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A Test for a Vanishing Tetrad: The Second Canonical Correlation Equals Zero

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Many four variable path models have the overidentifying restriction of a vanishing tetrad: $\rho_{13}\rho_{24} - \rho_{14}\rho_{23} = 0$. The literature offers but a 1924 test (Spearman and Holzinger) of a vanishing tetrad. We suggest that a canonical analysis can be set up to provide a test of a tetrad difference. Testing the hypothesis that the second canonical correlation of set X_1 and X_2 with set X_3 and X_4 is zero is identical to testing the hypothesis that $\rho_{13}\rho_{24} - \rho_{14}\rho_{23} = 0$.

Many of the overidentifying restrictions of path models with unmeasured variables are vanishing tetrads of the form

$$\rho_{13}\rho_{24} - \rho_{14}\rho_{23} = 0, \tag{1}$$

where the subscripts 1 through 4 refer to measured variables X_1 through X_4. Examples of such models are contained in Figs. 1 through 5. All five of these models contain four measured variables and have (1) as an overidentifying restriction. In Fig. 1 we have an unmeasured variable, F, that causes the four observables. The disturbances of X_1 and X_2 and the disturbances of X_3 and X_4 are correlated. In Fig. 2 we have a variant of a model discussed by Costner (1969) in the context of multiple indicators, and by Althauser and Heberlein (1970) in the context of a multitrait-multimethod matrix. In Fig. 3 we have a three-wave model discussed by Kenny (1973). In Fig. 4 we have a four-wave, one-variable model of Heise (1969). Finally in Fig. 5 we have a multiple causes and indicators of an unobserved variable, F (Hauser and Goldberger, 1971). All these models have the same overidentifying restriction: Eq. [1].

Duncan (1972) resurrected a test for the vanishing tetrad originally suggested by Spearman and Holzinger (1924). The test is not well known and yields a different value depending on the order of the products of correlations. We would like to suggest a test for the vanishing tetrad that can be

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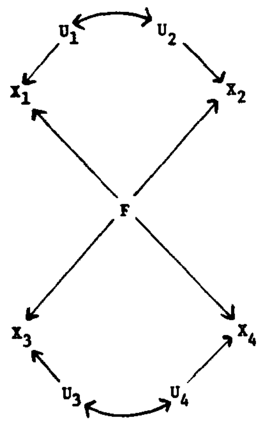

Fig. 1. Single factor model with correlated errors.

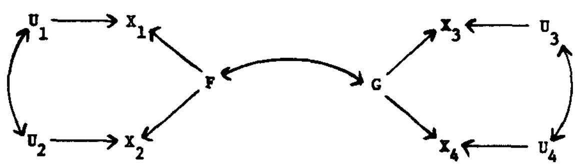

Fig. 2. Multiple indicator model.

made by using a canned program available at most installations: canonical correlation. (The use of canonical analysis for models with unmeasured variables has been suggested by numerous authors (e.g., Goldberger and Hauser, 1971) but none have explicitly suggested the use of canonical correlation as a test for the vanishing tetrad.)

For those who may be unfamiliar with canonical correlation, there are two sets of variables say $X_1, X_2, \ldots X_p$ and $Y_1, Y_2, \ldots Y_q$. The canonical correlation is the maximum correlation between a linear combination of set X with a linear combination of set Y. If both p and q are greater than one a second canonical correlation can be obtained such that the new linear combinations are orthogonal to the prior combinations. The total number of nonzero solutions is equal to the rank of the rectangular matrix of intercorrelations of set X and Y or R_{XY} (Tatsuoka, 1971, p. 86).

We can exploit this fact to obtain a test for a vanishing tetrad. Let X_1 and X_2 be set X and X_3 and X_4 be Y. The determinant of R_{XY} is then

$$r_{13}r_{24} - r_{14}r_{23}. \tag{2}$$

If [2] is zero then the rank of R_{XY} must be less than 2 making the second canonical correlation zero. Thus hypothesis [1] can be tested by the hypothesis that the second canonical correlation is zero. A significance test for this special case of $p = q = 2$ is given by Bartlett's χ^2 approximation (cf. Tatsuoka, 1971, p. 189)

$$\chi_1{}^2 df = -(N-3.5)\log_e(1-r_2{}^2), \tag{3}$$

where N is sample size and r_2 is the estimate of the second canonical correlation.

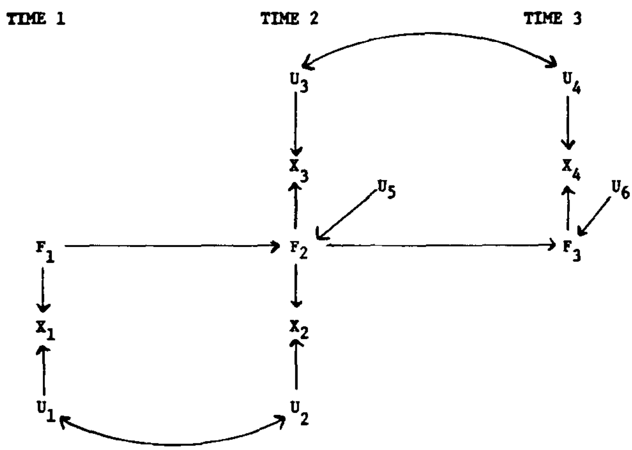

Fig. 3. Three-wave model with an underlying common factor.

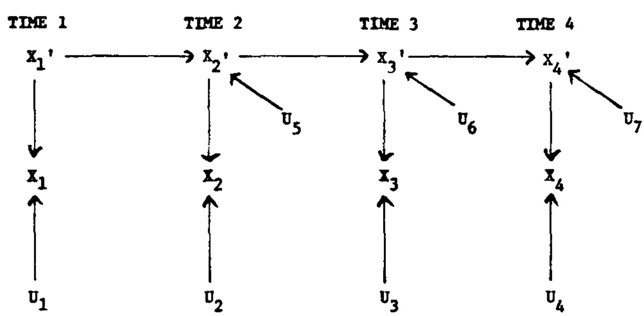

Fig. 4. Four-wave, one-variable model with measurement error (X_i' is the true score of X_i).

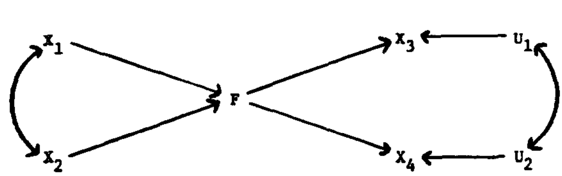

Fig. 5. Multiple cause with multiple indicators of an unobserved variable.

EXAMPLE

As an illustration[2] we consider the correlations from Duncan (1972, p. 55):

$$\begin{aligned} r_{12} &= .6508 \\ r_{13} &= .7304 \\ r_{14} &= .7548 \\ r_{23} &= .7100 \\ r_{24} &= .6999 \\ r_{34} &= .6642, \end{aligned}$$

where the sample size is 730. A test of the hypothesis $\rho_{13}\rho_{24} - \rho_{14}\rho_{23} = 0$ yields a second canonical correlation of .050 and a χ^2 with one degree of freedom of 1.82, which is not significant at the .05 level.

CONCLUSION

We have shown that the second canonical correlation can be used to test the hypothesis of a vanishing tetrad. The tetrad must be of such a form that the four variables involved in the first product of correlations must be the same as are involved in the second. To test hypothesis [1] the second canonical correlation between variables X_1 and X_2 and variables X_3 and X_4 is computed and is tested against the hypothesis that it is not significantly different from zero by formula [3]. This test is identical to the hypothesis of a vanishing tetrad.

We should note two limitations of this test. First, the tetrad may vanish even though an estimate of a path coefficient could be larger than one. This is an example of the Heywood case (cf. Harman, 1967, p. 117). Second, some path models may imply two or more tests of vanishing tetrads. The test we propose will ordinarily yield correlated tests of significance for the tetrads. A better test of the entire model can often be provided by the use of maximum likelihood estimation (Jöreskog, 1970).

APPENDIX

To test the hypothesis $\rho_{12}\rho_{34} - \rho_{14}\rho_{23} = 0$ we must solve the following equation

[2]The variable names have been changed from Duncan's to be consistent with notation in this paper.

$$\begin{vmatrix} \rho^2 & \rho^2 r_{12} & r_{13} & r_{14} \\ \rho^2 r_{12} & \rho^2 & r_{23} & r_{24} \\ r_{13} & r_{23} & 1 & r_{34} \\ r_{14} & r_{24} & r_{34} & 1 \end{vmatrix} = 0, \tag{4}$$

where ρ is the canonical correlation. The solution to [4] will be of the general form

$$a\rho^4 + b\rho^2 + c = 0, \tag{5}$$

where

$$\begin{aligned} a &= (1-r_{12}{}^2)(1-r_{34}{}^2) \\ b &= (r_{14}-r_{12}r_{24})(r_{13}r_{34}-r_{14}) + (r_{23}-r_{13}r_{12})(r_{24}r_{34}-r_{23}) + \\ &\quad (r_{13}-r_{12}r_{23})(r_{14}r_{34}-r_{13}) + (r_{24}-r_{14}r_{12})(r_{23}r_{34}-r_{24}) \\ c &= (r_{13}r_{24}-r_{14}r_{23})^2. \end{aligned}$$

If c equals zero then the smaller solution of ρ^2 from [5] will always be zero. Note that c exactly equals the square of the tetrad. For our example a = .3221476, b = -.2473426, and c = .0006101.

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