Assignment 2 – Kinematics

Assignment 2 concentrates on kinematics. Problem 1 will familiarize you with quaternions and how they can be used to track rigid bodies. Problem 2 concentrates on screw theory in the plane ($\mathbb{R}^2$). It is gives to the theoretical background to solve kinematics tasks for robot mechanisms that operate in a plane while helping you to revisit important concepts of screw theory. Finally, problem 3 focuses on programming the forward kinematics of the Barrett WAM™ arm robot. The assignment covers two weeks of lecture material. We recommend you work on the corresponding problems early on.

After completion of this assignment you should be able to use unit quaternions for tracking rigid body orientations, have a solid understanding of the foundational concepts of screw theory, and be able to identify and program the forward kinematics of serial manipulators.

1. Unit Quaternions (6+2pts)

Let $Q = (q_0, \vec{q})$ and $P = (p_0, \vec{p})$ be quaternions, where $q_0, p_0 \in \mathbb{R}$ are the scalar parts of Q and P and $\vec{q}, \vec{p}$ are the vector parts.

(a) (2pts) Let x be a point and let X be a quaternion whose scalar part is zero and whose vector part is equal to x (such a quaternion is called a *pure* quaternion). Show that if Q is a unit quaternion, the product QXQ^* is a pure quaternion and the vector part of QXQ^* satisfies

$$(q_0^2 - \vec{q}\cdot\vec{q})\vec{x} + 2\Big(q_0(\vec{q}\times\vec{x}) + (x\cdot\vec{q})\vec{q}\Big).$$

Verify that the vector part describes the point to which x is rotated under the rotation associated with Q.

(b) (2pts) Show that the set of unit quaternions is a two-to-one covering of $SO(3)$. That is, for each $R \in SO(3)$, there exist two distinct unit quaternions which can be used to represent this rotation.

(c) (2pts) Compare the number of additions and multiplications needed to perform the following operations:

i. Compose two rotation matrices.

ii. Compose two quaternions.

iii. Apply a rotation matrix to a vector.

iv. Apply a quaternion to a vector [as in part (a)].

Count a subtraction as an addition, and a division as a multiplication.

(d) (extra 2pts) Show that a rigid body rotating at unit velocity about a unit vector in $\omega \in \mathbb{R}^3$

can be represented by the quaternion differential equations

$$\dot{Q} \cdot Q^* = (0, \omega/2),$$

where $\cdot$ represents quaternion multiplication.

2. Planar Rigid Body Transformations (8pts)

A transformation $g = (p, R) \in SE(2)$ consists of a translation $p \in \mathbb{R}^2$ and a 2×2 rotation matrix R. We represent this in homogeneous coordinates as a 3×3 matrix:

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

A twist $\hat{\xi} \in se(2)$ can be represented by a 3×3 matrix of the form:

$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \qquad \hat{\omega} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \qquad \omega \in \mathbb{R}, v \in \mathbb{R}^2.$$

The twist coordination for $\hat{\xi} \in se(2)$ have the form $\xi = (v, w) \in \mathbb{R}^3$. Note that v is a vector in the plane and ω is a scalar.

(a) (2pts) Show that the exponential of a twist in $se(2)$ gives a rigid body transformation in $SE(2)$. Consider both the pure translation case, $\xi = (v, 0)$, and the general case, $\xi = (v, \omega), \omega \neq 0$. (2pts)

(b) (1pt) Show that the planar twists which correspond to pure rotation about a point q and pure translation in a direction v are given by

$$\xi = \begin{bmatrix} q_y \\ -q_x \\ 1 \end{bmatrix} \text{(pure rotation)} \qquad \xi = \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} \text{(pure translation)}.$$

(c) (2pts) Show that every planar rigid body motion can be described as either pure rotation about a point (called the *pole* of the motion) or pure translation.

(d) (2pt) Show that the matrices $\hat{V}^s = \dot{g}g^{-1}$ and $\hat{V}^b = g^{-1}\dot{g}$ are both twists. Define and interpret the spatial velocity $V^s \in \mathbb{R}^3$ and the body velocity $V^b \in \mathbb{R}^3$.

(e) (1pt) The adjoint transformation is used to map body velocities $V^b \in \mathbb{R}^3$ into spatial velocities $V^s \in \mathbb{R}^3$. Show that the adjoint transformation for planar rigid motions is given by

$$\mathrm{Ad}_g = \begin{bmatrix} R & \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

3. Forward Kinematics (10+2pts)

For this problem, you must create a forward kinematics implementation for a common 7-dof robotic arm: the Barrett WAM™. A description of the arm's kinematics is available on the Barrett website; grab the "Joints & Frames PDF" document for the "WAM and Wrist" model here: http://barrett.com/robot/products-arm-models.htm. We will use the joint conventions from this document. The arm is shown in its all-zeros configuration, and the red arrows show the positive directions for each joint J1-J7.

In our imagined scenario, the WAM is mounted upright to a tabletop whose surface is 1 meter above the floor. With respect to the origin (on the floor), the back-right corner of the robot (the location of the "Base" frame in the Barrett PDF) is located at $x = 0.75$m, $y = 0.5$m, $z = 1.0$m. The front of the robot is facing the positive y direction.

A whiteboard marker has been attached rigidly to the robot's end plate, such that the marker is vertical and centered on the end plate when all joints are at zero (as shown in the PDF). The marker is 12cm long, so the drawing marker tip is 12cm from the end plate.

A whiteboard is mounted nearby (in some unknown position and orientation).

a) (2pts) In the all-zeros configuration, what is the location of the marker tip, given with respect to the world origin? (You should be able to answer this without any code.)

The robot has decided to draw something! Its joint trajectory is given in the file `qdata.txt`. Each line is a point on the joint trajectory; the file should have 7835 points. Each point consists of seven space-separated numbers, one for each joint J1-J7. Joint values are given in radians. The robot is drawing for the entire trajectory.

b) (6pts) Create a program which implements the forward kinematics of the 7-dof Barrett WAM™. Use this program to convert the above joint trajectory into the x-y-z trajectory of the marker tip in the world origin frame. Save the marker tip trajectory in a similar format (one line per point, three space-separated values x, y, z). Submit your code as well.

c) (2pts) Where is the whiteboard? Give the location of any point on its surface, along with a vector normal to it (out of the board).

d) (extra 2pts) What did the robot write? Include a figure of the drawing.