MATH 4130 NOTES

Jan 22

hw due on Thursday

OH 4:10-5:00 Tue 413 Malott

Prelim 2 Last Thursday before the spring break
(2 prelims in total)

Final Monday May 13

Cover chapter 1-7 some 8.9

$\frac{m_1}{n_1} = \frac{m_2}{n_2}$

$(m_1, n_1) \sim (m_2, n_2)$ iff $m_1 n_2 = m_2 n_1$

$\left\{ \begin{array}{c} (m,n) \\ m,n \in \mathbb{Z} \\ n \neq 0 \end{array} \right\} = S$ $\mathbb{Z}$ produces $\mathbb{Q}$

$\{(\mathbb{N} \times \{0,1\}) \setminus (0,1)\}$ $\mathbb{N} \curvearrowright \mathbb{Z}$

Cartesian product

Ordered set

S a set

$<$ a binary relation on S

($<$ a subset of $S \times S$)

$<$ is a (linear, total) order

$x \in \underline{\overline{X}}$ $\quad \underline{\overline{X}} \cup \underline{\overline{Y}}$

$\underline{\overline{Y}} \underline{\subset} \underline{\overline{X}}$ $\quad \underline{\overline{X}} \cap \underline{\overline{Y}}$

$\underline{\overline{X}} \times \underline{\overline{Y}}$ $\quad \underline{\overline{X}} \setminus \underline{\overline{Y}}$

$\mathcal{P}(\underline{\overline{X}})$ = all subsets of $\underline{\overline{X}}$

If

1. for all $x, y \in S$, $x<y$ or $y<x$ or $x=y$
2. for all $x, y, z \in S$, if $x<y$ and $y<z$, then $x<z$

   ($(x,y) \in <$ and $(y,z) \in <$, then $(x,z) \in <$)

An ordered set

a set with an order on it.

S, $<$ an order on S

$(S, <)$ - an ordered set

Let S be an ordered set. (* a set never has repeated values)
Let $E \subseteq S$

E is bounded above if there is $y \in S$ s.t. $x \leq y$ for all $x \in E$. ($x < y$ or $x = y$) We say there $y \in S$ is an upper bound for E if $x \leq y$ for all $x \in E$

E bounded below ... A lower bound for E ...

$E \subseteq S$, $\alpha \in S$ is the least upper bound (lub) if for each upper bound y for E we have $\alpha \leq y$ and α is an upper bound for E ($\alpha = \sup E$)

greatest lower bound (glb) ... $\alpha = \inf E$

Examples

1. $(\mathbb{N}, <)$ natural $<$ on $\mathbb{N}$ is an order on $\mathbb{N}$

2. Consider $R \subseteq \mathbb{N} \times \mathbb{N}$

$(m, n) R (p, q)$ iff $m < p$ and $n < q$

① $(m, n) R (p, q)$ and $(p, q) R (r, s) \Rightarrow (m, n) R (r, s)$

② $(1,2), (2,1)$ $(1,2) \not R (2,1)$ $(2,1) \not R (1,2)$ so R is not an order on $\mathbb{N} \times \mathbb{N}$ ✓

3. $(\mathbb{Q}, <)$

the natural $<$ on $\mathbb{Q}$ is an order

Sup $\quad E \subseteq \mathbb{N}$

Case 1

E is unbounded

no upper bound

Case 2

E is bounded and nonempty

E bounded above

E is finite and non-empty

E has a largest elements which is its least upper bound

Case 3

$E = \emptyset$

E is bounded above

$\sup E = 0$

Inf

$E \subseteq \mathbb{N}$

Case 1

$E \neq \emptyset$

E has a smallest element, which is its greatest lower bound

$\inf E =$ the smallest element of E

Case 2

$E = \emptyset$

$\inf E$ does not exist

$A = \{r \in \mathbb{Q} \mid 0 < r \text{ and } r^2 < 2\}$

non-ϕ, bounded above by 2 and below by 0

$\inf A = 0$

$\sup A$ does not exist (in $\mathbb{Q}$)

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<u>Definition</u>

A linear ordered set S has the least upper bound property if for each $E \subseteq S$ s.t. E is bounded above and is non-ϕ, then E has a least upper bound (which is unique))

<u>Theorem</u> S an ordered set with the least upper bound property. Let $B \subseteq S$, $B \neq \phi$ and bounded below. Then L = all lower bounds of B's. Then L is bounded above, non-ϕ and so $\alpha = \sup L$ exists. And α is the greatest lower bound for B.

<u>Proof</u> $L \neq \phi$ since B bounded below

L bounded above since for each $b \in B$ we have $x \leq b$ for all $x \in L$. So each element of B is an upper bound for L and $B \neq \phi$, so L is bounded above. Indeed, $\alpha = \sup L$ exists.

α is a lower bound for B (If not, there exists $b \in B$, s.t. $b < \alpha$. Note b is not an upper bound for L since it is smaller than the least upper bound. So there is $x \in L$ s.t. $b < x$. But $x \in L$ so x is a lower bound for B -x.)

α is the greatest lower bound for B (if not, there is $x \in S$ s.t. $\alpha < x$ and x is a lower bound for B. so $x \in L$ and $\sup L = \alpha < x$. so α is not an upper bound for L, which is a contradiction).

Fields

F with 2 operations $+$ and $\cdot$ satisfying the following axioms:

(A) if $x, y \in F$, then $x+y \in F$

$$\left.\begin{array}{l} x+y = y+x \\ (x+y)+z = x+(y+z) \end{array}\right\} \text{for all } x, y \in F$$

F contains an element 0 s.t. $0+x = x+0 = x$

for each $x \in F$ there exists an element $-x \in F$ s.t. $x+(-x) = 0$

(M) if $x, y \in F$, then $xy \in F$

$$\left.\begin{array}{l} x \cdot y = y \cdot x \\ (x \cdot y) \cdot z = x \cdot (y \cdot z) \end{array}\right\} \text{for all } x, y \in F$$

F contains an element 1 s.t. $1 \neq 0$ and $1 \cdot x = x \cdot 1 = x$ for all $x \in F$

for each $x \in F$, $x \neq 0$, there is an element $\frac{1}{x}$ in F s.t. $x \cdot (\frac{1}{x}) = 1$

(D) $x(y+z) = xy + xz$ for all $x, y, z \in F$

$x - y = x + (-y)$

$\frac{x}{y} = x \cdot (\frac{1}{y})$

$x+y+z = x+(y+z) = (x+y)+z$

$xyz = (xy)z$

$x^2 = xx$

$x^n = \underbrace{x \cdots\cdots x}_{n}$ $\quad n \in \mathbb{N} \quad n > 0$

$nx = \underbrace{x + \cdots + x}_{n}$

<u>Proposition</u> (a) $x+y = x+z \Rightarrow y = z$ for all x, y, z

(b) $x+y = y \Rightarrow x = 0$

(c) $x+y = 0 \Rightarrow y = -x$

(d) $-(-x) = x$

Analysis prop. for $+$

<u>Proof</u>

(a) Assum $x+y = x+z$

$y = 0+y = (-x+x)+y = -x+(x+y)$

$= -x + (x+z)$

$= (-x+x)+z = 0+z = z$

(b) take $z = 0$ in (a)

(c) take $z = -x$ in (a)

(d) rewrite (c): $-x+y = 0 \Rightarrow y = -(-x)$

$-x+x = 0$ by an axiom

$x = -(-x)$

Prop $x, y, z \in F$

(a) $0x = 0$

(b) $x \neq 0, y \neq 0 \Rightarrow xy \neq 0$

(c) $(-x)y = -(xy) = x(-y)$

(d) $(-x)(-y) = xy$

Proof (a) $0x + 0x = (0+0)x = 0x$

so $0x = 0$ by last prop (b)

(b) Assum $xy = 0$ Then

$$\frac{1}{x}\frac{1}{y}xy = \frac{1}{x}\frac{1}{y}\cdot 0 = 0$$

$$\overset{\Vert}{\left(\frac{1}{x}x\right)\left(\frac{1}{y}y\right)} = 1\cdot 1 = 1$$ contradiction

(c) $(-x)y + xy = (-x+x)y = 0\cdot y = 0$

so $(-x)y = -(xy)$ by last prop (c)

$-(xy) = x(-y)$ same proof

(d) $(-x)(-y) = -(x(-y)) = -(-(xy)) = xy$

Definition An ordered (infinite) field is a field that is an ordered set s.t.

Axioms (a) $y < z \Rightarrow x+y < x+z$ for all $x, y, z \in F$

(b) $x > 0, y > 0 \Rightarrow xy > 0$ for all $x, y \in F$

Convention: $x > 0$ — x is positive

$x < 0$ — x is negative

Propositions

(a) $x > 0 \Rightarrow -x < 0$

$x < 0 \Rightarrow -x > 0$

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(b) $x>0, y<z \Rightarrow xy < xz$

(c) $x<0, y<z \Rightarrow xy > xz$

(d) $x \neq 0 \Rightarrow x^2 > 0$; implies $1>0$

(e) $0<x<y \Rightarrow 0<\frac{1}{y}<\frac{1}{x}$

Proof

(a) $x>0 \Rightarrow 0 = -x+x > -x+0 = -x$

$x<0 \Rightarrow 0 = -x+x < -x+0 = -x$

(b) $z>y \Rightarrow z-y > y-y = 0$

$$\Downarrow \text{ Axiom (a)}$$
$$x(z-y) > 0$$
$$\Downarrow$$
$$xz - xy > 0$$
$$\Downarrow \text{ Axiom (b)}$$
$$xz(-xy+xy) > 0 + xy$$
$$\Downarrow$$
$$xz + 0 > xy$$
$$\Downarrow$$
$$xz > xy$$

(c) $-(x(z-y)) = (-x)(z-y) > 0$ Axiom (b)

$$-(-(x(z-y))) < 0$$
$$\Downarrow$$
$$x(z-y) < 0$$
$$\Downarrow$$
$$xz - xy < 0 \text{ , proved in (b)}$$
$$\Downarrow$$
$$xz < xy$$

(d) $x>0 \overset{\text{Axiom (b)}}{\Longrightarrow} x \cdot x > 0 \Rightarrow x^2 > 0$

$x<0 \overset{(a)}{\Longrightarrow} -x>0 \Rightarrow (-x)\cdot(-x) > 0 \Rightarrow x \cdot x > 0$

$$\Downarrow$$
$$x^2 > 0$$

(e) Note: if $y>0$, $v\leq 0$, then $yv\leq 0$ from (c) and $y\cdot 0=0$

But $y\cdot\frac{1}{y}=1>0$ from (d)

so $\frac{1}{y}>0$ similarly $\frac{1}{x}>0$

Now: $x<y$

$\downarrow$ Axiom (b) and (b)

$(\frac{1}{x}\frac{1}{y})x<(\frac{1}{x}\frac{1}{y})y$

$\frac{1}{y}<\frac{1}{x}$

<u>Theorem</u> There exists an ordered field $\mathbb{R}$ that has the property of least upper bound which contains $\mathbb{Q}$ as an ordered subfield

($\mathbb{Q}$ is an ordered field: $<, \cdot, +, 0, 1, -, \div$

There is an injection: $f:\mathbb{Q}\to\mathbb{R}$

s.t $f(x\cdot y)=f(x)f(y)$, $f(x+y)=f(x)+f(y)$

$f(0_{\mathbb{Q}})=0_{\mathbb{R}}$ $f(1_{\mathbb{Q}})=1_{\mathbb{R}}$

$x<_{\mathbb{Q}}y$ iff $f(x)<_{\mathbb{R}}f(y)$

$f(-x)=-f(x)$, $f(\frac{1}{x})=\frac{1}{f(x)}$ if $x\neq 0_{\mathbb{Q}}$)

$\mathbb{Q}\underset{\text{embedded}}{\hookrightarrow}\mathbb{R}$ "$\mathbb{Q}\subseteq\mathbb{R}$"

<u>Theorem</u> (a) If $x,y\in\mathbb{R}$, $x>0$, then there exists a natural number n s.t. $nx>y$

$\mathbb{N}\ni n_{\geq 0}\longrightarrow\underbrace{1_{\mathbb{R}}+1_{\mathbb{R}}+\cdots+1_{\mathbb{R}}}_{n}\in\mathbb{R}$ $\mathbb{Z}\setminus\mathbb{N}\ni n\longrightarrow\underbrace{-(1+\cdots+1)}_{-n}$

$\downarrow$

(b) $x,y\in\mathbb{R}$, $x<y$, then there exists $p\in\mathbb{Q}$ s.t. $x<p<y$

$\mathbb{Z}\hookrightarrow\mathbb{R}$

$\mathbb{Q} \ni \frac{p}{q} \longrightarrow \frac{1}{\varphi(q)} \cdot \varphi(p) \in \mathbb{R}$

$q \neq 0$

$\Theta : \mathbb{R} \xrightarrow[\text{onto}]{1-1} \mathbb{R}'$

<u>Proof</u> (a) A = all number of the two f, nx.
where $n \in \mathbb{N}$

$A \neq \phi$. since $x \in A$

If (a) fails, A is bounded above, y is an upper bound for A. So let $\alpha = \sup A$

Since $x > 0$, we have $\alpha - x < \alpha$

Since α is the least upper bound of A, $\alpha - x$ is not an upper bound of A. So there is $n \in \mathbb{N}$ s.t. $\alpha - x < nx$

so $\alpha < nx + x = (n+1)x \in A$

α is not a least upper bound of A $\therefore$

(b) We have $x < y$. so $y - x > 0$

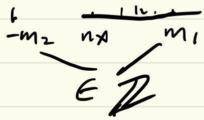
Find $n > 0 \in \mathbb{N}$ s.t. ...

Also there are $m_1, m_2 \in \mathbb{N}$ s.t.

$m_1 > nx$, $m_2 > -nx \Rightarrow nx > -m_1$

So there is an integer m s.t.

$m - 1 \leq nx < m$



so $nx < m \leq 1 + nx < ny$

so $nx < m < ny$

$x < \frac{m}{n} < y$

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<u>Theorem</u> For each real number $x>0$ and a natural number $n>0$. there exists a unique $y>0$ s.t. $y^n=x$

Note $y=x^{\frac{1}{n}}=\sqrt[n]{x}$

<u>Proof</u>

n is fixed ① Uniqueness:

induction

$0<y_1<y_2 \Rightarrow y_1^n<y_2^n$ ($y_1y_1<y_2y_1$
$y_2y_1<y_2y_2$)

$y_1\neq y_2$ s.t. $y_1,y_2>0$ $y_1^n<y_2^n$

if $y_1^n=x=y_2^n$ ※

② Existence $E=\{t>0:\ t^n<x\}$

$E\neq\varphi$: take $t=\frac{x}{1+x}$

Then: $0<t<x$, $t<1$

$0<t^n\leq t<x$

$t\in E$

E is bounded above: observe $t>1+x$ then

$t^n>t>x$

So $1+x$ is an upper bound for E

Since if $t>1+x$, then $t\notin E$

So if $t\in E$, then $t\leq 1+x$

Let $y=\sup E$. we claim this y works $y^n=x$

(i) Assume for the sake of contradiction that $y^n<x$

Fix $0<\varepsilon<1$

$$(y+\varepsilon)^n=y^n+\binom{n}{1}y^{n-1}\varepsilon+\binom{n}{2}y^{n-2}\varepsilon^2+\dots+\varepsilon^n$$

$$=y^n+\varepsilon\left(\binom{n}{1}y^{n-1}+\binom{n}{2}y^{n-2}\varepsilon+\dots+\varepsilon^{n-1}\right)$$

$$<y^n+\varepsilon\left(\binom{n}{1}y^{n-1}+\binom{n}{2}y^{n-2}+\dots+1\right)$$

$$<y^n+\varepsilon\left(y^n+\binom{n}{1}y^{n-1}+\binom{n}{2}y^{n-2}+\dots 1\right)$$

$$=y^n+\varepsilon(1+y)^n$$

Take ε s.t. $0<\varepsilon<1$ s.t. $\varepsilon(1+y)^n < x-y^n$

$$\varepsilon < \frac{x-y^n}{(1+y)^n}$$

Then $y^n + \varepsilon(1+y)^n < y^n + \varepsilon(1+y)^n < y^n + x - y^n = x$

So $(y+\varepsilon)^n < x$. So $y+\varepsilon \in E$ so y is not an upper bound for E since $y < y+\varepsilon$

[2] Assume for the sake of contradiction $y^n > x$

$0<\varepsilon<1$

$$(y-\varepsilon)^n = y^n - \binom{n}{1}y^{n-1}\varepsilon + \binom{n}{2}y^{n-2}\varepsilon^2 - \cdots$$
$$= y^n - \varepsilon\left(\binom{n}{1}y^{n-1} - \binom{n}{2}y^{n-2}\varepsilon + \cdots\right)$$
$$> y^n - \varepsilon\left(\binom{n}{1}y^{n-1} + \binom{n}{2}y^{n-2} + \cdots\right)$$
$$= y^n - \varepsilon(1+y)^n$$

Take $0<\varepsilon<1$, $\varepsilon(1+y)^n < y^n - x$

$$\varepsilon < \frac{y^n - x}{(1+y)^n}$$

So $(y-\varepsilon)^n > y^n - \varepsilon(1+y)^n > y^n - y^n + x = x$

So if $t > y-\varepsilon$, then $t^n > (y-\varepsilon)^n > x$

So if $0<t^n<x$, then $t \leq y-\varepsilon$

So $y-\varepsilon$ is an upper bound for E

But $y-\varepsilon < y$ and y is the least upper bound for E. Contradiction

Col Let $a, b > 0$, n is an integer, $n > 0$

Then $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$

Proof Let $\alpha = a^{\frac{1}{n}}$, $\beta = b^{\frac{1}{n}}$

Then $ab = \alpha^n \beta^n = (\alpha\beta)^n$

So by uniqueness, $\alpha\beta = (ab)^{\frac{1}{n}}$

Euclidean spaces

$\mathbb{R}$ = the reals

For $\varepsilon > 0$, $(\varepsilon \in \mathbb{Z})$ let $\mathbb{R}^\varepsilon = \{(x_1, \dots, x_\varepsilon) ; x_1, \dots x_\varepsilon \in \mathbb{R}\}$

Notation

$$\underline{x} = (x_1, \dots, x_\varepsilon)$$

$$\underline{x} + \underline{y} = (x_1, \dots, x_\varepsilon) + (y_1, \dots, y_\varepsilon)$$

$$= (x_1 + y_1, x_2 + y_2, \dots, x_\varepsilon + y_\varepsilon)$$

$$\alpha \underline{x} = \alpha (x_1, \dots, x_\varepsilon) = (\alpha x_1, \dots, \alpha x_\varepsilon) \quad \alpha \in \mathbb{R}$$

$$\underline{0} = (0, \dots, 0)$$

Inner Product: $\underline{x}, \underline{y} \in \mathbb{R}^\varepsilon$

$$\underline{x} \cdot \underline{y} = (x_1, \dots, x_\varepsilon) \cdot (y_1, \dots, y_\varepsilon)$$

$$= x_1 y_1 + x_2 y_2 + \dots + x_\varepsilon y_\varepsilon$$

$$= \sum_{i=1}^{\varepsilon} x_i y_i$$

$$|\underline{x}| = (\underline{x} \cdot \underline{x})^{\frac{1}{2}} = \left(\sum_{i=1}^{\varepsilon} x_i^2\right)^{\frac{1}{2}}$$

Theorem $x, y, z \in \mathbb{R}^\varepsilon$, α real

(a) $|\underline{x}| \geq 0$

(b) $|\underline{x}| = 0$ iff $\underline{x} = \underline{0}$

(c) $|\alpha \underline{x}| = |\alpha| \, |\underline{x}|$

(d) $|\underline{x} \cdot \underline{y}| \leq |\underline{x}| \, |\underline{y}|$

(e) $|\underline{x} + \underline{y}| \leq |\underline{x}| + |\underline{y}|$

$|\underline{x} - \underline{z}| \leq |\underline{x} - \underline{y}| + |\underline{y} - \underline{z}|$

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<u>Proof</u> d)
$$|x|^2 = x\cdot x = \sum a_i^2$$
$$|y|^2 = y\cdot y = \sum b_i^2$$

To prove $|x\cdot y|^2 \leq |x|^2|y|^2$

Suffice to see $(\sum a_i b_i)^2 \leq \sum a_i^2 \cdot \sum b_i^2$

$$(\sum a_i b_i)^2 = \sum_{i,j} a_i b_i a_j b_j$$
$$= \sum_i a_i^2 b_i^2 + \sum_{i<j} 2 a_i b_i a_j b_j$$
$$\sum a_i^2 \sum b_i^2 = \sum a_i^2 b_i^2 + \sum_{i<j} (a_i^2 b_j^2 + a_j^2 b_i^2)$$

Enough to show
$$\sum_{i<j} 2 a_i b_i a_j b_j \leq \sum_{i<j} (a_i^2 b_j^2 + a_j^2 b_i^2)$$
$$\Updownarrow$$
$$2 a_i b_i a_j b_j \leq a_i^2 b_j^2 + a_j^2 b_i^2$$
$$2(a_i b_j)(a_j b_i) \leq (a_i b_j)^2 + (a_j b_i)^2$$

Let $C = a_i b_j$, $D = a_j b_i$

Need $2CD \leq C^2 + D^2$

Note $(C-D)^2 \geq 0$
$$2CD \leq C^2 + D^2$$

<u>Check</u>
$$(\alpha x)\cdot y = x\cdot(\alpha y) = \alpha(x\cdot y)$$
$$x\cdot y = y\cdot x$$
$$x\cdot(y_1+y_2) = x\cdot y_1 + x\cdot y_2$$
$$(x_1+x_2)\cdot y = x_1\cdot y + x_2\cdot y$$

Construction of $\mathbb{R}$

$\mathbb{Q}$, $<$ on $\mathbb{Q}$, $+, \cdot, 0, 1$ on $\mathbb{Q}$

an ordered field of fractions

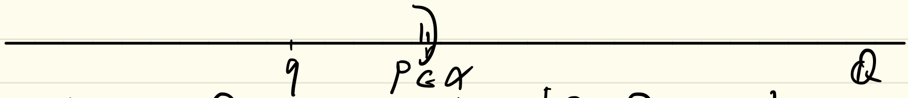
Definition

$\alpha \subseteq \mathbb{Q}$ is a cut if (Dedekind)

(1) $\alpha \neq \emptyset$ and $\alpha \neq \mathbb{Q}$

(2) if $p \in \alpha$, $q \in \mathbb{Q}$, $q < p$, then $q \in \alpha$

(3) if $p \in \alpha$, then there is $r \in \alpha$ s.t. $p < r$



For $r \in \mathbb{Q}$, let $r^* = \{p \in \mathbb{Q} : p < r\}$

$\mathbb{R}$ = the set of all cuts

$\mathbb{Q} \ni r \longrightarrow r^* \in \mathbb{R}$ 1-to-1

Define $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ iff $\left(\begin{array}{c}\alpha \neq \beta \\ \text{and} \\ \alpha \subseteq \beta\end{array}\right)$

For $r, s \in \mathbb{Q}$, $r < s$ iff $r^* < s^*$ in $\mathbb{R}$

$\mathbb{R}$ with $<$ has the least upper bound property

Let $A \subseteq \mathbb{R}$

a non-$\emptyset$ set of cuts bounded above

there is $\beta \in \mathbb{R}$ (a cut)

s.t. for each $\alpha \in A$ $\alpha \leq \beta$ i.e. $\alpha \subseteq \beta$

$\sup A = \bigcup A = \{p \in \mathbb{Q} : p \in \alpha \text{ for some } \alpha \in A\}$

$\sup A$ is a cut so in $\mathbb{R}$. $\sup A$ is the least upper bound for A

$\alpha, \beta \in \mathbb{R}$ (cuts)

$\alpha + \beta = \{p+q : p \in \alpha, q \in \beta\}$

a cut $\quad 0_{\mathbb{R}} = (0_{\mathbb{Q}})^*$

Note we can say

$0 < \alpha, \quad \beta < 0, \quad \alpha = 0$

$\alpha \in \mathbb{R}$

Let $\beta = \{p \in \mathbb{Q} :$ there is some $r \in \mathbb{Q}$, $r > 0$ s.t. $-p-r \notin \alpha\}$ $\quad -(p+r)$

we claim $\beta = -\alpha$ in $\mathbb{R}$

Multiplication is defined in two steps

For $\alpha > 0 \quad \beta > 0$ cuts

let $\alpha\beta = \{p : p \le r \cdot s$ for some $r \in \alpha, s \in \beta, r > 0, s > 0\}$

$\alpha \cdot 0 = 0 \cdot \alpha = 0$ for all $\alpha \in \mathbb{R}$

$$\underset{\text{a cut}}{\alpha\beta} = \begin{cases} (-\alpha)(-\beta) & \text{if } \alpha < 0\ \beta < 0 \\ -((-\alpha)\beta) & \text{if } \alpha < 0, \beta > 0 \\ -(\alpha(-\beta)) & \text{if } \alpha > 0, \beta < 0 \end{cases}$$

$1_{\mathbb{R}} = (1_{\mathbb{Q}})^*$

Check $\mathbb{R}$ with $+, \cdot, 0, 1$ is a field

together with $<$ is an ordered field

Sizes of Sets

$A \sim B$ if there is a bijection $f : A \to B$

Note: $A \sim A$

$A \sim B \Rightarrow B \sim A$

$(A \sim B$ and $B \sim C) \Rightarrow A \sim C$

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$J_n = \{1, \dots, n\}$, $n \in \mathbb{N}$ $(J_0 = \phi)$

A finite if $A \sim J_n$ for some n

A infinite if A is not finite

A countable if $A \sim \mathbb{N}$

A at most countable if A countable or finite

A uncountable if A not at most countable

A sequence is a function whose domain is $\mathbb{N}$

$f : \mathbb{N} \to X$ $\quad (f(0), f(1), \dots, f(n))$

Unions: Assume we have a family of sets E_α with $\alpha \in A$ $\quad \bigcup_{\alpha \in A} E_\alpha = \{x : x \in E_\alpha \text{ some } \alpha \in A\}$

At most countability:

Lemma: Let A be a set and let $f : \mathbb{N} \to A$ be a surjection. Then A is at most countable

Note: If A at most countable, then $A = \phi$ or there is a surjection $\mathbb{N} \to A$

Proof: Let $Y \subseteq \mathbb{N}$ defined by $n \in Y$ iff there is $a \in A$ s.t. n is the smallest in $f^{-1}(a) \neq \phi$

$(f \restriction Y) : Y \to A$ is a bijection

domain $= Y$

Take $n_1, n_2 \in Y$, $n_1 \neq n_2$

$n_1 = \min f^{-1}(a_1)$ $\quad a_1 \neq a_2$

$n_2 = \min f^{-1}(a_2)$

Claim: It will be enough to show if $Y \subseteq \mathbb{N}$, then Y is at most countable

Proof of claim:

Define a function h as follow

$h(0) = \min Y$ if $Y \neq \emptyset$

$h(n+1) = \min(Y \setminus \{h(0), \dots, h(n)\})$ if $Y \setminus \{h(0), \dots, h(n)\} \neq \emptyset$

$h(n)$ is defined for all $n \in \mathbb{N}$

Then $h: \mathbb{N} \to Y$ is a bijection

$(h(n) < h(n+1))$ so h is 1-to-1)

h is onto

if not there is $n_0 \in Y$ s.t. n_0 not a value of h

$h(n) < n_0$ for all n

So $\{h(n) : n \in \mathbb{N}\}$ is an infinite set of natural numbers all below n_0

Case 2: $h(n)$ is undefined for some n

let n_0 be s.t. $h(n_0)$ undefined and n_0 is smallest

$n_0 = 0 \Rightarrow Y = \emptyset$

$n_0 > 0 \Rightarrow Y \setminus \{h_0, h_1 \dots h(n_0)\} = \emptyset$ so Y is finite

$Y = \{h_0, h_1, \dots h(n_0 - 1)\}$

Lemma: $\mathbb{N} \times \mathbb{N}$ is countable

Proof: $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$

$$\begin{array}{c|cccc} \mathbb{N} & 6 & & & \\ & 3 & 7 & 8 & \\ & 1 & & & \\ & 0 & 2 & 5 & 9 \\ \hline & & & \mathbb{N} & \end{array}$$

$f: \mathbb{N}\times\mathbb{N} \xrightarrow[1-1]{\text{onto}} \mathbb{N}\setminus\{0\}$

$f(m,n) = 2^m(2n+1)$

$f(m,n) = f(m',n')$

$2^m(2n+1) = 2^{m'}(2n'+1)$

<u>Theorem</u> Let E_n be at most countable for $n\in\mathbb{N}$. The $\bigcup_n E_n$ is at most countable

<u>Proof</u> Each $E_n \neq \phi$ Then $\bigcup_n E_n \neq \phi$

Assume E_{n_0} is not empty

Consider $A_n = E_n \cup E_{n_0} \neq \phi$

Also A_n is at most countable $\leftarrow$ to be justified

Let $f_n: \mathbb{N} \to A_n$ surjection

Define $F: \mathbb{N}\times\mathbb{N} \to \bigcup_n A_n$

$F(m,n) = f_n(m)$ Then F is a surjection

Let $a \in \bigcup_n A_n$ then $a \in A_{n_0}$ for some n_0

so $a = f_{n_0}(m_0)$ for some $m_0\in\mathbb{N}$

Then $a = f_{n_0}(m_0) = F(m_0, n_0)$

$\mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}\times\mathbb{N} \xrightarrow{\text{onto}} \bigcup_n A_n$

$\bigcup_n E_n = E_0 \cup E_1 \cup \cdots \cup E_{n_0} \cup E_{n_0+1}$

$= (E_0 \cup E_{n_0}) \cup (E_1 \cup E_{n_0}) \cup \cdots \cup (E_{n_0} \cup E_{n_0})$

$= \bigcup_n A_n$

Need E_1, E_2 at most countable

$\downarrow$

$E_1 \cup E_2$ at most countable

$E_1 = \phi = E_2$ ✓ $E_1 = \phi \Rightarrow E_1\cup E_2 = E_2$ ✓ $E_2 = \phi \Rightarrow E_1 \cup E_2 = E_1$ ✓

$E_1 = \phi$, $E_2 \neq \phi$

Let $f_1 : \mathbb{N} \to E_1$ $f_2 : \mathbb{N} \to E_2$ surjection

$F : \mathbb{N} \xrightarrow[\text{onto}]{} \underbrace{E_1 \cup E_2}_{\text{most countable}}$ $F(2n) = f_2(n)$

$F(2n+1) = f_1(n)$

Corallary $\mathbb{Q}$ is countable

Proof $f : \mathbb{Z} \times (\mathbb{N} \setminus \{0\}) \to \mathbb{Q}^+$

$f(m,n) = \frac{m}{n}$

$\mathbb{N} \xrightarrow{g_1} \mathbb{Z}$ onto $\mathbb{N} \xrightarrow{g_2} \mathbb{N} \setminus \{0\}$ onto

$\mathbb{N} \times \mathbb{N} \to \mathbb{Z} \times (\mathbb{N} \setminus \{0\})$

$(m,n) \to (g_1(m), g_2(n))$

Feb 12

Corallary Let A, B be at most countable, Then $A \times B$ is at most countable

If $A = \phi$ or $B = \phi$, then $A \times B = \phi$, so at most countable.

Assume $A \neq \phi, B \neq \phi$ $f : \mathbb{N} \to A$, $g : \mathbb{N} \to B$ surj

So then $H : \mathbb{N} \times \mathbb{N} \to A \times B$ given by

$H(m,n) = (f(m), g(n))$ surj

So
$$\begin{tikzcd} \mathbb{N} \arrow[r, "surj"] \arrow[rr, bend right, "surj"'] & \mathbb{N} \times \mathbb{N} \arrow[r, "H"] & A \times B \end{tikzcd}$$

So $A \times B$ at most countable

Collary A at most countable
A^n at most countable

$n=1$ ✓ $A^1 = A$

$n \to n+1$

$A^{n+1} = A^n \times A$ at most countable by collary induction

Collary $A_n, n \in \mathbb{N}$ are at most countable

$\bigcup_{n \in \mathbb{N}} (A_0 \times \dots \times A_n)$ at most countable

for each n, $A_n \times \dots \times A_n$ is at most countable

$\bigcup_n (A_0 \times \dots \times A_n)$ is at most countable

$\{0,1\}^{\mathbb{N}}$ — all binary sequences

$\{x_n\}$ s.t. $x_n = 0$ or $x_n = 1$ for each n

Theorem (Cantor) Hartog's function

$\{0,1\}^{\mathbb{N}}$ is uncountable

Proof:

If $\{0,1\}^{\mathbb{N}}$ is at most countable, there is a surj

$f: \mathbb{N} \to \{0,1\}^{\mathbb{N}}$

$$\begin{array}{lcccccc}
f(0) & x_0^0 & x_1^0 & x_2^0 & x_3^0 & \cdots & x_n^0 \\
f(1) & x_0^1 & x_1^1 & x_2^1 & x_3^1 & \cdots & x_n^1 \\
f(2) & x_0^2 & x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\
f(3) & x_0^3 & x_1^3 & x_2^3 & x_3^3 & \cdots & x_n^3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots
\end{array}$$

Define $y_n = 1 - x_n^n$

$y_n \in \{0,1\}^n$

$f(k) = y_n$ for some k

So $y_n = x_n^k$ for all n

So $y_k = x_k^k$ But $y_k = 1 - x_k^k$

Then $x_k^k = \frac{1}{2}$ contradiction

Collory $\mathbb{R}$ is uncountable 回去看☆

$\mathbb{R}^k$ $|\underset{x}{x} - \underset{d}{y}|$

Df A metric space (X, d) is a set X and a function $d: X \times X \to \mathbb{R}$ s.t.

(1) $d(p,q) > 0$ if $p, q \in X$ and $p \neq q$

$d(p,p) = 0$

(2) $d(p,q) = d(q,p)$ if $p, q \in X$

(3) $d(p,q) + d(q,r) \geq d(p,r)$ for $p, q, r \in X$

Triangular Inequality

Ex

① $X = \mathbb{R}^k$, $d(\underline{x}, \underline{y}) = |\underline{x} - \underline{y}|$

$|\underline{x} + \underline{y}| \leq |\underline{x}| + |\underline{y}|$

② X any non-ϕ set

$d(x,y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$ check triangular inequality

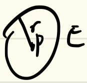
Df A neighborhood of $p \in X$ is a set of the form $N_r(p) = \{q \in X : d(p,q) < r\}$ for $r > 0$
(neighborhood is either the point or everything)
written by me

A point $p \in X$ is a limit point of a set $E \subseteq X$ if each neighb of p contains a point $q \in E$ with $p \neq q$

0 is the limit point of $E = \{\frac{1}{n+1} ; n \in \mathbb{N}\}$

No set has limit point for $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$
(X, d)-discrete

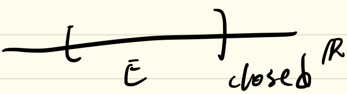
If $p \in E$ and p is not a limit point of E, then p is an isolated point of E

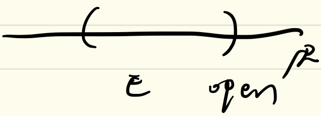


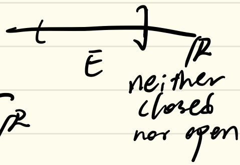
E only has p

- E closed if every limit point of E is in E
- p is an interior point of E if there is a neighborhood N of p s.t. $N \subseteq E$
- E is open if each of its points is an interior point of E

ϕ is both open and closed







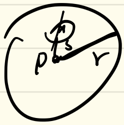
(X, d) - discrete
all sets are closed and all sets are open

- E is dense in X if each point in X in E or is a limit point of E.

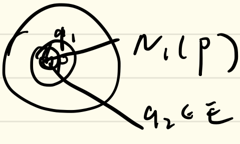
  $\mathbb{Q}$ is dense in $\mathbb{R}$

- E is bounded if $\exists M$ s.t. $d(p, q) < M$ for all $p, q \in E$
- E is perfect if E is closed and each point of E is a limit point of E.

Thm Each neighborhood is open



If p is a limit point of E, then each neighborhood of p contains infinitely many points from E



$q_m \in E$ s.t. $q_m \neq q_n$ if $m \neq n$

$r_2 = \frac{1}{2} d(p, q_1) > 0$ $\quad q_2 \neq p$, $q_2 \neq q_1$

$N_{r_2}(p)$ $\quad r_3 = \frac{1}{2} d(p, q_2)$

$q_3 \in N_{r_3}(p)$

$r_{n+1} = \frac{1}{2} d(p, q_n)$ $\quad N_{r_{n+1}}(p) \ni q_{n+1}$ s.t. $q_{n+1} \neq p$, $q_{n+1} \in E$, $q_{n+1} \neq q_1 \dots q_n$

Note: $r_n \leq \frac{1}{2^{n-1}}$

$1 = r_1 \leq \frac{1}{2^{1-1}} = 1$

Assume $r_n \leq \frac{1}{2^{n-1}}$

$r_{n+1} = \frac{1}{2}d(p, q_n) < \frac{1}{2}r_n \leq \frac{1}{2} \cdot \frac{1}{2^{n-1}} = \frac{1}{2^n}$

Let $r > 0$ consider $N_r(p)$

There exists n s.t. $\frac{1}{2^{n-1}} < r$ $\left(\frac{1}{r} < 2^{n-1}, \frac{1}{r} < n-1 \leq 2^{n-1}\right)$

So $N_{r_k}(p) \subseteq N_r(p)$ if $k \geq n$

$r_k \leq \frac{1}{2^{n-1}} < r$

So $q_k \in N_r(p)$ for all $k \geq n$

Theorem (X, d) a matrix space. (1) A complement of a closed set is open. (2) A complement of an open set is closed.

Proof (1) Let $E \subseteq X$ closed. $X \setminus E$ is open (to prove)

Take $p \in X \setminus E$. $r > 0$ s.t. $N_r(p) \subseteq X \setminus E$ (to prove)

Note: $p \notin E$ p is not a limit point of E

There is $r > 0$ s.t. $N_r(p)$ contains no point of E or p is the only point in E in $N_r(p)$

So $N_r(p) \subseteq X \setminus E$

Since $p \in X \setminus E$ was arbitrary in the argument above, we see that each point of $X \setminus E$ is its interior point, i.e. $X \setminus E$ is open

(2) $E \subseteq X$ open

Need: $X \setminus E$ closed

Let p be a limit point of $X \setminus E$

Note: $p \in X \setminus E$

Assume there was a contradiction that $p \notin X \setminus E$

i.e. $p \in E$

p is an interior point of $E \to N_r(p) \subseteq E$

Since p is a limit point of $X \setminus E$,

there is a $q \in X \setminus E$ s.t. $q \in N_r(p)$

Then $q \in E$, which is a contradiction.

Prop (1) Open sets are open under taking arbitrary unions and finite intersections.

(2) Closed sets are closed under taking arbitrary intersections and finite unions.

U_α, $\alpha \in A$, open $\to \bigcup_{\alpha \in \Lambda} U_\alpha$ is open

$U_1 \dots U_n$ are open $\to \bigcap_{i=1}^{n} U_i$ is open

(1) $\Rightarrow$ (2)

$$\left(X \setminus \bigcap_{\alpha \in A} E_\alpha = \bigcup_{\alpha \in A} (X \setminus E_\alpha) \right)$$

$$\left(X \setminus \bigcup_{\alpha \in A} E = \bigcap_{\alpha \in A} (X \setminus E_\alpha) \right)$$

(1) $p \in \bigcap_{i=1}^{n} U_i$

$p \in U_i \Rightarrow N_{r_i}(p) \subseteq U_i$ for some $r_i > 0$

Take $r = \min(r_1 \dots r_n) > 0$

$N_r(p) \subseteq N_{r_i}(p) \subseteq U_i$ for each i

$N_r(p) \subseteq \bigcap_{i=1}^{n} U_i$

$p \in \bigcap_{i=1}^{n} U_i$ arbitrary, $\bigcap_{i=1}^{n} U_i$ is open

(2) Similar

(X, d) matrix space, $E \subseteq X$, E' = all limit points of E

let $\bar{E} = E \cup E'$ — the closure of E'

Then $E \subseteq X$, (X, d) matrix

(1) $\bar{E}$ is closed

(2) For each $F \supseteq E$, F closed, we have $\bar{E} \subseteq F$

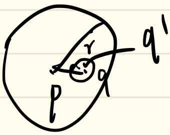
$\bar{E}$ is the smallest closure set

Need: all limit points of $\bar{E}$ are in $\bar{E}$

p a limit point of $\bar{E}$

we chose p is a limit point of E (so $p \in \bar{E}$)

$q \in \bar{E}$ s.t. $q \neq p \rightarrow q \in E$ then done



$q \in E' \rightarrow q' \in E$

$r_0 = \min(d(p,q), r_0 - d(p,q))$

$N_r(q) \subseteq N_r(p)$

$p \notin N_r(q)$

Find $q' \in N_{r_0}(q)$, $q' \in E$. q is a limit point of E. Then $q' \in N_r(p)$ $q' \neq p \to q' \in E, q' \in N_r(p)$
$q' \neq p$

(d) $\bar{E} \subseteq F \not\Rightarrow \bar{\bar{E}} \subseteq F$

<u>Theorem</u> $E \subseteq \mathbb{R}$ non-$\emptyset$ bounded above
then $\sup E \in \bar{E}$
if E closed, then $\sup E \in E$

<u>Proof</u> Assume $\sup E$ not in $\bar{E}$

$\sup E \notin E$ and $\sup E \notin E'$

For some $r > 0$, $N_r(\sup E) \cap E = \emptyset$

So $\sup E - r$ is an upper bound of E.

Contradiction since $\sup E - r < \sup E$

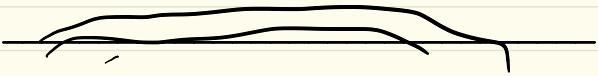
Ex $\bar{E} = \overset{E}{(0,1)} \cup \overset{E'}{\{0,1\}}$

<u>Open covers</u>/compactness

(X, d) is a matrix space $E \subseteq X$

A family $\{G_\alpha : \alpha \in I\}$ is an open cover of E if each G_α is an open subset of X and $E \subseteq \bigcup_{\alpha \in I} G_\alpha$

Ex ① $\mathbb{R}=X$ $\quad G_n=(-\infty,n),\ n\in\mathbb{N}$



$\bigcup_{n\in\mathbb{N}} G_n=\mathbb{R}$ $\quad \{G_n : n\in\mathbb{N}\}$ an open cover of $\mathbb{R}$

② $X=\mathbb{R}$ $\quad E=[0,1]$ $\quad G_n=(\frac{1}{n},2)$ $n\in\mathbb{N}, n\geq 2$

$G_1=(-1,\varepsilon)$ $\varepsilon>0$

$\{G_n : n\in\mathbb{N}, n\geq 1\}$ cover of $E=[0,1]$

Definition Let $E\subseteq X$, we say that E is compact if for each of open cover $\{G_\alpha : \alpha\in I\}$ of E there are $\alpha_1,\dots,\alpha_n$ s.t. $E\subseteq G_{\alpha_1}\cup\dots\cup G_{\alpha_n}$

Each open cover of E has a finite subcovers

(Y,d) a matrix space

$X\subseteq Y \to (X, d|(X\times X))$ subspace of (Y,d)

If $U\subseteq Y$ open in Y then $U\cap X$ is open in X

$x\in U\cap X$, $x\in U$ $\quad \exists r>0$, $N_r^Y(x)\subseteq U$

$N_r^Y(x)\cap X\subseteq U\cap X$

$N_r^Y(x)\cap X = N_r^X(x) \subseteq U\cap X$

有集合A, B, 若 $\forall x\in A$, x is an interior point of B, 同时 $N_r(x)\subseteq A$. 是否证明 A is open in B?

If $V \subseteq X$ is open in X then there exists $U \subseteq Y$ open in Y s.t. $V = U \cap X$

For $x \in V$, let $r_x > 0$ be s.t $N^X_{r_x}(x) \subseteq V$

let $U = \bigcup_{x \in V} N^Y_{r_x}(x) \subseteq Y$ open

Then $U \cap X \Rightarrow \left[\bigcup_{x \in V} N^Y_{r_x}(x)\right] \cap X$

$$= \bigcup_{x \in V} \left(N^Y_{r_x}(x) \cap X\right)$$

$$= \bigcup_{x \in V} N^X_{r_x}(x)$$

$$= V$$

<u>Theorem</u> (Y, d) a matrix space

X a subspace of Y

Let $E \subseteq X$ Then E is compact in Y iff E is compact in X

<u>Compactness</u>

$K \subseteq X \subseteq Y$

<u>Prop</u> X, Y matric spaces

K is compact in X iff K is compact in Y

Proof ① ($\Leftarrow$) Let $U_\alpha \subseteq Y$, $\alpha \in I$ be open subset of Y s.t.

$$\bigcup_{\alpha \in I} U_\alpha \supseteq K$$

Then consider $U_\alpha \cap X$ open in X

Note $K \subseteq \bigcup_{\alpha \in I} (U_\alpha \cap X)$

So there are $\alpha_1 \dots \alpha_n$ s.t. $K \subseteq \underbrace{(U_{\alpha_1} \cap X) \cup \dots \cup (U_{\alpha_n} \cap X)}_{\subseteq\, U_{\alpha_1} \cup \dots \cup U_{\alpha_n}}$

② $(\Leftarrow)$ Let $U_\alpha \subseteq X$, $\alpha \in I$ be open in X s.t.

$$K \subseteq \bigcup_{\alpha \in I} U_\alpha$$

For each $\alpha \in I$, there is $V_\alpha \subseteq Y$ open in Y s.t.

$U_\alpha = X \cap V_\alpha$

So $K \subseteq \bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} V_\alpha \cap X \subseteq \bigcup_{\alpha \in I} V_\alpha$

So there are $\alpha_1 \dots \alpha_n \in I$ s.t.

$$K \subseteq V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$$

$$K \subseteq K \cap X \subset (X \cap V_{\alpha_1}) \cup \dots \cup (X \cap V_{\alpha_n})$$
$$= U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$$

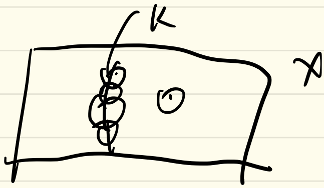
K is compact (in K) then K is compact in any space

Prop (X, d) a metric space

① Compact subsets of X are closed in X

② Closed subsets of compact metric spaces are compact

Proof ①



Let $K \subseteq X$ be compact

we show that $X \setminus K$ is open

so we need for each $x \in X \setminus K$,

there is $r > 0$ s.t. $N_r(x) \subseteq X \setminus K$

Fix $y \in K$ Find r_y s.t.

$N_{r_y}(y) \cap N_{r_y}(x) = \phi$

E.g. Take $r_y = \frac{d(x,y)}{3} > 0$

we do it for each $y \in K$.

Note $\bigcup_{y \in K} N_{r_y}(y) \supseteq K$ Since K is compact

there are $y_1 \ldots y_n$ s.t. $K \subseteq N_{r_{y_1}}(y_1) \cup \cdots \cup N_{r_{y_n}}(y_n)$

Note $N_{r_{y_i}}(y_i) \cap N_{r_{y_i}}(x) = \phi$ for $i = 1 \ldots n$

so $N_{\bar{r}}(y_i) \cap N_{\bar{r}}(x) = \phi$ where $\bar{r} = \min(r_{y_1}, \ldots, r_{y_n}) > 0$

So $(N_{r_{y_1}}(y_1) \cup \cdots \cup N_{r_{y_n}}(y_n)) \cap N_{\bar{r}}(x) = \phi$

So $K \cap N_{\bar{r}}(x) = \phi$ so $N_{\bar{r}}(x) \subseteq X \setminus K$

(2) X a compact matrix space

$K \subseteq X$ closed

Need K is compact in X

Let U_α, $\alpha \in I$, be open in X

s.t. $K \in \bigcup_{\alpha \in I} U_\alpha$

Consider $X \subseteq (X \setminus K) \cup \bigcup_{\alpha \in I} U_\alpha$?

So there are $\alpha_1 \ldots \alpha_n \in I$ s.t.

$X = (X \setminus K) \cup U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$

So $K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$ so K is compact

Prop Let K_α $\alpha \in I$ be a family of compact subsets of matrix space X.

Assume for all $\alpha_1 \ldots \alpha_n \in I$ we have

$K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \neq \phi$ Then $\bigcap_{\alpha \in I} K_\alpha \neq \phi$

Proof Pick $\alpha_0 \in I$

Assume towards a contradiction that

$\bigcap_{\alpha \in I} K_\alpha = \phi$. Then $K_{\alpha_0} \subseteq \bigcup_{\alpha \in I, \alpha \neq \alpha_0} (X \setminus \overset{\text{closed}}{K_\alpha})$

$$\| \quad X \setminus \bigcap_{\alpha \in I, \alpha \neq \alpha_0} K_\alpha$$

Since K_{α_0} is compact, there are $\alpha_1 \dots \alpha_n \in I$ s.t. $K_{\alpha_0} \subseteq (X \setminus K_{\alpha_1}) \cup \dots \cup (X \setminus K_{\alpha_n})$

$$\| \quad X \setminus (K_{\alpha_1} \cap \dots \cap K_{\alpha_n})$$

So $K_{\alpha_0} \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_n} = \phi$ contradiction

Cor Let $K_n, n \in \mathbb{N}$ be compact non-ϕ subsets of a matric space X. If $K_0 \supseteq K_1 \supseteq K_2 \supseteq \dots$, Then $\bigcap_n K_n \neq \phi$. Then consider $K_n, n \in \mathbb{N}$

Note $n_1 \dots n_\varphi \in \mathbb{N}$

$K_{n_1} \cap \dots \cap K_{n_\psi} = K_{\max(n_1, \dots n_\psi)} \neq \phi$

Prop Each infinite subset of a compact space has a limit point.

Proof Assume this is not the case. $(V_x = N_r(x)$ some $r>0)$

Take $x \in X$, we can find an open set V_x s.t. $x \in V_x$ and $|V_x \cap E| \leq 1$

Note $X = \bigcup_{x \in X} V_x$. By compactness there are $x_1 \dots x_n$ s.t. $X = V_{x_1} \cup \dots \cup V_{x_n}$

$E = E \cap X = (E \cap V_{x_1}) \cup \dots \cup (E \cap V_{x_n})$

So $|E| \leq n$ contradiction

Compactness in $\mathbb{R}^k$

Theorem Let (I_n) be a sequence of non-ϕ closed intervals in $\mathbb{R}$. If $I_0 \supseteq I_1 \supseteq \dots$ then $\bigcap_n I_n \neq \emptyset$

reformulation of lease upper bound property

proof $I_n = [a_n, b_n]$, $a_n \le b_n$

consider $A = \{a_n : n \in \mathbb{N}\}$ $A \neq \phi$

A bounded above by b_0 ($a_n \in [a_0, b_0]$)

Let $c = \sup A$ we claim that

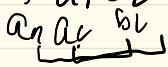
$c \in [a_n, b_n]$ for each n

Fix n. $a_n \le c$ since $c = \sup A \ni a_n$

$c \le b_n$ since b_n is an upper bound of A

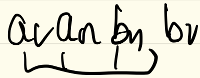
If $l \geq n$, then $[a_l, b_l] \subseteq [a_n, b_n]$, $a_l \in [a_n, b_n]$

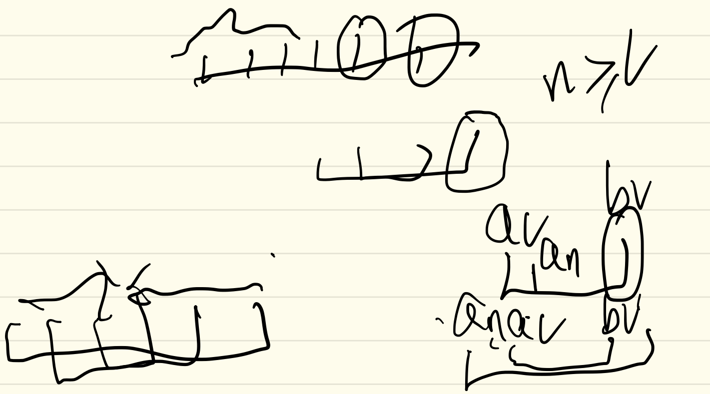
$a_l \le b_l$



If $l < n$, then $[a_n, b_n] \subseteq [a_l, b_l]$

So $a_l \le a_n \le b_n$





<u>Thm</u> Every cell in $\mathbb{R}^k$ is compact

A cell is a set of the form $[a_1, b_1] \times \cdots \times [a_k, b_k]$ where $a_1 \le b_1, \ldots, a_k \le b_k$, $a_i, b_i \in \mathbb{R}$

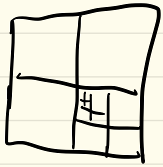
$\{x \in \mathbb{R}^k : a_1 \le x_1 \le b_1 \cdots a_k \le x_k \le b_k\}$

Proof: Let the cell be of the form $C = [a_1, b_1] \times \cdots \times [a_k, b_k]$

Need: Each open cover of C has a finite subcover

Fix U_α, open subsets of $\mathbb{R}^k$ s.t. $C \subseteq \bigcup_{\alpha \in A} U_\alpha$

Assume towards contradiction there's no finite subcover of $U_\alpha : \alpha \in A$



Set $a_i^0 = a_i$, $b_i^0 = b_i$

Assume we have a_i^n, b_i^n s.t.

- $[a_i^n, b_i^n] \subseteq [a_i^{n-1}, b_i^{n-1}] \subseteq \cdots \subseteq [a_i^0, b_i^0]$
- $b_i^n - a_i^n \le \frac{b_i - a_i}{2^n}$
- $[a_1^n, b_1^n] \times \cdots \times [a_k^n, b_k^n]$ cannot be covered by finitely many sets namely $U_{\alpha \in A}$

Consider the cells obtained by halving each interval $[a_i^n, b_i^n]$ and taking the product. We have 2^k such cells and they covers $[a_1^n, b_1^n] \times \cdots \times [a_k^n, b_k^n]$. So one of the cells cannot be covered by finitely many sets U_α, $\alpha \in A$. Pick one such cell be : $[a_1^{n+1}, b_1^{n+1}] \times \cdots \times [a_k^{n+1}, b_k^{n+1}]$

Clear : $[a_i^{n+1}, b_i^{n+1}] \subseteq [a_i^n, b_i^n]$ for $i = 1 \ldots k$

$0 \le b_i^{n+1} - a_i^{n+1} = \frac{b_i^n - a_i^n}{2} \le \frac{b_i - a_i}{\frac{2^n}{2}} = \frac{b_i - a_i}{2^{n+1}}$

So we maintain the three conditions in the construction

we have $[a_i^n, b_i^n] \subseteq \dots \subseteq [a_i^0, b_i^0]$ for each $1 \le i \le k$.

$\bigcap_n [a_i^n, b_i^n] \neq \phi$ (by theorem)

$\downarrow$

x_i

Note $\underline{x} = (x_1, \dots x_k) \in \bigcap_n ([a_1^n, b_1^n] \times \dots \times [a_k^n, b_k^n])$

Note $\bigcap_n ([a_1^n, b_1^n] \times \dots \times [a_k^n, b_k^n]) = \{\underline{x}\}$

If $\underline{y}$ was a point in the intersection with $\underline{x} \neq \underline{y}$ then

$x_i \neq y_i$ for some i

Then $x_i, y_i \in \bigcap_n [a_i^n, b_i^n]$ So $|x_i - y_i| \le b_i^n - a_i^n$

so $0 < |x_i - y_i| \le \dfrac{b_i - a_i}{2^n}$ so $n \le 2^n \le \dfrac{b_i - a_i}{|x_i - y_i|}$ Contradiction ?

Since $\underline{x} \in C$ there is $\alpha_0 \in A$ s.t. $\underline{x} \in U_{\alpha_0}$ so

There is $r > 0$ s.t. $N_r(\underline{x}) \subseteq U_{\alpha_0}$ since U_{α_0} is open

note $\underline{y}\ \underline{z} \in [c_1, d_1] \times \dots \times [c_k, d_k]$

$|\underline{y} - \underline{z}| = \sqrt{(z_1 - y_1)^2 + \dots + (z_k - y_k)^2} \le \sqrt{k} \cdot \max_{1 \le i \le k} |z_i - y_i| \le \sqrt{k} \cdot \max_{1 \le i \le k} |d_i - c_i|$

If $\underline{z} \in [a_1^n, b_1^n] \times \dots \times [a_k^n, b_k^n]$

then $|\underline{x} - \underline{z}| \le \sqrt{k} \max_{1 \le i \le k} |b_i^n - a_i^n| \le \sqrt{k} \dfrac{\max (b_i - a_i)}{2^n}$

So for n large enough, $\sqrt{k} \dfrac{\max (b_i - a_i)}{2^n} < r$

So $[a_1^n, b_1^n] \times \dots [a_k^n, b_k^n] \subseteq N_r(\underline{x})$

for large enough n

$\subseteq U_{\alpha_0}$ So for large enough n,

$[a_1^n, b_1^n] \times \dots \times [a_k^n, b_k^n]$ is covered by finitely many sets namely U_α, $\alpha \in A$

Thm A subset E of $\mathbb{R}^k$ is compact iff E is closed and bounded

Definition: (X, d) a metric space, $E \subseteq X$ is bounded if there is $k \geq 0$ s.t. $d(x,y) \leq k$ for all $x, y \in E$

$$\sup_{x,y \in E} d(x,y) < \infty$$

Proof $(\Rightarrow)$ compact $\Rightarrow$ closed (proved in general)

Pick $x \in E$, consider $N_n(x)$, $n \in \mathbb{N}$, $n > 0$

$$\bigcup_{n>0} N_n(x) = \mathbb{R}^k \supseteq E \quad \text{(open cover)}$$

Since E is compact, there are $n_1, \dots, n_m$ s.t.

$$E \subseteq N_{n_1}(x) \cup \dots \cup N_{n_m}(x)$$

So $E \subseteq N_{\bar{n}}(x)$ when $\bar{n} = \max(n_1, \dots, n_m)$

$$\underline{y}, \underline{z} \in E \Rightarrow d(\underline{y}, \underline{z}) \leq d(\underline{y}, \underline{x}) + d(\underline{x}, \underline{z})$$
$$\leq \underline{n} + \underline{n} = 2\underline{n} = k$$

$(\Leftarrow)$ If E is bounded, there are $a_1 \leq b_1, \dots, a_k \leq b_k$ s.t.

$$E \subseteq [a_1, b_1] \times \dots \times [a_k, b_k]$$

$\underline{x} \in E \qquad d(\underline{x}, \underline{y}) \leq k$

$$\sqrt{(x_1-y_1)^2 + \dots + (x_k-y_k)^2} \geq \sqrt{k} \max_{1 \leq i \leq k} |x_i - y_i|$$

?

So for each $i \leq k$, $|y_i - x_i| \leq k$ so

$$\underset{a_i}{x_i - k} \leq y_i \leq \underset{b_i}{x_i + k} \quad \leq \frac{\sqrt{k} \max(b_i - a_i)}{2}$$

Since E is closed set

proved $\rightarrow$ [a closed subset of a compact set is compact]

we have E is compact

Cor Each bounded infinite subset of $\mathbb{R}^k$ has a limit point.

Proof: $E \subseteq \mathbb{R}^k$ bounded. There is a cell $[a_1, b_1] \times \cdots \times [a_k, b_k] \supseteq E$, compact

Since infinite subset of a compact set has a limit point, we are done.

March 8

(X, d) a metric space, $p_n \in X$, $n \in \mathbb{N}$

(p_n) converges to $p \in X$ for each $\varepsilon > 0$

there is N s.t. for $n \geq N$

$$d(p_n, p) < \varepsilon$$

(p_n) converges if it converges to some $p \in X$

if (p_n) converges to p and p' then $p = p'$

If $p \neq p'$ then $d(p, p') > 0$. Let $\varepsilon = \frac{d(p,p')}{2} > 0$

For some N, for $n \geq N$

$$d(p, p_n) + d(p_n, p') \geq d(p, p')$$

$$\wedge$$

$$\varepsilon + \varepsilon$$

$$\shortparallel$$

$$2\varepsilon$$

$$\shortparallel$$

$$d(p, p') \geq 0$$

(p_n) diverges if it does not converge

Ex (1) $s_n = \frac{1}{n}$ $s_n \to 0$

Fix $\varepsilon > 0$. Find $N \in \mathbb{N}$ s.t.

$N > \frac{1}{\varepsilon}$ then $d(s_n, 0) = |s_n - 0| = |\frac{1}{n} - 0| = \frac{1}{n} \le \frac{1}{N} < \varepsilon$

for $n \geq N$

(2) $t_n = \frac{(-1)^3}{n}$

$d(t_n, 0) = |\frac{(-1)^3}{n} - 0| = |\frac{(-1)^3}{n}| = \frac{1}{n}$

(3) $s_n = n$ s_n diverges

Let $s \in \mathbb{R}$. Take $\varepsilon = 1$

Fix arbitrary $N \in \mathbb{N}$

Find $n \geq N$ s.t. $n > s+1$

$|s_n - s| = |n - s| = n - s > s+1-s = 1 = \varepsilon$ contradiction

Prop (X, d) metric space $p_n \in X$

(1) If (p_n) converges then (p_n) is bounded

$\{p_n : n \in \mathbb{N}\}$ is bounded

Proof: Let $p = \lim_n p_n$

Take N s.t. for $n \geq N$, $d(p, p_n) < 1$

Let $K = \max\left(2, \max_{i,j<N} d(p_i, p_j), \max_{i<N} (d(p_i, p) + 1)\right) < \infty$

$(p_m, p_n) \leq K$ $\forall m, n \in \mathbb{N}$

Prop (X, d) metric space, $E \subseteq X$ $p \in X$
p is a limit point of E iff there is a sequence
$p_n \in E$ s.t. $p_n \xrightarrow[n]{} p$ and $p_n \neq p$ all n

Proof: $(\Rightarrow)$ Let $p_0 \in E$ be s.t. $p_0 \in N_r(p)$ and $p_0 \neq p$

$$r_1 = \frac{d(p, p_0)}{2} > 0$$

Let $p_1 \in E$ be s.t. $p_1 \in N_{r_1}(p)$ and $p_1 \neq p$,

$r_2 = \frac{d(p, p_1)}{2} > 0$ Let $p_2 \in E$

$p_2 \in N_{r_2}(p)$, $p_2 \neq p$, Produce (p_n) in this fashion

Note $p_n \neq p$ all n. $p_n \in E$

Note $n \geq 1$ $\quad d(p_n, p) < r_n \leq \frac{d(p_0, p)}{2^n}$

$$r_1 = \frac{d(p_0, p)}{2}$$

$$r_{n+1} = \frac{d(p_n, p)}{2} < \frac{r_n}{2} \leq \frac{\frac{d(p_0, p)}{2^n}}{2} = \frac{d(p_0, p)}{2^{n+1}}$$

$p_n \xrightarrow[n]{} p$, $2^{n+1} > 2^n$, $2^n \geq 2^N$ if $n \geq N$

Fix $\varepsilon > 0$ Find N s.t. $\frac{d(p_0, p)}{2^N} < \varepsilon$

For $n \geq N$, $\frac{1}{\varepsilon} < \frac{2^N}{d(p_0, p)}$

$$d(p_n, p) < \frac{d(p_0, p)}{2^n} \leq \frac{d(p_0, p)}{2^N} < \varepsilon$$

(⇑) we have $p_n \in E$, $p_n \xrightarrow[n]{} p$, $p_n \neq p \Rightarrow p$ is a limit point of E.

Let $r > 0$ consider $N_r(p)$ need a point $q \in N_r(p)$, $q \in E$, $q \neq p$

Since $p_n \xrightarrow[n]{} p$ then there is N s.t.

$d(p_n, p) < r$ for all $n \geq N$

So $d(p_N, p) < r$, $p_N \in N_r(p)$

Let $q = p_N$ we are done

Theorem ~~TODO~~ ✓

(s_n), (t_n) sequences of reals s.t. $s_n \xrightarrow[n]{} s$, $t_n \xrightarrow[n]{} t$

(1) $\lim_n (s_n + t_n) = \lim_n s_n + \lim_n t_n = s+t$

(2) if $c \in \mathbb{R}$, then $\lim_n (cs_n) = c \lim_n (s_n) = cs$

(3) $\lim_n (s_n t_n) = \lim_n (s_n) \lim_n (t_n) = st$

(4) if $s_n \neq 0$ for all n, $s \neq 0$, then $\lim_n \left(\frac{1}{s_n}\right) = \frac{1}{\lim_n s_n} = \frac{1}{s}$

Proof: (3) $|s_n t_n - st| = |s_n t_n - s t_n + s t_n - st|$

$\leq |s_n t_n - s t_n| + |s t_n - st|$

$= |s_n - s| \cdot |t_n| + |s||t_n - t| \leq |s_n - s| C + |s||t_n - t|$

Notes since (t_n) converges, it is bounded so there is K s.t. $|t_m - t_n| \leq K$ all m, n

$|t_n| - |t_0| \leq |t_0 - t_n| \leq K$ all n

$|t_n| \le |t_0| + \overset{C}{K}$ all n

$|s_n - s| \cdot |t_n| + |s||t_n - t| \le |s_n - s| C + |s||t_n - t|$

Let $\varepsilon > 0$, find N s.t. for $n \ge N$, $|s_n - s| < \frac{\varepsilon}{2C}$

$|t_n - t| < \frac{\varepsilon}{2|s|+1}$

Then for $n \ge N$ $|s_n - s| C + |s||t_n - t| < \frac{\varepsilon}{2} + \frac{|s|}{2|s|+1} \varepsilon$

$\le \frac{\varepsilon}{2}(1+1) = \varepsilon$

(4) $\left| \frac{1}{s_n} - \frac{1}{s} \right| = \left| \frac{s - s_n}{s s_n} \right| = \frac{|s - s_n|}{|s||s_n|}$

for large n, $|s - s_n| < \frac{|s|}{2}$
$(n \ge N')$ $|s| - |s_n| \le$

$|s| - \frac{|s|}{2} \le |s_n|$

$\frac{|s|}{2} \le |s_n|$

$\frac{|s - s_n|}{|s||s_n|} \le \frac{|s - s_n|}{|s| \frac{|s|}{2}} = \frac{2}{|s|^2} |s - s_n|$

Fix $\varepsilon > 0$ for some $N \ge N'$

for $n \ge N$, $|s - s_n| < \frac{\frac{\varepsilon}{2}}{|s|^2}$

So $\frac{2}{|s|^2} |s - s_n| < \varepsilon$

March 12

Thm $\underline{x}_n \in \mathbb{R}^k$ $\underline{x}_n = (\alpha_{1n} \dots \alpha_{kn})$, $\underline{y}_n \in \mathbb{R}^k$, $\underline{x}, \underline{y} \in \mathbb{R}^k$

a) $(\underline{x}_n)$ converges to $\underline{x} = (\alpha_1, \dots \alpha_k)$ iff

$\lim_n \alpha_{in} = \alpha_i$ for $1 \le k$ — $\forall \varepsilon > 0\ \exists N\ \forall n \ge N$

$|\underline{x}_n - \underline{x}| < \varepsilon$

$\sqrt{(\alpha_{1n}-\alpha_1)^2 + \dots + (\alpha_{kn}-\alpha_k)^2} < \varepsilon$

b) $\underline{x}_n \to \underline{x}$, $\underline{y}_n \to \underline{y}$, $\beta_n \to \beta$, $\beta_n, \beta \in \mathbb{R}$

Then $\lim_n (\underline{x}_n + \underline{y}_n) = \underline{x} + \underline{y}$

$\lim_n \underline{x}_n \cdot \underline{y}_n = \underline{x} \cdot \underline{y}$

$\lim_n \beta_n \underline{x}_n = \beta \underline{x}$

$|\underline{x}_n - \underline{x}| \le \sqrt{(\alpha_{1n}-\alpha_1)^2 + \dots + (\alpha_{kn}-\alpha_k)^2} \le \sqrt{k} \max_{j=1\dots k} |\alpha_{jn} - \alpha_j|$

Subsequences

Let (x_n) be a sequence of elements of X. By a subsequence of (x_n), we mean any sequence (y_k) s.t. $y_k = x_{n_k}$ for some $n_0 < n_1 < n_2 < n_3 \dots$

Ex ① $x_n = (-1)^n$ $y_k = 1$ is a subsequence of x_n

$y_k = x_{2k} = (-1)^{2k} = 1$ so $n_k = 2k$

$z_k = -1$ $z_k = x_{2k+1}$

② $x_n = n$ $y_k = 1$ not a subsequence

Thm

(1) If (p_n) is a sequence in compact metric space (X, d), then (p_n) contains a convergent subsequence.

(2) Each bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence

Proof: (1) ① Consider $E = \{p_n : n \in \mathbb{N}\}$ E is finite

Then there is $p \in E$ s.t. $p = p_n$ for infinite many n.

i.e. there are $n_0 < n_1 < n_2 \cdots$

s.t. $p = p_{n_k}$ So (p_{n_k}) is a subsequence of (p_n) converges to p

② E is infinite. Re E has a limit point, say p

Inductively pick $n_0 < n_1 < n_2 < \cdots$ as follows. $n_0 = 0$

Assume $n_0 < \cdots < n_k$ constructed. Consider

$N_{\frac{1}{k+1}}(p)$ contains infinitely many elements of E

$p_{n_{k+1}} \in N_{\frac{1}{k+1}}(p) \cap E$ s.t. $n_{k+1} > n_k$

(suffice $p_{n_{k+1}} \neq p_i$ for $i \leq k$)

So (p_{n_k}) constructed with $n_0 < n_1 < \cdots$

Note $d(p_n, p) < \frac{1}{k}$ for $k \geq 1$ $\rightarrow$ So $p_{n_k} \xrightarrow[k]{} p$

$p_{n_k} \in N_{\frac{1}{k}}(p)$

(2) the values of (p_n) are contained in $\{\underline{x} \in \mathbb{R}^k ; |x| \leq K\}$ since x is compact (p_n) has a subsequent by (K)

Cauchy sequence

A sequence (p_n) of elements of (X, d) is Cauchy if for each $\varepsilon > 0$ there is an N s.t. for all $m, n \geq N$, $d(p_m, p_n) < \varepsilon$

Prop Convergence sequence are Cauchy

Proof Let (p_n) (is (X, d)) converge to p. Then take $\varepsilon > 0$
There is N s.t. for all $n \geq N$, $d(p_n, p) < \frac{\varepsilon}{2}$
Then for $m, n \geq N$, we have
$d(p_m, p_n) \leq d(p_m, p) + d(p, p_n) < \varepsilon$

Prop Each Cauchy sequence is bounded
Let $\varepsilon = 1$ find N s.t. for $m, n \geq N$, $d(p_m, p_n) < 1$
Let $k = \max d(p_i, p_j) + 1$
Then $d(p_m, p_n) \leq k$ for all m, n

Definition for $E \subseteq X$, (X, d) a metric space, $\mathrm{Diam}(E) = \sup\{d(x, y) \mid x, y \in E\}$
$\mathrm{diam}(\emptyset) = -\infty$

Proposition $E \subseteq X$, (X,d) is a metric space then
$\text{Diam}(E) = \text{Diam}(\bar{E})$

Proof $\text{Diam}(E) \leq \text{Diam}(\bar{E})$ as $E \subseteq \bar{E}$

Take $\varepsilon > 0$, choose $p \in \bar{E}$, $q \in \bar{E}$. by the definition of $\bar{E}$, there are points p', q' in E s.t.
$d(p,p') < \varepsilon$ and $d(q,q') < \varepsilon$, p', q' limit points

Hence

$$
\begin{aligned}
d(p,q) &\leq d(p,p') + d(p',q') + d(q',q) \\
&< 2\varepsilon + d(p',q') \\
&\leq 2\varepsilon + \text{Diam}(E)
\end{aligned}
$$

So $\text{Diam}(\bar{E}) \leq 2\varepsilon + \text{Diam}(E)$

Choose $\varepsilon = \dfrac{\text{Diam}(\bar{E}) - r}{2}$,

we have $\text{Diam}(\bar{E}) = \text{Diam}(E)$

Thm (X,d) compact metric Each Cauchy sequence converges

Proof Let (p_n) be a cauchy sequence in X

Consider $E_N = \{p_n : n \geq N\}$

$\mathrm{Diam}(E_N) \xrightarrow[N]{} 0$ (by Cauchy)

$\mathrm{Diam}(\overline{E_N}) \xrightarrow[N]{} 0$

$\overline{E_N} \supseteq \overline{E_{N+1}}$

$\overline{E_N}$ closed non-ϕ $(p_N \in E_N \subseteq \overline{E_N})$

$\bigcap_N \overline{E_N} \neq \phi = \{p\}$ note $\lim_n p_n = p$

March 14

Corollary In $\mathbb{R}^k$, Cauchy sequences converge

Proof Let (a_n) be Cauchy in $\mathbb{R}^k$. (a_n) is bounded

so (a_n) is contained in $\{\underline{x} \in \mathbb{R}^k ; |\underline{x}| \leq K\}$

so there is a s.t. $a_n \xrightarrow[n]{} a$ in $\mathbb{R}^k$! compact

so (a_n) converges.

Df Let (s_n) be a sequence in $\mathbb{R}$. If $s_n \leq s_{n+1}$, we say that (s_n) is monotonically increasing

Thm If (s_n) is a sequence in $\mathbb{R}$ that is monotonic and bounded, then (s_n) converges.

In fact, if (s_n) is increasing, then $s_n \xrightarrow[n]{} \sup\{s_n : n \in \mathbb{N}\}$

if (s_n) is decreasing, then $s_n \xrightarrow[n]{} \inf\{s_n : n \in \mathbb{N}\}$

Proof Assume (s_n) is increasing

Need $s_n \to \sup\{s_n : n \in \mathbb{N}\} = s$

let $\varepsilon > 0$ be given

let N be s.t. $s - \varepsilon < s_N$ (N exists since $s - \varepsilon < s$

$= \sup\{s_n : n \in \mathbb{N}\}$)

note for $n \geq N$, $s - \varepsilon < s_N \leq s_n \leq s < s + \varepsilon$

So for $n \geq N$, $|s - s_n| < \varepsilon$

Df (X, d) a metric space.

(X, d) is complete if each Cauchy sequence in (X, d) converges

E.g. $\mathbb{R}^k$ is complete

Every metric space lies in a complete metric space (only one way to "complete" it)

Upper/lower limits for sequences in $\mathbb{R}$

Note: $s_n \xrightarrow[n]{} \infty$ if $\forall C \in \mathbb{R}\ \exists N\ \forall n \geq N\ s_n \geq C$

$s_n \xrightarrow[n]{} -\infty$ if $\forall C \in \mathbb{R}\ \exists N\ \forall n \geq N\ s_n \leq C$

Fix (s_n)

Define E = set of all extended Reals ($[-\infty, \infty]$) of the form $\lim_k s_{n_k}$ for a subsequence of (s_n)

$s^* = \sup E$ (possibly ∞) Note $E \neq \emptyset$

$s_* = \inf E$ (possibly $-\infty$) $\subseteq [-\infty, \infty]$

Note $s^* = \lim_n \sup s_n$

$s_* = \lim_n \inf s_n$

Thm (a) $s^* \in E$

(b) If $x > s^*$, then there is N s.t. $s_n < x$ for all $n \geq N$

$x < s_*$ $\qquad$ $s_n > x$

Moreover, s^* is the unique element of $[-\infty, \infty]$ with (a) & (b)

Similarly for s_*

Proof

(a) Let $k < s^*$

There is a subsequence (s_{n_k}) s.t. $\lim_k s_{n_k} > k$

So there are arbitrarily large natural numbers n s.t. $s_n > k$

$s^* \in \mathbb{R}$ $\quad$ we pick $n_0 < n_1 < n_2 < \dots$

$$s_{n_k} > s^* - \frac{1}{k+1}$$

(s_{n_k}) has a subsequence with a limit $(s_{n_{k_i}})$

So $(s_{n_{k_i}})$ is a subsequence of (s_n)

Note $s_{n_{k_i}} > s^* - \frac{1}{k_i+1}$ $\qquad$ Infinitely many k $\neq$ $\forall i$ so we want to use $s_{n_{k_i}}$

So $\lim s_{n_{k_i}} \overset{(\in E)}{\geq} s^*$ (sup E)

So $s^* = \lim_i s_{n_{k_i}} \in E$

(b) Case 1 $\quad s^* \in \mathbb{R}$ $\quad$ Let $x > s^*$

Need: $\{n : s_n \geq x\}$ is finite. Assume for the contradiction, $\{n : s_n \geq x\}$ is infinite.

So there are $n_0 < n_1 < \dots n_k$ s.t. $s_{n_k} \geq x$ all k

Let $(s_{n_{k_i}})$ be a subsequence of (s_{n_k}). so of (s_n) s.t. $\lim_i s_{n_{k_i}}$ exists $\quad \lim_i s_{n_{k_i}} \geq x > s^*$ (sup E)

Contradiction

Uniqueness:

p, q with (a) & (b) distinct

Assume $p < q$

Choose x s.t. $p < x < q$

By (b) for p, there is N s.t. $s_n < x$ for all $n \geq N$

Then (a) fails for q since there cannot be a subsequence (s_{n_k}) s.t. $s_{n_k} \xrightarrow[k]{} q$

Ex on $\mathbb{Q}$

Let (s_n) be s.t. $\{s_n : n \in \mathbb{N}\} = \mathbb{Q}$

$E = [-\infty, \infty]$

$\lim_n \sup s_n = \infty \quad \lim_n \inf s_n = -\infty$

(2) $s_n = (-1)^n \quad E = \{-1, 1\}$

$\lim_n \sup s_n = 1 \quad \lim_n \inf s_n = -1$

(3) If $\lim s_n$ exists. Then $\lim_n \inf s_n = \lim_n s_n = \lim_n \sup s_n$

$E = \{\lim_n s_n\}$ iff

Example

(a) $p > 0 \Rightarrow \lim_n \frac{1}{n^p} = 0$

(b) $p > 0 \Rightarrow \lim_n \sqrt[n]{p} = 1$

① $p > 1$ Let $x_n = \sqrt[n]{p} - 1 > 0$

$p = (1 + x_n)^n \geq 1 + n x_n$

So $0 < x_n \leq \frac{p-1}{n} \to 0$ so $\sqrt[n]{p} \to 1$

② $p = 1$ obvious $\sqrt[n]{p} = 1$

③ $0 < p < 1$ $\frac{1}{p} > 1$ So $\sqrt[n]{\frac{1}{p}} \xrightarrow[n]{} 1 \quad \frac{1}{\sqrt[n]{p}} \xrightarrow[n]{} 1 \quad \sqrt[n]{p} \to 1$

Can we prove the limit by saying $\varepsilon \to 0$ without any further explanation?

March 19

(c) $\lim_n \sqrt[n]{n} = 1$

$x_n = \sqrt[n]{n} - 1 \geq 0$ for $n \geq 1$

$\frac{n(n-1)}{2} x_n^2 \leq (1+x_n)^n = n$

$0 \leq x_n \leq \sqrt{\frac{2}{n-1}} \quad n \geq 2$

So $x_n \xrightarrow[n]{} 0$ Achimedian property

(d) $p > 0$, α real $\Rightarrow \lim_n \frac{n^\alpha}{(1+p)^n} = 0$ ☆ Important

$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\dots(n-k+1)}{k!} p^k$

$> \frac{(\frac{n}{2})^k}{k!} p^k \quad (n > 2k)$

$= \frac{p^k}{2^k k!} n^k$

Take k s.t. $k > \alpha$

Then for $n > 2k$ we have $0 \leq \frac{n^\alpha}{(1+p)^n} < \frac{n^\alpha}{(\frac{p^k}{2^k k!}) n^k} = \frac{2^k k!}{p^k} n^{\alpha - k}$

where $\alpha - k < 0$ so by (a) $(\frac{1}{n^p}, p > 0)$, $\underbrace{\frac{2^k k!}{p^k}}_{\text{constant}} n^{\alpha-k} \xrightarrow[n]{} 0$

So $\frac{n^\alpha}{(1+p)^n} \xrightarrow[n]{} 0$

(e) $|x| < 1 \Rightarrow \lim_n x^n = 0$ Take $\alpha = 0$ in (d)

then $\lim_n x^n = \lim_n \frac{1}{(\frac{1}{x})^n} = \frac{1}{(1+p)^n} = 0$

Continuous functions

Limits

We have the following situations:

(X, d_X), (Y, d_Y), $E \subseteq X$, $f: E \to Y$

Let $p \in X$ be a limit point of E

We write $\lim_{x \to p} f(x) = q$ ($f(x) \to q$ as $x \to p$)

if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for all $x \in E$ with $0 < d(x,p) < \delta$ we have $d_Y(f(x), q) < \varepsilon$

Thm $\lim_{x \to p} f(x) = q$ iff $\lim_n f(p_n) = q$ for each sequence (p_n) in E s.t. $p_n \neq p$ and $\lim_n p_n = p$

Both can be definitions, but the first one is better as it can be easily generalized to a non-metric way

Proof ($\Rightarrow$)

Let (p_n) be a sequence in E s.t. $p_n \xrightarrow[n]{} p$, $p_n \neq p$

Need: $f(p_n) \xrightarrow[n]{} q$

Let $\varepsilon > 0$. Need N s.t. $d_Y(f(p_n), q) < \varepsilon$

Let $\delta > 0$ be s.t. $0 < d_X(x,p) < \delta$

Since $p_n \to p$, $\exists N$ s.t. $d_X(p_n, p) < \delta$ for $n \geq N$

$d_X(p_n, p) > 0$ as $p_n \neq p$

So for $n \geq N$ $d_Y(f(p_n), q) < \varepsilon$

($\Leftarrow$) Assume $f(x) \not\xrightarrow[x \to p]{} q$

So $\exists \varepsilon > 0$ $\forall \delta > 0$ $\exists x \in E$ s.t. $0 < d(x,p) < \delta$ and $d_Y(f(x), q) \geq \varepsilon$

Fix $\varepsilon > 0$ as in this sentence.

For $\delta = \frac{1}{n+1}$ there is $p_n \in E$

s.t. $0 < d_X(p_n, p) < \delta$ and $d_Y(f(p_n), q) \geq \varepsilon$

Since we need $\lim_n f(p_n) = q$, contradiction

<u>Corollary</u> If f has a limit, then the limit is unique.

Suppose $q \neq q'$

$\lim_{x \to p} f(x) = q$ $\lim_{x \to p} f(x) = q'$

I
went
to
the
restroom
for
this
part
⋮

Actuall I have filled all notes I missed!
I just left a larger blank than I expected Orz

Df $f, g : E \to \mathbb{R}$

$(f+g)(x) = f(x) + g(x)$

$(fg)(x) = f(x) \cdot g(x)$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ if $g(x) \neq 0$ all $x \in E$

$f, g : E \to \mathbb{R}^k$

$(f+g)(x) = f(x) + g(x)$

$(f \cdot g)(x) = f(x) \cdot g(x)$

$f, g : E \to \mathbb{R}$ or $f, g : E \to \mathbb{R}^k$

<u>Thm</u> $\lim_{x \to p} f(x) = A$ $\lim_{x \to p} g(x) = B$

Re $\lim_{x \to p} (f+g)(x) = A + B$

$\lim_{x \to p} (fg)(x) = AB$

$\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$ if $B \neq 0$

<u>Pf</u> X, Y metric spaces

$E \subseteq X$, $p \in E$ $f : E \to Y$

f is continuous at p if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

for all $x \in E$ with $d_X(x, p) < \delta$, we have $d_Y(f(x), f(p)) < \varepsilon$

<u>Thm</u> f as above. f is continuous at p iff $\lim_{x \to p} f(x) = f(p)$

If p is not a limit point, we can pick δ s.t. $d_X(x, x)$ we only have $p = x$

If p is a limit point ...

<u>Corollary</u> f continuous at p iff for each sequence $p_n \xrightarrow[n]{} p$

$f(p_n) \xrightarrow[n]{} f(p)$

We say that $f : E \to Y$ continuous if it is continuous at each $p \in E$

<u>Thm</u> $f : E \to Y$ $g : f(E) \to Z$, X, Y, Z metric space

Assume f, g continuous, then $g \circ f$ is continuous

proof: $g\circ f$:
$$\begin{tikzcd} \overset{\delta_2}{E} \arrow[rr, "g\circ f"] \arrow[dr, "f"'] & & Z \ \varepsilon_1 \\ & f(E) \arrow[ur, "g"'] & \end{tikzcd}$$
$\varepsilon_2 \leq \delta_1$

So $d_x(x,p) < \delta_2 \Rightarrow d_Z(f\circ g(x), f\circ g(p)) < \varepsilon_1$

Thm $f, g : X \to \mathbb{R}$ continuous
then $f+g$, $f\cdot g$, $\frac{f}{g}$ continuous Trivial

$f, g : X \to \mathbb{R}^k$ continuous, $f+g : X \to \mathbb{R}^k$
$f\cdot g : X \to \mathbb{R}^k$ continuous

$\underline{f}(x) = (f_1(x), \dots f_k(x))$ $f_i : X \to \mathbb{R}$

$\underline{f}$ continuous iff $f_1 \dots f_k$ continuous

March 21

Thm $f : X \to Y$
f is continuous iff $f^{-1}(V)$ is open for each $V \subseteq Y$ open

Cor $f : X \to Y$ then f continuous iff $f^{-1}(C)$ is closed for each $C \subseteq Y$ closed

Do we assume f is one-to-one here?

Proof of Cor from Thm: Assume f is continuous

$C \subseteq Y$ closed $\Leftrightarrow$ $Y \setminus C$ open $\Rightarrow f^{-1}(Y\setminus C)$ open

$\Rightarrow f^{-1}(Y) \setminus f^{-1}(C) \Rightarrow X \setminus f^{-1}(C) \Leftrightarrow f^{-1}(C)$ is closed

Assume $f^{-1}(C)$ is closed for $C \subseteq Y$ closed

$V \subseteq Y$ open $\Leftrightarrow Y \setminus V$ closed $\Rightarrow f^{-1}(Y\setminus V)$ closed

$= f^{-1}(Y) \setminus f^{-1}(V)$ closed $= X \setminus f^{-1}(V) = f^{-1}(V)$ open

Proof of the Theorem:

($\Rightarrow$) Let $V \subseteq Y$ open. Need $f^{-1}(V)$ open

Let $x \in f^{-1}(V)$ then $f(x) = V$. There is $\varepsilon > 0$ s.t. $N_\varepsilon(f(x)) \subseteq V$. There is $\delta > 0$ s.t. if $x' \in X$ and $d_X(x', x) < \delta$ then $d_Y(f(x'), f(x)) < \varepsilon$

$$\Updownarrow$$

$$f(N_\delta(x)) \subseteq N_\varepsilon(f(x))$$

So $N_\delta(x) \subseteq f^{-1}(N_\varepsilon(f(x))) \subseteq f^{-1}(V)$ So $f^{-1}(V)$ is open

($\Leftarrow$) Let $x \in X$. Let $\varepsilon > 0$ be given. Need $\delta > 0$ s.t. if $d_X(x, x') < \delta$ then $d_Y(f(x), f(x')) < \varepsilon$

$$\Updownarrow$$

$$f(N_\delta(x)) \subseteq N_\varepsilon(f(x))$$

Note $N_\varepsilon(f(x))$ is open. By assumption, $f^{-1}(N_\varepsilon(f(x)))$ open in X. There is $\delta > 0$ s.t.

$$N_\delta(x) \subseteq f^{-1}(N_\varepsilon(f(x)))$$

So $f(N_\delta(x)) \subseteq N_\varepsilon(f(x))$ $\square$

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[rr, bend left, "g\circ f"] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

$$(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$$

Another way to prove the composition of continuous functions is continuous

Image of open/closed set is not necessarily open/closed

pre-image is open/closed

Cor $f: X \to Y$ continuous, X compact then $f(X)$ is compact

Proof Need: each open covering of $f(X)$ has a finite subcovering.

Let $(V_\alpha)_{\alpha\in I}$ be an open covering of $f(X)$

Consider $U_\alpha = f^{-1}(V_\alpha)$ open in X so $(U_\alpha)_{\alpha \in I}$ cover X. $\bigcup_{\alpha\in I} U_\alpha = \bigcup_{\alpha\in I} f^{-1}(V_\alpha) = f^{-1}(\underbrace{\bigcup_{\alpha\in I} V_\alpha}_{\supseteq f(X)}) \supseteq X$

Let $\alpha_1, \dots \alpha_n \in I$ be s.t. $X = U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$ (since X is compact)

$X = \bigcup_{i=1}^{n} U_{\alpha_i} = \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i}) = f^{-1}(\bigcup_{i=1}^{n} V_{\alpha_i})$

$f(X) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$ so $V_{\alpha_1} \dots V_{\alpha_n}$ is a finite subcovering of $f(X)$

Df (X, d) is a metric s.t. $E \subseteq X$, $f: E \to \mathbb{R}^k$

f is bounded if there is a real number M s.t. $|f(x)| \leq M$ for all $x \in E$

Cor $f: X \to \mathbb{R}^k$, X compact. Then $f(X)$ is bounded and closed in $\mathbb{R}^k$. In particular, f is bounded.

Proof $f(X)$ is compact as X compact by last theorem. So closed & bounded by the characterization of compact subset of $\mathbb{R}^k$. So f bounded

X closed bdd $\to$ $f(X)$ closed bdd

Cor $f: X \to \mathbb{R}$ is continuous. X compact.

$M = \sup f(x)$ $\quad m = \inf f(x)$

Then there are $p, q \in X$ s.t. $f(p) = M$ and $f(q) = m$

Proof: $f(x)$ is bdd and closed

So $M = \sup f(x) \in f(x)$

$m = \inf f(x) \in f(x)$

So $\exists p, q \in X$ s.t. $f(p) = M$ and $f(q) = m$

Cor $f: X \to Y$ continuous bijection

X compact $\Rightarrow f^{-1}: Y \to X$ is continuous

f in ($\mathbb{N}$. discrete) is continuous $\quad f(n) = \begin{cases} 0 & \text{if } n = 0 \\ \frac{1}{n} & \text{if } n > 0 \end{cases}$

$f: X \to Y^{\text{compact}}$ bijection continuous

$f^{-1}: Y \to X$ disjoint

Proof Need under f^{-1} Preimage of closed set in X are closed in Y

Same as the image under $\underset{\text{compact}}{f}$ of $\underbrace{\text{closed sets}}_{\text{compact}}$ in X are closed in Y

Image under f compact subsets in X are compact in Y

Uniform continuous $f: X \to Y$

cont means $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for all $x' \in X$

if $d_X(x, x') < \delta$ then $d_Y(f(x), f(x')) < \varepsilon$

Proof If $f: X \to Y$ is uniformly distributions

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

for all $x, x' \in X$ if $d_X(x, x') < \delta$,

then $d_Y(f(x), f(x')) < \varepsilon$

Ex (1) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$ show it is not uniformly distributed

(2) $g \{x \in \mathbb{R} ; x \geq 0\} - \mathbb{R}$ $g(x) = \sqrt{x}$ show it is uniformly continuous

Thm $f: X \to Y$ continuous

X compact $\Rightarrow$ f unif continuous

Proof For $p \in X$, let $\varepsilon(p) > 0$

$$d_X(p, q) < \varepsilon(p) \Rightarrow d_Y(f(p), f(q)) < \frac{\varepsilon}{2}$$

$J(p) = \{ q \in X : d_X(p, q) < \frac{1}{2} \varepsilon(p) \}$

note $(J(p))_{p \in X}$ is an open coverage of X

Let $J(p_1) \dots J(p_n)$ be a finite subcover of X (X compact)

let $\delta = \frac{1}{2} \min(\varepsilon(p), \varepsilon(p'))>0$

Thus δ works: Let $p, p' \in X$ s.t. $d_X(p, p') < \delta$

need $d_Y(f(p), f(p')) < \varepsilon$

Then $p \in J(p_m)$ some $m \leq n$

so $d_X(p, p_m) < \frac{1}{2} \varepsilon(p_m)$

so $d_X(p', p_m) < d(p, p_m) + d(p, p') < \frac{1}{2}\varepsilon(p_m) + \frac{1}{2}\varepsilon(p_m) = \varepsilon(p_m)$

So $d_Y(f(p), f(p_n)) < \frac{\varepsilon}{2}$

$d_Y(f(p'), f(p_n)) < \frac{\varepsilon}{2}$

So $d_Y(f(p), f(p')) < d_Y(f(p), f(p_n)) + d_Y(f(p_n), f(p'))$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

March 26

Example: $f: \mathbb{R} \to \mathbb{R}$

$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ Dirichlet function

f is not continuous at every point

There is no function that is continous at every rational point but discontinuous at every irrational points

Df f defined on (a,b). $f(x+) = q$ if

$f(t_n) \to q$ for each $t_n \to x$ in (x, b)

$f(x-) = q$ if $f(t_n) \to q$ for each $t_n \to x$ in (a, x)

Note: $\lim_{t \to x} f(x)$ exists iff $f(x-)$ and $f(x+)$ exists and

$f(x-) = f(x+)$ Read the book

Df $f: (a,b) \to \mathbb{R}$

is monotonically increasing (decreasing) if $a < x < y < b$ then $f(x) \leq f(y)$ $(f(x) \geq f(y))$

Thm f monotonically increasing in (a,b) then $f(x+)$, $f(x-)$ exists at every $x \in (a,b)$.

In fact, $\sup_{a<t<x} f(t) = f(x-) \le f(x) \le f(x+) = \inf_{x<t<b} f(t)$

Proof $\sup_{a<t<x} f(t) = \sup A_x = l$

$A_x = \{f(t) : a<t<x\} \neq \phi$

since $f(\frac{a+x}{2}) \in A_x$. A_x bdd above by $f(x)$

Let $t_n \to x$ with $t_n \in (a,x)$ Need $f(t_n) \xrightarrow[n]{} l$

Let ε be given. Consider $l-\varepsilon$ not an upper bound of A_x. So that $a<\bar{t}<x$ s.t. $f(\bar{t}) > l-\varepsilon$

Then for large n, $\bar{t}< t_n < x$

$l-\varepsilon < f(\bar{t}) \le f(t_n) \le l < l+\varepsilon$

so $|f(t_n) - l| < \varepsilon$

Cor $f: (a,b) \to \mathbb{R}$ monotonically increasing

Then f has at most countably many discontinous points.

Proof Discont points $\ni x \xrightarrow{1\text{-to-}1} q_x \in \mathbb{Q}$ $f(x-) < q < f(x+)$

$x < x' \to q_x < f(x+) \le f(x') < q_{x'} \to q_x \neq q_{x'}$

Done since $\mathbb{Q}$ is countable

$\mathbb{Q} \ni q_n \to \frac{1}{2^n}$ $f(x) = \sum_{q_n < x} \frac{1}{2^n}$ uncountable discont

Thm $f:[a,b]\to\mathbb{R}$ continuous. let $f(a)<c<f(b)$
There is $x\in(a,b)$ s.t. $f(x)=c$

Proof let $A=\{r\in[a,b] : f([a,r])\subseteq(-\infty,c]\}$

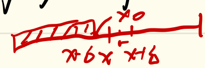
A bdd $A\neq\emptyset$ $a\in A$, $f([a,a])=\{f(a)\}\subseteq(-\infty,c]$

So $\sup A$ exists   let $x=\sup A$

we claim $f(x)=c$. If not, $f(x)<c$ or $f(x)>c$

① $f(x)>c$ consider $\varepsilon=f(x)-c>0$ Find $\delta>0$ s.t.
for all $y\in(x-\delta,x+\delta)$, we have $f(y)>f(x)-\varepsilon$
Because f is continuous at x
Take $y\in(x-\delta,x)$, then $f(y)>c$
So $f([a,y])\not\subseteq(-\infty,c]$
So for all $r\geq y$, $f([a,r])\not\subseteq(-\infty,c]$ so
$A\subseteq(a,y)$. So $\sup A\leq y<x$ Contradiction

② $f(x)<c$ Consider $\varepsilon=c-f(x)>0$ Find $\delta>0$ s.t.
for all $y\in(x-\delta,x+\delta)$, we have $f(y)<f(x)+\varepsilon$
Find $x_0\in(x-\delta,x]$ s.t. $x_0\in A$



Take $r\in(x,x+\delta)$, Then $f([a,r])\subseteq f([a,x_0])\cup f([x_0,r])$
$f([a,x_0])\subseteq(-\infty,c]$ because $x_0\in A$
$f([x_0,r])\subseteq(-\infty,c]$ So $f([a,r])\subseteq(-\infty,c]$
So $r\in A$ but $r>x$ contradiction □

Last subsection of ch 4 for study in the exam
Connectedness is not in the exam.

Differentiability

$f: [a,b] \to \mathbb{R}$, $x \in [a,b]$ Let $\varphi(t) = \frac{f(t)-f(x)}{t-x}$, $a<t<b$, $t \neq x$

f is differentiable at x if $\lim_{t \to x} \varphi(t)$ exists

we write $f'(x) = \lim_{t \to x} \varphi(t)$ – the derivative of f at x.

$E \subseteq [a,b]$, f differentiable on E if f is diff at each $x \in E$

<u>Thm</u> $f: [a,b] \to \mathbb{R}$ f diff at $x \in [a,b] \Rightarrow f$ is cont at x

<u>Proof</u> $f(t) - f(x) = \frac{f(t)-f(x)}{t-x}(t-x) \xrightarrow[t \to x]{} f'(x) \cdot 0 = 0$

so $f(t) \to f(x)$ when $t \to x$ $\square$

April 9

<u>Thm</u> (i) $(f+g)' = f' + g'$

(ii) $(fg)' = f'g + fg'$

(iii) $\left(\frac{f}{g}\right)' = \frac{gf' - g'f}{g^2}$ $g(x) \neq 0$

$\Bigg\}$ at x

Proof (ii) $h = f \cdot g$

$$h(t) - h(x) = f(t)(g(t) - g(x)) + g(x)(f(t) - f(x))$$

$$t \neq x \quad \frac{h(t)-h(x)}{t-x} = f(t) \cdot \frac{g(t)-g(x)}{t-x} + g(x) \cdot \frac{f(t)-f(x)}{t-x}$$

$$\downarrow t \to x$$

$$h'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

(iii) $h = \frac{f}{g}$

$$\frac{h(t)-h(x)}{t-x} = \frac{1}{g(t)\cdot g(x)}\left[g(x)\cdot\frac{f(t)-f(x)}{t-x} - f(x)\frac{g(t)-g(x)}{t-x}\right]$$

$\downarrow t\to x$

$$= \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g^2(x)}$$

<u>Thm</u> f continuous on $[a,b]$, $f'(x)$ exists at $x\in[a,b]$, g defined on interval containing the image of f. g differentiable at $f(x)$. If $h = g\circ f$ on $[a,b]$, then h is differentiable at x and $h'(x) = g'(f(x))\cdot f'(x)$

Proof $$\frac{h(t)-h(x)}{t-x} = \frac{g(f(t))-g(f(x))}{t-x} = \frac{g(f(t))-g(f(x))}{\boxed{f(t)-f(x)}}\cdot\frac{f(t)-f(x)}{t-x}$$

can be 0 can't work

$f(t)-f(x) = (t-x)\cdot(f'(x)+u(t))$ $\quad u(t)\to 0$ as $t\to x$

$g(s)-g(y) = (s-y)(g'(y)+v(s))$ $\quad v(s)\to 0$ as $s\to y$ where $y=f(x)$

$$\begin{aligned} h(t)-h(x) &= g(f(t)) - g(f(x)) \\ &= [f(t)-f(x)](g'(f(x)) + v(f(t))) \\ &= (t-x)(f'(x)+u(t))(g'(f(x))+v(f(t))) \\ &= (t-x)[f'(x)g'(f(x)) + f'(x)\underbrace{v(f(t))}_{\substack{\downarrow t\to x \\ f(t)\to f(x)=y \\ v(f(t))=0}} + g'(f(x))\underbrace{u(t)}_{\substack{\downarrow t\to x \\ u(t)=0}} + \underbrace{u(t)v(f(t))}_{\downarrow 0}] \end{aligned}$$

$$\frac{h(t)-h(x)}{t-x} = f'(x)g'(f(x))$$

Ex (I) $f(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

① f is continuous at all x

② f differentiable at each $x \neq 0$

$\underset{x\neq 0}{f'(x)} = x' \sin\frac{1}{x} + x \cdot (\sin\frac{1}{x})'$

$= \sin\frac{1}{x} + x \cdot \cos(\frac{1}{x}) \cdot (\frac{1}{x})'$

$= \sin\frac{1}{x} - \frac{1}{x}\cos(\frac{1}{x})$

$x=0$ $\quad \frac{f(t)-f(0)}{t-0} = \frac{t\sin\frac{1}{t}}{t} = \sin\frac{1}{t}$ - diverges as $t \to 0$

so $f'(0)$ does not exist

Ex (II) $g(x) = \begin{cases} x^2\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

① g is continuous at all x

② $g'(x)$ differentiable at $x \neq 0$

$\frac{g(t)-g(0)}{t-0} = \frac{t^2\sin\frac{1}{t}}{t} = t\sin\frac{1}{t} \xrightarrow[t\to 0]{} 0$

so $g'(0)$ exists $g'(0) = 0$

Df $f: X \to \mathbb{R}$, $p \in X$ — (X, d) metric space

f has a local maximum (minimum) at p if $\exists \delta > 0$ s.t.

$f(q) \underset{(\geq)}{\leq} f(p)$ for all $q \in X$ with $d(p,q) < \delta$

Thm $f: [a,b] \to \mathbb{R}$ f has a local maximum at $x \in [a,b]$ and $f'(x)$ exists, then $f'(x) = 0$. Same with local minimum

April 11

Proof: Fix $\delta > 0$ s.t. $a < x-\delta < x < x+\delta < b$
and $f(t) \le f(x)$ all $x-\delta < t < x+\delta$

$$\frac{f(t)-f(x)}{t-x} \ge 0 \quad \text{for } x-\delta < t < x$$

$$\frac{f(t)-f(x)}{t-x} \le 0 \quad \text{for } x < t < x+\delta$$

Since $\lim_{t \to x} \frac{f(t)-f(x)}{t-x} = \lim_{t \to x^+} \frac{f(t)-f(x)}{t-x} = \lim_{t \to x^-} \frac{f(t)-f(x)}{t-x}$

$$0 \le \lim_{t \to x} \frac{f(t)-f(x)}{t-x} \le 0 \quad \text{So} \quad f'(x) = 0$$

Thm $f, g : [a, b] \to \mathbb{R}$ continuous and differentiable on (a,b)
there is $x \in (a,b)$ s.t. $(f(b)-f(a))g'(x) = (g(b)-g(a)) \cdot f'(x)$
A particular case: take $g(x) = x$ mean value theorem

$$f(b) - f(a) = (b-a) \cdot f'(x)$$

$$f'(x) = \frac{f(b)-f(a)}{b-a}$$

$$f'(x^n) = \lim_{t \to x} \frac{t^n - x^n}{t-x} = \lim_{t \to x} \frac{(t-x)(t^{n-1} + t^{n-2}x + \dots + tx^{n-2} + x^{n-1})}{t-x}$$

$$= nx^{n-1}$$

$p(x) = a_0 + a_1 x + \dots + a_n x^n \qquad p'(x) = a_1 + 2a_2 x + \dots + n a_n x^{n-1}$

Proof Let $h(t) = (f(b)-f(a))g(t) - (g(b)-g(a))f(t)$
need: $h'(x) = 0$ for some $x \in (a,b)$
Note: ① h cont on $[a,b]$, diff on (a,b)

② $h(a) = (f(b) - f(a))g(a) - (g(b) - g(a))f(a)$

$= f(b)g(a) - f(a)g(a) - g(b)f(a) + g(a)f(a)$

$= f(b)g(a) - g(b)f(a)$

$h(b) = g(a)f(b) - f(a)g(b)$

So $h(a) = h(b)$

Case 1: $h(t)$ constant on $[a,b]$

So $h'(x) = 0$ for $x = \frac{a+b}{2}$

Case 2: $h(t) > h(a) = h(b)$ some $t \in (a,b)$

Then let $x \in [a,b]$ be s.t. $h(x) = \sup\{h(t) : t \in [a,b]\}$

Also x is a local max so $f'(x) = 0$

Case 3: $h(t) < h(a) = h(b)$ for some $t \in [a,b]$

Then similar to case 2.

Theorem f diff on (a,b)

(a) If $f'(x) \geq 0$ all $x \in (a,b)$ then f is monotonically increasing

(b) If $f'(x) = 0$ all $x \in (a,b)$ then f is constant

(c) If $f'(x) \leq 0$ all $x \in (a',b')$ then f is monotonically decreasing

$f(x_2) - f(x_1) = (x_2 - x_1) f'(x)$ for some $x_1 < x < x_2$

$x_1 < x_2$, $x_1, x_2 \in (a,b)$ arbitrary

(a) $f'(x) \geq 0 \Rightarrow f(x_2) - f(x_1) \geq 0$ $f(x_2) \geq f(x_1)$

(b) $f'(x) = 0 \Rightarrow f(x_2) - f(x_1) = 0$ $f(x_2) = f(x_1)$

(c) $f'(x) \leq 0 \Rightarrow f(x_2) - f(x_1) \leq 0$ $f(x_2) \leq f(x_1)$

Thm f diff on $[a,b]$, assume $f'(a) < \lambda < f'(b)$
There is $x \in (a,b)$ s.t. $f'(x) = \lambda$ (Intermidiate value theorem)

Proof Consider $g(t) = f(t) - \lambda t$

Then $g'(t) = f'(t) - \lambda$

Then $g'(a) = f'(a) - \lambda < 0$ $\qquad g'(a) = \lim_{t \to a} \frac{g(t)-g(a)}{t-a} < 0$

so $g(t_1) < g(a)$ for some $t_1 \in (a,b)$

$g'(b) = f'(b) - \lambda > 0$ $\qquad g'(b) = \lim_{t \to b} \frac{g(t)-g(b)}{t-b} > 0$

So $g(t_2) < g(b)$ for some $t_2 \in (a,b)$

Note: There is $x \in [a,b]$ s.t. $g(x) = \inf\{g(t) : t \in (a,b)\}$
note $x \in (a,b)$

Then $g'(x) = 0$ so $f'(x) - \lambda = 0$ $\quad f'(x) = \lambda$

Ch 3: 16(a, b), 20, 21, 22 (not turn in)
Ch 4: 4, 10, 12, 13, 14, 15, 18, 20, 4

I. Kaplansky "set theory and metric space" ch 4&5

April 16

Taylor's Theorem

$f : [a,b] \to \mathbb{R}$, $n > 0$, $n \in \mathbb{N}$

$f^{(n-1)}$ continuous on $[a,b]$. $f^{(n)}(t)$ exist at each $t \in (a,b)$

$\alpha, \beta \in [a,b]$, $\alpha \neq \beta$ Let $P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$

Then there exists a x between α and β s.t.

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n$$

Proof Let M s.t. $f(\beta) = P(\beta) + M(\beta-\alpha)^n$

so $M = \dfrac{f(\beta) - P(\beta)}{(\beta-\alpha)^n}$

Need: $M = \dfrac{f^{(n)}(x)}{n!}$ some x between α & β

i.e. $n!\,M = f^{(n)}(x)$

Df $g(t) = f(t) - P(t) - M(t-\alpha)^n \quad a \le t \le b$

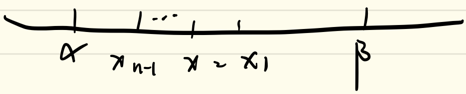
So suffice to show that $g^n(x) = 0$ some x between α & β [$g^n(t) = f^n(t) - 0 - n!\cdot M = f^n(t) - n!\cdot M$

$g^n(x) = 0 \Rightarrow f^n(x) = n!M \Rightarrow M = \dfrac{f^n(x)}{n!}$]

Note $p^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k<n$

so $g(\alpha) = g'(\alpha) \cdots = g^{n-1}(\alpha) = 0$

$g(\beta) = 0$ by our choice of M



$g(\alpha) = 0 \quad g(\beta) = 0$

$g'(\alpha) = 0 \quad g'(x_1) = 0 \quad \cdots$

$g''(\alpha) = 0 \quad g''(x_2) = 0$

$g^{(n-1)}(\alpha) = 0 - g^{(n-1)}(x_{n-1}) = 0 \quad g^n(x_n) = 0$

Thm (L'Hospital's rule)

f, g diff on (a,b), $g'(x) \neq 0$ all $x \in (a,b)$

Assume $\frac{f'(x)}{g'(x)} \xrightarrow[x \to a]{} A$

If $[f(x) \to 0$ and $g(x) \to 0$ as $x \to a]$ or

$[f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a]$

then $\frac{f(x)}{g(x)} \xrightarrow[x \to a]{} A$

Proof: Note $g(x) \neq g(y)$ all $x, y \in (a,b)$

$g(y) = 0$ for at most ony y in $[a,b]$

so we can, and will assume $g(y) \neq 0$ all y in (a,b)

Note: for all $x < y$ in (a,b) we want

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} \text{ some } t \text{ when } x \neq y$$

Fix $\varepsilon > 0$ Need $A - \varepsilon \leq \frac{f(x)}{g(x)} \leq A + \varepsilon$ if x is close to a

Find c s.t. $a < c < b$

$A - \varepsilon < \frac{f'(t)}{g'(t)} < A + \varepsilon$ if $t \in (a, c)$

if $a < x < y < c$, then $\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)}$ some t s.t. $x < t < y$

keep y fix and let $x \to a$ then

$$\frac{f(x) - f(y)}{g(x) - g(x)} \to \frac{f(y)}{g(y)} \quad \text{so } A - \varepsilon \leq \frac{f(y)}{g(x)} \leq A + \varepsilon$$

so for all $a < y < c$ we have $A - \varepsilon \leq \frac{f(y)}{g(y)} \leq A + \varepsilon$

for $[g(x) \to +\infty \quad f(x) \to +\infty \quad x \to a]$

$\frac{f(x)-f(y)}{g(x)-g(y)} \cdot \frac{g(x)-g(y)}{g(x)}$ we get $\frac{f(x)}{g(x)} - \frac{f(y)}{g(x)} = \frac{f'(t)}{g'(t)} \cdot \frac{g(x) - g(y)}{g(x)}$

$$\frac{f(x)}{g(x)} = \frac{f(y)}{g(x)} + \frac{f'(t)}{g'(t)} - \frac{f'(t)}{g'(t)} \frac{g(y)}{g(x)}$$

$\exists\ c$ s.t. $a < c < b$ &

$$A - \varepsilon \le \frac{f'(t)}{g'(t)} \le A + \varepsilon \text{ for } t \text{ in } (a, c)$$

keep y fixed s.t. y is in (a, c) e.g. $y = \frac{a+c}{2}$

Then $\frac{f(y)}{f(x)} \to 0 \qquad \frac{g(y)}{g(x)} \to 0$

$$A - \varepsilon \le \liminf_{x \to a} \frac{f(x)}{g(x)} \le \limsup_{x \to a} \frac{f(x)}{g(x)} \le A + \varepsilon$$

Then above holds for each $\varepsilon > 0$

So $\liminf_{x \to a} \frac{f(x)}{g(x)} = \limsup_{x \to a} \frac{f(x)}{g(x)} = A$

So $\lim_{x \to a} \frac{f(x)}{g(x)} = A$

Riemann – Streltres integral

$f : [a, b] \longrightarrow \mathbb{R}$ bounded

$\alpha : [a, b] \longrightarrow \mathbb{R}$ monothetically increasing

P a partition of $[a, b]$

$\{ \underset{a}{\overset{\shortparallel}{x_0}} < x_1 < \dots < x_{n-1} < \underset{b}{\overset{\shortparallel}{x_n}} \}$

$\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$

$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$ $M_i = \sup f(x)$ $x_{i-1} \le x \le x_i$

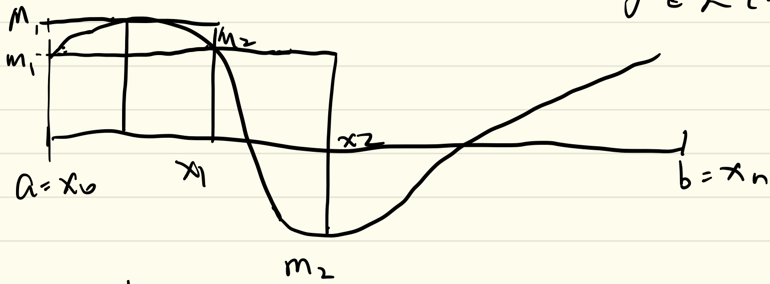
$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$ $m_i = \inf f(x)$ $x_{i-1} \le x \le x_i$

$\overline{\int}_a^b f d\alpha = \inf_P U(P, f, \alpha)$

$\underline{\int}_a^b f d\alpha = \sup_P L(P, f, \alpha)$

f is RS integral of α if $\overline{\int}_a^b = \underline{\int}_a^b$

$f \in R(\alpha)$



$$\int_a^b f d\alpha = f(x_0)$$

For $\alpha(x) = x$ all $x \in [a, b]$,

the integral is called the Riemann integral $\Delta\alpha_i = x_i - x_{i-1}$

$$\overline{\int}_a^b f d\alpha = \inf_P U(P, f, \alpha)$$

$$\underline{\int}_a^b f d\alpha = \sup_\alpha L(P, f, \alpha)$$

If $\overline{\int}_a^b f d\alpha = \underline{\int}_a^b f d\alpha$. we say f is R-S and

bounded out

Ex $\alpha(x) = x$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

$f: [0,1] \to \mathbb{R}$ $\quad \alpha: [0,1] \to [0,1]$

Check $\overline{\int_a^b} f\,d\alpha = 1$ $\quad \underline{\int_a^b} f\,d\alpha = 0$ So f is not integrable

$(M_i = 1)$

P, P^* partition of $[a,b]$, we say P^* is a refinement of P if $P \subseteq P^*$

Theorem If P^* is a refinement of P, then

$$L(P,f,\alpha) \le L(P^*,f,\alpha)$$

$$U(P,f,\alpha) \ge U(P^*,f,\alpha)$$

Corollary $\underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha$

Proof P_1, P_2 two partition Let $P^* = P_1 \cup P_2$ a refinement of P_1, P_2

$$L(P_1,f,\alpha) \le L(P^*,f,\alpha) \le U(P^*,f,\alpha) \le U(P_2,f,\alpha)$$

So for all partition P_1, P_2, we have

$$L(P_1,f,\alpha) \le U(P_2,f,\alpha)$$

So $\underline{\int}_a^b f\,d\alpha = \sup_{P_1} L(P_1, f, \alpha) \le U(P_2, f, \alpha)$

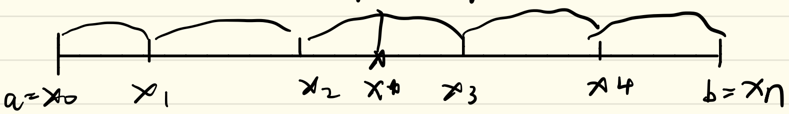
so $\underline{\int}_a^b f\,d\alpha \le \inf_{P_2} U(P_2, f, \alpha) = \overline{\int}_a^b f\,d\alpha$

Proof of the Theorem

we do only $L(P, f, \alpha) \le L(P^*, f, \alpha)$

By induction on the number of elements in P^*, P

Need to consider $P^* = P \cup \{x^*\}$, $x^* \notin P$



Let x^* s.t. $x_{i-1} < x^* < x_i$

$w_1 = \inf_{x_{i-1} \le x \le x^*} f(x)$ $\quad$ $w_2 = \inf_{x^* \le x \le x_i} f(x)$

Note: $w_1 \ge m_i$ $\quad$ $w_2 \ge m_i$

$$
\begin{aligned}
L(P^*, f, \alpha) - L(P, f, \alpha) &= w_1\big(\alpha(x^*) - \alpha(x_{i-1})\big) + w_2\big(\alpha(x_i) - \alpha(x^*)\big) \\
&\quad - m_i\big(\alpha(x_i) - \alpha(x_{i-1})\big) \\
&= (w_1 - m_i)\big(\alpha(x^*) - \alpha(x_{i-1})\big) + \\
&\quad (w_2 - m_i)\big(\alpha(x_i) - \alpha(x^*)\big) \\
&\ge 0
\end{aligned}
$$

Thm $f \in \mathcal{R}(\alpha)$ on $[a,b]$ iff $\forall \varepsilon > 0$ $\exists$ P a partition of $[a,b]$ s.t. $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$

Proof

$(\Leftarrow)$ $L(P,f,\alpha) \leq \underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha \leq U(P,f,\alpha)$ (*)

If $\exists$ P s.t. (*) holds, then we have

$$\overline{\int} - \underline{\int} < \varepsilon \quad (\varepsilon \text{ can be arbitrarilly small})$$

so $0 \leq \overline{\int} - \underline{\int} \leq 0$ so $\overline{\int} = \underline{\int}$, $f \in \mathcal{R}(\alpha)$

$(\Rightarrow)$ Assume $\overline{\int} = \underline{\int} = \int$

Find partitions P_1, P_2 s.t.

$$U(P_2, f, \alpha) - \int < \frac{\varepsilon}{2}$$

$$\int - L(P_1, f, \alpha) < \frac{\varepsilon}{2}$$

So $U(P_2, f, \alpha) < L(P_1, f, \alpha) + \varepsilon$

Let $P = P_1 \cup P_2$

Then $U(P, f, \alpha) \leq U(P_2, f, \alpha) < L(P_1, f, \alpha) + \varepsilon$

$L(P, f, \alpha) \geq L(P_1, f, \alpha) \leq L(P_1, f, \alpha) + \varepsilon$

So $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

Thm $f: [a,b] \to \mathbb{R}$ continuous, then $f \in \mathcal{R}(\alpha)$

Proof Pick P a partition of $[a,b]$

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

Given $\varepsilon > 0$, let $\delta > 0$ s.t.

$|f(x) - f(y)| < \varepsilon$ if $|x-y| < \delta$ for

$x, y \in [a,b]$ by uniformly continuous f on compact $[a,b]$

Let P s.t. $x_i - x_{i-1} < \delta$ for all i

Then for x, y s.t. $x_{i-1} \le x, y \le x_i$

we have $|f(x) - f(y)| < \varepsilon$

so $M_i - m_i < \varepsilon$

$$\text{So } \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i \le \sum_{i=1}^{n} \varepsilon \Delta\alpha_i$$

$$= \varepsilon \sum_{i=1}^{n} \Delta\alpha_i$$

$$= \varepsilon(\alpha(b) - \alpha(a)) \qquad \int f \cdot \alpha'(x)\,dx$$

<u>Thm</u> $f: [a,b] \to \mathbb{R}$ monotonic, α continuous $\Rightarrow f \in \mathcal{R}(\alpha)$

<u>Proof</u> $U(P, f, \alpha) - L(P, f, \alpha)$ Assume f is monotonically increasing

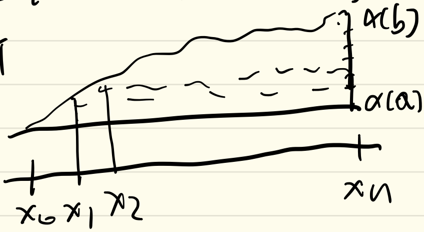
$$= \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

$$= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\Delta\alpha_i$$

Fix n Find $a = x_0 < x_1 \dots < x_n = b$

s.t. $\alpha(x_i) - \alpha(x_{i-1}) = \frac{\alpha(b) - \alpha(a)}{n}$

IMT



For this P,

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\left(\frac{\alpha(b) - \alpha(a)}{n}\right)$$

$$= (f(x_n) - f(x_0))\left(\frac{\alpha(b) - \alpha(a)}{n}\right)$$

$$= \frac{(f(b) - f(a))(\alpha(b) - \alpha(a))}{n}$$

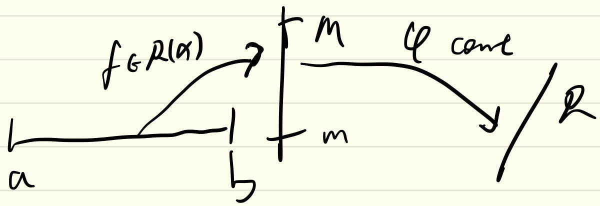
Given $\varepsilon > 0$, Find N s.t when $n \geq N$,

$$\frac{(f(b) - f(a))(\alpha(b) - \alpha(a))}{n} < \varepsilon$$

<u>Thm</u> $f \in R(\alpha)$ on $[a,b]$, $m \leq f(t) \leq M$ all $t \in [a,b]$

Let φ continuous on $[m,M]$ and let $h = \varphi \circ f$

$(h(x) = \varphi(f(x)))$ Then $h \in R(\alpha)$ on $[a,b]$



<u>Proof</u> Let $\varepsilon > 0$, Since $[m,M]$ is compact,

φ is uniformly continuous.

So there is $\delta > 0$ s.t. for all $s, t \in [m,M]$,

if $|t-s| < \delta$, $|\varphi(t) - \varphi(s)| < \varepsilon$

Can assume $\delta < \varepsilon$

Since $f \in R(\alpha)$, there is $P = \{\underset{a}{\overset{\shortparallel}{x_0}} < \dots < \underset{b}{x_n}\}$

s.t. $U(P,f,\alpha) - L(P,f,\alpha) < \delta^2$

$m_i = \inf\limits_{t \in [x_{i-1}, x_i]} f(t)$ $\quad M_i = \sup\limits_{t \in [x_{i-1}, x_i]} f(t)$

$m_i^* = \inf\limits_{t \in [x_{i-1}, x_i]} \varphi(f(t))$ $\quad M_i^* = \sup\limits_{t \in [x_{i-1}, x_i]} \varphi(f(t))$

Let $i \in [1, \dots, n]$ $\quad i \in A$ iff $M_i - m_i < \delta$

$i \in B$ iff $M_i - m_i \geq \delta$

$\{1, \dots, n\} = A \cup B$

If $i \in A$, then $M_i^* - m_i^* \le \varepsilon$

If $i \in B$, then ① $M_i^* - m_i^* \le 2K$ when $K = \sup_{m \le t \le M} |\varphi(t)|$

$= \sup_{m \le t \le M} \varphi(t) - \inf_{m \le t \le M} \varphi(t)$ and not 0

② $\delta \sum_{i \in B} \Delta\alpha_i \le \sum_{i \in B} (M_i - m_i) \Delta\alpha_i \le U(P, f, \alpha) - L(P, f, \alpha) < \delta^2$

So $\sum_{i \in B} \Delta\alpha_i < \delta$

So $U(P, h, \alpha) - L(P, h, \alpha)$

$= \sum_{i \in A} (M_i^* - m_i^*) \Delta\alpha_i + \sum_{i \in B} (M_i^* - m_i^*) \Delta\alpha_i$

$\le \sum_{i \in A} \varepsilon \Delta\alpha_i + 2k \sum_{i \in B} \Delta\alpha_i$

$\le \varepsilon \sum_{i=1}^{n} \Delta\alpha_i + 2k\delta$

$< \varepsilon(\alpha(b) - \alpha(a) + 2k)$

Thm (a) $f_1, f_2 \in R(\alpha)$ on $[a, b] \Rightarrow f_1 + f_2 \in R(\alpha)$

$f \in R(\alpha)$ on $[a, b] \Rightarrow cf \in R(\alpha)$ for each $c \in \mathbb{R}$ and $\int f_1 + f_2 \, d\alpha = \int f_1 \, d\alpha + \int f_2 \, d\alpha$. $\int cf \, d\alpha = c \int f \, d\alpha$

(b) $f_1, f_2 \in R(\alpha)$, $f_1 \le f_2 \Rightarrow \int f_1 \, d\alpha \le \int f_2 \, d\alpha$

(c) $f \in R(\alpha)$ on $[a, b]$, $a < c < b$ $\Rightarrow f \in R(\alpha)$ on $[a, c]$ and $[c, b]$ and $\int_a^c f \, d\alpha + \int_c^b f \, d\alpha = \int_a^b f \, d\alpha$

(d) $f \in R(\alpha)$ on $[a,b]$

$|f| \le M \Rightarrow \left|\int_a^b f\,d\alpha\right| \le M(\alpha(b)-\alpha(a))$

(e) $f \in R(\alpha_1)$, $f \in R(\alpha_2)$

$\Downarrow$

$f \in R(\alpha_1+\alpha_2)$ and $\int f\,d(\alpha_1+\alpha_2) = \int f\,d\alpha_1 + \int f\,d\alpha_2$

$f \in R(\alpha)$, $c>0$, $f \in R(c\alpha)$ and $\int f\,d(c\alpha) = c\int f\,d\alpha$

Proof

(a) see $f = f_1 + f_2$

P a partition of $[a,b] \in \mathbb{R}$

$L(P,f_1,\alpha) + L(P,f_2,\alpha) \le L(P,f,\alpha) \le U(P,f,\alpha)$

$\le U(P,f_1,\alpha) + U(P,f_2,\alpha)$

So $U(P,f,\alpha) - L(P,f,\alpha) \le \underbrace{U(P,f_1,\alpha) - L(P,f_1,\alpha)}_{<\varepsilon} + \underbrace{U(P,f_2,\alpha) - L(P,f_2,\alpha)}_{<\varepsilon} < 2\varepsilon$

P_1 $\quad$ P_2

$P = P_1 \cup P_2$

So $f \in R(\alpha)$

Same argument shows $\int_a^b f\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha$

Thm ① $f \in R(\alpha), g \in R(\alpha)$ on $[a,b] \Rightarrow fg \in R(\alpha)$

② $f \in R(\alpha) \Rightarrow |f| \in R(\alpha)$ & $|\int f d\alpha| = \int |f| d\alpha$

Proof ① $f, g \in R(\alpha)$

$$fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$$

So $fg \in R(\alpha)$ if

$(f+g)^2$ & $(f-g)^2 \in R(\alpha)$

Need: $h \in R(\alpha) \Rightarrow h^2 \in R(\alpha)$

but $h^2 = \varphi \circ h$ where $\varphi(y) = y^2$ so $h^2 \in R(\alpha)$

② $|f| = \varphi \circ f \quad \varphi(y) = |y|$

so $f \in R(\alpha) \Rightarrow |f| \in R(\alpha)$

Let $c = \pm 1$ so that $c\int f d\alpha \geq 0$ Then

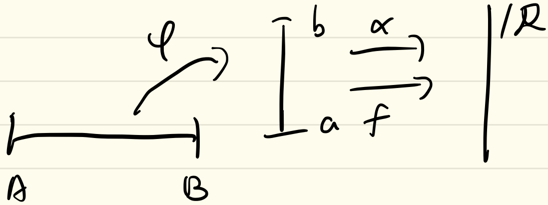
$$\left|\int f d\alpha\right| = c\int f d\alpha = \int cf d\alpha = \int |f| d\alpha$$

Thm $\varphi: [A, B] \to [a, b]$ strictly increasing (one-to-one) (just one-to-one?) and continuous and onto

$\alpha: [a,b] \to \mathbb{R}$ monotonically increasing

$f \in \mathcal{R}(\alpha)$ on $[a,b]$

Let $\beta = \alpha \circ \varphi$, $g = f \circ \varphi$



Then $g = \mathcal{R}(\beta)$ and

$$\int_A^B g \, d\beta = \int_a^b f \, d\alpha$$

Proof

$P = \{\underset{a}{x_0} \cdots \underset{b}{x_n}\}$ a part of $[a,b]$

$Q = \{\underset{A}{y_0} \cdots \underset{B}{y_n}\}$ a partition of $[A,B]$

s.t. $\varphi(y_i) = x_i$, $i = 0, \ldots n$

Every partition Q of $[A,B]$ is obtained in this way from some partitions P of $[a,b]$

$Q = \{y_0 \cdots y_n\} \to P = \{x_0 \cdots x_n\}$ by $x_i = \varphi(y_i)$

Note: $U(Q, g, \beta) = U(P, f, \alpha)$ — values of f on $[x_{i-1}, x_i]$
= values of g on $[y_{i-1}, y_i]$
$(f \circ \varphi)$

and $\alpha(x_i) - \alpha(x_{i-1}) = \alpha(\varphi(y_i)) - \alpha(\varphi(y_{i-1}))$
$= \beta(y_i) - \beta(y_{i-1})$

$$\left(\sup_{x \in [x_{i-1}, x_i]} f(x)\right)(\alpha(x_i) - \alpha(x_{i-1})) = \left(\sup_{y \in [y_{i-1}, y_i]} g(y)\right)(\beta(y_i) - \beta(y_{i-1}))$$

Similarly $L(Q, g, \beta) = L(P, f, \alpha)$
Since $f \in \mathcal{R}(\alpha)$, we have $g \in \mathcal{R}(\beta)$
Also $\overline{\int_a^b} f \, d\alpha = \overline{\int_A^B} g \, d\beta$
+ each Q comes from a P
+ integraltability
$\Downarrow$

$$\int_a^b f \, d\alpha = \int_A^B g \, d\beta$$

Notation: $\mathcal{R}(\alpha) = \mathcal{R}$ if $\alpha(x) = x$
If $\alpha(x) = x$, then $\beta = \varphi$ and

$$\int_A^B g \, d\varphi = \int_a^b f \, dx \quad \text{(special case)}$$

Thm $\alpha : [a, b] \to \mathbb{R}$ monotonically increasing differentiable
$\alpha' \in \mathcal{R}$ on $[a, b]$ f bdd on $[a, b]$
Then $f \in \mathcal{R}(\alpha)$ iff $f \cdot \alpha' \in \mathcal{R}$
and $\int_a^b f \, d\alpha = \int_a^b f(x) \alpha'(x) \, dx$

Proof Some $\alpha' \in \mathcal{R}$, given $\varepsilon > 0$, we can find

① $P = \{\underset{a}{\overset{\shortparallel}{x_0}} \ldots \underset{b}{\overset{\shortparallel}{x_n}}\}$ s.t.

$$U(P, \alpha') - L(P, \alpha') < \varepsilon$$

② By the mean value theorem, we get $t_i \in [x_{i-1}, x_i]$ s.t. $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$

③ (from ①)

$$\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i < \varepsilon \text{ for each } s_i \in [x_{i-1}, x_i]$$

($\alpha'(s_i) \leq M_i$, $\alpha'(t_i) \geq m_i$, so $|\alpha'(s_i) - \alpha'(t_i)| \leq M_i - m_i$)

for each $s_i \in [x_{i-1}, x_i]$

④ $$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i = \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i$$

$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i$$

$$= \sum f(s_i)\alpha'(t_i)\Delta x_i - \sum f(s_i)\alpha'(s_i)\Delta x_i$$

$$= \sum f(s_i)(\alpha'(t_i) - \alpha'(s_i))\Delta x_i$$

$$\leq \sum |f(s_i)| |\alpha'(t_i) - \alpha'(s_i)| \Delta x_i$$

$$\leq M \sum |\alpha'(t_i) - \alpha'(s_i)| \Delta x_i, \quad M = \sup_{x \in [a,b]} f(x_i) \text{ as } f \text{ bdd}$$

$$< M \cdot \varepsilon$$

So there is $P = \{x_0 < x_1 \ldots < x_n\}$ s.t.

$$\left| \sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \right| < M \cdot \varepsilon \quad \text{all } s_i \in [x_{i-1}, x_i]$$

So $\sum f(s_i)\Delta\alpha_i \leq \sum f(s_i)\alpha'(s_i)\Delta x_i + M\varepsilon \leq U(P, f, \alpha') + M \cdot \varepsilon$

$U(P, f, \alpha) \leq U(P, f, \alpha') + M\varepsilon$ since $s_i \in [x_{i-1}, x_i]$ arbitrary

By the same argument,

$$U(P, f\cdot\alpha') \leq U(P, f, \alpha) + M\varepsilon$$

so $|U(P, f, \alpha) - U(P, f, \alpha')| \leq M\varepsilon$

holds for all partitions finer than the partition with fixed

So $\left|\overline{\int} f\,d\alpha - \overline{\int} f\cdot\alpha'\,dx\right| \leq M\varepsilon$

So $\overline{\int} f\,d\alpha = \overline{\int} (f\cdot\alpha')(x)\,dx$

By an analogous argument,

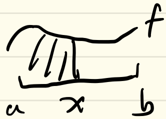
$$\underline{\int} f\,d\alpha = \underline{\int} (f\cdot\alpha')(x)\,dx$$

So $f \in \mathcal{R}(\alpha)$ iff $f\cdot\alpha' \in \mathcal{R}$

April 30

Thm $f \in \mathcal{R}$ on $[a,b]$

Let $F(x) = \int_0^x f(t)\,dt$ for $x \in [a,b]$



Then $F(x)$ is continuous. If f is continuous at $x_0 \in [a,b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$

(So if f continuous on $[a,b]$, then F differentiable on $[a,b]$ and $F' = f$)

Proof For $a \leq x \leq y \leq b$

$$F(y) - F(x) = \int_a^y f(t)\,dt - \int_a^x f(t)\,dt = \int_x^y f(t)\,dt$$

F continuous. Let $M = \sup_{t \in [a,b]} |f(t)| < \infty$

$$|F(y) - F(x)| = \left| \int_x^y f(t)\,dt \right| \le \int_x^y |f(t)|\,dt \le M(y-x)$$

Given $\varepsilon > 0$, Let $\delta = \frac{\varepsilon}{M}$

Then if $x \le y$ & $|y-x| < \delta$,

Then $|F(y) - F(x)| < M|y-x| < M\delta = \varepsilon$

If $y < x$, then $|F(y) - F(x)| = |F(x) - F(y)| < \varepsilon$ So F is cont

Assume f cont at x_0, Need $F'(x_0) = f(x_0)$

Let $a \le s \le t \le b$

$$\frac{F(t) - F(s)}{t-s} - f(x_0) = \frac{1}{t-s}\int_s^t f(u)\,du - \frac{1}{t-s}\int_s^t f(x_0)\,du$$

$$= \frac{1}{t-s}\int_s^t f(u) - f(x_0)\,du$$

Assume $s \le x_0 \le t$

$$\left| \frac{F(t) - F(s)}{t-s} - f(x_0) \right| = \left| \frac{1}{t-s}\int_s^t f(u) - f(x_0)\,du \right|$$

$$\le \frac{1}{t-s}\int_s^t |f(u) - f(x_0)|\,du$$

Given $\varepsilon > 0$, let $\delta > 0$ be s.t. for all $u \in (x_0 - \delta, x_0 + \delta)$

we have $|f(u) - f(x_0)| < \varepsilon$

Assume $x - \delta < s \le x_0 \le t < x + \delta$

$$\left| \frac{F(t) - F(s)}{t-s} - f(x_0) \right| \le \frac{1}{t-s}\int_s^t \varepsilon\,du = \frac{1}{t-s} \cdot \varepsilon \cdot (t-s) = \varepsilon$$

So $\left| \frac{F(t) - F(x_0)}{t-x_0} - f(x_0) \right| < \varepsilon$ & $\left| \frac{F(s) - F(x_0)}{s-x_0} - f(x_0) \right| < \varepsilon$

So $\lim_{t \to x_0^+} \frac{F(t) - F(x_0)}{t-x_0} = f(x_0)$ $\lim_{t \to x_0^-} \frac{F(t) - F(x_0)}{t-x_0} = f(x_0)$

So $\lim_{t \to x_0} \frac{F(t)-F(x_0)}{t-x_0} = f(x_0)$ i.e. $F'(x_0) = f(x_0)$

<u>Thm</u> $f \in \mathcal{R}$ on $[a,b]$ F diff on $[a,b]$ s.t. $F' = f$

Then $\int_a^b f(x)\,dx = F(b) - F(a)$

<u>Proof</u> $\varepsilon > 0$ be given

(Choose $P = \{x_0, \dots x_n\}$ a partition of $[a,b]$)

s.t. $U(P,f) - L(P,f) < \varepsilon$ ($f \in \mathcal{R}$)

By the mean value theorem:

$F(x_i) - F(x_{i-1}) = F'(t_i)\Delta x_i = f(t_i)\Delta x_i$

for some $t_i \in [x_{i-1}, x_i]$

Thus $\sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1} F(x_i) - F(x_{i-1}) = F(b) - F(a)$

$F(b) - F(a) - \sum f(t_i)\Delta x_i \geq F(b) - F(a) - U(P,f)$

$L(P,f) \leq \int_a^b f(x)\,dx \leq U(P,f)$

$U(P,f) < \int + \varepsilon$, $L(P,f) > \int - \varepsilon$

$F(b) - F(a) - \sum f(t_i)\Delta x_i = 0$

$$\begin{array}{ccc} & \vee\!\!| & \\ F(b) - F(a) - U(P,f) & & F(b) - F(a) - L(P,f) \\ \vee\!\!| & & \wedge\!\!| \\ F(b) - F(a) - \int_a^b f(x)\,dx - \varepsilon & & F(b) - F(a) - \int_a^b f(x)\,dx + \varepsilon \end{array}$$

So $\left| F(b) - F(a) - \int_a^b f(x)\,dx \right| < \varepsilon$

So since $\varepsilon > 0$ is arbitrary, $\int_a^b f(x)\,dx = F(b) - F(a)$

Thm (Integration by parts)

F, G differentiable on $[a,b]$, $F' = f \in \mathcal{R}$, $G' = g \in \mathcal{R}$

Then $\int_a^b F(x) g(x)\, dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x) G(x)\, dx$

Proof Consider $H(x) = F(x)G(x)$ - diff

$$H'(x) = F'(x)G(x) + F(x)G'(x)$$
$$= f(x)G(x) + F(x)g(x)$$

$f \in \mathcal{R}$, G diff $\to$ G cont $\to$ $G \in \mathcal{R}$

So $f \cdot G \in \mathcal{R}$, $F \cdot g \in \mathcal{R}$

So $fG + Fg \in \mathcal{R}$

So $\int_a^b H'(x)\, dx = H(b) - H(a)$

$$\int_a^b H'(x)\,dx = \int_a^b f(x)G(x) + \int_a^b F(x) g(x)$$

Uniform convergence

X a metric space

$f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$

$f : X \to \mathbb{R}$

$\{f_n\}$ converges uniformly to f if $\forall \varepsilon > 0$, $\exists N$ $\forall n \geq N$ $\forall x \in X$, $|f_n(x) - f(x)| < \varepsilon$

$$\sup_{x \in X} |f_n(x) - f(x)| < \varepsilon$$

$\varepsilon_n \xrightarrow[n]{} 0$ where $\varepsilon_n = \sup_{x \in X} |f_n(x) - f(x)|$

Thm $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$
$f_n \to f$ uniformly on $[a,b]$
Then $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and

$$\int_a^b f\, d\alpha = \lim_n \int_a^b f_n\, d\alpha$$

Proof Let $\varepsilon_n = \sup_x |f_n(x) - f(x)|$

Note $\underline{\int_a^b} f_n\, d\alpha - \varepsilon_n(\alpha(b) - \alpha(a)) \le \underline{\int_a^b} f\, d\alpha$ since $f_n \le f + \varepsilon_n$

$$\le \overline{\int_a^b} f\, d\alpha \le \int_a^b f_n\, d\alpha + \varepsilon_n(\alpha(b) - \alpha(a))$$

So $0 \le \overline{\int_a^b} f\, d\alpha - \underline{\int_a^b} f\, d\alpha \le 2\varepsilon_n(\alpha(b) - \alpha(a)) \xrightarrow[n]{} 0$

So $\underline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\, d\alpha$, $f \in \mathcal{R}(\alpha)$

$$\int f = \lim_n \int f_n$$

May 2

Derivatives

$f : [a,b] \to \mathbb{R}^k$

$f(x) \in \mathbb{R}^k = (f_1(x), f_2(x), \dots f_k(x))$

$f_i : [a,b] \to \mathbb{R}$ $\quad i = 1, \dots k$

Consider $\varphi(t) = \dfrac{f(t) - f(x)}{t - x} = \dfrac{1}{t-x}(f(t) - f(x)) \in \mathbb{R}^k$ $\quad t \ne x \quad t \in [a,b]$

$f'(x)$ exists if $\lim_{t \to x} \varphi(t)$ exists in $\mathbb{R}^k$

and we write $f'(x) = \lim_{t \to x} \varphi(t) \in \mathbb{R}^k$

$|f'(x) - \varphi(t)| \xrightarrow[t \to x]{} 0$

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

if $0 < |t - x| < \delta$, then $|f'(x) - \varphi(t)| < \varepsilon$

$x \in \mathbb{R}^k \quad |x| = \left(\sum_{i=1}^{k} x_i^2\right)^{\frac{1}{2}}$

Note: $f'(x) = \left(f_1'(x), \ldots, f_k'(x)\right)$

$$\max_{i=1\ldots k} |x_i| \le \left(\sum_{i=1}^{k} x_i^2\right)^{\frac{1}{2}} \le k \max_{i=1\ldots k} |x_i|$$

$$\frac{1}{t-x}\left(f(t) - f(x)\right) = \left(\frac{f_1(t) - f_1(x)}{t-x}, \ldots, \frac{f_k(t) - f_k(x)}{t-x}\right) = \left(f_1'(x) \ldots, f_k'(x)\right)$$

Thm $f : [a,b] \to \mathbb{R}^k$ cont f diff on (a,b)

Then there exists $x \in (a,b)$ s.t.

$$\frac{1}{b-a}|f(b) - f(a)| \le |f'(x)|$$

Proof: Let $z = f(b) - f(a)$

Let $\varphi(t) = z \cdot f(t)$

$\varphi : [a,b] \to \mathbb{R}$ cont on (a,b) & diff on (a,b)

By MVT, we get $x \in (a,b)$ $\varphi(b) - \varphi(a) = (b-a)\varphi'(x)$

$= (b-a)(z \cdot f'(x))$

So $\varphi(b) - \varphi(a) = z \cdot f(b) - z \cdot f(a)$

$= z \cdot (f(b) - f(a)) = z \cdot z = |z|^2$

So $|z|^2 = (b-a)|z \cdot f'(x)| \le (b-a)|z||f'(x)|$ (Schw Inequality)

$|z| \le (b-a)|f'(x)|$

Integration in $\mathbb{R}^k$

$f : [a,b] \to \mathbb{R}^k$

$f = (f_1, \dots f_k)$ Assume $f_1 \in \mathcal{R}(\alpha) \dots f_k \in \mathcal{R}(\alpha)$

Define $\int_a^b f \, d\alpha = \left(\int_a^b f_1 \, d\alpha, \dots \int_a^b f_k \, d\alpha \right)$

E.g. $f, F : [a,b] \to \mathbb{R}^k$. $f \in \mathcal{R}$ on $[a,b]$, $F' = f$

Then $\int_a^b f(t) \, dt = F(b) - F(a)$

Thm $f : [a,b] \to \mathbb{R}^k$, $f \in \mathcal{R}(\alpha)$ on $[a,b]$

Then $|f| \in \mathcal{R}(\alpha)$ on $[a,b]$ and

$$\left| \int_a^b f \, d\alpha \right| \le \int_a^b |f| \, d\alpha$$

Proof we have $f = (f_1, \dots f_k)$

So $|f| = \left(|f_1|^2 + \dots |f_k|^2 \right)^{\frac{1}{2}}$

Assume $f_1, \dots f_k \in \mathcal{R}(\alpha)$

then $f_1^2 + \dots + f_k^2 \in \mathcal{R}(\alpha)$

$(f_1^2 + \dots + f_k^2)^{\frac{1}{2}} \in \mathcal{R}(\alpha)$

$\psi : \underbrace{[0,k] \to [0,k]^2}_{\text{compact}}$ Review!

$\psi(x) = x^2$ cont, 1-to-1, onto

then $\psi^{-1} : [0,k^2] \to [0,k]$ is cont

So we have $f : [0,L] \to \mathbb{R}$ cont $\forall L > 0$

let $y_i = \int_a^b f_i \, d\alpha \in \mathbb{R}$

So $y = (y_1, \dots, y_k) = \int_a^b f \, d\alpha$

$$|y|^2 = \sum_{i=1}^{k} y_i \int_a^b f_i \, d\alpha = \int_a^b \left(\sum_{i=1}^{k} y_i f_i \right) d\alpha$$

By schwartz:

$$\Big| \underbrace{\sum_{i=1}^{k} y_i f_i(t)}_{y \cdot f(t)} \Big| \le |y| \, |f(t)|$$

So
$$|y|^2 \le \int_a^b |y| \, |f| \, d\alpha$$

$$|y| \le \int_a^b |f| \, d\alpha$$

$$\left| \int_a^b f \, d\alpha \right| \le \int_a^b |f| \, d\alpha$$

Curves

$\gamma : [a,b] \to \mathbb{R}^k$ cont

$\ast$ $\exists$ $\gamma_p : [0,1] \to \mathbb{R}^2$ cont

$\gamma_p([0,1]) = [0,1]^2$ Peano curve

let $P = \{ \underset{a}{x_0} < \dots < \underset{b}{x_k} \}$ a partition

$$\Lambda(P, \gamma) = \sum_{i=1}^{k} | \gamma(x_i) - \gamma(x_{i-1}) |$$

Def $\Lambda(\gamma) = \sup_P \Lambda(P, \gamma)$ (length of the curve)

Def γ rectifiable if $\Lambda(\gamma) < \infty$

Thm $\gamma : [a,b] \to \mathbb{R}^k$, γ' exists & is continuous on $[a,b]$

Then γ is rectifiable and $\Lambda(\gamma) = \int_a^b |\gamma'(t)| \, dt$

May 7

Proof We want to show $\Lambda(\gamma) \geq \int_a^b |\gamma'(t)| dt$ and (continuous)

$\Lambda(\gamma) \leq \int_a^b |\gamma'(t)| dt$

$P = \{ \overset{a}{x_0} < \cdots < \overset{b}{x_n} \}$

$|\gamma(x_i) - \gamma(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} \gamma'(t)\, dt \right|$ (FTC)

$\leq \int_{x_{i-1}}^{x_i} |\gamma'(t)| dt$

$\Lambda(P, \gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$

$\leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\gamma'(t)| dt$

$= \int_a^b |\gamma'(t)| dt$ so $\leq$ is proved

γ' is cont on $[a, b]$

So γ' is unif cont on $[a, b]$

given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall s, t \in [a, b]$

$(|s - t| < \delta \rightarrow |\gamma'(s) - \gamma'(t)| < \varepsilon)$

Need P a part of $[a, b]$ s.t

$\Lambda(P, \delta) \geq \int_a^b |\gamma'(t)| dt - \varepsilon$

(so $\forall \varepsilon > 0$ $\Lambda(\gamma) \geq \int_a^b |\gamma'(t)| dt - \varepsilon$

$\Lambda(\gamma) \geq \int_a^b |\gamma'(t)| dt$ as required)

Let $P = \{x_0, \dots x_n\}$ s.t $\Delta x_i < \delta$

So for each i, for $x_{i-1} < t < x_i$, we have

$|\gamma'(t)| \leq |\gamma'(x_i)| + \varepsilon$

$(|v - w| < \varepsilon \Rightarrow |v| < \varepsilon + |w|)$

$$\int_{x_{i-1}}^{x_i} |\gamma'(t)|\,dt \le (\gamma'(x_i)+\varepsilon)\Delta x_i$$

$$= |\gamma'(x_i)|\Delta x_i + \varepsilon \Delta x_i$$

$$= \left|\int_{x_{i-1}}^{x_i} (\gamma'(t)+\gamma'(x_i)-\gamma'(t))\,dt\right| + \varepsilon\Delta x_i$$

$$\le \left|\int_{x_{i-1}}^{x_i} \gamma'(t)\,dt\right| + \int_{x_{i-1}}^{x_i} |\gamma'(x_i)-\gamma'(t)|\,dt + \varepsilon \Delta x_i$$

$$\le |\gamma(x_i)-\gamma(x_{i-1})| + \int_{x_{i-1}}^{x_i} |\gamma'(x_i)-\gamma'(t)|\,dt + \varepsilon\Delta x_i$$

$$\le |\gamma(x_i)-\gamma(x_{i-1})| + 2\varepsilon\Delta x_i$$

So $\int_{x_{i-1}}^{x_i} |\gamma'(t)|\,dt \le |\gamma(x_i)-\gamma(x_{i-1})| + 2\varepsilon\Delta x_i$

$$\int_a^b |\gamma'(t)|\,dt = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |\gamma'(t)|\,dt \le \sum_{i=1}^n |\gamma(x_i)-\gamma(x_{i-1})| + 2\varepsilon\Delta x_i$$

$$= \Lambda(P,\gamma) + 2\varepsilon(b-a)$$

$\square$

Examples

Ch1: 5

ch 2: 22

ch 3: 21

ch 4: 8

ch5: 8 12

ch 6: 2 3

Office Hours:

Sunday 10-12 TA

1-3 Slawek

Exercise 4.7

$f, g: \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = \frac{xy^2}{x^2+y^4}$ $g(x,y) = \frac{xy^2}{x^2+y^6}$

WTS f bdd, g unbdd in each neighbourhood of $(0,0)$

- f, g continuous when restricted to each line in $\mathbb{R}^2$
- $f, g = 0$ at $(0,0)$

$$(x-y^2)^2 \geq 0$$

$$x^2 - 2xy^2 + y^4 \geq 0$$

$$x^2 + y^4 \geq 2xy^2$$

$$xy^2 \leq \frac{x^2+y^4}{2}$$

So $\frac{xy^2}{x^2+y^4} \leq \frac{1}{2}$ $x \neq 0$ or $y \neq 0$

$x = y = 0 \Rightarrow |f(0,0)| \leq \frac{1}{2}$

(y^3, y), $y > 0$ $\sqrt{(y^3)^2+y^2} = \sqrt{y^6+y^2}$

$= y\sqrt{y^4+1}$

$< \sqrt{2}\, y$ $0 < y < 1$

$((\frac{1}{n})^3, \frac{1}{n}) \to (0,0)$ $g((\frac{1}{n})^3, \frac{1}{n}) \to \infty$ $g(0,0) = 0$

$f(y^2, y) = \frac{y^4}{y^4+y^4} = \frac{1}{2}$ $f((\frac{1}{n})^2, \frac{1}{n}) \to (0,0)$

$f((\frac{1}{n})^2, \frac{1}{n}) \to \frac{1}{2}$ $f(0,0) = 0$

$x = ay + b \quad y = c$

$f(ay+b, y) = \dfrac{(ay+b)y^2}{(ay+b)^2+y^4}$, continuous unless $(ay+b)^2 = 0$ and $y^4 = 0$

So unless $b = 0$

$b=0$ $\quad f(ay, y) = \dfrac{ay^3}{a^2y^2+y^4} = \dfrac{ay}{a^2+y^2}$ (y is not 0, but it also holds when $y=0$)

Continuous unless $a = 0$

$f(0,y) = 0$ if $y \neq 0$ But $f(0,0) = 0$ So cont when $x=0$

$f(x,c) = \dfrac{xc^2}{x^2+c^2}$ cont unless $c=0$

$f(x,0) = 0$ if $x \neq 0$ $f(0,0) = 0$ if $x=0$ cont