Numerical Analysis HW10

1) Show that the initial value problem

$x' = x\cos(t), \quad 0 \le t \le 1, \quad x(0) = 1,$

has a unique solution & find this solution. Is the problem well-posed as well?

$f(t,x) = x\cos(t)$ continuous — There fore the IVP has a unique solution & exists

$\frac{\partial f}{\partial x}(t,x) = \cos(t)$ continuous ← Lipschitz constant > 0 at any t

$\frac{dx}{dt} = x\cos(t) \Rightarrow \frac{1}{x}\,dx = \cos(t)\,dt \Rightarrow \int \frac{1}{x}\,dx = \int \cos(t)\,dt \Rightarrow \ln|x| = \sin(t) + C$

$x = e^{\sin(t)+C} \Rightarrow 1 = e^{0+C} \Rightarrow 1 = e^0 + e^C \Rightarrow 1 = 1 + C \Rightarrow C = 0$

$\boxed{x(t) = e^{\sin(t)}}$ For well posed we know that there is a unique solution that exists.

$f(t,x) = x\cos(t) \qquad |f(t,x_1) - f(t,x_2)| = |x_1\cos(t) - x_2\cos(t)|$

$\cos(t)\,|x_1 - x_2| \qquad \cos(t)$ from 0 to 1 > 0 so

this problem has continuous dependene on the interval.

Therefore this problem is Well-Posed.

2) Consider the IVP

$x' = x^{\frac{1}{3}}$, $t \geq 0$, $X(0) = 0$

a) Show that the problem has a trivial solution $X(t) = 0$ for all $t \geq 0$.

$\boxed{\frac{dx}{dt} = 0 \quad , \quad x'(t) = 0 = 0^{\frac{1}{3}} \leftarrow \text{when } t \text{ changes, } x \text{ doesn't}}$

b) Find a non-trivial solution of this problem.

$\frac{dx}{dt} = x^{\frac{1}{3}} \Rightarrow \frac{dx}{x^{\frac{1}{3}}} = dt \Rightarrow \int x^{-\frac{1}{3}} dx = \int dt \Rightarrow \frac{3}{2} x^{\frac{2}{3}} = t + C$

$\Rightarrow x^{\frac{2}{3}} = \frac{2}{3}(t+c) \Rightarrow x = (\frac{2}{3}(t+c))^{\frac{3}{2}} \Rightarrow X(0) = (\frac{2}{3}(0+c))^{\frac{3}{2}} = 0$

so $c = 0$, $\boxed{X(t) = (\frac{2}{3}t)^{\frac{3}{2}}}$

c) Does this problem violate the uniqueness theorem for the IVPs? Explain.

$|x_1^{\frac{1}{3}} - x_2^{\frac{1}{3}}|$ with Lipschitz condition

when near 0, get $\frac{1}{3}x^{-\frac{2}{3}}$ (derivative) , $\frac{1}{3}\frac{1}{x^{\frac{2}{3}}}$ this means as the difference gets smaller, the number grows and therefore $\boxed{\text{Violates the uniqueness theorem}}$.

d) Which solution do you get by applying Eulers method?

$X(a) = \alpha$ or $X(0) = 0 = w_0$

$0 + x' \cdot h$ or $x^{\frac{1}{3}} \cdot h = w_1 \Rightarrow 0^{\frac{1}{3}} \cdot h = w_1 = 0 \Rightarrow w_2 = w_1 + x' \cdot h$

$w_2 = 0 + x' \cdot h$, same thing so $w_2 = 0$ therefore we get the

$\boxed{\text{trivial solution by applying Eulers method.}}$

3) Solve the IVP

$$x' = x - t^2 + 1, \quad 0 \le t \le 2, \quad x(0) = 0.5$$

Use this solution to evaluate the error that you get when you compute the approximation of $x(2)$ using Euler's method with $h = 0.5$. Find the best bound M for $|x''(t)|$ on interval $[0,2]$ (use the formula of the exact solution to do that). Use this bound to estimate the error of the Euler's method at $t=2$. Compare this estimate with the precise value of the error.

Eulers of $x(2)$. $w_0 = \alpha \Rightarrow w_0 = 0.5$, $w_1 = w_0 + f(x,t) \cdot h \Rightarrow w_1 = w_0 + x' \cdot h$
$\Rightarrow w_1 = 0.5 + (x - t^2 + 1) \cdot h \Rightarrow w_1 = 0.5 + (0.5 - 0^2 + 1) \cdot 0.5 = 0.5 + 0.75 = 1.25$,
$w_2 = w_1 + x' \cdot h \Rightarrow w_2 = 1.25 + (1.25 - (0.5)^2 + 1) \cdot 0.5 = 1.25 + (1.25 - 0.25 + 1) \cdot 0.5$
$= 1.25 + 2 \cdot 0.5 = 2.25$, $w_3 = w_1 + x' \cdot h \Rightarrow w_3 = 2.25 + (2.25 - (1)^2 + 1) \cdot 0.5 = 2.25 + (2.25) \cdot 0.5$
$2.25 + 1.125 = 3.375 = w_3$, $w_4 = w_4 + x' \cdot h \Rightarrow w_4 = 3.375 + (3.375 - (1.5)^2 + 1) \cdot 0.5$
$= 3.375 + (3.375 - 2.25 + 1) \cdot 0.5 = 3.375 + (1.125 + 1) \cdot 0.5 = 3.375 + (2.225) \cdot 0.5$
$= 3.375 + 1.0625 = \boxed{4.4375} = w_4$, now $t_i = 2$

$f(x,t) = x - t^2 + 1$ $\quad \frac{dx}{dt} = x - t^2 + 1 \Rightarrow \frac{dx}{dt} - x = -t^2 + 1$
$\frac{\partial f}{\partial x}(x,t) = 1 > 0$ ✓ $\quad \Rightarrow \frac{dx}{dt} + P(t)\,x = Q(t)$, $e^{\int P(t)\,dt} = e^{\int -1\,dt} = e^{-t}$,
$\cancel{e^{\int Q(t)\,dt} = e^{\int -t^2+1} = e^{-\frac{1}{3}t^3 + t}}$
$\Rightarrow e^{-t}\frac{dx}{dt} - e^{-t}x = e^{-t}(-t^2+1) \Rightarrow \int \frac{d}{dt}(x \cdot e^{-t})\,dt = \int e^{-t}(-t^2+1)\,dt$
$\Rightarrow x e^{-t} = e^{-t}(t^2 + 2t + 1) + C \Rightarrow x(t) = t^2 + 2t + 1 + e^{t} C \Rightarrow 0.5 = 1 + C$
$-0.5 = C$. $\boxed{x(t) = t^2 + 2t + 1 - \frac{e^t}{2}}$ $\quad x(2) = (2)^2 + 2(2) + 1 - \frac{e^2}{2} = 4 + 4 + 1 - \frac{e^2}{2}$
$x(2) \approx 5.30547$

$$5.30547 - 4.4375 = \boxed{0.86797 \text{ precise error value}}$$

$x(t) = t^2 + 2t + 1 - \frac{e^t}{2}$
$x'(t) = 2t + 2 - \frac{e^t}{2}$
$x''(t) = 2 - \frac{e^t}{2}$ $\quad [0,2]$
Largest error $\quad 2 - \frac{e^2}{2} \approx -1.6945$ so $M \ge 1.6945$ $\qquad L = 1$
$e_4 = |x(t_4) - w_4| \le \frac{hM}{2L}\left[e^{L(t_4 - a)} - 1\right] = \frac{0.5 \cdot 1.6945}{2 \cdot 1}\left[e^2 - 1\right]$
$\approx \boxed{2.70656 \text{ estimated error value}}$

4) Solve the IVP

$x' = -10x, \quad 0 \le t \le 2, \quad x(0) = 1$

What happens when Euler's method is applied to this problem with $h = 0.1$? Does this behavior violate Theorem 1 that we had in class. Give an explaination of your answer.

$w_0(a) = d \;;\; w_{i+1} = w_i + f(x,t) \cdot h$

$w_1 = w_0 - 10x \cdot h \Rightarrow w_1 = 1 - 10 \cdot 0.1 \Rightarrow w_1 = 1 - 1 = 0$ } The method converges

$w_2 = w_1 - 10x \cdot h \Rightarrow w_2 = 0 - 0 = 0$ } at 0.

Yes? The requirement for the THM is a Lipschitz constant of which $L > 0$. However, if $f(x) = -10x$ then $f'(x) = -10$; $-10 \neq 0$, so no Lipschitz constant which means that we cannot properly use the THM.

5) Consider the problem $x' = x$. If the initial condition is $x(0) = c$, then the solution is $x(t) = ce^t$. If a roundoff error of δ occurs in reading the value of c into the computer, what effect is there on the solution obtained by evaluating $x(t) = ce^t$ at the point $t = 10$? At $t = 20$? Do the same for $x' = -x$.

Solution for $x' = -x$ is $x(t) = ce^{-t}$

if $x' = x$ solution is $x(t) = ce^t$.

$x(t) = (c + \delta)e^t = ce^t + \delta e^t$

at $t = 10$ Error is δe^{10}

at $t = 20$ Error is δe^{20}

$x(t) = (c + \delta)e^{-t} = ce^{-t} + \delta e^{-t}$

at $t = 10$ Error is δe^{-10}

at $t = 20$ Error is δe^{-20}

For $x' = x$, δ error grows over time b/c of e^t where t grows & for $x' = -x$, δ error shrinks over time b/c of e^{-t} where t grows meaning $-t$ shrinks.

6) Suppose that there is a method which solves a differential equation on an interval $[a,b]$ and only involves local truncation errors. If the local truncation error is of order $O(h^{n+1})$, then show that the total error is of order $O(h^n)$.

$O(h^{n+1})$, Global Error $\le N \cdot O(h^{n+1}) = N \cdot Ch^{n+1}$

$\Rightarrow \left(\frac{b-a}{h}\right) \cdot Ch^{n+1} \Rightarrow (b-a) \cdot C \cdot h^n \Rightarrow Ch^n \Rightarrow O(h^n)$.

so Global Error $\le O(h^n)$