Numerical Analysis HW5

1) Use Bisection method to find the value of P_3 for
$f(x) = \sqrt{x} - \cos(x)$ on $[0, 1]$

Step 1 $P_1 = \frac{a_1 + b_1}{2} \longrightarrow a_1 = 0 ;\ b_1 = 1 \longrightarrow P_1 = \frac{0+1}{2} = \frac{1}{2}$

Now find $f(P_1) = \sqrt{\tfrac{1}{2}} - \cos(\tfrac{1}{2}) = \frac{\sqrt{2}}{2} - \cos(\tfrac{1}{2})$ = negative num

Now find $f(a_1) = \sqrt{0} - \cos(0) = 0 - 1 = -1$ = negative num

$f(P_1)$ and $f(a_1)$ share the same sign, so $a_2 = P_1 ;\ b_2 = b_1$

Step 2 $a_2 = \frac{1}{2} ;\ b_2 = 1 ;\ P_2 = \frac{\frac{1}{2}+1}{2} = \frac{1.5}{2} = 0.75$

Find $F(P_2) = \sqrt{0.75} - \cos(0.75)$ = positive num

Find $f(a_2) = \sqrt{\tfrac{1}{2}} - \cos(\tfrac{1}{2})$ = negative num

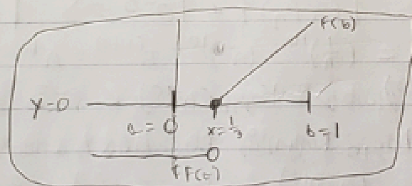
$f(P_1)$ and $f(a_1)$ don't share the same sign, so $a_3 = a_2 ;\ b_3 = P_2$

Step 3 $a_3 = \frac{1}{2} ;\ b_3 = \frac{3}{4} ;\ P_3 = \frac{\frac{1}{2} + \frac{3}{4}}{2} = \frac{\frac{2}{4} + \frac{3}{4}}{2} = \frac{(\frac{5}{4})}{2} = \frac{5}{8}$

Meaning $\boxed{P_3 = \frac{5}{8}}$

2) a) Draw a graph of a function that is discontinuous yet the bisection method converges.

"No empty points in function"

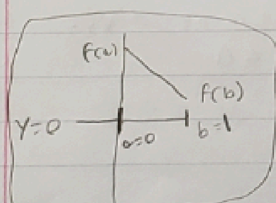


$\text{if} \begin{cases} x < \frac{1}{3} & y = -1 \\ \text{else} & y = x - \frac{1}{3} \end{cases}$ from $[0, 1]$

when $x = \frac{1}{3}$

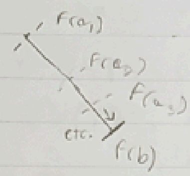
$\frac{1}{3} - \frac{1}{3} = 0$ ✓

b) Draw a graph of a function for which the bisection method fails to converge to a root. To be more precise it converges to some number z for which $f(z) \neq 0$. Explain why Algo doesnt converge to a root.



$y = -x + 3$ from $[0, 1]$ $3 = 0 + 3 ;\ 2 = -1 + 3$

In my example, I use the function $y = -x + 3$ from $[0, 1]$ this means $f(a) = 3$ & $f(b) = 2$ so $f(a) \cdot f(b) < 0$ or $6 < 0$ condition is not held for the bisection algo to guarantee a root. Also the bisection algorith checks to see if $f(a)$ is the same sign as $f(p)$. Because $f(a) \cdot f(b) < 0$ is not held and $f(a)$ & $f(b)$ are positive this statement will ALWAYS be called before the else statement. Hence $f(a_i)$ is inching towards $f(b)$.



Over all the algo will find the closest number to zero in the interval, but not 0 if the $f(a) \cdot f(b) < 0$ condition is violated.

3) Let $f(x) = (x-1)^{10}$, $p = 1$ & $p_n = 1 + \frac{1}{n}$. Show that $|f(p_n)| < 10^{-3}$ whenever $n > 1$ but that $|p_n - p| < 10^{-3}$ requires that $n > 1000$.

n represents the iteration number in the bisection algorithm meaning $n \in \mathbb{N}$ too. $|f(p_n)| < 10^{-3} \Rightarrow |f(1+\frac{1}{n})| < 10^{-3}$

$\Rightarrow f(1+\frac{1}{n}) = (1 + \frac{1}{n} - 1)^{10} = (\frac{1}{n})^{10} = (n^{-1})^{10} = n^{(-1 \cdot 10)} = n^{-10} \Rightarrow |n^{-10}| < 10^{-3}$

$\Rightarrow \log_{10}(n^{-10}) < \log_{10}(10^{-3}) \Rightarrow \log_{10}(n^{-10}) < -3 \Rightarrow -\log_{10}(n^{10}) < -3$

Now lets see $-\log_{10}((n+1)^{10})$ the next step, in this case it means $-\log_{10}((n+1)^{10}) < -\log_{10}(n^{10}) < -3$. Lets try $n=2$.

$-\log_{10}(2^{10}) = -3.01 < -3$ meaning by induction, this principall holds.

Let $|p_n - p| < 10^{-3}$. $|(1+\frac{1}{n}) - 1| < 10^{-3} \Rightarrow |\frac{1}{n}| < 10^{-3}$

$\Rightarrow |\frac{1}{n}| < \frac{1}{10^3}$. Because $n > 1$, n cannot be negative.

$\frac{1}{n} < \frac{1}{10^3} \Rightarrow 10^3 \cdot \frac{1}{n} < \frac{1}{10^3} \cdot 10^3 \Rightarrow \frac{10^3}{n} < 1 \Rightarrow 10^3 < n$

so $n > 1000$. $\boxtimes$

4) If the bisection method is applied with starting interval $[2^m, 2^{m+1}]$, where m is a positive or negative integer, how many steps should be taken to compute the root to full machine precision on a 32-bit word-length computer?

s	$c_0 \dots c_7$	$b_1 \dots b_{23}$

23 digits of precision. $|p_n - p| \leq \frac{1}{2}|b_i - a_i|$

2^{-24} is error unit.

$$|p_n - p| \leq \frac{1}{2^n}|2^{m+1} - 2^m| \leq 2^{-24}$$

$$\cancel{|2^{m+1} - 2^m| \leq 2^{-24} \cdot 2^n \Rightarrow |2^{m+1} - 2^m| \leq 2^{n-24}}$$

~~signs don't matter~~ $\cancel{\Rightarrow 2^{m+1} - 2^m \leq 2^{n-24}}$

Goal: Find n "number of iterations"

$$\frac{1}{2^n} \leq \frac{2^{-24}}{|2^{m+1} - 2^m|} \Rightarrow 1 \leq \frac{2^{-24} \cdot 2^n}{|2^{m+1} - 2^m|} \Rightarrow |2^{m+1} - 2^m| \leq 2^{-24} \cdot 2^n$$

$$\Rightarrow |2^{m+1} - 2^m| \leq 2^{n-24} \Rightarrow \log_2(|2^{m+1} - 2^m|) \leq n - 24$$

$$\Rightarrow \log_2(|2^{m+1} - 2^m|) + 24 \leq n \Rightarrow \boxed{\log_2(|2^{m+1} - 2^m|) + 24 = n}$$