

$$1) x = \sqrt{R} \Rightarrow x^2 = R \Rightarrow x^2 - R = 0 \Rightarrow f(x) = x^2 - R \quad \text{Formula 1)} f(x) = x^2 - R$$

Find Derivative,  $f'(x) = 2x \quad \text{Formula 2)} f'(x) = 2x$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)} = P_n - \frac{x^2 - R}{2x} \quad \text{is Equation for Newton's Method given } x = \sqrt{R}.$$

Newton's Method's approximation depends heavily on the initial guess  $x_0$ . The reason is due to the error terms growth in Newton's Method which is displayed by the formula  $|r - P_{n+1}| \leq C(f) |r - P_n|^2$ . Firstly we can see that picking a bad  $x_0$  can lead to  $|r - P_n| \geq 1$  which mean the error grows drastically in comparison to the bisection method  $|r - P_{n+1}| \leq \frac{1}{2} |r - P_n|$ . Picking a bad  $x_0$  can also lead to a high  $C(f)$  term which grows the error term more. Inversly a good  $x_0$  can lead to an exponentially decreasing error term with  $|r - P_n| < 1$  and  $C(f) < 1$  which can give you extremely precise results faster then the bisection method or false position methods. (Bad  $x_0$  can result in divergence).

$$2) f(x) = \frac{x^2}{1+x^2} \quad P_{n+1} = x - \frac{\frac{x^2}{1+x^2} - 0}{\frac{2x}{(1+x^2)^2}} = x - \frac{x(1+x^2)}{2} = x - \frac{x+x^3}{2}$$

$$f'(x) = \frac{2x}{(1+x^2)^2} \quad \Rightarrow \frac{x-x^3}{2} = \frac{x(1-x^2)}{2}$$

Find Roots of  $P_{n+1}$ , get  $x=0, 1, -1$ . (cannot Divide by 0)

Values which are not 0, 1, or -1 should converge to the root that also satisfy  $-1 < x_0 < 1$ . This range is pretty safe, and should converge well given the  $|r - P_n|^2 < 1$  property of this problem. Values outside the given range will diverge.

$$3) \quad f(x) = 2x^3 - 9x^2 + 12x + 15 \quad P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

$$f'(x) = 6x^2 - 18x + 12$$

$$\Rightarrow \frac{2x^3 - 9x^2 + 12x + 15}{6(x^2 - 3x + 2)} \Rightarrow \frac{x^3 - 9x^2 + 12x + 15}{6(x-2)(x-1)}$$

$$P_{n+1} = X - \frac{2x^3 - 9x^2 + 12x + 15}{6x^2 - 18x + 12}$$

$$X_0 = 3 \quad 3 - \frac{2(3)^3 - 9(3)^2 + 12(3) + 15}{6(3-2)(3-1)} = 3 - \frac{2 \cdot 27 - 9 \cdot 9 + 36 + 15}{6 \cdot 1 \cdot 2} = 3 - \frac{54 - 81 + 51}{12}$$

$$\Rightarrow 3 - \frac{105 - 81}{12} \Rightarrow 3 - \frac{24}{12} \Rightarrow 3 - 2 \Rightarrow 1 \quad X_1 = 1$$

$$1 - \frac{2(1)^3 - 9(1)^2 + 12(1) + 15}{6(1)^2 - 18(1) + 12} = 1 - \frac{\text{stuff}}{0} \quad X_0 = 3 \text{ doesn't lead to convergence.}$$

$$X_0 = 4 \quad (x > 3) \quad 4 - \frac{2(4)^3 - 9(4)^2 + 12(4) + 15}{6(4-2)(4-1)} = 4 - \frac{128 - 144 + 48 + 15}{96 - 72 + 12}$$

$$\Rightarrow 4 - \frac{47}{36} = \frac{97}{36} \Rightarrow \text{ugly number, division by 0 is unlikely, leading to convergence.}$$

$$X_0 = 2 \quad (x < 3) \quad 2 - \frac{\text{stuff}}{6(2-2)(2-1)} = 2 - \frac{\text{stuff}}{0} \quad X_0 = 2 \text{ doesn't lead to convergence.}$$

$X_0 = 3$  doesn't converge

$X_0 > 3$  Converges..

$X_0 < 3$  Diverges

~~(if  $x$  becomes 1 or 2)~~

4)  $f(x) = x^m - R \quad \text{vs} \quad f(x) = 1 - \frac{R}{x^m}$   
 $f'(x) = mx^{(m-1)} \quad \text{vs} \quad f'(x) = mR x^{(m-1)}$

$p_{n+1} = x - \frac{x^m - R}{mx^{(m-1)}} \quad \text{vs} \quad p_{n+1} = x - \frac{1 - \frac{R}{x^m}}{mR x^{(m-1)}}$

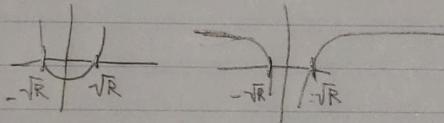
$R > 0, m \geq 2$

~~so  $R > 1, m \geq 2$~~

~~$p_{n+1} = x - \frac{x^2 - 1}{2x} \quad \text{vs} \quad p_{n+1} = x - \frac{1 - \frac{1}{x^2}}{2x^{(3)}} \quad \text{or} \quad x - 2x^3 + 2x$~~

The first formula  $(x^m - R)$  is better in this case because it's not prone to divergence. When  $m \geq 2$ .

Formula 1      Formula 2



In the first case, badly picked  $x_0$ 's can still converge. But badly picked  $x_0$ 's will result in divergence in the second formula.