

# Energy Based Control of Pendulum

Masafumi IWASHIRO \*, Katsuhisa FURUTA \* and Karl J. Åström \*\*

\* Department of Mechanical and Environmental Informatics, Tokyo Institute of Technology <sup>1</sup>

\*\* Department of Automatic Control, Lund Institute of Technology <sup>2</sup>

## Abstract

We have studied several control algorithms using pendulums. In this paper we consider a golf shot by a rotational type pendulum, that is, swinging a pendulum and hitting a golf ball by it. Firstly, we propose a control method based on an energy principle. Secondly, a control method in which the angle and energy of the pendulum are regulated is presented. The effectiveness of the proposed methods is shown by the experiments.

## 1. Introduction

The inverted-pendulum has been widely used in control laboratories to demonstrate the effectiveness of control system [1] [2] [4] in an analogy with the control of launching a rocket. The stabilization of not only a single inverted pendulum but also a double inverted pendulum and also multiple inverted pendulum using hinge control have also been studied. [5] [6] [8] Stabilization using feedback is one aspect of control, but there exists the other important issue to transfer the state from one equilibrium to the others. In contrast to the usual inverted pendulum, the swinging one cannot be regulated by a simple linear time-invariant controller. We have studied the problem to transfer a state of a double pendulum from a pendant state to the inverted one with controlling total kinematic energy of the system. [17]

In this paper, we propose a golf shot with a rotational type pendulum shown in Figure 1 in the sense of an application of energy based control that is used in the swinging pendulum. To shoot a golf ball, Energy is accumulated and is released it at the moment of impact. The non-actuation at the hinge of the pendulum makes it efficient because the free joint generally has less friction than actuated one. Besides this, good response data at the impact may be available. But it becomes more difficult to control it so as to shoot a ball accurately, so we adopt a control

method in which the angle of base and the energy of the pendulum are regulated. The proposed methods are applied to a TITech type pendulum, and experimental results will show that the proposed methods are effective and reliable for an actual implementation.

## 2. System Description

### 2.1. Special features of the rotate type

We have developed a rotationally driven pendulum of which a impact head is attached at the end shown in Figure 1. The pendulum has the following properties:

1. The first joint around which control input is exerted does not have hardware limitation about movement.
2. The pendulum is attached to a rigid arm fixed to a rotor of an actuator and there are less dynamic uncertainties than conventional one on a linear moving cart.
3. The actuator is a direct drive motor so it is free from backlash and flexibility of gear transmissions.

### 2.2. Modeling of the system

In order to derive a dynamic equation, coordinate frame systems shown in Figure 2 are introduced and parameters of the pendulum are defined in Table 1. Using the parameter in it Newton-Lagrangian formulation gives us the following dynamic equations:

$$M\ddot{q} + C\dot{q} + G = \tau \quad (1)$$

where

$$\begin{aligned} q &= [\theta_0, \theta_1]^T \\ M &= \begin{bmatrix} J_0 + m_1(L_0^2 + l_1^2 \sin^2 \theta_1) & m_1 L_0 l_1 \cos \theta_1 \\ m_1 L_0 l_1 \cos \theta_1 & J_1 + m_1 l_1^2 \end{bmatrix} \\ C &= \begin{bmatrix} m_1 l_1^2 \sin \theta_1 \cos \theta_1 \dot{\theta}_1 + d_0 & m_1 l_1^2 \sin \theta_1 \cos \theta_1 \dot{\theta}_0 \\ -m_1 l_1^2 \sin \theta_1 \cos \theta_1 \dot{\theta}_0 & d_1 \end{bmatrix} \\ G &= \begin{bmatrix} 0 \\ -m_1 l_1 g \sin \theta_1 \end{bmatrix} \\ \tau &= [u, 0]^T \quad (u \text{ is input torque by the motor}) \end{aligned}$$

<sup>1</sup>Department of Mechanical and Environmental Informatics, Tokyo Institute of Technology, 2-12-1 Oh-Okayama, Meguro-ku, Tokyo 152 Japan,  
E-mail: iwashiro@mei.titech.ac.jp

<sup>2</sup>Department of Automatic Control, Lund Institute of Technology, Box 118, S-221 00 LUND, Sweden

### 3. Energy Based Control in a Golf Swing

To shoot a golf ball by the pendulum, it accumulates energy by the control and releases it at the impact. We adopt two control methods for the golf swing by pendulum, acceleration and regulation of its energy and base angle  $\theta_0$  in order to shoot the golf ball accurately.

#### 3.1. Acceleration of pendulum

The basic idea is that the input is exerted toward the direction that energy of pendulum increases. Consider the motion of a single pendulum for which the input acceleration is horizontally affected. Then the dynamic equation is

$$ml^2\ddot{\theta} = mgl \sin \theta + mul \cos \theta$$

where  $\theta$  is its angle from a vertical line. The total energy of system and its time derivative are given by

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + mgl \cos \theta$$

$$\frac{dE}{dt} = mul \cos \theta \dot{\theta}.$$

Taking  $E_{ref}$  as the reference of energy, the control input is determined so that  $E_{ref} - E$  approaches to zero as follows.

$$u = \text{sat}_\alpha(k_e(E_{ref} - E)) \cdot \text{sign}(\cos \theta \cdot \dot{\theta})$$

where  $k_e > 0$  is a design parameter. In this expression,  $\text{sat}_\alpha(x)$  denotes a function which saturates at  $\alpha$  and  $\text{sign}(x)$  is following sign function.

$$\text{sign}(x) = \begin{cases} 1 & (x > 0) \\ 0 & (x = 0) \\ -1 & (x < 0) \end{cases}$$

In Figure 3, a simulation result of rotate type pendulum is shown as time series of the head speed of pendulum.

#### 3.2. Regulation of energy of pendulum and base

In a beginning, we show the simplification of dynamic equation of the pendulum. A general equation of pendulums with force or torque input  $u$  is

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix}$$

where  $x$  denotes position of a base to which a actuator is attached,  $\theta$  is angle of pendulum. Let  $f$  be acceleration of base. By substituting the input  $u$

$$u = \left( m_{11} - \frac{m_{12}m_{21}}{m_{22}} \right) f - \frac{m_{12}g_2}{m_{22}} + g_1$$

into the above equation, the follows are derived.

$$\ddot{x} = f$$

$$\ddot{\theta} = -\frac{g_2}{m_{22}} - \frac{m_{21}}{m_{22}}f$$

Dynamics equations of linear moving type pendulum can deform as below

$$\ddot{x} = f$$

$$\ddot{\theta} = -\frac{k_1 \sin \theta}{m_{22}} - \frac{k_2 \cos \theta}{m_{22}}f \quad (2)$$

where  $k_1, k_2, m_{22}$  are constants. There is no influence from the motion of pendulum to base position in this equation.

In order to explain concisely, we use this simplified model with  $k_1, k_2, m_{22} = 1$  for following discussion.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \sin x_1 + \cos x_1 u$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = u \quad (3)$$

where

- $x_1$  : angle of pendulum
- $x_2$  : angular velocity of pendulum
- $x_3$  : position of base
- $x_4$  : velocity of base.

Let energy of pendulum be  $\rho_1$ , position of base  $\rho_2$  and its velocity  $\rho_3$

$$\rho_1 = x_2^2 + 2 \cos x_1 + c$$

$$\rho_2 = x_3$$

$$\rho_3 = x_4$$

and we consider following Lyapunov function  $V$ .

$$V := k_1\rho_1^2 + k_2\rho_2^2 + k_3\rho_3^2 > 0 \quad (k_1, k_2, k_3 > 0)$$

Time derivative of  $V$  is given by

$$\frac{dV}{dt} = (4k_1\rho_1x_2 \cos x_1 + 2k_3\rho_3)u + 2k_2x_3x_4.$$

Substituting following controller  $u$

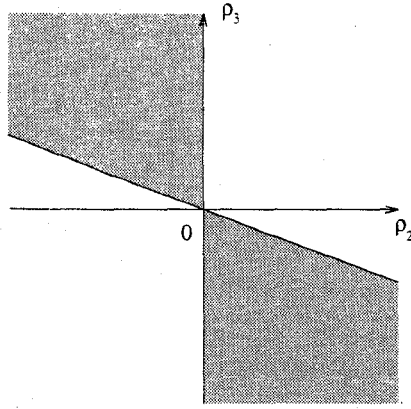
$$u = -4k_1\rho_1x_2 \cos x_1 - k_p\rho_2 - 2k_3\rho_3 \quad (k_p > 0)$$

into it, then

$$\frac{dV}{dt} = -\left(4k_1\rho_1x_2 \cos x_1 + \frac{k_p\rho_2}{2} + 2k_3\rho_3\right)^2$$

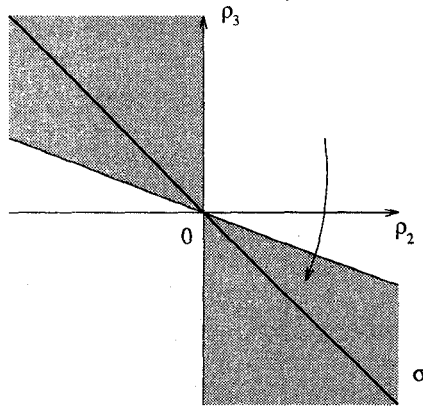
$$+ \frac{k_p^2}{4}\rho_2\left(\rho_2 + \frac{8k_2}{k_p^2}\rho_3\right) \quad (4)$$

If  $dV/dt$  is negative then  $\rho_1, \rho_2$  and  $\rho_3$  can converge to zero. The first term of (4) is always non-positive and the set  $(\rho_2, \rho_3)$  which second term of (4) is always non-positive is the gray part in the following figure.



The slant is equal to  $-k_p^2/8k_2 \leq 0$  and the gray part approaches to the second quadrant and the fourth one if we set  $k_2$  sufficiently bigger than  $k_p^2$ .

When  $(\rho_2, \rho_3)$  is in the first quadrant or the third one which  $dV/dt$  may become positive, the controller is switched to the other method so as to transfer state to the second quadrant or the fourth one. Choosing a switching line  $\sigma = 0$  which lies in the second quadrant and the fourth one, then reachable condition to the line is  $\sigma \cdot \dot{\sigma} < 0$ . While it may require an infinite time to reach the line by this condition, the state transfers into the second quadrant or the fourth one in a finite time before reaching the line.



We take a switching line  $\sigma : k\rho_2 + \rho_3 = 0 (0 < k < \infty)$  and the input  $u = -k_p\rho_2 - k_d\rho_3$ , then

$$\begin{aligned} \sigma \cdot \dot{\sigma} &= (k\rho_2 + \rho_3)(k\rho_4 + u) \\ &= -\frac{k_p}{k}(k\rho_2 + \rho_3)(k\rho_2 + k(k_d - k)/k_p\rho_3) \\ &\Rightarrow -\frac{k_p}{k}(kx_3 + x_4)^2 < 0 \\ &\quad [\text{If we choose } k \text{ as } k(k_d - k)/k_p = 1] \end{aligned}$$

If  $0 < 2\sqrt{k_p} < k_d$  then  $k > 0$  and the reachable condition is satisfied. In Figure 4, a simulation result of rotate type pendulum is shown as time series of the angle and the angular velocity of pendulum. In Figure 5, error of  $\rho_1, \rho_2, \rho_3$  from references are shown.

#### 4. Experimental Result

In Figure 6, a schematic drawing of the experimental systems is shown and parameters of the system are given in Table 2. The controller was realized on VME board with 68040 (25 MHz) and  $\theta_1$  and  $\theta_2$  were measured by optical encoders. The operating system was vxWorks and the control period was 1.5 [msec]. The head speed, trajectory, face angle and position of the club head just before impacts were measured by a golf training machine. The moments of impacts were observed by a photo sensor to switch control methods.

Physical conditions of the system are as follows.

- Position of a golf ball before impacts is fixed and known. It was placed at  $\theta_0 = 0, \theta_1 = \pi$ .
- If the angle between base and ball is smaller than 0.5 [rad], the pendulum contacts the golf ball.
- If the speed of a pendulum end becomes 50 [m/sec], then a period of rotation  $\theta_1$  is about 0.13[sec].

Golf swings by the rotate type pendulum were executed by three steps, acceleration of pendulum, shifting base to ball position with controlling energy and move base after the impact shown Figure 7. In Figure 8 the angle of base  $\theta_0$  of an experimental result is shown and the angle and the angular velocity of pendulum are shown in Figure 9. The time that the impact was observed is 3.0885 second and the states just before and right after it are as follows.

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \dot{\theta}_0 \\ \dot{\theta}_1 \end{bmatrix} : \begin{bmatrix} -6.306 \\ 3.203 \\ 2.645 \\ 35.081 \end{bmatrix} \Rightarrow \begin{bmatrix} -6.302 \\ 3.238 \\ 2.536 \\ 23.562 \end{bmatrix}$$

Figure 10 and Figure 11 are magnified plots near the impact and a picture right after impact is shown in Figure 12. From these experimental results as initial condition, the golf ball trajectories are given as shown Figure 13. We employ the aerodynamics of golf balls given by [16] and assume that the loft angle of golf club is 12 degree.

#### 5. Conclusion

In this paper we have proposed a golf shot with a rotate type pendulum by control methods based on an energy principle. The effectiveness of the proposed methods is shown by the experiments.

#### References

- [1] S. Mori, H. Nishihara and K. Furuta (1976) , "Control of unstable mechanical system Control of pendulum" *Int. J. Control*, vol.23, pp.673-692
- [2] N. Bakri, N. Becker and E. Ostertag (1988) , "Application of control disturbance observers for the

control and compensation of dry friction" *Automatisierungstechnik*, vol.36, pp.50-54

[3] W.R. Sturgeon, and M.V. Loscutoff (1972) , "Application of modal control and dynamic observers to control of a double inverted pendulum" *Proc. JACC. Stanford*, pp.857-865

[4] K. Furuta, H. Kajiwara and K. Kosuge (1980) , "Digital control of a double inverted pendulum on an inclined rail" *Int. J. Control*, vol.32, pp.907-924

[5] K. Furuta, T. Ochiai, and N. Ono (1984) , "Attitude control of a triple inverted of pendulum" *Int. J. Control*, vol.39, pp.1351-1365

[6] W. Maletinsky, M.F. Senning and F. Wiederkehr (1981) , "Observer based control of a double pendulum" *Proc. IFAC World Congress*, pp.3383-3387

[7] K. Furuta, A. Sano, D. Ahterton (1988) , "State variable methods in automatic control" *WILEY*

[8] H. Meier, Zu Farwig and H. Unbehauen (1990) , "Discrete computer control of a triple-inverted pendulum" *Opt. Cont. App. & Methods*, vol.11, pp.157-171

[9] K. Furuta, M. Yamakita, and S. Kobayashi (1992), "Swing up control of inverted pendulum using pseudo-state feedback", *Journal of Systems and Control Engineering*, Vol.206, pp. 263-269

[10] M. Wiklund, A. Kristenson and K. J. Astrom (1992), "A NEW STRATEGY FOR SWINGING UP AN INVERTED PENDULUM", *Proc. of IFAC Congress*, Vol. 9, pp. 151-154

[11] K. Kawashima, K. Kobayashi, N. Okubo, F. Sato, Y. Kaneko, S. Take , "Dynamics of Golf Goods" *Journal of the Japan Society for Precision Engineering*, Vol.59, No.10, pp.1593-1598, 1993.

[12] F. Sato , "Designing of Sporting Goods - Golf Club" *Journal of the Japan Society for Precision Engineering*, Vol.59, No.10, pp.1609-1613, 1993.

[13] C. C. Chung, J. Hauser, "Nonlinear Control of a Swinging Pendulum" *Automatica*, Vol.31, No.6, pp.851-862, 1995.

[14] J. Hauser, C. C. Chung , "Converse Lyapunov functions for exponentially stable periodic orbits" *Systems & Control Letters*, 23, pp.27-34, 1994.

[15] T. Iwatsubo , "A Study of Golf Ball Trajectory by Golf Swing" *Proceedings of 3rd Symposium of Sports Engineering*, pp.27-34, 1993.

[16] J. M. Davies , "The aerodynamic of golf balls" *Journal of Applied Physics*, 20-9, pp.821-828, 1949.

[17] M. Yamakita, M. Iwashiro, Y. Sugahara, K. Furuta , "Robust Swing Up Control of Double Pendulum" *Proceedings of the American Control Conference*, Seattle, Washington, pp.290-295, 1995.

$\theta_0$	angle of the actuated first joint
$\theta_1$	angle of the pendulum from a vertical line
$L_0$	length of the arm
$L_1$	length of the pendulum
$l_1$	length from free joint to the center of mass of the pendulum
$m_1$	mass of the pendulum
$J_0$	inertia of the arm around actuated first joint
$J_1$	inertia of the pendulum around the center of mass
$d_0, d_1$	viscosity of each joint

Table 1: Definitions of parameters

$L_0$	0.408 [m]	$L_1$	0.92 [m]
$l_1$	0.5 [m]	$m_1$	0.55 [kg]
$J_0$	1.2305 [N·m <sup>2</sup> ]	$J_1$	0.03 [N·m <sup>2</sup> ]
$d_0$	2.3214 [N/m·s]	$d_1$	$2.2558 \times 10^{-3}$ [N/m·s]

Table 2: Parameters of the system

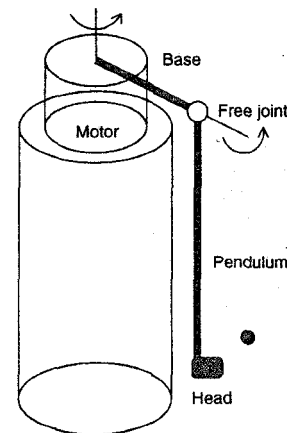


Figure 1: Rotate type pendulum

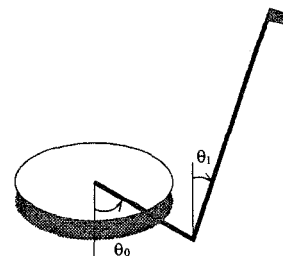


Figure 2: Coordinate frame systems of pendulum

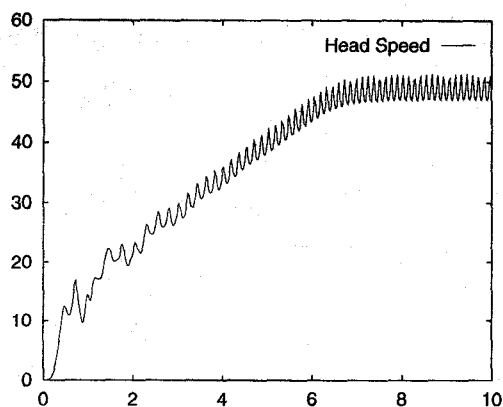


Figure 3: Head Speed at the acceleration of pendulum

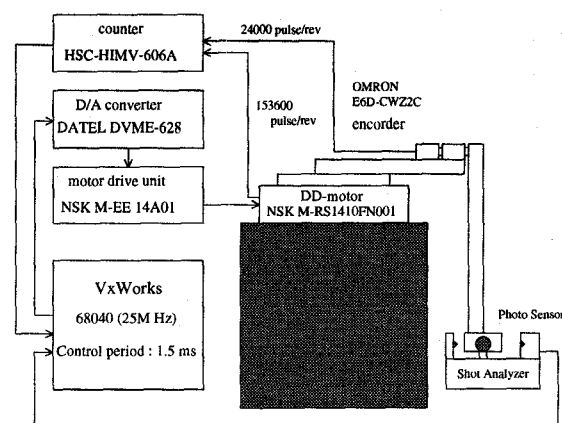


Figure 6: Schematic drawing of the experimental system

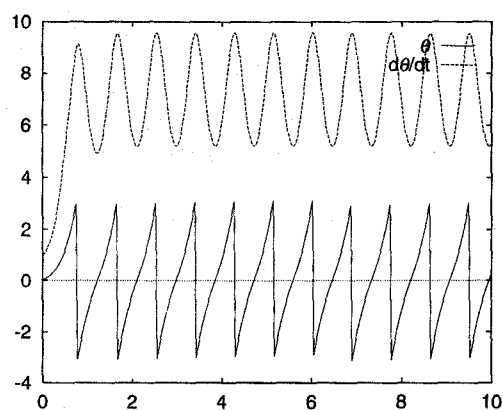


Figure 4: Angle and angular velocity of pendulum

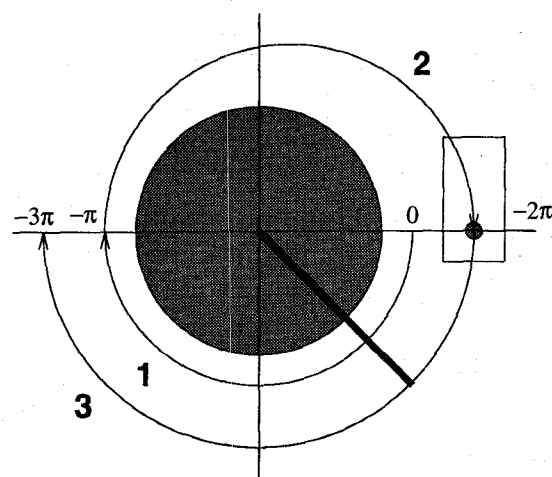


Figure 7: Three steps of golf swing in the experiment

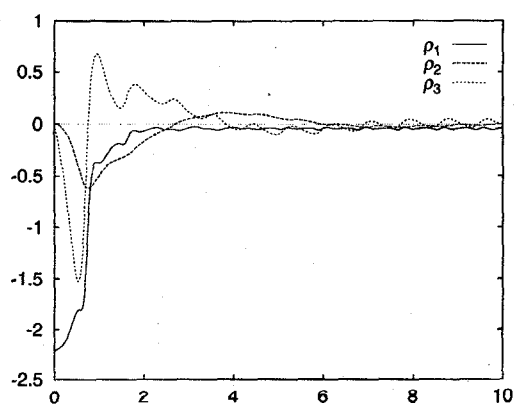


Figure 5: Response of the states ( $\rho_1, \rho_2, \rho_3$ )

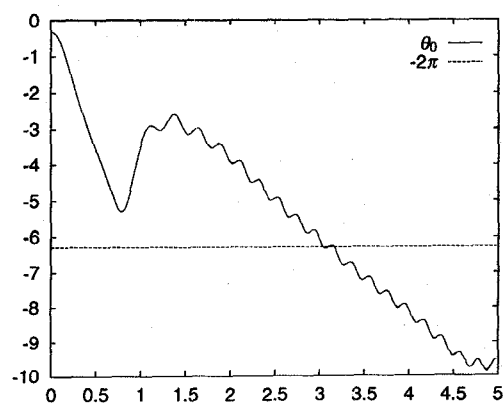
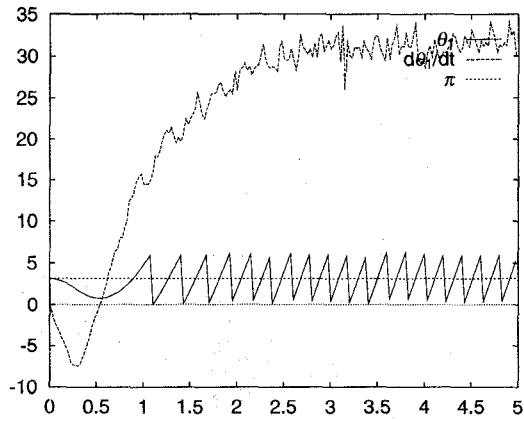
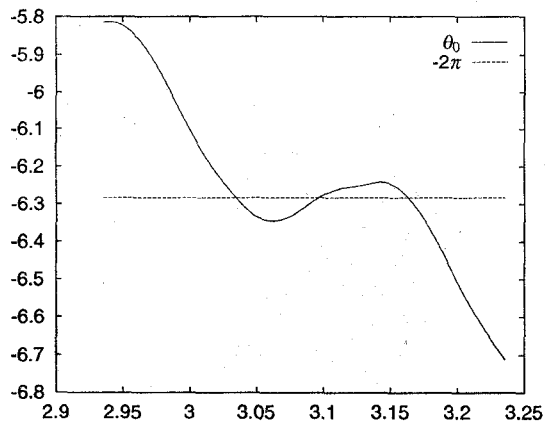


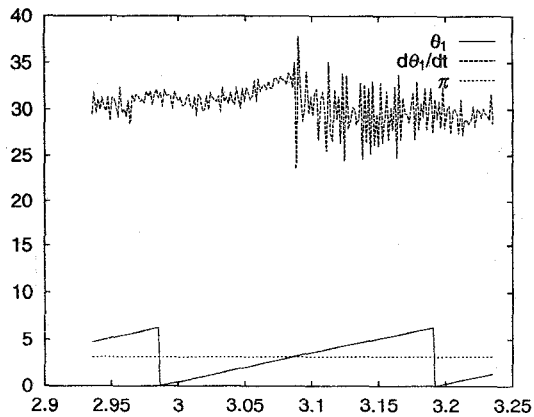
Figure 8: Angle of base in an experiment



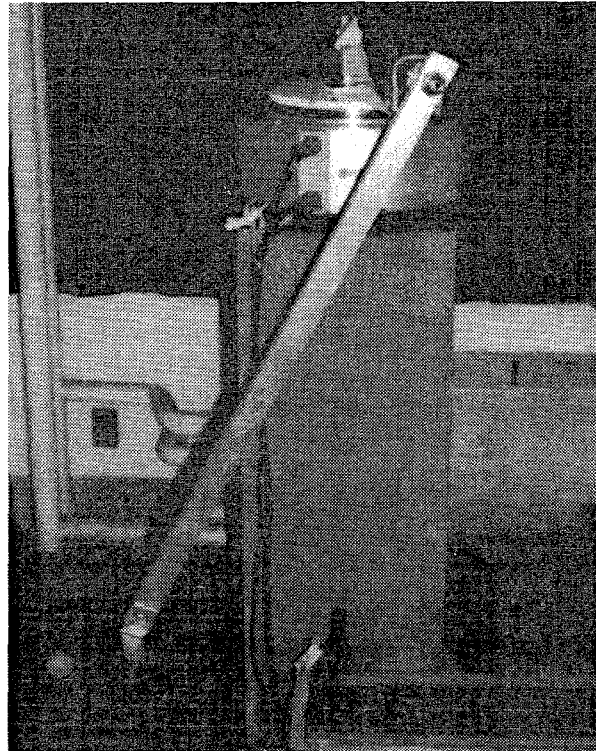
**Figure 9:** Angle and angular velocity of pendulum in an experiment



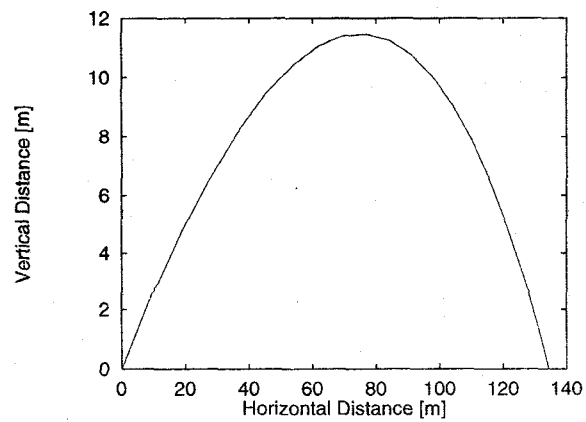
**Figure 10:** Angle of base close to impact



**Figure 11:** Angle and angular velocity of pendulum close to impact



**Figure 12:** A picture right after impact



**Figure 13:** Golf ball trajectory