Numerical Analysis HW 7

1) Determine how many terms of the Taylor series $\sum_{k=0}^{n} \frac{x^k}{k!}$ are necessary to evaluate e^1 correctly to 15 decimal places.
Hint: Use the Taylor's remainder theorem.

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \frac{x^k}{k!} + E_{n+1} \qquad E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$$

$$\underbrace{1}_{k=0} + \underbrace{x}_{k=1} + \underbrace{\frac{x^2}{2}}_{k=2} + \underbrace{\frac{x^3}{6}}_{k=3} + \underbrace{\frac{x^4}{24}}_{k=4} + \underbrace{\frac{x^5}{120}}_{k=5} + \underbrace{\frac{x^6}{720}}_{k=6} + \underbrace{\frac{x^7}{5040}}_{k=7} + \underbrace{\frac{x^8}{40320}}_{k=8} + \underbrace{\frac{x^9}{362880}}_{k=9} + \underbrace{\frac{x^{10}}{3628800}}_{k=10}$$

$$1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40320} + \frac{1}{362880} + \frac{1}{3628800}$$

$$+ \underbrace{\frac{x^{11}}{39916800}}_{k=11} + \underbrace{\frac{x^{12}}{479001600}}_{k=12} + \underbrace{\frac{x^{13}}{6227020800}}_{k=13} + \underbrace{\frac{x^{14}}{87178291200}}_{k=14} + \underbrace{\frac{x^{15}}{1307674363000}}_{k=15}$$

$$\frac{1}{39916800} + \frac{1}{479001600} + \frac{1}{6227020800} + \frac{1}{87178291200} + \frac{1}{1307674363000}$$

$$\underbrace{\frac{x^{16}}{20922789888000}}_{k=16} + \underbrace{\frac{x^{17}}{355687428096000}}_{k=17}$$

$$\frac{1}{20922789888000} + \frac{1}{355687428096000}$$ 15 digits

if $n=16$

$$\frac{f^{(17)}(\xi) \cdot 1^{n+1}}{355{,}687{,}428{,}096{,}000}$$

$c < \xi < x$

or

$0 < \xi < 1$

$\rightarrow$ worst case $\xi < 1$ $\rightarrow$ $\dfrac{1}{355{,}687{,}428{,}096{,}000}$

$\downarrow$

if $n=16$ from $k=0$ to 16, then we have $\boxed{\text{17 terms}}$

2) If $x < 0$, then the Taylor series of e^x is an alternating series. In this case the loss of significant digits can be a serious problem. Explain how you could use the formula $e^{-x} = \frac{1}{e^x}$ to mitigate this problem.

$\sum_{k=0}^{n} \frac{x^k}{k!} + E_{n+1}$ $\quad E_{n+1} = \frac{e^{\xi}}{(n+1)!} \cdot x^{n+1}$

if $x < 0$, then use $-x$

$\sum_{k=0}^{3} \frac{x^k}{k!} = 1 + (-x) + \frac{(-x)^2}{2} + \frac{(-x)^3}{6}$

$\Rightarrow 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$ is alternating.

Derivatives for e^{-x}: $f(x) = e^{-x}$, $f'(x) = -e^{-x}$, $f''(x) = e^{-x}$, $f'''(x) = -e^{-x}$

$\frac{e^0 \cdot x^0}{1} + \frac{-e^0 \cdot x}{1} + \frac{e^0 \cdot x^2}{2} + \frac{-e^0 \cdot x^3}{6}$, sub for $-x$, $\frac{(-x)^0}{1} + \frac{-(-x)^1}{1} + \frac{(-x)^2}{2} + \frac{-(-x)^3}{6}$

$\Rightarrow 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ is no longer alternating.

therefore this alternating problem is mitigated.

3) Show that if $E(h) = O(h^n)$, then $E(h) = O(h^m)$ for any nonnegative integer $m \leq n$.

Big $O(h)$ Thm states $|F(h)| \leq C|h^n|$, where $C>0$ & $C \in \mathbb{R}$.
Given our problem, get $|E(h)| \leq C|h^n|$ b/c $E \in O(h^n)$.
If $m \leq n$ & a nonnegative number (to satisfy $m \in \{0,1,\dots,n+1\}$).
Since $m \leq n$ and m a subset of n, we know $h^m \leq h^n$ and $c|h^m| \leq c|h^n|$. More importantly since values from 1 to m are present here, derivatives $\{0,1,\dots,m\}$, then we can plug this into the thm here to get $|E(h)| \leq c|h^m| \leq c|h^n|$. Further simplify, get $|E(h)| \leq c|h^m|$, which via the thm gives us $E(h) = O(h^m)$.

4) For small x, the approximation $\sin(x) \approx x$ is often used. For what range of x is this good to a relative accuracy of 10^{-16}?

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{(2k+1)}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

So $-\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots < 10^{-16}$ b/c $x - \underline{0} = x$ $k>0$

~~b/c $E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1} \Rightarrow$ 18! has 16 digits so $n=17$~~

$\cancel{\sum_{k=1}^{17} (-1)^k \frac{x^{(2k+1)}}{(2k+1)!} < 10^{-16}}$ $\longrightarrow$ (other terms are pretty negligable here)

If $n=1$, get $x - \frac{x^3}{3!} = x$ or $\frac{x^3}{3!} = 0$ or $\frac{x^3}{3!} < 10^{-16}$
$\Rightarrow \frac{x^3}{6} < 10^{-16} \Rightarrow \frac{(10^{-6})^3}{6} < 10^{-16} \Rightarrow \frac{10^{-18}}{6} < 10^{-16}$ meaning 10^{-6} is fine
now try $\frac{(10^{-5})^3}{6} < 10^{-16}$ — $\frac{10^{-15}}{6} < 10^{-16}$ ← not true. This tells us the bounds are between 10^{-5} & 10^{-6} to be safe we will use 10^{-6} as our bound term. So from $\boxed{[-10^{-6}, 10^{6}]}$ $x \approx \sin(x)$. We could get more accurate w/h a computer, but this will do for a pencil paper problem.