# Chapter 3
# Modeling of WIP Systems

## 3.1 Introduction

As shown in Fig. 3.1, the WIP vehicle consists of a pair of identical wheels, a chassis, two wheel actuators, an inverted pendulum, and a motion control unit. The chassis supports the pair of wheels and the inverted pendulum. The wheel actuators rotate the wheels with respect to the chassis. The motion control unit controls the wheel actuators so as to move the vehicle and stabilize the inverted pendulum. Under such a configuration, the vehicle is characterized by the ability to balance on the two wheels and spin on the spot.

Given the WIP robot, kinematics focus on the velocity relationships regarding the linear and angular velocities represented in the task space and in the joint space, while dynamics study the relationship between the forces applied on WIP and the resulted motion, i.e., how the forces acting on robots make these robots move by their kinematics. Mathematically, the kinematic equations establish a mapping between the Cartesian space positions/orientations and the joint space positions (translation or revolution). The velocity relationship is then determined by the Jacobian associated with this mapping, whereas each row of the Jacobian matrix is a vector of partial derivatives of a Cartesian position variable with respect to all the joint space position variables. It should be mentioned that the Jacobian is one of the most important quantities in the analysis and control of robot motion.

This chapter describes how to model the dynamic properties of chains of interconnected rigid bodies that form a WIP system, how to calculate the chains' time evolution under a given set of internal and external forces and/or desired motion specifications, and how to provide a means of designing prototype mobile inverted pendulum and testing control approaches without building the actual mobile inverted pendulum. The theoretical principle behind rigid body dynamics is that one extends Newton's laws for the dynamics of a point mass to rigid bodies. However, the interconnection of rigid bodies by means of revolute joints gives rise to new physical properties that do not exist for one single point mass or one single rigid body. More in particular, the topology of the kinematic chain determines to a large

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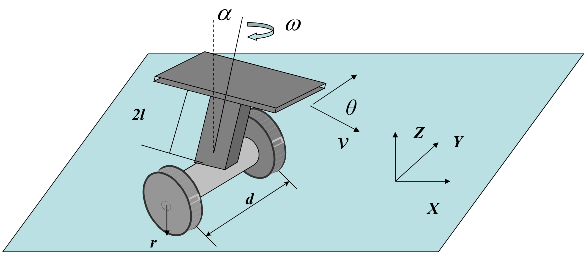


**Fig. 3.1** Model of wheeled inverted pendulum

extent the minimal complexity of the computational algorithms that implement these physical properties.

Kinematic and dynamic modeling of WIP deals with the mathematical formulation of the kinematic and dynamic equations. Modeling helps us understanding the physical properties and is the basis to develop relevant control algorithms. In this chapter, to derive dynamics model we assume that all the physical parameters of the robot are known, e.g., dimensions of links, relative positions and orientations of connected parts, mass distributions of the links, joints and motors. While in practice, it is not straightforward to obtain accurate values for all these parameters. In addition, in this chapter we assume an ideal system with rigid bodies, joints without backlash and accurately modeled friction.

For the easy computation of kinematics model for mobile inverted pendulum, we decomposed mobile inverted pendulum into the mobile platform and the inverted pendulum.

## 3.2 Kinematics of the WIP Systems

For convenience of kinematics modeling, we decompose the WIP into a mobile platform and an inverted pendulum. Let the generalized coordinates of the mobile platform be $q = (x_o, y_o, \theta, \theta_r, \theta_l)$, where $(x_o, y_o)$ is the center point of the driving wheels, $\theta$ is the heading angle of the mobile platform measured from $x$-axis, $\theta_r$ and $\theta_l$ are the angular positions of the two driving wheels, respectively. If the mobile platform of the WIP system satisfies the nonholonomic constraints without slipping, then the following constraint holds:

$$A(q)\dot{q} = 0 \tag{3.1}$$

where $A(q) \in \mathbb{R}^{3\times 5}$ is the matrix associated with the constraints.

As shown in Fig. 3.1, the motion and orientation of the WIP are achieved by two independent actuators which provide the torques to the wheels. The nonholonomic constraint states that the robot can only move in the direction normal to the axis of the driving wheels, i.e., the mobile base satisfies the conditions of pure rolling and nonslipping, therefore, the mobile platform is generally subject to three constraints. The first one is that the mobile robot can not move in the lateral direction, i.e.,

$$\dot{y}_o \cos\theta - \dot{x}_o \sin\theta = 0 \tag{3.2}$$

Equation (3.2) is a nonholonomic constraint that cannot be integrated analytically to result in a holonomic constraint among the configuration variables of the platform, namely $x_o$, $y_o$, and $\theta$. As well known, the configuration space of the system is three dimensional (completely unrestricted), while the velocity space is two-dimensional. This constraint becomes

$$\dot{x}_o \cos\theta + \dot{y}_o \sin\theta + \frac{d}{2}\dot{\theta} = r\dot{\theta}_r \tag{3.3}$$

$$\dot{x}_o \cos\theta + \dot{y}_o \sin\theta - \frac{d}{2}\dot{\theta} = r\dot{\theta}_l \tag{3.4}$$

where $d$ is the platform width and $r$ is the radius of wheel.

Combining Eqs. (3.2), (3.3) and (3.4), we see that matrix $A(q)$ can be written as follows

$$A(q) = \begin{bmatrix} -\sin\theta & \cos\theta & 0 & 0 & 0 \\ \cos\theta & \sin\theta & \frac{d}{2} & -r & 0 \\ \cos\theta & \sin\theta & -\frac{d}{2} & 0 & -r \end{bmatrix} \tag{3.5}$$

Let $m$ rank matrix $S(q) \in \mathbb{R}^{5\times 2}$ formed by a set of smooth and linearly independent vector fields spanning the null space of $A(q)$, i.e.,

$$S^T(q)A^T(q) = 0 \tag{3.6}$$

According to (3.1) and (3.6), it is possible to establish that

$$\dot{q} = S(q)v(t) \tag{3.7}$$

where $v(t) = [\dot{\theta}_r, \dot{\theta}_l]$ and matrix $S(q) \in \mathbb{R}^{5\times 2}$ is defined as follows

$$S(q) = \left[s_1(q)\; s_2(q)\right] = \begin{bmatrix} \frac{r}{2}\cos\theta & \frac{r}{2}\cos\theta \\ \frac{r}{2}\sin\theta & \frac{r}{2}\sin\theta \\ \frac{r}{d} & -\frac{r}{d} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.8}$$

It is obvious that the matrix $S(q)$ are in the null space of matrix $A(q)$, that is, $S^T(q)A^T(q) = 0$. A distribution spanned by the columns of $S(q)$ can be described as

$$\Delta = \text{span}\left\{s_1(q), s_2(q)\right\} \tag{3.9}$$

*Remark 3.1* The number of holonomic or nonholonomic constraints can be determined by the involutivity of the distribution $\Delta$. If the smallest involutive distribution containing $\Delta$ (denoted by $\Delta^*$) spans the entire 5-dimensional space, all the constraints are nonholonomic. If $\dim(\Delta^*) = 5 - k$, then $k$ constraints are holonomic and the others are nonholonomic.

To verify the involutivity of $\Delta$, we compute the Lie bracket of $s_1(q)$ and $s_2(q)$

$$s_3(q) = \left[s_1(q)\ s_2(q)\right] = \frac{\partial s_2}{\partial q} s_1 - \frac{\partial s_1}{\partial q} s_2 = \begin{bmatrix} -\frac{r^2}{d}\sin\theta \\ \frac{r^2}{d}\cos\theta \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.10}$$

which is not in the distribution $\Delta$ spanned by $s_1(q)$ and $s_2(q)$. Therefore, at least one of the constraints is nonholonomic. We continue to compute the Lie bracket of $s_1(q)$ and $s_3(q)$

$$s_4(q) = \left[s_1(q)\ s_3(q)\right] = \frac{\partial s_3}{\partial q} s_1 - \frac{\partial s_1}{\partial q} s_3 = \begin{bmatrix} -\frac{r^3}{d^2}\cos\theta \\ -\frac{r^3}{d^2}\sin\theta \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.11}$$

which is linearly independent of $s_1(q)$, $s_2(q)$, and $s_3(q)$. However, the distribution spanned by $s_1(q)$, $s_2(q)$, $s_3(q)$ and $s_4$ is involutive. Therefore, we have

$$\Delta^* = \text{span}\left\{s_1(q), s_2(q), s_3(q), s_4(q)\right\} \tag{3.12}$$

It follows that two of the constraints are nonholonomic and the other one is holonomic.

To obtain the holonomic constraint, we subtract Eq. (3.3) from Eq. (3.4)

$$d\dot{\theta} = r(\dot{\theta}_r - \dot{\theta}_l) \tag{3.13}$$

Integrating the above equation and properly choosing the initial condition of $\theta(0) = \theta_r(0) = \theta_l(0)$, we have

$$\theta = \frac{r}{d}(\theta_r - \theta_l) \tag{3.14}$$

which is obviously a holonomic constraint equation. Thus, $\theta$ may be eliminated from the generalized coordinates.

The two nonholonomic constraints are

$$\dot{x}_o \sin\theta - \dot{y}_o \cos\theta = 0 \tag{3.15}$$

$$\dot{x}_o \cos\theta + \dot{y}_o \sin\theta = \frac{r}{2}(\dot{\theta}_r + \dot{\theta}_l) \tag{3.16}$$

The second nonholonomic constraint equation in the above is obtained by adding Eq. (3.3) from Eq. (3.4). It is understood that $\theta$ is now a shorthand notation for $r/d(\theta_r - \theta_l)$ rather than an independent variable. We may write these two constraint equations in the matrix form

$$A(q)\dot{q} = 0 \tag{3.17}$$

where the generalized coordinate vector $q$ is now defined as

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} = \begin{bmatrix} x_o \\ y_o \\ \theta \\ \theta_r \\ \theta_l \end{bmatrix} \tag{3.18}$$

The kinematics of this mechanism can be written as

$$\begin{bmatrix} \dot{x}_o \\ \dot{y}_o \\ \dot{\theta} \\ \dot{\theta}_r \\ \theta_l \end{bmatrix} = \begin{bmatrix} \frac{r\cos(\theta)}{2} & \frac{r\cos(\theta)}{2} \\ \frac{r\sin(\theta)}{2} & \frac{r\sin(\theta)}{2} \\ \frac{r}{d} & -\frac{r}{d} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_r \\ \dot{\theta}_l \end{bmatrix} \tag{3.19}$$

From the relationship between linear and angular velocities of the vehicle and its wheels' velocities, we have

$$\begin{bmatrix} \varphi_r \\ \varphi_l \end{bmatrix} = \begin{bmatrix} \frac{1}{r} & \frac{1}{d} \\ \frac{1}{r} & -\frac{1}{d} \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \tag{3.20}$$

Substituting (3.20) into (3.19), we have

$$\begin{bmatrix} \dot{x}_o \\ \dot{y}_o \\ \dot{\theta} \\ \dot{\varphi}_r \\ \dot{\varphi}_l \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \\ \frac{1}{r} & \frac{1}{d} \\ \frac{1}{r} & -\frac{1}{d} \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \tag{3.21}$$

Skipping two equations from (3.21), we have

$$\begin{bmatrix} \dot{x}_c \\ \dot{y}_c \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \tag{3.22}$$

## 3.3 Dynamics of WIP Systems

In general, the dynamics of wheeled inverted pendulum can be derived by two traditional formulations: the closed-form Lagrange–Euler formulation and forward–backward recursive Newton–Euler formulation. The Lagrange–Euler approach treats the WIP as a whole and performs the analysis using the Lagrangian function (the difference between the kinetic energy and the potential energy of the mobile robotic system), which compose of each link of the wheeled inverted pendulum. While Newton–Euler approach describe the combined translational and rotational dynamics of a rigid body with respect to the each link's center of mass. The dynamics of the whole wheeled inverted pendulum can be described by the forward–backward recursive dynamic equations. Therefore, two different kinds of formulations provide different insights to the physical meaning of dynamics. Dynamic analysis is to find the relationship between the generalized coordinates $q$ and the generalized forces $\tau$. A closed-form equation like the Lagrange–Euler formulation is preferred such that we can conduct the controllers to obtain the time evolution of the generalized coordinates.

Thus, in the following section, the Lagrange–Euler formulation will be discussed in detail from Sect. 3.3.1 to Sect. 3.3.4, which follows the description of the previous work [44]. Section 3.3.5 comes from lots of the previous works, such as [44, 113], etc.

### *3.3.1 Lagrange–Euler Equations*

We briefly introduce the principle of virtual work since the Lagrange–Euler equations of motion are a set of differential equations that describe the time evolution of mechanical systems under holonomic constraints.

Consider a system consisting of $l$ particles, with corresponding coordinates $r_1, r_2, \ldots, r_l$ is subject to holonomic constraints as follows

$$f_i(r_1 \ldots r_l) = 0, \quad i = 1, 2, \ldots, m \tag{3.23}$$

The constraint implies a force (called constraint force) is produced, that hold this constraint forces. The system subject to constraints (3.23) has $m$ fewer degree of freedom than the unconstrained system, then the coordinates of the $l$ constraints are described in term of $n$ generalized coordinates $q_1, q_2, \ldots, q_n$ as

$$r_i = r_i(q), \quad i = 1, 2, \ldots, l \tag{3.24}$$

where $q = [q_1, q_2, \ldots, q_n]^T$ and $q_1, q_2, \ldots, q_n$ are independent. To keep the discussion simple, $l$ is assumed to be finite.

Differentiating the constraint function $f_i(\cdot)$ with respect to time, we obtain new constraint

$$\frac{d}{dt} f_i(r_1, r_2, \ldots, r_l) = \frac{q\partial f_i}{\partial r_1}\frac{dr_1}{dt} + \cdots + \frac{\partial f_i}{\partial r_l}\frac{dr_l}{dt} = 0 \tag{3.25}$$

The constraint of the form

$$\omega_1(r_1, \ldots, r_l)\, dr_1 + \cdots + \omega_k(r_1, \ldots, r_k)\, dr_k = 0 \tag{3.26}$$

is called nonholonomic if it can not be integrated back to $f_i(\cdot)$.

Given (3.25), by definition a set of infinitesimal displacements $\Delta r_1, \ldots, \Delta r_l$, that are consistent with the constraint

$$\frac{\partial f_i}{\partial r_1}\Delta r_1 + \cdots + \frac{\partial f_i}{\partial r_l}\Delta r_l = 0 \tag{3.27}$$

are called virtual displacements, which can be precisely defined as follows with Eq. (3.24) holding

$$\delta r_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j, \quad i = 1, 2, \ldots, l \tag{3.28}$$

where $\delta q_1, \delta q_2, \ldots, \delta q_n$ of the generalized coordinates are unconstrained.

Consider a system of $l$-particle with the total force $F_i$, suppose that

- the system has a holonomic constraint, that is some of particles exposed to constraint force $f_{ic}$;
- there are the external force $f_{ie}$ to the particles; and
- the constrained system is in equilibrium;

then the work done by all forces applied to $i$th-particle along each set of virtual displacement is zero,

$$\sum_{i=1}^{l} F_i^T \delta r_i = 0 \tag{3.29}$$

If the total work done by the constraint forces corresponding to any set of virtual displacement is zero, that is

$$\sum_{i=1}^{l} f_{ic}^T \delta r_i = 0 \tag{3.30}$$

Substituting Eq. (3.31) into (3.30), we have

$$\sum_{i=1}^{l} f_{ie}^T \delta r_i = 0 \tag{3.31}$$

which expresses the principle of virtual work: if satisfying (3.30), the work done by external forces corresponding to any set of virtual displacements is zero. Suppose that each constraint will be in equilibrium and consider the fictitious additional force $\dot{p}_i$ for each constraint with the momentum of the $i$th constraint $p_i$. By substituting $p_i$ with $F_i$ in Eq. (3.29), and the constraint forces are eliminated as before by using the principle of virtual work, we can obtain

$$\sum_{i=1}^{l} f_{ie}^T \delta r_i - \sum_{i=1}^{l} \dot{p}_i \delta r_i = 0 \tag{3.32}$$

The virtual work by the force $f_{ie}$ is expressed as

$$\sum_{i=1}^{l} f_{ie}^T \delta r_i = \sum_{j=1}^{n} \left( \sum_{i=1}^{l} f_{ie}^T \frac{\partial r_i}{\partial q_j} \right) \delta q_j = \sum_{j=1}^{n} \psi_j \delta q_j = \psi^T \delta q \tag{3.33}$$

where $\psi = [\psi_1, \psi_2 \ldots \psi_n]$, $\psi_j = \sum_{i=1}^{k} f_{ie}^T \frac{\partial r_i}{\partial q_j}$ is called the $j$th generalized force.

Considering $p_i = m_i \dot{r}_i$, the second summation in Eq. (3.32) becomes

$$\sum_{i=1}^{l} \dot{p}_i^T \delta r_i = \sum_{i=1}^{l} \sum_{j=1}^{n} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} \delta q_j = \sum_{j=1}^{n} \eta_j \delta q_j = \eta^T \delta q \tag{3.34}$$

where $\eta = [\eta_1, \eta_2 \ldots \eta_n]^T$, and $\eta_j = \sum_{i=1}^{k} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j}$.

Using the chain-rule, we can obtain

$$\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j} \tag{3.35}$$

since

$$v_i = \dot{r}_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \dot{q}_j \tag{3.36}$$

We can further obtain

$$\frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right] = \sum_{l=1}^{n} \frac{\partial^2 r_i}{\partial q_j \partial q_l} \dot{q}_l = \frac{\partial v_i}{\partial q_j} \tag{3.37}$$

Based on the product rule of differentiation, we have $m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \frac{d}{dt}[m_i \dot{r}_i^T] - m_i \dot{r}_i^T \frac{d}{dt}[\frac{\partial r_i}{\partial q_j}]$, and considering the above three relations, we can rewrite $\eta_j$ as

$$\eta_j = \sum_{i=1}^{l} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{l} \left( \frac{d}{dt}\left[ m_i v_i^T \frac{\partial v_i}{\partial \dot{q}_j}\right] - m_i v_i^T \frac{\partial v_i}{\partial q_j} \right) \tag{3.38}$$

Let $K = \sum_{i=1}^{l} \frac{1}{2} m_i v_i^T v_i$ be the kinetic energy, considering (3.38), we can obtain $\eta_j = \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j}$, $j = 1, 2, \ldots, n$, rewriting the above equation in a vector form, we have

$$\eta = \frac{d}{dt}\frac{\partial K}{\partial \dot{q}} - \frac{\partial K}{\partial q} \tag{3.39}$$

Summary equations from (3.32), (3.33), (3.34), and (3.39), we have

$$\left[\frac{d}{dt}\frac{\partial K}{\partial \dot{q}} - \frac{\partial K}{\partial q} - \psi\right]^T \delta q = 0 \tag{3.40}$$

Define a scalar potential energy $P(q)$ such that $\psi = -\frac{\partial P}{\partial q}$, since the virtual displacement vector $\delta q$ is unconstrained and its elements $\delta q_j$ are independent, which leads to $\frac{d}{dt}\frac{\partial K}{\partial \dot{q}} - \frac{\partial K}{\partial q} - \psi = 0$, it can be written as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{3.41}$$

where $L(q, \dot{q}) = K(q, \dot{q}) - P(q)$ is Lagrangian function.

*Remark 3.2* Given generalized coordinates, the choice of Lagrangian is not unique for a particular set of equations of motion.

*Remark 3.3* A necessary and sufficient condition that $F$ be the gradient of some scalar function $P$, i.e., $F = -\frac{\partial P(r)}{\partial q}$, which in turn means that the generalized force is derivable from $P$ by differentiating with respect to $q$.

If the generalized force $\psi$ includes an external applied force and a potential field force, suppose there exists a vector $\tau$ and a scalar potential function $P(q)$ satisfying $\psi = \tau - \frac{\partial P}{\partial q}$, then, Eq. (3.41) can be written in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau \tag{3.42}$$

Equations (3.41) and/or (3.42) are called the Lagrangian equations or Lagrange–Euler equations in the robotics literature.

### 3.3.2 Kinetic Energy

Consider the velocity of the point in base coordinates described by $v_i = \frac{dr_i}{dt} = \sum_{j=1}^{i}[\frac{\partial T_i^0}{\partial q_j}\dot{q}_j]r_i^i = \sum_{j=1}^{n}[\frac{\partial T_i^0}{\partial q_j}\dot{q}_j]r_i^i$ with $\frac{\partial T_i^0}{\partial q_j} = 0, \forall j > i$, we have

$$dK_i = \frac{1}{2}\,\mathrm{trace}\left[\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial T_i^0}{\partial q_j}\left(r_i^i r_i^{iT}\,dm\right)\frac{\partial T_i^{0T}}{\partial q_k}\dot{q}_j\dot{q}_k\right] \tag{3.43}$$

Define the $4 \times 4$ pseudo-inertia matrix for the $i$th link as

$$J_i = \int_{\text{link } i} r_i^i r_i^{iT}\,dm \tag{3.44}$$

The total kinetic energy for the $i$th link can be expresses as

$$K_i = \int_{\text{link } i} dK_i = \frac{1}{2}\,\mathrm{trace}\left[\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial T_i^0}{\partial q_j}J_i\frac{\partial T_i^{0T}}{\partial q_k}\dot{q}_j\dot{q}_k\right] \tag{3.45}$$

Consider the generalized coordinates as $r_i^i = [x\ y\ z\ 1]^T$, we can rewrite (3.44) as [44]

$$J_i = \begin{bmatrix} \frac{-I_{ixx}+I_{iyy}+I_{izz}}{2} & I_{ixy} & I_{ixz} & m_i\overline{x}_i \\ I_{ixy} & \frac{I_{ixx}-I_{iyy}+I_{izz}}{2} & I_{iyz} & m_i\overline{y}_i \\ I_{ixz} & I_{iyz} & \frac{I_{ixx}+I_{iyy}+I_{izz}}{2} & m_i\overline{z}_i \\ m_i\overline{x}_i & m_i\overline{y}_i & m_i\overline{z}_i & m_i \end{bmatrix} \tag{3.46}$$

where $I_{ixx} = \int(y^2+z^2)\,dm$, $I_{iyy} = \int(x^2+z^2)\,dm$, $I_{izz} = \int(x^2+y^2)\,dm$, $I_{ixy} = \int xy\,dm$, $I_{ixz} = \int xz\,dm$, $I_{iyz} = \int yz\,dm$, and $m_i\overline{x}_i = \int x\,dm$, $m_i\overline{y}_i = \int y\,dm$,

$m_i\bar{z}_i = \int z\,dm$ with $m_i$ as the total mass of the $i$th link, and $\bar{r}_i^i = [\bar{x}\ \ \bar{y}\ \ \bar{z}\ \ 1]^T$ as the center of mass vector of the $i$th link from the $i$th link coordinate frame and expressed in the $i$th link coordinate frame.

Therefore, the total kinetic energy can be written as

$$\begin{aligned} K(q,\dot{q}) &= \frac{1}{2}\sum_{i=1}^{n}\text{trace}\left[\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial T_i^0}{\partial q_j}J_i\frac{\partial T_i^{0T}}{\partial q_k}\dot{q}_j\dot{q}_k\right] \\ &= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}m_{jk}\dot{q}_j\dot{q}_k = \frac{1}{2}\dot{q}^T M(q)\dot{q} \end{aligned} \tag{3.47}$$

where the $jk$th element $m_{jk}$ of the $n\times n$ inertia matrix $M(q)$ is defined as

$$m_{jk}(q) = \sum_{i=1}^{n}\text{trace}\left[\frac{\partial T_i^0}{\partial q_j}J_i\frac{\partial T_i^{0T}}{\partial q_k}\right] \tag{3.48}$$

### 3.3.3 Potential Energy

The total potential energy of the robot is therefore expressed as

$$P(q) = -\sum_{i=1}^{n}P_i \tag{3.49}$$

where $P_i$ is the potential energy of the $i$th link with mass $m_i$ and center of gravity $\bar{r}_i^i$ expressed in the coordinates of its own frame, the potential energy of the link is given $P_i = -m_i g^T T_i^0 \bar{r}_i^i$, the gravity vector is expressed in the base coordinates as $g = [g_x\ g_y\ g_z\ 0]^T$.

### 3.3.4 Lagrangian Equations

Consider the kinetic energy $K(q,\dot{q})$ and the potential energy $P(q)$ can be expressed as $K(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q}$, $P(q) = -\sum_{i=1}^{n} m_i g T_i^0 \bar{r}_i^i$, so the Lagrangian function $L(q,\dot{q}) = K(q,\dot{q}) - P(q)$ is thus given by

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - P(q) \tag{3.50}$$

We can obtain

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_{j=1}^{n} m_{kj}\dot{q}_j \tag{3.51}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \sum_{j=1}^{n} m_{kj}\ddot{q}_j + \sum_{j=1}^{n} \frac{d}{dt} m_{kj}\dot{q}_j \tag{3.52}$$

$$= \sum_{j=1}^{n} m_{kj}\ddot{q}_j + \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial m_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j \tag{3.53}$$

$$\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial m_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial P}{\partial q_k} \tag{3.54}$$

for $k = 1, 2, \ldots, n$.

Considering the symmetry of the inertia matrix, we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial m_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial m_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j + \frac{\partial m_{ki}}{\partial q_i}\dot{q}_j\dot{q}_i\right] \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j}\right]\dot{q}_i\dot{q}_j \end{aligned} \tag{3.55}$$

The Lagrange–Euler equations can then be written as

$$\begin{aligned}&\sum_{j=1}^{n} m_{kj}\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial m_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial m_{ij}}{\partial q_k}\right)\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} \\ &\quad = \sum_{j=1}^{n} m_{kj}\ddot{q}_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial m_{ki}}{\partial q_j} + \frac{\partial m_{kj}}{\partial q_i} - \frac{\partial m_{ij}}{\partial q_k}\right]\dot{q}_i\dot{q}_j \\ &\quad = \sum_{j=1}^{n} m_{kj}\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}\dot{q}_i\dot{q}_j = \tau_k \end{aligned} \tag{3.56}$$

where $c_{ijk}$ is the Christoffel symbols (of the first kind) defined as

$$c_{ijk}(q) \triangleq \frac{1}{2}\left[\frac{\partial m_{kj}(q)}{\partial q_i} + \frac{\partial m_{ki}(q)}{\partial q_j} - \frac{\partial m_{ij}(q)}{\partial q_k}\right] \tag{3.57}$$

Define $g_k(q) = \frac{\partial P(q)}{\partial q_k}$, then the Lagrange–Euler equations can be written as

$$\sum_{j=1}^{n} m_{kj}(q)\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k, \quad k = 1, 2, \ldots, n \tag{3.58}$$

It is common to write the above equations in matrix form

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau \tag{3.59}$$

where $kj$th element of $C(q, \dot{q})$ defined as

$$c_{kj} = \sum_{i=1}^{n} c_{ijk}\dot{q}_i = \sum_{i=1}^{n} \frac{1}{2}\left[\frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{m_{ij}}{\partial q_k}\right]\dot{q}_i \tag{3.60}$$

To facilitate the understanding of the control problems, and to help design controllers for the above systems, it is essential to have a thorough study of the mathematical properties of the system.

### 3.3.5 Properties of Mechanical Dynamics

There are some properties summarized for the dynamics of wheeled inverted pendulum, which is convenient for controller design.

*Property 3.4* The inertia matrix $M(q)$ is symmetric, i.e., $M(q) = M^T(q)$.

*Property 3.5* The inertia Matrix $M(q)$ is uniformly positive definite, and bounded below and above, i.e. $\exists 0 < \alpha \leq \beta < \infty$, such that $\alpha I_n \leq M(q) \leq \beta I_n$, $\forall q \in \mathbb{R}^n$, where $I_n$ is the $n \times n$ identity matrix.

*Property 3.6* The inverse of inertia matrix $M^{-1}(q)$ exists, and is also positive definite and bounded.

*Property 3.7* Centrifugal and Coriolis forces $C(q, \dot{q})\dot{q}$ is quadratic in $\dot{q}$.

*Property 3.8* It may be written in $C(q, \dot{q})\dot{q} = C_1(q)C_2[\dot{q}\dot{q}] = C_3(q)[\dot{q}\dot{q}] + C_4(q)[\dot{q}^2]$, where $[\dot{q}\dot{q}] = [\dot{q}_1\dot{q}_2, \dot{q}_1\dot{q}_3, \ldots, \dot{q}_{n-1}\dot{q}_n]^T$ and $[\dot{q}^2] = [\dot{q}_1^2, \dot{q}_2^2, \ldots, \dot{q}_n^2]^T$.

*Property 3.9* Given two $n$-dimensional vectors $x$ and $y$, the matrix $C(q, \dot{q})$ defined by Eq. (3.59) implies that $C(q, x)y = C(q, y)x$.

*Property 3.10* The 2-norm of $C(q, \dot{q})$ satisfies the inequality $\|C(q, \dot{q})\| \leq k_c(q)\|\dot{q}\|$, where $k_c(q) = \frac{1}{2} \max_{q \in \mathbb{R}^n} \sum_{k=1}^{n} \|C_k(q)\|$. For revolute robots, $k_c$ is a finite constant since the dependence of $C_k(q)$, $k = 1, 2, \ldots, n$, on $q$ appears only in terms of sine and cosine functions of their entries.

*Property 3.11* Gravitational force $G(q)$ can be derived from the gravitational potential energy function $P(q)$, i.e. $G(q) = \partial P(q)/\partial q$, and is also bounded, i.e., $\|G(q)\| \leq k_{G(q)}$, where $k_{G(q)}$ is a scalar function which may be determined for any given WIP. For revolute joints, the bound is a constant independent of $q$ whereas for prismatic joints, the bound may depend on $q$.

*Property 3.12* The dependence of $M(q)$, $C(q, \dot{q})$ and $G(q)$ on q will appear only in terms of sine and cosine functions in their entries, so that $M(q)$, $C(q, \dot{q})$ and $G(q)$ have bounds that are independent of $q$.

*Property 3.13* By defining each coefficient as a separate parameter, the dynamics can be written in the linear in the parameters (LIPs) form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Y(q,\dot{q},\ddot{q})P \tag{3.61}$$

where $Y(q,\dot{q},\ddot{q})$ is an $n \times r$ matrix of known functions known as the regressor matrix, and $P$ is an $r$ dimensional vector of parameters.

*Remark 3.14* The above equation can also be written as

$$M(q)\ddot{q}_r + C(q,\dot{q})\dot{q}_r + G(q) = \Phi(q,\dot{q},\dot{q}_r,\ddot{q}_r)P \tag{3.62}$$

where $\dot{q}_r$ and $\ddot{q}_r$ are the corresponding $n$-dimensional vectors.

*Property 3.15* The matrix $N(q,\dot{q})$ defined by $N(q,\dot{q}) = \dot{M}(q) - 2C(q,\dot{q})$ is skew-symmetric, i.e., $n_{kj}(q,\dot{q}) = -n_{jk}(q,\dot{q})$, if $C(q,\dot{q})$ is defined using the Christoffel symbols.

*Property 3.16* Since $M(q)$ and therefore $\dot{M}(q)$ are symmetric matrices, the skew-symmetry of the matrix $\dot{M}(q) - 2C(q,\dot{q})$ can also be seen from the fact that $\dot{M}(q) = C(q,\dot{q}) + C^T(q,\dot{q})$.

*Property 3.17* The system is passive from $\tau$ to $\dot{q}$.

*Property 3.18* Even though the skew-symmetry property of $N(q,\dot{q})$ is guaranteed if $C(q,\dot{q})$ is defined by the Christoffel symbols, it is always true that $\dot{q}^T[\dot{M}(q) - 2C(q,\dot{q})]\dot{q} = 0$.

*Property 3.19* The system is feedback linearizable, i.e., there exists a nonlinear transformation such that the transformed system is a linear controllable system.

### 3.3.6 Dynamics of Wheeled Inverted Pendulum

Consider two-wheeled inverted pendulum shown in Fig. 3.1. The total kinetic energy can be written as

$$\begin{aligned} K = {} & \frac{1}{2}Mv^2 + M_w v^2 + \frac{1}{2}m(v + l\cos\alpha\dot{\alpha})^2 + \frac{1}{2}m(-l\sin\alpha\dot{\alpha})^2 \\ & + I_w\left(\frac{v}{r}\right)^2 + \frac{1}{2}I_M\dot{\alpha}^2 + \frac{1}{2}I_p\omega^2 + 2\left(M_w + \frac{I_w}{r^2}\right)d^2\omega^2 \end{aligned} \tag{3.63}$$

and the total potential energy is

$$U = mgl(1 - \cos\alpha) \tag{3.64}$$

which is subjected to the following constraints $\dot{x}\cos\theta - \dot{y}\sin\theta = 0$. Using the Lagrangian approach like the above section, we can choose $q = [x, \theta, \alpha]$, and $\dot{q} = [v, w, \dot{\alpha}]^T$ as generalized coordinations, then we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{\tau_l}{r} + \frac{\tau_r}{r} + d_l + d_r \tag{3.65}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 2d\left(\frac{\tau_l}{r} - \frac{\tau_r}{r} + d_l - d_r\right) \tag{3.66}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\dot{\alpha}}\right) - \frac{\partial L}{\partial \alpha} = 0 \tag{3.67}$$

Let

$$\begin{aligned} L = K - U = {} & \frac{1}{2}Mv^2 + M_w v^2 + \frac{1}{2}m(v + l\cos\alpha\dot{\alpha})^2 + \frac{1}{2}m(-l\sin\alpha\dot{\alpha})^2 \\ & + I_w\left(\frac{v}{r}\right)^2 + \frac{1}{2}I_M\dot{\alpha}^2 + \frac{1}{2}\left(I_p + 2\left(M_w + \frac{I_w}{r^2}\right)d^2\right)\omega^2 \\ & - mgl(1 - \cos\alpha) \end{aligned} \tag{3.68}$$

Then, we have

$$\frac{\partial L}{\partial v} = Mv + 2M_w v + m(v + l\dot{\alpha}\cos\alpha) + 2\frac{I_w}{r^2}v \tag{3.69}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) = \left(M + 2M_w + m + 2\frac{I_w}{r^2}\right)\dot{v} + ml\left(-\dot{\alpha}^2\sin\alpha + \ddot{\alpha}\cos\alpha\right) \tag{3.70}$$

$$\frac{\partial L}{\partial x} = 0 \tag{3.71}$$

Then, we have

$$\frac{\partial L}{\partial \omega} = \left(I_p + 2\left(M_w + \frac{I_w}{r^2}\right)d^2\right)\omega \tag{3.72}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \omega}\right) = I_p\dot{\omega} + 2\left(M_w + \frac{I_w}{r^2}\right)d^2\dot{\omega} \tag{3.73}$$

$$\frac{\partial L}{\partial \theta} = 0 \tag{3.74}$$

Moreover, we have

$$\frac{\partial L}{\partial \dot{\alpha}} = mlv\cos\alpha + ml^2\dot{\alpha} + I_M\dot{\alpha} \tag{3.75}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}}\right) = ml\dot{v}\cos\alpha + mlv(-\sin\alpha)\dot{\alpha} + ml^2\ddot{\alpha} + I_M\ddot{\alpha} \tag{3.76}$$

$$\begin{aligned} \frac{\partial L}{\partial \alpha} = {} & m(v + l\dot{\alpha}\cos\alpha)(-l\dot{\alpha}\sin\alpha) + m(l\dot{\alpha}\sin\alpha)(l\dot{\alpha}\cos\alpha) \\ & + mgl\sin\alpha \end{aligned} \tag{3.77}$$

Then we can obtain the dynamics equation as

$$\begin{aligned} & \left(M + 2M_w + m + 2\frac{I_w}{r^2}\right)\dot{v} + ml\ddot{\alpha}\cos\alpha - ml\dot{\alpha}^2\sin\alpha \\ & = \frac{\tau_l}{r} + \frac{\tau_r}{r} + d_l + d_r \end{aligned} \tag{3.78}$$

$$\left(I_p + 2\left(M_w + \frac{I_w}{r^2}\right)d^2\right)\dot{\omega} = 2d\left(\frac{\tau_l}{r} - \frac{\tau_r}{r} + d_l - d_r\right) \tag{3.79}$$

$$ml\dot{v}\cos\alpha + \left(ml^2 + I_M\right)\ddot{\alpha} - mgl\sin\alpha = 0 \tag{3.80}$$

## 3.4 Newton–Euler Approach

Consider the mobile inverted pendulum shown in Fig. 3.1, we give Newton–Euler to describe the time evolution of mechanical systems under nonholonomic constraints [35]. The table of forces and moments acting on the vehicle is shown in Table 3.1.

With an assumption that there is no slip between the wheels and the ground, balancing forces and moments acting on the left wheel results in the following equations of motion of the left wheel:

$$I_w\ddot{\theta}_l = F_l - H_l r \tag{3.81}$$

$$M_w\ddot{x}_l = d_l - F_l + H_l \tag{3.82}$$

Similarly, for the right wheel we have

$$I_w\ddot{\theta}_r = F_r - H_r r \tag{3.83}$$

$$M_w\ddot{x}_r = d_r - F_r + H_r \tag{3.84}$$

Balancing forces acting on the pendulum on the $x$-axis direction and moments about the center point of gravity $O$ results in

$$-ml\cos(\alpha)\ddot{\alpha} + ml\dot{\alpha}^2\sin(\alpha) - m\ddot{x} = F_p \tag{3.85}$$

$$ml^2\ddot{\alpha} + ml\cos(\alpha)\ddot{x} - mgl\sin(\alpha) = M_p \tag{3.86}$$

where $ml^2$ is the moment of inertia of the pendulum with respect to the $y$-axis.

Balancing forces acting on the platform along the $x$-axis direction, and moments about the $z$-axis results in

$$M\ddot{x} = F_l + F_r + F_p \tag{3.87}$$

$$I_M\ddot{\alpha} = -M_p \tag{3.88}$$

Balancing the moments acting on the platform and pendulum about the $z$-axis gives

$$I_M\ddot{\theta} = d(F_l - F_r) \tag{3.89}$$

The relationship between the displacement of the wheel along the $x$-axis and the rotational angle of the wheel about the $y$-axis is

$$\begin{cases} \theta_l = \frac{x_l}{r} \\ \theta_r = \frac{x_r}{r} \end{cases} \Rightarrow \quad \theta_l - \theta_r = \frac{x_l - x_r}{r} \tag{3.90}$$

On the other hand, the relationship between the rotational angle (heading angle) $\theta$

**Table 3.1** The parameters and variables of wheeled inverted pendulum

| | |
|---|---|
| $F_l, F_r$ | Interacting forces between the left and right wheels and the platform |
| $H_l, H_r$ | Friction forces acting on the left and right wheels |
| $\tau_l, \tau_r$ | Torques provided by wheel actuators acting on the left and right wheels |
| $d_l, d_r$ | External forces acting on the left and right wheels |
| $\theta_l, \theta_r$ | Rotational angles of the left and right wheels |
| $x_l, x_r$ | Displacements of the left and right wheels along the $x$-axis |
| $\alpha$ | Tilt angle of the pendulum |
| $\theta$ | Heading angle of the vehicle |
| $M_w$ | Mass of the wheel |
| $I_w$ | Moment of inertia of the wheel with respect to the $y$-axis |
| $r$ | Radius of the wheel |
| $m$ | Mass of the pendulum |
| $g$ | Gravity acceleration |
| $l$ | Distance from the point $O$ to the center of gravity, CG, of the pendulum |
| $d$ | Distance between the left and right wheels along the $y$-axis |
| $M$ | Mass of the platform |
| $I_M$ | Moment of inertia of the platform about the $y$-axis |
| $I_p$ | Moment of inertia of the platform and pendulum about the $z$-axis |
| $F_p$ | Interacting force between the pendulum and the platform on the $x$-axis |
| $M_p$ | Interacting moment between the pendulum and the platform about the $y$-axis |
| $v$ | the forward velocity of the mobile platform |
| $\omega$ | the rotation velocity of the mobile platform, and $\omega = \dot{\theta}$ |

of the vehicle about the $z$-axis and the displacement of the wheels along the $x$-axis is

$$\theta = \frac{x_l - x_r}{d} \tag{3.91}$$

By subtracting Eqs. (3.81) and (3.82) from Eqs. (3.83) and (3.84), respectively, we have

$$I_w(\ddot{\theta}_l - \ddot{\theta}_r) = \tau_l - \tau_r - (H_l - H_r)r \tag{3.92}$$

$$M_w(\ddot{x}_l - \ddot{x}_r) = d_l - d_r - (F_l - F_r) + H_l - H_r \tag{3.93}$$

By substituting Eqs. (3.90) and (3.97) into Eqs. (3.92) and (3.93), we have

$$I_w \frac{d}{r}\ddot{\theta} = \tau_l - \tau_r - (H_l - H_r)r \tag{3.94}$$

$$dM_w\ddot{\theta} = d_l - d_r - (F_l - F_r) + H_l - H_r \tag{3.95}$$

Dividing both sides of Eq. (3.94) by $r$ then adding with Eq. (3.95) result in

$$d\left(\frac{I_w}{r^2} + M_w\right)\ddot{\theta} = \frac{\tau_l - \tau_r}{r} - (F_l - F_r) + d_l - d_r \tag{3.96}$$

Multiplying both sides of Eq. (3.96) by $d$ then adding with Eq. (3.89), then we obtain

$$\ddot{\theta} = \frac{d}{rI_\theta}(\tau_l - \tau_r) + \frac{d}{I_\theta}(d_l - d_r) \tag{3.97}$$

where $I_\theta = I_p + d^2(M_w + \frac{I_w}{r^2})$.

The relationship between displacement of the wheel along the $x$-axis and the rotational angle of the wheel about the $y$-axis (see Eq. (3.97)) gives

$$\theta_l + \theta_r = \frac{x_l + x_r}{r} \tag{3.98}$$

On the other hand, the relationship between displacement of the vehicle and wheels along the $x$-axis is

$$x = \frac{x_l + x_r}{2} \tag{3.99}$$

Considering (3.81)–(3.84), we have

$$I_w(\ddot{\theta}_l + \ddot{\theta}_r) = \tau_l + \tau_r - (H_l + H_r)r \tag{3.100}$$

$$M_w(\ddot{x}_l + \ddot{x}_r) = d_l + d_r - (F_l + F_r) + H_l + H_r \tag{3.101}$$

By substituting Eqs. (3.98) and (3.99) into Eqs. (3.100) and (3.101), we obtain

$$\frac{2I_w}{r}\ddot{x} = \tau_l + \tau_r - (H_l + H_r)r \tag{3.102}$$

$$2M_w\ddot{x} = d_l + d_r - (F_l + F_r) + H_l + H_r \tag{3.103}$$

Integrating the above equations, we have

$$2\left(\frac{I_w}{r^2} + M_w\right)\ddot{x} = \frac{\tau_l + \tau_r}{r} - (F_l + F_r) + d_l + d_r \tag{3.104}$$

By subtracting Eq. (3.85) from Eq. (3.87), we obtain

$$(M + m)\ddot{x} + ml\cos(\alpha)\ddot{\alpha} - ml\dot{\alpha}^2\sin(\alpha) = F_l + F_r \tag{3.105}$$

Adding Eq. (3.104) with Eq. (3.105) results in

$$\begin{aligned} &ml\cos(\alpha)\ddot{\alpha} + \left(M + m + 2\left(\frac{I_w}{r^2} + M_w\right)\right)\ddot{x} \\ &= ml\dot{\alpha}^2\sin(\alpha) + \frac{\tau_l + \tau_r}{r} + d_l + d_r \end{aligned} \tag{3.106}$$

Adding Eq. (3.86) with Eq. (3.88) gives

$$\left(ml^2 + I_M\right)\ddot{\alpha} + ml\cos(\alpha)\ddot{x} = mgl\sin(\alpha) \tag{3.107}$$

From the above derivation, we can obtain the dynamics equation of motion of the two-wheeled mobile vehicle with an inverted pendulum as Eqs. (3.97), (3.106), (3.107), which are identical with Eqs. (3.78), (3.79), and (3.80).

## 3.5 Conclusion

In this chapter, we first describe the kinematics and dynamics model for wheeled inverted pendulum. The Lagrange–Euler equations of motion and Newton–Euler approach have been discussed respectively. Based on the Lagrange–Euler formulation, the dynamics for a general wheeled inverted pendulum have been presented, which incorporates the dynamic interactions between the mobile platform and the inverted pendulum. The structural properties of robots, which are useful for controller design, have also been briefly summarized. Finally, the dynamic equations for wheeled inverted pendulum have been derived in a step to step manner.