CSE253 Homework 1

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1 Problems from Bishop

1.1

From equation (1.42), we can get

$$\begin{aligned} S_d &= \frac{\Pi_{i=1}^{d} \int_{-\infty}^{\infty} e^{-x_i^2} dx_i}{\int_0^{\infty} e^{-r^2} r^{d-1} dr} && (1) \\ &= \frac{\Pi_{i=1}^{d} \int_{-\infty}^{\infty} e^{-\frac{2}{2}x_i^2} dx_i}{\int_0^{\infty} e^{-r^2} r^{d-1} dr} && (2) \\ & && (3) \end{aligned}$$

Then use equation(1.41) where $\lambda = 2$ to simplify $\int_{-\infty}^{\infty} e^{-\frac{2}{2}x_i^2} dx_i$ as $\pi^{\frac{1}{2}}$
Then the nominator of S_d is

$$\Pi_{i=1}^{d} \pi^{\frac{1}{2}} = \pi^{\frac{d}{2}}$$

For the denominator, let $u = r^2$ and $du = 2rdr$ then

$$\begin{aligned} \int_0^{\infty} e^{-r^2} r^{d-1} dr &= \int_0^{\infty} e^{-r^2} r^{d-2} r dr && (4) \\ &= \frac{1}{2} \int_0^{\infty} e^{-u} u^{\frac{d}{2}-1} du && (5) \\ &= \frac{1}{2} \Gamma(d/2) && (6) \end{aligned}$$

Therefore, we can get

$$\begin{aligned} S_d &= \frac{\pi^{\frac{d}{2}}}{\frac{1}{2}\Gamma(d/2)} && (7) \\ &= \frac{2\pi^{\frac{d}{2}}}{\Gamma(d/2)} \; (1.43 \; proved) && (8) \end{aligned}$$

When $d = 2$, $S_2 = \frac{2\pi}{\Gamma(1)} = 2\pi$, which is the perimeter of a circle
When $d = 3$, $S_3 = \frac{2\pi^{\frac{3}{2}}}{\Gamma(\frac{3}{2})} = 4\pi$, which is the surface area of a unit sphere in 3 dimensions.

1.2

(1) First get the surface area of a sphere of radius a in d dimensions. Consider the surface of a unit sphere in d dimensions is the sum of the area of many enough small cube. Then the area of the small cube is b^{d-1}. If we extend this problem into a sphere of radius a in d dimension, it is obvious that the small cube's area is $(ab)^{d-1}$ since each edge is a times as before. Thus, the area of a sphere of radius a in d dimensions is $S_d a^{d-1}$. Thus the volume of a sphere of radius a in d dimensions is

$$\begin{aligned} V_d &= \int_0^a S_d r^{d-1} dr \qquad (9)\\ &= \frac{S_d a^d}{d} \; (1.45 \; proved) \qquad (10) \end{aligned}$$

(2) The volume of a hypercube of side 2a is $(2a)^d$, so

$$\frac{V\ of\ hypersphere}{V\ of\ hypercude} = \frac{\frac{S_d a^d}{d}}{(2a)^d} = \frac{S^d}{d2^d} = \frac{2\pi^{\frac{d}{2}}}{d2^d\Gamma(\frac{d}{2})} = \frac{\pi^{\frac{d}{2}}}{d2^{d-1}\Gamma(\frac{d}{2})} \; (1.46 \; proved) \qquad (11)$$

(3) As $d \to \infty$, using equation 1.47, in which $x = \frac{d}{2}$,

$$\begin{aligned} \Gamma(\frac{d}{2}) &\simeq (2\pi)^{\frac{1}{2}} e^{-\frac{d}{2}+1}(\frac{d}{2}-1)^{\frac{d}{2}-\frac{1}{2}} \qquad (12)\\ &\simeq (2\pi)^{\frac{1}{2}} e^{-\frac{d}{2}}(\frac{d}{2})^{\frac{d}{2}} \qquad (13) \end{aligned}$$

Therefore,

$$\frac{V\ of\ hypersphere}{V\ of\ hypercude} \simeq \frac{(\pi e)^{\frac{d}{2}}}{d2^{d-1}(2\pi)^{\frac{1}{2}}(\frac{d}{2})^{\frac{d}{2}}} \xrightarrow{d\to\infty} 0 \qquad (14)$$

(4) With geometry knowledge,

$$d_{center\ to\ corner} = \frac{2a\sqrt{d}}{2} = a\sqrt{d} \qquad (15)$$

$$d_{center\ to\ edge} = \frac{2a}{2} = a \qquad (16)$$

Therefore, we have

$$\frac{d_{center\ to\ corner}}{d_{center\ to\ edge}} = \sqrt{d} \xrightarrow{d\to\infty} \infty \; (proved) \qquad (17)$$

1.3

(1) With equation (1.45), the volume of a sphere of radius a in d dimension is $V_{d,a} = \frac{S_d a^d}{d}$, and the volume of a sphere of radius $a-\epsilon$ in d dimension is $V_{d,a-\epsilon} = \frac{S_d(a-\epsilon)^d}{d}$. Therefore, the fraction of the volume of the sphere which lies at the value of the radius between $a-\epsilon$ and a is

$$\frac{V_{d,a}}{V_{d,a}-V_{d,a-\epsilon}} = \frac{a^d-(a-\epsilon)^d}{a^d} = 1-(1-\frac{\epsilon}{a})^d = f \; (1.48 \; proved) \qquad (18)$$

Since $0 < \epsilon < a$, we have $0 < 1-\frac{\epsilon}{a} < 1$, therefore, as $d \to \infty$, $(1-\frac{\epsilon}{a})^d \to 0$, $f = 1-(1-\frac{\epsilon}{a})^d \to 1$ (proved)

(2) when $\frac{\epsilon}{a} = 0.01$,

$$\begin{aligned} f_{d=2} &= 1-(1-0.01)^2 = 0.0199\\ f_{d=10} &= 1-(1-0.01)^1 0 = 0.0956\\ f_{d=1000} &= 1-(1-0.01)^1 000 = 0.999 \end{aligned}$$

When $\epsilon = \frac{a}{2}$, $\frac{\epsilon}{a} = 0.5$, the fraction lying inside ϵ is

$$f_{d=2} = (1-0.5)^2 = 0.25$$
$$f_{d=10} = (1-0.5)^{10} = 9.7e-4$$
$$f_{d=1000} = (1-0.5)^{1000} = 9.3e-302$$

1.4

(1) From equation (1.42), we can get

$$\int_{-\infty}^{+\infty} e^{-||x_i||^2} dx_i = S_d \int_0^{\infty} e^{-r^2} r^{d-1} dr \tag{19}$$

Since $\int p(x)dx = \int p(r)dr$, we can get

$$p(r) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{r^2}{2\sigma^2}} S_d r^{d-1} = \frac{S_d r^{d-1}}{(2\pi\sigma^2)^{1/2}} e^{-\frac{r^2}{2\sigma^2}} \ (1.50\ proved) \tag{20}$$

(2) Take the derivative of p(r)

$$\frac{dp(r)}{dr} = \frac{S_d(d-1)r^{d-2}}{(2\pi\sigma^2)^{1/2}} e^{-\frac{r^2}{2\sigma^2}} - \frac{S_d r^d}{(2\pi\sigma^2)^{1/2}\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \tag{21}$$
$$= \frac{S_d r^{d-2}}{(2\pi\sigma^2)^{1/2}\sigma^2} e^{-\frac{r^2}{2\sigma^2}} [(d-1)\sigma^2 - r^2] \tag{22}$$

when $\frac{dp(r)}{dr} = 0$, $r^2 = (d-1)\sigma^2 \xrightarrow{d\to\infty} r \simeq \sqrt{d}\sigma$, proved.

(3) With equation (1.50), we have

$$\frac{p(\hat{r}+\epsilon)}{p(\hat{r})} = (\frac{r+\epsilon}{r})^{d-1} e^{-\frac{(r+\epsilon)^2 - r^2}{2\sigma^2}} \tag{23}$$
$$= (1+\frac{\epsilon}{r})^{d-1} e^{-\frac{2r\epsilon+\epsilon^2}{2\sigma^2}} \tag{24}$$
$$= e^{(d-1)ln(1+\epsilon/r)} e^{-\frac{2r\epsilon+\epsilon^2}{2\sigma^2}} \tag{25}$$
$$= e^{\frac{\epsilon}{r}\frac{r^2}{\sigma^2} - \frac{\epsilon^2}{2r^2}\frac{r^2}{\sigma^2} - \frac{r\epsilon}{\sigma^2} - \frac{\epsilon^2}{2\sigma^2}} \ (ln(1+x) \simeq x - x^2/2,\ d-1 \simeq d) \tag{26}$$
$$= e^{-\frac{\epsilon^2}{\sigma^2}} \ (1.51\ proved) \tag{27}$$