Numerical Analysis HW9

1) Determine the error term for the formula.

$$f'(x) \approx \frac{1}{4h}[f(x+3h) - f(x-h)]$$

$$f(x+h) \approx f(x) + f'(x)\cdot h + \ldots + \frac{f''(\xi)}{2!}\cdot h^2$$
$$f(x+3h) \approx f(x) + f'(x)\cdot 3h + \ldots + \frac{f''(\xi)}{2!}\cdot 9h^2$$
$$f(x-h) \approx f(x) - f'(x)\,h + \ldots + \frac{f''(\xi)}{2!}\,h^2$$
$$f(x+3h) - f(x-h) \approx 4h\,f'(x) + f''(\xi)\,4h^2$$
$$\frac{1}{4h}[f(x+3h) - f(x-h)] \approx f'(x) + f''(\xi)\,h \quad \text{so } \boxed{O(h)}$$
$$\frac{1}{4h}[f(x+3h) - f(x-h)] - f''(\xi)\,h \approx f'(x)$$

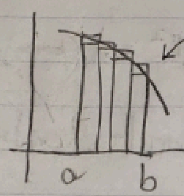
2) Derive the following formula.

$$f''(x) \approx \frac{1}{4h^2}[f(x+2h) - 2f(x) + f(x-2h)]$$

and determine the error term for the formula.

$$f(x+2h) \approx f(x) + f'(x)\cdot 2h + f''(x)\frac{4h^2}{2!} + f'''(x)\frac{8h^3}{3!} + \ldots + f''''(\xi)\frac{16h^4}{4!}$$
$$f(x-2h) \approx f(x) - f'(x)\cdot 2h + f''(x)\frac{4h^2}{2!} - f'''(x)\frac{8h^3}{3!} + \ldots + f''''(\xi)\frac{16h^4}{4!}$$
$$f(x+2h) + f(x-2h) \approx 2f(x) + 4h^2 f''(x) + f''''(\xi)\frac{4h^4}{6}$$
$$f(x+2h) - 2f(x) + f(x-2h) \approx 4h^2 f''(x) + f''''(\xi)\frac{4h^4}{6}$$
$$\frac{1}{4h^2}[f(x+2h) - 2f(x) + f(x-2h)] \approx f''(x) + f''''(\xi)\frac{h^2}{6} \quad \text{so } \boxed{O(h^2)}$$
$$\frac{1}{4h^2}[f(x+2h) - 2f(x) + f(x-2h)] - f''''(\xi)\frac{h^2}{6} \approx f''(x)$$

3) Let f be a decreasing function on the interval $[a,b]$. Consider the upper sum & the lower sum n uniform subintervals. Show that the difference between the upper sum & the lower sum is given by the expression $\dfrac{(f(a)-f(b))(b-a)}{n}$.



Upper and Lower Sums over decreasing function

$\Delta x = \frac{b-a}{n}$

Upper sum when f is decreasing $\Delta x[f(x_0)+f(x_1)+\ldots+f(x_{n-1})]$

lower sum when f is decreasing $\Delta x[f(x_1)+f(x_2)+\ldots+f(x_n)]$

Upper sum $-$ Lower sum $= \Delta x[f(x_0)-f(x_n)]$

In which x_0 is the first point and x_n is the second, so

$\Delta x[f(a)-f(b)]$ or $\frac{b-a}{n}(f(a)-f(b))$

4) If a trapezoid method is used to compute $\int_{-1}^{2} \sin(x)\,dx$ with $h=0.01$, give a realistic bound on the error.

$\Delta x = h$; $\int_a^b f(x)\,dx \approx \frac{h}{2}\sum_{i=0}^{n-1} f(x_i)+f(x_{i+1}) = T(h)$ "trapezoid rule"

$\int_a^b f(x)\,dx - T(h) = \frac{1}{12}(b-a)\cdot f''(\xi)\cdot h^2$; $\overset{0}{\sin(x)} \to \overset{1}{\cos(x)} \to -\overset{2}{\sin(x)}$

So $\frac{1}{12}(2-(-1))\cdot -\sin(\xi)\cdot(0.01)^2$; $\max_{[-1,2]} |-\sin(x)| = 1$

$\Rightarrow \frac{1}{12}\cdot 3\cdot 1\cdot(0.01)^2 \Rightarrow \boxed{\frac{1}{4}\cdot 10^{-4}}$

5) How large must n be to estimate $\int_0^{\pi} \sin(x)\,dx$ with error less than 10^{-20} using the trapezoid method?

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\sum_{i=0}^{n-1} f(x_i)+f(x_{i+1}) = T(h) \qquad \Delta x = h = \frac{b-a}{n}$$

$$\int_a^b f(x)\,dx - T(h) = \frac{1}{12}(b-a)\cdot f''(\xi)\,h^2 = \frac{1}{12}(b-a)\cdot f''(\xi)\left(\frac{b-a}{n}\right)^2$$

$$\overset{0}{\sin(x)} \rightarrow \overset{1}{\cos(x)} \rightarrow \overset{2}{-\sin(x)} \qquad \max_{[0,\pi]} |-\sin(x)| = 1$$

$$\frac{1}{12}(\pi-0)\cdot 1\cdot\left(\frac{\pi-0}{n}\right)^2 \Rightarrow \frac{\pi}{12}\cdot\left(\frac{\pi}{n}\right)^2$$

$$10^{-20} > \frac{\pi}{12}\cdot\left(\frac{\pi}{n}\right)^2 \Rightarrow 10^{-20}\cdot\frac{12}{\pi} > \left(\frac{\pi}{n}\right)^2 \Rightarrow \sqrt{10^{-20}\cdot\frac{12}{\pi}} > \frac{\pi}{n}$$

$$\Rightarrow \frac{\sqrt{10^{-20}\cdot\frac{12}{\pi}}}{\pi} > \frac{1}{n} \Rightarrow \boxed{\frac{\pi}{\sqrt{10^{-20}\cdot\frac{12}{\pi}}} < n}$$