

On Self Tuning Regulators*

Sur les Régulateurs Auto-Syntonisants

Über selbststellende Regler

О самонастраивающихся регуляторах

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Control laws obtained by combining a least squares parameter estimator and a minimum variance strategy based on the estimated parameters have asymptotically optimal performance.

Summary—The problem of controlling a system with constant but unknown parameters is considered. The analysis is restricted to discrete time single-input single-output systems. An algorithm obtained by combining a least squares estimator with a minimum variance regulator computed from the estimated model is analysed. The main results are two theorems which characterize the closed loop system obtained under the assumption that the parameter estimates converge. The first theorem states that certain covariances of the output and certain cross-covariances of the control variable and the output will vanish under weak assumptions on the system to be controlled. In the second theorem it is assumed that the system to be controlled is a general linear stochastic n th order system. It is shown that if the parameter estimates converge the control law obtained is in fact the minimum variance control law that could be computed if the parameters of the system were known. This is somewhat surprising since the least squares estimate is biased. Some practical implications of the results are discussed. In particular it is shown that the algorithm can be feasibly implemented on a small process computer.

1. INTRODUCTION

IT HAS been shown in several cases that linear stochastic control theory can be used successfully to design regulators for the steady state control of industrial processes. See Ref. [1]. To use this theory it is necessary to have mathematical models of the system dynamics and of the disturbances. In practice it is thus necessary to go through the steps of plant experiments, parameter estimation, computation of control strategies and implementation. This procedure can be quite time consuming in particular if the computations are made off-line. It might also be necessary to repeat the procedure if the system dynamics or the characteristics of the disturbances are changing as is often the case for industrial processes.

From a practical point of view it is thus meaningful to consider the control of systems with constant but unknown parameters. Optimal control problems for such systems can be formulated and solved using non-linear stochastic control theory. The solutions obtained are extremely impractical since even very simple problems will require computations far exceeding the capabilities of today's computers. For systems with constant but unknown parameters it thus seems reasonable to look for strategies that will converge to the optimal strategies that could be derived if the system characteristics were known. Such algorithms will be called *self-tuning* or *self-adjusting* strategies. The word adaptive is not used since adaptive, although never rigorously defined, usually implies that the characteristics of the process are changing. The problem to be discussed is thus simpler than the adaptive problem in the sense that the system to be controlled is assumed to have constant parameters.

The purpose of the paper is to analyse one class of self-adjusting regulators. The analysis is restricted to single-input single-output systems. It is assumed that the disturbances can be characterized as filtered white noise. The criterion considered is the minimization of the variance of the output. The algorithms analysed are those obtained on the basis of a separation of identification and control. To obtain a simple algorithm the identification is simply a least squares parameter estimator.

The main result is a characterization of the closed loop systems obtained when the algorithm is applied to a general class of systems. It is shown in Theorem 5.1 that if the parameter estimates converge the closed loop system obtained will be such that certain covariances of the inputs and the outputs of the closed loop system are zero. This is shown under weak assumptions on the system to be

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controlled. Moreover if it is assumed that the system to be controlled is a sampled finite dimensional linear stochastic system with a time delay in the control signal it is demonstrated in Theorem 5.2 that, if the parameter estimates converge, the corresponding regulator will actually converge to the minimum variance regulator. This is true, in spite of the fact that the least squares estimate is biased.

The major assumptions are that the system is minimum phase, that the time delay is known and that a bound can be given to the order of the system. The first two assumptions can be removed at the price of a more complicated algorithm.

The paper is organized as follows: sections 2 and 3 provide background and a motivation. The algorithm is given in section 4. Control strategies for systems with known parameters are given in section 2. Least squares parameter estimation is briefly reviewed in section 3. Some aspects on the notion of identifiability is also given in section 3. The algorithm presented in section 4 is obtained simply by fitting the parameters of a least squares structure as was described in section 3 and computing the corresponding minimum variance control strategy as was described in section 2. The possible difficulty with non-identifiability due to the feedback is avoided by fixing one parameter.

The main result is given as two theorems in section 5. We have not yet been able to prove that the algorithm converges in general. In section 6 it is, however, shown that a modified version of the algorithm converges for a first order system. The convergence properties of the algorithm are further illustrated by the examples in section 7. Some practical aspects of the algorithm as well as some problems which remain to be solved are given in section 8. In particular it is shown that the algorithm is easily implemented on a minicomputer.

2. MINIMUM VARIANCE CONTROL

This section gives the minimum variance strategy for a system with known parameters. Consider a system described by

$$\begin{aligned} y(t) + a_1 y(t-1) + \dots + a_n y(t-n) &= b_1 u(t-k-1) \\ &+ \dots + b_n u(t-k-n) + \lambda [e(t) + c_1 e(t-1) \\ &+ \dots + c_n e(t-n)], \\ t=0, \pm 1, \pm 2, \dots \end{aligned} \quad (2.1)$$

where u is the control variable, y is the output and $\{e(t), t=0, \pm 1, \pm 2, \dots\}$ is a sequence of independent normal $(0, 1)$ random variables. If the forward shift operator q , defined by

$$qy(t) = y(t+1)$$

and the polynomials

$$A(z) = z^n + a_1 z^{n-1} + \dots + a_n$$

$$B(z) = b_1 z^{n-1} + \dots + b_n, \quad b_1 \neq 0$$

$$C(z) = z^n + c_1 z^{n-1} + \dots + c_n$$

are introduced, the equation (2.1) describing the system can be written in the following compact form:

$$A(q)y(t) = B(q)u(t-k) + \lambda C(q)e(t). \quad (2.2)$$

It is well known that (2.1) or (2.2) is a canonical representation of a sampled finite dimensional single-input single-output dynamical system with time delays in the output and disturbances that are gaussian random processes with rational spectral densities.

The model (2.1) also admits a time delay τ in the system input which need not be a multiple of the sampling interval. The number k corresponds to the integral part of τ/h , where h is the sampling interval.

Let the criterion be

$$V_1 = E y^2(t) \quad (2.3)$$

or

$$V_2 = E \frac{1}{N} \sum_{t=1}^N y^2(t). \quad (2.4)$$

The optimal strategy is then

$$u(t) = - \frac{q^k G(q)}{B(q)F(q)} y(t) \quad (2.5)$$

where F and G are polynomials

$$F(z) = z^k + f_1 z^{k-1} + \dots + f_k \quad (2.6)$$

$$G(z) = g_0 z^{n-1} + g_1 z^{n-2} + \dots + g_{n-1} \quad (2.7)$$

determined from the identity

$$q^k C(q) = A(q)F(q) + G(q). \quad (2.8)$$

Proofs of these statements are given in [2]. The following conditions are necessary:

- The polynomial B has all zeroes inside the unit circle. Thus the system (2.1) is minimum phase.
- The polynomial C has all zeroes inside the unit circle.

These conditions are discussed at length in [1]. Let it suffice to mention here that if the system (2.1)

is non-minimum phase the control strategy (2.5) will still be a minimum variance strategy. This strategy will, however, be so sensitive that the slightest variation in the parameters will create an unstable closed loop system. Suboptimal strategies which are less sensitive to parameter variations are also well known. This paper will, however, be limited to minimum phase systems.

3. PARAMETER ESTIMATION

For a system described by (2.1) it is thus straight forward to obtain the minimum variance regulator, if the parameters of the model are known. If the parameters are not known it might be a possibility to try to determine the parameters of (2.1) using some identification scheme and then use the control law (2.5) with the true parameters substituted by their estimates. A suitable identification algorithm is the maximum likelihood method which will give unbiased estimates of the coefficients of the A , B and C polynomials. The maximum likelihood estimates of the parameters of (2.1) are, however, strongly non-linear functions of the inputs and the outputs. Since finite dimensional sufficient statistics are not known it is not possible to compute the maximum likelihood estimate of the parameters of (2.1) recursively as the process develops. Simpler identification schemes are therefore considered.

The least squares structure

The problem of determining the parameters of the model (2.1) is significantly simplified if it is assumed that $c_i = 0$ for $i = 1, 2, \dots, n$. The model is then given by

$$A(q)y(t) = B(q)u(t-k) + \lambda e(t+n). \quad (3.1)$$

The parameters of this model can be determined simply by the least squares method [3]. The model (3.1) is therefore referred to as a *least squares model*.

The least squares estimate has several attractive properties. It can easily be evaluated recursively. The estimator can be modified to take different model structures, e.g. known parameters, into account. The least squares estimates will converge to the true parameters e.g. under the following conditions.

- The output $\{y(t)\}$ is actually generated from a model (3.1).
- The residuals $\{e(t)\}$ are independent.
- The input is persistently exciting, see Ref. [3].
- The input sequence $\{u(t)\}$ is independent of the disturbance sequence $\{e(t)\}$.

These conditions are important. If the residuals are correlated the least squares estimate will be biased. If the input sequence $\{u(t)\}$ depends on

$\{e(t)\}$ it may not be possible to determine all parameters.

When the inputs are generated by a feedback they are correlated with the disturbances and it is not obvious that all the parameters of the model can be determined. Neither is it obvious that the input generated in this way is persistently exciting of sufficiently high order. A simple example illustrates the point.

Example 3.1

Consider the first order model

$$y(t) + ay(t-1) = bu(t-1) + e(t). \quad (3.2)$$

Assume that a linear regulator with constant gain

$$u(t) = \alpha y(t) \quad (3.3)$$

is used. If the parameters a and b are known the gain $\alpha = a/b$ would obviously correspond to a minimum variance regulator. If the parameters are not known the gain $\alpha = \hat{a}/\hat{b}$ where \hat{a} and \hat{b} are the least squares estimates of a and b could be attempted. The least squares parameter estimates are determined in such a way that the loss function

$$V(a, b) = \sum_{t=1}^N [y(t+1) + ay(t) - bu(t)]^2 \quad (3.4)$$

is minimal with respect to a and b . If the feedback control (3.3) is used the inputs and outputs are linearly related through

$$u(t) - \alpha y(t) = 0. \quad (3.5)$$

Multiply (3.5) by $-\gamma$ and add to the expression within brackets in (3.4).

Hence

$$\begin{aligned} V(a, b) &= \sum_{t=1}^N [y(t+1) + (a + \alpha\gamma)y(t) - (b + \gamma)u(t)]^2 \\ &= V(a + \alpha\gamma, b + \gamma). \end{aligned}$$

The loss function will thus have the same value for all estimates a and b on a linear manifold. Thus the two parameters a and b of the model (3.2) are not identifiable when the feedback (3.3) is used.

The simple example shows that it is in general not possible to estimate all the parameters of the model (3.1) when the input is generated by a feedback. Notice, however, that all parameters can be estimated if the control law is changed. In the particular example it is possible to estimate both parameters of the model, if the control law (3.3) is replaced by

$$u(t) = \alpha y(t-1)$$

or

$$u(t) = \alpha_1 y(t) + \alpha_2 y(t-1)$$

or if a time varying gain is used.

4. THE ALGORITHM

In order to control a system with constant but unknown parameters the following procedure could be attempted. At each step of time determine the parameters of the system (3.1) using least squares estimation based on all previously observed inputs and outputs as was described in section 3. Then determine a control law by calculating the minimum variance strategy for the model obtained. To compute the control law the identity (2.8) must be resolved in each step. The problem of computing the minimum variance regulator is simplified if it is observed that by using the identity (2.8) the system (3.1) can be written as

$$\begin{aligned} y(t+k+1) + \alpha_1 y(t) + \dots + \alpha_m y(t-m+1) \\ = \beta_0 [u(t) + \beta_1 u(t-1) + \dots + \beta_l u(t-l)] \\ + \varepsilon(t+k+1) \end{aligned} \quad (4.1)$$

where $m=n$ and $l=n+k-1$. The coefficients α_i and β_i are computed from the parameters a_i and b_i in (3.1) using the identity (2.8). The disturbance $\varepsilon(t)$ is a moving average of order k of the driving noise $e(t)$.

The minimum variance strategy is then simply

$$\begin{aligned} u(t) = \frac{1}{\beta_0} [\alpha_1 y(t) + \dots + \alpha_m y(t-m+1)] \\ - \beta_1 u(t-1) - \dots - \beta_l u(t-l). \end{aligned} \quad (4.2)$$

In order to obtain simple computation of the control strategy it could thus be attempted to use the model structure (4.1) which also admits least squares estimation. The trade-off for the simple calculation of the control law is that k more parameters have to be estimated.

As was shown in Example 3.1 all parameters of the model (4.1) can not necessarily be determined from input-output observations if the input is generated by a feedback (4.2) with constant parameters. In order to avoid a possible difficulty it is therefore assumed that the parameter β_0 is given. It will be shown in section 6 that the choice of β_0 is not crucial.

Summing up, the algorithm can be described as follows.

Step 1 parameter estimation

At the sampling interval t determine the parameters $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_l$ of the model

$$\begin{aligned} y(t) + \alpha_1 y(t-k-1) + \dots + \alpha_m y(t-k-m) \\ = \beta_0 [u(t-k-1) + \beta_1 u(t-k-2) \\ + \dots + \beta_l u(t-k-l-1)] + \varepsilon(t) \end{aligned} \quad (4.1)$$

using least squares estimation based on all data available at time t , i.e.

$$\sum_{k=0}^t \varepsilon^2(k)$$

minimum. The parameter β_0 is assumed known.

Step 2 control

At each sampling interval determine the control variable from

$$\begin{aligned} u(t) = \frac{1}{\beta_0} [\alpha_1 y(t) + \dots + \alpha_m y(t-m+1)] \\ - \beta_1 u(t-1) - \dots - \beta_l u(t-l) \end{aligned} \quad (4.2)$$

where the parameters α_i and β_i are those obtained in Step 1.

The control law (4.2) corresponds to

$$\begin{aligned} u(t) = \frac{\alpha_1 + \alpha_2 q^{-1} + \dots + \alpha_m q^{-m+1}}{\beta_0 [1 + \beta_1 q^{-1} + \dots + \beta_l q^{-l}]} y(t) \\ = \frac{q^{l-m+1} \mathcal{A}(q)}{\mathcal{B}(q)} y(t). \end{aligned} \quad (4.3)$$

Since the least squares estimate can be computed recursively the algorithm requires only moderate computations.

It should be emphasized that the algorithm is not optimal in the sense that it minimizes the criterion (2.3), or the criterion (2.4). It fails to minimize (2.3) because it is not taken into account that the parameter estimates are inaccurate and it fails to minimize (2.4) because it is not dual in FELDBAUM's sense [4]. These matters are discussed in [2]. It will however, be shown in section 5 that the algorithm has nice asymptotic properties.

The idea of obtaining algorithms by a combination of least squares identification and control is old. An early reference is KALMAN [5]. The particular algorithm used here is in essence the same as the one presented by PETERKA [6]. A similar algorithm where the uncertainties of the parameters are also considered is given in WIESLANDER-WITTENMARK [7].

5. MAIN RESULTS

The properties of the algorithm given in the previous section will now be analysed. We have

Theorem 5.1

Assume that the parameter estimates $\alpha_i(t)$, $i=1, \dots, m$, $\beta_i(t)$, $i=1, \dots, l$ converge as $t \rightarrow \infty$ and that the closed loop system is such that the output is *ergodic* (in the second moments). Then the closed loop system has the properties

$$Ey(t+\tau)y(t) = r_y(\tau) = 0 \quad \tau = k+1, \dots, k+m \quad (5.1)$$

$$Ey(t+\tau)u(t) = r_{yu}(\tau) = 0 \quad \tau = k+1, \dots, k+l+1. \quad (5.2)$$

Proof

The least squares estimates of the parameters $\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_l$ is given by the equations

Assume that the parameters converge. For sufficiently large N_0 the coefficients of control law (4.2) will then converge to constant values. Introduction of (4.2) into (5.3) gives

$$\begin{aligned} \Sigma y(t+k+1)y(t) &= 0 \\ \Sigma y(t+k+1)y(t-1) &= 0 \\ &\vdots \\ \Sigma y(t+k+1)y(t-m+1) &= 0 \\ \Sigma y(t+k+1)u(t-1) &= 0 \\ &\vdots \\ \Sigma y(t+k+1)u(t-l) &= 0. \end{aligned}$$

Using the control law (4.2) it also follows that

$$\Sigma y(t+k+1)u(t) = 0.$$

Under the ergodicity assumption the sums can furthermore be replaced by mathematical expectations and the theorem is proven.

$$\begin{bmatrix} \Sigma y(t)^2 & \Sigma y(t)y(t-1) & \dots & \Sigma y(t)y(t-m+1) & -\beta_0 \Sigma y(t)u & \dots & -\beta_0 \Sigma y(t)u(t-l) \\ \Sigma y(t)y(t-1) & & & \Sigma y(t-1)y(t-m+1) & \vdots & & \vdots \\ \vdots & & & \vdots & \vdots & & \vdots \\ \Sigma y(t)y(t-m+1) & & & \Sigma y^2(t-m+1) & -\beta_0 \Sigma y(t-m+1)u(t-1) & \dots & -\beta_0 \Sigma y(t-m+1)u(t-l) \\ \hline -\beta_0 \Sigma y(t)u(t-1) & & & & \beta_0^2 \Sigma u^2(t-1) & \dots & \beta_0^2 \Sigma u(t-1)u(t-l) \\ \vdots & & & & \vdots & & \vdots \\ -\beta_0 \Sigma y(t)u(t-l) & \dots & & & \dots & & \beta_0^2 \Sigma u^2(t-l) \end{bmatrix}.$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \\ \beta_1 \\ \vdots \\ \beta_l \end{bmatrix} = \begin{bmatrix} -\Sigma y(t+k+1)y(t) + \beta_0 \Sigma u(t)y(t) \\ \Sigma y(t+k+1)y(t-1) + \beta_0 \Sigma u(t)y(t-1) \\ \vdots \\ -\Sigma y(t+k+1)y(t-m+1) + \beta_0 \Sigma u(t)y(t-m+1) \\ -\beta_0 \Sigma y(t+k+1)u(t-1) - \beta_0^2 \Sigma u(t)u(t-1) \\ \vdots \\ \beta_0 \Sigma y(t+k+1)u(t-l) - \beta_0^2 \Sigma u(t)u(t-l) \end{bmatrix} \quad (5.3)$$

where the sums are taken over N_0 values. See Ref. [3].

Remark 1

Notice that the assumptions on the system to be controlled are very weak. In particular it is not necessary to assume that the system is governed by an equation like (2.1) or (3.1).

Remark 2

It is sufficient for ergodicity that the system to be controlled is governed by a difference equation of finite order, e.g. like (2.1), and that the closed loop system obtained by introducing the feedback law into (2.1) gives a stable closed loop system.

Remark 3

The self tuning algorithm can be compared with a PI-regulator. If the state variables of a deterministic system with a PI-regulator converge to steady state values, the control error must be zero irrespective of the properties of the system. Analogously theorem 5.1 implies that if the parameter estimates of the self tuning algorithm converge the covariances (5.1) and (5.2) are zero.

Remark 4

Theorem 5.1 holds even if the algorithm is modified in such a way that the parameter estimation (Step 1) is not done in every step.

If it is assumed that the system to be controlled is governed by an equation like (2.1) it is possible to show that the conditions (5.1) and (5.2) in essence imply that the self tuning regulator will converge to a minimum variance regulator. We have

Theorem 5.2

Let the system to be controlled be governed by the equation (2.1). Assume that the self tuning algorithm is used with $m=n$ and $l=n+k-1$. If the parameter estimates converge to values such that the corresponding polynomials \mathcal{A} and \mathcal{B} have no common factors, then the corresponding regulator (4.2) will converge to a minimum variance regulator.

Proof

Assume that the least squares parameter estimates converge. The regulator is then given by (4.3) i.e.

$$u(t) = \frac{q^{l-m+1}\mathcal{A}(q)}{\mathcal{B}(q)} y(t) = \frac{q^k\mathcal{A}(q)}{\mathcal{B}(q)}$$

where the coefficients of \mathcal{A} and \mathcal{B} are constant. Since the system to be controlled is governed by (2.1) the closed loop system becomes

$$[A(q)\mathcal{B}(q) - B(q)\mathcal{A}(q)]y(t) = \lambda C(q)\mathcal{B}(q)e(t). \quad (5.4)$$

The closed loop system is of order $r=n+l$. Introduce the stochastic process $\{v(t)\}$ defined by

$$v(t) = \lambda \frac{q^l C(q)}{A(q)\mathcal{B}(q) - B(q)\mathcal{A}(q)} e(t). \quad (5.5)$$

Then

$$y(t) = q^{-l}\mathcal{B}(q)v(t) \quad (5.6)$$

and

$$u(t) = q^{-m+1}\mathcal{A}(q)v(t). \quad (5.7)$$

Multiplying (5.6) and (5.7) by $y(t+\tau)$ and taking mathematical expectations gives

$$r_{yy}(\tau) = r_{yv}(\tau) + \beta_1 r_{yv}(\tau+1) + \dots + \beta_l r_{yv}(\tau+l) \quad (5.8)$$

$$r_{yu}(\tau) = \alpha_1 r_{yv}(\tau) + \alpha_2 r_{yv}(\tau+1) + \dots + \alpha_m r_{yv}(\tau+m-1). \quad (5.9)$$

Furthermore it follows from Theorem 5.1, equations (5.1) and (5.2), that the left member of (5.8) vanishes for $\tau=k+1, \dots, k+m$ and that the left member of (5.9) vanishes for $\tau=k+1, \dots, k+l+1$. We thus obtain the following equation for $r_{yv}(\tau)$.

$$\begin{bmatrix} 1 & \beta_1 & \dots & \beta_l & 0 & \dots & 0 \\ 0 & 1 & \beta_1 & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & 0 \\ 0 & \dots & 0 & 1 & \beta_1 & & \beta_l \\ 0 & \alpha_1 & \dots & \alpha_m & 0 & \dots & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & 0 \\ 0 & \dots & & \alpha_1 & \dots & \alpha_m & \end{bmatrix} \begin{bmatrix} r_{yv}(k+1) \\ r_{yv}(k+2) \\ \cdot \\ \cdot \\ r_{yv}(k+l+m) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad (5.10)$$

Since the polynomials \mathcal{A} and \mathcal{B} have no common factor it follows from an elementary result in the theory of equations [8, p. 145] that the $(l+m) \times (l+m)$ -matrix of the left member of (5.10) is non-singular.

Hence

$$r_{yv}(\tau) = 0 \quad \tau = k+1, \dots, k+l+m. \quad (5.11)$$

Since v is the output of a dynamical system of order $r = n + l = m + l$ driven by white noise it follows from (5.11) and the Yule-Walker equation that

$$r_{yv}(\tau) = 0 \quad \tau \geq k+1. \quad (5.12)$$

The equation (5.8) then implies

$$r_y(\tau) = 0 \quad \tau \geq k+1. \quad (5.13)$$

The output of the closed loop system is thus a moving average of white noise of order k . Denote this moving average by

$$y(t) = q^{-k} F(q) e(t) \quad (5.14)$$

where F is a polynomial of degree k . It follows from (5.4) and (5.14) that

$$q^k C = FA - \frac{BF\mathcal{A}}{\mathcal{B}}.$$

Since $q^k C$ and FA are polynomials $BF\mathcal{A}/\mathcal{B}$ must also be a polynomial.

Hence

$$q^k C = FA + G \quad (5.15)$$

where

$$G = -\frac{BF\mathcal{A}}{\mathcal{B}} \quad (5.16)$$

is of degree $n-1$ and F of degree k . A comparison with (2.8) shows, however, that (5.15) is the identity which is used to derive the minimum variance strategy. It thus follows from (2.5) that the minimum variance strategy is given by

$$u(t) = -\frac{q^k G}{BF} y(t).$$

The equation (5.16) then implies that

$$-\frac{q^k G}{BF} = \frac{q^k \mathcal{A}}{\mathcal{B}} \quad (5.17)$$

and it has thus been shown that (4.2) is a minimum variance strategy.

Remark

The conditions $m=n$ and $l=n+k-1$ mean that there are precisely the number of parameters in the model that are required in order to obtain the correct regulator for the process (2.1). Theorem 5.2 still holds if there are more parameters in the regulator in the following cases.

(i) Theorem 5.2 still holds if $m=n$ and $l \geq n+k-1$. In this case the order of the system is $r=n+l$ and since $m=n$ the equation (5.10) implies (5.11)–(5.13) and the equation (5.16) is changed to

$$G = -\frac{q^{l-k-m+1} BF\mathcal{A}}{\mathcal{B}}. \quad (5.16')$$

The rest of the proof remains unchanged.

(ii) If $m \geq n$ and $l = n+k-1$ the theorem will also hold. The closed loop system is of order $r \leq m+l$ but the equation (5.10) will still imply (5.11) and (5.16) is changed to (5.16'). The rest of the proof remains unchanged.

(iii) If $m > n$ and $l > n+k-1$ the theorem does not hold because \mathcal{A} and \mathcal{B} must have a common factor if the parameter estimates converge. It can, however, be shown that if the algorithm is modified in such a way that common factors of \mathcal{A} and \mathcal{B} are eliminated before the control signal is computed, Theorem 5.2 will still hold for the modified algorithm.

6. CONVERGENCE OF THE ALGORITHM

It would be highly desirable to have general results giving conditions for convergence of the parameter estimates. Since the system (2.1) with the regulator (4.2) and the least squares estimator is described by a set of nonlinear time dependent stochastic difference equations the problem of a general convergence proof is difficult. So far we have not been able to obtain a general result. It has, however, been verified by extensive numerical simulations that the algorithm does in fact converge in many cases. The numerical simulations as well as analysis of simple examples have given insight into some of the conditions that must be imposed in order to ensure that the algorithm will converge.

A significant simplification of the analysis is obtained if the algorithm is modified in such a way that the parameter estimates are kept constant over long periods of time. To be specific a simple example is considered.

Example 6.1

Let the system be described by

$$y(t) + ay(t-1) = bu(t-1) + e(t) + ce(t-1) \quad |c| < 1. \quad (6.1)$$

Assume that the control law

$$u(t) = \alpha_n y(t) \quad (6.2)$$

is used in the time interval $t_n < t < t_{n+1}$ where the parameter α_n is determined by fitting the least squares model

$$y(t+1) + \alpha y(t) = u(t) + \varepsilon(t+1) \quad (6.3)$$

to the data $\{u(t), y(t), t = t_{n-1}, \dots, t_n - 1\}$.

The least squares estimate is given by

$$\begin{aligned} \alpha_n &= - \frac{\sum_{t=t_{n-1}}^{t_n-2} y(t)[y(t+1) - u(t)]}{\sum_{t=t_{n-1}}^{t_n-2} y^2(t)} \\ &= \alpha_{n-1} - \frac{\sum_{t=t_{n-1}}^{t_n-2} y(t+1)y(t)}{\sum_{t=t_{n-1}}^{t_n-2} y^2(t)} \end{aligned} \quad (6.4)$$

where the last equality follows from (6.2). Assume that $t_n - t_{n-1} \rightarrow \infty$ and that

$$|a - b\alpha_{n-1}| < 1 \quad (6.5)$$

which means that the closed loop system used during the time interval $t_{n-1} < t < t_n$ is stable then

$$\alpha_n = \alpha_{n-1} - \frac{r_y(1)}{r_y(0)} \quad (6.6)$$

where $r_y(\tau)$ is the covariance function of the stochastic process $\{y(t)\}$ defined by

$$y(t) + (a - b\alpha_{n-1})y(t-1) = e(t) + ce(t-1). \quad (6.7)$$

Straightforward algebraic manipulations now give

$$\alpha_n = \alpha_{n-1} - \frac{(c - a + b\alpha_{n-1})(1 - ac + bc\alpha_{n-1})}{1 + c^2 - 2ac + 2bc\alpha_{n-1}}. \quad (6.8)$$

The problem is thus reduced to the analysis of the nonlinear difference equation given by (6.8).

Introduce

$$x_n = \alpha_n - \frac{a-c}{b} \quad (6.9)$$

the equation (6.8) then becomes

$$x_{n+1} = g(x_n) = (1-b)x_n + \frac{b^2 c x_n^2}{1 - c^2 + 2bcx_n}. \quad (6.10)$$

The point $x=0$ is a fixed point of the mapping g which corresponds to the optimal value of gain of the feedback loop, i.e. $\alpha = (a-c)/b$. The problem is thus to determine if this fixed point is stable. Since the closed loop system is assumed to be stable it is sufficient to consider

$$\frac{c-1}{b} < x < \frac{c+1}{b}. \quad (6.11)$$

Three cases have to be investigated

1. $c=0$
2. $c>0$ or $c<0$, $0 < b \leq 1$
3. $c>0$ or $c<0$, $1 < b < 2$.

For all cases $g(x) \approx (1-b)x$ if x is small. This implies that solutions close to 0 converge to $x=0$ if $0 < b < 2$.

Case 1

The equation (6.10) reduces to

$$x_{n+1} = (1-b)x_n$$

and the fixed point $x=0$ is stable if $|1-b| < 1$ i.e. $0 < b < 2$.

Case 2

The principal behavior of $g(x)$ when $c>0$ and $0 < b \leq 1$ is shown in Fig. 1. It is straightforward to verify that all initial values in the stability region $(c-1)/b < x < (c+1)/b$ will give solutions which converge to zero.

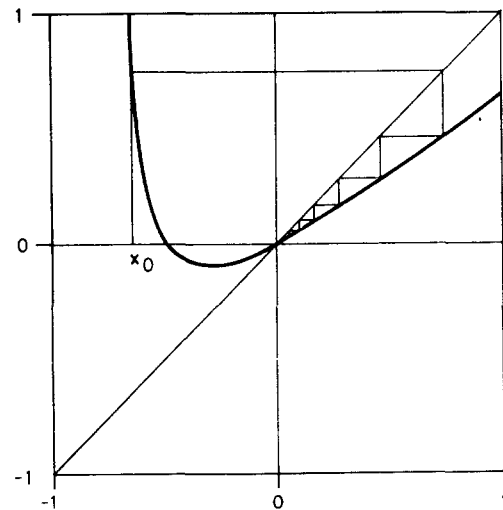


FIG. 1. Graph of the function g when $0 < b \leq 1$. The figure is drawn with the parameter values $a = -0.5$, $b = 0.5$ and $c = 0.7$.

Case 3

The function $g(x)$ for this case is shown in Fig. 2. It is not obvious that x will converge to zero, because there might exist a "limit cycle".

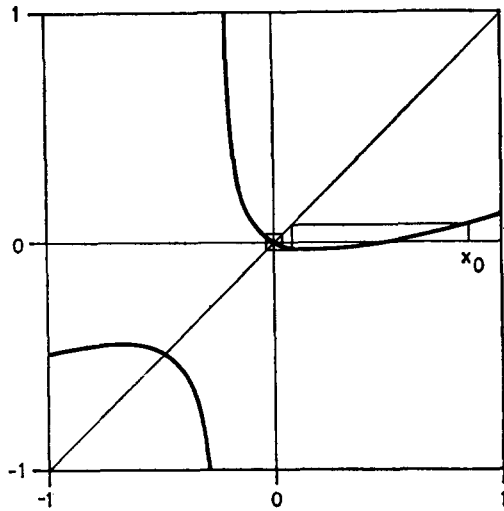


FIG. 2. Graph of the function g when $1 < b < 2$. The figure is drawn with the parameter values $a = -0.5$, $b = 1.5$ and $c = 0.7$.

If $c > 0$ and starting with $x_0 > 0$ then if $g(x_0) < 0$ it can be shown that after two iterations the new value x_2 will satisfy $0 < x_2 < x_0$ i.e.

$$0 < g(g(x)) < x \quad \text{if } x > 0, g(x) < 0.$$

If $(c-1)/b < x < 0$ then $g(x)$ will be positive and can be taken as a new initial value for which the condition above will be satisfied. If $c < 0$ then it can be shown that $g(g(x)) > x$ if $x < 0$ and $g(x) > 0$.

Summary

From the analysis above we can conclude that $x=0$ is a stable fixed point if

$$-1 < c < 1$$

and

$$0 < b < 2. \quad (6.12)$$

The example shows that under the condition (6.12) the version of the self-tuning algorithm where the parameters of the control law are kept constant over long intervals will in fact converge. In the analysis above $\beta_0 = 1$ was chosen. If $\beta_0 \neq 1$ then the condition (6.12) is replaced by

$$0 < b/\beta_0 < 2 \quad (6.12')$$

or

$$0.5b < \beta_0 < \infty. \quad (6.13)$$

The condition (6.13) implies that it is necessary to pick the parameter β_0 in a correct manner. The algorithm will always converge if β_0 is greater than b . Under-estimation may be serious and the value $\beta_0 < 0.5b$ gives an unstable algorithm.

The analysis presented in the simple example can be extended to give stability conditions for the modified algorithm in more complex cases. The analysis required is tedious.

7. SIMULATIONS

The results in section 5 are given under the assumption that the least squares estimator really converges, but yet we have not been able to give general conditions for convergence. But simulation of numerous examples have shown that the algorithm has nice convergence properties.

This section presents a number of simulated examples which illustrate the properties of the self-tuning algorithm.

Example 7.1

Let the system be

$$y(t) + ay(t-1) = bu(t-1) + e(t) + ce(t-1) \quad (7.1)$$

with $a = -0.5$, $b = 3$ and $c = 0.7$. The minimum variance regulator for the system is

$$u(t) = \frac{a-c}{b}y(t) = -0.4y(t). \quad (7.2)$$

A regulator with this structure can be obtained by using the self-tuning algorithm based on the model

$$y(t+1) + \alpha y(t) = \beta_0 u(t) + \varepsilon(t+1). \quad (7.3)$$

Figure 3 shows for the case $\beta_0 = 1$ how the parameter estimate converges to the value $\alpha = -0.4$ which corresponds to the minimum variance strategy (7.2).

In Figure 4 is shown the expected variance of the output if the current value of α should be used for all future steps of time. Notice that the algorithm has practically adjusted over 50 steps.

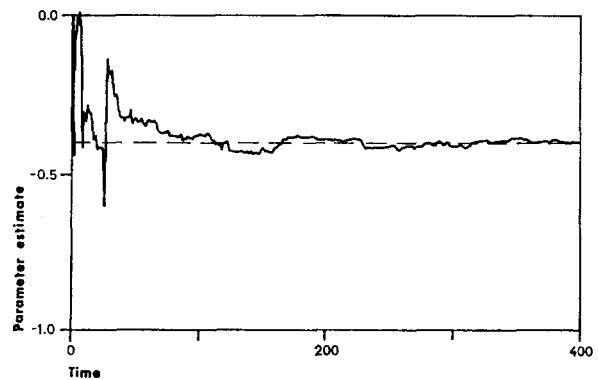


FIG. 3. Parameter estimate $\alpha(t)$ obtained when the self tuning algorithm based on the model (7.3) is applied to the system given by (7.1). The minimum variance regulator corresponds to $\alpha = -0.4$ and is indicated by the dashed line.

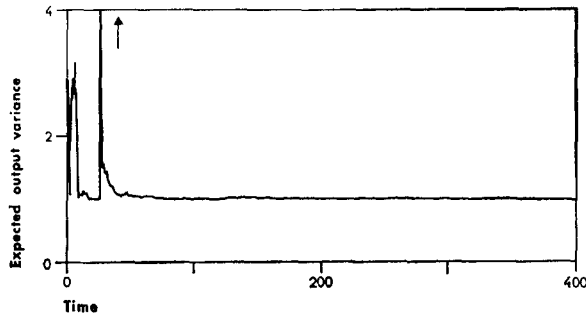


FIG. 4. Expected variance of the output of Example 7.1 if the control law obtained at time t is kept constant for all future times. Notice that the estimate at time $t=26$ would give an unstable system.

The analysis of Example 6.1 shows that, since $b > 2$, and $\beta_o = 1$ the modified self-tuning algorithm obtained when the parameters of the controller are kept constant over long intervals is unstable. The simulation in Example 7.1 shows that at least in the special case a conservative estimate of the convergence region is obtained by analysing the modified algorithm. If the value of b is increased further it has been shown that the algorithm is unstable. Unstable realizations have been found for $b=5$. In such cases it is of course easy to obtain a stable algorithm by increasing β_o . This requires, however, a knowledge of the magnitude of b .

The system of Example 7.1 is very simple. For instance, if no control is used the variance will still be reasonably small. The next example is more realistic in this aspect.

Example 7.2

Consider the system

$$y(t) - 1.9y(t-1) + 0.9y(t-2) = u(t-2) + u(t-3) + e(t) - 0.5e(t-1). \quad (7.4)$$

If no control is used the variance of the output is infinite. Also notice that $B(z) = z - 1$. The assumption that B has all zeroes inside the unit circle is thus violated. The minimum variance strategy for the system is

$$u(t) = -1.76y(t) + 1.26y(t-1) - 0.4u(t-1) + 1.4u(t-2). \quad (7.5)$$

A regulator with this structure is obtained by using the self-tuning algorithm with the model

$$y(t+2) + \alpha_1 y(t) + \alpha_2 y(t-1) = u(t) + \beta_1 u(t-1) + \beta_2 u(t-2) + e(t+2). \quad (7.6)$$

The convergence of the parameters is shown in Fig. 5. Figure 6 shows the accumulated losses when

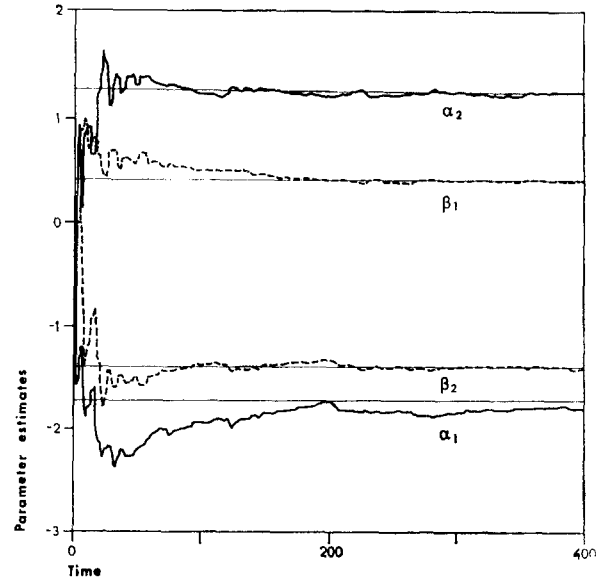


FIG. 5. Parameter estimates α_1 , α_2 , β_1 and β_2 obtained when the self tuning algorithm based on (7.6) is applied to the system given by (7.4). The thin lines indicate the parameter values of the minimum variance strategy.

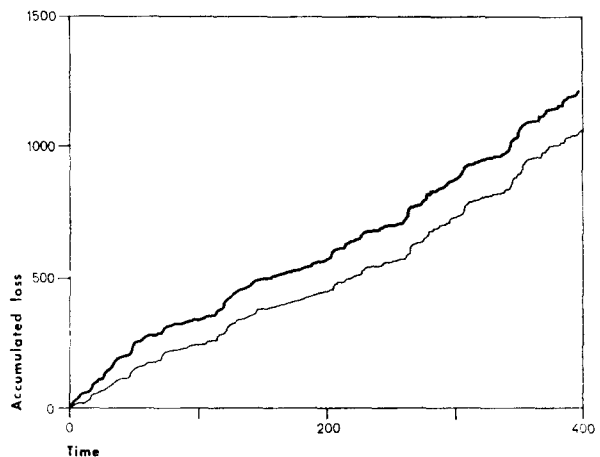


FIG. 6. Accumulated loss

$$\sum_{s=1}^t y^2(s)$$

for a simulation of the system (7.4) when using the self tuning algorithm (thick line) and when using the optimal minimum variance regulator (7.5) (thin line).

using the self-tuning algorithm and when using the optimal minimum variance regulator (7.5).

In both examples above, the models in the self-tuning algorithm have had enough parameters so it could converge to the optimal minimum variance regulator. The next example shows what happens when the regulator has not enough parameters.

Example 7.3

Consider the system

$$\begin{aligned} y(t) - 1.60y(t-1) + 1.61y(t-2) - 0.776y(t-3) \\ = 1.2u(t-1) - 0.95u(t-2) + 0.2u(t-3) + e(t) \\ + 0.1e(t-1) + 0.25e(t-2) + 0.87e(t-3). \end{aligned} \quad (7.7)$$

The polynomial $A(z)$ has two complex zeroes near the unit circle ($+0.4 \pm 0.9i$) and one real zero equal to 0.8.

If a self-tuning regulator is determined based on a model with $m=3$ and $l=2$ it will converge to the minimum variance regulator as expected. Figure 7(a) shows a small sample of the output together with the sample covariance of the output, $\hat{p}_y(\tau)$.

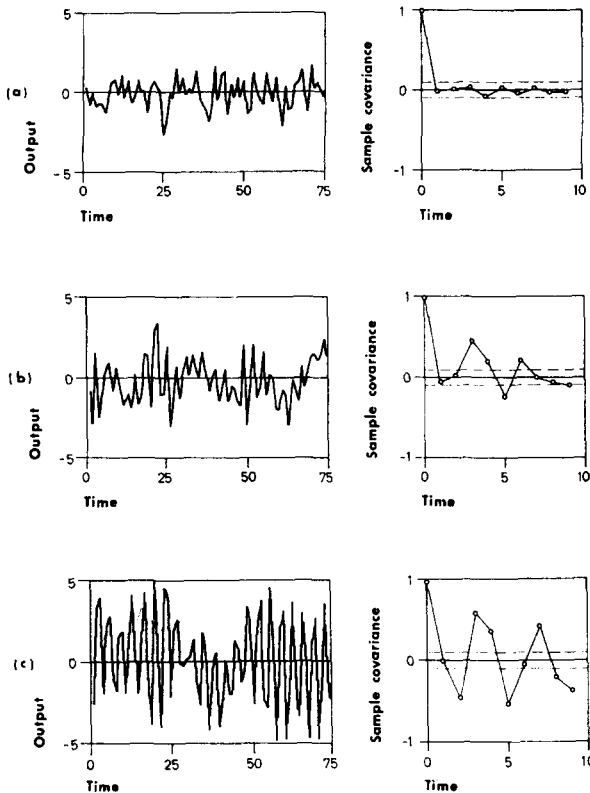


FIG. 7. Output of the system (7.7) and sample covariance of the output $\hat{p}_y(\tau)$ when controlling with self tuning regulator is having different number of parameters

(a) $m=3$ $l=2$ (b) $m=2$ $l=1$ (c) $m=1$ $l=0$.

The dashed lines show the 5 per cent confidence intervals for $\tau \neq 0$.

If the self-tuning algorithm instead is based on a model with $m=2$ and $l=1$ it is no longer possible to obtain the minimum variance regulator for the system since there are not parameters enough in the self-tuning regulator. Theorem 5.1 indicates, however, that if the self-tuning regulator converges, its parameters will be such that the covariances $r_y(1)$, $r_y(2)$, $r_{yu}(1)$ and $r_{yu}(2)$ are all zero. The simulation shows that the algorithm does in fact converge with $\beta_0 = 1.0$. The covariance function of the output is shown in Fig. 7(b). It is seen that the sample covariances $\hat{p}_y(1)$ and $\hat{p}_y(2)$ are within the 5 per cent confidence interval while $\hat{p}_y(3)$ is not as would be expected from Theorem 5.1.

If a self-tuning algorithm is designed based on a model with $m=1$, $l=0$ then Theorem 5.1 indicates that $r_y(1)$ should vanish. Again the simulation

shows that the algorithm does in fact converge and that the sample covariance $\hat{p}_y(1)$ does not differ significantly from zero. See Fig. 7(c).

When using regulators of lower order than the optimal minimum variance regulator, the parameters in the controller will not converge to values which for the given structure gives minimum variance of the output. In Table 1 is shown the variance of the output for the system above when using different regulators.

The loss when using the self-adjusting regulator is obtained through simulations. The optimal regulator is found by minimizing $r_y(0)$ with respect to the parameters in the controller.

TABLE 1

m	l	Loss $\frac{1}{N} \sum_{t=1}^N H y_2(t)$	
		Self-adjusting	Optimal
3	2	1.0	1.0
2	1	2.5	1.9
1	0	4.8	3.4

The previous examples are all designed to illustrate various properties of the algorithm. The following example is a summary of a feasibility study which indicates the practicality of the algorithm for application to basis weight control of a paper machine.

Example 7.4

The applicability of minimum variance strategies to basis weight control on a paper machine was demonstrated in [9]. In this application the control loop is a feedback from a wet basis weight signal to thick stock flow. The models used in [9] were obtained by estimating the parameters of (2.1) using the maximum likelihood method. In one particular case the following model was obtained.

$$y(t) = \frac{b_1 q^{-1} + b_2 q^{-2}}{1 + a_1 q^{-1} + a_2 q^{-2}} u(t-2) + v(t) \quad (7.8)$$

where the output y is basis weight in g/m^2 and the control variable is thick stock flow in g/m^2 . The disturbance $\{v(t)\}$ was a drifting stochastic process which could be modelled as

$$v(t) = \lambda \frac{1 + c_1 q^{-1} + c_2 q^{-2}}{(1 + a_1 q^{-1} + a_2 q^{-2})(1 - q^{-1})} e(t) \quad (7.9)$$

where $\{e(t)\}$ is white noise. The sampling interval was 36 sec and the numerical values of the parameters obtained through identification were as follows

$$a_1 = -1.283$$

$$a_2 = 0.495$$

$$b_1 = 2.307$$

$$b_2 = -2.025$$

$$c_1 = -1.438$$

$$c_2 = 0.550$$

$$\lambda = 0.382.$$

To investigate the feasibility of the self-tuning algorithm for basis weight control, the algorithm was simulated using the model (7.8) where the disturbance v was the actual realization obtained from measurements on the paper machine. The parameters of the regulator were chosen as $k=1$, $l=3$, $m=4$ and $\beta_0=2.5$ and the initial estimates were set to zero. The algorithm is thus tuning 7 parameters.

The results of the simulation are shown in Figs. 8–10. Figure 8 compares the output obtained when using the self-tuning algorithm with the result obtained when using the minimum variance regulator computed from the process model (7.8) with the disturbance given by (7.9). The reference value was 70 g/m^2 . In the worst case the self-tuning regulator gives a control error which is about 1 g/m^2 greater than the minimum variance regulator. This happens only at two sampling intervals.

After about 75 sampling intervals (45 min) the output of the system is very close to the output obtained with the minimum variance regulator.

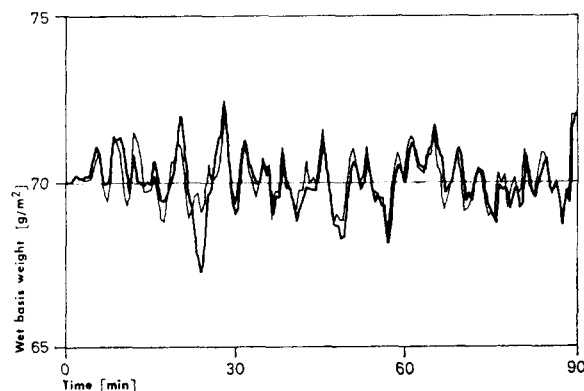


FIG. 8. Wet basis weight when using the self tuning regulator (thick line) and when using the minimum variance regulator based on maximum likelihood identification (thin line). The reference value for the controller was 70 g/m^2 .

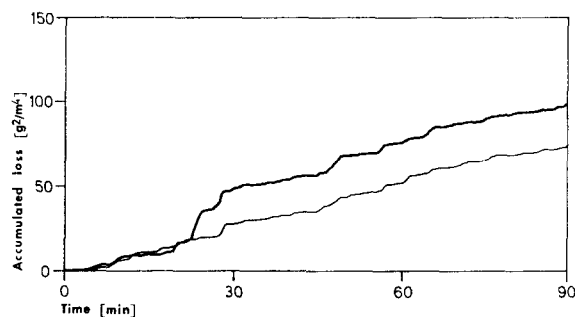


FIG. 9. Accumulated loss for Example 7.4 when using the self tuning regulator (thick line) and when using the minimum variance regulator (thin line).

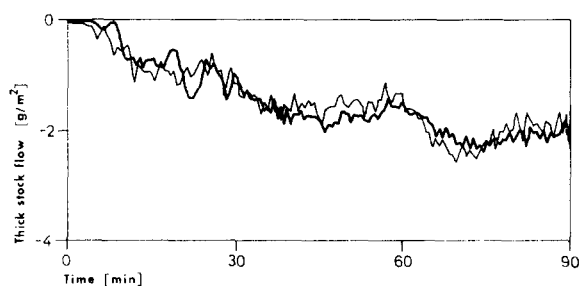


FIG. 10. The control signal in g/m^2 for Example 7.4 when using the self tuning regulator (thick line) and when using the minimum variance regulator (thin line).

Figure 9 compares the accumulated losses

$$V(t) = \sum_{n=0}^t y^2(n)$$

obtained with the minimum variance regulator and the self-tuning regulator. Notice that in the time interval (21, 24) minutes there is a rapid increase in the accumulated loss of the self-tuning regulator of about 17 units. The largest control error during this interval is 2.7 g/m^2 while the largest error of the minimum variance regulator is 1 g/m^2 . The accumulated losses over the last hour is 60 units for the self-tuning regulator and 59 units for the minimum variance regulator.

The control signal generated by the self-tuning algorithm is compared with that of the minimum variance regulator in Fig. 10. There are differences in the generated control signals. The minimum variance regulator generates an output which has more rapid variations than the output of the self-tuning regulator.

The parameter estimates obtained have not converged in 100 sampling intervals. In spite of this the regulator obtained will have good performance as has just been illustrated. The example thus indicates that the self-tuning algorithm could be feasible as a basis weight regulator.

8. PRACTICAL ASPECTS

A few practical aspects on the algorithm given in section 4 are presented in this section which also covers some possible extensions of the results

A priori knowledge

The only parameters that must be known *a priori* are k, l, m and β_0 . If the algorithm converges it is easy to find out if the *a priori* guesses of the parameters are correct simply by analyzing the sample covariance of the output. Compare Example 7.3. The parameter β_0 should be an estimate of the corresponding parameter of the system to be controlled. The choice of β_0 is not critical as was shown in the Examples 6.1 and 7.1. In the special cases studied in the examples an under-estimate led to a diverging algorithm while an over-estimate was safe.

Implementation on process computers

It is our belief that the self-tuning algorithm can be conveniently used in process control applications. There are many possibilities. The algorithm can be used as a tool to tune regulators when they are installed. It can be installed among the systems programs and cycled through different control loops repetitively to ensure that the regulators are always properly tuned. For critical loops where the parameters are changing it is also possible to use a dedicated version which allows slowly drifting parameters.

A general self-tuning algorithm requires about 40 FORTRAN statements. When compiled using the standard PDP 15/30 FORTRAN compiler the code consists of 450 memory locations. The number of memory locations required to store the data is $(l-1+m)^2 + 3(l-1+m) + 2k + 4$. Execution times on a typical process computer (PDP 15) without floating point hardware are given in the table below. The major part of the computing is to update the least squares estimate.

Number of parameters $l+m$	Execution time ms
1	5
3	16
5	34
8	69

Improved convergence rates

The results of this paper only shows that if the parameters converge the regulator obtained will tend to a minimum variance regulator. Nothing is said about convergence rates, which of course is of great interest from practical as well as theoretical points of view. There are in fact many algorithms

that have the correct asymptotic properties. Apart from the algorithm given in section 4 we have the algorithm which minimizes (2.4). But that algorithm is impossible to use due to the computational requirements. It is of interest to investigate if other possible algorithms have better convergence rates than the algorithm of section 4. No complete answer to this problem is yet known. A few possibilities will be indicated. It could be attempted to take into account that the parameter estimates are uncertain. See Refs. [2, 7 and 10]. The least squares identifier can be improved upon by introducing exponential weighting of past data. This has in some cases shown to be advantageous in simulations. Algorithms of this type have in simulations been shown to handle slowly drifting parameters.

Another possibility is to assume that the parameters are Wiener processes, which also can be incorporated fairly easily [2, 7]. It has been verified by simulation that the region of convergence can be improved by introducing a bound on the control signal.

Feed forward

In many industrial applications the control can be improved considerable if feed forward is used. The self tuning regulators in this paper can include feed forward control by changing the process model (4.1) to

$$\begin{aligned} y(t+k+1) + \alpha_1 y(t) + \dots + \alpha_m y(t-m+1) \\ = \beta_0 [u(t) + \beta_1 u(t-1) + \dots + \beta_l u(t-l)] + \gamma_1 s(t) \\ + \dots + \gamma_p s(t-p+1) + \varepsilon(t+k+1) \end{aligned} \quad (8.1)$$

where $s(t)$ is a known disturbance.

The parameters α_i, β_i and γ_i can be identified as before and the control law (4.2) will be changed to

$$\begin{aligned} u(t) = \frac{1}{\beta_0} [\alpha_1 y(t) + \dots + \alpha_m y(t-m+1) - \gamma_1 s(t) \\ - \dots - \gamma_p s(t-p+1)] - \beta_1 u(t-1) \\ - \dots - \beta_l u(t-l). \end{aligned} \quad (8.2)$$

Nonminimum phase systems

Difficulties have been found by a straightforward application of the algorithm to nonminimum phase systems, i.e. systems where the polynomial B has zeroes outside the unit circle.

Several ways to get around the difficulty have been found. By using a model with $B(z) = \beta_0$ it has in many cases been possible to obtain stable algorithms at the sacrifice of variance.

It is well-known that the minimum variance regulators are extremely sensitive to parameter variations for nonminimum phase systems [1]. This is usually overcome by using suboptimal strategies which are less sensitive [1]. The same idea can be used for the self-tuning algorithms as well. The drawback is that the computations increase because the polynomials F and G of an identity similar to (2.8) must be determined at each step of the iteration. An alternative is to solve a Riccati-equation at each step.

Multivariable and nonlinear systems

It is possible to construct algorithms that are similar to the one described in section 4 for multivariable and nonlinear systems as long as a model structure which is linear in the parameters [3, p. 131] is chosen. For multivariable systems the structure given in equation (3.2) of Ref. [3] can thus be attempted. Analyses of the properties of the algorithm obtained when applied to a multivariable or a nonlinear system are not yet done.

9. CONCLUSIONS

The paper has been concerned with control of systems with constant but unknown parameters. The analysis has been limited to single-input single-output systems with disturbances in terms of filtered white noise. A control algorithm based on least squares parameter estimation and a minimum variance regulator computed from the estimated parameters has been analysed. Assuming that the parameter estimates converge the closed loop system has been analysed. A characterization of the closed loop system has been given under weak assumption on the system to be controlled. Under stronger assumptions on the system to be controlled it has been shown that the regulator obtained will actually converge to the minimum variance regulator if the estimates converge.

Since the closed loop system is characterized as a nonlinear stochastic system it is very difficult to give general conditions that guarantee that the estimates converge. The convergence has only been treated for simple examples and under further assumptions as in section 6. But simulations of numerous examples indicate that the algorithm has nice convergence properties.

The simplicity of the algorithm in combination with its asymptotic properties indicate that it can be useful for industrial process control. The feasibility has also been demonstrated by experiments on real processes in the paper and mining industries.

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Résumé—On considère le problème du contrôle d'un système avec paramètres constants mais inconnus. L'analyse se limite aux systèmes discrets à entrée unique et sortie unique. Un algorithme obtenu en combinant un estimateur à carrés minimum avec un régulateur à variance minimum calculée du modèle estimé, est analysé. Les résultats principaux sont deux théorèmes qui caractérisent le système à boucle fermée obtenu en supposant que les estimations des paramètres convergent. Le premier théorème dit que certaines covariances de la sortie et certaines covariances transverses de la variable de contrôle et de la sortie disparaîtront avec des suppositions faibles du système contrôlé. Dans le second théorème il est supposé que le système à contrôler est un système général linéaire stochastique du même ordre. Il est montré que si les estimations des paramètres convergent, la loi de contrôle obtenue est en fait la loi de contrôle de variance minimale qui pourrait être calculée si les paramètres du système étaient connus. Ceci est quelque peu surprenant car l'estimation des carrés minimum est partielle. On discute certaines des implications pratiques des résultats. Il est montré en particulier qu'il est possible d'appliquer l'algorithme à un petit ordinateur.

Zusammenfassung—Betrachtet wird das Problem der Steuerung eines Systems mit konstanten, aber unbekannten Parametern. Die Analyse wird auf Systeme mit Diskretzeit und einem Eingang bzw. einem Ausgang beschränkt. Ein durch Kombination einer Schätzeinrichtung und der Methode der kleinsten Quadrate mit einer aus dem Schätzmodell berechneten Regeleinrichtung erhaltener Algorithmus wird analysiert. Die Hauptergebnisse sind zwei Theoreme, die unter der Annahme, daß die Parameterschätzungen konvergieren, den erhaltenen geschlossenen Regelkreis charakterisieren. Das erste Theorem konstatiert, daß bestimmte Kovarianzen des Ausgangs und bestimmte Kreuz-Kovarianzen der Steuervariablen und des Ausgangs unter schwachen Annahmen über das zu regelnde System verschwinden. Im zweiten Theorem wird angenommen, daß das zu regelnde System ein allgemeines lineares stochastisches System n -ter Ordnung ist. Gezeigt wird, daß bei Konvergenz der Parameterschätzung des erhaltenen Steuergesetzes in der Tat das Steuergesetz bei minimaler Varianz ist, das berechnet werden kann, wenn die Parameter des Systems bekannt wären. Das ist etwa überraschend, weil die Schätzung nach den kleinsten Quadraten angesteuert wird. Einige praktische Folgerungen aus den Ergebnissen werden diskutiert. Speziell wird gezeigt, daß der Algorithmus auf einem kleinen Prozeßrechner leicht verwirklicht werden kann.

Резюме—Рассматривается проблема регулирования систем при помощи постоянных но неизвестных параметров. Анализ ограничивается системами дискретного времени с одним вводом и с одним выводом. Анализируется алгоритм, полученный путем комбинирования

оценок наименьших квадратов с регулировкой минимальных колебаний, вычисленных из расчетной модели. Основными результатами являются две теоремы, характеризующие систему замкнутого контура, полученного в предположении, что оценки параметров сходятся. Первая теорема утверждает, что некоторая ковариантность вывода и некоторая пересекающаяся ковариантность контрольных переменных вывода исчезнут при небольших допущениях в регулируемой системе. Во второй теореме предполагается, что контролируемая система является общей линейной стохастической

системой порядка $n:th$. Показано, что, если оценки параметров сходятся, то полученный закон регулировки является фактически законом регулировки минимального колебания, которое могло бы быть вычислено при известных параметрах системы. Это до некоторой степени неожиданно, так как оценка наименьших квадратов является смещенной. Обсуждаются некоторые практические выводы из результатов. В частности, показано, что этот алгоритм может быть применен на небольшой счетно-вычислительной машине для продес-