dentally the only surge in a higher group to be recorded during this period, probably falls in the same category.

Except for these last three items (11, 12 and 15) it will be noted that the time difference between recorded surges and the switching operations to which they have been attributed is two minutes or less, and this difference was considered to be due to the usual variation between synchronous time, as employed by the surge counter, and Greenwich Mean Time.

To summarize these results: of the 18 surges recorded during this eight-day period, 13 were due to local switching, 2 were due to a known circuit-breaker fault, and 3 were due to unknown causes.

The instrument has now been installed in an area known to be subject to lightning, and it is hoped that evidence may be recorded of surges coincident with known lightning strokes. The voltage divider to which it is now connected, on a 66-kV line, is similar to that shown in Fig. 7, and includes a separate high-voltage capacitor unit. During the test previously detailed, however, this unit was dispensed with and a condenser-type bushing of approximately 300 $\mu\mu$F was used in its place. There appear to be no objections to the employment of such bushings for this purpose, and there is the great advantage that, where they are convenient to the desired installation, the cost of the high-voltage capacitor unit may be avoided.

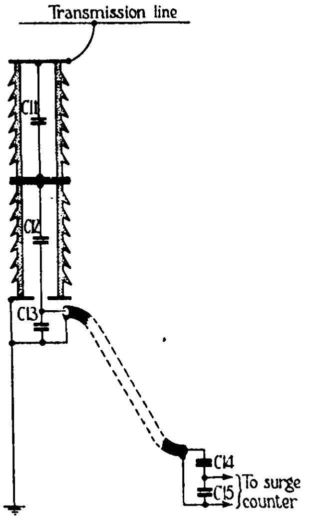

Fig. 7.—Voltage-divider circuit.

C1, C2	0·002 μF, 50 kV, r.m.s., working
C3	1·0 μF, 100 V, r.m.s., working
C4	0·001 μF
C5	0·02 μF

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DIGESTS OF PAPERS

A GENERAL THEORY OF SAMPLING SERVO SYSTEMS

621-526 Monograph No. 4 MEASUREMENTS SECTION

D. F. LAWDEN, M.A.

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The mathematical theory of sampling servo systems was initiated in America during the last war. Details of this development have been given by MacColl[1] and Hurewicz.[2] These authors largely confine themselves to a consideration of the system's behaviour at sampling instants, but recent work by Miller and Schwarz[3] has extended the analysis to deal with the behaviour between these instants. A system of a type different from that envisaged by all these workers has been built in this country by Porter and Stoneman,[4] and lately improvements in their design have been effected by Holt Smith and Lawden.[5] The methods of analysis available prove to be inadequate for a discussion of the characteristics of a system of this latter type, and it is the object of the paper to meet this deficiency and at the same time to provide a very general mathematical foundation upon which the theory of sampling servo systems of many diverse types may be based.

The flow diagram of the type of system to be analysed is shown in Fig. 1: $f(t)$ is the input and $g(t)$ the output quantity. The

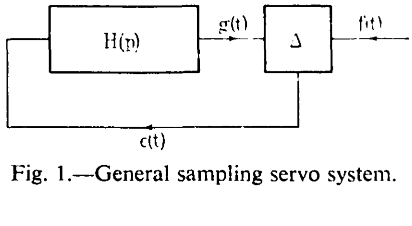

Fig. 1.—General sampling servo system.

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difference $(f-g)$ is sampled by the element Δ at times $t=0, \tau, 2\tau, \ldots$, so that the output from Δ is the sequence c_n, where

$$c_n = f(n\tau) - g(n\tau) = f_n - g_n \quad . \quad . \quad . \quad . \quad (1)$$

We shall suppose, for convenience, that the output from Δ is in the primitive form of a series of impulses, so that if $c(t)$ denotes this quantity,

$$c(t) = \sum_{n=0}^{\infty} c_n \delta(t - n\tau) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\delta(t)$ is the unit impulse function.

$c(t)$ is fed to a linear element (or series of elements) having overall transfer function $H(p)$, which serves to generate the output quantity $g(t)$. In general, $H(p)$ will not be a simple rational function of p, but will involve exponentials corresponding to time delays and possibly to digital manipulation of the sequence c_n. The object of the mathematical analysis is to establish a relationship between the two sequences c_n and f_n, the form of $H(p)$ being given. It proves to be a linear recurrence equation of the form

$$\sum_{r=0}^{p} a_r c_{n+r} = \sum_{s=0}^{q} b_s f_{n+s} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

To assist in the manipulation of such equations we define the transform $U(z)$ of a sequence u_n by means of the equation

$$U(z) = \sum_{n=0}^{\infty} u_n z^{-n} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and it may then be proved that, provided the initial conditions are suitably chosen, eqn. (3) is equivalent to the transformed equation

$$C(z) \sum_{r=0}^{p} a_r z^r = F(z) \sum_{s=0}^{q} b_s z^s \quad . \quad . \quad . \quad . \quad (5)$$

$C(z)$ being the transform of c_n, etc. Eqn. (5) is termed the *fundamental equation* of the system. Given the input sequence f_n, we may calculate $F(z)$, and then $C(z)$ is derivable from (5); and by expansion of this function in a series of inverse powers of z the sequence c_n is generated.

The characteristic equation of (3) is obtained by equating the left-hand side of (5) to zero, giving

$$\sum_{r=0}^{p} a_r z^r = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

To any root ρ of this equation corresponds a term $A\rho^n$ in the general solution of (3) for c_n. If the system is to be stable, all such terms must tend to zero as $n \to \infty$, and it follows that all roots of (6) must have moduli less than unity.

In practice $H(p)$ may be expressed as a sum of terms in the form

$$H(p) = \sum \epsilon^{-\lambda p\tau} J(\epsilon^{p\tau}) \frac{A}{\tau p(\tau p + \alpha)^{\beta}} \quad . \quad . \quad . \quad (7)$$

where $0 \leq \lambda < 1$, J is rational in its argument and β is a non-zero positive integer. Corresponding to (7) we can set up the equation

$$\tau R(z) = \sum A J(z) L_{\beta}(z, \alpha, \lambda) \quad . \quad . \quad . \quad . \quad (8)$$

defining the function $R(z)$, where

$$L_1(z, \alpha, \lambda) = \frac{1}{\alpha(z-1)} - \frac{\epsilon^{\lambda\alpha}}{\alpha(\epsilon^{\alpha} z - 1)} \quad . \quad . \quad . \quad (9)$$

$$L_{\beta}(z, \alpha, \lambda) = \frac{(-1)^{\beta-1}}{(\beta-1)!} \frac{\partial^{\beta-1}}{\partial \alpha^{\beta-1}} L_1(z, \alpha, \lambda) \quad . \quad . \quad (10)$$

It may now be shown that the system's fundamental equation is

$$(R+1)C = F \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In the particular case of $\alpha = \lambda = 0$ the above definitions of the L-functions fail and the appropriate definition is

$$L_{\beta}(z, 0, 0) = \frac{zP_{\beta}(z)}{(z-1)^{\beta+1}} \quad . \quad . \quad . \quad . \quad (12)$$

where $P_{\beta}(z)$ is a polynomial of degree $(\beta - 1)$ which may be calculated from the determinant

$$P_{\beta}(z) = \begin{vmatrix} 1 & 1-z & 0 & \ldots & 0 \\ \frac{1}{2!} & 1 & 1-z & \ldots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{\beta!} & \frac{1}{(\beta-1)!} & \frac{1}{(\beta-2)!} & \ldots & 1 \end{vmatrix} \quad . \quad (13)$$

P_0 is defined to be unity. The properties of these polynomials are discussed in Reference 6.

If $\alpha = 0$, $\lambda \neq 0$, we define $L_{\beta}(z, 0, \lambda)$ by the equation

$$L_{\beta}(z, 0, \lambda) = \frac{\phi_{\beta}(z, \lambda)}{(z-1)^{\beta+1}} \quad . \quad . \quad . \quad . \quad (14)$$

where

$$\phi_{\beta}(z, \lambda) = z \sum_{r=0}^{\beta} \frac{\lambda^r}{r!}(1-z)^r P_{\beta-r}(z) + \frac{\lambda^{\beta}}{\beta!}(1-z)^{\beta+1} \quad . \quad (15)$$

Other formulae involving the functions L_{β}, P_{β} and ϕ_{β}, which assist in their manipulation and calculation, are given in the paper, and an Appendix gives a list of the forms taken by these functions for small values of the integer β.

Application of the above results to single and multi-channel systems of practical interest will also be found in the paper, together with recommendations for future work in this field.

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