

MATH 4130 NOTES



Jan 22

hw due on Thursday

OH 4:10 - 5:00 Tue 4/3 Major

Prelim 2 (or Prelims in email) Last Thursday before the spring break

Final Monday May 13

Cover chapter 1-7 some 7-9

$$\frac{m_1}{n_1} = \frac{m_2}{n_2}$$

$(m_1, n_1) \sim (m_2, n_2)$ iff $m_1 n_2 = m_2 n_1$,

$\boxed{\begin{array}{l} (m, n) \\ m, n \in \mathbb{Z} \\ n \neq 0 \end{array}}$ \Rightarrow \mathbb{Z} produces $\not\sim$

$\{(N \times (0, 1)) \setminus (0, 1)\} / N \curvearrowright \mathbb{Z}$
Cone is a product

Ordered See

S a set

\subset a binary relation on S

(\subset a subset of $S \times S$)

\subset is a (linear, total) order

$$x \in \underline{X} \quad \underline{X} \cup \underline{Y}$$

$$\underline{Y} \subseteq \underline{X} \quad \underline{X} \cap \underline{Y}$$

$$\underline{X} \times \underline{Y} \quad \underline{X} \sim \underline{Y}$$

$P(\underline{X}) =$ all subsets of \underline{X}

- If
1. for all $x, y \in S$, $x \subset y$ or $y \subset x$ or $x = y$
2. for all $x, y, z \in S$, if $x \subset y$ and $y \subset z$, then $x \subset z$
($(x, y) \in \subset$ and $(y, z) \in \subset$, then $(x, z) \in \subset$)

An ordered see

a set with an order on it.

S , \subset an order on S

(S, \subset) - an ordered see

Let S be an ordered set. (* a set never has repeated values)
Let $E \subseteq S$

E is bounded above if there is $y \in S$ st. $x \leq y$ for all $x \in E$. We say there $y \in S$ is ($x < y$ or $x = y$) an upper bound for E if $x \leq y$ for all $x \in E$

E bounded below --- A lower bound for E ---

$E \subseteq S$, $\alpha \in S$ is the least upper bound (lub) if for each upper bound y for E we have $\alpha \leq y$ and α is an upper bound for E ($\alpha = \sup E$)
greatest lower bound (glb) --- $\alpha = \inf E$

examples

1. (\mathbb{N}, \leq) natural \leq on \mathbb{N} is an order on \mathbb{N}

2 Consider $R \subseteq \mathbb{N} \times \mathbb{N}$

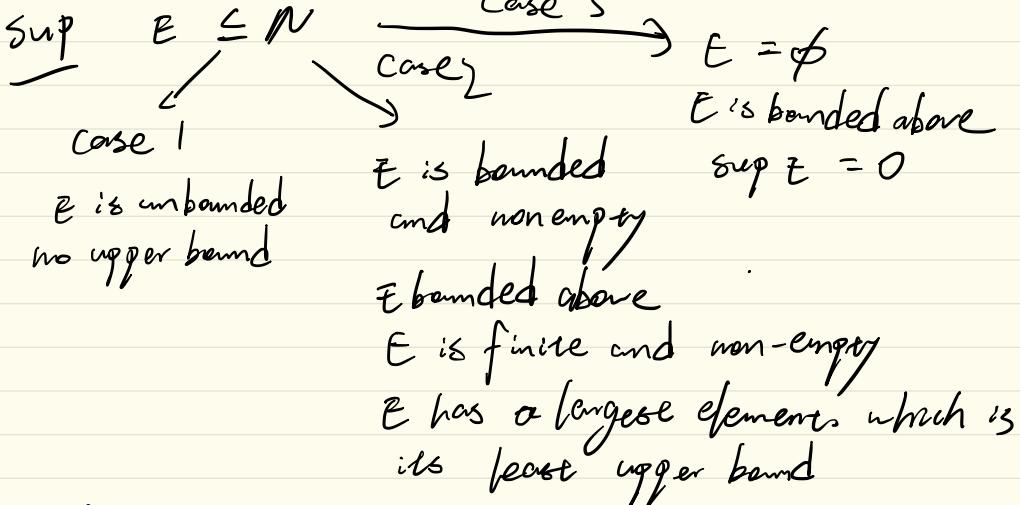
$(m, n) R (p, q)$ iff $m < p$ and $n < q$

① $(m, n) R (p, q)$ and $(p, q) R (r, s) \Rightarrow (m, n) R (r, s)$

② $(1, 2), (2, 1) \quad (1, 2) R (2, 1) \quad (2, 1) R (1, 2)$ so R is not an order on $\mathbb{N} \times \mathbb{N}$

3. (\mathbb{Q} , \subset)

The natural \subset on \mathbb{Q} is an order



Inf

$E \subseteq \mathbb{N}$

Case 1 $E \neq \emptyset$

Case 2 $E = \emptyset$

E has a smallest element, which

is its greatest lower bound

$\inf E =$ the smallest element of E

$\inf E$ does not exist

$$A = \{r \in \mathbb{Q} \mid 0 < r \text{ and } r^2 < 2\}$$

non- \emptyset , bounded above by 2 and below by 0

$$\inf A = 0$$

$\sup A$ does not exist (in \mathbb{Q})

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Definition

A linearly ordered set S has the least upper bound property if for each $E \subseteq S$ s.t. E is bounded above and is non- \emptyset , then E has a least upper bound (which is unique)

Theorem S an ordered set with the least upper bound property. Let $B \subseteq S$, $B \neq \emptyset$ and bounded below. Then $L = \text{all lower bounds of } B$. Then L is bounded above, non- \emptyset and so $\alpha = \sup L$ exists. And α is the greatest lower bound for B .

Proof

$L \neq \emptyset$ since B bounded below

L bounded above since for each $b \in B$ we have $x \leq b$ for all $x \in L$. So each element of B is an upper bound for L and $B \neq \emptyset$, so L is bounded above. Indeed, $\alpha = \sup L$ exists.

α is a lower bound for B (If not, there exists $b \in B$, s.t. $b < \alpha$. Note b is not an upper bound for L since it is

smaller than the least upper bound. So α is $\sup L$ s.t. $b < \alpha$. But $\alpha \in L$ so α is a lower bound for B .
 α is the greatest lower bound for B (if not, there is $x \in S$ s.t. $\alpha < x$ and x is a lower bound for B , so $x \in L$ and $\sup L = \alpha < x$. So α is not an upper bound for L , which is a contradiction).

Fields

F with 2 operations + and \cdot satisfying the following axioms:

(A) if $x, y \in F$, then $x+y \in F$

$$\begin{aligned} x+y &= y+x \\ (x+y)+z &= x+(y+z) \end{aligned} \quad \left\{ \text{for all } x, y \in F \right.$$

F contains an element 0 s.t. $0+x=x+0=x$

for each $x \in F$ there exists an element $-x \in F$ s.t.
 $x+(-x)=0$

(M) if $x, y \in F$, then $x \cdot y \in F$

$$\begin{aligned} x \cdot y &= y \cdot x \\ (x \cdot y) \cdot z &= x \cdot (y \cdot z) \end{aligned} \quad \left\{ \text{for all } x, y \in F \right.$$

F contains an element 1 s.t. $1 \neq 0$ and $1 \cdot x = x \cdot 1 = x$
 for all $x \in F$

for each $x \in F$, $x \neq 0$, there is an element $\frac{1}{x}$ in
 F s.t. $x \cdot (\frac{1}{x}) = 1$

(D) $x(y+z) = xy + xz$ for all $x, y, z \in F$

$$x - y = x + (-y)$$

$$\frac{x}{y} = x \cdot (\frac{1}{y})$$

$$x+y+z = x+(y+z) = (x+y)+z$$

$$xyz = (xy)z$$

$$x^2 = xx$$

$$x^n = \underbrace{xx\cdots x}_n$$

$n \in \mathbb{N}$ $n > 0$

$$nx = \underbrace{xx\cdots x}_n$$

Proposition (a) $x+y = x+z \Rightarrow y=z$ for all x, y, z

(b) $x+y = y \Rightarrow x=0$

(c) $x+y=0 \Rightarrow y=-x$

(d) $-(-x)=x$

Analysis prop.
for \neq

Proof

(a) Assume $x+y = x+z$

$$y=0+y = (-x+x)+y = -x+(x+y)$$

$$= -x+(x+z)$$

$$= (-x+x)+z = 0+z = z$$

(b) take $z=0$ in (a)

(c) take $z=-x$ in (a)

(d) rewrite (c) : $-x+y=0 \Rightarrow y=-(-x)$

$-x+x=0$ by an axiom

$$x = -(-x)$$

Prop $x, y, z \in F$

(a) $0x = 0$

(b) $x \neq 0, y \neq 0 \Rightarrow xy \neq 0$

(c) $(-x)y = -xy = x(-y)$

(d) $(-x)(-y) = xy$

Proof

(a) $0x + 0x = (0+0)x = 0x$

so $0x = 0$ by last prop (b)

(b) Assume $xy = 0$ then

$$\cancel{x} \cancel{y} xy = \cancel{x} \cancel{y} \cdot 0 = 0$$

$$(\cancel{x}x)(\cancel{y}y) = 1 \cdot 1 = 1 \text{ contradiction}$$

(c) $(-x)y + xy = (-x+x)y = 0 \cdot y = 0$

so $(-x)y = -xy$ by last prop (c)

$-xy = x(-y)$ same proof

(d) $(-x)(-y) = - (x(-y)) = -(-xy) = xy$

Definition An ordered field is a field that is an ordered set s.t.

Axioms (a) $y < z \Rightarrow xy < xz$ for all $x, y, z \in F$

(b) $x > 0, y > 0 \Rightarrow xy > 0$ for all $x, y \in F$

Convention: $x > 0$ - x is positive

$x < 0$ - x is negative

Propositions

(a) $x > 0 \Rightarrow -x < 0$

$x < 0 \Rightarrow -x > 0$

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(b) $x > 0, y < z \Rightarrow xy < xz$

(c) $x < 0, y < z \Rightarrow xy > xz$

(d) $x \neq 0 \Rightarrow x^2 > 0$; implies $1 > 0$

(e) $0 < x < y \Rightarrow 0 < \frac{1}{y} < \frac{1}{x}$

Proof

(a) $x > 0 \Rightarrow 0 = -x + x > -x + 0 = -x$

$x < 0 \Rightarrow 0 = -x + x < -x + 0 = -x$

(b) $z > y \Rightarrow z - y > y - y = 0$
 \downarrow
 $x(z - y) > 0$ Axiom (a)

\downarrow
 $xz - xy > 0$
 $xz + xy > 0 + xy$ Axiom (b)

\downarrow
 $xz + 0 > xy$

\downarrow
 $xz > xy$

(c) $- (x(z - y)) = (-x)(z - y) > 0$ Axiom (b)

$-(-x(z - y)) < 0$

\downarrow
 $x(z - y) < 0$

\downarrow
 $xz - xy < 0$, proved in (b)

\downarrow
 $xz < xy$

(d) $x > 0 \xrightarrow{\text{Axiom (b)}} x \cdot x > 0 \Rightarrow x^2 > 0$

$x < 0 \xrightarrow{\text{con}} -x > 0 \Rightarrow (-x) \cdot (-x) > 0 \nRightarrow x \cdot x > 0$
 \downarrow
 $x^2 > 0$

(e) Note: if $y > 0$, $v \leq 0$, then $vy \leq 0$ from (c)
and $y \cdot 0 = 0$

But $y \cdot \frac{1}{y} = 1 > 0$ from (d)

so $\frac{1}{y} > 0$ similarly $\frac{1}{x} > 0$

Now: $\begin{matrix} x < y \\ \downarrow \end{matrix}$ Axiom (b) and (d)

$$(\frac{1}{y})x < (\frac{1}{y})y$$

$$\frac{1}{y} < \frac{1}{x}$$

Theorem There exists an ordered field \mathbb{R} that has the property of least upper bound which contains \mathbb{Q} as an ordered subfield

(\mathbb{Q}) is an ordered field: $<, ., +, 0, 1, -, \div$

There is an injection: $f: \mathbb{Q} \rightarrow \mathbb{R}$

$$\text{s.t } f(x \cdot y) = f(x)f(y), f(x+y) = f(x) + f(y)$$

$$f(0_{\mathbb{Q}}) = 0_{\mathbb{R}}, f(1_{\mathbb{Q}}) = 1_{\mathbb{R}}$$

$$x < \mathbb{Q}y \text{ iff } f(x) < \mathbb{R}f(y)$$

$$f(-x) = -f(x), f(\frac{1}{x}) = \frac{1}{f(x)} \text{ if } x \neq 0_{\mathbb{Q}}$$

$$\mathbb{Q} \overset{\text{embedded}}{\hookrightarrow} \mathbb{R} \quad \mathbb{Q} \subseteq \mathbb{R}$$

Theorem (a) If $x, y \in \mathbb{R}$, $x > 0$. then there exists a natural number n s.t. $nx > y$

$$n > n_0 \rightarrow \underbrace{\frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2}}_{n} \in \mathbb{R} \quad \mathbb{Z} \ni n \xrightarrow{\text{count}} n$$

(b) $x, y \in \mathbb{R}$, $x < y$, then there exists $p \in \mathbb{Q}$ s.t. $x < p < y$

$$\mathbb{Z} \hookrightarrow \mathbb{R}$$

$$\mathbb{Q} \ni \frac{p}{q} \rightarrow \frac{1}{q(p)} \cdot q(p) \in \mathbb{R}$$

$q \neq 0$

$$\Theta : \mathbb{R} \xrightarrow{\text{onto}} \mathbb{R}^+$$

Proof (a) $A = \text{all numbers of the form } \frac{1}{n}x$, where $n \in \mathbb{N}$.

$A \neq \emptyset$. Since $x \in A$.
If case fails, A is bounded above, y is an upper bound for A . So let $\alpha = \sup A$

Since $x > 0$, we have $\alpha - x < \alpha$

Since α is the least upper bound of A , $\alpha - x$ is not an upper bound of A . So there is

$n \in \mathbb{N}$ s.t. $\alpha - x < nx$

$$\text{so } \alpha < nx + x = (n+1)x \in A$$

α is not a least upper bound of $A \Rightarrow$

(b) We have $x < y$. so $y - x > 0$

Find $m_1, m_2 \in \mathbb{N}$ s.t.

Also there are $m_1, m_2 \in \mathbb{N}$ s.t.

$$m_1 > nx, \quad m_2 > -nx \Rightarrow nx > -m_1$$

So there is an integer m s.t.

$$m-1 \leq nx < m$$

$$\frac{m-1}{m_2} < \frac{nx}{m_2} < \frac{m}{m_1}$$

$$\text{so } nx < m \leq 1 + nx < ny$$

$$\text{so } nx < m < ny$$

$$x < \frac{m}{n} < y$$

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Theorem

For each real number $x > 0$ and a natural number $n > 0$, there exists a unique $y > 0$ s.t. $y^n = x$

Note $y = x^{\frac{1}{n}} = \sqrt[n]{x}$

Proof

n is fixed. Uniqueness: induction
 $0 < y_1 < y_2 \Rightarrow y_1^n < y_2^n$ ($y_1 y_1 < y_2 y_1$)
 $y_2 y_1 < y_2 y_2$

$y_1 \neq y_2$ s.t. $y_1, y_2 > 0$ $y_1^n < y_2^n$
if $y_1^n = x = y_2^n$ \rightarrow

① Existence $E = \{t > 0 : t^n < x\}$

$E \neq \emptyset$: take $t = \frac{x}{1+x}$

then: $0 < t < x$, $t <$

$$0 < t^n \leq t < x$$

$$t \in E$$

E is bounded above: observe $t > 1+x$ then

$$t^n > t > x$$

so $1+x$ is an upper bound for E

Since if $t > 1+x$, then $t \notin E$

so if $t \in E$, then $t \leq 1+x$

Let $y = \sup E$. we claim this y works $y^n = x$

i) Assume for the sake of contradiction that $y^n < x$

Pix $0 < \varepsilon < 1$

$$\begin{aligned} (y + \varepsilon)^n &= y^n + \binom{n}{1} y^{n-1} \varepsilon + \binom{n}{2} y^{n-2} \varepsilon^2 + \dots + \varepsilon^n \\ &= y^n - \varepsilon (\binom{n}{1} y^{n-1} + \binom{n}{2} y^{n-2} \varepsilon + \dots + \varepsilon^{n-1}) \\ &< y^n + \varepsilon (\binom{n}{1} y^{n-1} + \binom{n}{2} y^{n-2} + \dots + 1) \\ &\leq y^n + \varepsilon (y^n + \binom{n}{1} y^{n-1} + \binom{n}{2} y^{n-2} + \dots + 1) \\ &= y^n + \varepsilon (1 + \dots) \end{aligned}$$

Take ε s.t. $0 < \varepsilon < 1$ s.t. $\varepsilon(1+y)^n < x - y^n$

$$\varepsilon < \frac{x-y^n}{(1+y)^n}$$

$$\text{then } y^n + \varepsilon(1+y)^n < y^n + \varepsilon(1+y)^n < y^n + x - y^n = x$$

so $(y+\varepsilon)^n < x$. So $y+\varepsilon \in E$ so y is not an upper bound for E since $y < y+\varepsilon$

(2) Assume for the sake of contradiction $y^n > x$

$$\begin{aligned} y - \varepsilon &< 1 \\ (y-\varepsilon)^n &= y^n - \binom{n}{1} y^{n-1} \varepsilon + \binom{n}{2} y^{n-2} \varepsilon^2 - \dots \\ &= y^n - \varepsilon \left(\binom{n}{1} y^{n-1} - \binom{n}{2} y^{n-2} \varepsilon + \dots \right) \\ &> y^n - \varepsilon \left(\binom{n}{1} y^{n-1} + \binom{n}{2} y^{n-2} + \dots \right) \\ &= y^n - \varepsilon (1+y)^n \end{aligned}$$

Take $0 < \varepsilon < 1$, $\varepsilon(1+y)^n < y^n - x$

$$\varepsilon < \frac{y^n - x}{(1+y)^n}$$

$$\text{so } (y-\varepsilon)^n > y^n - \varepsilon (1+y)^n > y^n - y^n + x = x$$

so if $t > y-\varepsilon$ then $t^n > (y-\varepsilon)^n > x$

so if $t^n < x$, then $t \leq y-\varepsilon$

so $y-\varepsilon$ is an upper bound for E

But $y-\varepsilon < y$ and y is the least upper bound for E . Contradiction

Col Let $a, b > 0$, n is an integer, $n > 0$
 Then $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$

Proof Let $\alpha = a^{\frac{1}{n}}$, $\beta = b^{\frac{1}{n}}$
 Then $ab = \alpha^n \beta^n = (\alpha\beta)^n$
 So by uniqueness, $\alpha\beta = r^{\frac{1}{n}}$

Euclidean spaces

\mathbb{R} = the reals

For $\varepsilon \geq 0$, let $\mathbb{R}^\varepsilon = \{(x_1, \dots, x_\varepsilon) : x_1, \dots, x_\varepsilon \in \mathbb{R}\}$
 $(\varepsilon \in \mathbb{Z})$

Notation

$$\underline{x} = (x_1, \dots, x_\varepsilon)$$

$$\underline{x} + \underline{y} = (x_1, \dots, x_\varepsilon) + (y_1, \dots, y_\varepsilon)$$

$$= (x_1 + y_1, x_2 + y_2, \dots, x_\varepsilon + y_\varepsilon)$$

$$\alpha \underline{x} = \alpha(x_1, \dots, x_\varepsilon) = (\alpha x_1, \dots, \alpha x_\varepsilon) \quad \alpha \in \mathbb{R}$$

$$\underline{0} = (0, \dots, 0)$$

Inner Product: $\underline{x}, \underline{y} \in \mathbb{R}^\varepsilon$

$$\underline{x} \cdot \underline{y} = (x_1, \dots, x_\varepsilon) \cdot (y_1, \dots, y_\varepsilon)$$

$$= x_1 y_1 + x_2 y_2 + \dots + x_\varepsilon y_\varepsilon$$

$$= \sum_{i=1}^{\varepsilon} x_i y_i$$

$$|\underline{x}| = (\underline{x} \cdot \underline{x})^{\frac{1}{2}} = \left(\sum_{i=1}^{\varepsilon} x_i^2 \right)^{\frac{1}{2}}$$

Theorem $x, y, z \in \mathbb{R}^\varepsilon$, α real

$$(a) |\underline{x}| \geq 0$$

$$(c) |\underline{x} + \underline{y}| \leq |\underline{x}| + |\underline{y}|$$

$$(b) |\underline{x}| = 0 \text{ iff } \underline{x} = \underline{0}$$

$$|\underline{x} - \underline{z}| \leq |\underline{x} - \underline{y}| + |\underline{y} - \underline{z}|$$

$$(c) |\alpha \underline{x}| = |\alpha| |\underline{x}|$$

$$(d) |\underline{x} \cdot \underline{y}| \leq |\underline{x}| |\underline{y}|$$

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Proof \Rightarrow

$$|x|^2 = x \cdot x = \sum a_i^2$$

$$|y|^2 = y \cdot y = \sum b_i^2$$

To prove $|x \cdot y|^2 \leq |x|^2 |y|^2$

Sufficient to see $(\sum a_i b_i)^2 \leq \sum a_i^2 \cdot \sum b_i^2$

$$(\sum a_i b_i)^2 = \sum_{i,j} a_i b_i a_j b_j$$

$$= \sum_i a_i^2 b_i^2 + \sum_{i < j} 2 a_i b_i a_j b_j$$

$$\sum a_i^2 \sum b_i^2 = \sum a_i^2 b_i^2 + \sum_{i < j} (a_i^2 b_j^2 + a_j^2 b_i^2)$$

Enough to show

$$\sum_{i < j} 2 a_i b_i a_j b_j \leq \sum_{i < j} (a_i^2 b_j^2 + a_j^2 b_i^2)$$

$$2 a_i b_i a_j b_j \leq a_i^2 b_j^2 + a_j^2 b_i^2$$

$$2(a_i b_j)(a_j b_i) \leq (a_i b_j)^2 + (a_j b_i)^2$$

Let

$$C = a_i b_j, D = a_j b_i$$

$$\text{Need } 2CD \leq C^2 + D^2$$

$$\text{Note } (CD)^2 \geq 0$$

$$2CD \leq C^2 + D^2$$

Check

$$(Ax) \cdot y = x \cdot (Ay) = A(x \cdot y)$$

$$x \cdot y = y \cdot x$$

$$x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2$$

$$(x_1 + x_2) \cdot y = x_1 \cdot y + x_2 \cdot y$$

Construction of \mathbb{R}

$\mathbb{Q}, <$ on \mathbb{Q} , $+, \cdot, 0, 1$ on \mathbb{Q}

\mathbb{Q} is an ordered field of fractions

Definition

$\alpha \subseteq \mathbb{Q}$ is a cut if (Dedekind)

(a) $\alpha \neq \emptyset$ and $\alpha \neq \mathbb{Q}$

(b) If $p \in \alpha$, $q \in \mathbb{Q}$, $q < p$, then $q \in \alpha$

(c) If $p \in \alpha$, then there is $r \in \alpha$ s.t. $p < r$

↓
q $p \in \alpha$ \mathbb{Q}

For $r \in \mathbb{Q}$, let $r^* = \{p \in \mathbb{Q} : p < r\}$
 $\mathbb{R} =$ the set of all cuts

$\mathbb{R} \ni r \rightarrow r^* \in \mathbb{R}$ 1-to-1

Define $\alpha, \beta \in \mathbb{R}$
 $\alpha < \beta$ iff $\left(\begin{array}{l} \alpha \neq \beta \\ \text{and} \\ \alpha \subseteq \beta \end{array} \right)$

For $r, s \in \mathbb{Q}$, $r < s$ iff $r^* < s^*$ in \mathbb{R}
 \mathbb{R} with $<$ has the least upper bound property

Let $A \subseteq \mathbb{R}$

a non- \emptyset set of cuts bounded above
there is $\beta \in \mathbb{R}$ (a cut)

s.t. for each $\alpha \in A$ $\alpha \leq \beta$ i.e. $\alpha \subseteq \beta$

$\sup A = \bigcup A = \{p \in \mathbb{Q} : p \in Q \text{ for some } \alpha \in A\}$

$\sup A$ is a cut so in \mathbb{R} . $\sup A$ is the least upper bound for A

$\alpha, \beta \in \mathbb{R}$ (cuts)

$$\alpha + \beta = \{p+q : p \in \alpha, q \in \beta\}$$

$$\text{a cut } 0_{\mathbb{R}} = (0_{\mathbb{Q}})^*$$

Note we can say

$$0 < \alpha, \beta < 0, \alpha = 0$$

$$\alpha \in \mathbb{R}$$

Let $\beta = \{p \in \mathbb{Q} : \text{there is some } r \in \mathbb{Q} \text{ s.t. } -p > r\}$

$$r > 0$$

$$-(p+r)$$

we claim $\beta = -\alpha$ in \mathbb{R}

Multiplication is defined in two steps

For $\alpha > 0, \beta > 0$ cuts

let $\alpha \beta = \{p : p \leq r \cdot s \text{ for some } r \in \alpha, s \in \beta, r > 0, s > 0\}$

$$\alpha \cdot 0 = 0 \cdot \alpha = 0 \text{ for all } \alpha \in \mathbb{R}$$

$$\alpha \beta = \begin{cases} (-\alpha)(-\beta) & \text{if } \alpha < 0, \beta < 0 \\ -(\alpha \beta) & \text{if } \alpha < 0, \beta > 0 \\ -(\alpha(-\beta)) & \text{if } \alpha > 0, \beta < 0 \end{cases}$$

$$1_{\mathbb{R}} = (1_{\mathbb{Q}})^*$$

Check \mathbb{R} with $+, -, 0, 1$ is a field
together with $<$ is an ordered field

Sizes of sets

$A \sim B$ if there is a bijection $f: A \rightarrow B$

Note: $A \sim \mathbb{R}$

$$A \sim B \Rightarrow B \sim A$$

$$(A \sim B \text{ and } B \sim C) \Rightarrow A \sim C$$

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$$J_n = \{1, \dots, n\}, n \in \mathbb{N} \quad (J_0 = \emptyset)$$

A finite if $A \sim J_n$ for some n

A infinite if A is not finite

A countable if $A \sim \mathbb{N}$

A at most ^{countable} if A countable or finite

A uncountable if A not at most countable

A sequence is a function whose domain is \mathbb{N}

$$f: \mathbb{N} \rightarrow X \quad (f(0), f(1), \dots, f(n))$$

Unions: Assume we have a family of sets E_α with
 $\alpha \in A$ $\bigcup_{\alpha \in A} E_\alpha = \{x : x \in E_\alpha \text{ some } \alpha \in A\}$

At most countability:

Lemma: Let A be a set and let $f: \mathbb{N} \rightarrow A$ be a surjection. Then A is at most countable

Note: If A at most countable, then $A = \emptyset$ or there is a surjection $\mathbb{N} \rightarrow A$

Proof: Let $y \subseteq \mathbb{N}$ defined by $n \in y$ iff there is $a \in A$ st. n is the smallest in $f^{-1}(a)$ $\neq \emptyset$

$(f|Y) : Y \rightarrow A$ is a bijection
domain = y

Take $n_1, n_2 \in y, n_1 \neq n_2$

$$n_1 = \min f^{-1}(a_1) \quad a_1 \neq a_2$$

$$n_2 = \min f^{-1}(a_2)$$

Claim: It will be enough to show if $\gamma \subseteq \mathbb{N}$, then γ is at most countable

Proof of claim:

Define a function h as follow

$$h(0) = \min \gamma \text{ if } \gamma \neq \emptyset$$

$$h(n+1) = \min (\gamma \setminus \{h(0), \dots, h(n)\}) \text{ if } \gamma \setminus \{h(0), \dots, h(n)\} \neq \emptyset$$

$h(n)$ is defined for all $n \in \mathbb{N}$

Then $h: \mathbb{N} \rightarrow \gamma$ is a bijection

($h(n) < h(n+1)$) so h is 1-to-1

h is onto

if not there is no $\gamma \subseteq \mathbb{N}$ s.t. no not a value of h

$h(n) < n_0$ for all n

So $\{h(n) : n \in \mathbb{N}\}$ is an infinite set of natural numbers all below n_0

Case 2: $h(n_0)$ is undefined for some n_0

Let n_0 be s.t. $h(n_0)$ undefined and n_0 is smallest

$$n_0 = 0 \Rightarrow \gamma = \emptyset$$

$$n_0 > 0 \Rightarrow \gamma \setminus \{h(0), h_1, \dots, h(n_0-1)\} = \emptyset \text{ so } \gamma \text{ is infinite}$$
$$\gamma = \{h_0, h_1, \dots, h(n_0-1)\}$$

Lemma: $\mathbb{N} \times \mathbb{N}$ is countable

Proof: $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$

\mathbb{N}	6
3	7
1	4
6	2
5	9

$$f: \mathbb{N} \times \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N} \setminus \{0\}$$

$$f(m, n) = 2^m(2n+1)$$

$$f(m, n) = f(m; n')$$

$$2^m(2n+1) = 2^{m'}(2n'+1)$$

Theorem Let E_n be at most countable for $n \in \mathbb{N}$
 The $\bigcup_n E_n$ is at most countable

Proof Each $E_n \neq \emptyset$. Then $\bigcup_n E_n \neq \emptyset$

Assume E_{n_0} is not empty

Consider $A_n = E_n \cup E_{n_0} \neq \emptyset$

Also A_n is at most countable \leftarrow to be justified

Let $f_n: \mathbb{N} \rightarrow A_n$ surjection

Define $F: \mathbb{N} \times \mathbb{N} \rightarrow \bigcup_n A_n$

$F(m, n) = f_n(m)$ Then F is a surjection

Let $a \in \bigcup_n A_n$ then $a \in A_{n_0}$ for some n_0

$\Rightarrow a = f_{n_0}(m_0)$ for some $m_0 \in \mathbb{N}$

Then $a = f_{n_0}(m_0) = F(m_0, n_0)$

$\mathbb{N} \xrightarrow{\text{onto}} \mathbb{N} \times \mathbb{N} \xrightarrow{\text{onto}} \bigcup_n A_n$

$\bigcup_n E_n = E_0 \cup E_1 \cup \dots \cup E_{n_0} \cup E_{n_0+1}$

$= (E_0 \cup E_{n_0}) \cup (E_1 \cup E_{n_0}) \cup \dots \cup (E_{n_0} \cup E_{n_0})$

$= \bigcup A_n$

Need E_1, E_2, \dots are at most countable

E_1, E_2 are at most countable

$E_1 = \emptyset \neq E_2 \vee E_1 = \emptyset \Rightarrow E_1 \cup E_2 = E_2 \vee E_2 = \emptyset \Rightarrow E_1 \cup E_2 = E_1 \vee$

$E_1 = \emptyset$, $E_2 \neq \emptyset$

Let $f_1: \mathbb{N} \rightarrow E_1$, $f_2: \mathbb{N} \rightarrow E_2$ surjection

$F: \mathbb{N} \rightarrow E_1 \cup E_2$ $F(2n) = f_2(n)$

more
countable $F(2n+1) = f_1(n)$

Corollary \mathbb{Q} is countable

Proof $f: \mathbb{Z} \times (\mathbb{N} \setminus \{0\}) \rightarrow \mathbb{Q}^+$

$f(m,n) = \frac{m}{n}$
 $\mathbb{N} \xrightarrow{\text{onto}} \mathbb{Z}$ onto $\mathbb{N} \xrightarrow{\text{onto}} \mathbb{N} \setminus \{0\}$ onto

$\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Z} \times (\mathbb{N} \setminus \{0\})$

$(m,n) \rightarrow (g_1(m), g_2(n))$

Feb 12
Corollary

Let A, B be at most countable, then $A \times B$ is at most countable

If $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$, so at most countable.

Assume $A \neq \emptyset, B \neq \emptyset$ $f: \mathbb{N} \rightarrow A$, $g: \mathbb{N} \rightarrow B$ surj

So there $h: \mathbb{N} \times \mathbb{N} \rightarrow A \times B$ given by

$h(m,n) = (f(m), g(n))$ surj

So $\mathbb{N} \xrightarrow{\text{surj}} \mathbb{N} \times \mathbb{N} \xrightarrow{h} A \times B$

So $A \times B$ at most countable

Corollary A at most countable

\mathbb{A}^n are most countable

$$n=1 \vee A' = A$$

$$n \rightarrow n+1$$

$n \rightarrow n+1$
 $A^{n+1} = A^n \times A$ are more countable by
 induction

Corollary $A_n, n \in \mathbb{N}$ are at most countable

$\bigcup_{n \in \mathbb{N}} (A_0 \times \dots \times A_n)$ at most countable

for each n , $A_1 \times \dots \times A_n$ is at most countable

$\bigcup_n (A_0 \times \dots \times A_n)$ is at most countable

$\{0,1\}^N$ — all binary sequences

$\{x_n\}$ s.t. $x_n = 0$ or $x_n = 1$ for each n

Theorem (Cantor)

Hartog's function

$\{0, 1\}^N$ is uncountable

Proof

If $\{0,1\}^N$ is at most countable, there is a surj
 $f: \mathbb{N} \rightarrow \{0,1\}^N$

So $y_n = x_n^k$ for all n

so $y_k = x_k^k$ But $y_k = 1 - x_k^k$

Then $x_k^k = \frac{1}{2}$ contradiction

(Corollary) \mathbb{R} is uncountable ~~因去看书~~

$$\mathbb{R}^k \quad |x-y|$$

Df A metric space (X, d) is a set X and a function

$d: X \times X \rightarrow \mathbb{R}$ s.t.

(1) $d(p, q) > 0$ if $p, q \in X$ and $p \neq q$

$$d(p, p) = 0$$

(2) $d(p, q) = d(q, p)$ if $p, q \in X$

(3) $d(p, q) + d(q, r) \geq d(p, r)$ for $p, q, r \in X$
Triangle Inequality

Ex

① $X = \mathbb{R}^k$, $d(x, y) = |x - y|$
 $|x + y| \leq |x| + |y|$

② X any non- \emptyset set

$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$ check triangle inequality

Df A neighborhood of $p \in X$ is a set of the form

$$N_r(p) = \{q \in X : d(p, q) < r\} \text{ for } r > 0$$

(neighborhood is either the point or everything)
written by me

A point $p \in X$ is a limit point of a set $E \subseteq X$ if each neighborhood of p contains a point $q \in E$ with $p \neq q$

0 is the limit point of $E = \{ \frac{1}{n+1} : n \in \mathbb{N} \}$

No set has limit point for $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$
(X, d) - discrete

If $p \in E$ and p is not a limit point of E , then p is an isolated point of E

① If E only has p

- E closed if every limit point of E is in E

- p is an interior point of E if there is a neighborhood N of p s.t. $N \subseteq E$

- E is open if each of its points is an interior point of E

\emptyset is both open and closed

$\overbrace{[\quad]}^E$ closed \mathbb{R}

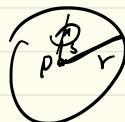
$\overbrace{(\quad)}^E$ open \mathbb{R}

$\overbrace{\quad}^E$ neither closed nor open

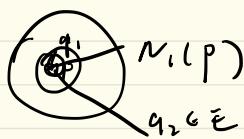
(X, d) - discrete
all sets are closed and all sets are open

- E is dense in X if each point in X is in E or is a limit point of E .
 \mathbb{Q} is dense in \mathbb{R}
- E is bounded if $\exists M$ s.t. $d(p, q) < M$ for all $p, q \in E$
- E is perfect if E is closed and each point of E is a limit point of E .

Thm Each neighborhood is open



If p is a limit point of E , then each neighborhood of p contains infinitely many points from E



$q_m \in E$ s.t.
 $q_m \neq q_n$ if
 $m \neq n$

$$r_1 = \frac{1}{2}d(p, q_1) > 0$$

$$\begin{matrix} q_2 \neq p \\ q_2 \neq q_1 \end{matrix}$$

$$N_{r_2}(p)$$

$$r_3 = \frac{1}{2}d(p, q_2)$$

$$q_2 \in N_{r_3}(p)$$

$$r_{mn} = \frac{1}{2}d(p, q_n)$$

$$N_{r_{mn}}(p) \ni q_{n+1}$$

$$\text{s.t. } q_{mn} \neq p$$

$$q_{mn} \neq q_n \dots q_n$$

$$\text{Note: } r_n \leq \frac{1}{2^{n-1}}$$

$$l = r_1 \leq \frac{1}{2^0} = 1$$

$$\text{Assume } r_n \leq \frac{1}{2^{n-1}}$$

$$r_{n+1} = \frac{1}{2} d(p, q_n) < \frac{1}{2} r_n \leq \frac{1}{2} \cdot \frac{1}{2^{n-1}} = \frac{1}{2^n}$$

Let $r > 0$ Consider $N_r(p)$

There exists n s.t. $\frac{1}{2^{n-1}} < r$ ($\frac{1}{r} < 2^{n-1}$, $\frac{1}{r} < n-1 \leq 2^{n-1}$)

So $N_{r_k}(p) \subseteq N_r(p)$ if $k \geq n$

$$r_k \leq \frac{1}{2^{n-1}} < r$$

So $q_k \in N_r(p)$ for all $k \geq n$

Theorem (x,d) a metric space. a) A complement of a closed set is open. b) A complement of an open set is closed.

Proof a) Let $E \subseteq X$ closed. $X \setminus E$ is open (to prove)

Take $p \in X \setminus E$. $\exists r > 0$ s.t. $N_r(p) \subseteq X \setminus E$ (to prove)

Note: $p \notin E$ p is not a limit point of E

There is $r > 0$ s.t. $N_r(p)$ contains no point of E or p is the only point in E in $N_r(p)$

so $N_r(p) \subseteq X \setminus E$

Since $p \in X \setminus E$ was arbitrary in the argument above, we see that each point of $X \setminus E$ is its interior point, i.e. $X \setminus E$ is open

(2) $E \subseteq X$ open

Need: $X \setminus E$ closed

Let p be a limit point of $X \setminus E$

Note: $p \in X \setminus E$

Assume there was a contradiction that $p \notin X \setminus E$
i.e. $p \in E$

p is an interior point of $E \rightarrow N_r(p) \subseteq E$

Since p is a limit point of $X \setminus E$,

there is a $q \in X \setminus E$ s.t. $q \in N_r(p)$

Then $q \in E$, which is a contradiction.

Prop a) Open sets are open under taking arbitrary unions and finite intersections.

b) Closed sets are closed under taking arbitrary intersections and finite unions.

U α . $\alpha \in A$, open $\rightarrow \bigcup_{\alpha \in A} u_\alpha$ is open

U i U n are open $\rightarrow \bigcap_{i=1}^n u_i$ is open
(1) \Rightarrow (2)

$$(X \setminus \bigcap_{\alpha \in A} E_\alpha = \bigcup_{\alpha \in A} (X \setminus E_\alpha))$$

$$(X \setminus \bigcup_{\alpha \in A} E_\alpha = \bigcap_{\alpha \in A} (X \setminus E_\alpha))$$

$$\text{a)} \quad p \in \bigcap_{i=1}^n U_i$$

$p \in U_i \Rightarrow N_{r_i}(p) \subseteq U_i$ for some $r_i > 0$

Take $r = \min(r_1, \dots, r_n) > 0$

$N_r(p) \subseteq N_{r_i}(p) \subseteq U_i$ for each i

$$N_r(p) \subseteq \bigcap_{i=1}^n U_i$$

$p \in \bigcap_{i=1}^n U_i$, arbitrary, $\bigcap_{i=1}^n U_i$ is open

b) Similar

(and) matrix space, $E \subseteq X$. $E' = \text{all limit points of } E$

let $\bar{E} = E \cup E'$ - the closure of E'

then $\bar{E} \subseteq X$, (x'd) matrix

c) \bar{E} is closed

c) For each $F \supseteq E$, F closed, we have $\bar{E} \subseteq F$
 \bar{E} is the smallest closure set

need: all limit points of \bar{E} are in \bar{E}

p a limit point of \bar{E}

we chose p is a limit point of E (so $p \in \bar{E}$)

$q \in \bar{E}$ s.t. $q \neq p \rightarrow q \in E$ then done



$$r_0 = \min(d(p, q), r_0 - d(p, q))$$

$$N_{r_0}(q) \subseteq N_p(p)$$

$$p \notin N_{r_0}(q)$$

Find $q' \in M_0(q)$, $q' \in E$. q is a limit point of E . Then $q' \in M_r(p)$ $q' \neq p \rightarrow q' \in E, q' \in M_r(p)$
 $q' \neq p$

$$\textcircled{2} \quad \bar{E} \subseteq F \Rightarrow \bar{E} \subseteq F$$

Theorem $E \subseteq \mathbb{R}$ non- \emptyset bounded above
 then $\sup E \in \bar{E}$
 if E closed, then $\sup E \in E$

Proof Assume $\sup E$ not in \bar{E}

$\sup E \notin E$ and $\sup E \notin E'$

For some $r > 0$. $M_r(\sup E) \cap E = \emptyset$

so $\sup E - r$ is an upper bound of E .

contradiction since $\sup E - r < \sup E$

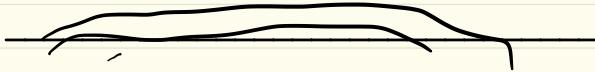
$$\text{Ex } \bar{E} = (0,1) \cup \left\{ \frac{1}{n} \right\}_{n=1}^{\infty}$$

Open covers/compactness

(X, d) is a metric space $E \subseteq X$

A family $\{G_\alpha : \alpha \in I\}$ is an open cover of E if each G_α is an open subset of X and $E \subseteq \bigcup_{\alpha \in I} G_\alpha$

Ex ① $\mathbb{R} = X$ $C_n = (-\infty, n), n \in \mathbb{N}$



$\bigcup_{n \in \mathbb{N}} C_n = \mathbb{R}$ $\{C_n : n \in \mathbb{N}\}$ an open cover of \mathbb{R}

② $X = \mathbb{R}$ $E = [0, 1]$ $C_n = [\frac{1}{n}, 2] n \in \mathbb{N}, n \geq 1$
 $C_1 = (-1, \varepsilon) \quad \varepsilon > 0$
 $\{C_n : n \in \mathbb{N}, n \geq 1\}$ cover of $E = [0, 1]$

Definition Let $E \subseteq X$, we say that E is compact if
 for each of open cover $\{G_\alpha : \alpha \in \Lambda\}$ of E
 there are $\alpha_1, \dots, \alpha_n$ s.t. $E \subseteq G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$
 Each open cover of E has a finite subcovers

(Y, d) a metric space

$x \in Y \rightarrow (x, d^Y(x, x))$ subspace of (Y, d)

If $U \subseteq Y$ open in Y then $U \cap X$ is open in X

$x \in U \cap X, x \in U \quad \exists r > 0, N_r^Y(x) \subseteq U$

$N_r^Y(x) \cap X \subseteq U \cap X$

$N_r^Y(x) \cap X = N_r^X(x) \subseteq U \cap X$

有集合 A, B . 若 $\forall x \in A$, x is an interior point of B ,
 同时 $N_r(x) \subseteq A$. 是否证明 A is open
 in B ?

If $V \subseteq X$ is open in X then there exists $U \subseteq Y$
open in Y s.t. $V = U \cap X$

for $x \in V$, let $r_x > 0$ be s.t. $N_{r_x}^{\bar{X}}(x) \subseteq V$

let $U = \bigcup_{x \in V} N_{r_x}^Y(x) \subseteq Y$ open

then $U \cap X \Rightarrow \left(\bigcup_{x \in V} N_{r_x}^Y(x) \right) \cap X$

$$= \bigcup_{x \in V} (N_{r_x}^Y(x) \cap X)$$

$$= \bigcup_{x \in V} N_{r_x}^{\bar{X}}(x)$$

$$= V$$

Theorem (Y, d) a metric space

\Rightarrow a subspace of Y

Let $E \subseteq X$ then E is compact in Y iff E
is compact in X

Compactness

$$K \subseteq X \subseteq Y$$

Prop \Rightarrow X, Y metric spaces

K is compact in X iff K is compact in Y

Proof \Rightarrow Let $U_\alpha \subseteq Y$, $\alpha \in I$ be open subset of Y s.t.

$$\bigcup_{\alpha \in I} U_\alpha \supseteq K$$

Then consider $\bigcup_{\alpha \in I} N_X^\alpha$ open in X

Note $K \subseteq \bigcup_{\alpha \in I} (\bigcup_{\alpha \in I} N_X^\alpha)$

So there are $\alpha_1, \dots, \alpha_n$ s.t. $K \subseteq \bigcup_{\alpha=1}^n (\bigcup_{\alpha \in I} N_X^\alpha) \cup \dots \cup (\bigcup_{\alpha=n+1}^m N_X^\alpha)$

$\bigcup_{\alpha=1}^n, \bigcup_{\alpha=n+1}^m, \dots, \bigcup_{\alpha=m}^m$

② (\Leftarrow) Let $\bigcup_{\alpha \in I} U_\alpha \subseteq X$, $\alpha \in I$ be open in X s.t.

$$K \subseteq \bigcup_{\alpha \in I} U_\alpha$$

for each $\alpha \in I$, there is $V_\alpha \subseteq Y$ open in Y s.t.

$$U_\alpha = X \cap V_\alpha$$

$$\text{so } K \subseteq \bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} V_\alpha \cap X \subseteq \bigcup_{\alpha \in I} V_\alpha$$

so there are $\alpha_1, \dots, \alpha_n \in I$ s.t.

$$K \subseteq V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$$

$$K \subseteq K \cap X \subset (X \cap V_{\alpha_1}) \cup \dots \cup (X \cap V_{\alpha_n}) \\ = U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$$

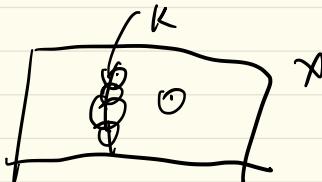
K is compact (in K) then K is compact in any space

Prop (X, d) a metric space

all compact subsets of X are closed in X

as closed subsets of compact metric spaces are compact

Proof of



Let $K \subseteq X$ be compact
we show that $X \setminus K$ is open
so we need for each $x \in X \setminus K$,
there is $r > 0$ s.t. $N_r(x) \subseteq X \setminus K$

Fix $y \in K$ Find r_y s.t.
 $N_{r_y}(y) \cap N_{r_y}(x) = \emptyset$

E.g. Take $r_y = \frac{d(x,y)}{3} > 0$
we do it for each $y \in K$.

Note $\bigcup_{y \in K} N_{r_y}(y) \supseteq K$ since K is compact
there are y_1, \dots, y_n s.t. $K \subseteq N_{r_{y_1}}(y_1) \cup \dots \cup N_{r_{y_n}}(y_n)$

Note $N_{r_{y_i}}(y_i) \cap N_{r_{y_j}}(x) = \emptyset$ for $i=1 \dots n$

so $N_{\bar{r}}(y_i) \cap N_{\bar{r}}(x) = \emptyset$ where $\bar{r} = \min(r_{y_1}, \dots, r_{y_n})$

so $(N_{r_{y_1}}(y_1) \cup \dots \cup N_{r_{y_n}}(y_n)) \cap N_{\bar{r}}(x) = \emptyset$

so $K \cap N_{\bar{r}}(x) = \emptyset$ so $N_{\bar{r}}(x) \subseteq X \setminus K$

(2) X a compact matrix space
 $K \subseteq X$ closed

Need K is compact in X

let U_α , $\alpha \in I$, be open in X

s.t. $K \subseteq \bigcup_{\alpha \in I} U_\alpha$

consider $X \subseteq (X \setminus K) \cup \bigcup_{\alpha \in I} U_\alpha$?

so there are $\alpha_1, \dots, \alpha_n \in I$ s.t.

$$X = (X \setminus K) \cup U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$$

so $K \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$ so K is compact

Prop Let K_α , $\alpha \in I$ be a family of compact subsets of matrix space X .

Assume for all $\alpha_1, \dots, \alpha_n \in I$ we have

$K_{\alpha_1} \cap \dots \cap K_{\alpha_n} \neq \emptyset$ Then $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$

Proof Pick $x_0 \in I$
 Assume towards a contradiction that
 $\bigcap_{\alpha \in I} K_\alpha = \emptyset$ Then $K_{x_0} \subseteq \bigcup_{\alpha \in I, \alpha \neq x_0} (X \setminus K_\alpha)$ closed

Since K_{x_0} is compact, there are $\alpha_1, \dots, \alpha_n \in I$ s.t. $K_{x_0} \subseteq (X \setminus K_{\alpha_1}) \cup \dots \cup (X \setminus K_{\alpha_n})$

$X \setminus \bigcap_{\alpha \in I, \alpha \neq x_0} K_\alpha$

So $K_{x_0} \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_n} = \emptyset$ contradiction

Cor Let $K_n, n \in \mathbb{N}$ be compact non- \emptyset subsets of a metric space X . If $K_0 \supseteq K_1 \supseteq K_2 \supseteq \dots$, then $\bigcap_n K_n \neq \emptyset$ Then consider $K_n, n \in \mathbb{N}$

Note $n_1, \dots, n_4 \in \mathbb{N}$

$$K_{n_1} \cap K_{n_2} \cap K_{n_3} \cap K_{n_4} = K_{\max(n_1, \dots, n_4)} \neq \emptyset$$

Prop Each infinite subset of a compact space has a limit point.

Proof Assume this is not the case. ($V_x = N(x)$ some $x \in X$)
 Take $x \in X$, we can find an open set V_x s.t.
 $x \in V_x$ and $|V_x \cap E| \leq 1$
 Note $x = \bigcup_{x \in X} V_x$ by compactness
 there are x_1, \dots, x_n s.t. $x = V_{x_1} \cup \dots \cup V_{x_n}$
 $E = E \cap X = (E \cap V_{x_1}) \cup \dots \cup (E \cap V_{x_n})$
 So $|E| \leq n$ contradiction

Compactness in \mathbb{R}^k

Theorem Let (I_n) be a sequence of non-empty closed intervals in \mathbb{R} . If $I_0 \supseteq I_1 \supseteq \dots$ then $\bigcap_n I_n \neq \emptyset$

reformulation of least upper bound property

proof

$$I_n = [a_n, b_n], a_n \leq b_n$$

$$\text{consider } A = \{a_n : n \in \mathbb{N}\} \quad A \neq \emptyset$$

A bounded above by b_0 ($a_n \in [a_0, b_0]$)

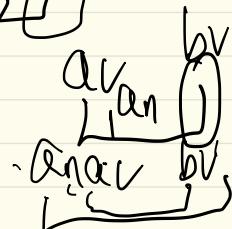
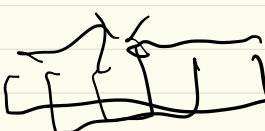
Let $c = \sup A$ we claim that
 $c \in [a_n, b_n]$ for each n

Fix n . $a_n \leq c$ since $c = \sup A \geq a_n$

$c \leq b_n$ since b_n is an upper bound of A

If $\exists n$, then $[a_1, b_1] \subseteq [a_n, b_n]$, $a_1 \in [a_n, b_n]$
 $a_1 \leq b_1$ $\underbrace{a_1, a_2, \dots, b_1, b_n}_{a_1 \leq a_2 \leq \dots \leq b_1 \leq b_n}$

If $l < n$, then $[a_n, b_n] \subseteq [a_l, b_l]$ $a_l \leq a_n \leq b_n$
so $a_l \leq a_n \leq b_n$



Thm Every cell in \mathbb{R}^k is compact

A cell is a set of the form $[a_1, b_1] \times \dots \times [a_k, b_k]$ where $a_i \leq b_1, \dots, a_k \leq b_k, a_i, b_i \in \mathbb{R}$

$$\{x \in \mathbb{R}^k : a_1 \leq x_1 \leq b_1, \dots, a_k \leq x_k \leq b_k\}$$

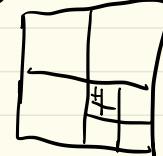
Proof: Let the cell be of the form $C = [a_1, b_1] \times \dots \times [a_k, b_k]$

Need: Each open cover of C has a finite subcover

Fix U_α open subsets of \mathbb{R}^k s.t. $C \subseteq \bigcup_{\alpha \in A} U_\alpha$

Assume towards contradiction there's no finite subcover of $U_\alpha : \alpha \in A$

$$\text{See } a_i^\circ = a_i, b_i^\circ = b_i$$



Assume we have a_i^n, b_i^n s.t.

$$[a_i^n, b_i^n] \subseteq [a_i^{n-1}, b_i^{n-1}] \subseteq \dots \subseteq [a_i^\circ, b_i^\circ]$$

$$b_i^n - a_i^n \leq \frac{b_i - a_i}{2^n}$$

$[a_i^n, b_i^n] \times \dots \times [a_k^n, b_k^n]$ cannot be covered by finitely many sets namely $U_\alpha : \alpha \in A$

Consider the cells obtained by halving each interval $[a_i^n, b_i^n]$ and taking the product. We have 2^k such cells and they covers $[a_1^n, b_1^n] \times \dots \times [a_k^n, b_k^n]$. So one of the cells cannot be covered by finitely many sets $U_\alpha : \alpha \in A$. Pick one such cell be : $[a_i^{n+1}, b_i^{n+1}] \times \dots \times [a_k^{n+1}, b_k^{n+1}]$

Clear : $[a_i^{n+1}, b_i^{n+1}] \subseteq [a_i^n, b_i^n]$ for $i = 1 \dots k$

$$\varrho \leq b_i^{n+1} - a_i^{n+1} = \frac{b_i^n - a_i^n}{2} \leq \frac{b_i - a_i}{2^n} = \frac{b_i - a_i}{2^{n+1}}$$

so we maintain the three conditions in the construction
 we have $[a_i^n, b_i^n] \subseteq \dots \subseteq [a_i^0, b_i^0]$ for each $1 \leq i \leq k$.

$\bigcap_n [a_i^n, b_i^n] \neq \emptyset$ (by theorem)

x_i

Note $\underline{x} = (x_1, \dots, x_k) \in \bigcap_n ([a_1^n, b_1^n] \times \dots \times [a_k^n, b_k^n])$

Note $\bigcap_n ([a_1^n, b_1^n] \times \dots \times [a_k^n, b_k^n]) = \{\underline{x}\}$

If \underline{y} was a point in the intersection with $\underline{x} \neq \underline{y}$ then
 $x_i \neq y_i$ for some i

Then $x_i, y_i \in \bigcap_n [a_i^n, b_i^n]$ so $|x_i - y_i| \leq b_i^n - a_i^n$

so $0 < |x_i - y_i| \leq \frac{b_i - a_i}{2^n}$ so $n \leq 2^n \leq \frac{b_i - a_i}{|x_i - y_i|}$ contradiction

Since $\underline{x} \in C$ there is $\forall r > 0$ s.t. $\underline{x} \in U_{r,0}$ so

There is $r > 0$ s.t. $N_r(\underline{x}) \subseteq U_{r,0}$ since $U_{r,0}$ is open
 more $\underline{y} \in G(c_1, d_1) \times \dots \times G(c_k, d_k)$

$$|\underline{y} - \underline{x}| = \sqrt{(y_1 - x_1)^2 + \dots + (y_k - x_k)^2} \leq \sqrt{k} \cdot \max_{1 \leq i \leq k} |y_i - x_i| \leq \sqrt{k} \cdot \max_{1 \leq i \leq k} (d_i - c_i)$$

If $\underline{z} \in [a_1^n, b_1^n] \times \dots \times [a_k^n, b_k^n]$

$$\text{then } |\underline{x} - \underline{z}| \leq \sqrt{k} \max_{1 \leq i \leq k} |b_i^n - a_i^n| \leq \sqrt{k} \frac{\max_{1 \leq i \leq k} (b_i - c_i)}{2^n}$$

so for n large enough, $\sqrt{k} \frac{\max_{1 \leq i \leq k} (b_i - a_i)}{2^n} < r$

$$\text{so } [a_1^n, b_1^n] \times \dots \times [a_k^n, b_k^n] \subseteq N_r(\underline{x})$$

for large enough n

$\subseteq U_{r,0}$ so for large enough n ,

$[a_1^n, b_1^n] \times \dots \times [a_k^n, b_k^n]$ is covered by finitely
 many sets namely U_α , $\alpha \in A$

Thm A subset E of \mathbb{R}^k is compact iff E is closed and bounded

↓
Definition: (X, d) a metric space, $E \subseteq X$ is bounded if there is $K \geq 0$ s.t. $d(x, y) \leq K$ for all $x, y \in E$
 $\sup_{x, y \in E} d(x, y) < \infty$

Proof (\Rightarrow) compact \Rightarrow closed (proved in general)

Pick $x \in E$, consider $N_n(x)$, $n \in \mathbb{N}$, $n > 0$

$\bigcup_{n>0} N_n(x) = \mathbb{R}^k \supseteq E$ (open cover)

Since E is compact, there are n_1, \dots, n_m s.t.

$E \subseteq N_{n_1}(x) \cup \dots \cup N_{n_m}(x)$

so $E \subseteq N_{\bar{n}}(x)$ when $\bar{n} = \max(n_1, \dots, n_m)$

$y, z \in E \Rightarrow d(y, z) \leq d(y, x) + d(x, z) \leq \underline{n} + \underline{n} = 2\underline{n} = k$

(\Leftarrow) If E is bounded, there are $a_1 \leq b_1, \dots, a_k \leq b_k$ s.t.

$E \subseteq [a_1, b_1] \times \dots \times [a_k, b_k]$

$\forall x \in E \quad d(x, y) \leq k$

$$\sqrt{(x_1 - y_1)^2 + \dots + (x_k - y_k)^2} \leq \sqrt{k} \max_{1 \leq i \leq k} |x_i - y_i| ?$$

so for each $i \leq k$, $|x_i - y_i| \leq k$ so

$$x_i - k \leq y_i \leq x_i + k \leq \frac{\sqrt{k} \max(b_i - a_i)}{2}$$

Since E is closed set

proved \rightarrow [a closed subset of a compact set]

We have E is compact

Cor Each bounded infinite subset of \mathbb{R}^k has a limit point.

Proof: $E \subseteq \mathbb{R}^k$ bounded. There is a cell $(a_1, b_1] \times \dots \times (a_k, b_k)$

$\begin{matrix} a_1 & \dots & a_k \\ \downarrow & \dots & \downarrow \\ E & \text{compact} \end{matrix}$

Since infinite subset of a compact set has a limit point, we are done.

March 8

(X, d) a metric space, $p_n \in X$, $n \in \mathbb{N}$

(p_n) converges to $p \in X$ for each $\varepsilon > 0$

there is N s.t. for $n \geq N$

$$d(p_n, p) < \varepsilon$$

(p_n) converges if it converges to some $p \in X$

if (p_n) converges to p and p' then $p = p'$

If $p \neq p'$ then $d(p, p') > 0$. Let $\varepsilon = \frac{d(p, p')}{2} > 0$

For some N , for $n \geq N$

$$d(p, p_n) + d(p_n, p') \geq d(p, p')$$

\wedge

$\varepsilon + \varepsilon$

$\frac{n}{2\varepsilon}$

$$\frac{d(p, p')}{2\varepsilon} \geq 0$$

(p_n) diverges if it does not converge

Ex a) $s_n = \frac{1}{n}$ $s_n \rightarrow 0$

Fix $\varepsilon > 0$. Find $N \in \mathbb{N}$ s.t.

$$N > \frac{1}{\varepsilon} \text{ then } d(s_n, 0) = |s_n - 0| = \left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

for $n \geq N$

a) $t_n = \frac{(-1)^n}{n}$

$$d(t_n, 0) = \left| \frac{(-1)^n}{n} - 0 \right| = \left| \frac{(-1)^n}{n} \right| = \frac{1}{n}$$

c) $s_n = n$ s_n diverges

Let $S \in \mathbb{R}$. Take $\varepsilon = 1$

Fix arbitrary $N \in \mathbb{N}$

Find $n \geq N$ s.t. $n > s+1$

$$|s_n - S| = |n - S| = n - S > s+1 - S = 1 = \varepsilon \quad \text{contradiction}$$

Prop (X, d) metric space $\{p_n\} \subset X$

a) If (p_n) converges then (p_n) is bounded

$\{p_n : n \in \mathbb{N}\}$ is bounded

Proof: Let $p = \lim_n p_n$

Take N s.t. for $n \geq N$, $d(p, p_n) < 1$

$$\text{let } K = \max \left(2, \max_{i,j \leq N} d(p_i, p_j), \max_{i \leq N} (d(p_i, p) + 1) \right) < \infty$$

$$(p_m, p_n) \in K \quad \forall m, n \in \mathbb{N}$$

Prop (X, d) metric space, $E \subseteq X$ $p \in X$

p is a limit point of E iff there is a sequence

$p_n \in E$ s.t. $p_n \xrightarrow{n} p$ and $p_n \neq p$ all n

Proof: (\Rightarrow) Let $p_0 \in E$ be s.t. $p_0 \in N_r(p)$ and $p_0 \neq p$

$$r_1 = \frac{d(p, p_0)}{2} > 0$$

Let $p_1 \in E$ be s.t. $p_1 \in N_{r_1}(p)$ and $p_1 \neq p$,

$$r_2 = \frac{d(p, p_1)}{2} > 0 \text{ let } p_2 \in E$$

$p_2 \in N_{r_2}(p)$, $p_2 \neq p$, produce (p_n) in this fashion

Note $p_n \neq p$ all n . $p_n \in E$

Note $n \geq 1$ $d(p_n, p) < r_n \leq \frac{d(p_0, p)}{2^n}$

$$r_1 = \frac{d(p_0, p)}{2}$$

$$r_{n+1} = \frac{d(p_n, p)}{2} < \frac{r_n}{2} \leq \frac{\frac{d(p_0, p)}{2^n}}{2} = \frac{d(p_0, p)}{2^{n+1}}$$

$p_n \xrightarrow{n} p$, $2^{n+1} > 2^n$, $2^n \geq 2^N$ if $n \geq N$

Fix $\varepsilon > 0$ find N s.t. $\frac{d(p_0, p)}{2^N} < \varepsilon$

For $n \geq N$, $\frac{1}{\varepsilon} < \frac{2^n}{d(p_0, p)}$

$$d(p_n, p) < \frac{d(p_0, p)}{2^n} \leq \frac{d(p_0, p)}{2^N} < \varepsilon$$

(P) we have $p_n \in E$, $p_n \xrightarrow{n} p$, $p_n \neq p \Rightarrow p$ is a limit point of E .

Let $r > 0$ consider $N_r(p)$ need a point $q \in N_r(p)$, $q \in E$, $q \neq p$

Since $p_n \xrightarrow{n} p$ then there is N s.t.

$d(p_n, p) < r$ for all $n \geq N$

So $d(p_N, p) < r$. $p_N \in N_r(p)$

Let $q = p_N$ we are done

Theorem

TODO ✓

$(s_n), (t_n)$ sequences of reals s.t. $s_n \xrightarrow{n} s$, $t_n \xrightarrow{n} t$

a) $\lim_n (s_n + t_n) = \lim_n s_n + \lim_n t_n = s + t$

b) if $c \in \mathbb{R}$, then $\lim_n (cs_n) = c(\lim_n s_n) = cs$

c) $\lim_n (s_n t_n) = \lim_n (s_n) \lim_n (t_n) = st$

d) if $s_n \neq 0$ for all n , $s \neq 0$, then $\lim_n \left(\frac{1}{s_n} \right) = \frac{1}{\lim_n s_n} = \frac{1}{s}$

Proof: b) $|s_n t_n - st| = |s_n t_n - s_n t_n + s_n t_n - st|$
 $\leq |s_n t_n - s_n t_n| + |s_n t_n - st|$
 $= |s_n - s| |t_n| + |s| |t_n - t| \leq |s_n - s| C + |s| |t_n - t|$

Notes since (t_n) converges, it is bounded so there is K s.t. $|t_m - t_n| \leq K$ all m, n
 $|t_0 - t_n| \leq |t_0 - t_m| + |t_m - t_n| \leq K$ all n

$$|t_n| \leq |t_0| + \overset{\curvearrowleft}{K} \text{ all } n$$

$$|\beta_n - s| \cdot |t_n| + |\beta| |t_n - t| \leq \beta |s| |t| + |\beta| |t_n - t|$$

Let $\varepsilon > 0$, find N s.t. for $n \geq N$, $|\beta_n - s| < \frac{\varepsilon}{2\beta}$

$$|t_n - t| < \frac{\varepsilon}{2|\beta| + 1}$$

Then for $n \geq N$ $|\beta_n - s| + |\beta| |t_n - t| < \frac{\varepsilon}{2} + \frac{|\beta|}{2|\beta| + 1} \varepsilon$
 $\leq \frac{\varepsilon}{2} (1 + 1) = \varepsilon$

$$(A) \quad \left| \frac{1}{s_n} - \frac{1}{s} \right| = \left| \frac{s - s_n}{ss_n} \right| = \frac{|s - s_n|}{|s| |s_n|}$$

for large n , $|\beta - s_n| < \frac{|\beta|}{2}$
 $(n \geq N')$ $|\beta| - |\beta_n| \leq$

$$|\beta| - \frac{|\beta|}{2} \leq |s_n|$$

$$\frac{|\beta|}{2} \leq |s_n|$$

$$\frac{|s - s_n|}{|\beta| |s_n|} \leq \frac{|s - s_n|}{|\beta| \frac{|\beta|}{2}} = \frac{2}{|\beta|^2} |s - s_n|$$

Fix $\delta > 0$ for some $N \geq N'$

for $n \geq N$, $|s - s_n| < \frac{\varepsilon}{\frac{2}{|\beta|^2}}$

$$\text{So } \frac{2}{|\beta|^2} |s - s_n| < \varepsilon$$

March 12

Thm $\underline{x}_n \in \mathbb{R}^k$ $x_n = (x_{1n}, \dots, x_{kn})$, $y_n \in \mathbb{R}^k$, $x, y \in \mathbb{R}^k$

or (x_n) converges to $\underline{x} = (x_1, \dots, x_k)$ iff

$$\lim_n x_{in} = x_i \text{ for } 1 \leq i \leq k - \forall \epsilon > 0 \exists N \forall n \geq N |x_{in} - x_i| < \epsilon$$

or $\underline{x}_n \rightarrow \underline{x}$, $y_n \rightarrow \underline{y}$, $\beta_n \rightarrow \beta$, $\beta_n, \beta \in \mathbb{R}$

Then $\lim_n (\underline{x}_n + \underline{y}_n) = \underline{x} + \underline{y}$

$$\lim_n \underline{x}_n \cdot \underline{y}_n = \underline{x} \cdot \underline{y}$$

$$\lim_n \beta_n \underline{x}_n = \beta \underline{x}$$

$$|\underline{x}_n - \underline{x}| \leq \sqrt{(x_{1n} - x_1)^2 + \dots + (x_{kn} - x_k)^2} \leq \sqrt{k} \max_{j=1 \dots k} |x_{jn} - x_j|$$

Subsequences

Let (x_n) be a sequence of elements of X . By a subsequence of (x_n) , we mean any sequence (y_k)

s.t. $y_k = x_{n_k}$ for some $n_0 < n_1 < n_2 < n_3 \dots$

Ex ① $x_n = (-1)^n$ $y_k = 1$ is a subsequence of x_n

$$y_k = x_{2k} = (-1)^{2k} = 1 \text{ so } n_k = 2k$$

$$z_k = -1 \quad z_k = x_{2k+1}$$

② $x_n = n$ $y_k = 1$ not a subsequence

Thm

① If (p_n) is a sequence in compact metric space (X, d) , then (p_n) contains a convergent subsequence.

② Each bounded sequence in \mathbb{R}^k contains a convergent subsequence

Proof: ① Consider $E = \{p_n : n \in \mathbb{N}\}$ E is finite

Then there is $p \in E$ s.t. $p = p_n$ for infinite many n .

i.e. there are $n_1 < n_2 < \dots$

s.t. $p = p_{n_k}$ so (p_{n_k}) is a subsequence of (p_n) converges to p

② E is infinite. If E has a limit point, say p

Inductively pick $n_0 < n_1 < n_2 < \dots$ as follows. $n_0 = 0$

Assume $n_0 < \dots < n_k$ constructed. Consider

$N_{n_k}^\perp(p)$ contains infinitely many elements of E

$p_{n_{k+1}} \in N_{n_k}^\perp(p)$ $n \in E$ s.t. $n_{k+1} > n_k$

(suffice $p_{n_{k+1}} \neq p_i$ for $i \leq k$)

so (p_{n_k}) constructed with $n_0 < n_1 < \dots$

Note $d(p_n, p) < \frac{1}{k}$ for $k \geq 1 \rightarrow$ so $p_{n_k} \rightarrow p$

$p_{n_k} \in N_k(p)$

(2) the values of (p_n) are contained in
 $\{x \in \mathbb{R}^k; |x| \leq k\}$ since x is compact
 (p_n) has a subsequence by UK)

Cauchy Sequence

A sequence (p_n) of elements of (X, d) is Cauchy if for each $\epsilon > 0$ there is an N s.t. for all $m, n \geq N$,

$$d(p_m, p_n) < \epsilon$$

Prop Convergence sequences are Cauchy

Proof Let (p_n) ($\in (X, d)$) converge to p . Then take $\epsilon > 0$
 There is N s.t. for all $n \geq N$, $d(p_n, p) < \frac{\epsilon}{2}$

Then for $m, n \geq N$, we have

$$d(p_m, p_n) \leq d(p_m, p) + d(p, p_n) < \epsilon$$

Prop Each Cauchy sequence is bounded

Let $s = 1$, find N s.t. for $m, n \geq N$, $d(p_m, p_n) < 1$
 Let $k = \max d(p_i, p_j) + 1$

Then $d(p_m, p_n) \leq k$ for all m, n

Definition for $E \subseteq X$, (X, d) a metric space, $\text{Diam}(E) =$
 $\sup \{d(x, y) \mid x, y \in E\}$
 $\text{diam } (\emptyset) = -\infty$

Proposition $E \subseteq X$, (X, d) is a metric space then
 $\text{Diam}(E) = \text{Diam}(\bar{E})$

Proof $\text{Diam}(E) \leq \text{Diam}(\bar{E})$ as $E \subseteq \bar{E}$

Take $\varepsilon > 0$, choose $p \in \bar{E}$, $q \in \bar{E}$. by the definition of \bar{E} , there are points p' , q' in E s.t.
 $d(p, p') < \varepsilon$ and $d(q, q') < \varepsilon$, p' , q' limit points

Hence

$$\begin{aligned} d(p, q) &\leq d(p, p') + d(p', q') + d(q', q) \\ &< 2\varepsilon + d(p', q') \\ &\leq 2\varepsilon + \text{Diam}(\bar{E}) \end{aligned}$$

$$\text{So } \text{Diam}(\bar{E}) \leq 2\varepsilon + \text{Diam}(E)$$

$$\text{choose } \varepsilon = \frac{\text{Diam}(\bar{E}) - r}{2},$$

$$\text{we have } \text{Diam}(\bar{E}) = \text{Diam}(E)$$

Thm (X, d) compact metric space Cauchy sequence converges

Proof Let (p_n) be a Cauchy sequence in X

Consider $E_N = \{p_n : n \geq N\}$

$\text{Diam}(E_N) \xrightarrow{n} 0$ (by Cauchy)

$\text{Diam}(\bar{E}_N) \xrightarrow{n} 0$

$\bar{E}_N \supseteq \bar{E}_{N+1}$

\bar{E}_N closed non- \emptyset ($p_n \in E_N \subseteq \bar{E}_N$)

$\bigcap_N \bar{E}_N \neq \emptyset = \{p\}$ note $\lim_n p_n = p$

March 1st

Corollary In \mathbb{R}^k , Cauchy sequences converge

Proof

Let (a_n) be Cauchy in \mathbb{R}^k . (a_n) is bounded

so (a_n) is contained in $\{x \in \mathbb{R}^k : |x| \leq k\}$

so there is a s.t. $a_n \xrightarrow{n} a$ in \mathbb{R}^k .

So (a_n) converges.

Df Let (s_n) be a sequence in \mathbb{R} . If $s_n \leq s_{n+1}$, we say that (s_n) is monotonically increasing

Thm If (s_n) is a sequence in \mathbb{R} that is monotonic and bounded, then (s_n) converges.

In fact, if (s_n) is increasing, then $s_n \xrightarrow{n} \sup\{s_n : n \in \mathbb{N}\}$

if (s_n) is decreasing, then $s_n \xrightarrow{n} \inf\{s_n : n \in \mathbb{N}\}$

Proof Assume (s_n) is increasing
 Need $s_n \rightarrow \sup \{s_n : n \in \mathbb{N}\} = s$
 Let $\epsilon > 0$ be given
 Let N be s.t. $s - \epsilon < s_N$ (N exists since $s - \epsilon < s$
 $= \sup \{s_n : n \in \mathbb{N}\}$)

Note for $n \geq N$, $s - \epsilon < s_N \leq s_n \leq s < s + \epsilon$

So for $n \geq N$, $|s - s_n| < \epsilon$

Def (X, d) a metric space.

(X, d) is complete if each Cauchy sequence in (X, d) converges

E.g. \mathbb{R}^k is complete

Every metric space lies in a complete metric space
 (only one way to "complete" it)

Upper/lower limits for sequences in \mathbb{R}

Note: $s_n \rightarrow \infty$ if $\forall c \in \mathbb{R} \exists N \in \mathbb{N} \forall n \geq N s_n \geq c$

$s_n \rightarrow -\infty$ if $\forall c \in \mathbb{R} \exists N \in \mathbb{N} \forall n \geq N s_n \leq c$

Fix (s_n)

Define $E = \text{set of all extended Reals } [-\infty, \infty] \text{ of the form } \lim_{k \rightarrow \infty} s_{n_k} \text{ for a subsequence of } (s_n)$

$s^* = \sup E$ (possibly ∞) Note $E \neq \emptyset$

$s^* = \inf E$ (possibly $-\infty$) $\subseteq [-\infty, \infty]$

Note $s^* = \lim_{n \rightarrow \infty} \sup s_n$

$s^* = \lim_{n \rightarrow \infty} \inf s_n$

Thm (a) $s^* \in E$

(b) If $x > s^*$, then there is N s.t. $s_n < x$ for all $n \geq N$
 $s_n > x$

Moreover, s^* is the unique element of $[-\infty, \infty]$
with (a) & (b)
Similarly for s_*

Proof

(a) Let $k < s^*$

There is a subsequence (s_{n_k}) s.t. $\lim_{k \rightarrow \infty} s_{n_k} > k$
so there are arbitrarily large natural numbers n
s.t. $s_n > k$

$s^* \in \mathbb{R}$ we pick $n_0 < n_1 < n_2 < \dots$

$$s_{n_k} > s^* - \frac{1}{k+1}$$

(s_{n_k}) has a subsequence with a limit $(s_{n_{k_i}})$

so $(s_{n_{k_i}})$ is a subsequence of (s_n)

Note $s_{n_{k_i}} > s^* - \frac{1}{k_i+1}$

so $\lim_{i \rightarrow \infty} s_{n_{k_i}} \geq s^* \quad (\text{Sup } \geq)$

so $s^* = \lim_{i \rightarrow \infty} s_{n_{k_i}} \in E$

infinitely many k

\neq

so we want
to use $s_{n_{k_i}}$

(b) Case 1 $s^* \in \mathbb{R}$ let $x > s^*$

need: $\{n : s_n \geq x\}$ is finite. Assume for the contradiction, $\{n : s_n \geq x\}$ is infinite.

so there are $n_0 < n_1 < \dots < n_k$ s.t. $s_{n_k} \geq x$ all k

let $(s_{n_{k_i}})$ be a subsequence of (s_{n_k}) . so if (s_n)

s.t. $\lim_{i \rightarrow \infty} s_{n_{k_i}}$ exists $\lim_{i \rightarrow \infty} s_{n_{k_i}} \geq x > s^* \quad (\text{Sup } >)$

contradiction

Uniqueness:

p, q with (a) & (b) distinct

Assume $p < q$

Choose x s.t. $p < x < q$

By (b) for p , there is N s.t. $s_n < x$ for all $n \geq N$

Then (a) fails for q since there cannot be a subsequence (s_{n_k}) s.t. $s_{n_k} \xrightarrow{k} q$

Ex on Q

Let (s_n) be s.t. $\{s_n : n \in \mathbb{N}\} = Q$

$$E = [-\infty, \infty]$$

$$\limsup_n s_n = \infty \quad \liminf_n s_n = -\infty$$

(2) $s_n = (-1)^n \quad E = \{-1, 1\}$

$$\limsup_n s_n = 1 \quad \liminf_n s_n = -1$$

(3) If $\lim s_n$ exists. Then $\liminf_n s_n = \lim_n s_n = \limsup_n s_n$
 $E = \{\lim_n s_n\}$ iff

Example

(a) $p > 0 \Rightarrow \lim_n \frac{1}{n^p} = 0$

(b) $p > 0 \Rightarrow \lim_n \sqrt[p]{n} = 1$

① $p > 1$ Let $x_n = \sqrt[p]{n} - 1 \geq 0$

$$P = (1 + x_n)^p \geq 1 + px_n$$

$$\text{So } 0 < x_n \leq \frac{p-1}{p} \rightarrow 0 \text{ so } \sqrt[p]{n} \rightarrow 1$$

② $p = 1$ obvious $\sqrt[p]{n} = 1$

③ $p < 1 \Rightarrow p > 1 \Rightarrow \sqrt[p]{n} \rightarrow 1 \quad \sqrt[p]{n} \rightarrow 1 \quad \sqrt[p]{n} \rightarrow 1$

Can we prove the limit by saying $\varepsilon \rightarrow 0$ without any further explanation?

March 19

(c) $\lim_n \sqrt[n]{n} = 1$

$$x_n = \sqrt[n]{n} + 30 \text{ for } n \geq 1$$

$$\frac{n(n+1)}{2} x_n^2 \leq (1+x_n)^n = n$$

$$0 < x_n \leq \sqrt[n+1]{2} \quad n \geq 2$$

so $x_n \xrightarrow{n \rightarrow \infty} 0$ *Achimedean property*

(d) $p > 0, \alpha \neq 0 \Rightarrow \lim_n \frac{n^\alpha}{(1+p)^n} = 0$ ★ Important

$$(1+p)^n > \binom{n}{k} p^k = \underbrace{n(n-1)\dots(n-k+1)}_{k!} p^k$$

$$> \frac{\left(\frac{n}{2}\right)^k}{k!} p^k \quad (n \geq 2k)$$

$$= \frac{p^k}{2^k k!} n^k$$

Take k s.t. $k > \alpha$

$$\text{then for } n \geq 2k \text{ we have } 0 < \frac{n^\alpha}{(1+p)^n} < \frac{n^\alpha}{\left(\frac{p^k}{2^k k!}\right) n^k} = \frac{2^k k!}{p^k} n^{\alpha-k}$$

where $\alpha - k < 0$ so by (a) $\left(\frac{1}{n^\alpha}, p > 0\right)$, $\frac{2^k k!}{p^k} n^{\alpha-k} \xrightarrow{n \rightarrow \infty} 0$

$$\therefore \frac{n^\alpha}{(1+p)^n} \xrightarrow{n \rightarrow \infty} 0$$

(e) $|x| < 1 \Rightarrow \lim_n x^n = 0 \quad \text{Take } \alpha = 0 \text{ in (d)}$

$$\text{then } \lim_n x^n = \lim_n \frac{1}{(\frac{1}{x})^n} = \frac{1}{(1+p)^n} = 0$$

Continuous functions

Units

We have the following situations:

$$(x, d_x), (y, d_y), E \subseteq X, f: E \rightarrow Y$$

Let $p \in X$ be a limit point of E

we write $\lim_{x \rightarrow p} f(x) = q$ ($f(x) \rightarrow q$ as $x \rightarrow p$)

if $\forall \varepsilon > 0, \exists \delta > 0$ st. for all $x \in E$ with $0 < d(x, p) < \delta$ we have $d_y(f(x), q) < \varepsilon$

Both can be definitions,
but the first

one is better as
it can be easily
generalized
to a non-metric
way

Thm $\lim_{x \rightarrow p} f(x) = q$ iff $\lim_n f(p_n) = q$ for each sequence

(p_n) in E s.t. $p_n \neq p$ and $\lim_n p_n = p$

Proof (\Rightarrow)

Let (p_n) be a sequence in E s.t. $p_n \rightarrow p$ $p_n \neq p$

Need: $f(p_n) \rightarrow q$

Let $\varepsilon > 0$ Need N s.t. $d_y(f(p_n), q) < \varepsilon$

Let $\delta > 0$ be s.t. $0 < d_X(x, p) < \delta$

Since $p_n \rightarrow p$, $\exists N$ s.t. $d_X(p_n, p) < \delta$ for $n \geq N$

$d_X(p_n, p) > 0$ as $p_n \neq p$

so for $n \geq N$ $d_Y(f(p_n), q) < \varepsilon$

(\Leftarrow) Assume $f(x) \xrightarrow{n \rightarrow p} q$

so $\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x \in E$ s.t. $0 < d(x, p) < \delta$

and $d_Y(f(x), q) \geq \varepsilon$

Fix $\epsilon > 0$ as in this sentence.

For $\delta = \frac{1}{n+1}$ there is $p_n \in E$

s.t. $0 < d_E(p_n, p) < \delta$ and $d_Y(f(p_n), q) \geq \epsilon$

Since we need $\lim_n f(p_n) = q$, contradiction

Corollary If f has a limit, then the limit is unique.

I
went
to
the
restroom
for
this
point
⋮

Suppose $q \neq q'$

$$\lim_{x \rightarrow p} f(x) = q \quad \lim_{x \rightarrow p} f(x) = q'$$

Actually I have filled all notes I missed!

I just left a larger blank than I expected orz

Def $f, g: E \rightarrow \mathbb{R}$

$$(f+g)(x) = f(x) + g(x)$$

$$(fg)(x) = f(x) \cdot g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ if } g(x) \neq 0 \text{ all } x \in E$$

$$f, g: E \rightarrow \mathbb{R}^k$$

$$(f+g)(x) = f(x) + g(x)$$

$$(f \cdot g)(x) = f(x) \circ g(x)$$

$f, g : E \rightarrow \mathbb{R}$ or $f, g : E \rightarrow \mathbb{R}^k$

Thm $\lim_{x \rightarrow p} f(x) = A \quad \lim_{x \rightarrow p} g(x) = B$

Re $\lim_{x \rightarrow p} (f + g)(x) = A + B$

$$\lim_{x \rightarrow p} (fg)(x) = AB$$

$$\lim_{x \rightarrow p} \left(\frac{f}{g}\right)(x) = \frac{A}{B} \text{ if } B \neq 0$$

Pf x, y metric spaces

$$E \subseteq X, p \in E \quad f: E \rightarrow Y$$

f is continuous at p if $\forall \varepsilon > 0 \exists \delta > 0$ s.t.

for all $x \in E$ with $d_x(x, p) < \delta$, we have $d_Y(f(x), f(p)) < \varepsilon$

Thm f as above. f is continuous at p iff $\lim_{x \rightarrow p} f(x) = f(p)$

If p is not a limit point, we can pick δ st. $d_n(x, x)$ we only have $p = x$

If p is a limit point - - .

Corollary f continuous at p iff for each sequence $p_n \rightarrow p$
 $f(p_n) \xrightarrow{n} f(p)$

we say that $f: E \rightarrow Y$ continuous if it is continuous at each $p \in E$

Thm $f: E \rightarrow Y, g: f(E) \rightarrow Z$, X, Y, Z metric space
Assume f, g continuous, then $g \circ f$ is continuous

Proof: $g \circ f: \delta_2 E \xrightarrow{g \circ f} Z \varepsilon_1$
 $f \xrightarrow{\quad} f(E) \xrightarrow{\quad} g$
 $\varepsilon_2 \leftarrow \delta_1$

so $d_x(x, p) < \delta_2 \Rightarrow d_Z(fog(x), fog(p)) < \varepsilon_1$,

Thm $f, g: X \rightarrow \mathbb{R}$ continuous

then $f+g, f \cdot g, \frac{f}{g}$ continuous Trivial

$f, g: X \rightarrow \mathbb{R}^k$ continuous, $f \cdot g: X \rightarrow \mathbb{R}^k$
 $f \cdot g: X \rightarrow \mathbb{R}^k$ continuous

$f(x) = (f_1(x), \dots, f_k(x))$ $f_i: X \rightarrow \mathbb{R}$

f continuous iff f_1, \dots, f_k continuous

March 21

Thm $f: X \rightarrow Y$

f is continuous iff $f^{-1}(V)$ is open for each $V \subseteq Y$ open

Cor $f: X \rightarrow Y$ then f continuous iff $f(C)$ is closed
 for each $C \subseteq Y$ closed

Do we assume f is one-to-one here?

Proof of Cor from Thm: Assume f is continuous

$C \subseteq Y$ closed $\Leftrightarrow Y \setminus C$ open $\Rightarrow f^{-1}(Y \setminus C)$ open
 $\Rightarrow f^{-1}(Y) \setminus f^{-1}(C) = X \setminus f^{-1}(C) \Leftrightarrow f^{-1}(C)$ is closed

Assume $f^{-1}(C)$ is closed for $C \subseteq Y$ closed

$V \subseteq Y$ open $\Leftrightarrow Y \setminus V$ closed $\Rightarrow f^{-1}(Y \setminus V)$ closed
 $\Rightarrow f^{-1}(Y) \setminus f^{-1}(V)$ closed $= X \setminus f^{-1}(V) = f^{-1}(V)$ open

Proof of the Theorem:

(\Rightarrow) Let $V \subseteq Y$ open. Need $f^{-1}(V)$ open

Let $x \in f^{-1}(V)$ then $f(x) = v$. There is $\varepsilon > 0$ s.t. $N_\varepsilon(f(x)) \subseteq V$. There is $\delta > 0$ s.t. if $x' \in X$ and $d_X(x', x) < \delta$ then $d_Y(f(x'), f(x)) < \varepsilon$

$$f(N_\delta(x)) \subseteq N_\varepsilon(f(x))$$

So $N_\delta(x) \subseteq f^{-1}(N_\varepsilon(f(x))) \subseteq f^{-1}(V)$ So $f^{-1}(V)$ is open

(\Leftarrow) Let $x \in X$. Let $\varepsilon > 0$ be given. Need $\delta > 0$ s.t. if $d_X(x, x') < \delta$ then $d_Y(f(x), f(x')) < \varepsilon$

$$f(N_\delta(x)) \subseteq N_\varepsilon(f(x))$$

Note $N_\varepsilon(f(x))$ is open. By assumption,

$f^{-1}(N_\varepsilon(f(x)))$ open in X . There is $\delta > 0$ s.t.

$$N_\delta(x) \subseteq f^{-1}(N_\varepsilon(f(x)))$$

$$\text{so } f(N_\delta(x)) \subseteq N_\varepsilon(f(x))$$

□

$$x \xrightarrow{f} y \xrightarrow{g} z$$

$$(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$$

Another way to prove the composition of continuous functions is continuous

Image of open/closed set is not necessarily open/closed
preimage is open/closed

Cor $f: X \rightarrow Y$ continuous . X compact then
 $f(X)$ is compact

Proof Need : each open covering of $f(X)$ has a finite
subcovering.

Let $(V_\alpha)_{\alpha \in I}$ be an open covering of $f(X)$

Consider $U_\alpha = f^{-1}(V_\alpha)$ open in X so $(U_\alpha)_{\alpha \in I}$ cover X . $\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} f^{-1}(V_\alpha) = f^{-1}\left(\bigcup_{\alpha \in I} V_\alpha\right) \supset X$

Let $\alpha_1, \dots, \alpha_n \in I$ be s.t. $X = U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$
(since X is compact)

$$X = \bigcup_{i=1}^n U_{\alpha_i} = \bigcup_{i=1}^n f^{-1}(V_{\alpha_i}) = f^{-1}\left(\bigcup_{i=1}^n V_{\alpha_i}\right)$$

$f(X) \subseteq \bigcup_{i=1}^n V_{\alpha_i}$ so $V_{\alpha_1} \dots V_{\alpha_n}$ is a finite subcovering
of $f(X)$

Df (X, d) is a metric s.t. $E \subseteq X$, $f: E \rightarrow \mathbb{R}^k$
 f is bounded if there is a real number M s.t.
 $|f(x)| \leq M$ for all $x \in E$

Cor $f: X \rightarrow \mathbb{R}^k$, X compact. Then $f(X)$ is bounded
and closed in \mathbb{R}^k . In particular, f is bounded.

Proof $f(X)$ is compact as X compact by last theorem.
So closed & bounded by the characterization of
compact subset of \mathbb{R}^k . So f bounded
 X closed bdd $\rightarrow f(X)$ closed bdd

Cor $f: X \rightarrow \mathbb{R}$ is continuous. X compact.

$$M = \sup f(x) \quad m = \inf f(x)$$

Then there are $p, q \in X$ s.t. $f(p) = M$ and $f(q) = m$

Proof: $f(X)$ is bounded and closed

$$\text{So } M = \sup f(x) \in f(X)$$

$$m = \inf f(x) \in f(X)$$

So $\exists p, q \in X$ s.t. $f(p) = M$ and $f(q) = m$

Cor $f: X \rightarrow Y$ continuous bijection

X compact $\Rightarrow f^{-1}: Y \rightarrow X$ is continuous

f in (\mathbb{N} , discrete) is continuous $f(n) = \begin{cases} 0 & \text{if } n=0 \\ n & \text{if } n>0 \end{cases}$

$f: X \rightarrow Y$ compact bijection continuous

$f^{-1}: Y \rightarrow X$ disjoint

Proof Need to show f^{-1} preimage of closed set in X are closed in Y

Same as the image under f of closed sets in X are closed in Y

Image under f compact subsets in X are compact in Y

Uniform continuity $f: X \rightarrow Y$

means $\forall \varepsilon > 0, \exists \delta > 0$ s.t. for all $x, x' \in X$

if $d_X(x, x') < \delta$ then $d_Y(f(x), f(x')) < \varepsilon$

Proof If $f: X \rightarrow Y$ is uniformly distributions

$\forall \varepsilon > 0 \exists \delta > 0$ st.

for all $x, x' \in X$ if $d_X(x, x') < \delta$,

then $d_Y(f(x), f(x')) < \varepsilon$

Ex on $f: \mathbb{R} \rightarrow \mathbb{R}$ $f(x) = x^2$ show it is not uniformly distributed

(2) $\mathcal{G} \{x \in \mathbb{R}; x^2 \leq 0\} = \emptyset$ $g(x) = f_x$ show it is uniformly continuous

Thm $f: X \rightarrow Y$ continuous

X compact $\Rightarrow f$ unif continuous

Proof For $p \in X$, let $\varepsilon(p) > 0$

$$d_X(p, q) < \varepsilon(p) \Rightarrow d_Y(f(p), f(q)) < \frac{\varepsilon}{2}$$

$$J(p) = \{q \in X: d_X(p, q) < \frac{1}{2}\varepsilon(p)\}$$

Note $(J(p))_{p \in X}$ is an open coverage of X

Let $J(p_1), \dots, J(p_n)$ be a finite subcover of X
(X compact)

$$\text{let } S = \frac{1}{2} \min(\varepsilon(p_1), \varepsilon(p_2), \dots, \varepsilon(p_n)) > 0$$

This S works Let $p, p' \in X$ st. $d_X(p, p') < S$

need $d_Y(f(p), f(p')) < \varepsilon$

Then $p \in J(p_m)$ some $m \in \mathbb{N}$

$$\text{so } d_X(p, p_m) < \frac{1}{2}\varepsilon(p_m)$$

$$\text{so } d_X(p', p_m) < d_X(p, p_m) + d_X(p, p') < \frac{1}{2}\varepsilon(p_m) + \frac{1}{2}\varepsilon(p_m) = \varepsilon(p_m)$$

$$\text{so } d_Y(f(p), f(p_n)) < \frac{\epsilon}{2}$$

$$d_Y(f(p_1), f(p_n)) < \frac{\epsilon}{2}$$

$$\text{so } d_Y(f(p), f(p')) < d_Y(f(p), f(p_n)) + d_Y(f(p_n), f(p')) \\ < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

March 26

Example: $f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \quad \text{Dirichlet function}$$

f is not continuous at every point

There is no function that is continuous at every rational point but discontinuous at every irrational points

If f defined on (a, b) , $f(x+) = q$ if

$f(t_n) \rightarrow q$ for each $t_n \rightarrow x$ in (x, b)

$f(x-) = q$ if $f(t_n) \rightarrow q$ for each $t_n \rightarrow x$ in (a, x)

Note: $\lim_{x \rightarrow x} f(x)$ exists iff $f(x-)$ and $f(x+)$ exists and

$f(x-) = f(x+)$ **Read the book**

If $f: (a, b) \rightarrow \mathbb{R}$

is monotonically increasing if $a < x < y < b$ then $f(x) \leq f(y)$
(decreasing) $(f(x) \geq f(y))$

Thm If monotonically increasing in (a, b) then $f(x_+), f(x_-)$ exists at every $x \in (a, b)$.

In fact, $\sup_{a < t < x} f(t) = f(x_-) \leq f(x) \leq f(x_+) = \inf_{x < t < b} f(t)$

Proof $\sup_{a < t < x} f(t) = \sup A_x = l$

$$A_x = \{f(t) : a < t < x\} \neq \emptyset$$

Since $f(\frac{a+x}{2}) \in A_x$. A_x bdd above by $f(x)$

Let $t_n \rightarrow x$ with $t_n \in (a, x)$. Need $f(t_n) \rightarrow l$

Let ϵ be given. Consider $l - \epsilon$ not an upper bound of A_x . So that $a < \bar{t} < x$ s.t. $f(\bar{t}) \rightarrow l - \epsilon$

Then for large n , $\bar{t} < t_n < x$

$$l - \epsilon < f(\bar{t}) \leq f(t_n) \leq l < l + \epsilon$$

$$\text{so } |f(t_n) - l| < \epsilon$$

Cor $f: (a, b) \rightarrow \mathbb{R}$ monotonically increasing

Then f has at most countably many discontinuous points.

Proof Discont points $\exists x \xrightarrow{\text{Hto1}} q_x \in \mathbb{Q} \quad f(x_-) < q < f(x_+)$
 $x < x^1 \rightarrow q_x < f(x^+) \leq f(x^1) < q_{x^1} \rightarrow q_x \neq q_{x^1}$

Done Since \mathbb{Q} is countable

$(\mathbb{Q} \ni q_n \rightarrow \sum_n \frac{1}{2^n} f(x) = \sum_{q_n < x} \frac{1}{2^n}$ uncountable discrete

Thm $f: [a,b] \rightarrow \mathbb{R}$ continuous. let $f(a) < c < f(b)$
 There is $x \in (a,b)$ s.t. $f(x) = c$

Proof let $A = \{r \in (a,b) : f([a,r]) \subseteq (-\infty, c]\}$
 A bdd $A \neq \emptyset$ $a \in A$, $f([a,a]) = \{f(a)\} \subseteq (-\infty, c]$
 so $\sup A$ exists let $x = \sup A$

We claim $f(x) = c$. If not, $f(x) < c$ or $f(x) > c$

\circlearrowleft $f(x) > c$ Consider $\varepsilon = f(x) - (\geq 0)$ find $\delta > 0$ s.t.

for all $y \in (x-\delta, x+\delta)$, we have $f(y) > f(x)-\varepsilon$
 Because f is continuous at x

Take $y \in (x-\delta, x)$, then $f(y) > c$

so $f([a,y]) \not\subseteq (-\infty, c]$

so for all $r \geq y$, $f([a,r]) \not\subseteq (-\infty, c]$ so

$A \subseteq (a,y)$. So $\sup A \leq y < x$ contradiction

\circlearrowleft $f(x) < c$ Consider $\varepsilon = c - f(x) > 0$ Find $\delta > 0$ s.t.

for all $y \in (x-\delta, x+\delta)$, we have $f(y) < f(x)+\varepsilon$

Find $x_0 \in (x-\delta, x)$ s.t. $x_0 \in A$

~~$x_0 \in A$~~
 $x_0 \in (x-\delta, x+\delta)$

Take $r \in (x, x+\delta)$, Then $f([a,r]) \subseteq f([a, x_0])$

$f([a, x_0]) \subseteq (-\infty, c)$ because $x_0 \in A$ $\cup f([x_0, r])$

$f([x_0, r]) \subseteq (-\infty, c)$ so $f([a, r]) \subseteq (-\infty, c)$

so $r \in A$ but $r > x$ contradiction

Leave subsection of ch 4 for study in the exam
 Connectedness is not in the exam.

Differentiability

$f: [a, b] \rightarrow \mathbb{R}$, $x \in [a, b]$ let $\varphi(t) = \frac{f(t) - f(x)}{t - x}$, $a < b$, $t \neq x$

f is differentiable at x if $\lim_{t \rightarrow x} \varphi(t)$ exists

we write $f'(x) = \lim_{t \rightarrow x} \varphi(t)$ — the derivative of f at x .

$E \subseteq [a, b]$, f differentiable on E if f is diff at each $x \in E$

Thm $f: [a, b] \rightarrow \mathbb{R}$ f diff at $x \in (a, b) \Rightarrow f$ is cont at x

Proof $f(t) - f(x) = \frac{f(t) - f(x)}{t - x} (t - x) \xrightarrow[t \rightarrow x]{} f'(x) \cdot 0 = 0$

so $f(t) \rightarrow f(x)$ when $t \rightarrow x$ \square

April 9

Thm $\left. \begin{array}{l} \text{(i)} (f+g)' = f' + g' \\ \text{(ii)} (fg)' = f'g + fg' \\ \text{(iii)} \left(\frac{f}{g}\right)' = \frac{g f' - g' f}{g^2} \end{array} \right\}$ at x

Proof (i) $h = f \cdot g$

$$h(t) - h(x) = f(t)(g(t) - g(x)) + g(x)(f(t) - f(x))$$

$$\begin{aligned} t \neq x \quad \frac{h(t) - h(x)}{t - x} &= f(t) \cdot \frac{g(t) - g(x)}{t - x} + g(x) \cdot \frac{f(t) - f(x)}{t - x} \\ h'(x) &= \downarrow_{t \rightarrow x} f(x) \cdot g'(x) + g(x) \cdot f'(x) \end{aligned}$$

$$(iii) \quad h = \frac{f}{g}$$

$$\begin{aligned} \frac{h(t) - h(x)}{t - x} &= \frac{1}{g(t) \cdot g(x)} \left[g(x) \cdot \frac{f(t) - f(x)}{t - x} - f(x) \frac{g(t) - g(x)}{t - x} \right] \\ &= \underbrace{g(x) \cdot f'(x)}_{g'(x)} - f(x) \cdot g'(x) \end{aligned}$$

Thm f continuous on $[a, b]$, $f'(x)$ exists at $x \in [a, b]$,
 g defined on interval containing the image of f .
 g differentiable at $f(x)$. If $h = g \circ f$ in $[a, b]$,
then h is differentiable at x and $h'(x) = g'(f(x)) \cdot f'(x)$

Proof

$$\frac{h(t) - h(x)}{t - x} = \frac{g(f(t)) - g(f(x))}{t - x} = \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \cdot \frac{f(t) - f(x)}{t - x}$$

can't be 0
can't work

$$f(t) - f(x) = (t - x) \cdot (f'(x) + u(t)) \quad u(t) \rightarrow 0 \text{ as } t \rightarrow x$$

$$g(s) - g(y) = (s - y) \cdot (g'(y) + v(s)) \quad v(s) \rightarrow 0 \text{ as } s \rightarrow y$$

where $y = f(x)$

$$\begin{aligned} h(t) - h(x) &= g(f(t)) - g(f(x)) \\ &= [f(x) - f(x)] (g'(f(x)) + v(f(t))) \\ &= (t - x) (f'(x) + u(t)) (g'(f(x)) + v(f(t))) \\ &= (t - x) \left[f'(x) g'(f(x)) + f'(x) v(f(t)) + g'(f(x)) u(t) + u(t) v(f(t)) \right] \\ \frac{h(t) - h(x)}{t - x} &= f'(x) g'(f(x)) \end{aligned}$$

$\downarrow t \rightarrow x$
 $\downarrow f(t) \rightarrow f(x) = y$
 $v(f(t)) = 0$
 $u(t) = 0$
 \downarrow

$$\underline{\text{Ex (1)}} \quad f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

- ① f is continuous at all x
 ② f differentiable at each $x \neq 0$

$$\begin{aligned} f'(x) &= x' \sin \frac{1}{x} + x \cdot (\sin \frac{1}{x})' \\ &\stackrel{x \neq 0}{=} \sin \frac{1}{x} + x \cdot \cos \left(\frac{1}{x}\right) \cdot \left(\frac{1}{x}\right)' \\ &= \sin \frac{1}{x} - \frac{1}{x^2} \cos \left(\frac{1}{x}\right) \end{aligned}$$

$$x=0 \quad \frac{f(t)-f(0)}{t-0} = \frac{t \sin \frac{1}{t}}{t} = \sin \frac{1}{t} \rightarrow \text{diverges as } t \rightarrow 0$$

so $f'(0)$ does not exist

$$\underline{\text{Ex (2)}} \quad g(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

- ① g is continuous at all x
 ② $g'(x)$ differentiable at $x \neq 0$

$$\frac{g(t)-g(0)}{t-0} = t \frac{\sin \frac{1}{t}}{t} = t \sin \frac{1}{t} \xrightarrow[t \rightarrow 0]{} 0$$

so $g'(0)$ exists $\lim_{t \rightarrow 0} g'(t) = 0$

Df $f: X \rightarrow \mathbb{R}$, $p \in X$ — (X, d) metric space
 f has a local maximum at p if $\exists \delta > 0$ s.t.
 $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$

Thm $f: [a, b] \rightarrow \mathbb{R}$ f has a local maximum at $x \in [a, b]$ and $f'(x)$ exists, then $f'(x) = 0$. Same with local minimum

April 11

Proof

Fix $\delta > 0$ s.t. $a - \delta < x < x + \delta < b$
and $f(t) \leq f(x)$ all $x < t < x + \delta$

$$\frac{f(t) - f(x)}{t - x} \geq 0 \text{ for } x < t < x + \delta$$

$$\frac{f(t) - f(x)}{t - x} \leq 0 \text{ for } x < t < x + \delta$$

Since $\lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} = \lim_{t \rightarrow x^+} \frac{f(t) - f(x)}{t - x} = \lim_{t \rightarrow x^-} \frac{f(t) - f(x)}{t - x}$

$$0 \leq \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} \leq 0 \quad \text{So } f'(x) = 0$$

Thm $f, g : [a, b] \rightarrow \mathbb{R}$ continuous and differentiable on (a, b)
there is $x \in (a, b)$ s.t. $(f(b) - f(a))g'(x) = (g(b) - g(a)) \cdot f'(x)$
A particular case: take $g(x) = x$ mean value theorem

$$f(b) - f(a) = (b - a) \cdot f'(x)$$

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

$$f'(x^n) = \lim_{t \rightarrow x} \frac{t^n - x^n}{t - x} = \lim_{t \rightarrow x} \frac{(t-x)(t^{n-1} + t^{n-2}x + \dots + x^{n-2} + x^{n-1})}{t - x} = nx^{n-1}$$

$$P(x) = a_0 + a_1 x + \dots + a_n x^n \quad P'(x) = a_1 + 2a_2 x + \dots + na_n x^{n-1}$$

Proof Let $h(t) = (f(b) - f(a))g(t) - (g(b) - g(a))f(t)$

need: $h'(x) = 0$ for some $x \in (a, b)$

Note: h cont on $[a, b]$, diff on (a, b)

$$\begin{aligned} \textcircled{2} \quad h(a) &= (f(b) - f(a))g(a) - (g(b) - g(a))f(a) \\ &= f(b)g(a) - f(a)g(a) - g(b)f(a) + g(a)f(a) \\ &= f(b)g(a) - g(b)f(a) \end{aligned}$$

$$h(b) = g(a)f(b) - f(a)g(b)$$

$$\text{So } h(a) = h(b)$$

Case 1: $h(t)$ constant on $[a, b]$

$$\text{So } h'(x) = 0 \text{ for } x = \frac{a+b}{2}$$

Case 2: $h(t) \rightarrow h(a) = h(b)$ some $t \in [a, b]$

Then let $x \in [a, b]$ be s.t. $h(x) = \sup \{h(t) : t \in [a, b]\}$

Also x is a local max so $f'(x) = 0$

Case 3: $h(t) < h(a) = h(b)$ for some $t \in [a, b]$

Then similar to case 2.

Theorem f diff on (a, b)

a) If $f'(x) \geq 0$ all $x \in (a, b)$ then f is monotonically increasing

b) If $f'(x) = 0$ all $x \in (a, b)$ then f is constant

c) If $f'(x) \leq 0$ all $x \in (a, b)$ then f is monotonically decreasing

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(x) \text{ for some } x_1 < x < x_2$$

$x_1 < x_2, x_1, x_2 \in (a, b)$ arbitrary

$$\text{a) } f'(x) \geq 0 \Rightarrow f(x_2) - f(x_1) \geq 0 \quad f(x_2) \geq f(x_1)$$

$$\text{b) } f'(x) = 0 \Rightarrow f(x_2) - f(x_1) = 0 \quad f(x_2) = f(x_1)$$

$$\text{c) } f'(x) \leq 0 \Rightarrow f(x_2) - f(x_1) \leq 0 \quad f(x_2) \leq f(x_1)$$

Thm f diff on $[a, b]$, assume $f'(a) < \lambda < f'(b)$

There is $x \in (a, b)$ s.t. $f'(x) = \lambda$ (Intermediate Value theorem)

Proof Consider $g(t) = f(t) - \lambda t$

Then $g'(t) = f'(t) - \lambda$

Then $g'(a) = f'(a) - \lambda < 0$

so $g(t_1) < g(a)$ for some $t_1 \in (a, b)$

$$g'(a) = \lim_{t \rightarrow a^+} \frac{g(t) - g(a)}{t - a} < 0$$

$$g'(b) = f'(b) - \lambda > 0 \quad g'(b) = \lim_{t \rightarrow b^-} \frac{g(t) - g(b)}{t - b} > 0$$

so $g(t_2) < g(b)$ for some $t_2 \in (a, b)$

Note: There is $x \in [a, b]$ s.t. $g(x) = \inf \{g(t) : t \in [a, b]\}$
note $x \in (a, b)$

Then $g'(x) = 0$ so $f'(x) - \lambda = 0 \quad f'(x) = \lambda$

Ch 3: 1b(a,b), 20, 21, 22 (not turn in)

Ch 4: 4, 10, 12, 13, 14, 15, 17, 20, 24

I. Kaplansky "set theory and metric space" ch 4&5

April 16

Taylor's Theorem

$f : [a, b] \rightarrow \mathbb{R}, n \geq 0, n \in \mathbb{N}$

$f^{(n+1)}$ continuous on $[a, b]$. $f^{(n)}$ exist at each $t \in (a, b)$
 $\alpha, \beta \in [a, b], \alpha \neq \beta$ Let $P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t-\alpha)^k$

Then there exists a x between α and β s.t.

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!} (\beta - \alpha)^n$$

Proof Let M s.t. $f(\beta) = P(\beta) + M(\beta-\alpha)^n$

$$\text{so } M = \frac{f(\beta) - P(\beta)}{(\beta-\alpha)^n}$$

Need: $M = \frac{f^{(n)}(x)}{n!}$ some x between α & β

i.e. $n!M = f^{(n)}(x)$

Def $g(t) = f(t) - P(t) - M(t-\alpha)^n \quad \alpha \leq t \leq b$

so suffice to show that $g^n(x) = 0$ some x between α & β

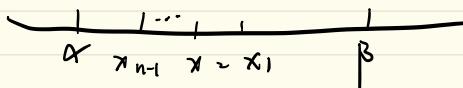
$$g^n(t) = f^n(t) - 0 - n! \cdot M = f^n(t) - n! \cdot M$$

$$g^n(x) = 0 \Rightarrow f^n(x) = n!M \Rightarrow M = \frac{f^n(x)}{n!}]$$

Note $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k < n$

so $g(\alpha) = g'(\alpha) \dots = g^{n-1}(\alpha) = 0$

$g(\beta) = 0$ by our choice of M



$$g(\alpha) = 0 \quad g(\beta) = 0$$

$$g'(x) = 0 \quad g'(x_1) = 0 \quad \dots$$

$$g^{n-1}(x) = 0 \quad g^{n-1}(x_2) = 0$$

$$g^{n-1}(x) = 0 - g^{n-1}(x_{n-1}) = 0 \quad g^n(x_{n-1}) = 0$$

Thm (L'Hospital's rule)

f, g diff on (a, b) , $g'(x) \neq 0$ all $x \in (a, b)$

Assume $\frac{f'(x)}{g'(x)} \xrightarrow{x \rightarrow a} A$

If $f(x) \rightarrow 0$ and $g(x) \rightarrow 0$ as $x \rightarrow a$] or

[$f(x) \rightarrow \infty$ and $g(x) \rightarrow \infty$ as $x \rightarrow a$]

then $\frac{f(x)}{g(x)} \xrightarrow{x \rightarrow a} A$

Proof: Note $g(x) \neq g(y)$ all $x, y \in (a, b)$

$g(y) = 0$ for at most one y in (a, b)

so we can. and will assume $g(y) \neq 0$ all $y \in (a, b)$

Note: for all $x < y$ in (a, b) we have

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} \text{ some } t \text{ when } x \neq y$$

Fix $\varepsilon > 0$ Need $A - \varepsilon \leq \frac{f(x)}{g(x)} \leq A + \varepsilon$ if x is close to a

Find c.s.t. $a < c < b$

$$A - \varepsilon < \frac{f'(t)}{g'(t)} < A + \varepsilon \text{ if } t \in (a, c)$$

if $a < x < y < c$, then $\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)}$ some t s.t. $x < t < y$

keep y fix and let $x \rightarrow a$ then

$$\frac{f(x) - f(y)}{g(x) - g(y)} \rightarrow \frac{f(y)}{g(y)} \text{ so } A - \varepsilon \leq \frac{f(y)}{g(y)} \leq A + \varepsilon$$

so for all $a < y < c$ we have $A - \varepsilon < \frac{f(y)}{g(y)} < A + \varepsilon$

for $[g(x) \rightarrow +\infty \text{ } f(x) \rightarrow +\infty \text{ } x \rightarrow a]$

$$\frac{f(x)-f(y)}{g(x)-g(y)} \cdot \frac{g(x)-g(y)}{g(x)} \text{ we get } \frac{f(x)}{g(x)} - \frac{f(y)}{g(x)} = \frac{f'(t)}{g'(t)} \cdot \frac{g(x) \cdot g(y)}{g(x)}$$

$$\frac{f(x)}{g(x)} = \frac{f(y)}{g(x)} + \frac{f'(t)}{g'(t)} - \frac{f'(t)}{g'(t)} \frac{g(y)}{g(x)}$$

$\exists c \text{ s.t. } a < c < b \text{ &}$

$$A - \varepsilon \leq \frac{f(t)}{g(t)} \leq A + \varepsilon \text{ for } t \text{ in } (a, c)$$

keep y fixed s.t. y is in (a, c) e.g. $y = \frac{a+c}{2}$

$$\text{then } \frac{f(y)}{f(x)} \rightarrow 0 \quad \frac{g(y)}{g(x)} \rightarrow 0$$

$$A - \varepsilon \leq \liminf_{x \rightarrow a} \frac{f(x)}{g(x)} \leq \limsup_{x \rightarrow a} \frac{f(x)}{g(x)} \leq A + \varepsilon$$

Then above holds for each $\varepsilon > 0$

$$\text{So } \liminf_{x \rightarrow a} \frac{f(x)}{g(x)} = \limsup_{x \rightarrow a} \frac{f(x)}{g(x)} = A$$

$$\text{So } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = A$$

Riemann - Stieltjes integral

$f: [a, b] \rightarrow \mathbb{R}$ bounded

$g: [a, b] \rightarrow \mathbb{R}$ monotonically increasing

P a partition of $[a, b]$

$$\{x_0 < x_1 < \dots < x_{n-1} < x_n\}$$

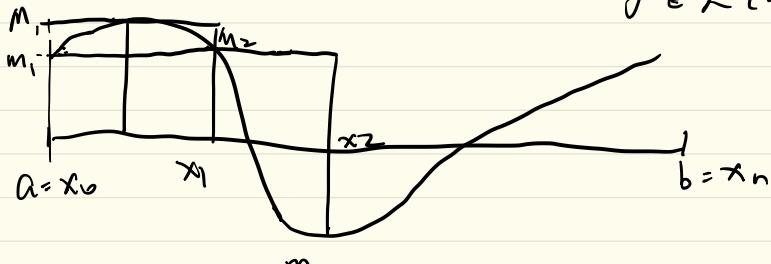
$$\Delta x_i = g(x_i) - g(x_{i-1})$$

$$U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta x_i \quad M_i = \sup f(x) \quad x_{i-1} \leq x \leq x_i$$

$$L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta x_i \quad m_i = \inf f(x) \quad x_{i-1} \leq x \leq x_i$$

$$\overline{\int}_a^b f d\alpha = \inf_P U(P, f, \alpha) \quad f \text{ is RS integral of } \alpha$$

$$\underline{\int}_a^b f d\alpha = \sup_P L(P, f, \alpha) \quad \text{if } \overline{\int}_a^b = \underline{\int}_a^b \\ f \in R(\alpha)$$



$$\underline{\int}_a^b f d\alpha = f(x_0)$$

For $\alpha(x) = x$ all $x \in [a, b]$,

The integral is called the Riemann integral $\Delta x_i = x_i - x_{i-1}$

$$\overline{\int}_a^b f d\alpha = \inf_P U(P, f, \alpha)$$

$$\underline{\int}_a^b f d\alpha = \sup_{\alpha} L(P, f, \alpha)$$

If $\overline{\int}_a^b f d\alpha = \underline{\int}_a^b f d\alpha$, we say f is R-S and bounded on

Ex $\alpha(x) = x$

$$f(x) = \begin{cases} 1 & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$$

$$f: [0,1] \rightarrow \mathbb{R} \quad \alpha: [0,1] \rightarrow [0,1]$$

Check $\int_a^b f d\alpha = 1 \quad \int_a^b f d\alpha = 0 \quad \text{so } f \text{ is not integrable}$
 $(m_i = 1)$

P, P^* partition of $[a,b]$, we say P^* is a refinement of P
if $P \subseteq P^*$

Theorem If P^* is a refinement of P . then

$$L(P, f, \alpha) \leq L(P^*, f, \alpha)$$

$$U(P, f, \alpha) \geq U(P^*, f, \alpha)$$

Corollary $\int_a^b f d\alpha \leq \int_a^b f d\alpha$

Proof P_1, P_2 two partition. Let $P^* = P_1 \cup P_2$ a refinement
of P_1, P_2 $L(P, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha)$
 $\leq U(P_2, f, \alpha)$

So for all partition P_1, P_2 , we have

$$L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$$

$$\text{So } \underline{\int_a^b} f d\alpha = \sup_{P_i} L(P_i, f, \alpha) \leq U(P_i, f, \alpha)$$

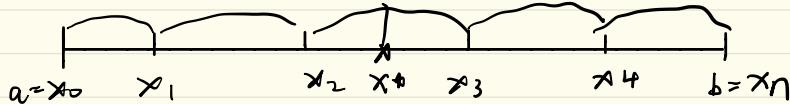
$$\text{So } \underline{\int_a^b} f d\alpha \leq \inf_{P_i} U(P_i, f, \alpha) = \overline{\int_a^b} f d\alpha$$

Proof of the Theorem

$$\text{we do only } L(P, f, \alpha) \leq L(P^*, f, \alpha)$$

By induction on the number of elements in P^* , P

Need to consider $P^* = P \cup \{x^*\}$, $x^* \notin P$



Let x^* st. $x^{i-1} < x^* < x^i$

$$w_1 = \inf_{x_i \leq x \leq x^*} f(x) \quad w_2 = \inf_{x^* \leq x \leq x_i} f(x)$$

Note: $w_1 \geq m_i$; $w_2 \geq m_i$

$$\begin{aligned} L(P^*, f, \alpha) - L(P, f, \alpha) &= w_1 (\alpha(x^*) - \alpha(x_{i-1})) + w_2 (\alpha(x_i) - \alpha(x^*)) \\ &\quad - m_i (\alpha(x_i) - \alpha(x_{i-1})) \\ &= (w_1 - m_i)(\alpha(x^*) - \alpha(x_{i-1})) + \\ &\quad (w_2 - m_i)(\alpha(x_i) - \alpha(x^*)) \end{aligned}$$

Thm $f \in R(\alpha)$ on $[a, b]$ iff $\forall \varepsilon > 0 \exists P$ a partition of $[a, b]$ s.t. $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

Proof \leftarrow $L(P, f, \alpha) \leq \int_a^b f d\alpha = \bar{\int}_a^b f d\alpha \leq U(P, f, \alpha)$ (*)
 If $\exists P$ s.t. (*) holds, then we have

$$\bar{\int} - \underline{\int} < \varepsilon \quad (\varepsilon \text{ can be arbitrarily small})$$

$$\text{so } 0 \leq \bar{\int} - \underline{\int} \leq 0 \quad \text{so } \bar{\int} = \underline{\int}. \quad f \in R(\alpha)$$

\Rightarrow Assume $\bar{\int} = \underline{\int} = \int$

Find partitions P_1, P_2 s.t.
 $U(P_2, f, \alpha) - \int < \frac{\varepsilon}{2}$
 $\int - L(P_1, f, \alpha) < \frac{\varepsilon}{2}$

$$\text{so } U(P_2, f, \alpha) < L(P_1, f, \alpha) + \varepsilon$$

$$\text{Let } P = P_1 \cup P_2$$

$$\text{Then } U(P, f, \alpha) \leq U(P_2, f, \alpha) < L(P_1, f, \alpha) + \varepsilon$$

$$L(P, f, \alpha) \geq L(P_1, f, \alpha) \leq L(P_1, f, \alpha) + \varepsilon$$

$$\text{so } U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

Thm $f: [a, b] \rightarrow \mathbb{R}$ continuous, then $f \in R(\alpha)$

Proof Pick P a partition of $[a, b]$

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i$$

Given $\varepsilon > 0$, let $\delta > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon \text{ if } |x - y| < \delta \text{ for}$$

$x, y \in [a, b]$ by uniformly continuous f on compact $[a, b]$

Let P s.t. $x_i - x_{i-1} < \delta$ for all i

Then for x, y s.t. $x_{i-1} \leq x, y \leq x_i$

$$\text{we have } |f(x) - f(y)| < \varepsilon$$

$$\text{so } m_i - m_{i-1} < \varepsilon$$

$$\begin{aligned} \text{so } \sum_{i=1}^n (m_i - m_{i-1}) \Delta x_i &\leq \sum_{i=1}^n \varepsilon \Delta x_i \\ &= \varepsilon \sum_{i=1}^n \Delta x_i \\ &= \varepsilon (q(b) - q(a)) \quad \int f d\alpha \end{aligned}$$

Thm $f: [a, b] \rightarrow \mathbb{R}$ monotonic, α continuous $\Rightarrow f \in R(\alpha)$

Proof $U(P, f, \alpha) - L(P, f, \alpha)$ Assume f is monotonically increasing

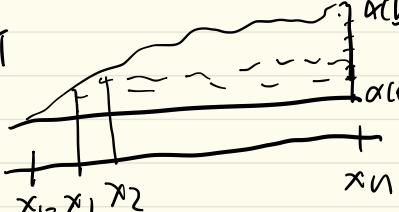
$$= \sum_{i=1}^n (m_i - m_{i-1}) \Delta x_i$$

$$= \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \Delta x_i$$

Fix n . Find $a = x_0 < x_1 \dots < x_n = b$

$$\text{s.t. } \alpha(x_i) - \alpha(x_{i-1}) = \frac{\alpha(b) - \alpha(a)}{n}$$

IMT



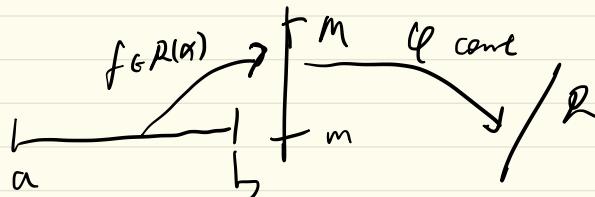
For this P ,

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \frac{\alpha(b) - \alpha(a)}{n} \\ &= (f(x_n) - f(x_0)) \frac{\alpha(b) - \alpha(a)}{n} \end{aligned}$$

Given $\varepsilon > 0$, find N s.t. when $n \geq N$,

$$\frac{(f(b) - f(a))(x(b) - x(a))}{n} < \varepsilon$$

Thm $f \in R(\alpha)$ on $[a, b]$, $m \leq f(t) \leq M$ all $t \in [a, b]$
 Let φ continuous on $[m, M]$ and let $h = \varphi \circ f$
 $(h(x) = \varphi(f(x)))$ Then $h \in R(\alpha)$ on $[a, b]$



Proof Let $\varepsilon > 0$. Since $[m, M]$ is compact,
 φ is uniformly continuous.
 So there is $\delta > 0$ s.t. for all $s, t \in [m, M]$,
 if $|t - s| < \delta$, $|\varphi(t) - \varphi(s)| < \varepsilon$
 (can assume $\delta < \varepsilon$)
 Since $f \in R(\alpha)$, there is $P = \{x_0 \underset{\alpha}{\llcorner} \dots \llcorner x_b\}$

$$\text{s.t. } U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon^2$$

$$m_i = \inf_{t \in [x_{i-1}, x_i]} f(t) \quad M_i = \sup_{t \in [x_{i-1}, x_i]} f(t)$$

$$m_i^* = \inf_{t \in [x_{i-1}, x_i]} \varphi(f(t)) \quad M_i^* = \sup_{t \in [x_{i-1}, x_i]} \varphi(f(t))$$

Let $i \in \{1, \dots, n\}$
 $i \in A : \text{iff } m_i - m_i^* < \delta$
 $i \in B : \text{iff } M_i - m_i^* \geq \delta$

$$\{1, \dots, n\} = A \cup B$$

If $i \in A$, then $m_i^* - m_i^* \leq \varepsilon$

If $i \in B$, then ① $m_i^* - m_i^* \leq 2K$ when $K = \sup_{a \leq t \leq b} |f(t)|$
 $= \sup_{m \leq t \leq M} f(t) - \inf_{m \leq t \leq M} f(t)$ and not 0

$$\textcircled{2} \quad \delta \sum_{i \in B} \Delta \alpha_i \leq \sum_{i \in B} (M_i - m_i) \Delta \alpha_i \leq U(P, f, \alpha) - L(P, f, \alpha) \leq \delta^2$$

$$\text{So } \sum_{i \in B} \Delta \alpha_i < \delta$$

$$\text{So } U(P, h, \alpha) - L(P, h, \alpha)$$

$$= \sum_{i \in A} (M_i^* - m_i^*) \Delta \alpha_i + \sum_{i \in B} (M_i^* - m_i^*) \Delta \alpha_i$$

$$\leq \sum_{i \in A} \varepsilon \Delta \alpha_i + 2K \sum_{i \in B} \Delta \alpha_i$$

$$\leq \varepsilon \sum_{i=1}^n \Delta \alpha_i + 2K \delta$$

$$< \varepsilon (\alpha(b) - \alpha(a)) + 2K \delta$$

Thm (a) $f_1, f_2 \in R(\alpha)$ on $[a, b] \Rightarrow f_1 + f_2 \in R(\alpha)$

$f \in R(\alpha)$ on $[a, b] \Rightarrow cf \in R(\alpha)$ for each

$$c \in \mathbb{R}$$
 and $\int f_1 + f_2 d\alpha = \int f_1 d\alpha + \int f_2 d\alpha$. $\int c f d\alpha = c \int f d\alpha$

(b) $f_1, f_2 \in R(\alpha)$, $f_1 \leq f_2 \Rightarrow \int f_1 d\alpha \leq \int f_2 d\alpha$

(c) $f \in R(\alpha)$ on $[a, b] \Rightarrow f \in R(\alpha)$ on $[a, c]$ and $[c, b]$
 $a < c < b$ and $\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha$

(d) $f \in R(\alpha)$ on $[a, b]$

$$|f| \leq M \Rightarrow \left| \int_a^b f d\alpha \right| \leq M(\alpha(b) - \alpha(a))$$

(e) $f \in R(\alpha_1), f \in R(\alpha_2)$

↓

$$f \in R(\alpha_1 + \alpha_2) \text{ and } \int f d(\alpha_1 + \alpha_2) = \int f d\alpha_1 + \int f d\alpha_2$$

$$f \in R(\alpha) - c > 0, f \in R(c\alpha) \text{ and } \int f d(c\alpha) = c \int f d\alpha$$

Proof

(a) see $f = f_1 + f_2$

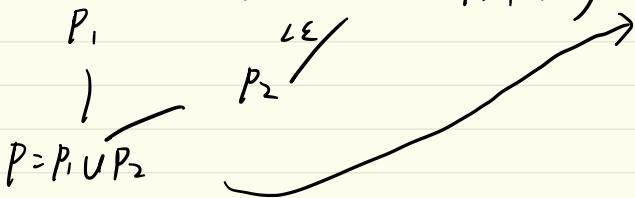
P a partition of $[a, b] \in \mathbb{R}$

$$L(P, f_1, \alpha) + L(P, f_2, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha)$$

$$\leq U(P, f_1, \alpha) + U(P, f_2, \alpha)$$

$$\text{so } U(P, f, \alpha) - L(P, f, \alpha) \leq U(P, f_1, \alpha) - L(P, f_1, \alpha) +$$

$$\cancel{U(P, f_2, \alpha) - L(P, f_2, \alpha)} < 2\varepsilon$$



so $f \in R(\alpha)$

same argument shows $\int_a^b f d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$

Thm ① $f \in R(\alpha), g \in R(\alpha)$ on $[a, b] \Rightarrow fg \in R(\alpha)$

② $f \in R(\alpha) \Rightarrow |f| \in R(\alpha) \text{ & } |\int f d\alpha| = \int |f| d\alpha$

Proof of $f, g \in R(\alpha)$

$$fg = \frac{1}{4} ((f+g)^2 - (f-g)^2)$$

so $fg \in R(\alpha)$ if

$$(f+g)^2 \text{ & } (f-g)^2 \in R(\alpha)$$

Need: $h \in R(\alpha) \Rightarrow h^2 \in R(\alpha)$

but $h^2 = \varphi \circ h$ where $\varphi(y) = y^2$ so $h^2 \in R(\alpha)$

② If $|f| = \varphi \circ f$ $\varphi(y) = |y|$

so $f \in R(\alpha) \Rightarrow |f| \in R(\alpha)$

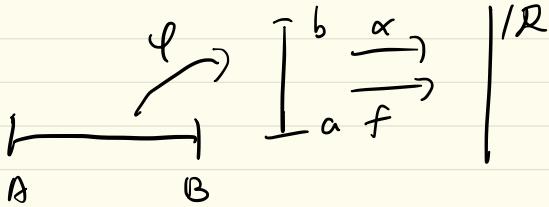
Let $c = \pm 1$ so that $c \int f d\alpha \geq 0$ then

$$|\int f d\alpha| = c \int f d\alpha = \int cf d\alpha = \int |f| d\alpha$$

Thm $\varphi: [A, B] \rightarrow [a, b]$ strictly increasing and continuous (one-to-one) (just one-to-one?) and onto
 $\alpha: [a, b] \rightarrow \mathbb{R}$ monotonically increasing

$f \in \mathcal{R}(A)$ on $[a, b]$

let $\beta = \alpha \circ \varphi$. $g = f \circ \varphi$



Then $g = R(\beta)$ and

$$\int_A^B g d\beta = \int_a^b f d\alpha$$

Proof

$P = \{x_0, \dots, x_n\}$ a part of $[a, b]$

$Q = \{y_0, \dots, y_n\}$ a partition of $[A, B]$

s.t. $\varphi(y_i) = x_i$. $i = 0, \dots, n$

Every partition Q of $[A, B]$ is obtained in this way
 from some partitions P of $[a, b]$

$Q = \{y_0, \dots, y_n\} \rightarrow P = \{x_0, \dots, x_n\}$ by $x_i = \varphi(y_i)$

Note: $U(Q, g, \beta) = U(P, f, \alpha)$ - values of f on $[x_{i-1}, x_i]$
 $\qquad\qquad\qquad$ = values of g on $[y_{i-1}, y_i]$
 $\qquad\qquad\qquad$ ($f \circ \varphi$)

$$\text{and } \alpha(x_i) - \alpha(x_{i-1}) = \varphi(\varphi(y_i)) - \varphi(\varphi(y_{i-1})) \\ = \beta(y_i) - \beta(y_{i-1})$$

$$(\sup_{x \in [x_{i-1}, x_i]} f(x))(\alpha(x_i) - \alpha(x_{i-1})) = (\sup_{y \in [y_{i-1}, y_i]} g(y))(\beta(y_i) - \beta(y_{i-1}))$$

Similarly $L(Q, g, \beta) = L(P, f, \alpha)$

Since $f \in \mathcal{L}(\alpha)$, we have $g \in \mathcal{L}(\beta)$

Also $\int_a^b f d\alpha = \int_A^B g d\beta$

+ each α comes from a P
+ integrability
 \Downarrow

$$\int_a^b f d\alpha = \int_A^B g d\beta$$

Notation: $\mathcal{R}(x) = \mathbb{R}$ if $\alpha(x) = x$

If $\alpha(x) = x$, then $\beta = \varphi$ and

$$\int_A^B g d\varphi = \int_a^b f dx \quad (\text{special case})$$

Thm $\alpha: [a, b] \rightarrow \mathbb{R}$ monotonically increasing differentiable
 $\alpha' \in \mathcal{R}$ on $[a, b]$ if b dd on $[a, b]$

Then $f \in \mathcal{L}(\alpha)$ iff $f \cdot \alpha'$ $\in \mathcal{R}$

and $\int_a^b f d\alpha = \int_a^b f(x) \alpha'(x) dx$

Proof Some $\alpha' \in \mathbb{R}$. given $\varepsilon > 0$. we can find

$$\textcircled{1} \quad P = \{x_0^0, \dots, x_n^0\} \text{ s.t.}$$

$$U(P, \alpha') - L(P, \alpha') < \varepsilon$$

② By the mean value theorem, we get

$$t_i \in [x_{i-1}, x_i] \text{ s.t. } \Delta x_i = \alpha'(t_i) \Delta x_i$$

③ (From ①)

$$\sum_{i=1}^n |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i < \varepsilon \text{ for each } s_i \in [x_{i-1}, x_i]$$

$$(\alpha'(s_i) \leq M_i, \alpha'(t_i) \geq m_i, \text{ so } |\alpha'(s_i) - \alpha'(t_i)| \leq M_i - m_i)$$

for each $s_i \in [x_{i-1}, x_i]$

$$\begin{aligned} \textcircled{4} \quad \sum_{i=1}^n f(s_i) \Delta x_i &= \sum_{i=1}^n f(s_i) \alpha'(t_i) \Delta x_i \\ &= \sum_{i=1}^n f(s_i) \Delta x_i - \sum_{i=1}^n f(s_i) \alpha'(s_i) \Delta x_i \\ &= \sum_{i=1}^n f(s_i) (\alpha'(t_i) - \alpha'(s_i)) \Delta x_i \\ &\leq \sum_{i=1}^n |f(s_i)| |\alpha'(t_i) - \alpha'(s_i)| \Delta x_i \\ &\leq M \sum_{i=1}^n |\alpha'(t_i) - \alpha'(s_i)| \Delta x_i, \quad M = \sup_{x \in [a, b]} f(x) \text{ as } f \text{ bdd} \end{aligned}$$

$$< M \cdot \varepsilon$$

So there is $P = \{x_0 < x_1 \dots < x_n\}$ s.t.

$$\left| \sum_{i=1}^n f(s_i) \Delta x_i - \sum_{i=1}^n f(s_i) \alpha'(s_i) \Delta x_i \right| < M \cdot \varepsilon \text{ all } s_i \in [x_{i-1}, x_i]$$

$$\begin{aligned} \text{So } \sum_{i=1}^n f(s_i) \Delta x_i &\leq \sum_{i=1}^n f(s_i) \alpha'(s_i) \Delta x_i + M \varepsilon \leq U(P, f, \alpha') + M \varepsilon \\ U(P, f, \alpha) &\leq U(P, f, \alpha') + M \varepsilon \text{ since } s_i \in [x_{i-1}, x_i] \text{ arbitrary} \end{aligned}$$

By the same argument,

$$U(P, f \cdot \alpha') \leq U(P, f, \alpha) + M\epsilon$$

$$\text{so } |U(P, f, \alpha) - U(P, f, \alpha')| \leq M\epsilon$$

holds for all partitions finer than the partition with fixed

$$\text{so } |\bar{\int} f d\alpha - \bar{\int} f \cdot \alpha' dx| \leq M\epsilon$$

$$\text{so } \int f d\alpha = \int (f \cdot \alpha')(x) dx$$

By an analogous argument,

$$\int f d\alpha = \int (f \cdot \alpha')(x) dx$$

$$\text{so } f \in \mathcal{R}(\alpha) \text{ iff } f \cdot \alpha' \in \mathcal{R}$$

April 30

Thm $f \in \mathcal{L}$ on $[a, b]$

$$\text{Let } F(x) = \int_a^x f(t) dt \text{ for } x \in [a, b] \quad \begin{array}{c} f \\ \int_a^x \\ a \quad x \quad b \end{array}$$

Then $F(x)$ is continuous. If f is continuous at $x_0 \in [a, b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$ (so if f continuous on $[a, b]$, then F differentiable on $[a, b]$ and $F' = f$)

Proof For $a \leq x \leq y \leq b$

$$F(y) - F(x) = \int_a^y f(t) dt - \int_a^x f(t) dt = \int_x^y f(t) dt$$

F continuous. Let $M = \sup_{t \in [a, b]} |f(t)| < \infty$

$$|F(y) - F(x)| = \left| \int_x^y f(t) dt \right| \leq \int_x^y |f(t)| dt \leq M(y-x)$$

Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{M}$

Then if $x \leq y$ & $|y-x| < \delta$,

$$\text{Then } |F(y) - F(x)| < M|y-x| < M\delta = \epsilon$$

If $y < x$, then $|F(y)-F(x)| = |F(x)-F(y)| < \epsilon$ So F is cont

Assume f cont at x_0 . Need $F'(x_0) = f(x_0)$

Let $a \leq s \leq t \leq b$

$$\begin{aligned} \left| \frac{F(t) - F(s)}{t-s} - f(x_0) \right| &= \left| \frac{1}{t-s} \int_s^t f(u) du - \frac{1}{t-s} \int_s^t f(x_0) du \right| \\ &= \frac{1}{t-s} \int_s^t |f(u) - f(x_0)| du \end{aligned}$$

Assume $s \leq x_0 \leq t$

$$\begin{aligned} \left| \frac{F(t) - F(s)}{t-s} - f(x_0) \right| &= \left| \frac{1}{t-s} \int_s^t |f(u) - f(x_0)| du \right| \\ &\leq \frac{1}{t-s} \int_s^t |f(u) - f(x_0)| du \end{aligned}$$

Given $\epsilon > 0$, let $\delta > 0$ be s.t. for all $u \in (x_0-\delta, x_0+\delta)$ we have $|f(u) - f(x_0)| < \epsilon$

Assume $x_0-\delta < s \leq x_0 \leq t < x_0+\delta$

$$\left| \frac{F(t) - F(s)}{t-s} - f(x_0) \right| \leq \frac{1}{t-s} \int_s^t \epsilon du = \frac{1}{t-s} \cdot \epsilon \cdot (t-s) = \epsilon$$

$$\text{So } \left| \frac{F(t) - F(x_0)}{t-x_0} - f(x_0) \right| < \epsilon \text{ & } \left| \frac{F(s) - F(x_0)}{s-x_0} - f(x_0) \right| < \epsilon$$

$$\text{So } \lim_{t \rightarrow x_0^+} \frac{F(t) - F(x_0)}{t-x_0} = f(x_0) \quad \lim_{t \rightarrow x_0^-} \frac{F(t) - F(x_0)}{t-x_0} = f(x_0)$$

$$\text{So } \lim_{t \rightarrow x_0} \frac{F(t) - F(x_0)}{t - x_0} = f(x_0) \text{ i.e. } F'(x_0) = f(x_0)$$

Thm $f \in \mathbb{R}$ on $[a, b]$ F diff on $[a, b]$ s.t. $F' = f$

$$\text{Then } \int_a^b f(x) dx = F(b) - F(a)$$

Proof $\varepsilon > 0$ be given

(choose $P = \{x_0, \dots, x_n\}$ a partition of $[a, b]$)

$$\text{s.t. } U(P, f) - L(P, f) < \varepsilon \quad (f \in \mathbb{R})$$

By the mean value theorem:

$$F(x_i) - F(x_{i-1}) = F'(t_i) \Delta x_i = f(t_i) \Delta x_i$$

for some $t_i \in [x_{i-1}, x_i]$

$$\text{thus } \sum_{i=1}^n f(x_i) \Delta x_i = \sum_{i=1}^n (F(x_i) - F(x_{i-1})) = F(b) - F(a)$$

$$F(b) - F(a) - \sum f(t_i) \Delta x_i \geq F(b) - F(a) - U(P, f)$$

$$L(P, f) \leq \int_a^b f(x) dx \leq U(P, f)$$

$$U(P, f) < \int_a^b f(x) dx + \varepsilon, \quad L(P, f) > \int_a^b f(x) dx - \varepsilon$$

$$F(b) - F(a) - \sum f(t_i) \Delta x_i = 0$$

$$F(b) - F(a) - U(P, f) = \overbrace{F(b) - F(a) - L(P, f)}^{\Delta}$$

$$F(b) - F(a) - \int_a^b f(x) dx - \varepsilon$$

$$F(b) - F(a) - \int_a^b f(x) dx + \varepsilon$$

$$\text{So } |F(b) - F(a) - \int_a^b f(x) dx| < \varepsilon$$

$$\text{so since } \varepsilon > 0 \text{ is arbitrary, } \int_a^b f(x) dx = F(b) - F(a)$$

Thm (Integration by parts)

f, g differentiable on $[a, b]$, $f' \in \mathcal{L}$, $g' \in \mathcal{L}$

$$\text{Then } \int_a^b f(x) g(x) dx = F(b) G(b) - F(a) G(a) - \int_a^b f(x) G'(x) dx$$

Proof consider $H(x) = F(x) G(x) - \text{diff}$

$$\begin{aligned} H'(x) &= f'(x) G(x) + f(x) G'(x) \\ &= f(x) G(x) + f(x) g(x) \end{aligned}$$

$f \in \mathcal{L}$, G diff $\rightarrow G$ cont $\rightarrow G \in \mathcal{L}$

$\therefore f, G \in \mathcal{L}$, $F, g \in \mathcal{L}$

$\therefore fG + FG \in \mathcal{L}$

$$\therefore \int_a^b H'(x) dx = H(b) - H(a)$$

$$\int_a^b f(x) G(x) + \int_a^b F(x) g(x)$$

Uniform convergence

X a metric space

$f_n: X \rightarrow \mathbb{R}$, $n \in \mathbb{N}$

$f: X \rightarrow \mathbb{R}$

$\{f_n\}$ converges uniformly to f if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\sup_{x \in X} |f_n(x) - f(x)| < \varepsilon$$

$$\varepsilon_n \rightarrow 0 \text{ where } \varepsilon_n = \sup_{x \in X} |f_n(x) - f(x)|$$

Thm $f_n \in \mathcal{R}(x)$ on $[a, b]$
 $f_n \rightarrow f$ uniformly on $[a, b]$
 Then $f \in \mathcal{R}(x)$ on $[a, b]$ and

$$\int_a^b f d\alpha = \lim_n \int_a^b f_n d\alpha$$

Proof Let $\varepsilon_n = \sup_x |f_n(x) - f(x)|$

$$\text{Note } \int_a^b f_n d\alpha - \varepsilon_n (q(b) - q(a)) \leq \int_a^b f d\alpha \quad \text{since } f_n \leq f + \varepsilon_n$$

$$\text{So } 0 \leq \int_a^b f d\alpha - \left(\int_a^b f d\alpha - \varepsilon_n (q(b) - q(a)) \right) \xrightarrow{n} 0$$

$$\text{So } \int_a^b f d\alpha = \int_a^b f d\alpha, \quad f \in \mathcal{R}(x)$$

$$\int f = \lim_n \int f_n$$

May 2 Derivatives

$$f: [a, b] \rightarrow \mathbb{R}^k$$

$$f(x) \in \mathbb{R}^k = (f_1(x), f_2(x), \dots, f_k(x))$$

$$f_i: [a, b] \rightarrow \mathbb{R} \quad i=1, \dots, k$$

Consider $\varphi(t) = \frac{f(t) - f(x)}{t-x} = \frac{1}{t-x} (f(t) - f(x)) \in \mathbb{R}^k \quad t \neq x \in G(a, b)$
 $f'(x)$ exists if $\lim_{t \rightarrow x} \varphi(t)$ exists in \mathbb{R}^k

and we write $f'(x) = \lim_{t \rightarrow x} \varphi(t) \in \mathbb{R}^k$

$$|f'(x) - \varphi(t)| \xrightarrow[t \rightarrow x]{} 0$$

$\forall \varepsilon > 0 \exists \delta > 0$ s.t.

: if $0 < |t-x| < \delta$, then $|f'(x) - \varphi(t)| < \varepsilon$

$$x \in \mathbb{R}^k \quad |x| = \left(\sum_{i=1}^k x_i^2 \right)^{\frac{1}{2}}$$

Note: $f'(x) = (f'_1(x), \dots, f'_k(x))$

$$\max_{i=1 \dots k} |x_i| \leq \left(\sum_{i=1}^k x_i^2 \right)^{\frac{1}{2}} \leq \sqrt{k} \max_{i=1 \dots k} |x_i|$$

$$\frac{1}{\sqrt{k}} (f(b) - f(a)) = \left(\frac{f_1(b) - f_1(a)}{\sqrt{k}}, \dots, \frac{f_k(b) - f_k(a)}{\sqrt{k}} \right) = (f'_1(x) \dots, f'_k(x))$$

Thm $f: [a, b] \rightarrow \mathbb{R}^k$ concave f diff on (a, b)

Then there exists $x \in (a, b)$ s.t.

$$\frac{1}{b-a} |f(b) - f(a)| \leq |f'(x)|$$

Proof: Let $z = f(b) - f(a)$

$$\text{let } \varphi(t) = z \cdot f(t)$$

$\varphi: [a, b] \rightarrow \mathbb{R}$ concave on (a, b) & diff on (a, b)

By MVT, we get $x \in (a, b)$ $\varphi(b) - \varphi(a) = (b-a)\varphi'(x)$
 $= (b-a)(z \cdot f'(x))$

$$\text{so } \varphi(b) - \varphi(a) = z \cdot f(b) - z \cdot f(a)$$

$$= z \cdot (f(b) - f(a)) = z \cdot z = |z|^2$$

$$\text{so } |z|^2 = (b-a) |z \cdot f'(x)| \leq (b-a) |z| |f'(x)| \quad (\text{Schw Inequality})$$
$$|z| \leq (b-a) |f'(x)|$$

Integration in \mathbb{R}^k

$$f: [a, b] \rightarrow \mathbb{R}^k$$

(f_1, \dots, f_k) Assume $f_1, \dots, f_k \in \mathcal{Q}(x)$

define $\int_a^b f d\alpha = (\int_a^b f_1 d\alpha, \dots, \int_a^b f_k d\alpha)$

E.g. $f, F: [a, b] \rightarrow \mathbb{R}^k$, $f \in \mathcal{Q}$ on $[a, b]$, $F' = f$

Then $\int_a^b f(x) dx = F(b) - F(a)$

Then $f: [a, b] \rightarrow \mathbb{R}^k$, $f \in \mathcal{Q}(x)$ on $[a, b]$

then $|f| \in \mathcal{Q}(x)$ on $[a, b]$ and

$$|\int_a^b f d\alpha| \leq \int_a^b |f| d\alpha$$

Proof we have $f = (f_1, \dots, f_k)$

$$\text{so } |f| = (\|f_1\|^2 + \dots + \|f_k\|^2)^{\frac{1}{2}}$$

Assume $f_1, \dots, f_k \in \mathcal{Q}(x)$

$$\text{then } f_1^2 + \dots + f_k^2 \in \mathcal{Q}(x)$$

$$(f_1^2 + \dots + f_k^2)^{\frac{1}{2}} \in \mathcal{Q}(x)$$

$$\psi: [0, k] \rightarrow [0, k]^2$$

compose

Review!

$$\psi(x) = x^2 \text{ cont. } 1 \text{-to-1, onto}$$

$$\text{then } \psi^{-1}: [0, k^2] \rightarrow [0, k] \text{ is cont}$$

So we have $f: [0, L] \rightarrow \mathbb{R}$ cont $\forall L > 0$

Let $\gamma_i = \int_a^b f_i dt \in \mathbb{R}$

$$\text{So } \gamma = (\gamma_1, \dots, \gamma_k) = \int_a^b f dt$$

$$|\gamma|^2 = \sum_{i=1}^k \gamma_i \int_a^b f_i dt = \int_a^b \left(\sum_{i=1}^k \gamma_i f_i \right) dt$$

By Schwartz:

$$\left| \underbrace{\sum_{i=1}^k \gamma_i f_i(t)}_{\gamma \cdot f(t)} \right| \leq |\gamma| \|f(t)\|$$

$$\text{So } |\gamma|^2 \leq \int_a^b |\gamma| \|f\| dt$$

$$|\gamma| \leq \int_a^b \|f\| dt$$

$$|\int_a^b f dt| \leq \int_a^b \|f\| dt$$

Curves

$\gamma: [a, b] \rightarrow \mathbb{R}^k$ cont

* $\exists \gamma_p: [0, 1] \rightarrow \mathbb{R}^2$ cont

$$\gamma_p([0, 1]) = [0, 1]^2$$
 Peano curve

Let $P = \{x_0^a < \dots < x_k^b\}$ a partition

$$\Lambda(P, \gamma) = \sum_{i=1}^k |\gamma(x_i) - \gamma(x_{i-1})|$$

Def $\Lambda(\gamma) = \sup_P \Lambda(P, \gamma)$ (length of the curve)

Def γ rectifiable if $\Lambda(\gamma) < \infty$

Thm $\gamma: [a, b] \rightarrow \mathbb{R}^k$, γ' exists & is continuous on $[a, b]$

$$\text{Then } \gamma \text{ is rectifiable and } \Lambda(\gamma) = \int_a^b |\gamma'(t)| dt$$

continuous

May 7 Proof we want to show $\Lambda(\gamma) \geq \int_a^b |\gamma'(t)| dt$ and $\Lambda(\gamma) \leq \int_a^b |\gamma'(t)| dt$

$$P = \{x_0 < \dots < x_n\}$$

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} \gamma'(t) dt \right| \quad (\text{FTC})$$

$$\leq \int_{x_{i-1}}^{x_i} |\gamma'(t)| dt$$

$$\begin{aligned} \Lambda(P, \gamma) &= \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})| \\ &\leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |\gamma'(t)| dt \\ &= \int_a^b |\gamma'(t)| dt \quad \text{so } \leq \text{ is proved} \end{aligned}$$

γ' is cont on $[a, b]$
 so γ' is unif cont on $[a, b]$
 given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall s, t \in [a, b]$
 $(|s-t| < \delta \rightarrow |\gamma'(s) - \gamma'(t)| < \varepsilon)$

Need P a part of $[a, b]$ s.t.

$$\begin{aligned} \Lambda(P, \delta) &\geq \int_a^b |\gamma'(t)| dt - \varepsilon \\ (\text{so } \forall \varepsilon > 0 \quad \Lambda(\gamma) &\geq \int_a^b |\gamma'(t)| dt - \varepsilon) \end{aligned}$$

$$\Lambda(\gamma) \geq \int_a^b |\gamma'(t)| dt \text{ as required)$$

Let $P = \{x_0, \dots, x_n\}$ s.t. $\Delta x_i < \delta$

so for each i , for $x_{i-1} < t < x_i$, we have

$$\begin{aligned} |\gamma'(t)| &\leq |\gamma'(x_i)| + \varepsilon \\ (|v-w| < \varepsilon \Rightarrow |v| &< \varepsilon + |w|) \end{aligned}$$

$$\begin{aligned}
\int_{x_{i-1}}^{x_i} |\varphi'(t)| dt &\leq (\varphi'(x_i) + \varepsilon) \Delta x_i \\
&= |\varphi'(x_i)| \Delta x_i + \varepsilon \Delta x_i \\
&= \left| \int_{x_{i-1}}^{x_i} (\varphi'(t) + \varphi'(x_i) - \varphi'(t)) dt \right| + \varepsilon \Delta x_i \\
&\leq \left| \int_{x_{i-1}}^{x_i} \varphi'(t) dt \right| + \int_{x_{i-1}}^{x_i} |\varphi'(x_i) - \varphi(t)| dt + \varepsilon \Delta x_i \\
&\leq |\varphi(x_i) - \varphi(x_{i-1})| + \int_{x_{i-1}}^{x_i} |\varphi'(x_i) - \varphi(t)| dt + \varepsilon \Delta x_i \\
&\leq |\varphi(x_i) - \varphi(x_{i-1})| + 2\varepsilon \Delta x_i \\
\text{So } \int_{x_{i-1}}^{x_i} |\varphi'(t)| dt &\leq |\varphi(x_i) - \varphi(x_{i-1})| + 2\varepsilon \Delta x_i \\
\int_a^b |\varphi'(t)| dt &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |\varphi'(t)| dt \leq \sum_{i=1}^n |\varphi(x_i) - \varphi(x_{i-1})| + \\
&\quad 2\varepsilon \sum_{i=1}^n \Delta x_i \\
&= \Lambda(P, \varphi) + 2\varepsilon(b-a)
\end{aligned}$$

□

Examples

affine Hours:

- Ch 1: 5
- Ch 2: 22
- Ch 3: 21
- Ch 4: 28
- Ch 5: 8 12
- Ch 6: 2 3

Sunday	10 - 12	TA
	1 - 3	Stamek

Exercise 4.7

$$f, g: \mathbb{R}^2 \rightarrow \mathbb{R} \quad f(x, y) = \frac{xy^2}{x^2+y^4}, \quad g(x, y) = \frac{xy^2}{x^2+y^6}$$

WTS f bdd, g unbdd in each neighbourhood of $(0, 0)$

- f, g continuous when restricted to each line in \mathbb{R}^2
- $f, g = 0$ at $(0, 0)$

$$(x - y^2)^2 \geq 0$$

$$x^2 - 2xy^2 + y^4 \geq 0$$

$$x^2 + y^4 \geq 2xy^2$$

$$xy^2 \leq \frac{x^2 + y^4}{2}$$

$$\text{So } \frac{xy^2}{x^2+y^4} \leq \frac{1}{2} \quad x \neq 0 \text{ or } y \neq 0$$

$$x=y=0 \Rightarrow |f(0, 0)| \leq \frac{1}{2}$$

$$(y^3 - y), y > 0 \quad \sqrt{(y^3 - y)^2} = \sqrt{y^6 + y^2}$$

$$= y \sqrt{y^4 + 1}$$

$$< \pi y \quad 0 < y < 1$$

$$((\frac{1}{n})^3, \frac{1}{n}) \rightarrow (0, 0) \quad g((\frac{1}{n})^3, \frac{1}{n}) \rightarrow \infty \quad g(0, 0) = 0$$

$$f(y^2, y) = \frac{y^4}{y^4 + y^4} = \frac{1}{2} \quad f((\frac{1}{n})^2, \frac{1}{n}) \rightarrow (0, 0)$$

$$f((\frac{1}{n})^2, \frac{1}{n}) \rightarrow \frac{1}{2} \quad f(0, 0) = 0$$

$$x = ay + b \quad y = c$$

$$f(ay+b, y) = \frac{(ay+b)y^2}{(ay+b)^2 + y^4} - \text{continuous unless } (ay+b)^2 = 0 \text{ and } y^4 = 0$$

so unless $b = 0$

($b=0$) $f(ay, y) = \frac{ay^3}{a^2y^2 + y^4} = \frac{ay}{a^2 + y^2}$ [y is not 0,
but it also holds
when $y=0$)

continuous unless ($a=0$)

$$f(0, y) = 0 \text{ if } y \neq 0 \text{ But } f(0, 0) = 0 \text{ So } \underset{\text{cont when } x=0}{\text{cont}}$$

$$f(x, c) = \frac{x^2}{x^2 + c^2} \text{ cont unless } c=0$$

$$f(x, 0) = 0 \text{ if } x \neq 0 \quad f(0, 0) = 0 \text{ if } x=0 \quad (\text{cont})$$