Rigid Body Motions

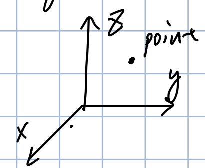
1.1 Fundamentals

Kinematics describes motion of a rigid body in space

(A) Concept of Space.

What is the space in which rigid bodies move?

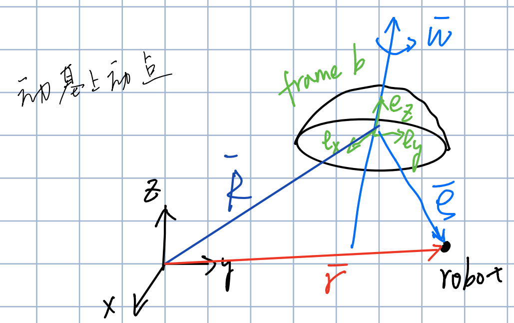
R^3 euclidean space



3 degrees of freedom

(B) Concept of world or Inertial frame: Where is the origin of R^3?

accelerating frames "create" constraint forces



velocity of "robot"

$$\frac{d\bar{r}}{dt} = \dot{\bar{r}} = \frac{d}{dt}(\xi \bar{e}) + \dot{\bar{R}}$$

相对速度

$$= \dot{\bar{R}} + \bar{V}_\xi + \bar{w} \times \bar{\xi}$$

牵连速度

$$\ddot{\bar{r}} = \ddot{\bar{R}} + \bar{a}_\xi + \bar{w} \times (\bar{w} \times \bar{\xi}) + \dot{\bar{w}} \times \bar{\xi} + 2\bar{w} \times \bar{V}_\xi$$

动系的加速度 acceleration of moving frame

相对加速度

centripetal accel. correction

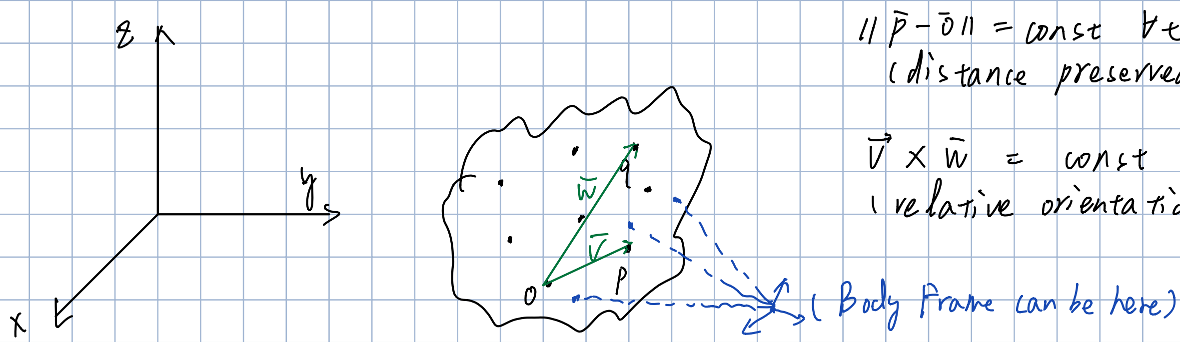
angular accel correction

Coriolis acceleration correction. 科氏.

+ 牵连加速度 +

(C) Concept of rigid body. (RB).

A (RB) is a completely undisturbable object that consists of point masses with distances and relative orientations preserved throughout time



$\|\bar{p} - \bar{o}\| = const \quad \forall t$
(distance preserved)

$\vec{v} \times \bar{w} = const$
(relative orientation preserved).

in R^3. a rigid body has 6 dof.

(D) Body Frame: We don't need to track all points of a rigid body

Learning Objectives

1). Every theory only as good as its underlying assumptions.

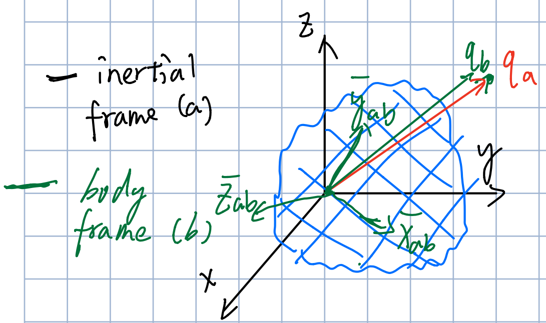
2). You understand that $\mathbb{R}^3$ ($\mathbb{R}^2$), Inertial Frame, rigid body, and body frame are purely math concepts

3). Explain the abstractions they represent.

1.2 Tracking RB in Space

(means to track position $\wedge$ orientation of body frame attached to RB).

(A). Rotation Matrix: How can we track orientation of RB in $\mathbb{R}^3$?



knowing orientation $\Leftrightarrow$ knowing $\bar{x}_{ab}, \bar{y}_{ab}, \bar{z}_{ab}$

unit vectors, orthonormal

<u>Def</u>: $R_b^a = [\bar{x}_{ab}, \bar{y}_{ab}, \bar{z}_{ab}]$ is the Rotation matrix $\in \mathbb{R}^{3\times3}$

Rotation matrix not only describes orientation of a RB, but also transforms points between coordinate systems

$\bar{q}_a = R_b^a \bar{q}_b$

R_b^a $a \Leftarrow$ under which coor-system, $b \Leftarrow$ body frame

Tracking across multiple frames: $\bar{q}_a = \underbrace{R_b^a R_c^b}_{R_c^a} \cdot \bar{q}_c$

$R_c^a = R_b^a R_c^b$

$R_b^a R_c^b \neq R_c^b R_b^a$ (Do not commute!).

(B) Algebraic Properties of R

like a vector, R is mathematical object.

R is member of special Orthognal group (SO_3)

$SO(n) = \{R \in \mathbb{R}^{n\times n}, RR^T = I, \det(R) = +1\}$

R 的列代表了 body frame 的 axis 在 inertial frame 的坐标.

① $RR^T = I$:

$R = [\bar{r}_1, \bar{r}_2, \bar{r}_3]$, $r_i^T r_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$ $\Rightarrow R^TR = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \Rightarrow I.$

$\Rightarrow R^TR = I = R^{-1}R.$

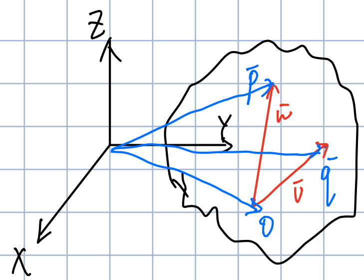
$\Rightarrow RR^T = RR^{-1} = I.$

② $\det R = +1$:

$$\det(R) = \bar{r}_1^T \underbrace{(\bar{r}_2 \times \bar{r}_3)}_{\bar{r}_1} \Rightarrow = \bar{r}_1^T \bar{r}_1 = 1$$?

③ group: $R_1 \cdot R_2 \in SO(3)$

Is R applied to RB really preserving distances and relative orientations?



$$\underset{\text{inertial frame}}{\|\bar{p} - \bar{q}\|} = \underset{\text{body frame}}{\|R\bar{p} - R\bar{q}\|} \; ?$$

$$\|R\bar{p} - R\bar{q}\|^2 = (R\bar{p} - R\bar{q})^T (R\bar{p} - R\bar{q})$$
$$= (\bar{p} - \bar{q})^T R^T R (\bar{p} - \bar{q})$$
$$= (\bar{p} - \bar{q})^T (\bar{p} - \bar{q})$$
$$= \|\bar{p} - \bar{q}\|^2$$

$\Rightarrow$ does not change distances

relative orientations?

$R(\bar{v} \times \bar{w}) = (R\bar{v}) \times (R\bar{w})$ (need to show).

$\Uparrow$ derivation.

$$(\widehat{R\bar{v}}) = R\hat{\bar{v}}R^T$$

$$(R\bar{v}) \times (R\bar{w}) = (\widehat{R\bar{v}})R\bar{w} = R\hat{\bar{v}}R^T R\bar{w} = R\hat{\bar{v}}\bar{w} = R(\bar{v} \times \bar{w})$$

$$\bar{a} \times \bar{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -a_1 b_3 + a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}_{\hat{a}} \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}_{\bar{b}} = \hat{a} \cdot \bar{b}$$

$\hat{a}^T = -\hat{a}$: skew symmetric matrix

(c) Euler Angles: What are least parameter representations of R?

How many params does $R \in \mathbb{R}^{3\times 3}$ have? $= 9$.

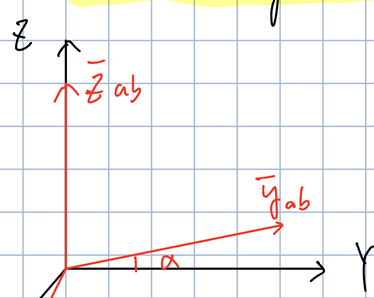
$$\left.\begin{matrix} RR^T = I \\ R^T R = I \end{matrix}\right\} \text{ 6 constraints}$$

$9 - 6 = 3$ free params!

$$\left[\,\diagdown\, b\,\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ Representations of R which reduce # of free parameters:

Elementary Rotations:



$$R_z(\alpha) = [\bar{x}_{ab}(\alpha) \;\; \bar{y}_{ab}(\alpha) \;\; \bar{z}_{ab}(\alpha)]$$
$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

arbitrary rotation: $R = R_Z(\alpha) \cdot R_Y(\beta) \cdot R_X(\gamma)$.

angles α, β, γ. Euler angles.

need to define: current frame or fixed rotations in addition to particular composition rule

Z Y X = fixed frame
Roll Pitch Yaw.

$(\alpha, \beta, \gamma) \underset{\text{not always}}{\overset{\text{always}}{\rightleftarrows}} R$ (singular problem)

$R_{ab} = R_X(\alpha) \cdot R_Y(\beta)\, R_Z(\gamma)$

$= \begin{bmatrix} C\alpha C\beta C\gamma - S\alpha S\gamma & \cdots \\ \vdots & \ddots \end{bmatrix} \xrightarrow{R_{ab} \to \alpha\beta\gamma?}$

$\beta = \operatorname{atan2}\left(\sqrt{r_{31}^2 + r_{33}^2},\ r_{33}\right)$

$\alpha = \operatorname{atan2}\left(\frac{r_{23}}{\sin\beta},\ \frac{r_{13}}{\sin\beta}\right)$

$\gamma = \operatorname{atan2}\left(\frac{r_{32}}{S_\beta},\ \frac{-r_{31}}{S_\beta}\right)$

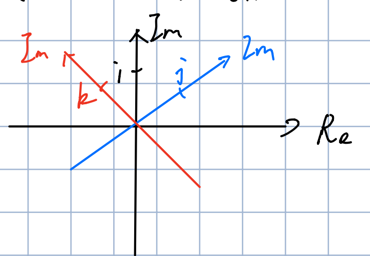
r_{ij}: matrix elements.

if $\sin\beta = 0 \to \alpha, \gamma$ cannot be resolved

$\Rightarrow$ Gimbal Lock

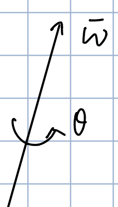
(D) Quaternions: A safe parameteritation of R (used in mechanics, computer vision, computer graphics to represent relations in 3D)

Quaternions extend complex numbers: (Hamilton 1850s)



$Q: = q_0 + \underbrace{q_1 i + q_2 j + q_3 k}_{=\bar{q}}$ 4 parameters

$Q = (q_0, \bar{q})$ where $\bar{q}$



$(\theta, \bar{\omega}) \to Q = (\cos\frac{\theta}{2},\ \bar{\omega}\sin\frac{\theta}{2})$

$\bar{\omega}$: axis of rotation in space.
θ: amount of rotation

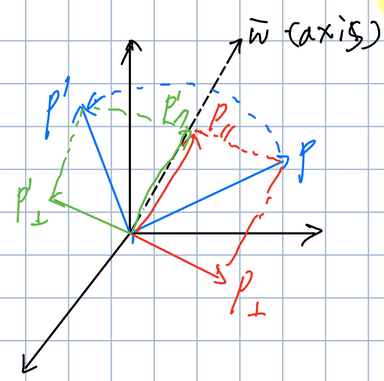
Composition rule: $Q_1 \circ Q_2$
$= (q_0^1 q_0^2 - \bar{q}_1 \bar{q}_2,\ q_0^1 \bar{q}_2 + q_0^2 \bar{q}_1 + \bar{q}_1 \times \bar{q}_2)$

and $Q_c^a = Q_b^a Q_c^b$

Inversion: $Q = (q_0, \bar{q}) \Rightarrow \theta = 2\operatorname{acos} q_0,\ \bar{\omega} = \begin{cases} \frac{\bar{q}}{\sin(\theta/2)}, & \theta \neq 0 \\ 0, & \theta = 0 \end{cases}$

No singular problem.

Axis-angle / Exponential Co-ordinates.



$$\dot{p} = \bar{w} \times \bar{p}$$
$$\downarrow$$
$$p(\theta) = \int_0^\theta (w \times p)$$
$$\downarrow$$
$$p(\theta) = e^{\hat{w}\theta} p(0)$$

$\hat{w}$ is a skew-sym matrix from $\bar{w}$

$$\boxed{\begin{array}{c} \dot{x} = ax \\ \Downarrow \\ x(t) = e^{at} x_0 \end{array}}$$

$p'_\parallel = p_\parallel$, $p'_\perp$ is $p_\perp$ rotated by θ.

$p'_\perp = p_\perp (\hat{p}_\perp \cos\theta + w \times \hat{p}_\perp \sin\theta)$ ($\hat{p}_\perp$ is a vector)

$$\Rightarrow p' = p'_\parallel + p'_\perp = p_\parallel + p_\perp(\hat{p}_\perp \cos\theta + w \times \hat{p}_\perp \sin\theta)$$

$$= \overset{p_\parallel}{w \underbrace{(w^T p)}_{\text{scalar}}} + \overset{p_\perp}{(p_\perp \cos\theta + w \times p_\perp \sin\theta)}$$

$$= (ww^T)p + ((p - (ww^T)p)\cos\theta + w \times p \sin\theta)$$

$$= (ww^T + (I - ww^T)\cos\theta + \hat{w}\sin\theta)p$$

$$e^{\hat{w}\theta} = ww^T + (I - ww^T)\cos\theta + \hat{w}\sin\theta + I - I \qquad (\hat{w})^2 = ww^T - I$$

w is a unit vector

$$\boxed{e^{\hat{w}\theta} = I_3 + \hat{w}\sin\theta + \hat{w}^2(1-\cos\theta)}$$ Rodrigues Formula

Any orientation $R \in SO(3)$ is equivalent to a rotation around a fixed axis $\bar{w} \in \mathbb{R}^3$ through an angle $\theta \in [0, 2\pi)$

$$R(\bar{w}, \theta) = e^{\hat{w}\theta}$$

Given $R = [r_{ij}]$ $R \in SO(3)$, $\Rightarrow$

$$\theta = \cos^{-1}\left(\frac{\operatorname{trace}(R) - 1}{2}\right)$$

$$w = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

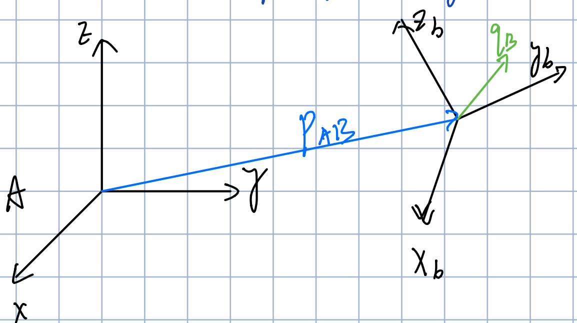
$\Rightarrow$ not unique for $w = -w$, $\theta' = 2\pi - \theta$ gives same result.

Rotation matrix R

$\downarrow$

Exponential Coordinates. $e^{\hat{w}\theta}$

Full see of rigid body motions: Homogeneous Transforms



P_{AB}: vector joining O_B to O_A.

R_{AB}: Orientation frame B w.r.t. A

$$q_A = P_{AB} + R_{AB} q_B \qquad \bar{q}_A$$

$$\bar{q}_A = \begin{bmatrix} q_A \\ 1 \end{bmatrix} = \begin{bmatrix} R_{AB} & P_{AB} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_B \\ 1 \end{bmatrix} \leftarrow \bar{q}_B \qquad \bar{q}_A = M \cdot \bar{q}_B$$

Homogeneous Coords.

$$\bar{q} = \begin{bmatrix} q \\ 1 \end{bmatrix} \qquad \bar{v} = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

1) Sum of a Diff of vectors are vectors

2) Sum of a point and a vector is a point.

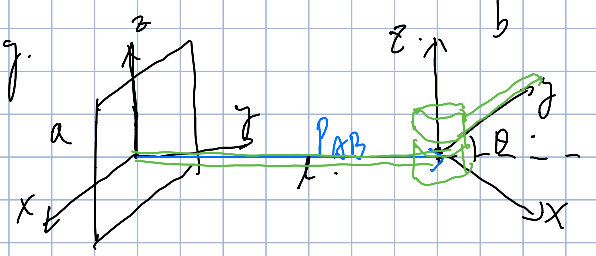
3) Diff of points is a vector.

4) Adding points is not defined.

$$g_{Ha}^{c} = g_{Hb}^{c} \cdot g_{Ha}^{b}, \quad g_{Ha}^{b} = \begin{bmatrix} R_{ab} & P_{ab} \\ 0 & 1 \end{bmatrix} \quad g_{Hb}^{c} = \begin{bmatrix} R_{bc} & P_{bc} \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow g_{Ha}^{c} = \begin{bmatrix} R_{bc} R_{ab} & R_{bc} P_{ab} + P_{bc} \\ 0 & 1 \end{bmatrix} \qquad g_{Ha}^{b\,-1} = \begin{bmatrix} R_{ab}^{T} & -R_{ab}^{T} P_{ab} \\ 0 & 1 \end{bmatrix}$$

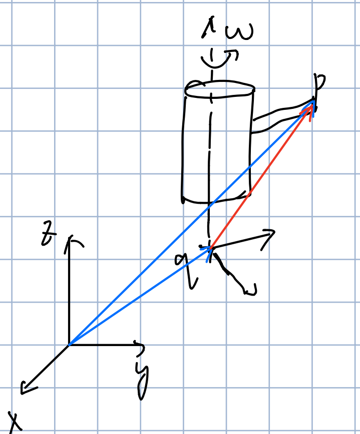
e.g.



$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow g_{a}^{b} = \left[\begin{array}{ccc|c} c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & l \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$\vec{P} = [0, l, 0]$

How do exponential co-ords generalize to full motion? : Twist.



$$\dot{p} = \omega \times (p - q) \qquad \bar{p} = \begin{bmatrix} p \\ 1 \end{bmatrix}$$

$$\dot{\bar{p}} = \underbrace{\begin{bmatrix} \hat{\omega} & -\hat{\omega} q \\ 0 & 0 \end{bmatrix}}_{\hat{\xi}} \bar{p} \Rightarrow \dot{\bar{p}} = \hat{\xi} \bar{p} \Rightarrow \bar{p}(\theta) = e^{\hat{\xi}\theta} \bar{p}(0)$$

translation component $v = -\omega \times q$

$$\dot{\bar{p}} = \underbrace{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}}_{\hat{\xi}} \bar{p}$$ ← translation

twist ⇐ (v, w)

v: translation velocity

w: angular velocity

$$\dot{p} = \omega \times p + v$$

$$\boxed{\dot{\bar{p}} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \bar{p}}$$

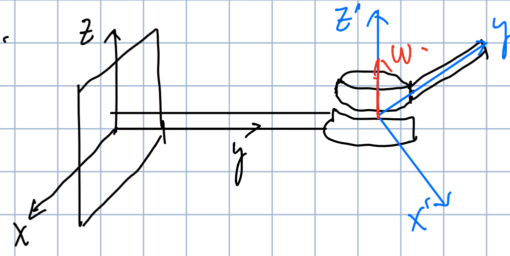
$$e^{\hat{\xi}\theta} = I + \hat{\xi}\theta + \frac{(\hat{\xi}\theta)^2}{2!} + \frac{(\hat{\xi}\theta)^3}{3!} + \dots$$

<u>Math</u>

$$\begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})(\omega \times v) + (\omega\omega^{T}) v\theta \\ 0 & 1 \end{bmatrix}$$

$\bar{\omega} \neq 0$

e.g.



$w = [0 \ \ 0 \ \ 1]$
$q = [0 \ \ l_1 \ \ 0]$
$v = -w \times q = [l_1 \ \ 0 \ \ 0]$

$$g_{AB}(\theta) = e^{\hat{\xi}\theta} g_{AB}(0), \qquad g_{AB}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})(w \times v) + (ww^T)v\theta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})(w \times v) \\ 0 & 1 \end{bmatrix}$$

$w^T v = [0 \ 0 \ 1]\begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} = 0.$

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\hat{\omega}} \sin\theta + \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\hat{\omega}^2} (1-\cos\theta)$$

$$e^{\hat{\omega}\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad e^{\hat{\xi}\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & l_1\sin\theta \\ \sin\theta & \cos\theta & 0 & l_1(1-\cos\theta) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$g_{AB}(0) = \begin{bmatrix} I & \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}, \quad \Rightarrow \quad g_{AB}(\theta) = \begin{bmatrix} e^{\hat{\omega}\theta} & \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

? Given R, P. what is (v, w) that generate R, P.

$g = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}$. calculate $\hat{\omega}, \theta$ given R.

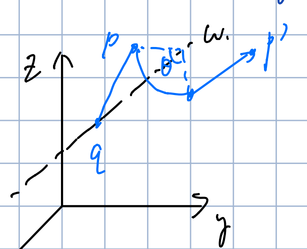
$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})(w \times v) + (ww^T)v\theta \\ 0 & 1 \end{bmatrix} \ (\hat{\omega} \neq 0). \qquad = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} \ (\hat{\omega} = 0)$$

$$\Downarrow$$

$$[\underbrace{(I - e^{\hat{\omega}\theta})\hat{\omega} + ww^T\theta}_{A}]v = p.$$

A is invertible, $w \neq 0$. $\theta \neq 0$

Screws: What is the geometric meaning of twists?



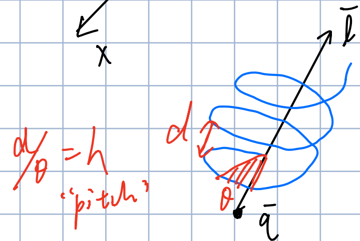
$P_{int} = q + e^{\hat{\omega}\theta}(p-q)$

$p' = q + e^{\hat{\omega}\theta}(p-q) + \|d\| w.$

Pitch = translation / rotation.

Chasles Theorem

Every rigid body motion can be realized by a rotation along an axis combined with



$h = d/\theta$

Axis: $\{q + \lambda w : \lambda \in \mathbb{R}\}$.
Magnitude: θ.

a translation parallel to that axis
$\Leftrightarrow$ Screw motion

Screw motion $$g_{AB}(\theta) = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta w \\ 0 & 1 \end{bmatrix} g_{AB}(0)$$

$\xi = (v, w)$ twist. $\quad v = -w \times q + hw$

Given a twist $\xi = (v, w)$.

$$\text{Axis} \Rightarrow \begin{cases} \dfrac{w \times v}{\|w\|^2} + \lambda w & \lambda \in \mathbb{R} \text{ if } w \neq 0. \\ \lambda v & \lambda \in \mathbb{R} \text{ if } w = 0 \end{cases}$$

$$\text{Pitch} = \frac{w^T v}{\|w\|^2}, \quad \text{Magnitude} = \begin{cases} \|w\| & \text{if } w \neq 0 \\ \|v\| & \text{if } w = 0 \end{cases}$$

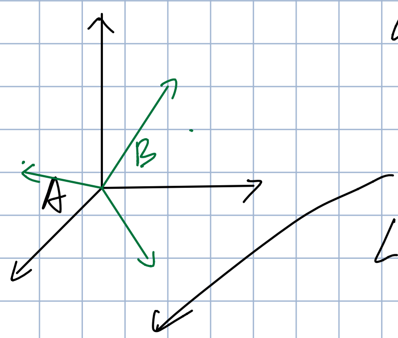
a) Rotational velocity of rigid bodies.

b) full rigid body velocity

c) adjoint transformation

$$g_{ab}(t) = \begin{bmatrix} R_{ab}(t) & P_{ab}(t) \\ 0 & 1 \end{bmatrix} \qquad \text{velocity ?} \quad \cancel{\dot{g}_{ab}(t)}$$

a) Rotational Velocity of RB



Let q be a point on RB
q_a is q in A. q_b is q in B.

$q_a(t) = R_{ab}(t)\, q_b$ $\xrightarrow{\text{differentiate}}$ $\bar{V}_{q_a}(t) = \frac{d}{dt}(\bar{q}_a(t)) = \dot{R}_{ab}(t)\,\bar{q}_b$

$= \dot{R}_{ab}(t)[R_{ab}^{-1}(t)\, R_{ab}(t)]\,\bar{q}_b$

$= (\dot{R}_{ab}(t)\, R_{ab}^{-1}(t))\, R_{ab}\,\bar{q}_b.$

$\Rightarrow \boxed{\bar{V}_{q_a}(t) = \dot{R}_{ab}(t)\, R_{ab}^{-1}\,\bar{q}_a}$

Let $\bar{w}$ be a (constant) velocity,

$\dot{\bar{q}}_a = \bar{w} \times \bar{q}_a = \hat{w}\,\bar{q}_a$

$q_a(t) = R_{ab}(t)\, q_b(t)$

$$\dot{\bar{q}}_a = \bar{V}_{q_a} = \hat{w}\,\bar{q}_a = \dot{R}_{ab}\, R_{ab}^{-1}\,\bar{q}_a$$

Def: instantaneous spatial angular velocity

$\hat{W}_{ab}^{s} = \dot{R}_{ab}\, R_{ab}^{-1}$

" body "

$\hat{W}_{ab}^{b} = R_{ab}^{-1}\,\dot{R}_{ab}$ $\qquad \bar{W}_{ab}^{b} = R_{ab}^{-1}\,\bar{W}_{ab}^{s}$

proof that $\dot{R}_{ab} R_{ab}^{-1}$ is symmetric:

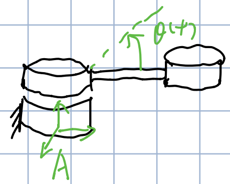
$\frac{dI}{dt} = \frac{d}{dt}(RR^T) = 0$

$= \dot{R}R^T + R\cdot\dot{R}^T$

$\Rightarrow \dot{R}R^{-1} = -R\dot{R}^{-1}$

$\dot{R}R^T = -(\dot{R}R^T)^T$

Ex: 1 DoF (rotational) manipulator



$$R(t) = \begin{bmatrix} c\theta(t) & -s\theta(t) & 0 \\ s\theta(t) & c\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{w}^s = \dot{R}R^T = \begin{bmatrix} -\dot{\theta}s\theta & -\dot{\theta}c\theta & 0 \\ \dot{\theta}c\theta & -\dot{\theta}s\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} R^T(t) = \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ Screw symetric

$$\bar{w}^s = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$

b) Full Rigid Body Velocity. $SO(3) \to \underline{SE(3)}$

recall $\hat{\xi}$ extends $\hat{w}$

we have:

$$\tilde{q}_a = \begin{bmatrix}\bar{q}_a\\ 1\end{bmatrix} = g_{ab}\begin{bmatrix}\bar{q}_b\\ 1\end{bmatrix} = g_{ab}\tilde{q}_b$$

$$\dot{\tilde{q}}_a = \frac{d}{dt}(g_{ab}\tilde{q}_b) = \dot{g}_{ab}g_{ab}^{-1}g_{ab}\tilde{q}_b = \dot{g}_{ab}g_{ab}^{-1}\tilde{q}_a$$

$$\begin{bmatrix}\dot{R} & \dot{p}\\ 0 & 0\end{bmatrix}\begin{bmatrix}R^T & -R^Tp\\ 0 & 1\end{bmatrix} = \dot{g}_{ab}g_{ab}^{-1}$$

$$= \begin{bmatrix}\dot{R}R^T & -\dot{R}R^Tp + \dot{p}\\ 0 & 0\end{bmatrix}$$ is twist with

$$V^s_{ab\,6\times 1} = \begin{pmatrix}\bar{v}^s_{ab}\\ \bar{w}^s_{ab}\end{pmatrix} = \begin{pmatrix}-\dot{R}R^Tp+\dot{p}\\ (\dot{R}R^T)^\vee\end{pmatrix} = \begin{bmatrix}-\bar{w}^s_{ab}\times p_{ab}+\dot{p}_{ab}\\ w^s_{ab}\end{bmatrix}$$

instantaneous spatial velocity $\hat{V}^s_{ab} = \dot{g}_{ab}g_{ab}^{-1}$

$\bar{w}^s_{ab}$ instantaneous angular velocity of RB viewed from inertial frame

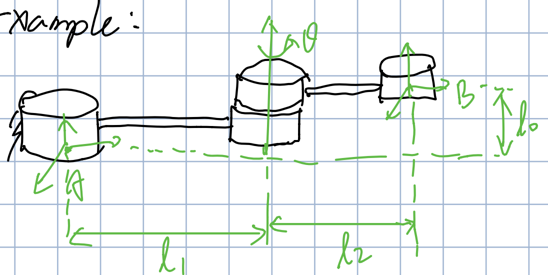
$\bar{v}^s_{ab}$ not velocity of B
it's velocity of B [$\dot{p}_{ab}$]
with a rotational correction about origin of A

$-\bar{w}\times\bar{p}+\dot{\bar{p}}$ is velocity of RB origin corrected by "false" rotation about inertial origin

$$\hat{V}^b_{ab} = g_{ab}^{-1}\dot{g}_{ab} = \begin{bmatrix}R^T\dot{R} & R^T\dot{p}\\ 0 & 0\end{bmatrix}$$

$$V^b_{ab} = \begin{pmatrix}\bar{v}^b_{ab}\\ \bar{w}_{ab}\end{pmatrix} = \begin{bmatrix}R^T\dot{p}\\ (R^T\dot{R})^\vee\end{bmatrix}$$ velocity of origin
angular velocity in body frame

Example:



$$g_{ab}(\theta) = \begin{bmatrix}c\theta & -s\theta & 0 & -l_2 s\theta\\ s\theta & c\theta & 0 & l_1+l_2c\theta\\ 0 & 0 & 1 & l_0\\ 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}R & P\\ 0 & 1\end{bmatrix}$$

$$\bar{V}^s_{ab} = (\dot{g}g^{-1})^\vee = \begin{pmatrix}-\dot{R}R^Tp+\dot{p}\\ (\dot{R}R^T)^\vee\end{pmatrix}$$

$$= \begin{pmatrix}\begin{bmatrix}0 & \dot{\theta} & 0\\ -\dot{\theta} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}-l_2 s\theta\\ l_1+l_2c\theta\\ l_0\end{bmatrix} + \begin{bmatrix}-l_2c\theta\,\dot{\theta}\\ -l_2 s\theta\,\dot{\theta}\\ 0\end{bmatrix}\\ (\dot{R}R^T)^\vee\end{pmatrix}$$

$$V^b_{ab} = [-l_2\dot{\theta}\ 0\ 0\ 0 \dots 0] \qquad = [\,l_1\dot{\theta}\quad 0\quad 0\ |\ 0\quad 0\quad \dot{\theta}\,]^T$$

c) Adjoint Transformation — how do $\hat{V}^s$ and $\hat{V}^b$ relate?

g_{ab} transforms motion between frames
Ad_g transforms velocities

$$\bar{w}^s_{ab} = R_{ab}\bar{w}^b_{ab}$$

$$V^s_{ab} = -w^s_{ab}\times p_{ab} + \dot{p}_{ab} \qquad (V^b_{ab} = R^T\dot{p}_{ab})$$

$\hat{V}^s_{ab} = \dot{g}_{ab}g_{ab}^{-1}$ $\quad$ $\hat{V}^b_{ab} = g_{ab}^{-1}\dot{g}_{ab}$

$$\hat{V}^s_{ab} = [g_{ab}g_{ab}^{-1}]\dot{g}_{ab}g_{ab}^{-1} = g_{ab}(g_{ab}^{-1}\dot{g}_{ab})g_{ab}^{-1}$$

$$= \hat{p}_{ab} \times w^s_{ab} + \dot{p}_{ab} \quad \rightarrow p_{ab} = R\, v_{ab}$$

$$= \hat{p}_{ab} \times (R_{ab} w^b_{ab}) + \dot{p}_{ab}$$

$$= g_{ab} \hat{V}^b_{ab} g_{ab}^{-1}$$

$$V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ w^s_{ab} \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & \hat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix}}_{\text{Adjoint transformation}} \begin{bmatrix} V^b_{ab} \\ W^b_{ab} \end{bmatrix}$$

$\Rightarrow$ all twists $\bar{\xi}_a = g_{ab} \bar{\xi}_b g_{ab}^{-1}$

Adjoint transformation : transforms the corresponding velocities

$$(Ad\, g)^{-1} = Ad(g^{-1})$$

$$\bar{V}^s_{ac} = \bar{V}^s_{ab} + Ad_{g_{ab}} \bar{V}^s_{bc}$$

1.3. Differential Kinematics

$$\hat{V}^s_{ab} = \dot{g}_{ab} g_{ab}^{-1} \qquad Ad_g = \begin{bmatrix} R & \hat{P} R \\ 0 & R \end{bmatrix}$$

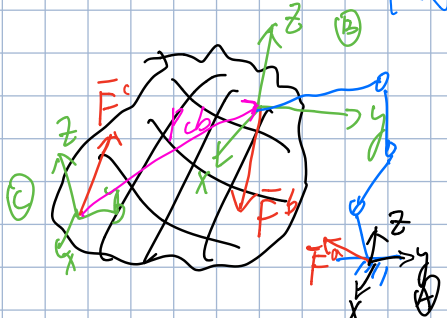
$$\hat{V}^B_{ab} = g^{-1}_{ab} \dot{g}_{ab}$$

$$\hat{\xi}' = g \hat{\xi} g^{-1} \qquad \bar{\xi}' = Ad_g \bar{\xi}$$

D) Forces and moments applied to RBs: what is a wrench?

wrench $\bar{F}$ is generalization of forces $\bar{f}$ and torques $\bar{\tau}$ applied to a RB:

$$\bar{F} := \begin{bmatrix} \bar{f} \\ \bar{\tau} \end{bmatrix}$$



Remarks:

a) Value of $\bar{F}$ depends on coord-sys within which it is represented

$(\bar{\tau}' = \bar{\tau} + \bar{r} \times \bar{f})$

b) By Definition, $\bar{F}$ is applied to the origin of frame within which it is represented

– what is equivalant wrench F_a that generates same effect as F_b?

Def: 2 wrenches are equivalant if they generate the same work for every rigid body motion.

$$W = \int \bar{V}^T \bar{F}^b \, dt = \int (\bar{v}^T \bar{f} + \bar{w}^T \tau) \, dt \qquad \bar{\xi}' = Ad_g \bar{\xi}$$

$$\delta W = \bar{V}^T \bar{F} \quad (\text{for every possible motion})$$

body frame

therefore: $\bar{V}^{b\,T}_{ac} \bar{F}^c \overset{!}{=} \bar{V}^{b\,T}_{ab} \cdot \bar{F}^b$

$$\bar{V}^b_{ab} = Ad_{g_1} \bar{V}^b_{ac}$$

$\bar V_{ac}^{b\,T} \bar F^c = \bar V_{ac}^{b\,T} Ad^T_{g_{bc}} \bar F^b$

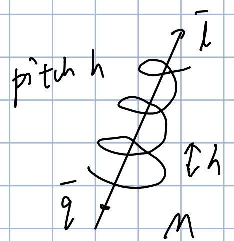
$\Rightarrow \boxed{\bar F^c = Ad^T_{g_{bc}} \bar F^b}$

$$\begin{bmatrix} \bar f_c \\ \bar \tau_c \end{bmatrix} = \begin{bmatrix} R_{bc}^T & 0 \\ R_{bc}^T(-\hat{\bar p}_{bc}) & R_{bc}^T \end{bmatrix} \begin{bmatrix} \bar f_b \\ \bar \tau_b \end{bmatrix}$$

$-\bar p_{bc} \times \bar f_b = \bar p_{cb} \times \bar f_b$ f_b 转换后 加在 C上 torque

Points of Theorem: Every collection of wrenches applied to a RB is equivalant to a force applied along a fixed axis plus a pure torque about the same axis.

$$\bar F = \begin{cases} M \cdot \begin{bmatrix} \bar\omega \\ -\bar\omega \times \bar q + h\bar\omega \end{bmatrix}, & h \text{ finite} \\ M \begin{bmatrix} 0 \\ \bar\omega \end{bmatrix}, & h \text{ infinite} \end{cases}$$



Key Points (1.2 ∧ 1.3):

1) homogeneous transformation $g = \begin{bmatrix} R & \bar P \\ 0 & 1 \end{bmatrix}$

$\bar q_a = g_{ab} \bar q_b$ (across frames)

2) representation of homogeneous transformation by exponentials of twists:

$g(\theta) = e^{\hat\xi\theta} g(0)$ (same frame)

3) Velocity of rigid body specified as $\hat V^s = \dot g g^{-1}$ (spacial) $\hat V^b = g^{-1}\dot g$ (body)

$\dot{\bar q}_a = \hat V^s_{ab} \bar q_a$ $\bar V_b = \hat V^b_{ab} \bar q_b$

4) Transformation btw velocities:

$\bar V^s = Ad_g \bar V^b$ w/ $Ad_g := \begin{bmatrix} R & \hat P R \\ 0 & R \end{bmatrix}$

5) Forces generalize to wrenches: $F = \begin{bmatrix} \bar f \\ \bar \tau \end{bmatrix} \in \mathbb{R}^{6\times 1}$

$\bar F_c = Ad^T_{g_{bc}} \bar F_b$

for a RB with config g_{ab}. $\bar F^a = \bar F^s$ (spatial wrench)
$\bar F^b = \bar F^b$ (body wrench)