Hw 2

2020年2月3日 11:56

Q1 (a)

$Q \times Q^*$

$= (q_0, \vec{q}) \cdot (0, \vec{x}) \cdot (q_0, -\vec{q})$

$= (q_0 \cdot 0 - \vec{q} \cdot \vec{x},\ q_0\vec{x} + 0\vec{q} + \vec{q} \times \vec{x}) \cdot (q_0, -\vec{q})$

$= (-\vec{q} \cdot \vec{x},\ q_0\vec{x} + \vec{q} \times \vec{x}) \cdot (q_0, -\vec{q})$

$= (0, (\vec{q} \cdot \vec{x})\vec{q} + q_0^2\vec{x} + q_0(\vec{q} \times \vec{x}) - q_0(\vec{x} \times \vec{q}) - \vec{q} \times \vec{x} \times \vec{q})$

$\because\ \vec{q} \times \vec{x} \times \vec{q} = (\vec{q} \cdot \vec{q})\vec{x} - (\vec{q} \cdot \vec{x})\vec{q}$

$\therefore\ Q \times Q^*$

$= (0,\ (q_0^2 - \vec{q} \cdot \vec{q})\vec{x} + 2(q_0(\vec{q} \times \vec{x}) + (\vec{x} \cdot \vec{q})\vec{q}))$

let $q_0 = \cos\frac{\theta}{2}$ $\vec{q} = \sin\frac{\theta}{2}\vec{u}$ $\vec{u}$ is unit vector

$(q_0^2 - \vec{q} \cdot \vec{q}) = \cos\theta$

$2q_0\vec{q} = \sin\theta\,\vec{u}$

$2(\vec{x} \cdot \vec{q})\vec{q} = (1 - \cos\theta)(\vec{u} \cdot \vec{x}) \cdot \vec{u}$

$\therefore$ $(q_0^2 - \vec{q} \cdot \vec{q})\vec{x} + 2(q_0(\vec{q} \times \vec{x}) + (\vec{x} \cdot \vec{q})\vec{q}))$

$= \cos\theta\,\vec{x} + \sin\theta\,\vec{u} \times \vec{x} + (1 - \cos\theta)(\vec{u} \cdot \vec{x})\vec{u}$

$= \vec{x_{\parallel}} + \cos\theta\,\vec{x_{\perp}} + \sin\theta\,(\vec{u} \times \vec{x_{\perp}})$

(b) Because $Q \times Q^*$ can be regard as $\vec{x}$ rotate (θ) with respect to $\vec{u}$, which Q is a rotation wrt $\vec{u}$ with angle (θ)

Then it is also a rotation wrt $-\vec{u}$ with angle $(2\pi - \theta)$

Thus $(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{u})$ and $(\cos\frac{2\pi-\theta}{2}, -\sin\frac{2\pi-\theta}{2}\vec{u})$ can be used to represent R

$(\cos\frac{2\pi-\theta}{2}, -\sin\frac{2\pi-\theta}{2}\vec{u}) = (-\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\vec{u})$

(c)

i $R_{ab} \cdot R_{bc}$ need 27 "x", 18 "+"

ii two quaternion 16 "x", 12 "+"

iii $R\vec{v}$ need 9 "x", 6 "+"

iv $(q_0^2 - \vec{q} \cdot \vec{q})\vec{x} + 2(q_0(\vec{q} \times \vec{x}) + (\vec{x} \cdot \vec{q})\vec{q}))$

need 23 "x", 14 "+"

(d)

Q2 (a)

case 1 : $w = 0$

$$\hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \qquad \hat{\xi}^2 = \hat{\xi}^3 = \dots = 0$$

$$e^{\hat{\xi}} = I + \hat{\xi} = \begin{bmatrix} I & v \\ 0 & 1 \end{bmatrix} \in SE(2)$$

case2 : $(w \neq 0)$

$$g = \begin{bmatrix} I & \hat{w} v \\ 0 & 1 \end{bmatrix} \qquad g^{-1} = \begin{bmatrix} I & -\hat{w} v \\ 0 & 1 \end{bmatrix}$$

$$\hat{\xi}' = g^{-1} \hat{\xi} g$$

$$= \begin{bmatrix} I & -\hat{w} v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & \hat{w} v \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{w} & \hat{w}\hat{w} v + v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{w} & h v \\ 0 & 0 \end{bmatrix}$$

$$h = \hat{w}^2 + I$$

$$\hat{\xi}'^2 = \begin{bmatrix} \hat{w}^2 & (\hat{w}^3 + \hat{w}) v \\ 0 & 0 \end{bmatrix}$$

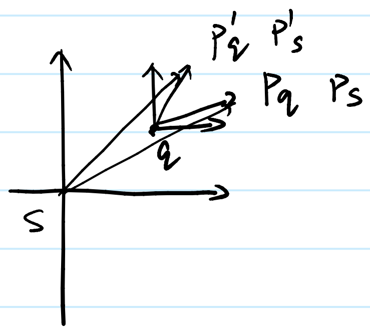
if $w = 1$ $\quad \hat{w}^3 + \hat{w} = 0 \quad \hat{\xi}'^2 = [\hat{w}^2 \;\; 0]$

if $w=1$ $\quad \hat{w}^3+\hat{w}=0 \quad \hat{\xi}'^2 = \begin{bmatrix} \hat{w}^2 & 0 \\ 0 & 0 \end{bmatrix}$

$$e^{\hat{\xi}'} = \begin{bmatrix} e^{\hat{w}} & hv \\ 0 & 1 \end{bmatrix}$$

$$e^{\hat{\xi}} = g\, e^{\hat{\xi}'} g^{-1} = \begin{bmatrix} e^{\hat{w}} & (I-e^{\hat{w}})\hat{w}v \\ 0 & 1 \end{bmatrix} \in SE(2)$$

(b)



First we create a new frame on q

$$g_{sq} = \begin{bmatrix} I & q \\ 0 & 1 \end{bmatrix}$$

if a twist on this new frame is $\xi' = (0, 0, w)$.

assume the twist on spacial fram is ξ,

$$p_q' = e^{\hat{\xi}'\theta} p_q$$

$$p_s' = e^{\hat{\xi}\theta} p_s$$

$$g_{sq} p_q' = e^{\hat{\xi}\theta} g_{sq} p_q$$

$$e^{\hat{\xi}'\theta} = g_{sq}^{-1} e^{\hat{\xi}\theta} g_{sq}$$

$$\hat{\xi} = g_{sq} \hat{\xi}' g_{sq}^{-1}$$

$$= \begin{bmatrix} \hat{w} & -\hat{w}\vec{q} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{w} & -\hat{w}\vec{q} \\ 0 & 0 \end{bmatrix}$$

$$\xi = \begin{bmatrix} w q_y \\ -w q_x \\ w \end{bmatrix} = \begin{bmatrix} q_y \\ -q_x \\ 1 \end{bmatrix}$$

if a twist on this new frame is $\xi' = (v_x, v_y, 0)$

$$\hat{\xi} = g_{sq} \hat{\xi}' g_{sq}^{-1}$$

$$= \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$$

$$\xi = \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix}$$

(c) From (b) we can see For any $\hat{\xi} = \begin{bmatrix} \hat{w} & \vec{v} \\ 0 & 0 \end{bmatrix}$

if $w = 0$ it's pure translation.

if $w \neq 0$ we can find a g_{sq} such that.

$$\hat{\xi}' = g_{sq}^{-1} \hat{\xi} g_{sq} \qquad g_{sq} = \begin{bmatrix} I & q \\ 0 & 1 \end{bmatrix}$$

$$-\hat{w} q = \vec{v}$$

$$\xi' = (0, 0, w)$$

Thus, it's pure rotation about a point.

(d) $$\hat{V}^s = \dot{g} g^{-1} = \begin{bmatrix} \dot{R} & \dot{P} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T P \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dot{R} R^T & -\dot{R} R^T P + \dot{P} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \dot{R}R^T & -\dot{R}R^Tp+\dot{p} \\ 0 & 0 \end{bmatrix}$$

$$\frac{dI}{dt} = \frac{d(RR^T)}{dt} = 0$$

$$= \dot{R}R^T + R\dot{R}^T$$

$$\dot{R}R^T = -R\dot{R}^T = -(\dot{R}R^T)^T$$

Thus $\dot{R}R^T$ is skew-symmetric.

$$\hat{V}^b = g^{-1}\dot{g} = \begin{bmatrix} R^T & -R^Tp \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0 & 0 \end{bmatrix}$$

$$\frac{dI}{dt} = \frac{d(R^TR)}{dt} = 0$$

$$= \dot{R}^TR + R^T\dot{R} = 0$$

$$R^T\dot{R} = -(\dot{R}^TR) = -(R^T\dot{R})^T$$

Thus $\hat{V}^s$ and $\hat{V}^b$ are both twists.

$$\hat{V}^s = \begin{bmatrix} \hat{w}^s & \vec{v}^s \\ 0 & 0 \end{bmatrix} \qquad \hat{V}^b = \begin{bmatrix} \hat{w}^b & \vec{v}^b \\ 0 & 0 \end{bmatrix}$$

$\hat{w}^s$ is instantaneous angular velocity of rigid body viewed from spacial frame

$\hat{w}^b$ is instantaneous angular velocity of rigid body viewed from body frame

$\vec{v}^s$ is instantaneous velocity of rigid body at the origin viewed from spacial frame

$\vec{v}^b$ is instantaneous velocity of rigid body at the origin viewed from spacial frame

(e) let $g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$

$$\hat{\xi} = g \hat{\xi}' g^{-1}$$

$$= \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^{-1} & -R^{-1}p \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R\hat{w} & Rv \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^{-1} & -R^{-1}p \\ 0 & 1 \end{bmatrix}$$

Because at planar $R\hat{w}R^{-1} = \hat{w}$

$$= \begin{bmatrix} R\hat{w}R^{-1} & -R\hat{w}R^{-1}p + Rv \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{w} & -\hat{w}p + Rv \\ 0 & 0 \end{bmatrix}$$

$$\xi = (-\hat{w}p + Rv, w)$$

$$\xi' = (v, w)$$

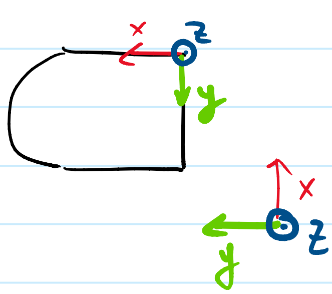
$$\begin{bmatrix} R & \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ w \end{bmatrix} = \begin{bmatrix} -\hat{w}\vec{p} + R\vec{v} \\ w \end{bmatrix} = \xi$$

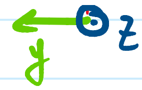
Q3 (a) I am not sure in the question.

"The front of robot is facing the positive y direction"

means vector on front face pointing outside is the same with y direction or not

Thus, I assume a same world coordinate with the one used in recitation.





Tip (0.61 , 0.72 , 2.196)

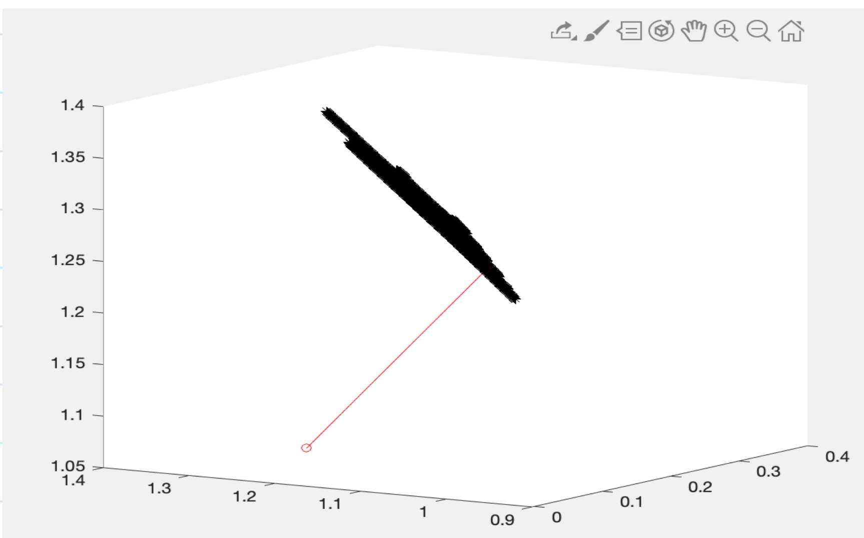
(b) see hw2.m and xyz.txt file.

(c) White board is a plain go through

a = (0.0754 0.9817 1.2337)
b = (0.1136 1.0459 1.2521)
c = (0.2648 1.2136 1.2638)

b is on the white board

V = (0.1167 -0.1167 0.1650) is a norm vector.



(d) kdc.