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On the Theory of Discrete Systems*

Sur la théorie des systèmes discrets

Über die Theorie diskreter Systeme

О теории дискретных систем

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A survey of discrete systems theory involves a variety of topics: Description, Analysis and Synthesis of Pulsed Systems, Analysis of Non-linear Pulsed Systems, Synthesis of Optimal Pulsed Systems, Random Sampling, Adaptive Pulsed Systems, Digital Filtering, Sequential Machines and Biocontrol Systems.

Summary—The authors in this paper have joined their efforts in writing a survey paper on the theory of discrete systems. In this combined effort, several important topics are covered: description of pulsed systems, analysis of pulsed systems, synthesis of pulsed systems, analysis of non-linear pulsed systems, synthesis of optimal non-linear systems, random sampling, adaptive pulsed systems, digital filtering, sequential machines, and biocontrol systems. For each of the mentioned topics, sections giving a brief evaluation of the work done are presented, combined with references to research performed both in the East and the West. Because of the limited space available, the authors have omitted many other references which are pertinent to this survey. However, the references which are included are familiar to the authors and can be used for further study and research.

The aim of this paper is to present to control scientists the wide spectrum of the research done in this important area within the last two decades. It is hoped that the objectives of the authors will be fulfilled.

DISCRETE AUTOMATIC SYSTEMS

DISCRETE automatic systems were the first to appear as the technology of automatic control was developed. The first controllers to control the number of revolutions of steam machinery were controllers of the relay and intermittent, or discontinuous, type, because of the specific nature of the controlled units, which permitted a change in the number of revolutions per unit time by varying the amount of steam, on–off regulation, only at definite time intervals. Other early applications of discrete automatic systems can be actually traced back to 1897, when GOUY [1] designed an oven regulator that is based on intermittent applications of information. Actually, such a system can be described as a Pulse Width Modulating Feedback System. In connection with a gradual moving towards new, more modern steam machinery design, at the beginning of this century, and also because of the appearance of steam turbines, discontinuous controllers lost their importance.

After a lapse of several years, in the thirties, discontinuous controllers again appeared in control systems for thermal-energetic processes, as controllers with a tapered profile closing, at one or another time instant, of the circuit of an electric servomotor as a function of the deviation of the galvonometer pointer. Such tapered profile controllers solved the problem of obtaining arbitrarily large power amplifications, and hence enabled them to convert fairly small power thermocouple signals into signals powerful enough to control the electric servomotor. The simplicity of design and the effectiveness of such controllers have earned them a wide popularity and application in the control of slow processes. The appearance of automatic potentiometers has slightly restricted their use, and today the construction of these controllers is virtually unchanged.

Other applications of discrete automatic systems also include the use of sampling in electronic feedback amplifiers [2], radio transmitters, or receivers for improving stability. Essential development of controllers of the discrete type was a consequence of pulse radio-coupling, radio-location, and especially those computers in which the transfer and transformation of information have a discrete character. Many biological and economic models reduce to discrete automatic systems.

All the above has exercised and essential influence and stimulation on the development of the theory of discrete automatic systems, which are characterized by the fact that at least one of the variables

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describing the state of the automatic systems may be made discrete and quantized. Depending on the type of quantization, in level, in time, or in both, discrete automatic systems are subdivided in three types, namely: relay automatic systems, (*i*m)pulse automatic systems,* and digital automatic systems. Simpler relay systems with two or three quantization levels, and quasi-relay-systems or systems with a variable structure, in which system parameters vary discretely in jumps, are typical representatives of essentially non-linear systems. Their theory and specifications are considered in a section on the theory of non-linear systems. It is of historical interest to note that HAZEN [3] in an early article has pointed out the applications of discrete automatic systems to industrial processes.

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1. DESCRIPTION OF PULSED SYSTEMS† [1.1–1.26]

Unlike continuous systems, whose behaviour is described by differential equations, pulsed systems, as has already been established in classical treatises [1.1, 1.2], are described by finite difference equations or difference equations. A description and study of pulsed systems using linear difference equations has been exhaustively carried out in [1.3] and has been systematized so well that it formed part of one of the first textbooks on control theory [1.4].

Classical methods of studying and solving difference equations and methods of their formation were associated with tedious computations; thus robbing the classical methods of intuitive geometrical insight and making it impossible to arrive at general dynamic properties of pulsed systems. Today, it is necessary to find an essentially different approach, which would permit a tractable approach to describing and studying sample data systems, with equal simplicity and ease as in the theory of continuous systems, using operational methods, and in particular Laplace transforms. As a result of searching for such a method, a new mathematical device was created to cope with problems occurring in the theory of discrete-time systems, which is called the discrete Laplace transform [1.5] or the equivalent Z-transform [1.6, 1.7].

If in the ordinary Laplace transform we have the relation

$$L\{f(t)\}=\int_0^\infty e^{-pt}f(t)\mathrm{d}t=F(p), \qquad (1)$$

then in the discrete Laplace transform we have

$$D\{f(nT)\}=\sum_{n=0}^{\infty} e^{-pnT}f(nT)=F^*(p); \qquad (2)$$

the discrete Laplace transform (DLT) acts on the discrete function $f(nT)$ that are obtained from the continuous function $f(t)$ by setting $t=nT$, where $n=0, 1, 2, \ldots$, and T is the period of repetition, usually a constant.

It follows from equation (2) that $F^*(p)$, unlike $F(p)$, is not simply a function of p, but a function of e^{pT}, thus, frequently and especially in Western literature, in order to simplify the notation, one can let $e^{pT}=z$, and thus define the Z-transform

$$F(z)=\sum_{n=0}^{\infty} z^{-n}f(nT). \qquad (3)$$

Using a discrete Laplace transform one is able to avoid the difficulties that one encounters in setting up difference equations, to formalize this labor-consuming stage, and to compile a theory of pulsed systems that is formally similar to the theory of continuous systems. Thus an equation of a typical pulsed equation consisting of pulsed element an continuous linear part may be shown in the familiar form:

$$Z^*(p)=\frac{W^*(p)}{1+W^*(p)}F^*(p), \qquad (4)$$

where $Z^*(p)$, $F^*(p)$ are discrete Laplace transforms of the output and input variables respectively and $W^*(p)$ is the transfer function of an open loop pulsed system which is determined from the transfer function or the time characteristic of the continuous part, by following some simple rules. Such formal analogy was important not only because it enabled the introduction of the usual notions and representations into the theory of pulse automatic systems, as for example: transfer function, time and frequency characteristics, steady state and transient processes, etc., but also because it permitted, taking into account the specific nature of discrete time systems, to use a number of ways and methods to study that were used in the case of continuous systems. The introduction of shift discrete functions $f(nT, \Delta t)$ and the corresponding transforms $F^*(p, \Delta t)$ [1.8], permitted the description of the behavior of sampled data systems not only at the discrete time $t=nT$, but at any time $t=nT+\Delta t$, where $0\leq\Delta t\leq T$. This method can be

* Also known as Sampled-Data Systems.

† These systems are also known as Sampled-Data or Discrete-Time Systems.

extended to the z-transform and referred to as the modified z-transform [1.9, 1.10]. Extensive treatment of both Z-Transform and the modified z-transform is given in Refs. [1.11, 1.12].

Other forms of transform methods have been applied to pulsed systems [1.13–1.15] that in certain cases have been applied to analysis and synthesis problems. Special rules have been devised, simplifying the setting up of multiloop systems [1.16–1.25]. It is of interest to note that the mentioned Discrete Laplace transform has been applied to pulsed systems by MacColl [1.26].

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2. THE ANALYSIS OF PULSED SYSTEMS [2.1–2.76]

The analysis of pulsed systems embraces the methods of studying the stability and properties of processes, for a given deterministic, as well as random, actions. In order that a system be operational, it is, first of all, necessary to guarantee its stability. With Discrete Laplace transforms criteria analogous to the stability criteria of Mihailov and Nyquist [2.1, 2.2] may be established. The fundamental difference of the new stability criteria from that used for continuous systems consists in the fact that the roots of the characteristic equation, i.e. p_v or $z_v = e^{p_v T}$, $(v=1, 2, \ldots, n)$ must be in the left-half region [Re $p<0$; $-(\pi/T)<\omega\leq(\pi/T)$], or in the interior of the unit circle $|z|=1$, not in the entire left-half plane as is the case for continuous systems. Frequency characteristics of pulsed systems are fully defined by changing the frequency in the range $0<\omega<(\pi/T)$, but with the exception of this particular detail, formulations of frequency stability criteria for pulsed systems in no other respect differ from the known formulations of the theory of continuous systems. Analytical tests for stability are also available in the literature [2.3–2.6] that constitute the counterpart of Routh–Hurwitz criteria for the continuous case. A special characteristic of pulsed systems is that they have a finite critical amplification, which is independent of the order of the system. In order to estimate the quality of a process, the notions of degree of stability and cumulative estimate were introduced [2.7, 2.8]. Steady state process characteristics may be studied by means of error coefficients [2.9].

Fundamental results of pulsed system analyses, both open and closed loop, obtained up to 1951 were communicated in [2.10]. Subsequent development of the theory went in the direction of working out frequency methods for deterministic [2.11] and random [2.12] actions. Extensive study has been achieved for random inputs and several results have been reported [2.13–2.19]. See also references [3.23–3.45]. The frequency method was used for the construction of a transient process, on the basis of typical trapezoidal characteristics [2.20, 2.21]; however, the frequency method in its present form, when applied to pulsed systems, does not yield similar simplifications to those which were obtained in the theory of continuous systems using by logarithmic asymptotic frequency characteristics. The bilinear transformation

$$e^{j\omega T}=\frac{1-v}{1+v} \quad \text{or} \quad v=\tan\frac{\omega T}{2} \tag{2.1}$$

enables one to use, approximately, logarithmic characteristics [2.22–2.24].

Many authors were concerned with systems in which sampled data arrive instantly at discrete times, but many applications, especially in radiolocation automatic systems, sampled data arrive over a finite period of time. The presence of this condition renders the study of this class of systems much more difficult. Using the apparatus of difference equations and discrete Laplace transforms, one succeeds in describing these systems and working out methods for studying their stability and process establishment [2.25, 2.26] in the simplest form possible. Later, by applying some theorems involving discrete Laplace transforms, it became possible to reduce the study of this class of systems to the study of ordinary pulsed systems with instant data sampling [2.27, 2.28]. Several methods have been proposed to account for finite pulse width which are both approximate and exact [2.29–2.48]. For specific computations and the design of similar systems, it appeared advisable to use approximate frequency methods [2.49].

The use of sampling for improving the characteristic of continuous systems has been recently demonstrated by an example [2.50]. The already developed analysis techniques permitted the peculiarities and properties particular to linear pulsed systems to be established, namely: the possibility of stabilizing continuous systems with delay and unstable components by introducing a pulsed element [2.11] or a source [2.25] and the realization of finite duration processes, with infinite degree of stability, in pulsed systems [2.51]. This last fact, throughout the years, has been at the basis of the important notion of "controllability" in the general theory of control [2.52] and also one of the fundamentals of construction of invariant pulsed systems [2.53, 2.54]. The state variable approach has been successfully applied [2.55–2.60] for both finite pulse-width and impulse approximations of pulsed systems invariance of pulsed systems has a somewhat different character than for continuous systems—invariance may refer to discrete time instances which are sampling times, or time intervals between the sampling times [2.53]. Because discrete time systems, except for the dependence on their order, do not admit an increase of the amplification coefficient, the unique possibility of constructing invariant pulsed systems is based on changing and compensating external perturbation effects [2.53, 2.54. 2.61, 2.62]. It is to be understood that all difficulties encountered in constructing continuous invariant systems also are to be contended with when considering pulsed systems.

Sensitivity problems connected with sampled data systems have been proposed and tackled in the literature [2.63–2.65]. The apparatus of discrete Laplace transforms is successfully applicable in the analysis and description of pulsed systems with extrapolators [2.66–2.68] and with variable parameters, varying smoothly [2.69, 2.70] and discontinuously [2.71, 2.72]. The z-transform method has also been applied to systems with extrapolators [2.73, 2.74] (and [4.93]). Furthermore, analysis problems connected with time varying pulsed systems have been presented in various publications [2.75, 2.76] and references [4.43–4.46b].

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3. SYNTHESIS OF PULSED SYSTEMS [3.1–3.65]

When designing and constructing automatic systems, an important role is played by synthesis methods, i.e. the construction of controlling equipment to guarantee a given, and frequently optimal, process the quality of designed systems. In pulsed systems, the synthesis may be based on continuous pulsed correction [3.1–3.7] or consist of changing the form of control pulses [3.8, 3.9]. Continuous correction implies changing the characteristics of the continuous part by additional placement of continuous circuits. This correction method is identical to the method used in continuous systems.

Pulsed correction* corresponds to introducing a discrete filter that transforms the law of varying the pulse sequence, which may be regarded as a model of a program of a control digital computer [3.10–3.19]. Devised synthesis methods permit the definition of the structure of pulsed automatic systems which compensate for lag [3.20] and also the structure of pulsed systems which are best with respect to speed of action, by the standard deviation or root-mean-square estimates [2.28, 3.21–3.45]. Simultaneously it was clarified, that the best chance of success, when synthesizing optimal pulsed systems, is afforded when one combines a discrete correction with changing the shape of control pulses [3.46] and by external action control. Changing the pulse shape if desired, is the unique means of removing nonminimal-phase properties of the invariant part of the system, and these nonminimal-phase properties are a considerable restriction when striving to satisfy high system quality indices.

Appreciable difficulties have been encountered in the synthesis of optimal systems when the invariant part is nonminimal-phase and is not stable. For the solution of this type of problem, a method based on the theory of polynomial equations has proved to be fairly good [3.47]. This method will simultaneously consider the general conditions and will yield explicitly an expression for the optimal transfer function of the control setup [2.28, 3.47]. This method is especially good for solving problems arising in the synthesis of pulsed automatic systems using the least mean square error, taking into account the characteristics of the invariant part of the system and a number of additional conditions [3.48, 3.49]. This type of problem includes analogs of problems of KOLMOGOROV, WIENER, RAGAZZINI-ZADEH, etc. [2.28, 3.47, 3.50–3.52].

It is interesting to notice that unlike continuous systems, the optimal transfer functions obtained for pulsed systems, are always physically feasible. Assumptions on which the obtained results are based, consist of selecting an admissible form for the transfer function of the closed system and zero initial conditions. This leads to minimizing the functional by removing the left poles and the zero transfer function of the unit, using the control system [3.47]. It is also possible to employ another approach which is based on the arbitrary nature of initial conditions and the dependence of the functional—total square estimate—on all state coordinates of the system. Depending on the choice of the coefficients at the coordinates, one is able to obtain any process from among the multitude of stable processes, described by a difference equation of a given order [3.53]. In particular, this was the method used to establish that system of finite, the least, process duration, represent an appreciable part of pulsed systems and are optimal with respect to the total quadratic estimate [3.53].

Methods of synthesis optimal systems, without any special difficulty, couched in matrix terminology, can be extended into multidimensional, multi-connected pulsed systems [3.54–3.58]. Quite complex, on the other hand, is the problem of synthesis of pulsed systems with variable parameters, especially when the structure of the given part is fixed. Some synthesis methods have been worked out using a special algebra of difference operators [3.59–3.62].

* Also known as discrete compensation.

The theory of linear pulsed systems, which embraces systems with amplitude modulation and small intensity width modulation, has been satisfactorily described. It enables the researcher to carry out a complete analysis and synthesis and provides answers to interesting problems concerning the properties and possibilities of linear pulsed systems. Methods and graphs worked out in [3.63–3.65] are very useful for specific computations when choosing optimal parameters for typical control systems.

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4. ANALYSIS OF NON-LINEAR PULSED SYSTEMS [4.1–4.108]

The existence of unavoidably present or deliberately introduced non-linearities, limits the applicability of the linear theory of pulsed systems. This is especially true in systems employing width- or frequency pulse modulation, as well as systems which, as an element, contain digital computer units, where one has to take into account the limitations imposed by the memory and the number of digits available. The theory of non-linear pulsed automatic systems began to develop relatively recently. Unlike continuous non-linear systems, in non-linear pulsed systems we do not encounter the difficulties associated with an exact formulation of the control process at discrete points, as the equations of these systems in the time domain, may be regarded as recurrence relations, enabling us to define sequentially the values of the process [4.1, 4.5], numerically, as in tabular or graphic forms, as well as the multidimensional Z-Transform [4.6] one can obtain the response of such systems at the sampling instants. The application of these methods has proved easier than their continuous counterpart to ordinary non-linear continuous systems. However, this feature does not allow us to solve problems associated with the study of the stability of the system, the quality of the process, the existence and stability of admissible periodic operations and a synthesis of a system such that, according to some philosophy, optimal processes are guaranteed.

Using the ideas of methods for the study of absolute stability, which are based on the direct method of A. M. Liapunov, and using the approach of V. M. Popov, sufficient conditions for absolute stability of equilibrium positions of non-linear pulsed systems as a solvable system of quadratic equations [4.7–4.28] and frequency criteria for absolute stability [4.29–4.46b] have been found. The latter conditions actually proved to be unusually simple. For absolute stability we require that the frequency characteristic of the linear pulsed part satisfy the Nyquist criterion, and that it does not intersect the circle whose center lies on the abscissa axis. Coordinates of diameter ends of this circle determine the slopes of the sector radii, wherein lies the characteristic of the non-linear element [4.47]. If the derivative of this characteristic is contained between those radii which form the sector, then, together with the stability of the equilibrium position, the stability of any process induced by bounded external action is also guaranteed [4.48]. Such frequency approaches enable the estimation of the bounds of quality indices of pulsed systems by means of the degree of stability and total estimates [4.49]. Frequency criteria of absolute stability may be generalized for the case of multi-non-linear pulsed systems [4.34, 4.50], although in those cases, the criteria are available in a less elegant and more complex form, and a positivity of a certain Hermitian matrix is then a requirement. The fact that the bounds for quality indices, within defined limits, are not sensitive to changes of the forms of non-linear characteristics, is important, because it gives the opportunity to solve the control problem in terms of devices which do not have exactly known non-linearities, without having to use complex self-steering systems.

When stability criteria are not fulfilled in pulsed and digital automatic systems, various periodic and aperiodic phenomena may appear. For the determination of periodic oscillations the phase plane and the incremental phase plane can be used as well as other analytical techniques [4.51–4.65]. The character of these manifestations is often more complex than in continuous systems, especially for pulsed systems which are not autonomous. The study of periodic systems, historically speaking, preceded the establishment of stability criteria, and was based on the adoption of ideas of harmonic balance.† The increasing use of the harmonic balance method permitted effective methods to be devised for determining periodic regimes with a period that is a multiple of the repetition period in relay-pulsed [4.66], non-linear amplitude-pulsed [4.67], and other modulated systems [4.68–4.80]. This approach is fairly good and has been used successfully for the determination of low frequency periodic behavior. For high frequency periodic behavior, it transpires that a simple substitution of continuous frequency characteristic $W(j\omega)$ by the impulse characteristic $W^*(j\omega)$, gives us an approximate solution defining the existence of high frequency regimes [4.81]. For periodic regimes, whose period is not a multiple of the repetition period and for complex periodic regimes, the only possibility of determining them, which today is at our disposal, is to develop the method of harmonic balance by the dominant harmonic [4.82].‡ In all those cases it is necessary to use harmonic linearization coefficients, which may be computed beforehand [4.83–4.85]. The problem of studying stability of periodic regimes is reduced to the study of local stability of a linear pulsed system having some pulse elements in [4.76a, 4.86].

Digital automatic systems may be considered as a special case of non-linear pulse systems in which the non-linearity defining quantization by level has a step-like character. Such a system may be represented as a linear pulsed system with a supplemental external action, bounded by an absolute quantity and quantized by level. Deterministic and probabilistic estimates of this effect are possible [4.87–4.97]. With digital automatic systems one may use directly the method of stability studies and pulse regime of non-linear pulsed systems. An important problem in the design of digital systems is the problem of information transmittal on the basis of an incremental method and a full transmission of levels. Therefore, it was necessary to explain the various methods for increasing the effectiveness and to compare the noise stability of the various methods of discrete information transmittal such as delta-modulation, difference-discrete and pulse-code modulation. The analysis of these types of modulations permits the researcher to make a judicious choice of types of modulation for the various conditions under which they are employed [4.98, 4.99] and to introduce simplifications in the analysis of periodic regimes [4.100–4.102].

Non-linear pulse systems contain also a wide class of extremal pulse systems. Using the discrete Laplace transform, general equations for such systems were obtained [4.103], which are the fundamentals for the study of transient and steady state regimes of extremal pulse systems with independent search [4.104]. In order to minimize losses incurred in searching, such systems may employ the same correction schemes which are used in pulse relay systems. Other pertinent general relationships are discussed in [4.105–4.108 and 9.12].

† Also known as Describing Function.

‡ See also: J. Ackermann: Beschreibungsfuektimen für die Analyse und Synthese von nichlinearen Abstast—Regelkreison, Regelungstechnik 14, Heft 11, pp. 497-504 (1966).

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5. SYNTHESIS OF OPTIMAL NON-LINEAR SYSTEMS [5.1–5.57]

Recently the problem of synthesis of optimal systems, for which one is able to obtain a minimum for any functional characterizing the quality of the process under the given constraints imposed on the control action, has become an important one. Here two approaches are distinguished: The first approach is based on the pulse system in the time domain by means of an equation of the Convolution Sum type. In this case the problem of synthesis may be reduced to finding the extremum of a function of several variables having also available additional constraints in the form of equations and inequalities. The second method uses the method of indeterminate Lagrange multipliers and dynamic programming to determine the best control law and to demonstrate various ways of synthesizing optimal systems [5.1].

When using inequalities for constraints, it is convenient to use mathematical programming. In order to solve the problems that arise in this connection special algorithms were proposed [5.2]. Many partial synthesis problems may be effectively solved by employing linear programming methods [5.3, 5.4]. Dynamic Programming [5.5, 5.6] has also been extensively applied in the optimum synthesis of non-linear discrete systems. Another approach is based on describing the pulse system by a family of difference equations of first order. For non-linear systems of the general type, necessary conditions for optimality were obtained in [5.7, 5.8], which, unlike those for continuous systems, have a local character. For the discrete case one can find conditions for which the maximum principle holds [5.8]. Also interesting are the sufficiency conditions for optimality [5.9]. For a certain defined class of systems, the discrete maximum principle gives the necessary and sufficient conditions of optimality [5.10]. Other relevant research on this method is given in [5.11–5.21]. Considerable attention has been devoted to the synthesis of linear discrete systems, which are optimal, relative to rapidity of action, when the control action is saturated [5.22–5.27]. Extension of these techniques to non-linear discrete systems also has been attempted [5.28]. In order to find optimal control law corresponding to the time function in such systems, the *L*-problem of M. G. KREIN [5.29, 5.30] was employed for systems of third order, which yielded a direct solution of the synthesis problem, i.e. one can find the optimal control, as a function of phase coordinates [5.31]. These results may be used also for the synthesis of continuous fast-acting systems. Considerable interest has been aroused by the algorithmic approach to the synthesis problem in which a function is not sought but rather an algorithm that yields the best control action on the basis given values of phase coordinates to be found [5.32].

For simpler optimal relay-pulse systems, which are described by a difference equation of the second order, unusually elegant results have been obtained, and optimal control laws were established in the absence of noise [5.33] and in the presence of noise [5.34, 5.35]. Optimal control of pulse width modulation systems [5.36–5.39] as well as pulse frequency modulation [5.40, 5.41] has been studied and implemented. Optimal control of discrete systems with time delay has been tackled in the literature [5.42, 5.43]. For a thorough discussion of Discrete Optimal Control Theory, the reader is referred to [5.44]. A unified treatment of algorithms for discrete optimal control and for mathematical programming is presently being developed [5.45]. In this reference, the development and exploration of convergence theory is presented. Furthermore, the relationship between the type of points different algorithms will compute for a given problem is also included in this exhaustive study. Other forms Mathematical Programming methods are described in the literature [5.46–5.57].

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6. RANDOM SAMPLING [6.1–6.26]

Random sampling may occur because of uncertainty regarding the exact sampling time, or because of failure in the sampling mechanism. Random sampling can also be introduced intentionally, when for economy a digital computer is time-shared in a random rather than deterministic manner. Moreover, the sampling period may also be designed to be random in order to prevent jamming and to eliminate hidden oscillations between sampling instants. Finally, a major area of application where non-linear randomly sampled systems appear is in the field of biological systems. This is discussed in detail in Section (10).

The early work dealing explicitly with randomly sampled systems is Kalman's dissertation [6.1]. In his research he describes a linear sampled-data system by the state variable approach and then recalls the different types of stochastic stability and proposes a synthesis procedure via dynamic programming. Later on in [6.1], Kalman generalizes this synthesis procedure for systems with quadratic cost. Many authors have attempted to compute the spectral density of the output of a randomly sampled system. BERGEN [6.2] provides an integral equation to determine the spectral density of the output of a random sampler with independent identically distributed sampling intervals. ADOMIAN [6.3] performs the same computations for a random sampler followed by zero-order hold. Lately, SIBUL [6.4] tried to apply Adomian's results to compute the spectral density of the output of a closed-loop randomly sampled system; he uses the Neumann series and obtains results on the form of infinite series. However, this method is limited to white noise input and Poisson sampling. In a series of papers, LENEMAN [6.5–6.10] tries to apply results of his dissertation on stationary point procedures [6.8]. In [6.7], he studies a first-order randomly sampled system with independent identically distributed sampling intervals; he retrieves previous results and computes spectral densities for a zero input. In [6.6] LENEMAN studies the properties of various randomly spaced trains of impulses. Furthermore, LENEMAN and LEWIS [6.11–6.14], have studied the mean-square error of various sampling schemes. An independent contribution was made by S. S. L. CHANG [6.15], who re-examined the application of state variables to filtering in the Kalman–Bucy filter and extended it to linear randomly sampled model.

The problem of stochastic stability has been a continuing research effort. The basic idea of stochastic Lyapunov functions was given by BUCY [6.16]; he pointed out that they had to be supermartingales. Recent extension of discrete stochastic stability to non-linear randomly sampled systems has been independently discussed by AGNIEL–JURY [6.17], and KUSHNER–TOBIAS [6.18]. Further extension of stochastic stability is presently discussed by AGNIEL and JURY [6.19]. Other closely related problems based on jittered sampling is discussed in [6.20–6.22]. Further studies on stability and random pulse trains are reported in [6.23–6.26].

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7. ADAPTIVE PULSED SYSTEMS [7.1–7.28]

A considerable amount of work has been performed in the study of adaptive pulsed systems. Some of the work is concerned with the adaptive features of the sampling frequency [7.1–7.4], and several adaptive laws have been advocated. In the works of TAIT [7.5–7.8] and in his survey [7.9], many features of such adaptive schemes and laws are discussed in detail. The sensitivity of adaptive sampling also has been investigated [7.2, 7.7]. In addition to the adaptation of the sampling frequency, adaptation of system parameters of pulsed systems has been investigated [7.10–7.17]. Many uses of adaptive pulsed systems both in space exploration and in biological systems are presently being pursued. Closely related to adaptive theory is the problem of identification. Several algorithms for identification of pulsed systems have been proposed and evaluated [7.18–7.28]. In most of these investigations the plant identification was the major area of concern. Most of the techniques developed are also applicable to continuous systems where the digital computer is needed for the purpose of calculations. Several books and many survey articles have been written on this subject and certainly this brief survey constitutes only a minor description of the problem. Only a few references are listed pertinent to the discrete case. Further studies of signal dependent sampling applications are also mentioned in Sections (4) and (10) of this survey.

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8. DIGITAL FILTERING [8.1–8.25]

The engineering origin of digital filtering can be traced to the earliest efforts to simulate analog signal processing schemes on general-purpose digital computers. The development of many speech processing systems has relied on this type of simulation. The broad applications of the digital filtering techniques are in such areas as audio-electrostatics, speech processing, radar systems and data communication. Both the z-transform method and sampled-data synthesis techniques are used extensively in the design of digital filters.

In general digital filters [8.1–8.13] can be classified as recursive and nonrecursive. Recursive filters are distinguished by feedback or delayed outputs or intermediate computational variables. They have transfer functions in the z-plane that are a ratio of polynomials in z with poles located other than at $z=0$. Design and implementation of such digital filters are discussed in part (3) of this survey. In general, implementation of such recursive filters can be performed using the standard z-transform, the bilinear z-transform and the matched z-transform [8.4].

Some design methods yield the digital filters $D(z)$ as a polynomial in z^{-1} rather than a ratio of polynomials. These are called nonrecursive digital filters [8.1, 8.6]. The nonrecursive filter is characterized by the absence of feedback; that is, past values of the output sequence y_j are not used in computations of the current y_j. The nonrecursive filter relationship may be written as:

$$y_j = \sum_{n=0}^{M-1} h_n x_{j-n} \tag{8.1}$$

where x_n and h_n are the input and the impulsive response sequences respectively.

When M is large enough, it is computationally efficient to implement the filter by means of the technique called "high-speed convolution" [8.14, 8.15 and 8.16]. This technique is based on these observations: (a) The discrete convolution of y_j given above may be replaced by using the respective z-transforms of y_j, h_n and x_n. (b) The z-transform may be evaluated at uniformly spaced points on the unit circle in the z-plane. The resulting transform is called the discrete Fourier transform (DFT), which is defined as follows:

$$V_k = \sum_{n=0}^{L-1} v_n e^{-j2\pi nk/L}, \qquad k=0, 1, \ldots L-1$$

where L is the number of samples in the array being transformed. Also the inverse discrete Fourier transform is given by:

$$v_n = \frac{1}{L}\sum_{k=0}^{L-1} V_k e^{+j2\pi nk/L}, \qquad n=0, 1, \ldots, L-1 .$$

(c) The discrete Fourier transform and its inverse may be computed by means of the fast Fourier transform algorithm, which requires approximately $L\log_2 L$ operations (multiply-adds). The fast Fourier transform is discussed in the literature [8.14, 8.16 and 8.17].

In the actual implementation of the digital filter, recursive or nonrecursive, infinite accuracy of representation and computation are not possible. There are three primary sources of error that arise from the use of a finite word length computer. These errors are due to what is called "Quantization Effects". One source of error is incurred when the input to the filter is quantized to a finite number of bits. The second source of error arises in the evaluation of the arithmetic products and their sums. And finally, the last source of error arises in the representation of each of the digital filter coefficients by a fixed number of bits and by pole location accuracy. As indicated in Section 4 of the paper, the theory of quantization developed in the literature, so far as been concentrated on rough estimates, such as upper bounds and mean square errors [8.18, 8.19].

In summarizing the importance and impact of digital filters, one may mention the following advantages:

(a) Digital filters are flexible; changes in performance characteristics can be easily made.
(b) Filters not realizable with analog computers, such as transversal filters can be designed by digital means.
(c) Digital filters can be time-shared, thus servicing a number of inputs.
(d) With modular large scale integrated circuits (LSI), the design procedure for digital filters is faster and easier.

Another form of a discrete filter, known as the "Kalman Filter", [8.20] has found many practical applications for optimizing the performance of modern terrestrial and space navigation systems. The Kalman filter is quite different than the earlier discussed digital filter, because it optimizes recursively the least squared error of the system dynamics and applies correction to the system, while the digital filters, recursive or nonrecursive, are designed to particular frequency response characteristics. The Discrete Kalman Filter has been extended, modified and applied by many authors [8.21–8.25]. It is useful in many applications and hence its brief inclusion in this survey is warranted.

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9. SEQUENTIAL MACHINES [9.1–9.15]

One of the particular cases of discrete automatic systems is the sequential machine, which can be characterized by the fact that it has a finite number of discrete states that change at discrete times [9.1]. Sequential machines may be represented as ordinary pulsed systems with a special type of non-linearity performing the operation of comparing absolute values. Such an approach permits one to use the known theory pulsed systems for the analysis and synthesis of sequential machines [9.2]. However, a more promising approach for the analysis and synthesis of sequential machines, has proved to be the approach based on introducing the notions of the discrete Boolean function and the Laplace–Galois transform, which use operations on time functions which belong to a Galois field [9.3, 9.4].

A discrete Boolean function

$$y[n]=f(x_1[n],\dots,x_v[n]) \qquad (5)$$

is a function whose independent and dependent variables belong to the finite Galois field. This function enables us to express all logical operations by addition and multiplication over the elements of the field. The Laplace–Galois transform of some discrete Boolean function $y[n]$ is described by the relation

$$Y^*(q)=\sum_{n=0}^{\infty} e^{-qn}y[n]=D\{y[n]\},\ GF(2^k). \qquad (6)$$

Addition and multiplication are over the element of field $GF(2^k)$. Not all discrete Boolean functions have Laplace–Galois transforms, those which do not constitute irregular events in the sense of Kleene. To a direct Laplace–Galois transform there corresponds the inverse, as below:

$$y[n]=\sum_{q=-N}^{N} Y^*(q)\varepsilon^{qn}=D^{-1}\{Y^*(q)\},\ GF(2^k). \quad (7)$$

From the definitions follow several theorems that establish the relations between $y[n]$ and $Y^*(q)$ and operations over them. Such correspondences permit the introduction for sequential machines of important notions: transfer functions, time characteristics, and frequency characteristics. These notions enable one to carry out an analysis and a synthesis of linear sequential machines by regarding them as ordinary continuous or impulse dynamic systems. A similar approach to the one described above is based on using the p-adic number formalism [9.5].

Linear sequential machines have been also analysed using the Z-Transform Theory [9.6–9.12]. Recent extension of the multidimensional Z-Transform Theory [9.13, 9.14] have shown further explication to non-linear sequential machines. In particular, non-linear binary sequential circuits and their inverses have recently been studied using the above techniques [9.15].

† $GF(2^k)$, indicates Galois-Field of modulo—2^k.

Thus, the problems of coding, error correction, synthesis of encoders and decoders, which before were far removed from the general theory of control, today may be considered from the same point of view. In sequential machine theory it makes sense to talk about control, optimality, and stability. Today, there is a tendency to establish a direct connection between control systems, sequential machines, and finite automata.

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10. BIOCONTROL SYSTEMS [10.1–10.22]

In recent years one notices more and more application of pulsed system theory into the area of biology. The discrete model simulation of certain biological systems is being more emphasized and utilized because of the evidence of the discrete behavior of certain biological systems. In particular, the study of the human operator in control systems was recently approached using periodic sampled-data systems [10.1–10.5, 10.8, and 10.10–10.12]. As more physiological evidence was introduced the concept of periodic sampling was generalized to include Pulse Amplitude Pulse Width Modulation [10.6] and random sampling [10.7, 10.9 and 10.13]. Closely connected with the human operator modeling was the modeling of eye movement [10.10–10.13] as well as hand movement [10.14]. In both of these studies the concept of sampling or intermittancy is well emphasized. Recently more sophistication is being injected into the eye movement model by Hurthy and Deekshatulu [10.21]. Another major area of application of pulsed systems theory is that of neuron modeling. Both the concepts of Integral Pulse Frequency Modulation [10.15, 10.18, 10.20, 10.22] and sigma pulse frequency modulation has been applied to neural modeling [10.16, 10.17]. In a recent survey of biological control systems, the application of pulsed systems is extensively mentioned and discussed [10.19]. It appears from this brief description, that many more applications of the discrete theory and concepts will insure the usefulness and significance of pulsed system theory and application in the coming decades.

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CONCLUSION

The theory of discrete automatic systems plays an important role in the theory of continuous systems largely due to a widespread use of digital computers. In order to use digital computers, it is necessary, first of all, to construct a discrete model of the corresponding continuous problems. The results of discrete system theory are used as a basis for the compilation of approximate methods for the computation of transient processes in linear and non-linear continuous systems, estimate of errors of mathematical operations in digital differential analyzers, the determination of optimal characteristics of large scale control systems with the use of digital computers, etc. Discrete system theory finds wide application in the study of a number of processes in biological and economic systems, that have a discrete character. Quite recently one begins to notice the formation of a general approach which embraces various types of discrete systems; however, as yet, no end of this task is in sight.

Résumé—Les auteurs ont conjugué leurs efforts dans cet article pour écrire une revue de la théorie des systèmes discrets. Dans cet effort conjugué plusieurs sujets importants sont couverts: la description des systèmes à impulsions, l'analyse des systèmes à impulsions, la synthèse des systèmes à impulsions non-linéaires optimaux, l'échantillonnage aléatoire, les systèmes adaptatifs à impulsions, le filtrage numérique, les machines séquentielles et les systèmes de commande biologiques. Pour chacun de ces sujets des chapitres donnant une brève évaluation du travail accompli sont donnés et combinés à des références de la recherche réalisée aussi bien à l'Est qu'à l'Ouest. Étant donne l'espace limité disponible, les auteurs ont omis beaucoup d'autres références pertinentes pour cette revue. Toutefois, les références qui sont incluses sont bien connues des auteurs et peuvent être utilisées pour des études et des recherches ultérieures.

Le but de cet article consiste à présenter aux savants de l'Automatique un large aspect de la recherche réalisée dans le domaine important durant les deux dernières décennies. Il est espéré que les objectifs des auteurs auront été atteints.

Zusammenfassung—Die Autoren der Arbeit vereinigten ihre Anstrengungen, um eine Übersicht über die Theorie diskreter Systeme zu geben. Dabei werden eine ganze Reihe wichtiger Themen behandelt: Beschreibung von Impulssystemen, Analyse von Impulssystemen, Synthese von Impulssystemen, Analyse von nichtlinearen Impulssystemen, Synthese von optimalen nichtlinearen Systemen, zufällige Abtastung, adaptive Impulssysteme, digitale Filterung, sequentielle Maschinen und biologische Reglungssysteme. Für jedes der erwähnten Gebiete wird eine kurze Einschätzung des bisher Erreichten gebracht und zwat mit Hinweisen auf die in Ost und West durch geführte Forschung kombiniert. Wegen des beschränkten verfügbaren Raumes unterdrückten die Autoren viele andere dahin gehörige Quellen.

Die Arbeit soll den Wissenschaftlern auf dem Gebiet der Regelungstechnik das breite Spektrum der in den letzten zwei Jahrzehnten auf diesem wichtigen Gebiete durchgeführten Forschung nahebringen. Es ist zu hoffen, daß die Ziele der Verfasser erreicht werden.

Резюме—Авторы соединили свои усилия в этой статье чтобы написать обзор теории дискретных систем. В этом соединенном усилии, затронуты многие важные темы: описание импульсных систем, анализ импульсных систем, синтез нелинейных оптимальных систем, случайное квантование, адаптативные импульсные системы; цифровое фильтрирование, циклические машины и системы биологического управления. Для каждой из этих тем, даются главы кратко оценивающие проведенную работу и дающие референции изисканий осуществленных как на Востоке так и на Западе. Ввиду ограниченного доступного пространства, авторы опустили немало других референций полезных для этого обзора. Однако, включаемые референции хорошо известны авторам и могут быть использованы для дальнейших изучений и изысканий.

Целью этой статьи является желание дать ученым Автоматики широкий обзор изысканий осуществленных в этой важной области за два последных десятилетия. Имеется надежда что цели авторов смогут быть достигнуты.