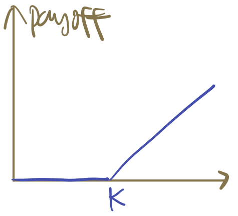
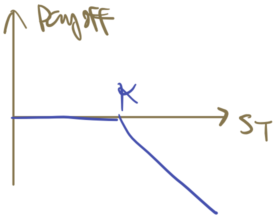
☆ PPT #7. options.

Long call



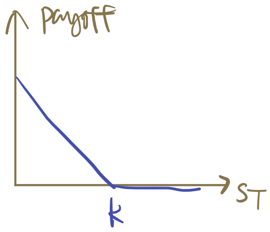
$\max(S_T - K, 0)$

Short call



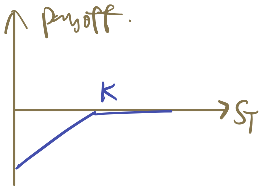
$\min(K - S_T, 0)$

Long put



$\max(K - S_T, 0)$

Short put.



$\min(S_T - K, 0)$

☆☆ n-for-m stock split, K reduce to $\frac{m}{n}(K)$, # of options increased to $(\frac{n}{m})(\#)$.

$\frac{\text{Before}}{\text{after}}(K)$.

☆☆ Initial mar m is the greater of :

1) (# of calls × shares per call) × (call unit price + 0.2 × share price − out of money).
2) (# of calls × shares per call) × (call unit price + 0.1 × share price)
   OR (# of puts × shares per put) × (put unit price + 0.1 × strike price).

☆☆ Effect of variables on option pricing

	c	p	C	P
S_0	+	−	+	−
K	−	+	−	+
T			+	+
σ	+	+	+	+
r	+	−	+	−
D	−	+	−	+

→ r ↑ then return from stock ↑, c ↑.

c p Europ. call, put
C P U.S. call, put.

☆☆ American vs. Europ. options.

$C \geq c$

$P \geq p$

In U.S. have more choices and opportunities.

$\max(S_0 - Ke^{-rT}, 0) \leq c \leq S_0$ $\quad$ $\max(Ke^{-rT} - S_0, 0) \leq p \leq K$ $\quad$ $p \leq Ke^{-rT}$

$S_0 - K. \quad < C \leq S_0$ $\quad$ $\max(K - S_0, 0) \leq P \leq K$ $\quad$ $p \leq Ke^{-rT} \leq P \leq K$.

$c + Ke^{-rT} = p + S_0$

$c - p = S_0 - Ke^{-rT}$ = forward contract value.

$\because$ $p = c - S_0 + Ke^{-rT}$

$P \geq p,\ C = c$

$\therefore$ $P \geq c - S_0 + Ke^{-rT}$

$\therefore$ $C - P \leq S_0 - Ke^{-rT}$

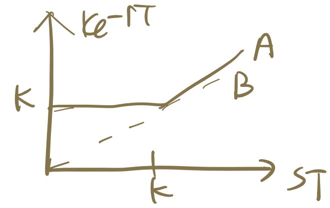
$\because$ $c + K \geq P + S_0$.

$c = C$

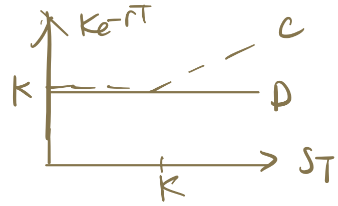
$\therefore$ $C - P \geq S_0 - K$.

$\therefore$ $S_0 - K \leq C - P \leq S_0 - Ke^{-rT}$

Portfolio A : 1 c + Ke^{-rT} (cash)
B : 1 share



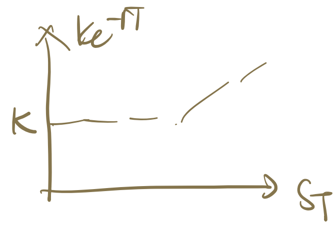
Portfolio C : 1 p + 1 share
D : Ke^{-rT} (cash)



PF A = PF C, $c + Ke^{-rT} = p + S_0$.

Portfolio E : 1 P + 1 share

PF A ≥ PF E. $c + K \geq P + S_0$

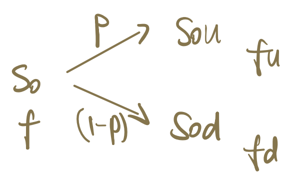


① No dividend, no early American call option exercise.

with dividend, $S_0 - D - K \leq C - P \leq S_0 - Ke^{-rT}$
w/o dividend, $S_0 - K \leq C - P \leq S_0 - Ke^{-rT}$

★ PPT #8 Binomial Trees.

Stock price S_0
expiration T
option current price f.



$$\Delta = \frac{f_u - f_d}{S_0u - S_0d}$$

Value of the riskless portfolio at T is : $S_0u\Delta - f_u = S_0d\Delta - f_d$
value of the portfolio today is : $S_0\Delta - f = (S_0u\Delta - f_u)e^{-rT} = (S_0d\Delta - f_d)e^{-rT}$
value of the option : $f = S_0\Delta - (S_0u\Delta - f_u)e^{-rT} = S_0\Delta - (S_0d\Delta - f_d)e^{-rT}$
$f = (p f_u + (1-p) f_d)e^{-rT}$

$p = \frac{e^{rT} - d}{u - d}$ risk neutral prob. up prob. ; $E[f_T] = [p f_u + (1-p) f_d] = S_0 e^{rT}$

$u = e^{\sigma\sqrt{\Delta t}}$
$d = e^{-\sigma\sqrt{\Delta t}}$

$p^* = \frac{e^{\mu\Delta t} - d}{u - d}$ (real world prob.)

$p = \frac{a - d}{u - d}$, $a = e^{r\Delta t}$ non-yield.
$a = e^{(r-q)\Delta t}$ q → yield
$a = e^{(r-r_f)\Delta t}$ r_f → foreign currency.
$a = 1$ for future contract

☆ PPT #9 Stochastic Process

$\Delta z = \epsilon\sqrt{\Delta t}$, $\epsilon \sim \phi(0,1)$, $\Delta z \sim N(0, \Delta t)$.

Wiener process :

$$dx = a\,dt + b\,dz \qquad \text{若 } G = \ln S,\ dG = (\mu - \sigma^2/2)\,dt + \sigma\,dz$$

Itô process :

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

Itô process for stock prices :

$$dS = \mu S\,dt + \sigma S\,dz$$

$$\frac{dS}{S} = \mu\,dt + \sigma\,dz. \qquad \frac{\Delta S}{S} \sim \phi(\mu\Delta t, \sigma^2\Delta t)$$

Itô's Lemma :

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz.$$

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}(\sigma S)\,dz.$$

Forward price :

$$F_0 = S_0 e^{rT} = F(S_0, 0)$$

$$F(S_t, t) = S_t e^{r(T-t)}$$

Trading days : 252. days.

☆ PPT #10 Black-Scholes Model.

Price is lognormal:

$$E(S_T) = S_0 e^{\mu T}$$

$$Var(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$$

$$\ln S_T - \ln S_0 = \ln\frac{S_T}{S_0} \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

Derivation of BSM :

$$\Delta S = \mu S\Delta t + \sigma S\Delta z$$

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}\right)\Delta t + \sigma S\frac{\partial f}{\partial S}\Delta z$$

Portfolio : $-1 \times$ derivative, $+\frac{\partial f}{\partial S} \times$ shares

$$\Pi = -f + \frac{\partial f}{\partial S}S$$

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S$$

Plug in :

$$\Delta\Pi = -\left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}\right)\Delta t - \sigma S\frac{\partial f}{\partial S}\Delta z + \frac{\partial f}{\partial S}(\mu S\Delta t + \sigma S\Delta z).$$

$$= -\cancel{\frac{\partial f}{\partial S}\mu S\Delta t} - \frac{\partial f}{\partial t}\Delta t - \frac{1}{2}\sigma^2 S^2\Delta t\frac{\partial^2 f}{\partial S^2} - \cancel{\sigma S\frac{\partial f}{\partial S}\Delta z} + \cancel{\frac{\partial f}{\partial S}\mu S\Delta t} + \cancel{\frac{\partial f}{\partial S}\sigma S\Delta z}$$

$$= -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}\right)\Delta t$$

$$\because \Delta\Pi = r\Pi\Delta t, \quad r\left(-f + \frac{\partial f}{\partial S}S\right)\Delta t = -\frac{\partial f}{\partial t}\Delta t - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}\Delta t$$

$$rS\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf$$

☆☆ boundary conditions.

c: $f = \max(S-K, 0)$

p: $f = \max(K-S, 0)$

☆☆ forward contract.

$f = S_0 - Ke^{-rT}$.

☆☆ Black-Scholes Formula.

$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$

$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$.

$$d_1 = \frac{\ln(S_0/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r-\sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

$N(d_2) \to$ Prob. call will be exercised.

Properties: $S_0 \uparrow$, $c \to S_0 - Ke^{-rT}$, $p \to 0$.

$d_1, d_2 \uparrow$; $N(d_1), N(d_2) \to 1$; $N(-d_1), N(-d_2) \to 0$.

$S_0 \downarrow$, $c \to 0$, $p \to Ke^{-rT} - S_0$.

☆☆ With dividend yield.

$c \geq \max(S_0 e^{-qT} - Ke^{-rT}, 0)$

$p \geq \max(Ke^{-rT} - S_0 e^{-qT}, 0)$

Put call parity: $c + Ke^{-rT} = p + S_0 e^{-qT}$

$c = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2)$

$p = Ke^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1)$.

$$d_1 = \frac{\ln(S_0 e^{-qT}/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln(S_0/K) + (r-q+\sigma^2/2)T}{\sigma\sqrt{T}}$$

☆☆ Futures

Drift is zero.

$dF = \sigma F dz$, $q = r$ $\quad dS = (r-q)dt + \sigma S dz$

$c = e^{-rT}(F_0 N(d_1) - K N(d_2))$

$p = e^{-rT}(K N(-d_2) - F_0 N(-d_1))$

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}}.$$

☆ PPT #11 Greek.

The position required in futures for delta hedging is $e^{-(r-q)T}$ times the position required in the corresponding spot contract.

$H_F = e^{-(r-q)T} H_A$.

Gamma neutral quantity: $w_T = -\Gamma/\Gamma_T$

e.g. European call:

$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$

delta: $\Delta = \frac{\partial c}{\partial S} = N(d_1)$

theta: $\Theta = \frac{\partial c}{\partial t} = -rKe^{-r(T-t)} N(d_2) - S N'(d_1) \frac{\sigma}{2\sqrt{T-t}}$

gamma: $\Gamma = \frac{\partial^2 c}{\partial S^2} = \frac{1}{S\sigma\sqrt{T-t}} N'(d_1)$

vega: $V = \frac{\partial c}{\partial \sigma} = S N'(d_1)\sqrt{T}$

rho: $\rho = \partial c/\partial r = TKe^{-rT} N(d_2)$

☆ Section Review

☆☆ Put-call parity: $C + Ke^{-rT} = P + S_0$ without dividend.

$C + Ke^{-rT} + D = P + S_0$ with dividend.

例: HW#7, 11.6, 11.13.

11.6. No dividend. $S_0 = 19$, $T = 3/12$, $K = 20$, $c = 1$, $r = 0.04$

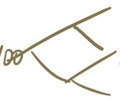
$1 + 20e^{-0.04(3/12)} = p + 19$, 求 p.

11.13. 含2次分红. $T = 6/12$, $K = 30$, $c = 2$, $S_0 = 29$, $D_1 = 0.5$, $D_2 = 0.5$, $r = 0.10$.

$t = 2/12$ $\quad t = 5/12$

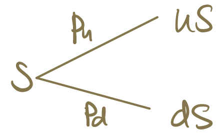
$$29 + p - 0.5e^{-0.1\times 2/12} - 0.5e^{-0.1\times 5/12} = 2 + 30e^{-0.1\times 6/12}$$, 求 p.

☆☆ Binomial Tree: $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$ ← 服从这个的话, 树长这样: 100 … 100.



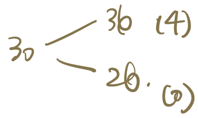
risk-neutral probability: $p_u = \dfrac{e^{rT} - d}{u - d}$

$\mathbb{E}(S_T) = Se^{rT} = pSu + (1-p)Sd$



例: Clicker Question.

Short 1 call option



$K = 32$ $\quad \Delta = \dfrac{f_u - f_d}{Su - Sd} = \dfrac{4 - 0}{36 - 26} = 0.4$.

∴需要 long 0.4 stock 去 hedge.

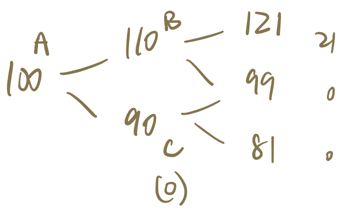
☆ BSM $\quad \Delta c = N(d_1) > 0$, $\Delta p = -N(-d_1)$

$c - p = S - Ke^{-rT}$

$\partial/\partial S$: $\Delta c - \Delta p = 1$ $\quad \Delta p = \Delta c - 1$

$= -(1 - N(d_1)) = -N(-d_1)$

例: $S_0 = 100$, $r = 0.08$, $T = 1$, $K = 100$. $c = ?$



$u = 1.1$

$d = 0.9$

$p = \dfrac{e^{0.08(0.5)} - 0.9}{1.1 - 0.9}$

$B = (p(21) + (1-p)(0))e^{-0.08(0.5)}$

$C = (p(0) + (1-p)(0))e^{-0.08(0.5)} = 0$.

$A = (p(B) + (1-p)C)e^{-0.08(0.5)}$

$= p^2(e^{-0.08\times 1})$

option on future

⇓

$p = \dfrac{1 - 0.9}{1.1 - 0.9}$

☆☆ Stochastic Process.

Ito Lemma $\begin{cases} dS = a\,dt + b\,dz \\ dG = \left(\left(\frac{\partial G}{\partial S}\right)a + \frac{\partial G}{\partial t} + \frac{1}{2}\left(\frac{\partial^2 G}{\partial S^2}\right)b^2\right)dt + \left(\frac{\partial G}{\partial S}\right)b\,dz. \end{cases}$

$dS = \mu S\,dt + \sigma S\,dt$

HW10, 15.11. $G = \ln S$, $\frac{\partial G}{\partial S} = \frac{1}{S}$, $\frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$, $\frac{\partial G}{\partial t} = 0$, $a = \mu S$, $b = \sigma S$.

$dG = \left(\frac{1}{S}\cdot \mu S + 0 + \frac{1}{2}\left(-\frac{1}{S^2}\right)\sigma^2 S^2\right)dt + \frac{1}{S}\sigma S\,dz = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma\,dz$.

$$d\ln S = (\mu - \tfrac{1}{2}\sigma^2)\,dt + \sigma\,dz \sim N((\mu-\tfrac{1}{2}\sigma^2)dt, \sigma^2 dt).$$

drift term　　　brownian motion.
跟着时间走的.　　随机项.　diffusion term.
deterministic term.

$$\ln S_T = \ln S_0 + \Delta \ln S_T \sim N((\mu-\tfrac{1}{2}\sigma^2)T + \ln S_0, \sigma^2 T).$$

assume payoff of option f at time T is $\ln S_T$.
S_0 is stock price at time t

$$f = e^{-r(T-t)}\,\mathbb{E}(\ln S_T) = e^{-r(T-t)}\left[(\mu-\tfrac{1}{2}\sigma^2)(T-t) + \ln S_0\right]$$

$$\partial f/\partial t + rS\,\partial f/\partial S + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \Rightarrow \text{BSM diff. eqn.}$$

☆ BSM

$$c = S_0 N(d_1) - Ke^{-rT}N(d_2)$$
$$p = Ke^{-rT}N(-d_2) - S_0 N(-d_1)$$

$$d_1 = \frac{\ln S/K + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$\Delta c = N(d_1)$$
$$\Delta p = -N(-d_1).$$

If cont. dividend, q applied.

$$c = Se^{-qT}N(d_1) - Ke^{-rT}N(d_2)$$
$$p = Ke^{-rT}N(-d_2) - S_0 e^{-qT}N(-d_1)$$

$$d_1 = \frac{\ln S/K + (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

☆ Boundary condition.

$$S \geq C \geq c \geq (S - Ke^{-rT})^+$$
$$K \geq P \geq p \geq (Ke^{-rT} - S)^+$$

American call, no dividend. never exercise early.
　　　　　　　if dividend, may exercise early.
American put, can exercise early.

☆ Greek letters.

K
S — delta, gamma　　　　$V_c = V_p > 0$.
T — theta　　　　$\rho = \partial V/\partial r$　call >0, put <0.
σ — vega　　Violatility.
r — rho.

Bn 13.11.　$S_0 = 40$.　$r = 0.08$ quarterly comp.　calculate 3 mth. put. use no-arbitrage & risk-neutral.
$K = 40$.

$$40 \begin{cases} 45 & f_u = 0 \\ 35 & f_d = 5 \end{cases}$$

portfolio : $\begin{cases} -\Delta : \text{shares} \\ +1 : \text{option put} . \end{cases}$

$$\Delta = \frac{f_u - f_d}{S_{0}u - S_{0}d} = \frac{0-5}{45-35} = -0.5$$

买0.5 shares
买一个 put options.

$40 \times 0.5 = 35 \times 0.5 + 5 = 22.5$

$(40 \times 0.5 + f) \times (1 + 8\% \div 4)$ 解 f.

risk neutral

$$\underset{f}{40} \begin{cases} \xrightarrow{p} & f_u = 0 \\ \xrightarrow{1-p} & f_d = 5 \end{cases}$$

$$p = \frac{e^{rT} - d}{u - d}$$

$$p = \frac{(1 + r_q/4) - d}{u - d} .$$

$u = 45/40$, $d = 35/40$.

$r_q \rightarrow r_c$

553.444/644 Introduction to Financial Derivatives

Module 12: Final Exam Review and End-of-Course Wrap-up

JOHNS HOPKINS WHITING SCHOOL of ENGINEERING

Where we are

- Last Day of Classes: Wednesday, December 6th
- No Office Hours Thursday, December 7
  - Email for Alternate
- Sections will be held on Friday, December 8
  - See Note, Next …

JOHNS HOPKINS WHITING SCHOOL of ENGINEERING

Schedule

- Section:
  - 644/S2: Hodson 211 (29) 10:00am Friday
    - TA: Andrew Searns
  - 644/S3: Hodson 216 (15) 11:00am Friday
    - TA: Kaiwen Huang
  - 644/S4: Hodson 216 (13) 12:00 Noon Fri
    - TA: Linyi Xiao
  - 644/S5: Hodson 211 (7) 1:30pm Friday
    - TA: Linyi Xiao

JOHNS HOPKINS WHITING SCHOOL of ENGINEERING

Where we are

- Final Exam: Tuesday, December 19; 9am – Noon
  - Location: Maryland 110
- Note on Final Exam
  - Close Book
  - NO CHEAT SHEET
  - Calculators will be needed
  - Standard Normal Distribution table will be provided

JOHNS HOPKINS WHITING SCHOOL of ENGINEERING

Assignment

- For November 13th, 15th and 27th
- Read: Hull Chapter 15 (Black-Scholes-Merton)
- Read: Hull Chapter 17-18 (Options on Indices & Currencies + Futures Options)
- Homework 10: Problems (Due December 4th)
  - Posted in Canvas, also
    - Chapter 15 (15, 10e): 4, 5, 11, 13, 15, 17; 28
    - Chapter 17 (17, 10e): 4, 22; 23*
    - Chapter 18 (18, 10e): 7, 8, 15; 22

* Problem Data varies from edition to edition

JOHNS HOPKINS WHITING SCHOOL of ENGINEERING

Assignment

- For November 29th & December 4th
- Read: Hull Chapter 19 (The Greeks)
- Homework 11: Problems (Due December 6th)
  - Chapter 19 (18, 8e; 17, 7e): 1 (2), 6 (10), 7 (18); 20 (24)
- Course Wrap-up and Final Exam Review (Dec 6)

JOHNS HOPKINS WHITING SCHOOL of ENGINEERING

Schedule

Date 2023	Module	Due	Comments
Wednesday, October 18	Midterm		M1 – M6: TA
Monday, October 23	Options (M7)		Hull: 11 - 12
Wednesday, October 25	Options (M7)		
Monday, October 30	Binomial (M8)	HW 7	Hull: 13
Wednesday, November 01	Binomial (M8)		
Monday, November 06	Stoch Process (M9)	HW 8	Hull: 14
Wednesday, November 08	Stoch Process (M9)		
Monday, November 13	BSM Models (M10)		Hull: 15, 17 – 18
Wednesday, November 15	BSM Models (M10)		
November 20 – 24	Thanksgiving Break		
Monday, November 27	BSM Models (M10)	HW 9	
Wednesday, November 29	Greeks (M11)		
Monday, December 04	Greeks (M11)	HW10	Hull: 19
Wednesday, December 06	Final Review	HW11	
Tuesday, December 19: 9am-Noon	**Final Exam**		**M7 – M11**

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Final Review

- Know the properties of stock options;
- be able to explain/develop (prove?) the “bounding” conditions for European-style option value and the condition of put-call parity
  - Know the put-call parity relation
- Explain the considerations when expanding this analysis to American-style options and for dividend-paying (/-yielding) stock

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Lower Bound for European Call Option Price (No Dividends)

$$\boxed{c \geq \max\left(S_0 - Ke^{-rT}, 0\right)}$$

- Consider the two portfolios

  Portfolio A: One European Call option plus cash = Ke^{-rT}

  Portfolio B: One Share
- For PF A at <u>expiration</u>, T
  - If the cash is invested, it will grow to K at T
  - If $S_T > K$, then the option is exercised; PF A is worth S_T
    - We pay for the share with the cash and have a share worth S_T
  - If $S_T < K$, then the option is worthless; PF A is worth K
  - Hence PF A is worth max (S_T, K)

JOHNS HOPKINS WHITING SCHOOL of ENGINEERING

Lower Bound for European Call Option Price (No Dividends)

- PF A is worth max (S_T, K) at expiration
- PF B at <u>expiration</u> is worth S_T
- At expiration, PF A has value of at least PF B
- In the absence of arbitrage, this must also be true today (Discuss)
- Hence $c + Ke^{-rT} \geq S_0$ or $c \geq S_0 - Ke^{-rT}$
- Finally, at worst the call will expire worthless, so

$$\boxed{c \geq \max\left(S_0 - Ke^{-rT}, 0\right)}$$

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European Option Put-Call Parity (No Dividends)

- Consider the two portfolios

  Portfolio A: One European Call option plus cash = Ke^{-rT}

  Portfolio C: One European Put option plus One Share
- PF A at <u>expiration</u>, T
  - If the cash is invested, it will grow to K at T
  - If $S_T > K$, then the option is exercised; PF A is worth S_T
    - We pay for the share with the cash and have a share worth S_T
  - If $S_T < K$, then the option is worthless; PF A is worth K
  - Hence PF A is worth max (S_T, K)

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European Option Put-Call Parity (No Dividends)

- PF A is worth max (S_T, K) at expiration
- PF C at <u>expiration</u>, T
  - If $S_T < K$, then the option is exercised; PF C is worth K
    - We deliver the share and receive K
  - If $S_T > K$, then the option is worthless; PF C is worth S_T
  - Hence PF C is worth max (S_T, K)
- At <u>expiration</u>, PF A has value equal to PF C
- In the absence of arbitrage, this is also true today
- Hence $c + Ke^{-rT} = p + S_0$
- This is known as ***Put-Call Parity***

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Final Review

- Be prepared to construct a binomial tree to evaluate an option on a given underlying asset.
- Understand the relationship of the probability for an up/down movement to volatility
- Know the principle of risk-neutral valuation and be prepared to describe the risk-neutral valuation rules.
- Know Girsanov's Result
- Be able to apply the binomial tree result to indexes, currencies, dividend yielding assets, and futures

Generalization of the Binomial Model

- Consider the portfolio that is
  - Long Δ shares and
  - Short 1 option

$$\begin{tikzcd} & S_0 u\Delta - f_u \\ S_0\Delta - f \arrow[ur] \arrow[dr] & \\ & S_0 d\Delta - f_d \end{tikzcd}$$

- The portfolio is riskless when $S_0 u\Delta - f_u = S_0 d\Delta - f_d$ or

$$\boxed{\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}}$$

Generalization of the Binomial Model

- Substituting for Δ, where $\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$
- We obtain

$$\boxed{f = [p f_u + (1 - p) f_d] \times e^{-rT}}$$

- Where we define p as

$$\boxed{p = \frac{e^{rT} - d}{u - d}}$$

- This is the **Generalized Binomial Model** of Option Value

One Last Detail (about the real world)

- The expected stock price at the end of the first time step Δt:

$$E(S_T) = p^* S_0 u + (1 - p^*) S_0 d = p^* S_0 (u - d) + S_0 d = S_0 e^{\mu \Delta t}$$

  where μ is the expected return on the stock
- And the probability, p^*, can be found as $p^* = \frac{e^{\mu\Delta t} - d}{u - d}$
- The volatility σ of price is defined so that

$$\sigma\sqrt{\Delta t} \ = std\ dev\ (stock\ price\ return;\ \Delta t)$$

- There are two points in our space after Δt, $S_0\, u$ and $S_0\, d$
- Making use of the probability result for p^* then, since

Variance = mean of the "squares" minus the "square" of the mean (of return)

$$= p^* \left(\frac{S_0 u - S_0}{S_0}\right)^2 + (1 - p^*)\left(\frac{S_0 d - S_0}{S_0}\right)^2 - \left[p^*(u-1) + (1-p^*)(d-1)\right]^2$$

One Last Detail

$$p^* = \frac{e^{\mu\Delta t} - d}{u - d}$$

- So the variance of the stock price return is

$$p^* u^2 + (1 - p^*) d^2 - [p^* u + (1 - p^*) d]^2 = \sigma^2 \Delta t$$

- Substituting for p^* from above, we get

$$e^{\mu\Delta t}(u + d) - ud - e^{2\mu\Delta t} = \sigma^2 \Delta t$$

  and one solution for u and d is:

$$\boxed{\begin{aligned} u &= e^{\sigma\sqrt{\Delta t}} \\ d &= e^{-\sigma\sqrt{\Delta t}} \end{aligned}}$$

  where σ is the volatility and Δt is the length of the time step (and when terms in Δt^2 and higher powers of Δt are ignored) and as Δt gets small (use $e^x = 1 + x + \frac{x^2}{2!} + \ldots$)
- This is the approach used by <u>Cox, Ross, and Rubinstein</u>
- Standard practice for constructing the lattice

Girsanov's World

- The variance of <u>this</u> stock price return model is

$$p(u-1)^2 + (1-p)(d-1)^2 - [p(u-1) + (1-p)(d-1)]^2 = [e^{r\Delta t}(u+d) - ud - e^{-2r\Delta t}] = \sigma^2 \Delta t$$

- And when substituting $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$ from before, we find <u>this **too** equals</u> $\sigma^2 \Delta t$ (when terms in Δt^2 and higher powers of Δt are ignored)!
- *Hence, we see that when moving from the real world to the risk-neutral world, the expected return on the stock changes, from μ to r , but the stock price <u>volatility remains the same</u> (at least in the limit as the time step becomes arbitrarily small) – **Girsanov's Theorem***
- Use u & d from above to replicate observed volatility for both the risk-neutral world and real world!

The General Case with Other Assets

- Use u & d to replicate observed volatility of the asset prices for the lattice
- For up probabilities, use $\boxed{p = \frac{a - d}{u - d}}$
  - Where $a = e^{r\Delta t}$ for a <u>non-dividend paying stock</u>
  - Where $a = e^{(r-q)\Delta t}$ for a <u>dividend yielding stock</u> or for a <u>stock index</u> where <u>q is the yield</u> on the index
    - Indeed, $S_0 \to S_0 e^{\mu \times \Delta t} = S_0 e^{(r-q)\times\Delta t} \Rightarrow pS_0 u + (1-p) S_0 d = S_0 e^{(r-q)\times\Delta t}$
    - And $\Rightarrow p = \frac{e^{(r-q)\times\Delta t} - d}{u - d}$
  - Where $a = e^{(r - r_f)\Delta t}$ for a <u>currency</u> where <u>r_f is the foreign risk-free rate</u>
  - Where $a = 1$ for a <u>futures contract</u> (more about this, latter)

Final Review

- Know the form for the Ito Process of a stock price for a non dividend paying stock
- Know Ito's Lemma and be able to apply to form a process model for a function of an underlying process
- Know the distribution for the mean (and variance) of the state of this Ito Process and what it means for it to be a Log-Normal Process (i.e., (14.18)-(14.19) and/or (15.2)-(15.3))

Ito's Lemma for a Stock Price Process

- When the **stock price process** is an Ito Process

$$dS = \mu S\, dt + \sigma S\, dz$$

- Then for a function G of S and t Ito's Lemma gives that

$$\boxed{dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + ½ \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S\, dz}$$

- So …

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Lognormal Property of Stock Prices

- From Ito's Lemma, letting $G = \ln S$, gives $\boxed{\frac{\partial G}{\partial S} = \frac{1}{S};\ \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2} \quad \partial G/\partial t = 0}$

$$dG = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma\, dz$$

- So the change in $\ln S$ between 0 and some future T is normally distributed with mean $(\mu - \sigma^2/2)T$ and variance $\sigma^2 T$
- Which we can express as

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2} \right) T,\ \sigma^2 T \right]$$

or

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right) T,\ \sigma^2 T \right]$$

- Since $\ln S_T$ is normal, S_T is lognormally distributed
- And

$$\ln \frac{S_T}{S_0} \sim \phi\left[\left(\mu - \frac{\sigma^2}{2} \right) T,\ \sigma^2 T \right]$$

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Final Review

- Be prepared to give an expression for the Black-Scholes-Merton differential equation and to show an understanding of its properties
- Know the principle of risk-neutral valuation and be prepared to describe the risk-neutral valuation rules. Be prepared to apply the principle of risk neutral valuation to determine the value of a forward contract

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Summary

- Black-Scholes-Merton Differential Equation
  - Derived from the Stock Price process and Ito's lemma, knowing the derivative is a function of the Stock Price process
  - Form riskless portfolio: short one derivative & long $\Delta \triangleq \frac{\partial f}{\partial S}$ x shares
  - Gives the differential equation: $\boxed{\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf}$
  - With boundary conditions, it's solution describes the derivative
- Risk-Neutral Valuation applies as the BSM Differential Equation is independent of any variable affected by risk-preference – only $S, t, \sigma,$ and r – no expected return, μ
- Principle of Risk-Neutral Valuation
  - Assume the expected return of the underlying asset is r , i.e. $\mu = r$
  - Calculate the expected payoff from the derivative $\boxed{f = e^{-rT} \hat{E}[f_T]}$
  - Discount the expected payoff at the risk-free rate, r

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Final Review

- Be prepared to exhibit the Black-Scholes formulas for both a Euro-style call- and put-option on a non-dividend paying stock. Know the meaning of its terms and parameters

The Black-Scholes Formulas

- Black-Scholes Formulas for the Present Value of a European Call, c (Put, p) with expiration T and strike K on a non-dividend paying stock with price S_0

$$\boxed{\begin{aligned} c &= S_0\, N(d_1) - K\, e^{-rT} N(d_2) \\ p &= K\, e^{-rT}\, N(-d_2) - S_0\, N(-d_1) \end{aligned}}$$

where

$$\boxed{d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}}$$

and

$$\boxed{\begin{aligned} d_2 &= \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \\ &= d_1 - \sigma\sqrt{T} \end{aligned}}$$

Final Review

- Understand and be prepared to analyze the situation for a dividend yielding stock and its relationship to the value of an option on that stock
- Know how the model for a dividend yielding stock can adapted to indexes, currencies, and futures
- Know how the **binomial model** is applied **for futures**

Option Results for a Stock paying a known Dividend Yield

- Extending Black-Scholes formulas to a stock paying a dividend yield q (replace S_0 by $S_0\, e^{-qT}$)

$$c = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$$

$$p = K e^{-rT}\, N(-d_2) - S_0 e^{-qT} N(-d_1)$$

where $$d_1 = \frac{\ln(S_0/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - q - \sigma^2/2)T}{\sigma\sqrt{T}}$$

since $$\ln\left(\frac{S_0 e^{-qT}}{K}\right) = \ln\left(S_0 e^{-qT}\right) - \ln(K) = \ln(S_0) - \ln(K) + \ln\left(e^{-qT}\right)$$

$$= \ln\left(\frac{S_0}{K}\right) - qT$$

Summary of Key Results

- We can treat stock indices, currencies, and futures like a stock paying a dividend yield of q
  - For stock indices, q = average dividend yield on the index over the option life
  - For currencies, $q = r_f$
  - For futures, $q = r$ and $\hat{E}(F_T) = F_0$

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Final Review

- Know the key option risk-parameters (Greeks) of Delta, Gamma and Vega
- In particular, know the formula for Gamma and Vega and how they are derived; and
- know the technique for simultaneously hedging these for an option- laden portfolio

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Final Review

- BOTTOM LINE:
  - Binomial Tree – all forms -- formulas
  - Ito's Lemma Formula
  - Black-Scholes-Merton differential equation
  - Black Scholes formula (including d's)
  - Futures Options

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