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ZEROS OF SAMPLED SYSTEMS

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The zeros of the discrete time system obtained when sampling a continuous time system are explored. Theorems for the limiting zeros for large and small sampling periods are given. A condition which guarantees that the sampled system only has stable zeros is also presented.

1. Introduction

Poles and zeros are fundamental properties of linear time-invariant systems. The poles reflect the internal couplings in the system and thus its autonomous behaviour. The zeros reflect the way the internal variables are coupled to the inputs and the outputs. It is well known that unstable zeros limit the performance that can be achieved when controlling a system. Many techniques for design of control systems are based on the cancellation of process zeros. Such methods will not work when the process has unstable zeros. Several of the adaptive algorithms that are currently investigated belong to this category [1-5].

When a continuous time system is sampled the poles p are transformed as

$$p \to e^{ph} \tag{1.1}$$

where h is the sampling period. The transformation (1.1) maps the left half plane onto the unit disc. This means that stability is preserved. There is unfortunately no simple transformation which shows how the zeros of a continuous time system are transformed by sampling. For example, it is not true that a continuous time system with zeros in the left half plane will transform to a sampled system with zeros inside the unit disc or vice versa. Design methods for sampled systems which are based on cancellation of process zeros can thus work well for certain sampling periods and fail for other.

The purpose of this paper is to characterize the zeros of a system obtained by sampling a continuous time system. The paper is organized as follows. Some preliminaries are given in Section 2. The main results are limit theorems, which give the zero locations for small and large sampling periods. These results are given in Section 3. It is shown that all continuous time systems with pole excess larger than 2 will always give sampled systems with unstable zeros provided that the sampling period is sufficiently small. Contrary to popular belief, discrete time systems which have unstable zeros are thus quite common [6].

The property that a system has stable zeros is crucial for many design procedures which are based on pole zero cancellation. Criteria which ensure stable zeros are discussed in Section 4. Zeros of systems obtained by sampling continuous time systems with time delays or transfer functions, which are not strictly proper, are briefly discussed in Section 5.

2. Preliminaries

Consider a system composed of a sample and hold circuit followed by a linear time invariant continuous time system with a strictly proper rational transfer function G(s). See Fig. 1.

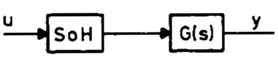

Fig. 1. Block diagram of the system.

The input output behaviour of the system at times which are synchronized with the internal clock of the sample and hold circuit can be characterized by the pulse transfer function H(z).

Representation of the pulse transfer function

To obtain the main results it is useful to have different representations of the pulse transfer function. Such representations will be given in this section. These representations are minor modifications of well known results for sampled data systems [7-10]. We have

$$H(z) = (1 - z^{-1}) \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{sh}}{z - e^{sh}} \cdot \frac{G(s)}{s} \, ds \tag{2.1}$$

where h is the sampling interval and where the real number γ is such that all poles of $G(s)/s$ have real parts less than γ.

The formula (2.1) is derived by using the inverse Laplace transform to calculate the z-transform of the step response and dividing it by the z-transform of a step.

Since the transfer function G is strictly proper, the integration path can be closed and the integral can be evaluated using residue calculus. Closing the integration path with a large semicircle on the left gives

$$H(z) = (1 - z^{-1}) \sum_i \frac{e^{p_i h}}{z - e^{p_i h}} \operatorname{Res}_{p_i} \frac{1}{s} G(s) \tag{2.2}$$

where p_i are the poles of $G(s)/s$. Equation (2.2) only holds for single poles. This is, however, no serious restriction, since the case of multiple poles is easily handled by the standard continuity argument.

If the integration path is closed by a semicircle to the right the following equation is obtained:

$$H(z) = (1 - z^{-1}) \sum_{k=-\infty}^{\infty} \frac{G[(\log z + 2\pi i k)/h]}{\log z + 2\pi i k}. \tag{2.3}$$

The representation (2.1) holds primarily for z such that $|z| > \exp(\gamma h)$. The representation can, however, be extended by analytic continuation. For transfer functions with finitely many poles equation (2.2) defines a rational function. This function can thus be extended to a function which is analytic in the whole plane except at finitely many poles $z_i = \exp(p_i h)$. From the formula (2.1) it may appear that H has a pole at $z = 0$. Using a state space representation it can, however, be shown that H(0) is finite. It also follows that

$$H(1) = G(0)$$

which may be infinite. When G(s) is rational and strictly proper it also follows that the terms in the series (2.3) go to zero as k^{-2} for large k. The series is thus uniformly convergent, and (2.3) is analytic in the whole plane except at a finite number of poles.

A special case

The pulse transfer function of a special system plays a crucial role in the argument. We have

Lemma 1. The pulse transfer function corresponding to $G(s) = s^{-n}$ is given by

$$H(z) = \frac{h^n}{n!} \cdot \frac{B_n(z)}{(z-1)^n} \tag{2.4}$$

where

$$B_n(z) = b_1^n z^{n-1} + b_2^n z^{n-2} + \ldots + b_n^n \tag{2.5}$$

and

$$b_k^n = \sum_{\ell=1}^{k} (-1)^{k-\ell} \ell^n \binom{n+1}{k-\ell}, \quad k = 1, \ldots, n. \tag{2.6}$$

Proof. It follows from (2.2) that the pulse transfer function is rational of the form

$$H(z) = \frac{b_1 z^{n-1} + \ldots + b_n}{a_0 z^n + \ldots + a_n}. \tag{2.7}$$

Since $G(s) = s^{-n}$ it follows from (1.1) that

$$a_0 z^n + \ldots + a_n = (z-1)^n.$$

Hence

$$a_k = \binom{n}{k} (-1)^k. \tag{2.8}$$

Let the input signal be a step function. The output is then

$$y(t) = \frac{t^n}{n!}. \tag{2.9}$$

When the input is a step it follows from (2.7) that

$$\sum_{\ell=1}^{k} a_{k-\ell}\, y(\ell h) = \sum_{\ell=1}^{k} b_\ell, \quad k = 1, \ldots, n.$$

Insertion of (2.8) and (2.9) then gives

$$b_k = \frac{(kh)^n}{n!} + \sum_{\ell=1}^{k-1} \left\{ \binom{n}{k-\ell} (-1)^{k-\ell} - \binom{n}{k-\ell-1} (-1)^{k-\ell-1} \right\} \frac{(\ell h)^n}{n!}$$

which implies (2.6). □

Remark. Notice that the coefficients can be computed recursively as follows:

$$b_k^n = k\, b_k^{n-1} + (n-k+1)\, b_{k-1}^{n-1}, \quad k = 2, \ldots, n-1$$
$$b_1^n = b_n^n = 1. \quad □ \tag{2.10}$$

The polynomials B_n are listed below for a few values of n.

$B_1(z) = 1$

$B_2(z) = z + 1$

$B_3(z) = z^2 + 4z + 1$

$B_4(z) = z^3 + 11z^2 + 11z + 1$

$B_5(z) = z^4 + 26z^3 + 66z^2 + 26z + 1$

$B_6(z) = z^5 + 57z^4 + 302z^3 + 302z^2 + 57z + 1.$

The polynomials B_n have zeros outside or on the unit circle for $n \geq 2$. The unstable zeros are listed below:

n	Unstable zero of B_n
2	-1
3	-3.732
4	-1, -9.899
5	-2.322, -23.20
6	-1, -4.542, -51.22
7	-1.868, -8.160, -109.3
8	-1, -3.138, -13.96, -228.5
9	-1.645, -4.957, -23.14, -471,4

It thus follows that there are continuous time systems with stable zeros such that the corresponding pulse transfer function has unstable zeros.

Example 1. The pulse transfer function

$$G(s) = \frac{1}{s^3}$$

has a stable inverse, but its corresponding pulse transfer function

$$H(z) = \frac{h^3}{6} \cdot \frac{z^2 + 4z + 1}{(z-1)^3}$$

has an unstable zero $z = -3.732$. ■

3. Limiting zeros

If the transfer function G is rational it follows from (2.2) that H is also a rational function. There are in general no simple closed form expressions for the zeros of H. The limiting cases for small or large sampling periods can, however, be characterized. We have

Theorem 1. Let G be a rational function

$$G(s) = K \frac{(s-z_1)(s-z_2) \ldots (s-z_m)}{(s-p_1)(s-p_2) \ldots (s-p_n)} \tag{3.1}$$

and H the corresponding pulse transfer function. Assume that $m < n$. Then as the sampling period $h \to 0$, m zeros of H go to 1 as $\exp(z_i h)$ and the remaining $n-m-1$ zeros of H go to the zeros of $B_{n-m}(z)$, where $B_k(z)$ is the polynomial defined by Lemma 1.

Proof. The major steps in the proof are given below. It is an exercise in elementary analysis to fill in the details. Consider the representation (2.1), i.e.

$$H(z) = (1 - z^{-1}) \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{sh}}{z - e^{sh}} G(s) \frac{ds}{s} =$$

$$= (1 - z^{-1}) \frac{1}{2\pi i} \int_{\gamma h - i\infty}^{\gamma h - i\infty} \frac{e^w}{z - e^w} G\left(\frac{w}{h}\right) \frac{dw}{w}$$

where the last equality is obtained by the variable substitution $w = sh$. It follows from (3.1) that

$$G\left(\frac{w}{h}\right) = K \cdot \left(\frac{h}{w}\right)^{n-m} \frac{(1 - z_1 h / w) \ldots (1 - z_m h / w)}{(1 - p_1 h / w) \ldots (1 - p_n h / w)}.$$

Hence

$$\lim_{h \to 0} h^{m-n} H(z) = (1 - z^{-1}) \frac{1}{2\pi i} \int_\Gamma \frac{e^w}{z - e^w} \cdot \frac{K}{w^{n-m}} \cdot \frac{dw}{w} =$$

$$= K \frac{B_{n-m}(z)}{(n-m)!(z-1)^{n-m}} = \frac{K}{(n-m)!} \cdot \frac{(z-1)^m B_{n-m}(z)}{(z-1)^n},$$

where the integration path is the imaginary axis with a small detour to the right around the origin. The transfer function G is a rational function whose denominator is of degree n. The function H thus has n poles which go to $z = 1$ as h goes to zero. Since the pulse transfer function is continuous for small h it follows that it has m zeros close to $z = 1$. The remaining zeros are close to the zeros of the polynomial $B_{n-m}(z)$. A better resolution of the zeros close to $z = 1$ is obtained from the representation (2.3). For small h and $|z-1| \ll 1$ the first term dominates and we get

$$H(z) \approx (1 - z^{-1}) \frac{G(\frac{1}{h} \log z)}{\log z}.$$

The right hand side vanishes for

$$z = \exp(z_i h), \qquad i = 1, 2, \ldots, m. \quad \square$$

Remark 1. The results of the theorem were partially conjectured by Edmunds[11]. □

Remark 2. Notice that the limiting zeros of a pulse transfer function depend critically on the pole excess of the corresponding continuous time system. □

Remark 3. Notice that a continuous time system with a pole excess larger than 2 will always give a pulse transfer function with zeros outside the unit disc provided that the sampling period is sufficiently short. This may happen for quite reasonable sampling periods (cf. Example 2), and sampled data systems with unstable inverses are thus quite common. □

Example 2. Consider a system with the transfer function

$$G(s) = \frac{1}{(s+1)^3}.$$

The corresponding pulse transfer function is

$$H(z) = \frac{b_1 z^2 + b_2 z + b_3}{(z - e^{-h})^3}$$

where

$$b_1 = 1 - (1 + h + h^2/2)\, e^{-h}$$

$$b_2 = (-2 + h + h^2/2)\, e^{-h} + (2 + h - h^2/2)\, e^{-2h}$$

$$b_3 = (1 - h + h^2/2)\, e^{-2h} - e^{-3h}.$$

This function has a zero outside the unit disc if $0 < h < 1.8399$. The step response of the system is shown in Fig. 2 where the shortest sampling period giving stable zeros is indicated. □

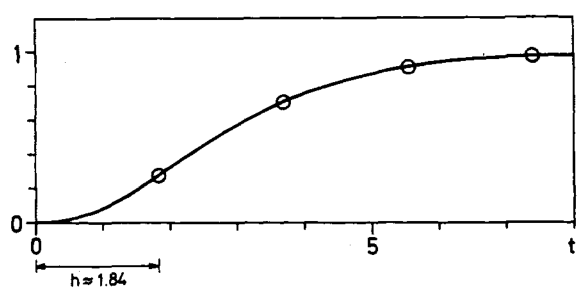

Fig. 2. Step response of a system with $G(s) = (1+s)^{-3}$.

It is thus possible to give a complete characterization of the zeros of the pulse transfer function for small sampling periods. A similar result for large sampling periods is given by

Theorem 2. Let G be a strictly proper rational transfer function with $G(0) \neq 0$ and $\mathrm{Re}\, p_i < 0$. Then all zeros of the pulse transfer function go to zero as the sampling period h goes to infinity.

Proof. Consider the representation (2.2). The function $G(s)/s$ has a pole at the origin with residue $G(0)$ since $G(0) \neq 0$. Let

$$\frac{1}{s} G(s) = \frac{G(0)}{s} + \sum_{i=1}^{n} \frac{A_i}{s - p_i}.$$

It follows from (2.2) that

$$H(z) = G(0)\, z^{-1} + (1 - z^{-1}) \sum_{i=1}^{n} A_i \frac{e^{p_i h}}{z - e^{p_i h}}. \qquad (3.2)$$

Since $\mathrm{Re}\, p_i < 0$, it follows that the sum in the right hand side goes to zero as h tend to infinity. For large h all the poles and all the zeros of the pulse transfer function will thus go to zero and the pulse transfer function is close to

$$H(z) = G(0)\, z^{-1}. \quad \square$$

Remark. If $G(0) = 0$ it follows from (3.2) that for any h, the corresponding pulse transfer function has one zero at $z = 1$. The behaviour of the rest of the zeros may be more complex as is shown by the following example. □

Example 3. Let

$$G(s) = \frac{s}{[(s+1)^2 + 1](s+2)}.$$

Then the corresponding pulse transfer function is

$$H(z) = \frac{1}{2} e^{-h} \frac{(z-1)\{(e^{-h} + \sin h - \cos h)\, z + e^{-h}[1 - e^{-h}(\sin h + \cos h)]\}}{(z - e^{-2h})\,(z^2 - 2z\, e^{-h} \cos h + e^{-2h})} \qquad (3.3)$$

The zeros are $z_1 = 1$ and

$$z_2 = \frac{e^{-2h}(\sin h + \cos h) - e^{-h}}{e^{-h} + \sin h - \cos h} \approx \frac{-1}{1 + e^{h}(\sin h - \cos h)}$$

for large h. Fig. 3 shows how the zero z_2 varies with h. □

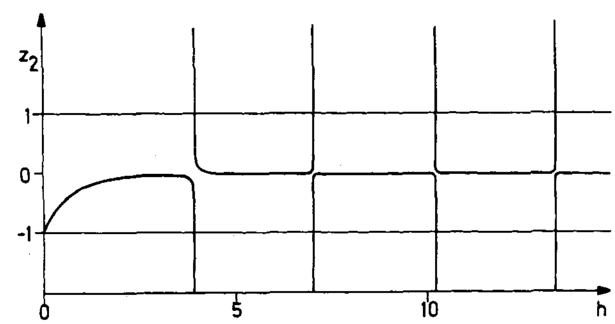

Fig. 3. The zero z_2 of (3.3).

It is in principle easy to find the properties of the pulse transfer function for large h from the representation (2.2) also when $\mathrm{Re}\, p_i \geq 0$ for some p_i. If there is one pole of $G(s)/s$ whose real part is larger than the real part of the other poles, there is one term in the series (2.2) which dominates. If there are several poles with the same real parts, then the behaviour is more complex. Two examples with poles on the imaginary axis illustrate what can happen.

Example 4. Consider a transfer function of the form

$$G(s) = \frac{Q(s)}{s^{\ell}(s - p_1) \ldots (s - p_{n-\ell})}$$

where $\mathrm{Re}\, p_i < 0$ for all i. Hence

$$G(s) = \sum_{k=1}^{\ell} \frac{A_k}{s^k} + \sum_{k=1}^{n-\ell} \frac{R_k}{s - p_k}.$$

The corresponding pulse transfer function is

$$H(z) = \sum_{k=1}^{\ell} \frac{A_k h^k}{k!} \cdot \frac{B_k(z)}{(z-1)^k} - \sum_{k=1}^{n-\ell} \frac{R_k}{p_k} \cdot \frac{1-e^{p_k h}}{z - e^{p_k h}}$$

where Lemma 1 is used to obtain the first sum. For h sufficiently large the term

$$\frac{A_\ell h^\ell}{\ell!} \cdot \frac{B_\ell(z)}{(z-1)^\ell}$$

dominates and the zeros are thus close to the zeros of $B_\ell(z)$ for large h. □

Example 5. Consider

$$G(s) = \frac{1}{(s+1)(s^2+1)}. \tag{3.4}$$

The corresponding pulse transfer function H is

$$H(z) = \frac{1}{2}\left(\frac{1-a}{z-a} - \frac{(-1+\sin h + \cos h)z + (-1 - \sin h + \cos h)}{z^2 - 2z\cos h + 1}\right) \tag{3.5}$$

where $a = \exp(-h)$. For small h the zeros of H are given by

$$z_{1,2} = -2 \pm \sqrt{3}.$$

Compare Theorem 1. For large values of h the zeros are approximately given by

$$z^2(2 - \sin h - \cos h) + z(1 + \sin h - 3\cos h) + 1 = 0.$$

It is easily seen that the zeros are periodic in h with period 2π. As h increases the zeros will thus go to periodic orbits in the complex plane. The orbits are shown in Fig. 4. □

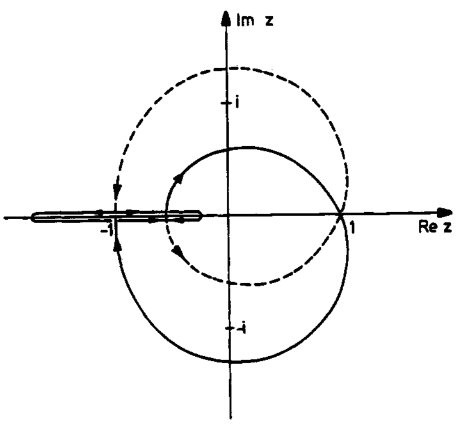

Fig. 4. Asymptotic orbits of zeros of the pulse transfer function (3.5).

It follows from Theorem 2 that there are continuous time transfer functions which have unstable zeros which give sampled systems with stable zeros. One example is given below.

Example 6. The transfer function

$$G(s) = \frac{6(1-s)}{(s+2)(s+3)}$$

has an unstable zero $s = 1$. The transfer function satisfies the conditions of Theorem 2. It thus follows that the corresponding pulse transfer function does not have any zeros outside the unit disc if the sampling period is sufficiently large. Simple calculations show that the zero of the pulse transfer function is

$$z_1 = -\frac{8e^{-2h} - 9e^{-3h} + e^{-5h}}{1 - 9e^{-2h} + 8e^{-3h}}.$$

For $h = 1.2485$ we get $z_1 = -1.0000$ and for larger h the zero is always in the unit disc. □

4. A condition for stable zeros

For the control systems design it is important to separate the case when all zeros of the pulse transfer function are inside the unit disc from the case when the pulse transfer function has zeros outside the unit disc. When all zeros are inside the unit disc the discrete system has a stable inverse and all its zeros can be cancelled†. It is therefore of interest to find criteria which guarantee that all zeros of a sampled transfer function are inside the unit disc. We have

Lemma 2. Let G be a strictly proper transfer function such that $G(0) \neq 0$ and

$$-\pi < \arg G(i\omega) < 0, \qquad 0 < \omega < \infty, \tag{4.1}$$

then the zeros of the corresponding pulse transfer function H will all stay either inside or outside the unit disc for all sampling periods h.

Proof. It follows from (2.3) that

$$H(z) = \frac{1 - z^{-1}}{h} F(z) \tag{4.2}$$

where

$$F(z) = \sum_{k=-\infty}^{\infty} G(s_k) / s_k$$

and

$$s_k = (\log z + 2\pi i k) / h.$$

Evaluation of F on the unit circle gives

$$F(e^{i\varphi}) = \sum_{k=-\infty}^{\infty} G(i\omega_k) / i\omega_k$$

where

$$\omega_k = (\varphi + 2\pi k) / h.$$

Hence

$$\mathrm{Re}\, F(e^{i\varphi}) = \sum_{k=-\infty}^{\infty} \mathrm{Re}[G(i\omega_k)/i\omega_k] = \sum_{k=-\infty}^{\infty} \mathrm{Im}[G(i\omega_k)/\omega_k].$$

Condition (4.1) implies that

$$\mathrm{Im}[G(i\omega)/\omega] < 0 \tag{4.3}$$

for $0 < \omega < \infty$. Since

$$G(-i\omega) = \overline{G(i\omega)}$$

it follows that (4.3) holds for $\omega < 0$. The function $F(e^{i\varphi})$ thus does not vanish for any $\varphi \neq 0$. Since $H(1) = G(0) \neq 0$, it also follows that $z = 1$ is not a zero of H. □

Remark 1. The condition (4.1) is very strong since it implies that the Nyquist curve of G is strictly below the real axis. □

Remark 2. The same result holds if the Nyquist curve of G is strictly above the real axis. □

†This is of course a mathematical idealization. In practice it must be required that the zeros are well inside the unit disc.

Remark 3. Notice that the condition (4.1) does not imply that all zeros of the pulse transfer function are inside the unit disc. Consider for example

$$G(s) = \frac{s-a}{s(s-1)} .$$

The condition (4.1) holds all positive a. It follows, however, from Theorem 1 that the corresponding pulse transfer function has a zero close to $z = \exp(ah)$ for small h. □

For stable systems Lemma 2 and Theorem 2 can be combined to give a condition for the discrete time zeros to be inside the unit disc. We have

Theorem 3. Let G be a strictly proper, rational transfer function with

i) $\operatorname{Re} p_i < 0$

ii) $G(0) \neq 0$

iii) $-\pi < \arg G(i\omega) < 0 \quad \text{for} \quad 0 < \omega < \infty$

then all the zeros of the corresponding pulse transfer function H are stable.

Proof. It follows from Theorem 2 that all zeros are inside the unit disc for sufficiently large h. Lemma 2 implies that no zero can be on the unit circle for any sampling period h. The finite zeros depend continuously on h, and they can thus never cross the unit circle when varying h. □

Analysis of several examples have indicated that the zeros of H(z) often pass the unit circle for sampling period h such that $\operatorname{Im} G(i\pi/h) \approx 0$. It can therefore be expected that the condition

$$-\pi < \arg G(i\omega) < 0 \quad \text{for} \quad \omega_0 < \omega < \omega_2$$

together with some additional conditions on the sampling period h will guarantee that the pulse transfer function has only stable zeros. Such conditions have not been found.

5. Time delays and improper transfer functions

So far it has been assumed that the transfer function is strictly proper. This means that $G(s) \to 0$ as $|s| \to \infty$. It is easy to see how the results should be modified if $G(s) \to a$ as $|s| \to \infty$. Rewrite G as

$$G(s) = G(\infty) + [G(s) - G(\infty)] = G(\infty) + G_1(s)$$

where G_1 now is strictly proper. The corresponding pulse transfer function is then

$$H(z) = G(\infty) + H_1(z)$$

where H_1 is the pulse transfer function corresponding to G_1. Since the results of the previous section now apply to H_1 it is not difficult to find the zeros of H. Notice, however, that the results become more involved as is shown by the following examples.

Example 7. If the transfer function G is stable we get for large h

$$H(z) \approx G(\infty) + [G(0) - G(\infty)] / z.$$

This transfer function has a zero

$$z = 1 - G(\infty) / G(a)$$

which is outside the unit disc if $G(\infty)$ and $G(a)$ have different signs or if they have the same signs and $G(\infty) > 2G(0)$. □

It follows from (2.2) that H is rational if G is rational. It also follows from (2.2) that H is rational if G is of the form

$$G(s) = ke^{-s\tau} \frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

which correspond to a system with time delays. In this case it is easy to obtain unstable zeros in H as is shown by the following example.

Example 8. Consider

$$G(s) = e^{-s\tau}\frac{1}{s}, \qquad 0 < \tau < h.$$

Then it is easily found that

$$H(z) = (1-z^{-1})\frac{1}{2\pi i}\int \frac{e^{s(h-\tau)}}{z-e^{sh}}\cdot\frac{1}{s^2}\,ds = \frac{(h-\tau)z+\tau}{z(z-1)}.$$

The pulse transfer function thus has a zero outside the unit disc if $h/2 < \tau < h$. For larger values of the time delay the transfer function is

$$H(z) = \frac{(h-\tau_1)\,z + \tau_1}{z^n(z-1)}$$

where

$$\tau = \tau_1 + nh.$$

The zero of the pulse transfer function is thus

$$z_1 = \frac{\tau \bmod h}{\tau \bmod h - h}.$$

If the zero is regarded as a function of τ it is thus periodic with period h. □

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