.81].$$
The desired response of the system can be obtained by using the MATLAB step response function, called step. The MATLAB statements
```
t = 0:0.02:10;         % vector of times for output, 0 to 10 at 0.02 increments
num = 1;
den = [1 0 9.81];
sys = tf(num,den);     % defines the system by its numerator and denominator
y = step(sys,t);       % computes step responses at times given by t for step
                         at t = 0
plot(t, 57.3*y)        % converts radians to degrees and plots step response
```
will produce the desired time history shown in Fig. 2.15.

SIMULINK

As we saw in this example, the resulting equations of motion are often nonlinear. Such equations are much more difficult to solve than linear ones, and the kinds of possible motions resulting from a nonlinear model are much more difficult to categorize than those resulting from a linear model. It is therefore useful to linearize models in order to gain access to linear analysis methods. It may be that the linear models and linear analysis are used only for the design of the control system (whose function may be to maintain the system in the linear region). Once a control system is synthesized and shown to have desirable performance based on linear analysis, it is then prudent to carry out further analysis or an accurate numerical simulation of the system with the significant nonlinearities in order to validate that performance. SIMULINK is an expedient way to carry out these simulations and can handle most

[3]In a grandfather clock it is desired to have a pendulum period of exactly 2 sec. Show that the pendulum should be approximately 1 m in length.

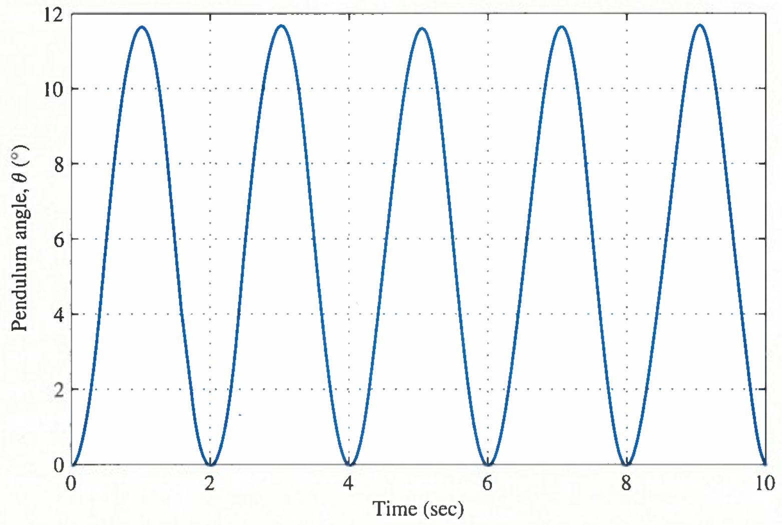


Figure 2.15 Response of the pendulum to a step input of 1 N-m in the applied torque

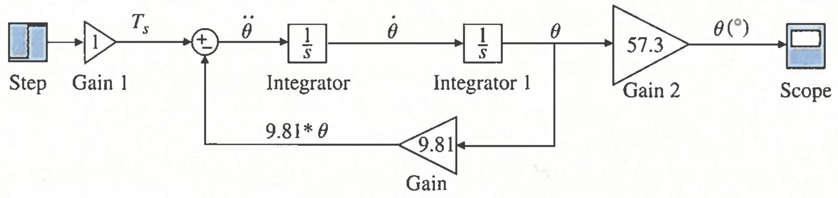


Figure 2.16 The SIMULINK block diagram representing the linear equation (2.26)

nonlinearities. Use of this simulation tool is carried out by constructing a block diagram[4] that represents the equations of motion. The linear equation of motion for the pendulum with the parameters as specified in Example 2.5 can be seen from Eq. (2.23) to be

$$\ddot{\theta} = -9.81 * \theta + 1, \tag{2.26}$$

and this is represented in SIMULINK by the block diagram in Fig. 2.16. Note that the circle on the left side of the figure with the + and − signs indicating addition and subtraction implements the equation above.

The result of running this numerical simulation will be essentially identical to the linear solution shown in Fig. 2.15 because the solution is for relatively small angles where $\sin\theta \cong \theta$. However, using SIMULINK to solve for the response enables us to simulate the nonlinear equation so that we could analyze the system for larger motions. In this case, Eq. (2.26) becomes

$$\ddot{\theta} = -9.81 * \sin\theta + 1, \tag{2.27}$$

[4]A more extensive discussion of block diagrams is contained in Section 3.2.1.

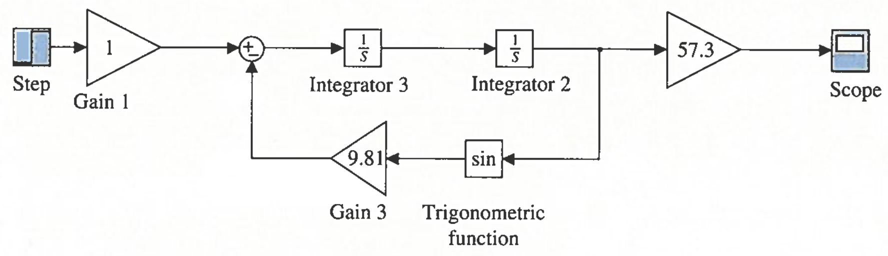


Figure 2.17 The SIMULINK block diagram representing the nonlinear equation (2.27)

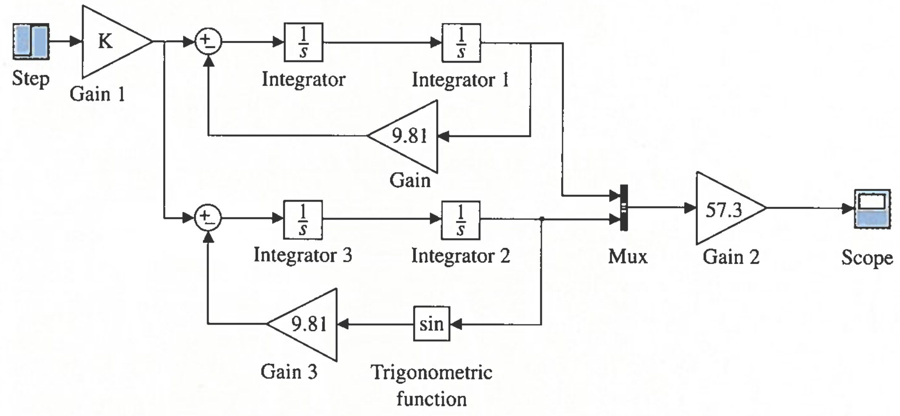


Figure 2.18 Block diagram of the pendulum for both the linear and nonlinear models

and the SIMULINK block diagram shown in Fig. 2.17 implements this nonlinear equation.

SIMULINK is capable of simulating all commonly encountered nonlinearities, including deadzones, on–off functions, stiction, hysteresis, aerodynamic drag (a function of v^2), and trigonometric functions. All real systems have one or more of these characteristics in varying degrees.

EXAMPLE 2.6 *Use of SIMULINK for Nonlinear Motion: Pendulum*

Use SIMULINK to determine the time history of θ for the pendulum in Example 2.5. Compare it against the linear solution for T_c values of 1 N·m and 4 N·m.

Solution. Time history: The SIMULINK block diagrams for the two cases discussed above are combined and both outputs in Fig. 2.16 and 2.17 are sent via a "multiplexer block (Mux)" to the "scope" so they can be plotted on the same graph. Fig. 2.18 shows the combined block diagram where the gain, K, represents the values of T_c. The outputs of this system for T_c values of 1 N · m and 4 N· m are shown in Fig. 2.19. Note that for $T_c = 1$ N·m, the outputs at the top of the figure remain at 12° or less and the linear approximation is extremely close to the nonlinear output. For $T_c = 4$ N·m, the output angle grows to near 50° and a substantial difference in the response magnitude and frequency is apparent due to θ being a poor approximation to $\sin\theta$ at these magnitudes.

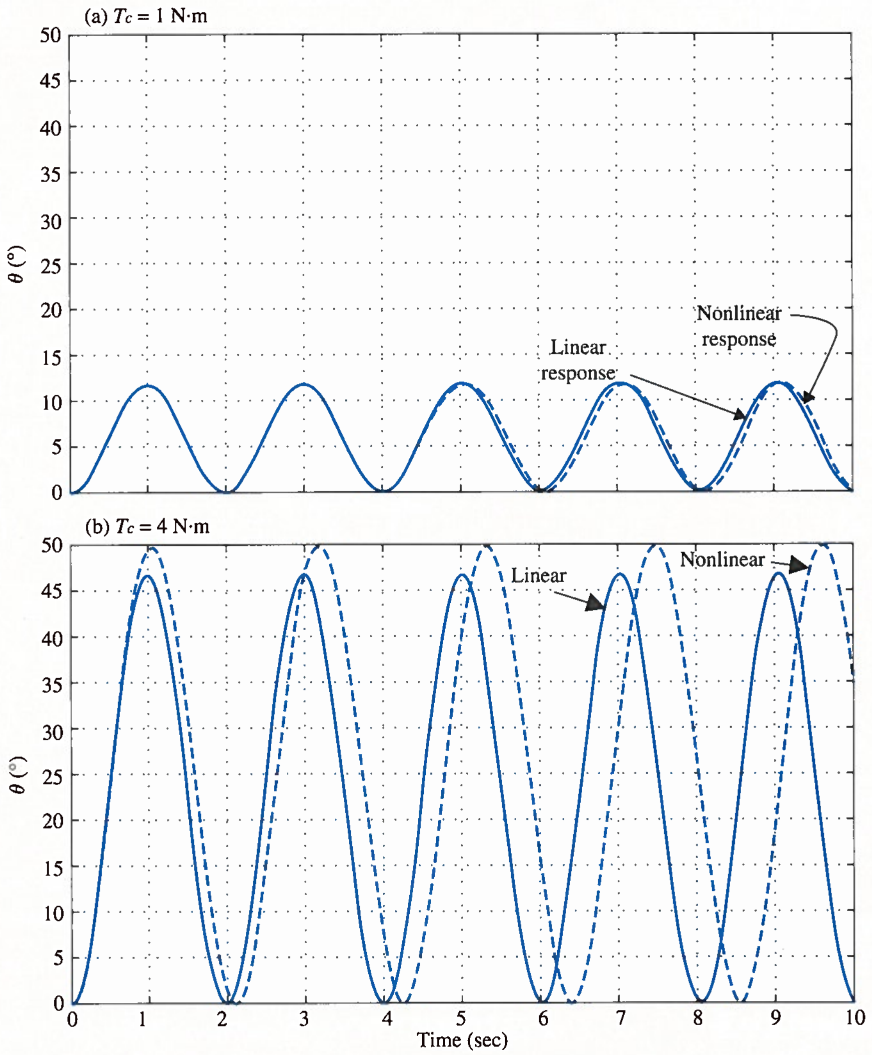


Figure 2.19
Response of the pendulum SIMULINK numerical simulation for the linear and nonlinear models. (a) for $T_c = 1$ N·m and (b) $T_c = 4$ N·m

Chapter 9 is devoted to the analysis of nonlinear systems and greatly expands on these ideas.

2.1.3 Combined Rotation and Translation

In some cases, mechanical systems contain both translational and rotational portions. The procedure is the same as that described in Sections 2.1.1 and 2.1.2: sketch the free-body diagrams, define coordinates and positive directions, determine all forces and moments acting, and apply Eqs. (2.1) and/or (2.14). An exact derivation of the equations for these systems can become quite involved; therefore, the complete analysis for the following examples are contained in Appendix W2 and only the linearized equations of motion and their transfer functions are given here.

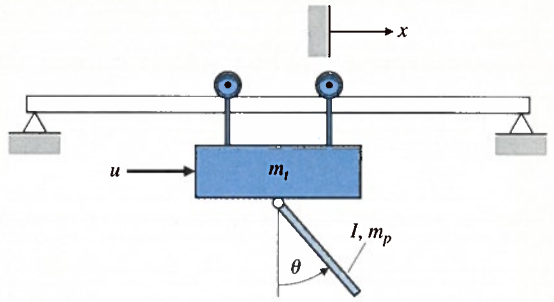


Figure 2.20
Schematic of the crane with hanging load

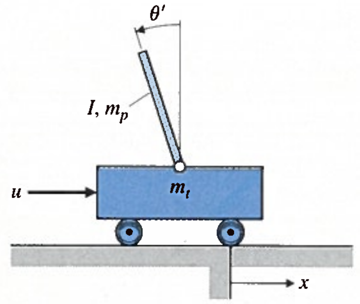


Figure 2.21
Inverted pendulum

EXAMPLE 2.7 *Rotational and Translational Motion: Hanging Crane*

Write the equations of motion for the hanging crane shown schematically in Fig. 2.20. Linearize the equations about $\theta = 0$, which would typically be valid for the hanging crane. Also linearize the equations for $\theta = \pi$, which represents the situation for the inverted pendulum shown in Fig. 2.21. The trolley has mass, m_t, and the hanging crane (or pendulum) has mass, m_p, and inertia about its mass center of I. The distance from the pivot to the mass center of the pendulum is l; therefore, the moment of inertia of the pendulum about the pivot point is $(I + m_p l^2)$.

Solution. Free-body diagrams need to be drawn for the trolley and the pendulum and the reaction forces considered where the two attach to one another. We carry out this process in Appendix W2. After Newton's Laws are applied for the translational motion of the trolley and the rotational motion of the pendulum, it will be found that the reaction forces between the two bodies can be eliminated, and the only unknowns will be θ and x. The results are two coupled second-order nonlinear differential equations in θ and x with the input being the force applied to the trolley, u. They can be linearized in a similar manner that was done for the simple pendulum by assuming small angles. For small motions about $\theta = 0$, we let $\cos\theta \cong 1$, $\sin\theta \cong \theta$, and $\dot{\theta}^2 \cong 0$; thus the equations are approximated by

$$(I + m_p l^2)\ddot{\theta} + m_p g l\theta = -m_p l\ddot{x},$$
$$(m_t + m_p)\ddot{x} + b\dot{x} + m_p l\ddot{\theta} = u. \tag{2.28}$$

Note that the first equation is very similar to the simple pendulum, Eq. (2.21), where the applied torque arises from the trolley accelerations. Likewise, the second equation representing the trolley motion, x, is very similar to the car translation in

Eq. (2.3) where the forcing term arises from the angular acceleration of the pendulum. Neglecting the friction term b leads to the transfer function from the control input u to hanging crane angle θ:

$$\frac{\theta(s)}{U(s)} = \frac{-m_p l}{((I + m_p l^2)(m_t + m_p) - m_p^2 l^2)s^2 + m_p g l(m_t + m_p)}. \tag{2.29}$$

For the inverted pendulum in Fig. 2.21, where $\theta \cong \pi$, assume $\theta = \pi + \theta'$, where θ' represents motion from the vertical *upward* direction. In this case, $\sin\theta \cong -\theta'$, $\cos\theta \cong -1$, and the nonlinear equations become[5]

Inverted pendulum equations

$$(I + m_p l^2)\ddot{\theta}' - m_p g l\theta' = m_p l\ddot{x},$$
$$(m_t + m_p)\ddot{x} + b\dot{x} - m_p l\ddot{\theta}' = u. \tag{2.30}$$

As noted in Example 2.2, a stable system will always have the same signs on each variable, which is the case for the stable hanging crane modeled by Eqs. (2.28). However, the signs on θ and $\ddot{\theta}$ in the top Eq. (2.30) are opposite, thus indicating instability, which is the characteristic of the inverted pendulum.

The transfer function, again without friction, is

$$\frac{\theta'(s)}{U(s)} = \frac{m_p l}{((I + m_p l^2) - m_p^2 l^2)s^2 - m_p g l(m_t + m_p)}. \tag{2.31}$$

In Chapter 5 you will learn how to stabilize systems using feedback and will see that even unstable systems like an inverted pendulum can be stabilized providing there is a sensor that measures the output quantity and a control input. For the case of the inverted pendulum perched on a trolley, it would be required to measure the pendulum angle, θ', and provide a control input, u, that accelerated the trolley in such a way that the pendulum remained pointing straight up. In years past, this system existed primarily in university control system laboratories as an educational tool. However, more recently, there is a practical device in production and being sold that employs essentially this same dynamic system: The Segway. It uses a gyroscope so that the angle of the device is known with respect to vertical, and electric motors provide a torque on the wheels so that it balances the device and provides the desired forward or backward motion. It is shown in Fig. 2.22.

2.1.4 Distributed Parameter Systems

All the preceding examples contained one or more rigid bodies, although some were connected to others by springs. Actual structures—for example, satellite solar panels, airplane wings, or robot arms—usually bend, as shown by the flexible beam in Fig. 2.23(a). The equation describing its motion is a fourth-order *partial* differential equation that arises because the mass elements are continuously distributed along the beam with a small amount of flexibility between elements. This type of system is called a **distributed parameter system.** The dynamic analysis methods presented

[5]The inverted pendulum is often described with the angle of the pendulum being positive for *clockwise* motion. If defined that way, then reverse the sign on all terms in Eqs. (2.30) in θ' or $\ddot{\theta}'$.

Figure 2.22
The Segway, which is similar to the inverted pendulum and is kept upright by a feedback control system
Source: Photo courtesy of David Powell

in this section are not sufficient to analyze this case; however, more advanced texts (Thomson and Dahleh, 1998) show that the result is

$$EI\frac{\partial^4 w}{\partial x^4} + \rho\frac{\partial^2 w}{\partial t^2} = 0, \tag{2.32}$$

where

E = Young's modulus,

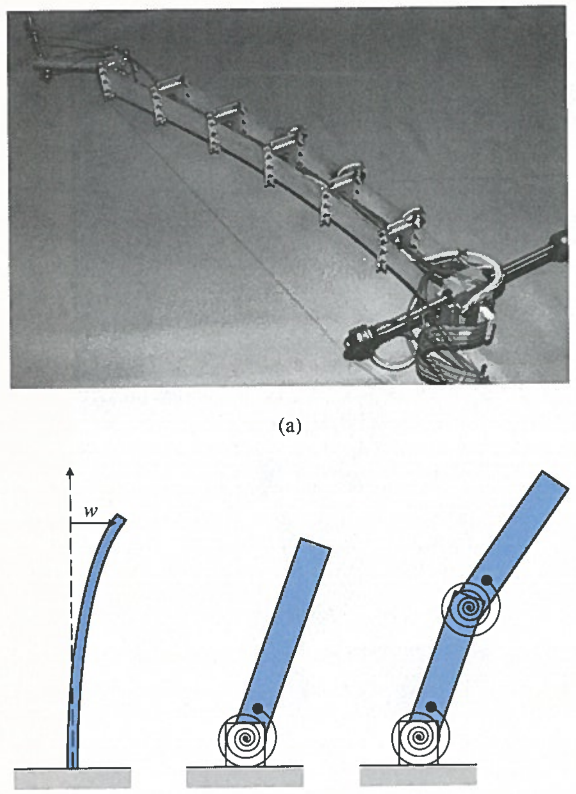
I = beam area moment of inertia,

ρ = beam density,

w = beam deflection at length x along the beam.

The exact solution to Eq. (2.32) is too cumbersome to use in designing control systems, but it is often important to account for the gross effects of bending in control systems design.

The continuous beam in Fig. 2.23(b) has an infinite number of vibration-mode shapes, all with different frequencies. Typically, the lowest-frequency modes have the largest amplitude and are the most important to approximate well. The simplified model in Fig. 2.23(c) can be made to duplicate the essential behavior of the first



(a) (b) (c) (d)

Figure 2.23
(a) Flexible robot arm used for research at Stanford University; (b) model for a continuous flexible beam; (c) simplified model for the first bending mode; (d) model for the first and second bending modes
Source: Photo courtesy of E. Schmitz

bending mode shape and frequency and would usually be adequate for controller design. If frequencies higher than the first bending mode are anticipated in the control system operation, it may be necessary to model the beam as shown in Fig. 2.23(d), which can be made to approximate the first two bending modes and frequencies. Likewise, higher-order models can be used if such accuracy and complexity are deemed necessary (Thomson and Dahleh, 1998; Schmitz, 1985). When a continuously bending object is approximated as two or more rigid bodies connected by springs, the resulting model is sometimes referred to as a **lumped parameter model**.

A flexible structure can be approximated by a lumped parameter model

2.1.5 Summary: Developing Equations of Motion for Rigid Bodies

The physics necessary to write the equations of motion of a rigid body is entirely given by Newton's laws of motion. The method is as follows:

1. Assign variables such as x and θ that are both necessary and sufficient to describe an *arbitrary* position of the object.
2. Draw a free-body diagram of each component. Indicate *all* forces acting on each body and their reference directions. Also indicate the accelerations of the center of mass with respect to an inertial reference for each body.

3. Apply Newton's law in translation [Eq. (2.1)] and/or rotation [Eq. (2.14)] form.
4. Combine the equations to eliminate internal forces.
5. The number of independent equations should equal the number of unknowns.

2.2 Models of Electric Circuits

Electric circuits are frequently used in control systems largely because of the ease of manipulation and processing of electric signals. Although controllers are increasingly implemented with digital logic, many functions are still performed with analog circuits. Analog circuits are faster than digital and, for very simple controllers, an analog circuit would be less expensive than a digital implementation. Furthermore, the power amplifier for electromechanical control and the anti-alias prefilters for digital control must be analog circuits.

Electric circuits consist of interconnections of sources of electric voltage and current, and other electronic elements such as resistors, capacitors, and transistors. An important building block for circuits is an operational amplifier (or op-amp),[6] which is also an example of a complex feedback system. Some of the most important methods of feedback system design were developed by the designers of high-gain, wide-bandwidth feedback amplifiers, mainly at the Bell Telephone Laboratories between 1925 and 1940. Electric and electronic components also play a central role in electromechanical energy conversion devices such as electric motors, generators, and electrical sensors. In this brief survey we cannot derive the physics of electricity or give a comprehensive review of all the important analysis techniques. We will define the variables, describe the relations imposed on them by typical elements and circuits, and describe a few of the most effective methods available for solving the resulting equations.

Symbols for some linear circuit elements and their current–voltage relations are given in Fig. 2.24. Passive circuits consist of interconnections of resistors, capacitors, and inductors. With electronics, we increase the set of electrical elements by adding active devices, including diodes, transistors, and amplifiers.

Kirchhoff's laws

The basic equations of electric circuits, called Kirchhoff's laws, are as follows:

1. **Kirchhoff's current law (KCL):** The algebraic sum of currents leaving a junction or node equals the algebraic sum of currents entering that node.
2. **Kirchhoff's voltage law (KVL):** The algebraic sum of all voltages taken around a closed path in a circuit is zero.

With complex circuits of many elements, it is essential to write the equations in a careful, well organized way. Of the numerous methods for doing this, we choose for description and illustration the popular and powerful scheme known as **node analysis**. One node is selected as a reference and we assume the voltages of all other nodes to be unknowns. The choice of reference is arbitrary in theory, but in actual electronic

[6]Oliver Heaviside introduced the mathematical operation p to signify differentiation so that $pv = dv/dt$. The Laplace transform incorporates this idea, using the complex variable s. Ragazzini et al. (1947) demonstrated that an ideal, high-gain electronic amplifier permitted one to realize arbitrary "operations" in the Laplace transform variable s, so they named it the operational amplifier, commonly abbreviated to op-amp.

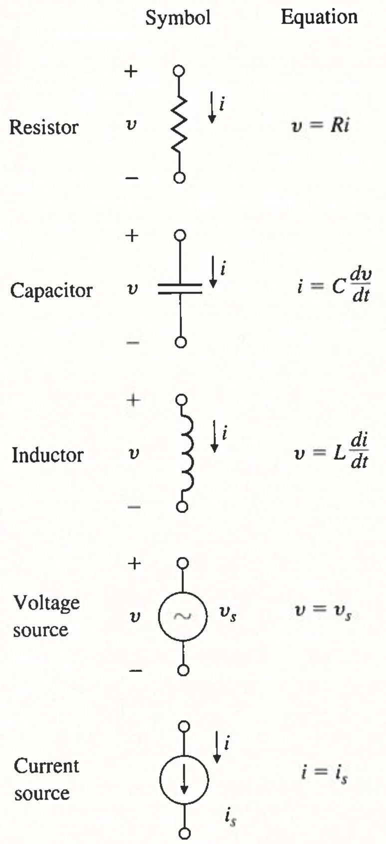


Figure 2.24
Elements of electric circuits

circuits the common, or ground, terminal is the obvious and standard choice. Next, we write equations for the selected unknowns using the current law (KCL) at each node. We express these currents in terms of the selected unknowns by using the element equations in Fig. 2.24. If the circuit contains voltage sources, we must substitute a voltage law (KVL) for such sources. Example 2.8 illustrates how node analysis works.

EXAMPLE 2.8 *Equations for the Bridged Tee Circuit*

Determine the differential equations for the circuit shown in Fig. 2.25.

Solution. We select node 4 as the reference and the voltages v_1, v_2, and v_3 at nodes 1, 2, and 3 as the unknowns. We start with the degenerate KVL relationship

$$v_1 = v_i. \tag{2.33}$$

At node 2 the KCL is

$$-\frac{v_1 - v_2}{R_1} + \frac{v_2 - v_3}{R_2} + C_1\frac{dv_2}{dt} = 0, \tag{2.34}$$

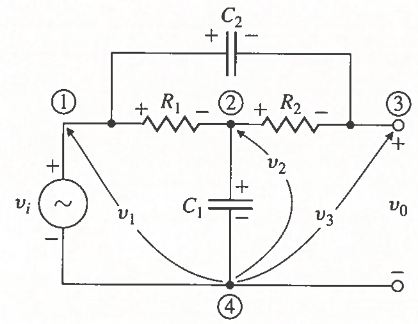


Figure 2.25
Bridged tee circuit

and at node 3 the KCL is

$$\frac{v_3 - v_2}{R_2} + C_2\frac{d(v_3 - v_1)}{dt} = 0. \tag{2.35}$$

These three equations describe the circuit.

Operational amplifier

Kirchhoff's laws can also be applied to circuits that contain an **operational amplifier**. The simplified circuit of the op-amp is shown in Fig. 2.26(a) and the schematic symbol is drawn in Fig. 2.26(b). If the positive terminal is not shown, it is assumed to be connected to ground, $v_+ = 0$, and the reduced symbol of Fig. 2.26(c) is used. For use in control circuits, it is usually assumed that the op-amp is *ideal* with the values $R_1 = \infty$, $R_0 = 0$, and $A = \infty$. The equations of the ideal op-amp are extremely simple, being

$$i_+ = i_- = 0, \tag{2.36}$$

$$v_+ - v_- = 0. \tag{2.37}$$

The gain of the amplifier is assumed to be so high that the output voltage becomes v_{out} = *whatever it takes* to satisfy these equations. Of course, a real amplifier only

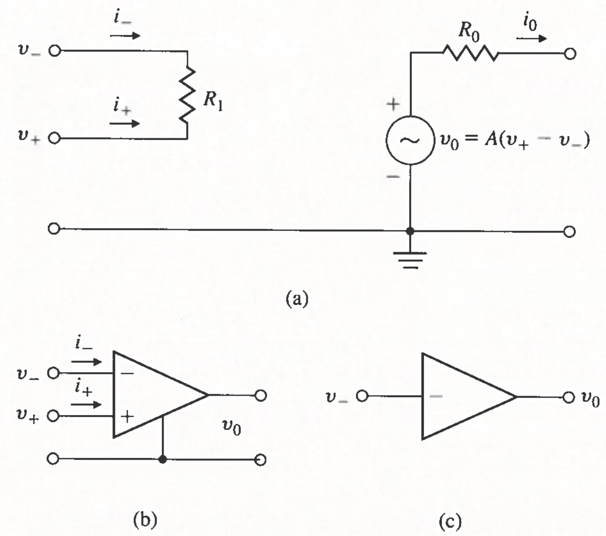
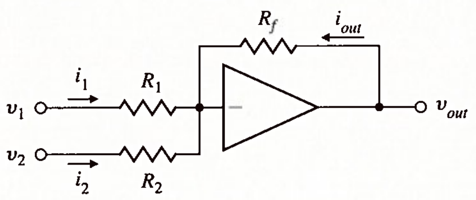


Figure 2.26
(a) Op-amp simplified circuit; (b) op-amp schematic symbol; (c) reduced symbol for $v_+ = 0$

Figure 2.27
The op-amp summer



approximates these equations, but unless they are specifically described, we will assume all op-amps are ideal. More realistic models are the subject of several problems given at the end of the chapter.

EXAMPLE 2.9 *Op-Amp Summer*

Find the equations and transfer functions of the circuit shown in Fig. 2.27.

Solution. Equation (2.37) requires that $v_- = 0$, and thus the currents are $i_1 = v_1/R_1$, $i_2 = v_2/R_2$, and $i_{out} = v_{out}/R_f$. To satisfy Eq. (2.36), $i_1 + i_2 + i_{out} = 0$, from which it follows that $v_1/R_1 + v_2/R_2 + v_{out}/R_f = 0$, and we have

$$v_{out} = -\left[\frac{R_f}{R_1}v_1 + \frac{R_f}{R_2}v_2\right]. \tag{2.38}$$

The op-amp summer

From this equation we see that the circuit output is a weighted sum of the input voltages with a sign change. The circuit is called a **summer**.

A second important example for control is given by the op-amp integrator.

EXAMPLE 2.10 *Integrator*

Op-amp as integrator

Find the transfer function for the circuit shown in Fig. 2.28.

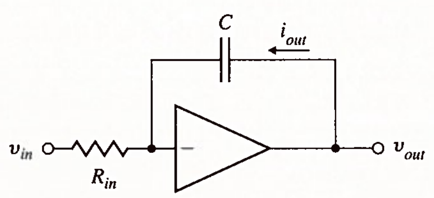
Solution. In this case the equations are differential and Eqs. (2.36) and (2.37) require

$$i_{in} + i_{out} = 0, \tag{2.39}$$

so that

$$\frac{v_{in}}{R_{in}} + C\frac{dv_{out}}{dt} = 0. \tag{2.40}$$

Figure 2.28
The op-amp integrator



Eq. (2.40) can be written in integral form as

$$v_{out} = -\frac{1}{R_{in}C}\int_0^t v_{in}(\tau)\,d\tau + v_{out}(0). \tag{2.41}$$

Using the operational notation that $d/dt = s$ in Eq. (2.40), the transfer function (which assumes zero initial conditions) can be written as

$$V_{out}(s) = -\frac{1}{s}\frac{V_{in}(s)}{R_{in}C}. \tag{2.42}$$

Thus the ideal op-amp in this circuit performs the operation of integration and the circuit is simply referred to as an **integrator**.

2.3 Models of Electromechanical Systems

Electric current and magnetic fields interact in two ways that are particularly important to an understanding of the operation of most electromechanical actuators and sensors. If a current of i amperes in a conductor of length l meters is arranged at right angles in a magnetic field of B tesls, then there is a force on the conductor at right angles to the plane of i and B, with magnitude

Law of motors

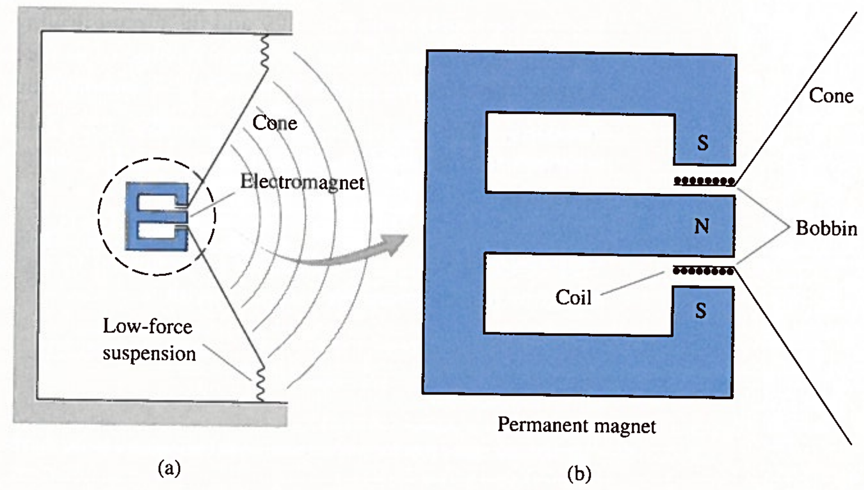
$$F = Bli \text{ newtons}. \tag{2.43}$$

This equation is the basis of conversion of electric energy to mechanical work and is called the **law of motors**.

EXAMPLE 2.11 *Modeling a Loudspeaker*

A typical geometry for a loudspeaker for producing sound is sketched in Fig. 2.29. The permanent magnet establishes a radial field in the cylindrical gap between the poles of the magnet. The force on the conductor wound on the bobbin causes the

Figure 2.29
Geometry of a loudspeaker: (a) overall configuration; (b) the electromagnet and voice coil



voice coil to move, producing sound.[7] The effects of the air can be modeled as if the cone had equivalent mass M and viscous friction coefficient b. Assume that the magnet establishes a uniform field B of 0.5 tesla and the bobbin has 20 turns at a 2-cm diameter. Write the equations of motion of the device.

Solution. The current is at right angles to the field, and the force of interest is at right angles to the plane of i and B, so Eq. (2.43) applies. In this case the field strength is $B = 0.5$ tesla and the conductor length is

$$l = 20 \times \frac{2\pi}{100} m = 1.26\, m.$$

Thus, the force is

$$F = 0.5 \times 1.26 \times i = 0.63 i \text{ N}.$$

The mechanical equation follows from Newton's laws, and for a mass M and friction coefficient b, the equation is

$$M\ddot{x} + b\dot{x} = 0.63 i. \qquad (2.44)$$

This second-order differential equation describes the motion of the loudspeaker cone as a function of the input current i driving the system. Substituting s for d/dt in Eq. (2.44) as before, the transfer function is easily found to be

$$\frac{X(s)}{I(s)} = \frac{0.63/M}{s(s + b/M)}. \qquad (2.45)$$

The second important electromechanical relationship is the effect of mechanical motion on electric voltage. If a conductor of length l meters is moving in a magnetic field of B teslas at a velocity of v meters per second at mutually right angles, an electric voltage is established across the conductor with magnitude

Law of generators

$$e(t) = Blv \text{ V}. \qquad (2.46)$$

This expression is called the **law of generators.**

EXAMPLE 2.12 *Loudspeaker with Circuit*

For the loudspeaker in Fig. 2.29 and the circuit driving it in Fig. 2.30, find the differential equations relating the input voltage v_a to the output cone displacement x. Assume the effective circuit resistance is R and the inductance is L.

Solution. The loudspeaker motion satisfies Eq. (2.44), and the motion results in a voltage across the coil as given by Eq. (2.46), with the velocity $\dot{x}$. The resulting voltage is

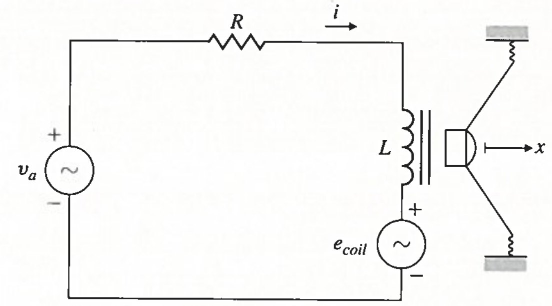
$$e_{coil} = Bl\dot{x} = 0.63\dot{x}. \qquad (2.47)$$

This induced voltage effect needs to be added to the analysis of the circuit. The equation of motion for the electric circuit is

$$L\frac{di}{dt} + Ri = v_a - 0.63\dot{x}. \qquad (2.48)$$

[7] Similar voice-coil motors are commonly used as the actuator for the read/write head assembly of computer hard-disk data access devices.

Figure 2.30
A loudspeaker showing the electric circuit



These two coupled equations, (2.44) and (2.48), constitute the dynamic model for the loudspeaker.

Again substituting s for d/dt in these equations, the transfer function between the applied voltage and the loudspeaker displacement is found to be

$$\frac{X(s)}{V_a(s)} = \frac{0.63}{s\left[(Ms + b)(Ls + R) + (0.63)^2\right]}. \qquad (2.49)$$

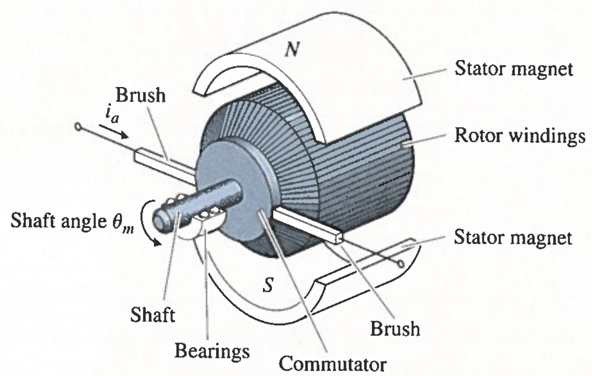
DC motor actuators

A common actuator based on these principles and used in control systems is the DC motor to provide rotary motion. A sketch of the basic components of a DC motor is given in Fig. 2.31. In addition to housing and bearings, the nonturning part (stator) has magnets, which establish a field across the rotor. The magnets may be electromagnets or, for small motors, permanent magnets. The brushes contact the rotating commutator, which causes the current always to be in the proper conductor windings so as to produce maximum torque. If the direction of the current is reversed, the direction of the torque is reversed.

Back emf

The motor equations give the torque T on the rotor in terms of the armature current i_a and express the back emf voltage in terms of the shaft's rotational velocity $\dot{\theta}_m$.[8]

Figure 2.31
Sketch of a DC motor



[8] Because the generated electromotive force (emf) works against the applied armature voltage, it is called the **back emf**.

Thus

$$T = K_t\, i_a, \tag{2.50}$$

$$e = K_e\, \dot{\theta}_m. \tag{2.51}$$

Torque

In consistent units, the torque constant K_t equals the electric constant K_e, but in some cases the torque constant will be given in other units, such as ounce-inches per ampere, and the electric constant may be expressed in units of volts per 1000 rpm. In such cases the engineer must make the necessary translations to be certain the equations are correct.

EXAMPLE 2.13

Modeling a DC Motor

Find the equations for a DC motor with the equivalent electric circuit shown in Fig. 2.32(a). Assume that the rotor has inertia J_m and viscous friction coefficient b.

Solution. The free-body diagram for the rotor, shown in Fig. 2.32(b), defines the positive direction and shows the two applied torques, T and $b\dot{\theta}_m$. Application of Newton's laws yields

$$J_m\ddot{\theta}_m + b\dot{\theta}_m = K_t i_a. \tag{2.52}$$

Analysis of the electric circuit, including the back emf voltage, shows the electrical equation to be

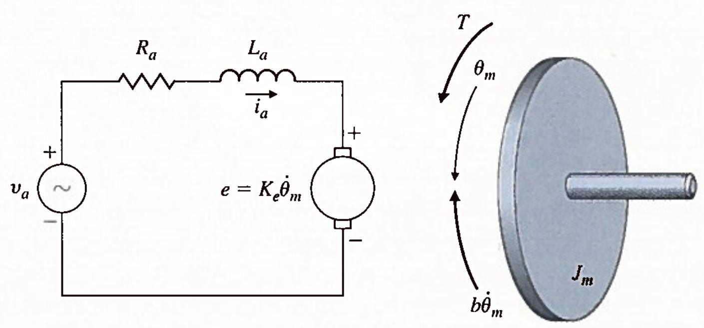
$$L_a\frac{di_a}{dt} + R_a i_a = v_a - K_e\dot{\theta}_m. \tag{2.53}$$

With s substituted for d/dt in Eqs. (2.52) and (2.53), the transfer function for the motor is readily found to be

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{K_t}{s[(J_m s + b)(L_a s + R_a) + K_t K_e]}. \tag{2.54}$$

In many cases the relative effect of the inductance is negligible compared with the mechanical motion and can be neglected in Eq. (2.53). If so, we can combine Eqs. (2.52) and (2.53) into one equation to get

$$J_m\ddot{\theta}_m + \left(b + \frac{K_t K_e}{R_a}\right)\dot{\theta}_m = \frac{K_t}{R_a} v_a. \tag{2.55}$$



(a) (b)

Figure 2.32
DC motor: (a) electric circuit of the armature; (b) free-body diagram of the rotor

From Eq. (2.55) it is clear that in this case the effect of the back emf is indistinguishable from the friction, and the transfer function is

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{\frac{K_t}{R_a}}{J_m s^2 + \left(b + \frac{K_t K_e}{R_a}\right)s} \tag{2.56}$$

$$= \frac{K}{s(\tau s + 1)}, \tag{2.57}$$

where

$$K = \frac{K_t}{bR_a + K_t K_e}, \tag{2.58}$$

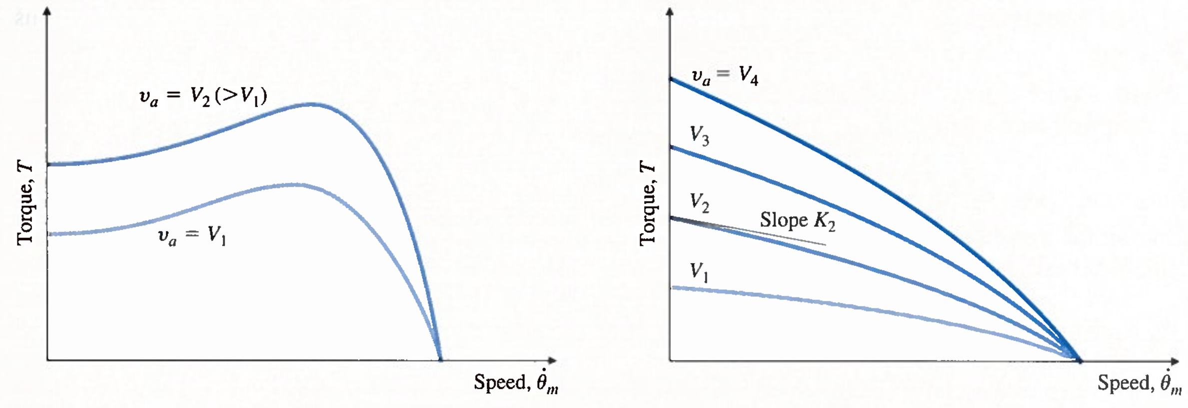
$$\tau = \frac{R_a J_m}{bR_a + K_t K_e}. \tag{2.59}$$

In many cases, a transfer function between the motor input and the output speed ($\omega = \dot{\theta}_m$) is required. In such cases, the transfer function would be

$$\frac{\Omega(s)}{V_a(s)} = s\frac{\Theta_m(s)}{V_a(s)} = \frac{K}{\tau s + 1}. \tag{2.60}$$

AC motor actuators

Another device used for electromechanical energy conversion is the alternating current (AC) induction motor invented by N. Tesla. Elementary analysis of the AC motor is more complex than that of the DC motor. A typical experimental set of curves of torque versus speed for fixed frequency and varying amplitude of applied (sinusoidal) voltage is given in Fig. 2.33. Although the data in the figure are for a constant engine speed, they can be used to extract the motor constants that will provide a dynamic model for the motor. For analysis of a control problem involving an AC



(a) (b)

Figure 2.33
Torque–speed curves for a servo motor showing four amplitudes of armature voltage: (a) low-rotor-resistance machine; (b) high-rotor-resistance machine showing four values of armature voltage, v_a

motor such as that described by Fig. 2.33, we make a linear approximation to the curves for speed near zero and at a midrange voltage to obtain the expression

$$T = K_1 v_a - K_2 \dot{\theta}_m. \tag{2.61}$$

The constant K_1 represents the ratio of a change in torque to a change in voltage at zero speed and is proportional to the distance between the curves at zero speed. The constant K_2 represents the ratio of a change in torque to a change in speed at zero speed and a midrange voltage; therefore, it is the slope of a curve at zero speed as shown by the line at V_2. For the electrical portion, values for the armature resistance R_a and inductance L_a are also determined by experiment. Once we have values for K_1, K_2, R_a, and L_a, the analysis proceeds as the analysis in Example 2.13 for the DC motor. For the case in which the inductor can be neglected, we can substitute K_1 and K_2 into Eq. (2.55) in place of K_t/R_a and K_tK_e/R_a, respectively.

In addition to the DC and AC motors mentioned here, control systems use brushless DC motors (Reliance Motion Control Corp., 1980) and stepping motors (Kuo, 1980). Models for these machines, developed in the works just cited, do not differ in principle from the motors considered in this section. In general, the analysis, supported by experiment, develops the torque as a function of voltage and speed similar to the AC motor torque–speed curves given in Fig. 2.33. From such curves one can obtain a linearized formula such as Eq. (2.61) to use in the mechanical part of the system and an equivalent circuit consisting of a resistance and an inductance to use in the electrical part.

△ 2.4 Heat and Fluid-Flow Models

Thermodynamics, heat transfer, and fluid dynamics are each the subject of complete textbooks. For purposes of generating dynamic models for use in control systems, the most important aspect of the physics is to represent the dynamic interaction between the variables. Experiments are usually required to determine the actual values of the parameters and thus to complete the dynamic model for purposes of control systems design.

2.4.1 Heat Flow

Some control systems involve regulation of temperature for portions of the system. The dynamic models of temperature control systems involve the flow and storage of heat energy. Heat energy flows through substances at a rate proportional to the temperature difference across the substance; that is,

$$q = \frac{1}{R}(T_1 - T_2), \tag{2.62}$$

where

q = heat energy flow, joules per second (J/sec), or British Thermal Unit/sec (BTU/sec),

R = thermal resistance, °C/J·sec or °F/BTU·sec,

T = temperature, °C or °F.

The net heat-energy flow into a substance affects the temperature of the substance according to the relation

$$\dot{T} = \frac{1}{C}q, \tag{2.63}$$

where C is the thermal capacity. Typically, there are several paths for heat to flow into or out of a substance, and q in Eq. (2.63) is the sum of heat flows obeying Eq. (2.62).

EXAMPLE 2.14 *Equations for Heat Flow*

A room with all but two sides insulated ($1/R = 0$) is shown in Fig. 2.34. Find the differential equations that determine the temperature in the room.

Solution. Application of Eqs. (2.62) and (2.63) yields

$$\dot{T}_I = \frac{1}{C_I}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)(T_O - T_I),$$

where

C_I = thermal capacity of air within the room,

T_O = temperature outside,

T_I = temperature inside,

R_2 = thermal resistance of the room ceiling,

R_1 = thermal resistance of the room wall.

Normally the material properties are given in tables as follows:

Specific heat

1. The specific heat at constant volume c_v, which is converted to heat capacity by

$$C = mc_v, \tag{2.64}$$

where m is the mass of the substance;

Thermal conductivity

2. The thermal conductivity[9] k, which is related to thermal resistance R by

$$\frac{1}{R} = \frac{kA}{l},$$

where A is the cross-sectional area and l is the length of the heat-flow path.

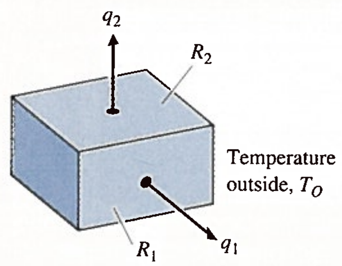


Figure 2.34 Dynamic model for room temperature

[9] In the case of insulation for houses, resistance is quoted as R-values; for example, R-11 refers to a substance that has a resistance to heat flow equivalent to that given by 11 in. of solid wood.

In addition to flow due to transfer, as expressed by Eq. (2.62), heat can also flow when a warmer mass flows into a cooler mass, or vice versa. In this case,

$$q = wc_v(T_1 - T_2), \tag{2.65}$$

where w is the mass flow rate of the fluid at T_1 flowing into the reservoir at T_2. For a more complete discussion of dynamic models for temperature control systems, see Cannon (1967) or textbooks on heat transfer.

EXAMPLE 2.15 *Equations for Modeling a Heat Exchanger*

A heat exchanger is shown in Fig. 2.35. Steam enters the chamber through the controllable valve at the top, and cooler steam leaves at the bottom. There is a constant flow of water through the pipe that winds through the middle of the chamber so that it picks up heat from the steam. Find the differential equations that describe the dynamics of the measured water outflow temperature as a function of the area A_s of the steam-inlet control valve when open. The sensor that measures the water outflow temperature, being downstream from the exit temperature in the pipe, lags the temperature by t_d seconds.

Solution. The temperature of the water in the pipe will vary continuously along the pipe as the heat flows from the steam to the water. The temperature of the steam will also reduce in the chamber as it passes over the maze of pipes. An accurate thermal model of this process is therefore quite involved because the actual heat transfer from the steam to the water will be proportional to the local temperatures of each fluid. For many control applications it is not necessary to have great accuracy because the feedback will correct for a considerable amount of error in the model. Therefore, it makes sense to combine the spatially varying temperatures into single temperatures T_s and T_w for the outflow steam and water temperatures, respectively. We then assume that the heat transfer from steam to water is proportional to the difference in these temperatures, as given by Eq. (2.62). There is also a flow of heat into the chamber

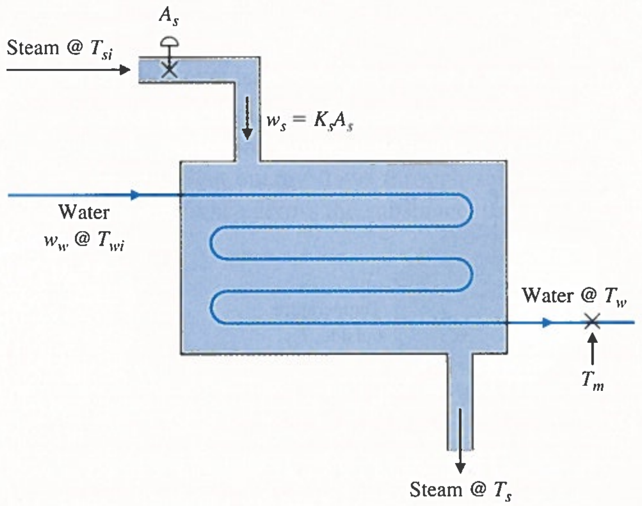


Figure 2.35 Heat exchanger

from the inlet steam that depends on the steam flow rate and its temperature according to Eq. (2.65),

$$q_{in} = w_s c_{vs}(T_{si} - T_s),$$

where

$w_s = K_s A_s$, mass flow rate of the steam,
A_s = area of the steam inlet valve,
K_s = flow coefficient of the inlet valve,
c_{vs} = specific heat of the steam,
T_{si} = temperature of the inflow steam,
T_s = temperature of the outflow steam.

The net heat flow into the chamber is the difference between the heat from the hot incoming steam and the heat flowing out to the water. This net flow determines the rate of temperature change of the steam according to Eq. (2.63),

$$C_s\dot{T}_s = A_sK_sc_{vs}(T_{si} - T_s) - \frac{1}{R}(T_s - T_w), \tag{2.66}$$

where

$C_s = m_s c_{vs}$ is the thermal capacity of the steam in the chamber with mass m_s,
R = the thermal resistance of the heat flow averaged over the entire exchanger.

Likewise, the differential equation describing the water temperature is

$$C_w\dot{T}_w = w_w c_{cw}(T_{wi} - T_w) + \frac{1}{R}(T_s - T_w), \tag{2.67}$$

where

w_w = mass flow rate of the water,
c_{cw} = specific heat of the water,
T_{wi} = temperature of the incoming water,
T_w = temperature of the outflowing water.

To complete the dynamics, the time delay between the measurement and the exit flow is described by the relation

$$T_m(t) = T_w(t - t_d),$$

where T_m is the measured downstream temperature of the water and t_d is the time delay. There may also be a delay in the measurement of the steam temperature T_s, which would be modeled in the same manner.

Equation (2.66) is nonlinear because the quantity T_s is multiplied by the control input A_s. The equation can be linearized about T_{so} (a specific value of T_s) so that $T_{si} - T_s$ is assumed constant for purposes of approximating the nonlinear term, which we will define as ΔT_s. In order to eliminate the T_{wi} term in Eq. (2.67), it is convenient to measure all temperatures in terms of deviation in degrees from T_{wi}. The resulting equations are then

$$C_s\dot{T}_s = -\frac{1}{R}T_s + \frac{1}{R}T_w + K_s c_{vs}\Delta T_s A_s,$$

$$C_w\dot{T}_w = -\left(\frac{1}{R} + w_w c_{vw}\right)T_w + \frac{1}{R}T_s,$$

$$T_m = T_w(t - t_d).$$

Although the time delay is not a nonlinearity, we will see in Chapter 3 that operationally, $T_m = e^{-t_d s}T_w$. Therefore, the transfer function of the heat exchanger has the form

$$\frac{T_m(s)}{A_s(s)} = \frac{Ke^{-t_d s}}{(\tau_1 s + 1)(\tau_2 s + 1)}. \qquad (2.68)$$

2.4.2 Incompressible Fluid Flow

Fluid flows are common in many control systems components. One example is the hydraulic actuator, which is used extensively in control systems because it can supply a large force with low inertia and low weight. They are often used to move the aerodynamic control surfaces of airplanes, to gimbal rocket nozzles, to move the linkages in earth-moving equipment, farm tractor implements, snow-grooming machines, and to move robot arms.

The continuity relation

The physical relations governing fluid flow are continuity, force equilibrium, and resistance. The continuity relation is simply a statement of the conservation of matter; that is,

$$\dot{m} = w_{in} - w_{out}, \qquad (2.69)$$

where

m = fluid mass within a prescribed portion of the system,

w_{in} = mass flow rate into the prescribed portion of the system,

w_{out} = mass flow rate out of the prescribed portion of the system.

EXAMPLE 2.16

Equations for Describing Water Tank Height

Determine the differential equation describing the height of the water in the tank in Fig. 2.36.

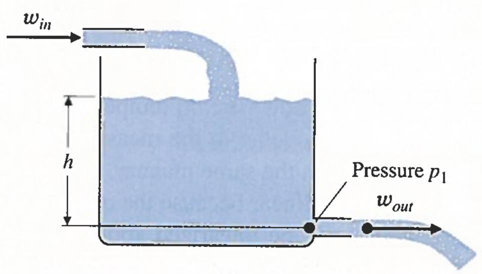


Figure 2.36
Water tank example

Solution. Application of Eq. (2.69) yields

$$\dot{h} = \frac{1}{A\rho}(w_{in} - w_{out}), \qquad (2.70)$$

where

A = area of the tank,

ρ = density of water,

$h = m/A\rho$ = height of water,

m = mass of water in the tank.

Force equilibrium must apply exactly as described by Eq. (2.1) for mechanical systems. Sometimes in fluid-flow systems some forces result from fluid pressure acting on a piston. In this case the force from the fluid is

$$f = pA, \qquad (2.71)$$

where

f = force,

p = pressure in the fluid,

A = area on which the fluid acts.

EXAMPLE 2.17

Modeling a Hydraulic Piston

Determine the differential equation describing the motion of the piston actuator shown in Fig. 2.37, given that there is a force F_D acting on it and a pressure p in the chamber.

Solution. Equations (2.1) and (2.71) apply directly, where the forces include the fluid pressure as well as the applied force. The result is

$$M\ddot{x} = Ap - F_D,$$

where

A = area of the piston,

p = pressure in the chamber,

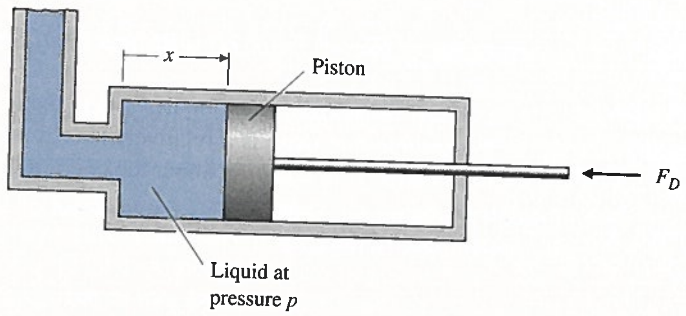


Figure 2.37
Hydraulic piston actuator

$$M = \text{mass of the piston,}$$

$$x = \text{position of the piston.}$$

In many cases of fluid-flow problems the flow is resisted either by a constriction in the path or by friction. The general form of the effect of resistance is given by

$$w = \frac{1}{R}(p_1 - p_2)^{1/\alpha}, \tag{2.72}$$

where

$$w = \text{mass flow rate,}$$

$$p_1, p_2 = \text{pressures at ends of the path through which flow is occurring,}$$

$$R, \alpha = \text{constants whose values depend on the type of restriction.}$$

Or, as is more commonly used in hydraulics,

$$Q = \frac{1}{\rho R}(p_1 - p_2)^{1/\alpha}, \tag{2.73}$$

where

$$Q = \text{volume flow rate, where } Q = {}^{w}/_{\rho},$$

$$\rho = \text{fluid density.}$$

The constant α takes on values between 1 and 2. The most common value is approximately 2 for high flow rates (those having a Reynolds number Re $> 10^5$) through pipes or through short constrictions or nozzles. For very slow flows through long pipes or porous plugs wherein the flow remains laminar (Re $\lesssim 1000$), $\alpha = 1$. Flow rates between these extremes can yield intermediate values of α. The Reynolds number indicates the relative importance of inertial forces and viscous forces in the flow. It is proportional to a material's velocity and density and to the size of the restriction, and it is inversely proportional to the viscosity. When Re is small, the viscous forces predominate and the flow is laminar. When Re is large, the inertial forces predominate and the flow is turbulent.

Note that a value of $\alpha = 2$ indicates that the flow is proportional to the square root of the pressure difference and therefore will produce a nonlinear differential equation. For the initial stages of control systems analysis and design, it is typically very useful to linearize these equations so that the design techniques described in this book can be applied. Linearization involves selecting an operating point and expanding the nonlinear term to be a small perturbation from that point.

EXAMPLE 2.18 *Linearization of Water Tank Height and Outflow*

Find the nonlinear differential equation describing the height of the water in the tank in Fig. 2.36. Assume that there is a relatively short restriction at the outlet and that $\alpha = 2$. Also linearize your equation about the operating point h_o.

Solution. Applying Eq. (2.72) yields the flow out of the tank as a function of the height of the water in the tank:

$$w_{out} = \frac{1}{R}(p_1 - p_a)^{1/2}. \tag{2.74}$$

Here,

$$p_1 = \rho g h + p_a, \text{ the hydrostatic pressure,}$$

$$p_a = \text{ambient pressure outside the restriction.}$$

Substituting Eq. (2.74) into Eq. (2.70) yields the nonlinear differential equation for the height:

$$\dot{h} = \frac{1}{A\rho}\left(w_{in} - \frac{1}{R}\sqrt{p_1 - p_a}\right). \tag{2.75}$$

Linearization involves selecting the operating point $p_o = \rho g h_o + p_a$ and substituting $p_1 = p_o + \Delta p$ into Eq. (2.74). Then we expand the nonlinear term according to the relation

$$(1+\varepsilon)^{\beta} \cong 1 + \beta\varepsilon, \tag{2.76}$$

where $\varepsilon \ll 1$. Equation (2.74) can thus be written as

$$\begin{aligned} w_{out} &= \frac{\sqrt{p_o - p_a}}{R}\left(1 + \frac{\Delta p}{p_o - p_a}\right)^{1/2} \\ &\cong \frac{\sqrt{p_o - p_a}}{R}\left(1 + \frac{1}{2}\frac{\Delta p}{p_o - p_a}\right). \end{aligned} \tag{2.77}$$

The linearizing approximation made in Eq. (2.77) is valid as long as $\Delta p \ll p_o - p_a$; that is, as long as the deviations of the system pressure from the chosen operating point are relatively small.

Combining Eqs. (2.70) and (2.77) yields the following linearized equation of motion for the water tank level:

$$\Delta\dot{h} = \frac{1}{A\rho}\left[w_{in} - \frac{\sqrt{p_o - p_a}}{R}\left(1 + \frac{1}{2}\frac{\Delta p}{p_o - p_a}\right)\right].$$

Because $\Delta p = \rho g \Delta h$, this equation reduces to

$$\Delta\dot{h} = -\frac{g}{2AR\sqrt{p_o - p_a}}\Delta h + \frac{w_{in}}{A\rho} - \frac{\sqrt{p_o - p_a}}{\rho A R}, \tag{2.78}$$

which is a linear differential equation for $\Delta\dot{h}$. The operating point is not an equilibrium point because some control input is required to maintain it. In other words, when the system is at the operating point ($\Delta h = 0$) with no input ($w_{in} = 0$), it will move from that point because $\Delta\dot{h} \neq 0$. So if no water is flowing into the tank, the tank will drain, thus moving it from the reference point. To define an operating point that is also an equilibrium point, we need to require that there be a nominal flow rate,

$$\frac{w_{in_o}}{A\rho} = \frac{\sqrt{p_o - p_a}}{\rho A R},$$

and define the linearized input flow to be a perturbation from that value.

Hydraulic actuators

Hydraulic actuators obey the same fundamental relationships we saw in the water tank: continuity [Eq. (2.69)], force balance [Eq. (2.71)], and flow resistance [Eq. (2.72)]. Although the development here assumes the fluid is perfectly incompressible, in fact, hydraulic fluid has some compressibility due primarily to entrained air. This feature causes hydraulic actuators to have some resonance because the compressibility of the fluid acts like a stiff spring. This resonance limits their speed of response.

EXAMPLE 2.19 *Modeling a Hydraulic Actuator*

1. Find the nonlinear differential equations relating the movement θ of the control surface to the input displacement x of the valve for the hydraulic actuator shown in Fig. 2.38.
2. Find the linear approximation to the equations of motion when $\dot{y}$ = constant, with and without an applied load—that is, when $F \neq 0$ and when $F = 0$. Assume that θ motion is small.

Solution

1. **Equations of motion:** When the valve is at $x = 0$, both passages are closed and no motion results. When $x > 0$, as shown in Fig. 2.38, the oil flows clockwise as shown and the piston is forced to the left. When $x < 0$, the fluid flows counterclockwise. The oil supply at high pressure p_s enters the *left* side of the large piston chamber, forcing the piston to the right. This causes the oil to flow out of the valve chamber from the rightmost channel instead of the left.

   We assume that the flow through the orifice formed by the valve is proportional to x; that is,

$$Q_1 = \frac{1}{\rho R_1}(p_s - p_1)^{1/2}x. \tag{2.79}$$

Similarly,

$$Q_2 = \frac{1}{\rho R_2}(p_2 - p_e)^{1/2}x. \tag{2.80}$$

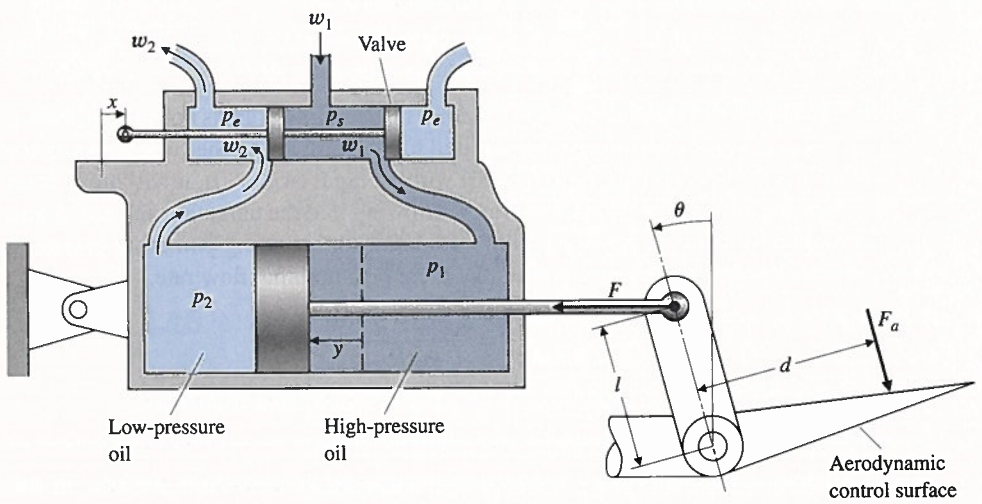


Figure 2.38
Hydraulic actuator with valve

The continuity relation yields

$$A\dot{y} = Q_1 = Q_2, \tag{2.81}$$

where

A = piston area.

The force balance on the piston yields

$$A(p_1 - p_2) - F = m\ddot{y}, \tag{2.82}$$

where

m = mass of the piston and the attached rod,

F = force applied by the piston rod to the control surface attachment point.

Furthermore, the moment balance of the control surface using Eq. (2.14) yields

$$I\ddot{\theta} = Fl\cos\theta - F_a d, \tag{2.83}$$

where

I = moment of inertia of the control surface and attachment about the hinge,

F_a = applied aerodynamic load.

To solve this set of five equations, we require the following additional kinematic relationship between θ and y:

$$y = l\sin\theta. \tag{2.84}$$

The actuator is usually constructed so that the valve exposes the two passages equally; therefore, $R_1 = R_2$, and we can infer from Eqs. (2.79) to (2.81) that

$$p_s - p_1 = p_2 - p_e. \tag{2.85}$$

These relations complete the nonlinear differential equations of motion; they are formidable and difficult to solve.

2. **Linearization and simplification:** For the case in which $\dot{y}$ = a constant ($\ddot{y} = 0$) and there is no applied load ($F = 0$), Eqs. (2.82) and (2.85) indicate that

$$p_1 = p_2 = \frac{p_s + p_e}{2}. \tag{2.86}$$

Therefore, using Eq. (2.81) and letting $\sin\theta = \theta$ (since θ is assumed to be small), we get

$$\dot{\theta} = \frac{\sqrt{p_s - p_e}}{\sqrt{2}A\rho Rl}x. \tag{2.87}$$

This represents a single integration between the input x and the output θ, where the proportionality constant is a function only of the supply pressure and the fixed parameters of the actuator. For the case $\dot{y}$ = constant but $F \neq 0$, Eqs. (2.82) and (2.85) indicate that

$$p_1 = \frac{p_s + p_e + F/A}{2}$$

and

$$\dot{\theta} = \frac{\sqrt{p_s - p_e - F/A}}{\sqrt{2}A_p Rl}x. \tag{2.88}$$

This result is also a single integration between the input x and the output θ, but the proportionality constant now depends on the applied load F.

As long as the commanded values of x produce θ motion that has a sufficiently small value of $\ddot{\theta}$, the approximation given by Eqs. (2.87) or (2.88) is valid and no other linearized dynamic relationships are necessary. However, as soon as the commanded values of x produce accelerations in which the inertial forces ($m\ddot{y}$ and the reaction to $I\ddot{\theta}$) are a significant fraction of $p_s - p_e$, the approximations are no longer valid. We must then incorporate these forces into the equations, thus obtaining a dynamic relationship between x and θ that is much more involved than the pure integration implied by Eqs. (2.87) or (2.88). Typically, for initial control system designs, hydraulic actuators are assumed to obey the simple relationship of Eqs. (2.87) or (2.88). When hydraulic actuators are used in feedback control systems, resonances have been encountered that are not explained by using the approximation that the device is a simple integrator as in Eqs. (2.87) or (2.88). The source of the resonance is the neglected accelerations discussed above along with the additional feature that the oil is slightly compressible due to small quantities of entrained air. This phenomenon is called the "oil-mass resonance."

2.5 Historical Perspective

Newton's second law of motion (Eq. 2.1) was first published in his *Philosophiae Naturalis Principia Mathematica* in 1686 along with his two other famous laws of motion. The first: A body will continue with the same uniform motion unless acted on by an external unbalanced force, and the third: To every action there is an equal and opposite reaction. Isaac Newton also published his law of gravitation in this same publication, which stated that every mass particle attracts all other particles by a force proportional to the inverse of the square of the distance between them and the product of their two masses. His basis for developing these laws was the work of several other early scientists, combined with his own development of the calculus in order to reconcile all the observations. It is amazing that these laws still stand today as the basis for almost all dynamic analysis with the exception of Einstein's additions in the early 1900s for relativistic effects. It is also amazing that Newton's development of calculus formed the foundation of our mathematics that enable dynamic modeling. In addition to being brilliant, he was also very eccentric. As Brennan writes in *Heisenberg Probably Slept Here*, "He was seen about campus in his disheveled clothes, his wig askew, wearing run-down shoes and a soiled neckpiece. He seemed to care about nothing but his work. He was so absorbed in his studies that he forgot to eat." Another interesting aspect of Newton is that he initially developed the calculus and the now famous laws of physics about 20 years prior to publishing them! The incentive to publish them arose from a bet between three men having lunch at a pub in 1684: Edmond Halley, Christopher Wren, and Robert Hooke. They all had the opinion that Kepler's elliptical characterization of planetary motion could be explained by the inverse square law, but nobody had ever proved it, so they "placed a bet as to who could first prove the conjecture."[10] Halley went to Newton for help due to his fame as a mathematician, who responded he had already done it many years ago and would forward the papers to him. He not only did that shortly afterwards, but followed it up with the *Principia* with all the details two years later.

The basis for Newton's work started with the astronomer Nicholas Copernicus more than a hundred years before the *Principia* was published. He was the first to speculate that the planets revolved around the sun, rather than everything in the skies revolving around the earth. But Copernicus' heretical notion was largely ignored at the time, except by the church who banned his publication. However, two scientists did take note of his work: Galileo Galilei in Italy and Johannes Kepler in Austria. Kepler relied on a large collection of astronomical data taken by a Danish astronomer, Tycho Brahe, and concluded that the planetary orbits were ellipses rather than the circles that Copernicus had postulated. Galileo was an expert telescope builder and was able to clearly establish that the earth was not the center of all motion, partly because he was able to see moons revolving around other planets. He also did experiments with rolling balls down inclined planes that strongly suggested that $F = ma$ (alas, it's a myth that he did his experiments by dropping objects out of the Leaning Tower of Pisa). Galileo published his work in 1632, which raised the ire of the church who then later banned him to house arrest until he died.[11] It was not until 1985 that the church recognized the important contributions of Galileo! These men laid the groundwork for Newton to put it all together with his laws of motion and the inverse square gravitational law. With these two physical principles, all the observations fit together with a theoretical framework that today forms the basis for the modeling of dynamic systems.

The sequence of discoveries that ultimately led to the laws of dynamics that we take for granted today were especially remarkable when we stop to think that they were all carried out without a computer, a calculator, or even a slide rule. On top of that, Newton had to invent calculus in order to reconcile the data.

After publishing the *Principia*, Newton went on to be elected to Parliament and was given high honors, including being the first man of science to be knighted by the Queen. He also got into fights with other scientists fairly regularly and used his powerful positions to get what he wanted. In one instance, he wanted data from the Royal Observatory that was not forthcoming fast enough. So he created a new board with authority over the Observatory and had the Astronomer Royal expelled from the Royal Society. Newton also had other less scientific interests. Many years after his death, John Maynard Keynes found that Newton had been spending as much of his time on metaphysical occult, alchemy, and biblical works as he had been on physics.

More than a hundred years after Newton's *Principia*, Michael Faraday performed a multitude of experiments and postulated the notion of electromagnetic lines of force in free space. He also discovered induction (Faraday's Law), which led to the

[10] Much of the background on Newton was taken from *Heisenberg Probably Slept Here*, by Richard P. Brennan, 1997. The book discusses his work and the other early scientists that laid the groundwork for Newton.

[11] Galileo's life, accomplishments, and house arrest are very well described in Dava Sobel's book, *Galileo's Daughter.*

electric motor and the laws of electrolysis. Faraday was born into a poor family, had virtually no schooling, and became an apprentice to a bookbinder at age 14. There he read many of the books being bound and became fascinated by science articles. Enthralled by these, he maneuvered to get a job as a bottle washer for a famous scientist, eventually learned enough to be a competitor to him, and ultimately became a professor at the Royal Institution in London. But lacking a formal education, he had no mathematical skills, and lacked the ability to create a theoretical framework for his discoveries. Faraday became a famous scientist in spite of his humble origins. After he had achieved fame for his discoveries and was made a Fellow of the Royal Society, the prime minister asked him what good his inventions could be.[12] Faraday's answer was, "Why Prime Minister, someday you can tax it." But in those days, scientists were almost exclusively men born into privilege; so Faraday had been treated like a second-class citizen by some of the other scientists. As a result, he rejected knighthood as well as burial at Westminster Abbey. Faraday's observations, along with those by Coulomb and Ampere, led James Clerk Maxwell to integrate all their knowledge on magnetism and electricity into Maxwell's Equations. Against the beliefs of most prominent scientists of the day (Faraday being an exception), Maxwell invented the concepts of fields and waves that explained magnetic and electrostatic forces and was the key to creating the unifying theory. Although Newton had discovered the spectrum of light, Maxwell was also the first to realize that light was one type of the same electromagnetic waves, and its behavior was explained as well by Maxwell's Equations. In fact, the only constant in his equations are μ and ε. The constant speed of light is $c = 1/\sqrt{\mu\varepsilon}$.

Maxwell was a Scottish mathematician and theoretical physicist. His work has been called the second great unification in physics, the first being that due to Newton. Maxwell was born into the privileged class and was given the benefits of an excellent education and he excelled at it. In fact, he was an extremely gifted theoretical and experimental scientist as well as a very generous and kind man with many friends and little vanity. In addition to unifying the observations of electromagnetics into a theory that still governs our engineering analyses today, he was the first to present an explanation of how light travels, the primary colors, the kinetic theory of gases, the stability of Saturn's rings, and the stability of feedback control systems! His discovery of the three primary colors (red, green, and blue) forms the basis of our color television to this day. His theory showing the speed of light is a constant was difficult to reconcile with Newton's laws and led Albert Einstein to create the special theory of relativity in the early 1900s. This led Einstein to say, "One scientific epoch ended and another began with James Clerk Maxwell."[13]

SUMMARY

Mathematical modeling of the system to be controlled is the first step in analyzing and designing the required system controls. In this chapter we developed models

[12] $E = MC^2$, *A Biography of the World's Most Famous Equation*, by David Bodanis, Walker and Co., New York, 2000.

[13] *The Man Who Changed Everything: The Life of James Clerk Maxwell*, Basil Mahon, Wiley, 2003.

TABLE 2.1 Key Equations for Dynamic Models

System	Important Laws or Relationships	Associated Equations	Equation Number
Mechanical	Translation motion (Newton's law)	$F = ma$	(2.1)
	Rotational motion	$M = I\alpha$	(2.14)
Electrical	Operational amplifier		(2.36), (2.37)
Electromechanical	Law of motors	$F = Bli$	(2.43)
	Law of the generator	$e(t) = Blv$	(2.46)
	Torque developed in a rotor	$T = K_t i_a$	(2.50)
Back emf	Voltage generated as a result of rotation of a rotor	$e = K_e \dot{\theta}_m$	(2.51)
Heat flow	Heat-energy flow	$q = {}^1/_R(T_1 - T_2)$	(2.62)
	Temperature as a function of heat-energy flow	$\dot{T} = {}^1/_C q$	(2.63)
	Specific heat	$C = mc_v$	(2.64)
Fluid flow	Continuity relation (conservation of matter)	$\dot{m} = w_{in} - w_{out}$	(2.69)
	Force of a fluid acting on a piston	$f = pA$	(2.71)
	Effect of resistance to fluid flow	$w = {}^1/_R(p_1 - p_2)^{1/\alpha}$	(2.72)

for representative systems. Important equations for each category of system are summarized in Table 2.1.

REVIEW QUESTIONS

1. What is a "free-body diagram"?
2. What are the two forms of Newton's law?
3. For a structural process to be controlled, such as a robot arm, what is the meaning of "collocated control"? "Noncollocated control"?
4. State Kirchhoff's current law.
5. State Kirchhoff's voltage law.
6. When, why, and by whom was the device named an "operational amplifier"?
7. What is the major benefit of having zero input current to an operational amplifier?
8. Why is it important to have a small value for the armature resistance R_a of an electric motor?
9. What are the definition and units of the electric constant of a motor?
10. What are the definition and units of the torque constant of an electric motor?

International Edition

Feedback Control of Dynamic Systems

Sixth Edition

Gene F. Franklin
J. David Powell
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