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PID Control Design and $\mathcal{H}_\infty$ Loop Shaping

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Abstract This paper shows that traditional methods for design of PID controllers can be related to robust $\mathcal{H}_\infty$ control. In particular it is shown that the specifications in terms of maximum sensitivity and maximum complementary sensitivity are related to the weighted $\mathcal{H}_\infty$ norm introduced by Glover and McFarlane (1989). It is also shown that the Vinniecomb metric can be used to classify the classes of systems that can be stabilized with the design methods in Åström *et al.* (1998) and Panagopoulos (1998).

Keywords $\mathcal{H}_\infty$ Controller Design. PID Controller Design. Specifications. Robustness.

1. Introduction

Many different methods have been proposed for designing PID controllers compromising between robustness and performance, see Åström and Hägglund (1995). Therefore it is of interest to relate traditional methods for design of PID controllers to the $\mathcal{H}_\infty$ loop shaping method.

The idea of the $\mathcal{H}_\infty$ loop shaping method is to design a controller that minimizes the signal transmission from load disturbances and measurement noise to process input and output. This can be expressed by the $\mathcal{H}_\infty$ norm, γ, of a two-by-two transfer function matrix.

Traditional methods to design PID controllers make a compromise between robustness and performance. For example, in Shinskey (1990) robustness is expressed by requiring that the Nyquist curve is outside a square which encloses the critical point, and performance is expressed as maximization of integral gain. The design method for PID controllers in Åström *et al.* (1998) and Panagopoulos (1998) are based on this idea with the robustness measure expressed as circles of constant sensitivity and constant complementary sensitivity.

The first aim of the present paper is to show how the design method for PID controllers presented in Åström *et al.* (1998) and Panagopoulos (1998) are related to the $\mathcal{H}_\infty$ loop shaping method developed in Glover and McFarlane (1989) and Vinnicombe (1998). In particular it is shown how the specifications for the PID design should be chosen to guarantee that the weighted $\mathcal{H}_\infty$ norm of the transfer function from load disturbances to process inputs and outputs is less than a specified value γ.

The paper shows how the condition that γ is small can be expressed in terms of requirements on the Nyquist curve of the loop transfer function. In particular the curve should be outside a contour which encloses the critical point. An explicit formula is given for the contour of the region. The contour can be bounded internally and externally by the circles representing constant sensitivity and constant complementary sensitivity. This establishes the relation between classical design conditions and $\mathcal{H}_\infty$ robust control. The results can also be used to develop design methods, see Åström *et al.* (1998).

The second aim of the present paper is to show that a consequence of the proven relation between traditional methods for design of PID controllers and the $\mathcal{H}_\infty$ loop shaping method gives a nice way to classify the class of systems a PID controller stabilizes. This gives a possibility to design a PID controller with good robustness of the closed loop system.

2. $\mathcal{H}_\infty$ Control

To begin with an overview of the relevant parts of the $\mathcal{H}_\infty$ loop shaping method is presented. Typically the design objective of a control problem express requirements on:

- Attenuation of load disturbances.
- Effects of measurement noise.
- Robustness with respect to model uncertainties.
- Set point response.

The primary issue for process controllers are: load disturbances, measurement noise and robustness, since they mostly operate as regulators. Consequently, in $\mathcal{H}_\infty$ control a multivariable transfer function is first derived describing the influence of all disturbance to the deviations of interesting variables. A controller which minimizes the $\mathcal{H}_\infty$-norm of this transfer function is then determined. This method is well suited for dealing with the first three design criteria given above. Set point weighting and filtering is then used to obtain a good set point response using the two degrees of freedom structure proposed in Horowitz (1963).

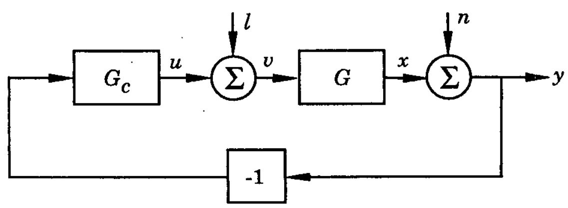

Figure 1 Block diagram describing the design problem.

The idea of the $\mathcal{H}_\infty$ loop shaping method is illustrated by considering the block diagram shown in Figure 1. The inputs are the load disturbance l and the measurement noise n. The outputs of interest are the process output x and the signal v which represents the combined action of the load disturbance and the control signal. The signals are related through

$$\begin{bmatrix} x \\ v \end{bmatrix} = H \begin{bmatrix} n \\ l \end{bmatrix}$$

where

$$H = \begin{bmatrix} G \\ I \end{bmatrix} (I + GG_c)^{-1} [-G_c \quad I]. \tag{1}$$

The block diagonal elements of H are the complementary sensitivity function

$$T = G(I + GG_c)^{-1} G_c$$

and the sensitivity function

$$S = (I + GG_c)^{-1}.$$

The off-diagonal elements are the transfer functions

$$G_c(I + GG_c)^{-1},$$
$$G(I + GG_c)^{-1}.$$

Good control requires both x and v to be small with respect to the disturbances l and n. This can be expressed quantitatively by requiring that the $\mathcal{H}_\infty$-norm

$$\gamma = \|H\|_\infty \tag{2}$$

should be small. The use of the parameter γ as a criterion for loop shaping was suggested by Glover and McFarlane (1989) and McFarlane and Glover (1992) where they recommended the values in the range $[2, 10]$. They also showed that their design method has many nice properties:

- The design methods gives a good controller if one exist.
- The obtained closed loop system is robustly stable against coprime factor uncertainties, Vidyasagar (1985).
- The parameter γ is a good design variable.

Furthermore, it is shown in Vinnicombe (1998) that γ is closely related to the metric $\|G, -1/G_c\|$.

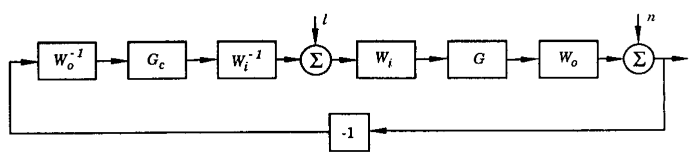

Figure 2 Block diagram describing the design problem when introducing frequency weighting.

Frequency Weighting

Frequency weighting may be introduced in $\mathcal{H}_\infty$ loop shaping to emphasize the response at certain frequencies, see Figure 2. In this case the design of the $\mathcal{H}_\infty$-controller is solved for the transformed system $\bar{G}$ and the transformed controller $\bar{G}_c$, i.e.,

$$\bar{G} = W_o G W_i,$$
$$\bar{G}_c = W_i^{-1} G_c W_o^{-1},$$

where W_i is the input weight and W_o is the output weight. Consequently, the transformed system $\bar{H}$ becomes

$$\bar{H} = \begin{bmatrix} W_o G W_i \\ I \end{bmatrix} W_o (I + GG_c)^{-1} W_o^{-1} [-W_i^{-1} G_c W_o^{-1} \quad I]. \tag{3}$$

Note, that the transformation is equivalent to design for the disturbances

$$\bar{l} = W_i l,$$
$$\bar{n} = W_o^{-1} n.$$

Single Input Single Output Systems

In the case of single input single output systems the transfer function H in Equation (1) becomes

$$H = \frac{1}{1+GG_c} \begin{bmatrix} -GG_c & G \\ -G_c & 1 \end{bmatrix}. \qquad (4)$$

The matrix H is of rank 1, and its largest singular value is given by

$$\sigma^2(H) = \frac{(1+G_cG_c^*)(1+GG^*)}{(1+GG_c)(1+G^*G_c^*)}.$$

It follows from Equation (2) that

$$\gamma^2 = \sup_\omega \frac{(1+G_cG_c^*)(1+GG^*)}{(1+GG_c)(1+G^*G_c^*)}. \qquad (5)$$

Also, in the case of frequency weighting of single input single output systems is simplified since it is sufficient to use only one weight, i.e., $W_i = W$ and $W_o = 1$. The transformed system matrix $\bar{H}$ in Equation (3) becomes

$$\bar{H} = \frac{1}{1+GG_c} \begin{bmatrix} -GG_c & GW \\ -G_cW^{-1} & 1 \end{bmatrix} \qquad (6)$$

3. PID Control

A design method for PID controller based on constraints on maximum sensitivity and complementary sensitivity is described in Åström *et al.* (1998) and Panagopoulos (1998). This method is briefly summarized in this section.

The PID controller is described by the transfer function

$$G_c(s) = k + \frac{k_i}{s} + k_d s,$$

where k is the controller gain, k_i is the integral gain and k_d is the derivative gain. In reality its structure is more complicated, because of filtering of the derivative term and set point weighting, see Åström and Hägglund (1995). Both design methods in Åström *et al.* (1998) and Panagopoulos (1998) are based on the idea of maximizing the integral gain k_i subject to a robustness constraint, i.e., the Nyquist curve of the loop transfer function should lie outside a specified circle. This idea, which goes back to Shinskey (1990) is discussed in detail in Åström *et al.* (1998) and

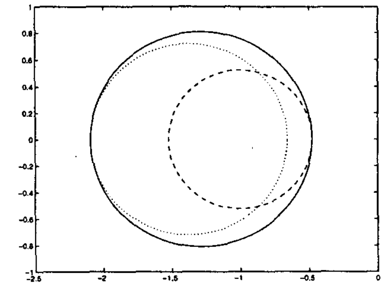

Figure 3 The M_s-circle (dashed line), the M_p-circle (dotted line) and the combined M_sM_p-circle (full line) for $M_s = M_p = 2.0$.

Panagopoulos (1998), where the robustness is measured in the classical terms of the maximum of the sensitivity function, M_s, and the maximum of the complementary sensitivity function, M_p. Thus, the robustness measure provides a transparent design variable. The robustness constraint is expressed as either

$$M_s = \|(1+GG_c)^{-1}\|_\infty,$$

i.e., the Nyquist curve of the loop transfer function avoids the circle with center at $C = -1$ and radius $R = 1/M_s$, see Figure 3, or as

$$M_p = \|GG_c(1+GG_c)^{-1}\|_\infty,$$

i.e., the Nyquist curve of the loop transfer function avoids the circle with center at $C = -M_p^2/(M_p^2-1)$ and radius $R = |M_p/(M_p^2-1)|$, see Figure 3. It is possible to replace the constraints on M_s and M_p with a combined one which reduces the computational effort substantially, see Åström *et al.* (1998). For the combined constraint the Nyquist curve of the loop transfer function avoids the circle with center at

$$C = -\frac{2M_sM_p - M_s - M_p + 1}{2M_s(M_p-1)},$$

and radius at

$$R = \frac{M_s + M_p - 1}{2M_s(M_p-1)},$$

see Figure 3. The circle has its diameter on the interval $[-M_p/(M_p-1),\ -(M_s-1)/M_s]$. For practical purposes it is not much more stringent than combining the two constraints on M_s and M_p respectively.

The combined constraint is simplified if both the sensitivity and the complementary sensitivity function are less than or equal to M, i.e., the Nyquist curve of the loop transfer function avoids the circle with center

$$C = -\frac{2M^2 - 2M + 1}{2M(M-1)},$$

and radius

$$R = \frac{2M-1}{2M(M-1)}.$$

The circle has its diameter on the interval $[-M/(M-1), -(M-1)/M]$.

4. Comparisons

The relation between the $\mathcal{H}_\infty$ design problem and the methods for designing PID controllers will now be presented. In particular it is shown how the specifications of the PID designs should be chosen to guarantee the $\mathcal{H}_\infty$-norm of the frequency weighted transfer function $\bar{H}$ in Equation (6) to be less than a specified value γ.

The minimization of the robustness measure γ in the $\mathcal{H}_\infty$ design gives a controller that compromises between attenuation of the disturbances l and n. By introducing frequency weighting it is possible to emphasize the weighting of the two disturbances by a proper choice of the weighting function W, which will serve as a design variable. For reasonable choices of W the largest value of γ will occur in the neighborhood of the crossover frequency. Note that the rejection of low frequency disturbances can be influenced by the weighting function but it is not particularly critical.

Consequently, W is regarded as a design variable. What will be the best choice of it? For the frequency weighted transfer function $\bar{H}$ in Equation (6), the robustness measure γ is given by

$$\gamma^2 = \sup_\omega \frac{(1 + G_c G_c^* W^{-1} W^{-1*})(1 + GG^* WW^*)}{(1 + GG_c)(1 + G^* G_c^*)}. \quad (7)$$

The most favorable frequency weighting is the one that minimizes the numerator of Equation (7). Let $X = WW^*$, then the numerator of Equation (7) becomes

$$1 + GG^* X + G_c G_c^* X^{-1} + GG^* G_c G_c^*,$$

which has its minimum for $X = \sqrt{G_c G_c^* / GG^*}$. The weighting factor becomes then,

$$W = \sqrt[4]{G_c G_c^* / GG^*}. \quad (8)$$

Consequently, the weighting will typically enhance low and high frequencies. Recall that the low frequency gain of the PID controller is proportional to $1/\omega$. By introducing the weight W given by (8) into (7) we find that γ is given by

$$\gamma^2 = \sup_\omega \frac{(1 + |GG_c|)^2}{|1 + GG_c|^2}. \quad (9)$$

This implies that

$$\gamma = \max_\omega(|S(i\omega)| + |T(i\omega)|). \quad (10)$$

Consequently, Equation (10) shows that the robustness measure γ is related to the values M_s and M_p which are also used as robustness constraints in the PID design presented in Section 3. Notice that even if the weight W in Equation (8) depends on the transfer functions G_c and G, the quantity γ depends only on the loop transfer function GG_c.

Although the robustness measure of the $\mathcal{H}_\infty$ design and the robustness constraint of the PID designs are related, there are some fundamental differences between the two. For example, in the $\mathcal{H}_\infty$ design γ is insensitive to k_i for low frequencies and the requirements on the transfer functions $G_c/(1 + GG_c)$ and $G/(1 + GG_c)$ appear explicitly in Equation (4) compared to the PID design. In the PID design it is attempted to maximize k_i explicitly.

The Gamma Contour

A graphical interpretation of the robustness measure γ in Equation (9) will now be given. Let $L = GG_c$ be the loop transfer function. It follows from Equation (9), that

$$\gamma = \sup_\omega \frac{1 + |L|}{|1 + L|}. \quad (11)$$

Thus, the condition $\gamma \le \gamma_0$ can be interpreted graphically that the Nyquist curve of the loop transfer function should lie outside the contour

$$\frac{1 + |L|}{|1 + L|} = \gamma_0.$$

The level curves of the function

$$f(L) = \frac{1 + |L|}{|1 + L|},$$

in the complex plane are therefor of interest. By introducing $L = re^{i\varphi}$ Equation (11) is rewritten as

$$\gamma = \frac{1 + r}{|1 + re^{i\varphi}|} = \frac{1 + r}{\sqrt{1 + r^2 + 2r\cos\varphi}},$$

and the γ-curve will be represented by

$$r(\varphi) = -\frac{\gamma^2 \cos\varphi - 1}{\gamma^2 - 1} \pm \sqrt{\left(\frac{\gamma^2 \cos\varphi - 1}{\gamma^2 - 1}\right)^2 - 1}.$$

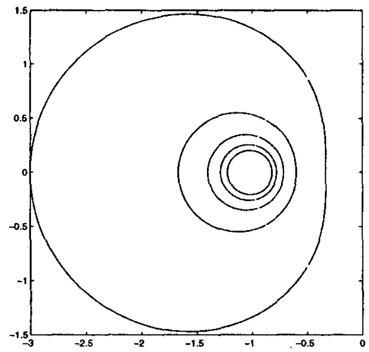

Figure 4 The loci of $(1+|L|)/(|1+L|)=\gamma$ for $\gamma=2$ (outer curve), 4, 6, 8, 10 (inner curve).

Typical contours are given in Figure 4.

By performing straight forward calculations it is shown in Figure 5 that

$$OA = 1, \qquad OB = -\frac{\gamma+1}{\gamma-1},$$
$$OC = \frac{\gamma^2}{2-\gamma^2}, \qquad OD = -\frac{\gamma-1}{\gamma+1},$$
$$AC = \frac{2\sqrt{\gamma^2-1}}{\gamma^2-2}, \qquad OB \cdot OD = 1.$$

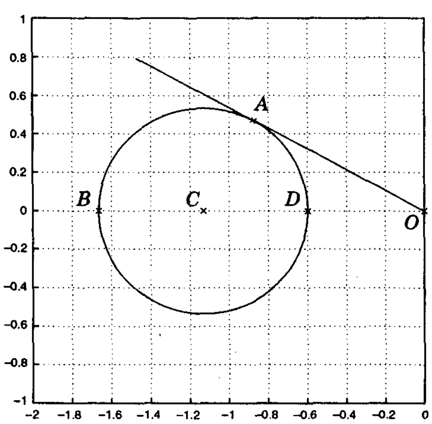

Figure 5 The γ-curve.

Relations Between M and γ

Several relations will now be presented between M and γ. It follows from Equation (10), that

$$\gamma \le \max|S| + \max|T| = 2M,$$

and

$$\gamma \ge \max(2|S|-1) = 2M-1.$$

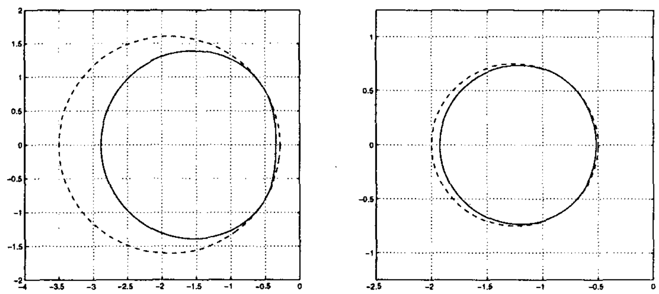

Figure 6 γ-curve (full line) for $\gamma = 2.06$ (left), 3.16 (right) enclosed by the combined M-circle (dashed line) for $M = 1.4$ (left), 2.0 (right).

These inequalities are obtained from the relation $S+T=1$ and the triangular inequality. Consequently, the following inequality has been established

$$2M-1 \le \gamma \le 2M,$$

which gives an indication of how to chose M to guarantee a certain γ.

Sharper results are obtained from tedious calculations which shows that the γ-curve is inside the combined M-circle if M is chosen as

$$M(\gamma) = \frac{\gamma^2 + 2\sqrt{\gamma^2-1}}{2+2\sqrt{\gamma^2-1}}. \tag{12}$$

The inverse relation is

$$\gamma(M) = \sqrt{4M^2-4M+2}. \tag{13}$$

Figure 6 shows contours for constant γ and the combined M-circles that enclose them. The figure show that the contours are very close for $\gamma = 3.16$ and $M = 2.0$.

Consequently, a controller designed for the combined constraint $M_s = M_p = M$ guarantees that γ larger than the value given by Equation (13). Table 1 and 2 gives numerical values of corresponding values of M and γ. The choice $M = 2.0$ guarantees a γ less than $\sqrt{10}$ which according to Vinnicombe (1998) gives good robustness. Lower values of M give even better robustness. Figure 6 shows corresponding γ- and M-curves for $M = 1.4$ and $M = 2.0$. The figure indicates that design based on combined M-curves are not much more conservative than design based on γ, particularly for $M = 2.0$. However, the calculations for constraint on γ are much more complicated.

Classification of Stabilizable Systems

A way to determine for which class of systems a PID controller is stabilizing will now be presented. Vinnicombe (1993) and Vinnicombe (1998) has introduced

Table 1 Numerical values of corresponding M and γ.

M	1.4	1.5	1.6	1.7	1.8	1.9	2.0
γ	2.06	2.24	2.42	2.60	2.79	2.97	3.16

Table 2 Numerical values of corresponding M and γ.

γ	2.0	2.5	3.0	3.5	4.0	4.5	5.0
M	1.37	1.65	1.91	2.18	2.44	2.69	2.95

the generalized stability margin b defined as

$$b := \begin{cases} \dfrac{1}{\gamma} & \text{if } [G, G_c] \text{ is stable,} \\ 0 & \text{otherwise.} \end{cases}$$

The parameter b is in the range $0 \leq b \leq 1$. It follows from the definition that the system is unstable for $b = 0$ and performance improves with increasing values of b. The solutions of many design examples, see Vinnicombe (1998) and McFarlane and Glover (1992), indicates a value $b > 1/\sqrt{10}$ to have reasonable robustness and performance.

Vinnicombe has derived very interesting results relating the generalized stability margin to robustness and model uncertainty. He introduced the metric

$$\delta(P, Q) = |(I + Q^*Q)^{-1/2}(Q - P)(1 + P^*P)^{-1/2}|_\infty \tag{14}$$

which measures the distance between two stable systems P and Q subject to a constraint on the winding number. He showed that if the closed loop system (G, G_c) is stable with $b(G, G_c) \geq \beta$ then the closed loop system $(\bar{G}, G_c)$ is stable for all $\bar{G}$ such that $\delta(G, \bar{G}) \leq \beta$. Due to symmetry the system $(G, \bar{G}_c)$ is also stable for all $\bar{G}_c$ such that $\delta(G, \bar{G}_c) \leq \beta$.

A PID controller which is designed to satisfy the constraints $M_p < M_0$ and $M_s < M_0$ thus guarantees that $\gamma \leq \gamma(M_0)$. The results of Vinnecombe then gives a complete characterization of the class of systems which are stabilized by the controller.

5. Conclusions

Traditional methods for designing PID controllers were related to robust $\mathcal{H}_\infty$ control, in particular the specifications of PID design in Åström *et al.* (1998) and Panagopoulos (1998) to the one of the weighted $\mathcal{H}_\infty$ norm in Glover and McFarlane (1989) denoted γ. The requirement of a sufficiently small γ was expressed as, the Nyquist curve of the loop transfer function should lie outside a certain region. The region showed to be bounded internally and externally by circles closely related to the ones of constant sensitivity and constant complementary sensitivity, which is of use for efficient computations, see Åström *et al.* (1998). A method for classification of the class of systems a PID controller would stabilize was also presented.

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