Problem 1

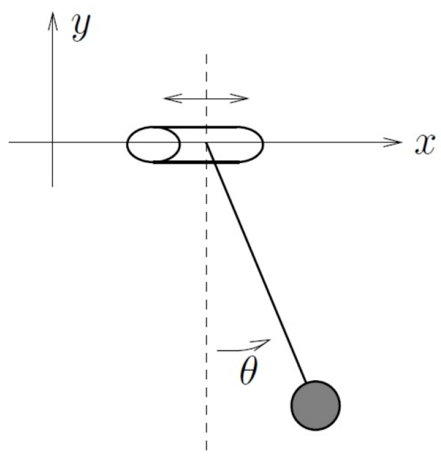


Figure 1: Pendulum on a wire.

$$L = T - V$$
$$T = \frac{1}{2} m \dot{\vec{r}}^T \dot{\vec{r}}$$
$$V = -mglcos\theta$$
$$\vec{r} = \begin{bmatrix} x + lsin\theta \\ -lcos\theta \end{bmatrix}$$
$$\dot{\vec{r}} = \begin{bmatrix} \dot{x} + l\dot{\theta}cos\theta \\ l\dot{\theta}sin\theta \end{bmatrix}$$
$$T = \frac{1}{2} m(\dot{x}^2 + 2\dot{x}\dot{\theta}lcos\theta + l^2\dot{\theta}^2)$$
$$L = T - V = \frac{1}{2} m(\dot{x}^2 + 2\dot{x}\dot{\theta}lcos\theta + l^2\dot{\theta}^2) + mglcos\theta$$

$for\ x:$

$$\frac{\partial L}{\partial x} = 0$$
$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + m\dot{\theta}l\cos\theta$$
$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = m\ddot{x} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$$

$for\ \theta:$

$$\frac{\partial L}{\partial \theta} = -ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta$$
$$\frac{\partial L}{\partial \dot{\theta}} = ml\dot{x}\cos\theta + ml^2\dot{\theta}$$
$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) = ml(\ddot{x}\cos\theta - \dot{x}\dot{\theta}\sin\theta) + ml^2\ddot{\theta}$$
$$\begin{bmatrix} m & ml\cos\theta \\ ml\cos\theta & ml^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & -ml\dot{\theta}\sin\theta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ mlg\sin\theta \end{bmatrix} = 0$$

Problem 2

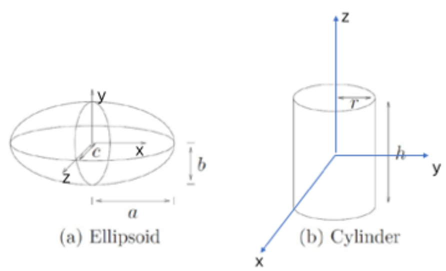


Figure 2: Parameterized ellipsoid and cylinder.

$$
\begin{aligned}
I &= \int \hat{q}^T \hat{q} dm \\
&= \int \begin{bmatrix} q_2^2 + q_3^2 & -q_1 q_2 & -q_1 q_3 \\ -q_1 q_2 & q_1^2 + q_3^2 & -q_2 q_3 \\ -q_1 q_3 & -q_2 q_3 & q_1^2 + q_2^2 \end{bmatrix} dm
\end{aligned}
$$

Part(a)

because the symmetrical property of Ellipsoid, $\int -q_i q_j dm$ is 0 when i is not equal to j.
Let x = aX, y = bY and z = cZ

$$\int\int\int_{x^2/a^2+y^2/b^2+z^2/c^2\leq 1} x^2 dV = abc\int\int\int_{X^2+Y^2+Z^2\leq 1} a^2X^2 dXdYdZ$$

$$= abc\int_{-1}^{1}(\int_{-\sqrt{1-X^2}}^{\sqrt{1-X^2}}(\int_{-\sqrt{1-X^2-Y^2}}^{\sqrt{1-X^2-Y^2}} dZ))a^2X^2dX$$

$$= abc\int_{-1}^{1}(\int_{-\sqrt{1-X^2}}^{\sqrt{1-X^2}} 2\sqrt{1-X^2-Y^2}dY)a^2X^2dX$$

$$= abc\int_{-1}^{1}\pi(1-X^2)a^2X^2dX$$

$$since\ \rho = \frac{3m}{4\pi abc}$$

$$I_{11} = \int\int\int(y^2+z^2)\rho dV = \frac{m}{5}(b^2+c^2)$$

$$Similarly\ \ we\ have$$

$$I_{22} = \frac{m}{5}(a^2+c^2)$$

$$I_{33} = \frac{m}{5}(a^2+b^2)$$

$$Thus,$$

$$I = \begin{bmatrix} \frac{m}{5}(b^2+c^2) & 0 & 0 \\ 0 & \frac{m}{5}(a^2+c^2) & 0 \\ 0 & 0 & \frac{m}{5}(a^2+b^2) \end{bmatrix}$$

Part(b)

If the origin of coordinate is collocated on the center of the geometrical center of cylinder, then due to the symmetrical property, $\int -q_i q_j dm$ is 0 when i is not equal to j.

$$\rho\int_{\frac{h}{2}}^{-\frac{h}{2}}\int_{-r}^{r}\int_{\sqrt{r^2-y^2}}^{-\sqrt{r^2-y^2}}(y^2+z^2)dxdydz = \rho\int_{\frac{h}{2}}^{-\frac{h}{2}}\int_{-r}^{r}\int_{\sqrt{r^2-y^2}}^{-\sqrt{r^2-y^2}}(x^2+z^2)dxdydz$$

$$= \frac{1}{12}Mh^2 + \frac{1}{4}Mr^2$$

$$\rho\int_{\frac{h}{2}}^{-\frac{h}{2}}\int_{-r}^{r}\int_{\sqrt{r^2-y^2}}^{-\sqrt{r^2-y^2}}(x^2+y^2)dxdydz = \frac{1}{2}Mr^2$$

$$Thus,\ the\ inertia\ matrix\ for\ cylinder\ is\ :$$

$$\begin{bmatrix} \frac{1}{12}Mh^2+\frac{1}{4}Mr^2 & 0 & 0 \\ 0 & \frac{1}{12}Mh^2+\frac{1}{4}Mr^2 & 0 \\ 0 & 0 & \frac{1}{2}Mr^2 \end{bmatrix}$$

Problem 3

Part 1

The velocity of center of mass is:

$$v_x = 0.01$$
$$v_y = 0.03$$
$$v_z = 0.02$$

Part 2

The inertia tensor of the asteroid in the body frame is:

$$\begin{bmatrix} 0.268599053182904 & -1.04508577415866e-07 & 2.04117731179916e-07 \\ -1.04508577415866e-07 & 0.724459663605225 & -0.0287803958765721 \\ 2.04117731179916e-07 & -0.0287803958765721 & 0.634180126818389 \end{bmatrix}$$

where we set the body frame have the same direction of beacon's frame and the origin on the center of mass of asteroid.

Part 3

After doing Eigen value decomposition, we find the Diagonal matrix is:

$$\begin{bmatrix} 0.268599053182773 & 0 & 0 \\ 0 & 0.625785701677064 & 0 \\ 0 & 0 & 0.732854088746681 \end{bmatrix}$$

The Eigen vector is:

$$\hat{x} = [0.999999999999834, -4.66674495755628e-07, -3.39214333363641e-07]$$
$$\hat{y} = [1.94974414628373e-07, -0.280004396274166, 0.959998717742426]$$
$$\hat{z} = [-5.42988422149526e-07, -0.959998717742332, -0.280004396274029]$$

Therefore, the asteroid's shape may be like a flat Ellipsoid

Part 4

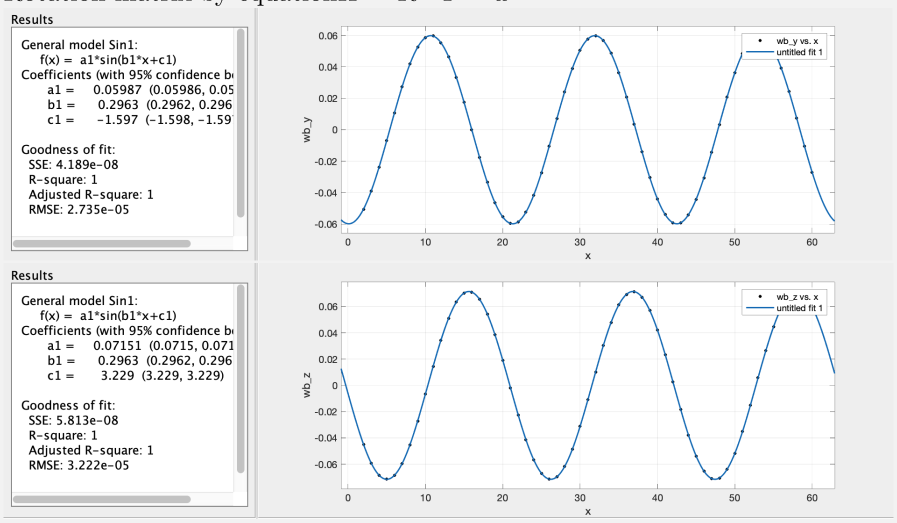
In the beacon's frame the coordinate of center of mass of asteroid is $[-4.88848366165825e-06, -0.0839991329138293, -0.288001208351013]$
The distance is 0.3

Part 5

The angular momentum is:
$[0.0124, 0.1329, 0.0338]$

Part 6

After plotting the angular velocity of beacon in body frame, we find it is periodic, we can calculate the angular velocity at 120s, because the angular momentum is not changed then, we can get the Rotation matrix by equation$H = R * I^b * w^b$



At 120s, the pose is:
$[121, 1.69, 3.29, 2.42, 0.01, -0.072, -0.72, 0.69]$