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MEASUREMENTS SECTION

THE PULSE TRANSFER FUNCTION AND ITS APPLICATION TO SAMPLING SERVO SYSTEMS

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SUMMARY

The manipulation of samples in a linear system may be studied by a transform method closely analogous to the Laplace transform. The pulse transfer function relates a sequence of samples at the output of a system to the sequence producing it. The theory is presented in language familiar to the engineer and without the formalities of mathematical rigour.

Special attention is paid to servo systems, and particularly to those containing a finite time-delay within the feedback loop. The problem of synthesis is considered and a new method of stabilization is described.

A short list of transforms is given in an appendix.

LIST OF SYMBOLS

t = Time.
τ = Sampling period.
k = Integer identifying the instant of sampling.
$\mathscr{L}$ = Symbol for the Laplace transform.
p = Operator of the Laplace transform.
$s(t)$ = Function indicating the form of a sample.
$w(t)$ = Response of a filter to unit sample.
$Y(p)$ = Transfer function of a filter.
w_k = Weighting sequence of a filter.
z = Operator of the sequence transform.
$W(z)$ = Sequence transform of w_k, the pulse transfer function of a filter.
f_k = Input sequence.
g_k = Output sequence.
c_k = Correction sequence.
s_k = Output sequence of stabilizer.
U, V = Series-connected filters (and their pulse transfer functions).
θ, ϕ = Transfer functions.
Θ, Φ = Pulse transfer functions corresponding to θ and ϕ.
α, β, γ = Reciprocals of time constants.
$\gamma, \bar{\gamma}$ = Characteristic root and its complex conjugate.
d = $\epsilon^{-\alpha\tau}$.
Δ = Difference operator.
λ = Time delay measured in sampling periods.
δ = Time delay measured as a fraction of a sampling period.
m = $1 - \delta$.
n = Next integer $\geqslant \lambda$.
$Y(z)$ = Pulse transfer function of a servo system.
A, B = System constants.
A, B, C, D = Polynomials in z used in the appendix.
P, Q = Polynomials in z such that $W(z) = P(z)/Q(z)$.
x = Sinusoidal signal.
ω = Angular frequency.
ϕ = Phase.
ζ = Complex or cissoidal input signal.
C = Modulus of ζ.
ξ = Complex or cissoidal output signal.
M = Noise power-gain.
Q = Magnification.
K = Amplifier gain.
ρ = Attenuation constant.
a = Staleness weighting constant.

(1) INTRODUCTION

The mathematical device of interpolation has long been used to enable a limited number of samples to convey adequately all the essential information contained in a continuous function. Applications are to be found in many branches of engineering. Examples include the quality control of a product by sampling, the transmission of a speech waveform by pulse or pulse-code modulation, and certain servo systems used in radar.

The paper is concerned with linear systems involving the manipulation of samples, and in particular with a method of analysis based upon a transformation closely analogous to the well-known Laplace transform. The method was developed by MacColl,[1] Hurewicz[2] and Lawden[3] primarily in connection with pulsed servo-mechanisms, but it is of more general application. The so-called "sequence transform" takes the place of the Laplace transform and a function known as the "pulse transfer function" completely defines the performance of the system as judged at the sampling instants. No attempt is made to discuss the subject with mathematical rigour. The object is rather to introduce it as much from a physical standpoint as is possible, and to illustrate its usefulness in dealing with a number of engineering problems.

(1.1) The Sequence Transform Method

Fig. 1 shows a filter the input to which is the sequence of samples obtained by closing the switch S1 for a brief instant at moments defined by $t = k\tau$, where τ is the sampling interval.

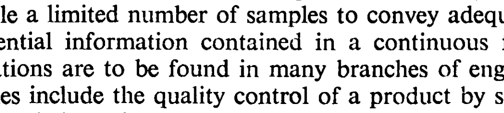

Fig. 1.—Filter with sampling switches.

Suppose that the sample taken at $t = 0$ is of unit magnitude and that at other times it is zero. The filter output is the continuous function of time, $w(t)$, given by the Laplace-transform equation

$$\mathscr{L}[w(t)] = Y(p)\mathscr{L}[s(t)] = Y(p)S(p) \quad . \quad . \quad . \quad (1)$$

where $s(t)$ is the driving function representing the unit sample and $S(p)$ is its transform. The form of $s(t)$ depends upon the physical arrangements. If, for example, the interval for which the switch is held closed is negligible in comparison with the sampling interval and with other time constants of the system, the unit sample may be treated as a unit impulse, for which $S(p) = 1$.

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The output of the filter is sampled by a second switch S2 synchronous with S1 and has at those instants the sequence of values w_k given by

$$w_k = w(k\tau) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is known as the weighting sequence of the filter. It is logical to treat the two switches as being similar in their modes of operation, in which case the sample obtained at $t = 0$ is $w_0 s(t)$—that is of magnitude w_0 and of the same form $s(t)$ as the original driving function. At $t = \tau$ a second sample will become available, of magnitude w_1 and of the same form, namely $s(t - \tau)$ referred to the same origin of time. The complete output is the sum of all such samples, namely

$$w_0 s(t) + w_1 s(t - \tau) + \ldots + w_k s(t - k\tau) + \ldots$$

The Laplace transform of this is

$$w_0 S(p) + w_1 S(p)\epsilon^{-p\tau} + \ldots + w_k S(p)\epsilon^{-pk\tau} + \ldots$$

$$= S(p)\sum_{k=0}^{\infty} w_k \epsilon^{-kp\tau} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It is convenient to make the substitution $z = \epsilon^{p\tau}$ and to use an abbreviated notation for the resulting function of z, thus

$$W(z) = \sum_{k=0}^{\infty} w_k z^{-k} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This equation defines a transformation in which $W(z)$ is said to be the sequence transform of w_k. It is closely analogous to the Laplace transform

$$Y(p) = \int_0^{\infty} y(t)\epsilon^{-pt}dt \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and, in fact, reverts to it as the sampling interval approaches zero. A formula for the inverse transform also exists[3] but will not be used in the paper.

The input and the output sequences both possess transforms defined in a similar manner. When the input is the unit sample, the output is simply the weighting sequence. By the application of the superposition principle the output corresponding to any other input can be deduced. At the time $t = k\tau$ the output sample g_k is the sum of the effects of all input samples f_k, f_{k-1}, etc., up to and including that at $t = k\tau$.

$$g_k = w_0 f_k + w_1 f_{k-1} + w_2 f_{k-2} + \ldots \quad . \quad . \quad (6)$$

To obtain the corresponding equation in terms of the transform, multiply each term by z^{-k}.

$$g_k z^{-k} = w_0 z^0 f_k z^{-k} + w_1 z^{-1} f_{k-1} z^{-(k-1)} + \ldots$$

This represents an infinity of equations corresponding to all values of k. It is convenient, however, to choose the origin of time so that the input signal is zero prior to $t = 0$. Then f_k and hence g_k are both zero when k is negative and the equations corresponding to these cases disappear. Summation of the remainder gives

$$\sum_{k=0}^{\infty} g_k z^{-k} = \sum_{k=0}^{\infty} w_k z^{-k} \sum_{k=0}^{\infty} f_k z^{-k} \quad . \quad . \quad . \quad . \quad (7)$$

or, using corresponding capital letters to denote the transforms,

$$G(z) = W(z)F(z) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$W(z)$ is therefore the ratio of the transforms of the output sequence and the input sequence producing it, and it is known as the pulse transfer function. This term was first introduced by Hurewicz[2] and is so called because it relates to the transfer of samples which are often represented by impulse functions.

To be precise in the interpretation of eqn. (8) we should multiply both sides by $S(p)$ as in eqn. (3), and then, as a final operation, take the inverse Laplace transform to yield the output as a function of time. Since $S(p)$ is a constant, however, it may be omitted from the calculations provided that the output sequence g_k is regarded as a series of numbers indicating the numerical values of the successive samples.

(1.2) Applications

In a number of simple cases it is possible to carry through the required transformations analytically. Some examples which have been found useful are given in Appendix 9. Where this is not possible—e.g. if the transfer function has been determined empirically—one can always resort to the multiplication or division of the sequences themselves. This is basically the same as the method of time-series analysis proposed by Tustin[4] for the approximate solution of certain problems in the continuous domain. Approximate solutions to differential equations may be obtained in a similar manner. In this connection, Brown[5] has shown that the differential operator p can be expressed to any desired degree of accuracy as a rational fraction of z, the simplest expression being the one suggested by Tustin, namely $\frac{1}{2}\tau(z - 1)/(z + 1)$.

It is possible to extend the notion of the pulse transfer function to include the case when the operation of the output-sampling switch S2 is delayed by an amount $m\tau$, say, relative to S1. The pulse transfer function then depends upon both m and z and may be written as $W(z, m)$. The same function results if both samples are taken simultaneously but the filter has an associated finite time-delay $\delta\tau$ where $\delta = 1 - m$. Besides its obvious use in connection with servo mechanisms involving transmission lags, the function $W(z, m)$ may be applied with expediency for determining the way in which the output of a system varies between the sampling instants.

There are a number of other applications of the sequence-transform technique which, owing to lack of space, cannot be discussed here. These include the statistical treatment of filters and servo mechanisms for discrimination against noise and for prediction, the design of time equalizers for the correction of distortion in digital transmission systems, and the calculation of the impulse response of a minimum-phase-shift system when the autocorrelation function of the response to random noise is known. The last of these is a useful approximate method when applied to continuous systems.

(2) CALCULATION OF THE PULSE TRANSFER FUNCTION

The pulse transfer function (p.t.f.) has already been defined. The general procedure for calculating it from first principles is summarized in the following rules:

(a) Determine the transfer function $Y(p)$ of the system by conventional operational methods.

(b) Find the inverse Laplace transform of $Y(p)$; this is the impulse response of the system.

(c) Substitute $t = k\tau$ in the impulse response so as to obtain the weighting sequence w_k. Note that, if the output is to be sampled at a time different from that at which the input is sampled, the appropriate substitution, for example $t = (k - \delta)\tau$, must be used.

(d) Perform the summation $\sum_{k=0}^{\infty} w_k z^{-k}$. The series will always converge for a physically realizable system provided that z is assumed to be small.

The usefulness of the method depends upon the possibility of performing the final summation. Fortunately a considerable number of applications involve exponential functions for which the summation is possible. Appendix 9 has been constructed with a view to simplifying the work in such cases as it enables the p.t.f. to be written down directly $Y(p)$ is known. More complicated expressions may often be dealt with by splitting them up into partial fractions.

(2.1) Pulse Transfer Function of Two Filters in Series

It is often desirable to make use of the overall p.t.f. of two filters in series. The simplest case is that in which only samples from the output of the first filter are applied to the second. The overall p.t.f. is then simply the product of the separate p.t.f.'s, $U(z)$ and $V(z)$, of the components, and may be written as

$$W(z) = U(z)V(z) \quad . \quad . \quad . \quad . \quad . \quad (9)$$

A more complicated situation arises if no sampling device separates the filters. That is to say, the input to the second filter V is the continuous function of time present at the output terminal of the first filter U. The overall p.t.f. must then be evaluated by considering the system as a whole. This is indicated by the formal expression

$$W(z) = [UV](z) \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The order of the filters is now important since $[UV](z)$ is not necessarily equal to $[VU](z)$. The following method has been found useful in dealing with the situation:

The first filter U is imagined to be split into two parts in series in such a manner that only samples pass between them. $\theta(p)$ and $\phi(p)$ are the transfer functions of these parts. If the latter is associated with the second filter V, transfer function $v(p)$, the system becomes one to which eqn. (9) applies. The p.t.f.'s corresponding to $\theta(p)$ and to $\phi(p)v(p)$ can be determined separately and then multiplied together. The method is suitable only if $\phi(p)$ is a fairly simple function. Fortunately this is often the case as the inverse Laplace transform of $\phi(p)$ indicates the way in which the output of V varies with time between sampling instants and is not likely to be complicated. Table 1 shows some simple examples. Others may readily be obtained from Appendix 9.

Table 1

$\phi(t)$	$\phi(p)$	$\Phi(z)$
Constant	p^{-1}	$\dfrac{z}{z-1}$
Linear	p^{-2}	$\dfrac{z}{(z-1)^2}$
Quadratic	p^{-3}	$\dfrac{z(z+1)}{2(z-1)^3}$
Exponential	$(p+\alpha)^{-1}$	$\dfrac{z}{z-d}$

Denote the p.t.f. corresponding to the first part, $\theta(p)$, by $\Theta(z)$ and that of the second part by $[\Phi V](z)$. Then

$$[UV](z) = \Theta(z)[\Phi V](z) = \frac{U(z)}{\Phi(z)}[\Phi V](z)$$

This leads to the idea of expressing the effect of the first filter by means of an operational instruction which may be abbreviated as $\overline{\Theta(z)}\phi(p)$. The interpretation is that the transfer function of whatever it operates upon is multiplied by $\phi(p)$ and the p.t.f. corresponding to the product is then multiplied by $\Theta(z)$. More generally the operational instruction will be the sum of a number of such and may be written as $\Sigma\overline{\Theta_r(z)}\phi_r(p)$.

The method will be illustrated by an example in which the first filter is a clamp and the second is an integrator having an exponential lag. The clamp performs the function of extending each sample for the duration of the ensuing sampling interval. It may also perform the sampling operation but this is not at the moment relevant. The time response of the clamp to a unit sample at $t = 0$ is a unit step starting then and lasting until just before $t = \tau$. The transfer function is therefore $p^{-1}(1 - \epsilon^{-p\tau})$ in which p^{-1} will be identified with $\phi(p)$ so that the operational instruction for the clamp is $\overline{(1 - z^{-1})}p^{-1}$.

The transfer function of the second filter is assumed to be $v(p) = \dfrac{\alpha}{p(p+\alpha)}$. Although the p.t.f. corresponding to $p^{-1}v(p)$ may be found in Appendix 9, it will be derived here from first principles. First, split up the expression by partial fractions:

$$\frac{\alpha}{p^2(p+\alpha)} = \frac{1}{\alpha}\left(\frac{\alpha}{p^2} - \frac{1}{p} + \frac{1}{p+\alpha}\right)$$

The inverse Laplace transform is

$$w(t) = \frac{1}{\alpha}(\alpha t - 1 + \epsilon^{-\alpha t})$$

The weighting sequence is obtained by substituting $t = k\tau$

$$w(k\tau) = \frac{1}{\alpha}(\alpha k\tau - 1 + d^k) \text{ where } d = \epsilon^{-\alpha\tau}$$

The p.t.f. is therefore

$$[\Phi V](z) = \tau\sum_0^\infty kz^{-k} - \frac{1}{\alpha}\sum_0^\infty z^{-k} + \frac{1}{\alpha}\sum_0^\infty d^k z^{-k}$$

$$= \frac{\tau z}{(z-1)^2} - \frac{z}{\alpha(z-1)} + \frac{z}{\alpha(z-d)}$$

$$= \tau\left[\frac{z}{(z-1)^2} - \frac{1-d}{\alpha\tau}\frac{z}{(z-1)(z-d)}\right] \quad . \quad (11)$$

The operational instruction will have been carried out completely when this is multiplied by $1 - z^{-1}$.

$$[UV](z) = \tau\left(\frac{1}{z-1} - \frac{1-d}{\alpha\tau}\frac{1}{z-d}\right) \quad . \quad . \quad (12)$$

(2.2) Finite Time-Delay

A filter may possess a finite time-delay, or distance/velocity lag as it is sometimes called, equal to $\lambda\tau$. If λ is an integer, say n, each sample is delayed by n periods and any term w_k of the weighting sequence becomes w_{k-n} for $k \geqslant n$ and zero otherwise. The p.t.f. is $\sum_{k=n}^{\infty} w_{k-n}z^{-k}$ which is identical with $z^{-n}\sum_{k=0}^{\infty} w_k z^{-k}$. The effect of delay by n sampling periods is therefore to multiply the p.t.f. by z^{-n}.

If the delay is a fraction, say δ, of a sampling period, a different method of treatment is necessary. The weighting sequence becomes

$$0, \quad w(\tau - \delta\tau), \quad w(2\tau - \delta\tau), \quad w(3\tau - \delta\tau), \quad \text{etc.}$$

The transform of this sequence is the p.t.f. $\sum_{k=1}^{\infty} w_{(k\tau-\delta\tau)}z^{-k}$. This may also be written as the function of z and m

$$W(z, m) = z^{-1}\sum_{k=0}^{\infty} w_{(k\tau+m\tau)}z^{-k} \quad . \quad . \quad . \quad (13)$$

where $m = 1 - \delta$. In the case of a general delay $\lambda\tau$ such that $(n-1) < \lambda \leqslant n$ the p.t.f. is as given by eqn. (13) except that z^{-1} is replaced by z^{-n} and m equals $n - \lambda$.

Column (iv) in Appendix 9 lists some pulse transfer functions calculated on the assumption that a finite time-delay $\delta\tau$ is associated with the filter. As δ approaches unity $W(z, m)$ approaches $z^{-1}W(z)$. As δ approaches zero $W(z, m)$ approaches $W(z)$ provided that the impulse response, $\mathscr{L}^{-1}[Y(p)]$, of the filter is continuous over the range of variation of δ and through zero. Where this condition is not fulfilled the indication that $\delta > 0$ is added.

(2.3) The Pulse Transfer Function of a System with Feedback

In Fig. 2 is shown a simple sampling servo system, a common application of feedback.

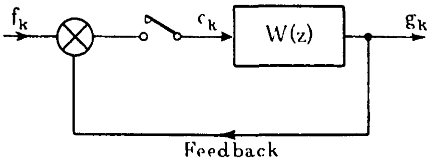


Fig. 2.—A simple sampling servo system.

Let f_k be the input sequence, g_k be the output sequence, and c_k be the correction sequence, defined by

$$c_k = f_k - g_k$$

$$\frac{g_k}{f_k} = \frac{g_k}{c_k}\frac{1}{1 + g_k/c_k}$$

If the overall p.t.f. of the system is $Y(z)$, defined by the transform equation

$$G(z) = Y(z)F(z)$$

then

$$Y(z) = \frac{W(z)}{1 + W(z)} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The effect of a second filter $U(z)$ in the feedback path is to modify the denominator of this expression. As in Section 2.1, two cases must be considered, namely:

(*a*) Samples only of the output are fed back via the filter $U(z)$. The open-loop p.t.f. is $W(z)U(z)$.

$$Y(z) = \frac{W(z)}{1 + W(z)U(z)} \quad . \quad . \quad . \quad . \quad (15)$$

(*b*) The complete function $g(t)$, which is the output of the filter W before sampling, is fed back via the filter $U(z)$. The open-loop p.t.f. is $[WU](z)$.

$$Y(z) = \frac{W(z)}{1 + [WU](z)} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Eqns. (14) and (15) have their analogues in the theory of continuous servo mechanisms. In general the p.t.f. of a system with negative feedback is equal to the p.t.f. of the forward path divided by one plus the p.t.f. of the complete loop.

In the work of Porter and Stoneman[6] and later of Holt Smith, Lawden and Bailey,[7] integrators and summators have played an important role in the synthesis of pulsed servo systems. The p.t.f. of a set of $n + 1$ integrators is $\frac{zD_n(z)}{(z - 1)^{n+1}}$ with no delay included, or $\frac{B_n(z, m)}{(z - 1)^{n+1}}$ with a delay $(1 - m)\tau$. $D_n(z)$ and $B_n(z, m)$ are polynomials which are given in Appendix 9. They have been discussed mathematically in some detail by Lawden.[8]

A summator, as its name implies, is a device for generating a sequence, each term of which is the sum of the present and all preceding inputs. A single unit sample gives rise to the sequence 1, 1, 1, 1, etc., and the p.t.f. is therefore

$$\sum_{k=0}^{\infty} 1z^{-k} = \frac{z}{z - 1}$$

(2.4) Derivation of the Pulse Transfer Function from Difference Equations

The classical approach to problems involving functions which change only at regular intervals is via the solution of difference equations. If g_k represents a sequence, the first difference may be defined as $\Delta g_k = g_k - g_{k-1}$, the second difference as $\Delta^2 g_k = \Delta g_k - \Delta g_{k-1}$ and so on. The symbol Δ does, in fact, represent an operator which can be manipulated algebraically in a similar manner to the operator p or the operator z. Its relationship to the latter is simple. The sequence transform $G(z)$, when written in full, is

$$G(z) = g_0 + g_1 z^{-1} + g_2 z^{-2} + \ldots + g_k z^{-k} + \ldots$$

Also $\quad z^{-1}G(z) = g_0 z^{-1} + g_1 z^{-2} + \ldots + g_{k-1}z^{-k} + \ldots$

Subtracting

$$(1 - z^{-1})G(z) = \Delta g_0 + \Delta g_1 z^{-1} + \Delta g_2 z^{-2} + \ldots$$

Dividing by $G(z)$ leads to the operational relation

$$\frac{z - 1}{z} = \Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

As one would expect, the operation of taking a difference is exactly the inverse of that of summation.

A typical linear difference equation would be

$$a_0 g_k + a_1 \Delta g_k + a_2 \Delta^2 g_k + \ldots = f_k$$

where f_k is the excitation sequence. The transform is obtained by putting $\Delta = 1 - z^{-1}$

$$[a_0 + a_1(1 - z^{-1}) + a_2(1 - z^{-1})^2 + \ldots]G(z) = F(z)$$

More generally, if differences in f_k are also included one would obtain a relation such as

$$G(z) = \frac{P(z)}{Q(z)}F(z) \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where both P and Q are polynomials in z. Here, $P(z)/Q(z)$ obviously corresponds to the pulse transfer function. The difference equation may be solved by taking the inverse transform of the right-hand side of eqn. (18). The solution obtained by this method is not a general one but fulfils the particular condition that it is zero for negative values of k.

As an example, consider the p.t.f. of the clamp-and-integrator system discussed at the end of Section 2.1. The differential equation of the system is

$$\frac{d^2g}{dt^2} + \alpha\frac{dg}{dt} = \alpha f(t)$$

where $f(t)$ is the output of the clamp and is constant except for sudden changes that occur at the sampling instants. Integrate this equation once and apply it to the interval beginning at $(k - 1)\tau$ for which the initial value of $\int f(t)$ is $\tau\sum_{x=0}^{k-2} f_x$.

Then $$\frac{dg}{dt} + \alpha g = \alpha\tau\sum_{x=0}^{k-2} f_x + \alpha t f_{k-1}$$

The solution which fulfils the initial condition $g = g_{k-1}$ is

$$g(t) = \epsilon^{-\alpha t}g_{k-1} + tf_{k-1} + (1 - \epsilon^{-\alpha t})(\tau\sum f_x - \alpha^{-1}f_{k-1})$$

$$\Delta g_k = g_k - g_{k-1}$$

$$= \tau f_{k-1} + (1 - d)(g_{k-1} + \tau\sum f_x - \alpha^{-1}f_{k-1})$$

where $d = \epsilon^{-\alpha\tau}$. $\quad$ Since $\Delta\left(\sum_{x=0}^{k-2} f_x\right) = f_{k-2}$

$$\Delta^2 g_k + (1-d)\Delta g_{k-1} = \tau\left[\left(1-\frac{1-d}{\alpha\tau}\right)\Delta f_{k-1} + (1-d)f_{k-2}\right] \quad . \quad . \quad . \quad . \quad (19)$$

This is the difference equation of the system. The corresponding transform equation is obtained by putting $\Delta = 1 - z^{-1}$

$$[(z-1)^2 + (1-d)(z-1)]G(z)$$

$$= \tau\left[\left(1-\frac{1-d}{\alpha\tau}\right)(z^2 - z) + (1-d)\right]F(z)$$

or $$W(z) = \frac{G(z)}{F(z)} = \tau\left(\frac{1}{z-1} - \frac{1-d}{\alpha\tau}\,\frac{1}{z-d}\right)$$

which is identical with eqn. (12) obtained by the more direct approach. The inverse transform of $W(z)F(z)$ yields directly a solution to the difference equation (19). If an analytical expression is not available, it is always possible to obtain the sequence solution by expansion in inverse powers of z.

(3) THE OUTPUT SEQUENCE

The output sequence depends upon the weighting sequence of the filter and upon the driving function in the manner described by the transform equation (8).

$$G(z) = W(z)F(z)$$

Some criterion of performance must be adopted when a system is being studied, and the driving function $F(z)$ must be chosen to yield information relevant to it. The response of a system to unit driving functions, to sinusoidal and to random sequences will be considered in more detail. In addition, a method for finding the output between sampling instants will be described.

(3.1) Unit Driving Functions

An input signal commonly used in the analysis of servo-system performance is the unit step of position. It gives direct information concerning rise time and overshoot. The transforms of this and similar functions are given below. In each case $f_k = 0$ for negative values of k.

Table 2

Description	Sequence	Transform $F(z)$
Unit impulse or unit sample ..	$f_0 = 1$ $f_k = 0,\ k \geqslant 1$	1
Unit step-function	$f_k = 1$	$\frac{z}{z-1}$
Unit velocity- or ramp-function ..	$f_k = k$	$\frac{z}{(z-1)^2}$
Unit acceleration-function	$f_k = k^2$	$\frac{z(z+1)}{2(z-1)^3}$

In a servo system it is often important to know the error in the output when some derivative of the input is held constant. The transform of the error sequence is

$$E(z) = [Y(z) - 1]F(z) \quad . \quad . \quad . \quad . \quad (20)$$

$Y(z)$ being the overall p.t.f. of the system. If the numerator of $Y(z) - 1$ contains $(z-1)$ as a factor the system will have zero static error in response to a unit step—that is after the initial transient disturbance has died out. For the static error to be zero for driving functions whose higher derivatives are constant, the factor $(z-1)$ must appear raised to the same power as that to which it occurs in the driving-function transform.

If the system has a velocity lag, its value may be determined by putting $F(z) = \frac{z}{(z-1)^2}$ in eqn. (20). $E(z)$ then contains $(z-1)$ as a factor of its denominator and may be split up by a process of partial fractionization.

$$E(z) = A\frac{z}{z-1} + B\,(\text{some function of } z)$$

The first term is responsible for the steady-state error and the second for the transient part of the error sequence. The velocity lag is the error per unit rate of change of input and is equal to A. The theory of partial fractions can be used to show that A is obtained by multiplying the right-hand side of eqn. (20) by $(z-1)$ and then substituting $z = 1$.

This procedure may be generalized. The static error when the nth derivative is constant and of unit magnitude is obtained by substituting $z = 1$ in

$$(z-1)[1 - Y(z)]F(z), \text{ where } F(z) = \sum_0^{\infty} k^n z^{-k} \quad . \quad (21)$$

(3.2) Sinusoidal Sequences

Methods of frequency analysis have their analogues in sampling systems. The result of sampling a continuous sine wave is a sinusoidal sequence. The sine wave may be written as $x = A\cos(\omega t + \phi)$ which is the real part of $A\epsilon^{j(\omega t+\phi)}$. It is more convenient here, as with continuous systems, to use a complex input, say $\zeta = C\epsilon^{j\omega t}$ where $C = A\epsilon^{j\phi}$. After sampling, this function provides the sequence $\zeta_k = C\epsilon^{j\omega k\tau}$. If the weighting sequence of the filter is w_k the output sequence will be

$$\xi_k = w_0 C\epsilon^{j\omega k\tau} + w_1 C\epsilon^{j\omega(k-1)\tau} + \ldots$$

$$= C\epsilon^{j\omega k\tau}(w_0 + w_1\epsilon^{-j\omega\tau} + w_2\epsilon^{-2j\omega\tau} + \ldots)$$

$$= \zeta_k \sum_{k=0}^{\infty} w_k \epsilon^{-j\omega k\tau}$$

Therefore $$\xi_k = \zeta_k W(\epsilon^{j\omega\tau}) \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$W(\epsilon^{j\omega\tau})$ is just the p.t.f. of the filter with $j\omega\tau$ substituted for z. Eqn. (22) shows that the output sequence is also sinusoidal and of angular frequency ω. The modulus and phase may be determined in the usual way. The frequency of the input sine wave may of course be greater than $1/2\tau$, but the series of samples obtained from it cannot reveal that this is so. In fact the sequence obtained is the same for any of the frequencies $n/\tau \pm f$ where n is an integer and f less than $1/2\tau$. A sampling system can therefore act as a frequency changer. Consider for example a signal whose frequency spectrum is confined to the band $1/\tau$ to $3/2\tau$. If samples taken at intervals τ are applied to a low-pass filter cutting off at $1/2\tau$, the original signal will be obtained but with all component frequencies lowered by $1/\tau$.

(3.3) Random-Noise Sequences

The input signal to a filter or servo system is often contaminated by noise, the effect of which must be calculated. The noise may be regarded as random if the sampling interval τ is large compared to the periods of the principal components of the noise spectrum. There is then no correlation between successive samples and the noise spectrum will, moreover, be uniform up to frequencies higher than the cut-off frequency of the filter concerned. In a sampling system this cannot exceed $1/2\tau$. The ratio of the power associated with an output sequence to that of the random-noise sequence producing it is known as the noise-power gain.

At a single frequency $\omega/2\pi$ the output power is, from eqn. (22),

$$\overline{\xi_k^2} = \overline{\zeta_k^2} W(\epsilon^{j\omega\tau}) W(\epsilon^{-j\omega\tau}) \quad . \quad . \quad . \quad . \quad (23)$$

Since the spectrum of the noise is flat to beyond the limits $\omega = \pm \pi/\tau$, the total power is obtained by integrating eqn. (23) over the range.

$$\text{Noise-power gain} = M^2 = \int_{-\pi/\tau}^{\pi/\tau} W(\epsilon^{j\omega\tau}) W(\epsilon^{-j\omega\tau})$$

If the substitution $\epsilon^{j\omega\tau} = z$ is made

$$M^2 = \frac{1}{2\pi j} \oint W(z) W(z^{-1}) \frac{dz}{z} \quad . \quad . \quad . \quad . \quad (24)$$

where the contour of integration is the circle $|z| = 1$. The integration may be performed by calculating the residues of $z^{-1} W(z) W(z^{-1})$ at its poles within the contour. $W(z)$ may sometimes be expressed as a power series with a finite number of terms.

$$W(z) = w_0 + w_1 z^{-1} + w_2 z^{-2} + \ldots + w_r z^{-r}$$

Similarly $W(z^{-1}) = w_0 + w_1 z + w_2 z^2 + \ldots + w_r z^r$

The only pole of $z^{-1} W(z) W(z^{-1})$ within the circle $|z| = 1$ is at $z = 0$ and is of order $r + 1$. The residue is equal to the constant term in the product $W(z) W(z^{-1})$ and is $w_0^2 + w_1^2 + \ldots + w_r^2$. By a more rigorous treatment is it possible to show that this result is still true when r tends to infinity. Hence an alternative expression for the noise-power gain is

$$M^2 = \sum_{k=0}^{\infty} w_k^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

(3.4) The Output between Sampling Instants

Calculations based upon the sequence transform yield only information about the output function at the instants that it is sampled. The law of interpolation between points calculated in this way involves the impulse response of the system as a continuous function of time. The delayed pulse transfer function $W(z, m)$, however, is a convenient device for dealing with the situation. If the filter possessed a delay $\delta\tau = (1 - m)\tau$ the output sequence would be obtained by using $W(z, m)$ instead of $W(z)$. Use of the former function in relation to a filter that does not have any delay therefore yields the output sequence which would have been obtained if the samples were taken at $t = (k + m)\tau$ instead of at $t = k\tau$. By varying m from zero to just less than unity, the output function is explored over the sampling interval.

A similar method is applicable to servo systems. Imagine the system of Fig. 2 to be replaced by that of Fig. 3, in which there

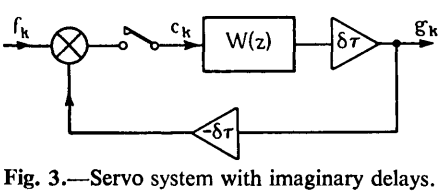


Fig. 3.—Servo system with imaginary delays.

is a delay $\delta\tau$ in the forward path and an equal but negative delay in the feedback path. The performance will be identical with that of the original system except that the calculated output sequence will refer to the times $t = (k + m)\tau$. The overall p.t.f. is then

$$Y(z) = \frac{W(z, m)}{1 + W(z)} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

If a real delay is present within the servo loop, it must be included in the p.t.f. used in the denominator.

(4) STABILITY

The output of a linear system that is subjected to a finite disturbance will either come to rest, oscillate indefinitely, or increase to the limits of the range of linearity. Only in the first instance can the system be regarded physically as stable.

When the disturbance is the unit step function the resulting output at $t = n\tau$ is $\sum_{k=0}^{n} w_k$, where w_k is the weighting sequence. If this is to remain finite, then $\sum_{k=0}^{\infty} w_k$ must be finite, a test easily applied by substituting $z = 1$ in the pulse transfer function. This alone is insufficient, however, since it does not normally detect a state of continuous oscillation.

A random-noise sequence contains frequency components up to $1/2\tau$ and is more suitable as a test signal. The system can be regarded as stable if the noise-power gain, $\sum_{k=0}^{\infty} w_k^2$, is finite. Since all the weighting-sequence terms are real numbers, an equivalent condition is that $\sum_{k=0}^{\infty} |w_k|$ is finite, and it is possible to show that this condition is both necessary and sufficient. Yet another way of expressing it is that $\sum_{k=0}^{\infty} w_k z^{-1}$ must be finite for all values of the complex variable z such that $|z| < 1$. The p.t.f., $W(z)$ must, in fact, contain no poles or other singular points within the unit circle $|z| = 1$.

(4.1) The Characteristic Roots

The pulse transfer function can usually be expressed as the ratio of two polynomials in z

$$W(z) = \frac{P(z)}{Q(z)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

in which the order of P cannot be greater than, and is usually one less than, the order of Q. The equation $Q(z) = 0$ is known as the characteristic equation, and its roots as the characteristic roots, denoted here by $\gamma_1, \gamma_2 \ldots \gamma_q$. Hence

$$W(z) = \frac{P(z)}{(z - \gamma_1)(z - \gamma_2) \ldots (z - \gamma_q)} \quad . \quad . \quad (28)$$

Since $W(z)$ is real the roots are either real or occur in complex conjugate pairs. The right-hand side may be split up by means of partial fractions into a number of terms such as

$$W(z) = a_1 \frac{z^2 + b_1 z}{(z - \gamma_1)(z - \bar{\gamma}_1)} + \ldots + \frac{a_q z}{z - \gamma_q}$$

The inverse transform of this is the output sequence generated by a unit sample input. Each real root gives rise to an exponential sequence such as γ^k which increases indefinitely if $|\gamma| > 1$. Each pair of conjugate complex roots is responsible for an oscillating sequence of the form $d^k \sec\phi \cos(k\beta + \phi)$ where

$$d^2 = \gamma\bar{\gamma}$$

$$2d\cos\beta = \gamma + \bar{\gamma}$$

$$d\sec\phi\cos(\beta - \phi) = b_1$$

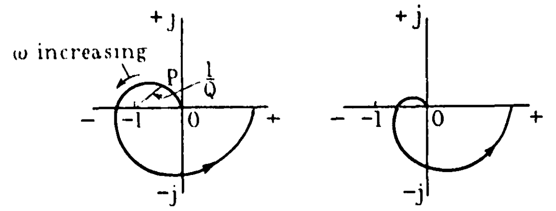
The oscillation dies away if $|\gamma| < 1$, remains constant if $|\gamma| = 1$ and grows indefinitely if $|\gamma| > 1$. The practical condition of stability is that none of the characteristic roots shall have a modulus greater than or equal to one.

(4.2) The Modified Nyquist Criterion

There is a well-known criterion due to Nyquist for determining the stability of a continuous servo-system or feedback amplifier. It is based upon the open-loop frequency response. A modified technique is applicable to pulsed systems such as that shown in Fig. 2. The overall p.t.f. is given by eqn. (14) from which it follows that

$$\frac{1}{W(z)} = \frac{1}{Y(z)} + 1$$

The response to a sinusoidal sequence of frequency $\omega/2\pi$ is obtained by putting $z = \epsilon^{j\omega\tau}$, and may be represented on an Argand diagram by a point P whose co-ordinates are the real and imaginary parts of $1/W(\epsilon^{j\omega\tau})$. A typical locus of P as ω varies from 0 to π/τ is shown in Fig. 4. Between π/τ and $2\pi/\tau$ the locus is the mirror image in the real axis. If the frequency is further increased the same path is again traversed.



(a) (b)

Fig. 4.—Modified Nyquist diagram.

(a) Stable condition.
(b) Unstable condition.

The magnification, or Q, is equal to the modulus of $Y(\epsilon^{j\omega\tau})$ and is equal to the reciprocal of the length of the vector joining the point $P(\omega)$ to the point $-1, j0$. The system will be stable if the vector sweeps out a net positive angle as ω increases from zero to $2\pi/\tau$. Fig. 4 shows examples of stable and unstable conditions. The length of the vector gives a very good indication of the degree of stability and whether undue enhancement of any particular frequencies is to be expected.

Similar considerations apply to a servo system containing a delay element or a modifying filter $U(z)$ in the feedback path. The open-loop p.t.f. will be $W(z, m)$ if there is a delay $(1 - m)\tau$ and $W(z)U(z)$ or $[WU](z)$ if eqns. (15) or (16) apply. The vector from $-1, j0$ to P does not in this case give the overall Q of the system. It does, however, still indicate the margin of stability.

Kochenburger[9] and Linvill[10] have studied in some detail the behaviour of sampling servo-systems by methods involving frequency analysis of this general type.

(4.3) High-Frequency Instability

An analysis based upon the sequence transform only gives information about the performance at the sampling instants. As judged at these instants the system will always appear to possess a cut-off frequency not exceeding $1/2\tau$. The possibility of high-frequency oscillations taking place between sampling instants must not, however, be overlooked. If it occurs at a frequency which is an integral multiple of τ^{-1} it would not be revealed by the samples making up the output sequence.

Since the sequence-transform technique provides a short cut to the solution of many problems involving sampling, it is important when applying it to make sure that no hidden instability of this type can exist. Two examples will illustrate the danger.

Example 1.

The transfer function is $\dfrac{p^2 + ap + b}{(p + \gamma)(p^2 - 2\alpha p + \beta^2)}$

If the constants have certain values, namely

$$\beta^2 = \alpha^2 + \pi^2/\tau^2$$
$$a = \pi/\tau - 2\alpha$$
$$b = \pi^2/\tau^2 + \gamma\pi/\tau + \alpha^2$$

the p.t.f. reduces to the simple form $\dfrac{z}{z - d_1}$ where $d_1 = \epsilon^{-\gamma\tau}$.

There is only one characteristic root and that is at $z = d_1$. The system is therefore stable if $|d| < 1$ or if $\gamma > 0$. However, if the performance midway between sampling instants is investigated by computing $W(z, m)$ with $m = \frac{1}{2}$, then

$$W(z, \tfrac{1}{2}) = \frac{\sqrt{d_1}}{z - d_1} + \frac{\sqrt{d_2}}{z + d_2}$$

where $d_2 = \epsilon^{\alpha\tau}$. There are now two characteristic roots and the system will not be stable unless $|d_2| < 1$. The second term gives rise to an oscillating component which grows exponentially in amplitude if α is positive. The frequency is $1/2\tau$ and the phase such that the instantaneous value is zero at the normal sampling times.

Example 2.

Example 2 was investigated in an attempt to secure the stable operation of a servo mechanism having a delay τ in the feedback loop. The open-loop transfer function was

$$Y(p) = \frac{(p + \alpha)\epsilon^{-p\tau}}{(p + \alpha)^2 + \pi^2/\tau^2}\left[\frac{B_1\tau^2(1 - \epsilon^{-p\tau})}{p^2} - \frac{B_2\tau + B_2'\tau\epsilon^{-p\tau}}{p}\right]$$

One would expect some tendency to instability by reason of the delay term. If, however, certain values are given to the constants, namely

$$\epsilon^{-\alpha\tau} = 2,\ \tfrac{1}{2}B_1\tau^2 = 374,\ B_2\tau = 396 \cdot 1, \text{ and } B_2'\tau = 350 \cdot 6$$

the p.t.f. of the servo system reduces simply to

$$Y(z) = 3z^{-2} - 2z^{-3}$$

This would appear to be reasonably stable, the response to a unit step being the sequence

0, 0, 3, 1, 1, 1, 1, etc.

However, if the complete output function is examined the response obtained is as shown in Fig. 5. At the sampling instants it passes through the points indicated in the above sequence, but in between it has an overshoot of approximately 70 times. A practical system operating on this basis is obviously out of the question.

These examples emphasize the danger of relying only upon the information available at the sampling instants. Such cases are seldom encountered in practice, and if they are, there is usually some significant feature to give a warning. In the second example the transfer function $Y(p)$ depends mainly upon the small difference between two rather large quantities.

(4.4) Control of Stability

System synthesis is nearly always much more difficult than system analysis. In the regime of continuous operation, root determination and Nyquist diagrams are sometimes used for the former but, on account of the labour, very seldom for the latter. A very much more powerful technique has been devised involving

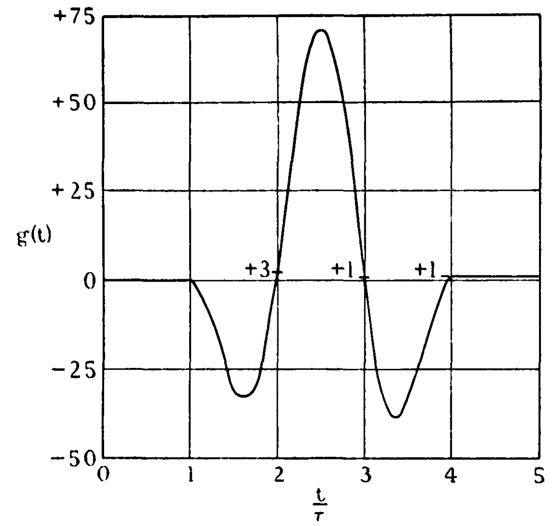


Fig. 5.—Response of a sampling servo system with hidden instability.

the construction of asymptotic diagrams in which log modulus and phase are plotted against log frequency. It depends upon the fact that the effects of various components of the system are additive on such a diagram. It is unfortunately not available for the study of sampling systems since the p.t.f. of the whole is not usually equal to the product of the separate p.t.f.'s of the components.

Nyquist diagrams involving the substitution $z = \epsilon^{j\omega\tau}$ have been mentioned but they are not very useful as an aid to synthesis, and one is driven to methods dependent upon a knowledge of the characteristic roots. A possible approach is to synthesize a system having a characteristic equation chosen beforehand and known to possess a suitable degree of stability. The particular case in which all the roots are real and equal has been studied by Holt Smith, Lawden and Bailey.[7] The characteristic equation of such a system is

$$(z - a)^n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where a has been called the "staleness weighting constant" and n is the order of the equation.

The open-loop p.t.f. of a system with feedback can often be expressed in the form

$$W(z) = \sum_r B_1^{(r)} z^{-r} W_1(z) + \sum_r B_2^{(r)} z^{-r} W_2(z) + \text{etc.} \quad . \quad (30)$$

The constants B are assumed to be under the control of the designer, whereas the functions $W_1(z)$, $W_2(z)$, etc., are supposed to be particular to the system under consideration and invariant as far as this part of the problem is concerned. The overall p.t.f. is, from eqn. (14),

$$Y(z) = \frac{W(z)}{1 + W(z)}$$

and the characteristic equation is therefore $1 + W(z) = 0$. The procedure is to choose the constants B so that it is identical with eqn. (29). To illustrate this, associate with the filter in Section 2.1 an amplifier of gain K and provide the necessary feedback connection to make it into a servo system as in Fig. 6.

The p.t.f. has already been found from eqn. (12).

$$W(z) = \tau K\left(\frac{1}{z - 1} - \frac{1 - d}{\alpha}\frac{1}{z - d}\right)$$

$$= B_1\frac{1}{z - 1} + B_2\frac{1}{z - d} \quad \text{(say)} \quad . \quad . \quad (31)$$

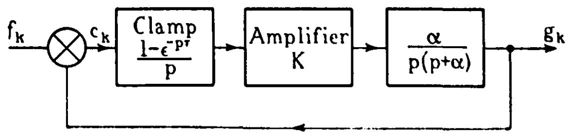


Fig. 6.—A simple servo system.

The characteristic equation is

$$(z - 1)(z - d) + B_1(z - d) + B_2(z - 1) = 0 \quad . \quad (32)$$

which is identical with $(z - a)^2 = 0$ if

$$B_1 = \frac{(1 - a)^2}{1 - d} \text{ and } B_2 = -\frac{(d - a)^2}{1 - d}$$

In terms of the original parameters K and α, the necessary conditions are

$$\tau K = \frac{(1 - a)^2}{1 - \epsilon^{-\alpha\tau}} \text{ and } (1 - \epsilon^{-\alpha\tau})(1 - a)^2 = \alpha\tau(a - \epsilon^{-\alpha\tau})^2$$

The latter equation may easily be solved for a if a value of α is assumed: α is infinity when $a = 0$ and steadily decreases as a increases, reaching zero at $a = 1$. Hence, by suitable choice of the two parameters both roots may be made equal to any chosen staleness weighting constant.

The open-loop p.t.f. may now be written as

$$W(z) = \frac{(z - a)^2}{(z - 1)(z - d)} - 1$$

and the overall p.t.f. as

$$Y(z) = 1 - \frac{(z - 1)(z - d)}{(z - a)^2} \quad . \quad . \quad . \quad . \quad (33)$$

Application of eqn. (21) shows that the system will have a velocity lag equal to $1/\tau K$.

It has been assumed above that the time constant, α^{-1}, of the integrator is one of the parameters adjustable in the design stage. This, however, is not necessarily the case, and if the desired value of a is less than $\epsilon^{-\alpha\tau}$ it will be necessary to introduce additional complexity. The problem is then to synthesize a control system which, when operating upon the unit having the transfer function $\frac{\alpha}{p(p + \alpha)}$, yields $(z - a)^2 = 0$ as the characteristic equation.

The system can be treated as two filters in series, the first one being unknown though its operational instruction must contain two adjustable constants. In most physical applications it would be undesirable to apply impulses to the second filter and the simplest alternative is for its input to be held constant between sampling instants. This will be so if $\phi(p) = p^{-1}$. The operational instruction can now be expressed as $\overline{\Theta(z)}p^{-1}$ where $\Theta(z) = \frac{z - 1}{z}U(z)$. The p.t.f. corresponding to $\frac{\alpha}{p^2(p + \alpha)}$ has already been found [eqn. (11)].

$$[\Phi V](z) = \tau\left[\frac{z}{(z - 1)^2} - \frac{1 - d}{\alpha\tau}\frac{z}{(z - 1)(z - d)}\right]$$

$$= \left(1 - \frac{1 - d}{\alpha\tau}\right)\frac{\tau z(z + b)}{(z - 1)^2(z - d)} \quad . \quad . \quad . \quad (34)$$

where $\quad b = \dfrac{\alpha\tau - 1 + d}{1 - d - \alpha\tau d}, d = \epsilon^{-\alpha\tau}$

$$W(z) = \frac{z-1}{z}U(z)[\Phi V](z)$$

$$= \tau\left(1 - \frac{1-d}{\alpha\tau}\right)\frac{z+b}{(z-1)(z-d)}U(z) \quad . \quad . \quad (35)$$

$U(z)$ must contain two adjustable constants if the form of the characteristic equation is to be completely under control. Furthermore it must be such that the order of the denominator of $W(z)$ is not increased. The simplest solution is

$$U(z) = B_1\frac{(z-B_2)}{z+b} \quad . \quad . \quad . \quad . \quad . \quad (36)$$

The first filter must have this value for its p.t.f. and must be such that its output does not change between sampling instants. Provided that α is positive the constant b lies between 0 and 1. It will be shown in Section 5 that a p.t.f. of the form $\frac{z}{z+b}$ is always associated with a unit having an oscillatory response such as is produced by delayed negative feedback. One possible arrangement is indicated in Fig. 7. This is similar to Fig. 6 except that

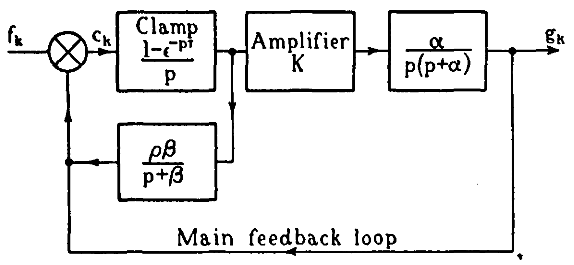


Fig. 7.—Servo system with oscillatory control.

there is a subsidiary loop including a unit having the transfer function $\frac{\rho\beta}{p+\beta}$. Eqn. (16) may be applied to determine the p.t.f. of the clamp with feedback. It is $\frac{z-\epsilon^{-\beta\tau}}{z-\epsilon^{-\beta\tau}+\rho(1-\epsilon^{-\beta\tau})}$ and is equal to $\frac{z-B_2}{z+b}$ if

$$B_2 = \epsilon^{-\beta\tau}$$

and
$$b = \rho(1-\epsilon^{-\beta\tau}) - \epsilon^{-\beta\tau}$$

The other constant, B_1, is the amplifier gain K. If the system is to have $(z-a)^2 = 0$ as its characteristic equation the necessary values of the constants are

$$B_1 = \frac{\alpha(1+d-2a)}{\alpha-1+d}$$

$$B_2 = \frac{d-a^2}{1+d-2a}$$

The system will then have exactly the same performance as that shown in Fig. 6, except that the staleness weighting constant a may now be made as small as desired.

(5) SYSTEMS WITH DELAYED FEEDBACK

It is well known that any form of delay in the feedback path of a servo system is detrimental to good performance. Such a delay may, however, be inevitable, as in some process controls where a sample must be analysed or tested before corrective action can be taken or where the plant itself possesses a distance/velocity lag. It then remains to mitigate the undesirable effects as much as possible One of the most important of these is that the stability is reduced.

(5.1) Stabilization of Systems with Delayed Feedback

The system to be discussed is shown in Fig. 8. If there were no delay in the feedback path there would be no need for the stabilizer. The plant is assumed to be invariant in the design stage and the controller is assumed to incorporate such parameters as must necessarily be adjusted to give the desired performance.

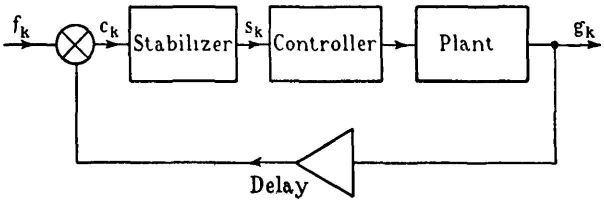


Fig. 8.—Stabilized servo system having delayed feedback.

The stabilizer modifies the correction sequence c_k to produce a new input sequence s_k for the controller. The latter may be of the same form as when there is no delay, though in general the values of the adjustable parameters will be different. It is possible to design the stabilizer so that the effective time constants of the servo system are the same as if delay were not present, and when this is done the input to the controller is the same in both cases. That is to say, the stabilizer generates a sequence s_k which is the same as the correction sequence would have been in the simple case of no delay and no stabilizer.

Consider first the operation of the system without the stabilizer. The p.t.f. of controller and plant taken together may be expressed as the ratio of two polynomials in z, namely

$$W(z) = \frac{P(z)}{Q(z)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

The order of P can never be greater than that of Q and in most practical cases is at least one less. Similarly the p.t.f. of controller, plant and delay unit may be written as

$$W(z, m) = z^{-n}\frac{P(z, m)}{Q(z)} \quad . \quad . \quad . \quad . \quad (38)$$

where $n - m$ is the delay measured in sampling periods. Note that $Q(z)$ is assumed to be the same for both eqns. (37) and (38). This is so in all the examples given in Section 9 and is, in fact, true for any system that is characterized by a linear differential equation with constant coefficients. The characteristic equation is

$$z^nQ(z) + P(z, m) = 0 \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Let the order of $Q(z)$ be N and that of $P(z, m)$ be $N-1$. It is obviously not possible by adjustment of the controller to influence the coefficients of z^N and higher powers. The characteristic equation cannot be put into the desired form and stability can be secured only at the expense of a very slow response.

The function of the stabilizer is to replace z^n in eqn. (39) by $z^n + A_1z^{n-1} + A_2z^{n-2} + \ldots + A_n$, where the A's are design constants which enable the coefficients of the higher powers of z to be controlled. To do this, the stabilizer must have the pulse transfer function $z^n/(z^n + A_1z^{n-1} + \ldots + A_n)$.

A comparison may be made between a system stabilized in this manner and a similar one without any feedback delay. The former will be distinguished where necessary by the subscript s. Let the characteristic equations be $R_s(z) = 0$ and $R(z) = 0$. The correction sequence transforms when the input is $F(z)$ are

$$C_s(z) = \frac{Q_s(z)(z^n + A_1z^{n-1} + \ldots + A_n)}{R_s(z)}F(z)$$

and $$C(z) = \frac{Q(z)}{R(z)}F(z)$$

The output of the stabilizer is

$$S_s(z) = \frac{C_s(z)z^n}{z^n + A_1z^{n-1} + \ldots + A_n} = \frac{z^nQ_s(z)}{R_s(z)}F(z) \quad . \quad (40)$$

Since $Q_s(z) = Q(z)$ the correction sequence $C(z)$ is identical with the stabilizer output $S_s(z)$ if $R(z) = z^{-n}R_s(z)$. This is the condition that the two characteristic equations shall have the same roots, apart from additional ones at $z = 0$. It can serve as a basis of design when it is desirable that the response time shall not be increased by the inclusion of delay in the feedback loop. The disadvantages are that overshoot and noise-power gain are increased and it may, in fact, be preferable to arrange for all the characteristic roots to be equal.

The p.t.f. of the stabilizer is always oscillatory. If, as is often the case, any roots of the equation $z^n + A_1z^{n-1} + A_2z^{n-2} + \ldots + A_n = 0$ lie outside the unit circle $|z| = 1$, the stabilizer is itself an unstable element. Feedback via the main loop, however, prevents oscillations from building up.

The physical realization of the stabilizer obviously depends upon the application. The general approach is to provide a subsidiary negative-feedback loop in which components with amplitudes $A_1, A_2, \ldots A_n$ are fed back with delays of $\tau, 2\tau, \ldots n\tau$ respectively, as in Fig. 9. An amplifier will be needed if any component exceeds unity.

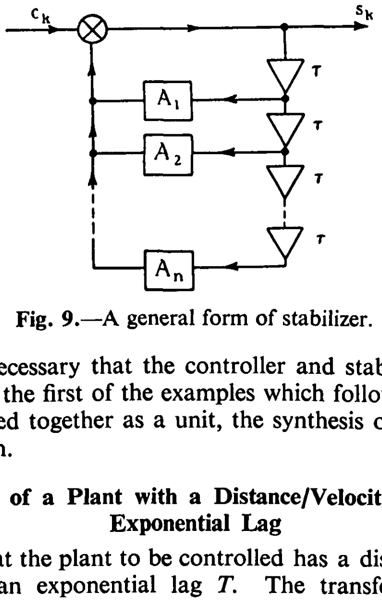


Fig. 9.—A general form of stabilizer.

It is not necessary that the controller and stabilizer shall be separate. In the first of the examples which follow they will, in fact, be treated together as a unit, the synthesis of which is the main problem.

(5.2) Control of a Plant with a Distance/Velocity Lag and an Exponential Lag

Assume that the plant to be controlled has a distance/velocity lag $\lambda\tau$ and an exponential lag T. The transfer function is $\frac{1}{1 + pT}\epsilon^{-\lambda p\tau}$. The problem consists of finding the best form of controller (including the function of stabilizer) and of choosing the best sampling interval τ. The operational instruction of the controller will be written as

$$\overline{\Theta(z)p^{-1}} = \overline{(1 - z^{-1})U(z)p^{-1}} \quad . \quad . \quad . \quad (41)$$

where $U(z)$ is the p.t.f. of the controller. The p^{-1} term is introduced in order that no impulse functions shall be applied directly to the plant. The sequence transform corresponding to $p^{-1}\frac{1}{1 + pT}\epsilon^{-\lambda p\tau}$ is now required. Assuming the sampling interval to be an integral fraction of the distance/velocity lag it is $\frac{1 - d}{(z - 1)(z - d)}z^{1-n}$, where $d = \epsilon^{-T/\tau}$ and $n = \lambda$. Use of the operational instruction enables a completely artificial sub-division to be created between plant and controller, leading to the equivalent system shown in Fig. 10. This equivalent system is easier to deal with because only samples are fed back.

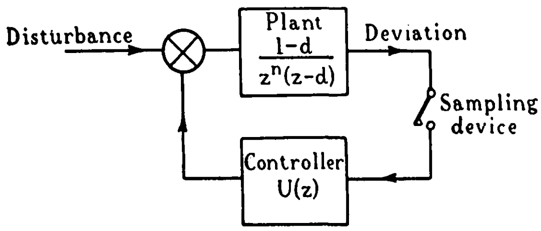


Fig. 10.—A system equivalent to that in Fig. 8.

From eqn. (15) the overall p.t.f. may be obtained

$$Y(z) = \frac{1 - d}{z^n(z - d) + (1 - d)U(z)} \quad . \quad . \quad . \quad (42)$$

Setting the denominator equal to zero gives the characteristic equation, the order of which obviously cannot be less than $n + 1$. The controller must therefore be such that it includes $n + 1$ design parameters. Bearing in mind the form of unit required for stabilization, it is easy to see that the simplest expression for the p.t.f. of the controller is

$$U(z) = \frac{Bz^n}{z^n + A_1z^{n-1} + A_2z^{n-2} + \ldots + A_n} \quad . \quad (43)$$

As a criterion of performance calculate the deviation of the output when a unit step of disturbance is applied to the plant at $t = 0$. This corresponds to a unit impulse applied to the equivalent system of Fig. 10. On account of the distance/velocity lag, $\lambda\tau$, there is no deviation of output until $t = n\tau$, when it changes in a manner governed only by the plant equation

$$g(t) = 1 - \epsilon^{-(t-\lambda\tau)/T}, \; t \geqslant \lambda\tau \quad . \quad . \quad . \quad (44)$$

This continues until the feedback becomes effective at $t = 2n\tau$ after which the behaviour depends upon the value of the staleness weighting constant a. Recovery is most rapid if $a = 0$, and the characteristic equation reduces to $z^{n+1} = 0$ if the design parameters of the controller are given the values

$$A_r = d^r \text{ and } B = d^{n+1}/1 - d$$

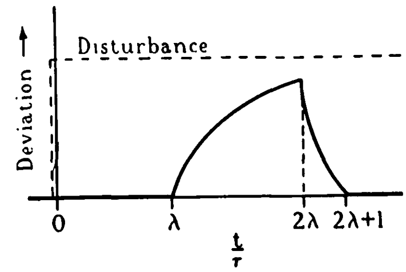


Fig. 11.—Deviation caused by unit-step disturbance.

The deviation function is as shown in Fig. 11. The deviation remains at zero from time $t = (2\lambda + 1)\tau$ onwards because the controller operates to cancel out all correction signals after the first. The special case of $\lambda \leqslant 1$ has been dealt with by Sartorius[11] and Oldenbourg[12] whose design criterion is the minimization of the deviation integral consequent upon a unit-step disturbance. This corresponds to the condition that the staleness weighting constant shall be zero.

The response of the plant to random noise is likewise minimized when $a = 0$, and is given by

$$M^2 = \frac{1-d}{1+d}(1 - d^{2n+2}) \quad . \quad . \quad . \quad . \quad (45)$$

The minimum is very flat, however, and increasing a up to $\frac{1}{2}(1 + d)$ causes M to increase by only a few per cent.

(5.3) A Predicting Servo-System

A servo mechanism provides a means of obtaining a representation of the input quantity to a much higher degree of accuracy than the individual components would yield in an open-loop system. Only the error-measuring device need be really accurate. The same principle may be applied to prediction by using a servo mechanism in the feedback path of which is a time-delay. The system operates by comparing the input at $t = k\tau$ with what the output was $\lambda\tau$ previously and by applying corrections to keep this difference as small as possible. A system with zero velocity-lag will predict according to a linear law, one with zero acceleration-lag according to a quadratic law and so on. That is to say, so long as the velocity or acceleration in these cases is constant there will be no error in prediction. This type of predictor operates well on exponential functions since the higher derivatives change fairly slowly.

$$W(z, m) = \frac{B_1\tau^3}{6z^n}\left[\frac{z^2+4z+1}{(z-1)^3} + \frac{3m(z+1)}{(z-1)^2} + \frac{3m^2}{z-1} + m^3\right]$$
$$+ \frac{\tau^2(Bz_2 + B_2')}{2z^n}\left[\frac{z+1}{(z-1)^3} + \frac{2m}{(z-1)^2} + \frac{m^2}{z-1}\right]$$
$$+ \frac{\tau(B_3z + B_3')}{z^n}\left[\frac{1}{(z-1)^2} + \frac{m}{z-1}\right] \quad . \quad . \quad . \quad (47)$$

where $n = \lambda + m$.

If the characteristic equation is identified with $z^{n+3} = 0$ the prediction will depend only on the last three known samples. This case will be considered first. The stabilizer is, as usual, oscillatory and operates to correct the delayed feedback. The design parameters of the stabilizer may be determined by equating the coefficients of z^{n+2} and lower powers to zero in the characteristic equation $1 + S(z)W(z, m) = 0$. The stabilizer p.t.f which is then obtained is

$$S(z) = \frac{z^n}{z^n + (1+2)z^{n-1} + (1+2+3)z^{n-2} + \ldots + \frac{1}{2}n(n+1)z + c}$$

where c is a constant . . . . . (48)

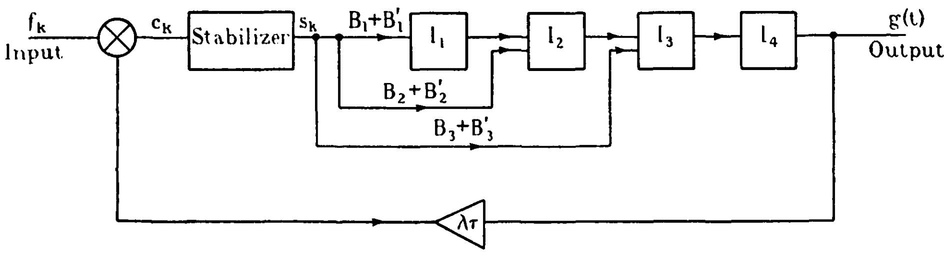


Fig. 12.—A predicting servo system based upon integrators.

Fig. 12 shows the system to be discussed. It is based on the type suggested by Porter and Stoneman, the principal components being the integrators. In addition the stabilizer is required. Six design parameters have been included, and these specify the proportions of the control impulse to be applied to the various stages of integration. The application of impulses B_1', B_2' and B_3' is delayed by the sampling interval τ.

The transfer function relating to the forward path is

$$(B_1 + B_1'\epsilon^{-p\tau})p^{-4} + (B_2 + B_2'\epsilon^{-p\tau})p^{-3} + (B_3 + B_3'\epsilon^{-p\tau})p^{-2}$$

The corresponding p.t.f. is

$$W(z) = \frac{\tau^3(B_1z + B_1')(z^2+4z+1)}{6(z-1)^4}$$
$$+ \frac{\tau^2(B_2z + B_2')(z+1)}{2(z-1)^3} + \frac{\tau(B_3z + B_3')}{(z-1)^2} \quad . \quad (46)$$

It is apparent that the order of $W(z)$ can be decreased by one and the number of design parameters decreased also by one if $B_1 = -B_1'$. That is to say, the first integrator in Fig. 12 operates as a clamp. This simplification imposes no limitations and will be adopted. The first term of eqn. (46) reduces to

$$B_1\tau^3(z^2 + 4z + 1)/6(z-1)^3$$

The loop p.t.f. involves the delay $\lambda\tau$, in this case not necessarily an integral number of sampling periods.

The relations to be obeyed by the other design parameters are

$$\left.\begin{aligned}
&B_1\tau^3 + B_2\tau^2 + B_2'\tau^2 = 1\\
&c + \tfrac{1}{6}B_1\tau^3m^3 + \tfrac{1}{2}B_2\tau^2m^2 + B_3\tau m = \tfrac{1}{2}(n+1)(n+2)\\
&c + \tfrac{1}{6}B_1\tau^3\delta^3 + \tfrac{1}{2}B_2'\tau^2\delta^2 - B_3'\tau\delta = 0\\
&3c - \tfrac{1}{6}B_1\tau^3(1 + 3m + 3m^2 - 3m^3) - \tfrac{1}{2}B_2(1 + 2m - 2m^2)\\
&\quad - \tfrac{1}{2}B_2'\tau^2m^2 - B_3\tau(1-2m) - B_3'\tau = n(n+2)
\end{aligned}\right\} (49)$$

Note that there are only four of these equations, but that six constants, including c, are to be determined. Two of the constants are therefore redundant and may be chosen arbitrarily. The output function between sampling instants depends upon this choice but the sequence obtained at the sampling instants is independent of it. This does not apply generally but is so in this example because only ideal integrators are used in the system.

The overall p.t.f. [see eqn. (16)] is

$$Y(z) = \tfrac{1}{2}(\lambda+2)(\lambda+3)z^{-1} - (\lambda+1)(\lambda+3)z^{-2}$$
$$+ \tfrac{1}{2}(\lambda+1)(\lambda+2)z^{-3} \quad . \quad (50)$$

and the response to a unit-step input is the sequence

$$0, \ \tfrac{1}{2}(\lambda+2)(\lambda+3), \ -\tfrac{1}{2}\lambda(\lambda+3), \ 1, \ 1, \ \text{etc.}$$

This is shown in Fig. 13, in which the response is delayed by $\lambda\tau$ in order to illustrate the quadratic law of prediction. The overshoot peak is predicted by passing a parabola through the last three known values, those when t is -2τ, $-\tau$ and 0. The

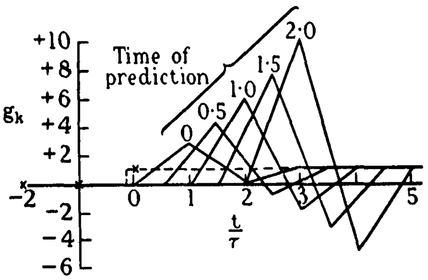


Fig. 13.—The output sequence illustrated for quadratic predictor (unit-step input).

undershoot peak is obtained by passing a parabola through the points when t is $-\tau$, 0 and τ. Eqn. (50) enables the noise-power gain to be calculated. It is

$$M^2 = 3\lambda^4 + 24\lambda^3 + 69\lambda^2 + 84\lambda + 38 \quad . \quad . \quad (51)$$

As one would expect, the noise increases rapidly with the time of prediction, and some degree of smoothing is required if it is not to be excessive, i.e. the predicted values must depend not merely on the last three known values but upon many which precede them. This is accomplished by introducing the staleness weighting constant a and identifying the characteristic equation of the system with $(z - a)^{n+3} = 0$. Expressions for the performance are complicated by the additional parameter, but in the particular case of $\lambda = 1$, corresponding to prediction over one sampling period, the p.t.f. is

$$Y(z) = 1 - \frac{(z + 3 - 4a)(z - 1)^3}{(z - a)^4}$$

and the output sequence produced by a unit-step input is

$$g_k = 1 - a^k\left[1 + k\left(\frac{1-a}{a}\right) - \tfrac{5}{2}k(k-1)\left(\frac{1-a}{a}\right)^2 + \tfrac{1}{2}k(k-1)(k+2)\left(\frac{1-a}{a}\right)^3\right] \quad . \quad . \quad . \quad (52)$$

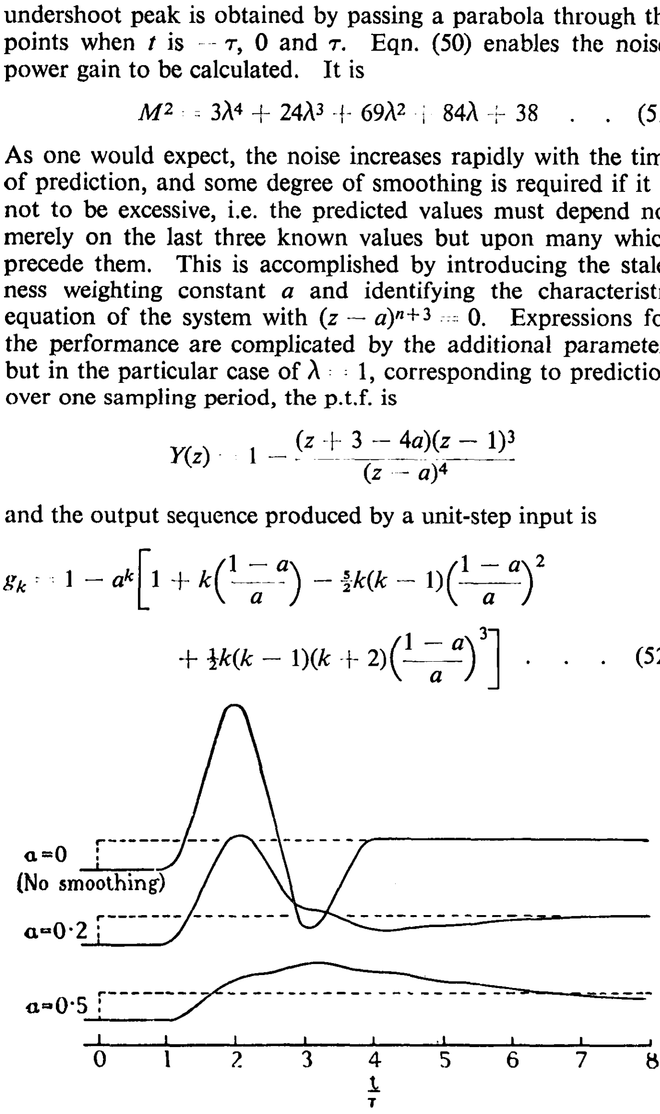


Fig. 14.—The output function of a quadratic predictor with various degrees of smoothing (unit-step input).

The output between sampling instants depends upon the values allotted to the two redundant control constants. An important case occurs when $B_3 = B_3' = 0$, for then the output function will have a continuous first derivative. The curves of Fig. 14, illustrating eqn. (52), have been calculated on this assumption. They also give a good idea of the effect of the smoothing constant a.

(6) CONCLUSION

The principal object of the paper will have been served if it attracts attention to the usefulness of the sequence-transform theory and, more particularly, of the pulse transfer function, in dealing with problems involving sampling. Two new concepts have been introduced—the use of the operational instruction as an aid to synthesis and the method for stabilizing systems with delayed feedback. The theory is reasonably straightforward so long as the time-delays included are a discrete number of sampling periods. Delays of fractional sampling periods involve considerable complication and a general treatment does not seem to be practicable.

A pulsed servo mechanism receives less information upon which to operate than does its continuous counterpart, and one would therefore expect its performance to be inferior. This is not necessarily the case in practice, however, since pulsed servo-systems of high order may be stabilized without the need for generating smooth derivatives. Furthermore, there are some applications in which the available data are themselves intermittent and the problem then is to make the best use of them. The pulsed servo system is less efficient than the continuous one as a smoothing device and, whenever circumstances permit, any smoothing should be carried out before the sampling operation. This would apply particularly to a communication system where, for reasons of economy in channel utilization, samples of the information are transmitted only at intervals.

(7) ACKNOWLEDGMENTS

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(11) "Automatic and Manual Control" (Butterworth, 1952), p. 421.
(12) *Ibid.*, p. 435.

(9) APPENDIX

The quantities given in columns (ii) and (iii) are transform pairs. In column (i) is the transfer function $Y(p)$ corresponding to the pulse transfer function $W(z)$ of column (iii). The effect of a delay $\delta\tau$ is to multiply $Y(p)$ by $\epsilon^{-p\delta\tau}$, and the pulse transfer function corresponding to this is given in column (iv) as $W(z, m)$ where $m = 1 - \delta$. The sampling period τ has throughout been taken as unity. This will cause no difficulty if all time constants and response times are calculated in terms of τ.

(i) $Y(p)$	(ii) w_k	(iii) $W(z)$. Eqn. (4)	(iv) $W(z, m)$. Eqn. (13)
$\frac{1}{p}$	1	$\frac{z}{z-1}$	$\frac{1}{z-1}$ $[(\delta > 0)]$
$\frac{1}{p^2}$	k	$\frac{z}{(z-1)^2}$	$\frac{1}{(z-1)^2} + \frac{m}{z-1}$, $m = 1 - \delta$
$\frac{1}{p^3}$	$\frac{1}{2}k^2$	$\frac{1}{2}(z+1)\frac{z}{(z-1)^3}$	$\frac{1}{(z-1)^3} + \frac{1+2m}{2(z-1)^2} + \frac{m^2}{2(z-1)}$
$\frac{1}{p^{n+1}}$	$\frac{k^n}{n!}$	$D_n(z)\frac{z}{(z-1)^{n+1}}$ $D_0 = D_1 = 1$ $D_2 = (z+1)/2!$ $D_3 = (z + 4z + 1)/3!$ $D_4 = (z + 11z + 11z + 1)/4!$ $D_5 = (z + 26z + 66z + 26z + 1)/5!$	$\sum_{r=0}^{n} \frac{m^{n-r}D_r(z)}{(n-r)!(z-1)^{r+1}}$ $= \frac{B_n(z, m)}{(z-1)^{n+1}}$
$\frac{1}{p+\alpha}$	d^k, $d = \epsilon^{-\alpha}$	$\frac{z}{z-d}$, $d = \epsilon^{-\alpha}$	$\frac{d^m}{z-d}$, $(\delta > 0)$
$\frac{1}{(p+\alpha)^2}$	kd^k	$\frac{zd}{(z-d)^2}$	$d^m\left[\frac{d + m(z-d)}{(z-d)^2}\right]$
$\frac{1}{(p+\alpha)^3}$	$\frac{1}{2}k^2d^k$	$\frac{1}{2}(z+d)\frac{zd}{(z-d)^3}$	$d^m\left[\frac{z+d}{2(z-1)^3} + \frac{m}{(z-d)^2} + \frac{m^2}{2(z-d)}\right]$
$\frac{1}{(p+\alpha)^{n+1}}$	$\frac{k^nd^k}{n!}$	$d^nD_n(z/d)\frac{z}{(z-d)^{n+1}}$	$d^m\sum_{r=0}^{n}\frac{m^{n-r}d^{r-1}D_r(z/d)}{(n-r)!(z-d)^{r+1}}$ $= d^{n+m}\frac{B_n(z/d, m)}{(z-d)^{n+1}}$

$$D_n(z/d) = \begin{vmatrix} 1 & 1 - z/d & 0 & . & . & 0 \\ \frac{1}{2!} & 1 & 1 - z/d & . & . & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & . & . & 0 \\ . & . & . & . & . & . \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & . & . & 1 \end{vmatrix}$$

n rows and columns

$$B_n(z/d, m) = \begin{vmatrix} 1 & 1 - z/d & 0 & . & . & 0 \\ m & 1 & 1 - z/d & . & . & 0 \\ \frac{m^2}{2!} & \frac{1}{2!} & 1 & . & . & 0 \\ . & . & . & . & . & . \\ \frac{m^n}{n!} & \frac{1}{n!} & \frac{1}{(n-1)!} & . & . & 1 \end{vmatrix}$$

$n + 1$ rows and columns

(i) $Y(p)$	(ii) w_k	(iii) $W(z)$. Eqn. (4)	(iv) $W(z, m)$. Eqn. (13)
$\dfrac{\alpha}{p(p+\alpha)}$	$1 - d^k$	$\dfrac{(1-d)z}{(z-1)(z-d)}$	$\dfrac{1}{z-1} - \dfrac{d^m}{z-d}$
$\dfrac{\alpha}{p^2(p+\alpha)}$	$\dfrac{1}{\alpha}(\alpha k - 1 + d^k)$	$\dfrac{z}{(z-1)^2} - \dfrac{(1-d)z}{\alpha(z-1)(z-d)}$	$\dfrac{1}{(z-1)^2} + \dfrac{m - \alpha^{-1}}{z-1} + \dfrac{d^m}{\alpha(z-d)}$
$\dfrac{\alpha}{p^3(p+\alpha)}$	$\dfrac{1}{\alpha^2}(\tfrac{1}{2}\alpha^2k^2 - \alpha k + 1 - d^k)$	$\dfrac{(z+1)z}{2(z-1)^3} - \dfrac{z}{\alpha(z-1)^2} + \dfrac{(1-d)z}{\alpha^2(z-1)(z-d)}$	$\dfrac{B_2}{(z-1)^3} - \dfrac{B_1}{\alpha(z-1)^2} + \dfrac{1}{\alpha^2(z-1)} - \dfrac{d^m}{\alpha^2(z-d)}$
$\dfrac{\alpha}{p^{n+1}(p+\alpha)}$		$\sum\limits_{r=1}^{n} \dfrac{zD_r(z)}{(-\alpha)^{n-r}(z-1)^{r+1}} + \dfrac{(1-d)z}{(-\alpha)^n(z-1)(z-d)}$ $= \dfrac{zA_n}{(z-1)^{n+1}} - \dfrac{z}{(-\alpha)^n(z-d)}$	$\sum\limits_{r=0}^{n} \dfrac{B_r(z,m)}{(-\alpha)^{n-r}(z-1)^{r+1}} - \dfrac{d^m}{(-\alpha)^n(z-d)}$ $= \dfrac{C_n(z,m)}{(z-1)^{n+1}} - \dfrac{d^m}{(-\alpha)^n(z-d)}$

$$A_n(z) = \begin{vmatrix} 1 & 1-z & 0 & \cdot & \cdot & 0 \\ (-\alpha)^{-1} & 1 & 1-z & \cdot & \cdot & 0 \\ (-\alpha)^{-2} & \dfrac{1}{2!} & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (-\alpha)^{-n} & \dfrac{1}{n!} & \dfrac{1}{(n-1)!} & \cdot & \cdot & 1 \end{vmatrix}$$

$n + 1$ rows and columns

$$C_n(z, m) = \begin{vmatrix} 1 & 1-z & 0 & \cdot & \cdot & 0 & 0 \\ m & 1 & 1-z & \cdot & \cdot & 0 & 0 \\ \dfrac{m^2}{2!} & \dfrac{1}{2!} & 1 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \dfrac{m^n}{n!} & \dfrac{1}{n!} & \dfrac{1}{(n-1)!} & \cdot & \cdot & 1 & 1-z \\ 0 & (-\alpha)^{-n} & (-\alpha)^{-(n-1)} & \cdot & \cdot & (-\alpha)^{-1} & 1 \end{vmatrix}$$

$n + 2$ rows and columns

(i) $Y(p)$	(ii) w_k	(iii) $W(z)$. Eqn. (4)	(iv) $W(z, m)$. Eqn. (13)
$\dfrac{p}{(p+\alpha)^2}$	$d^k(1 - \alpha k)$	$\dfrac{z^2 - zd(1+\alpha)}{(z-d)^2}$	$d^m\left[\dfrac{1-\alpha m}{z-d} - \dfrac{\alpha d}{(z-d)^2}\right], \quad \delta > 0$
$\dfrac{\alpha^2}{p(p+\alpha)^2}$	$1 - d^k(1 + \alpha k)$	$\dfrac{1}{z-1} - \dfrac{zd(1-\alpha) - d^2}{(z-d)^2}$	$\dfrac{1}{z-1} - d^m\left[\dfrac{1+\alpha m}{z-d} + \dfrac{\alpha d}{(z-d)^2}\right]$
$\dfrac{\alpha^3}{p^2(p+\alpha)^2}$	$2(d^k - 1) + \alpha k(d^k + 1)$	$\dfrac{z(\alpha-2)+2}{(z-1)^2} + \dfrac{zd(\alpha+2) - 2d^2}{(z-d)^2}$	$\dfrac{z(\alpha m - 2) + (\alpha\delta + 2)}{(z-1)^2} + d^m\left[\dfrac{z(\alpha m + 2) + d(\alpha\delta - 2)}{(z-d)^2}\right]$
$\dfrac{\gamma - \alpha}{(p+\alpha)(p+\gamma)}$	$d^k - c^k$ $d = \epsilon^{-\alpha} : c = \epsilon^{-\gamma}$	$\dfrac{z(d-c)}{(z-d)(z-c)}$	$\dfrac{z(d^m - c^m) + dc^m - cd^m}{(z-d)(z-c)}$
$\dfrac{(\gamma - \alpha)p}{(p+\alpha)(p+\gamma)}$	$\gamma c^k - \alpha d^k$	$\dfrac{z^2(\gamma - \alpha) - z(\gamma d - \alpha c)}{(z-d)(z-c)}$	$\dfrac{z(\gamma c^m - \alpha d^m) + \alpha c d^m - \gamma d c^m}{(z-d)(z-c)}, \quad \delta > 0$

(i)	(ii)	(iii)	(iv)
$Y(p)$	w_k	$W(z)$. Eqn. (4)	$W(z, m)$. Eqn. (13)
$\dfrac{\beta}{p^2 + \beta^2}$ β is any real constant	$\sin k\beta$	$\dfrac{z \sin \beta}{z^2 - 2z \cos \beta + 1}$	$\dfrac{z \sin m\beta + \sin \delta\beta}{z^2 - 2z \cos \beta + 1}$
$\dfrac{p}{p^2 + \beta^2}$	$\cos k\beta$	$\dfrac{z^2 - z \cos \beta}{z^2 - 2z \cos \beta + 1}$	$\dfrac{z \cos m\beta - \cos \delta\beta}{z^2 - 2z \cos \beta + 1}, \quad \delta > 0$
$\dfrac{\beta}{p^2 - \beta^2}$	$\sinh k\beta$	$\dfrac{z \sinh \beta}{z^2 - 2z \cosh \beta + 1}$	$\dfrac{z \sinh m\beta + \sinh \delta\beta}{z^2 - 2z \cosh \beta + 1}$
$\dfrac{p}{p^2 - \beta^2}$	$\cosh k\beta$	$\dfrac{z^2 - z \cosh \beta}{z^2 - 2z \cosh \beta + 1}$	$\dfrac{z \cosh m\beta - \cosh \delta\beta}{z^2 - 2z \cosh \beta + 1}, \quad \delta > 0$
$\dfrac{p + \alpha}{(p + \alpha)^2 + \beta^2}$	$d^k \cos k\beta$	$\dfrac{z^2 - zd \cos \beta}{z^2 - 2zd \cos \beta + d^2}$	$d^m\left(\dfrac{z \cos m\beta - d \cos \delta\beta}{z^2 - 2zd \cos \beta + d^2}\right), \quad \delta > 0$
$\dfrac{p + a_0}{(p + \alpha)^2 + \pi^2}$	$(-d)^k$	$\dfrac{z}{z + d}$	
$\dfrac{\beta}{(p + \alpha)^2 + \beta^2}$	$d^k \sin k\beta$	$\dfrac{zd \sin \beta}{z^2 - 2zd \cos \beta + d^2}$	$d^m\left(\dfrac{z \sin m\beta + d \sin \delta\beta}{z^2 - 2zd \cos \beta + d^2}\right)$
$\dfrac{\alpha\gamma}{p(p + \alpha)(p + \gamma)}$	$1 - \dfrac{\gamma d^k - \alpha c^k}{\gamma - \alpha}$	$\dfrac{1}{z - 1} - \dfrac{z(\gamma d - \alpha c) - dc(\gamma - \alpha)}{(\gamma - \alpha)(z - d)(z - c)}$ $= \dfrac{\gamma(1 - d)(z^2 - zc) - \alpha(1 - c)(z^2 - zd)}{(\gamma - \alpha)(z - 1)(z - d)(z - c)}$	$\dfrac{1}{z - 1} - \dfrac{z(\gamma d^m - \alpha c^m) + \alpha d c^m - \gamma c d^m}{(\gamma - \alpha)(z - d)(z - c)}$
$\dfrac{\alpha^2\gamma^2}{p^2(p + \alpha)(p + \gamma)}$	$\alpha\gamma k - (\alpha + \gamma)$ $+ \dfrac{\gamma^2 d^k - \alpha^2 c^k}{\gamma - \alpha}$	$\dfrac{z(\alpha\gamma - \alpha - \gamma) + (\alpha + \gamma)}{(z - 1)^2}$ $+ \dfrac{z(\gamma^2 d - \alpha^2 c) - cd(\gamma^2 - \alpha^2)}{(\gamma - \alpha)(z - d)(z - c)}$	$\dfrac{z(m\alpha\gamma - \alpha - \gamma) + (\delta\alpha\gamma + \alpha + \gamma)}{(z - 1)^2}$ $+ \dfrac{z(\gamma^2 d^m - \alpha^2 c^m) + \alpha^2 d c^m - \gamma^2 c d^m}{(\gamma - \alpha)(z - d)(z - c)}$
$\dfrac{\pi}{p^2 + \pi^2}$	0	0	$\dfrac{\sin \delta\pi}{z + 1}$
$\dfrac{p}{p^2 + \pi^2}$	$(-1)^k$	$\dfrac{z}{z + 1}$	$-\dfrac{\cos \delta\pi}{z + 1}, \quad \delta > 0$
$\dfrac{\pi}{(p + \alpha)^2 + \pi^2}$	0	0	$d^m\left[\dfrac{z \sin m\pi + d \sin \delta\pi}{(z + d)^2}\right]$
$\dfrac{p + \alpha}{(p + \alpha)^2 + \pi^2}$ α is any real constant	$(-d)^k$	$\dfrac{z}{z + d}$	$d^m\left[\dfrac{z \cos m\pi - d \cos \delta\pi}{(z + d)^2}\right]$

(i) $Y(p)$	(ii) w_k	(iii) $W(z)$. Eqn. (4)	(iv) $W(z, m)$. Eqn. (13)
$\dfrac{p + a_0}{(p+\alpha)^2 + \beta^2}$	$d^k \sec\phi \cos(k\beta + \phi)$ $\tan\phi = \dfrac{\alpha - a_0}{\beta}$	$\dfrac{z^2 - zd\sec\phi\cos(\beta - \phi)}{z^2 - 2zd\cos\beta + d^2}$	$d^m \sec\phi \dfrac{z\cos(m\beta + \phi) - d\cos(\delta\beta - \phi)}{z^2 - 2zd\cos\beta + d^2}$ $\delta > 0$
$\dfrac{\alpha^2 + \beta^2}{p[(p+\alpha)^2 + \beta^2]}$	$1 - d^k \sec\phi\cos(k\beta + \phi)$ $\tan\phi = \dfrac{-\alpha}{\beta}$	$\dfrac{1}{z-1} - \dfrac{zd\sec\phi\cos(\beta + \phi) - d^2}{z^2 - 2zd\cos\beta + d^2}$	$\dfrac{1}{z-1} - d^m \sec\phi \dfrac{z\cos(m\beta + \phi) - d\cos(\delta\beta - \phi)}{z^2 - 2zd\cos\beta + d^2}$
$\dfrac{(\alpha^2 + \beta^2)(p + a_0)}{p[(p+\alpha)^2 + \beta^2]}$	$a_0[1 - d^k \sec\phi\cos(k\beta + \phi)]$ $\tan\phi = \dfrac{\alpha^2 + \beta^2 - \alpha a_0}{a_0\beta}$	$a_0\left[\dfrac{1}{z-1} - \dfrac{zd\sec\phi\cos(\beta + \phi) - d^2}{z^2 - 2zd\cos\beta + d^2}\right]$ $= a_0 \dfrac{z^2[1 - d\sec\phi\cos(\beta + \phi)] + zd[d - \sec\phi\cos(\beta - \phi)]}{(z-1)(z^2 - 2zd\cos\beta + d^2)}$	$a_0\left[\dfrac{1}{z-1} - d^m \sec\phi \dfrac{z\cos(m\beta + \phi) - d\cos(\delta\beta - \phi)}{z^2 - 2zd\cos\beta + d^2}\right]$
$\dfrac{\beta_0^4}{p^2[(p+\alpha)^2 + \beta^2]}$ $\beta_0^2 = \alpha^2 + \beta^2$	$-2\alpha + \beta_0^2 k$ $+ 2\alpha d^k \sec\phi\cos(k\beta + \phi)$ $\tan\phi = \dfrac{\beta^2 - \alpha^2}{2\alpha\beta}$	$\dfrac{z(\beta_0^2 - 2\alpha) + 2\alpha}{(z-1)^2} + 2\alpha d \dfrac{z\sec\phi\cos(\beta + \phi) - d}{z^2 - 2zd\cos\beta + d^2}$ $= 2\alpha d \dfrac{z^3\sec\phi\cos(\beta + \phi) + 2z^2\tan\phi\sin\beta - z\sec\phi\cos(\beta - \phi)}{(z-1)^2(z^2 - 2zd\cos\beta + d^2)}$	$\dfrac{z(m\beta_0^2 - 2\alpha) + \delta\beta_0^2 + 2\alpha}{(z-1)^2}$ $+ 2\alpha d^m \sec\phi \dfrac{z\cos(m\beta + \phi) - d\cos(\delta\beta - \phi)}{z^2 - 2zd\cos\beta + d^2}$
$\dfrac{\beta_0^4(p + a_0)}{p^2[(p+\alpha)^2 + \beta^2]}$	$\beta_1^2 + \alpha_0\beta_0^2 k$ $- \beta_1^2 d^k \sec\phi\cos(k\beta + \phi)$	$\dfrac{z(\beta_1^2 + a_0\beta_0^2) - \beta_1^2}{(z-1)^2}$ $- \beta_1^2\left[\dfrac{zd\sec\phi\cos(\beta + \phi) - d^2}{z^2 - 2zd\cos\beta + d^2}\right]$	$\dfrac{z(\beta_1^2 + ma_0\beta_0^2) - \beta_1^2 + \delta a_0\beta_0^2}{(z-1)^2}$ $- \beta_1^2 d^m \sec\phi \dfrac{z\cos(m\beta + \phi) - d\cos(\delta\beta - \phi)}{z^2 - 2zd\cos\beta + d^2}$

$$\beta_0^2 = \alpha^2 + \beta^2; \quad \beta_1^2 = \beta_0^2 - 2\alpha a_0; \quad \tan\phi = -\frac{\alpha\beta_1^2 + a_0\beta_0^2}{\beta\beta_1^2}; \quad \text{when } a_0 = \alpha,\ \tan\phi = -\frac{2\alpha\beta}{\beta^2 - \alpha^2}$$