Introduction to Probability

Adeniyi, I.A

E-mail: isaac.adeniyi@fulokoja.edu.ng

Department of Statistics

Faculty of Science

Federal University Lokoja

LESSON ONE

1.0 Probability as a Measure of Uncertainty

Probability as a general concept can be defined as the chance of an event occurring. It is the chance that something will happen. Probability is the basis of inferential statistics because predictions are based on probability, and hypotheses are tested by using probability. Probability is used to quantify the likelihood, or chance, that an outcome of a random experiment will occur. “The chance of rain today is 30%’’ is a statement that quantifies our feeling about the possibility of rain. The likelihood of an outcome is quantified by assigning a number from the interval [0, 1] to the outcome (or a percentage from 0 to 100%). Higher numbers indicate that the outcome is more likely than lower numbers. A 0 indicates an outcome will not occur. A probability of 1 indicates an outcome will occur with certainty.

Generally, the three approaches to probability include classical approach, empirical approach and axiomatic approach respectively. These approaches are usually referred to as the basic interpretations of probability.

1.1 Classical Probability

This is based on the assumption that the outcomes of an experiment are equally likely. It uses sample spaces to determine the numerical probability that an event will happen. You do not actually have to perform the experiment to determine that probability. It is so named because it was the first type of probability studied formally by mathematicians in the 17th and 18th centuries. Let E be any event in the sample space S. The probability of any event E is defined as:

$$\mathrm{P(E)} = \frac{number\ of\ outcomes\ in\ E}{Total\ number\ of\ outcomes\ in\ the\ sample\ space\ S}$$

$$\mathrm{P(E)} = \frac{n(E)}{n(S)}$$

Where n(E) is the number of favourable outcomes, and n(S) is the number of possible outcomes.

Definitions

1.1.1 Outcome

An outcome is the result of a single trial of a probability experiment.

1.1.2 Trial

A trial means flipping a coin once, rolling one die once, or the like. When a coin is tossed, there are two possible outcomes: head or tail. (Note: We exclude the possibility of a coin landing on its edge.) In the roll of a single die, there are six possible outcomes: 1, 2, 3, 4, 5, or 6.

1.1.3 Sample Point

Each element or outcome in the sample space of a probability experiment is called a sample point.

1.1.4 Sample Space

The set of all possible outcomes of a random experiment is called the sample space of the experiment. The sample space is denoted as S. A sample space is often defined based on the objectives of the analysis. A sample space is discrete if it consists of a finite or countable infinite set of outcomes. A sample space is continuous if it contains an interval (either finite or infinite) of real numbers.

1.1.5 Equally Likely Events

Equally likely events are events that have the probability of occurring.

1.1.6 Rounding Rule for Probabilities

Probabilities should be expressed as reduced fractions or rounded to two or three decimal places. When the probability of an event is an extremely small decimal, it is permissible to round the decimal to the first nonzero digit after the point. For example, 0.0000587 would be 0.00006.

1.1.7 Tree diagram

A tree diagram is a device consisting of line segments emanating from a starting point and also from the outcome point. It is used to determine all possible outcomes of a probability experiment.

Example 1-1

Consider throwing a die once. What is the probability of the event of a prime number?

Solution

Let E denote event of a prime number, and S denote the sample space

$S = \{1,2,3,4,5,6\}, \quad n(S) = 6$

$E = \{2,3,5\}; \quad n(E) = 3$

$$\text{P(E)} = \frac{n(E)}{n(S)} = \frac{3}{6} = 0.5$$

Example 2-1

Find the probability of getting a black 10 when drawing a card from a deck.

Solution

Let E denote event of a black 10 from a deck of playing cards, and S denote the Sample space. Hence

$S = \{Total\ number\ of\ cards\ in\ a\ deck\ of\ playing\ cards\}$

$n(S) = 52$

$E = \{the\ 10\ of\ spades, the\ 10\ of\ clubs\}$

$n(E) = 2$

$P(a\ black\ 10) = P(E)$

$$P(E) = \frac{2}{52}$$

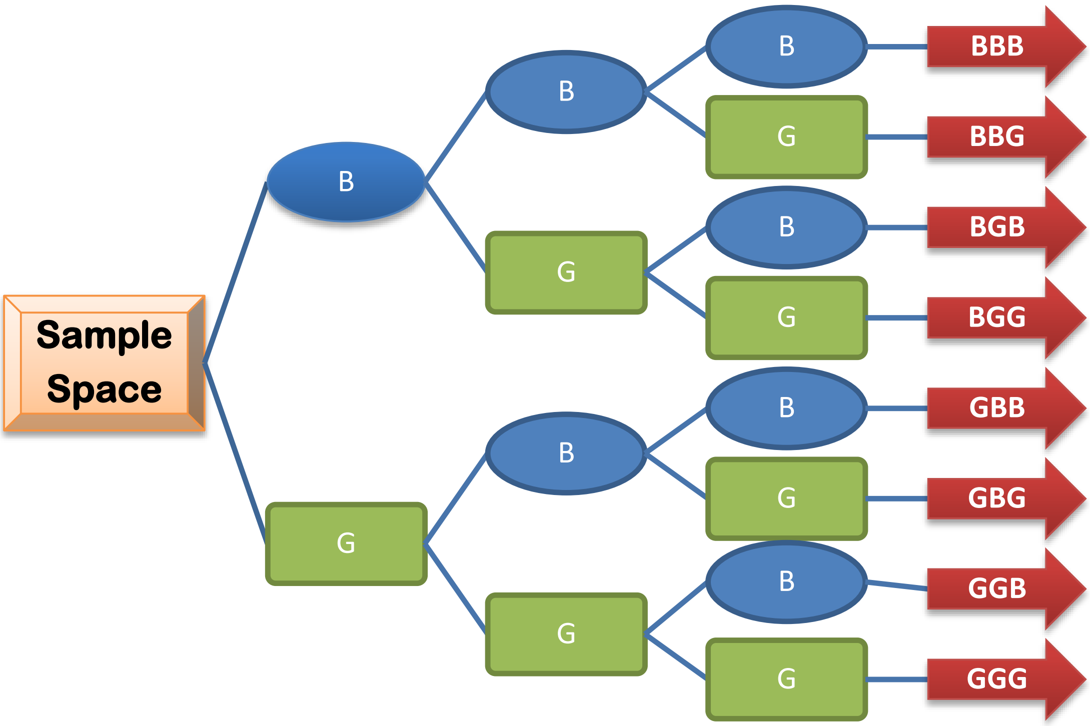
$$P(E) = \frac{1}{26}$$

Example 3-1

If a family has three children, find the probability that two of the three children are girls.

Solution

Let B and G denote boys and girls respectively. You can obtain sample space S using a tree diagram as shown below



$S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$

$n(S) = 8$

Let E = Event of two girls

$E = \{BGG, GBG, GGB\}$

$n(E) = 3$

$P(two\ girls) = \frac{3}{8}$

Exercise 1-1

If an octagonal fair die is rolled once, determine

a) the sample space
b) probability of obtaining an even number
c) probability of obtaining a multiple of 3
d) probability of obtaining a prime number and
e) state the appropriate probability approach

1.2 Empirical or Relative Frequency Approach

Here, the probability of an event occurring is given by the fraction of the time similar events happened in the past. The difference between classical and empirical probability is that classical probability assumes that certain outcomes are equally likely (such as the outcomes when a die is rolled), while empirical probability relies on actual experience to determine the likelihood of outcomes. In empirical probability, one might actually roll a given die 6000 times, observe the various frequencies, and use these frequencies to determine the probability of an outcome. Therefore,

$$Probability\ of\ an\ event\ occuring = \frac{number\ of\ times\ events\ in\ the\ past\ occure}{total\ number\ of\ observations}$$

Example 4-1

A survey of a class of 30 FUL graduates showed that 4 graduated with First Class Honours, 10 graduated with Second Class Honours and 16 graduated with Third Class Honours. What is the probability that a graduate selected from this group graduated with First Class Honours.

Solution

$$P\{1^{st}\ calss\ honours\} = \frac{number\ of\ 1^{st} classes\ on\ records}{total\ number\ of\ degrees\ awarded}$$

$$P\{1^{st}\ class\ honours\} = \frac{4}{30}$$

$$P\{1^{st}\ class\ honours\} = \frac{2}{15}$$

Example 5-1

Suppose that a researcher for the American Automobile Association (AAA) asked 50 people who plan to travel over the Thanks giving holiday how they will get to their destination. The results can be categorized in a frequency distribution as shown.

Method	Frequency
Drive	41
Fly	6
Train or Bus	3
Total	50

Probabilities can be computed for various categories for the three categories are:

$$P(Driving) = \frac{41}{50}$$

$$P(Flying) = \frac{3}{25}$$

$$P(Train) = P(Bus) = \frac{3}{50}$$

Example 6-1

In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities.

a. A person has type O blood.

b. A person has type A or type B blood.

c. A person has neither type A nor type O blood.

d. A person does not have type AB blood.

Source: The American Red Cross

Solution

Blood Type	**Frequency**

O	21
A	22
B	5
AB	2
Total	50

a. $P(Type\ O\ blood) = \frac{21}{50}$

b. $P(Type\ A\ or\ type\ Bblood) = \frac{22}{50} + \frac{5}{50} = \frac{27}{50}$

c. $P(Neither\ type\ A\ nor\ type\ O\ blood) = \frac{5}{50} + \frac{2}{50} = 7$

d. $P(Type\ AB\ blood) = \frac{2}{50}$

e. $P(Not\ AB\ blood) = 1 - P(Type\ AB\ blood)$

$$P(Not\ AB\ blood) = 1 - \frac{2}{50}$$

$$P(Not\ AB\ blood) = \frac{24}{25}$$

Alternatively,

$$P(Not\ AB\ blood) = P(TypeO + Type\ A + Type\ AB\ blood)$$

$$P(Not\ AB\ blood) = \frac{21}{50} + \frac{22}{50} + \frac{5}{50}$$

$$P(Not\ AB\ blood) = \frac{24}{25}$$

Exercise 2-1

Suppose it is on records that out 25 fresh students who offered STA 101 in the 2013/2014 academic session at Federal University Lokoja, 10 are from Statistics department, 8 are from Electrical Engineering department while the remaining 7 were from Physics department respectively. What are the chances of having STA 101 students in the 2014/2015 academic session from

a) Statistics department
b) Physics department
c) Electrical engineering department.
d) Hence, what approach of probability is this?

1.3 Axiomatic Approach

This is an approach based on the four basic laws of probability. It is such that probabilities problems are solved using understanding of nature of probability as measure of uncertainty, in deciding correctness of answers to such problems.

1.3.1 The Axioms of Probability

Let E denote an event in sample space S such that $P(E) = Probability\ of\ event\ E$. Then $P(E)$ is said to be a true probability function if and only if the following axioms are satisfied

1. $\boldsymbol{0 \leq p(E) \leq 1}$
   Interpretation: The probability of any event E is a number (either a fraction or decimal) between and including 0 and 1. The simple implication of this is that probabilities cannot be negative or greater than 1
2. $\boldsymbol{P(\emptyset) = 0}$
   Interpretation: If an event E cannot occur (i.e., the event contains no members in the sample space), its probability is 0.
3. $\boldsymbol{P(E) = Possibility = 1}$
   Interpretation: If an event E is certain, then the probability of E is 1.
4. $\boldsymbol{P(S) = 1}$
   Interpretation: The sum of the probabilities of all the outcomes in the sample space is 1.

1.4 Events: An event is defined as any subset of the outcome set S in a probability experiment. It is a subset of the sample space of a random experiment.

1.4.1 Combination of Events

In probability theory, events are usually combined using the basic set operations; union, intersection and complements. Interpretations of these basic set operations are summarized in what follows:

- The union of two events is the event that consists of all outcomes that are contained in either of the two events. We denote the union as $E_1 \cup E_2$.
- The intersection of two events is the event that consists of all outcomes that are contained in both of the two events. We denote the intersection as $E_1 \cap E_2$.
- The complement of an event in a sample space is the set of outcomes in the sample space that are not in the event. We denote the component of the event E as E'.

It should be noted that diagrams are often used to portray relationships between sets, and these diagrams are also used to describe relationships between events. We can use Venn diagrams to represent a sample space and events in a sample space.

1.4.2 Types of Events

1. **Independent Events**
   Two events E_1 and E_2 are said to be independent if the occurrence of E_1 does not prevent that of E_2 from happening. This simply means that both events can occur simultaneously.
   Mathematically, E_1 and E_2 independent if and only if
   $$P_r(E_1 and\ E_2) = P_r(E_1) \times P_r(E_2)$$
2. **Dependent Events**
   Events $E_1$and E_2 are said to be dependent if the occurrence of E_1 prevents that of E_2 from happening or if they depend on each other. This implies that occurrence of one depends on the other.
   Mathematically, E_1 and E_2 are dependent if an only if
   $$P_r(E_1 and\ E_2) = P_r(E_1) \times P_r(E_2|E_1)$$
3. **Mutually Exclusive Events**
   Two or more events are said to be mutually exclusive if they are disjoint. E_1 and E_2, for instance are said to be mutually exclusive if $E_1 \cap E_2 = \emptyset$

Example 7-1

Consider rolling a die once with the following events,

$E_1 = (even\ numbers)$

$= \{2,4,6\},$

$E_2 = (odd\ numbers)$

$= \{1,3,5\},$

$E_3 = (perfect\ square\ greater\ than\ 1)$

$= \{4\}$

Events E_1 and E_2 are mutually exclusive because $E_1 \cap E_2 = \emptyset$ which implies that in a throw of a single die, we cannot have event of an even and an odd numbers together. But events $E_1$and E_3 are independent because $E_1 \cap E_3 = 4$. This means that an event can contain an outcome which is both even and a perfect square greater than1.

LESSON TWO

2.0 Interpretation of Probability

Based on the axioms of probability presented in previous section, the following interpretation can be given to probability values

1. When the probability of an event is close to 0, its occurrence is highly unlikely.
2. When the probability of an event is near 0.5, there is about a 50-50 chance that the event will occur; and
3. when the probability of an event is close to 1, the event is highly likely to occur.

Example 2-1

Consider an experiment of tossing a coin 3 times. What is the probability of

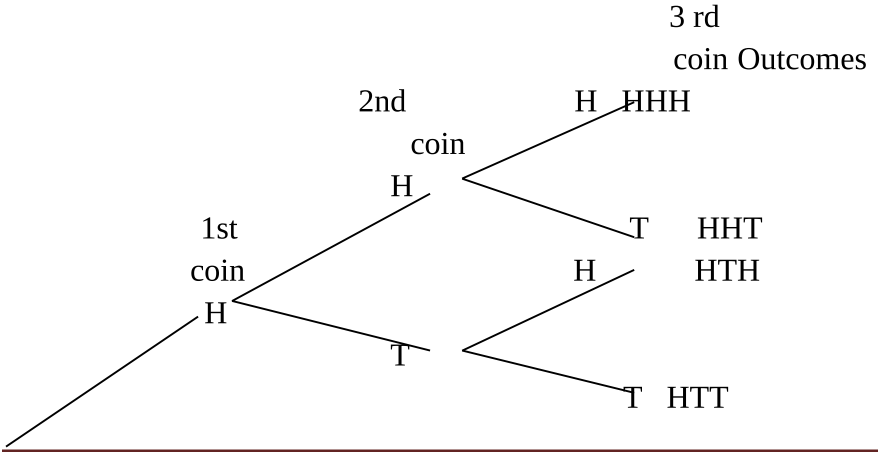
1. obtaining 4 heads
2. obtaining at most 4 heads
3. obtaining at least 1 tail

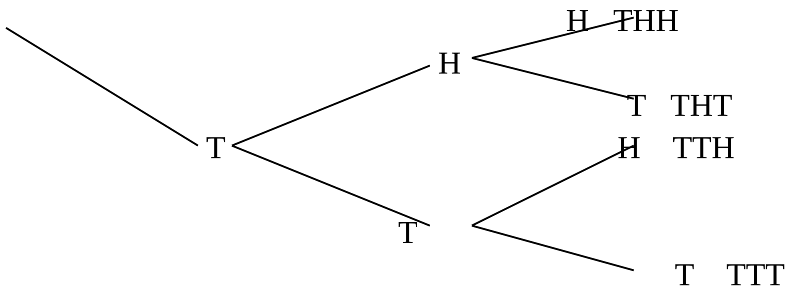
Solution

Since a single coin has two possible outcomes of H and T, the sample space for tossing 3 coins together can be obtained using tree diagram,

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$n(S) = 8$





1. Let E_1 denote event of obtaining 4 heads. Since S does not contain HHHH, $E_1 = \emptyset$ and $P\{\emptyset\} = 0$
2. Let E_2 denote event of at most 4 heads. Hence
   $E_2 = \{H \leq 4\}$
   $E_2 = \{\emptyset, 3H, 2H \; or \; 1H \;\}$
   $P(E_2) = P(\emptyset) + P(3H) + P(2H) + P(1H)$
   $P(E_2) = \frac{0}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$
   $P(E_2) = \frac{7}{8}$
3. Let E_3 denote event of at least 1 tail. Hence
   $E_3 = \{T \geq 1\}$
   $E_3 = \{1T, 2T, or \; 3T\}$
   $P(E_3) = P(1T) + P(2T) + P(3T)$
   $P(E_3) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$
   $P(E_3) = \frac{7}{8}$
   Alternatively,
   $P(E_3) = 1 - P(no \; T)$
   $P(E_3) = 1 - \frac{1}{8}$
   $P(E_3) = \frac{7}{8}$

2.1 Interpretation of "OR" and "AND" in Probability Problems

In probability theory, the word "or" simply means addition. It is synonymous to words such as "either", "neither" and "union operation in set theory". On the other hand, the word "and" simply means product (multiplication) and it is synonymous to words like "both", "together" and "intersection operation in set theory". Hence

$P_r(E_1 \cup E_2) = P_r(E_1 or E_2)$ and

$P_r(E_1 \cap E_2) = P_r(E_1 \; and \; E_2)$

2.2 Addition and Multiplication Rules of Probability

2.2.1 Addition Rule Non-mutually Exclusive Events

For any two events $E_1$and E_2,

$$P_r(E_1 \cup E_2) = P_r(E_1) + P_r(E_2) - P_r(E_1 \cap E_2)$$

For any three events E_1, E_2 and E_3,

$$P_r(E_1 \cup E_2 \cup E_3) = P_r(E_1) + P_r(E_2) - P_r(E_1 \cap E_2) + P_r(E_3) - P_r(E_1 \cap E_3)$$
$$-P_r(E_2 \cap E_3) + P_r(E_1 \cap E_2 \cap E_3)$$

$$P_r(E_1 \cup E_2 \cup E_3) = P_r(E_1) + P_r(E_2) + P_r(E_3) - P_r(E_1 \cap E_2) - P_r(E_1 \cap E_3)$$
$$-P_r(E_2 \cap E_3) + P_r(E_1 \cap E_2 \cap E_3)$$

2.2.2 Addition Rule for mutually exclusive events

For any two independent events $E_1$and E_2,

$$P_r(E_1 \cup E_2) = P_r(E_1) + P_r(E_2) \qquad [since\ P_r(E_1 \cap E_2) = 0]$$

For any three independent events E_1, E_2 and E_3,

$$P_r(E_1 \cup E_2 \cup E_3) = P_r(E_1) + P_r(E_2) + P_r(E_3)$$

Example 2-2

In a single throw of a fair die, what is the probability that an odd number or a perfect square greater than1 shows up?

Solution

$S = \{1,2,3,4,5,6\}$

Let $E_1 = Event\ that\ odd\ number\ \ shows\ up$

$E_1 = \{1,3,5\}$

$E_2 = Event\ that\ a\ perfect\ square\ greater\ than\ 4\ shows\ up$

$E_2 = \{4\}$

$n(S) = 6,$

$n(E_1) = 3$ and

$n(E_3) = 1$

$P_r(E_1 \cap E_2) = 0$

Since E_1 and E_2 are mutually exclusive,

$$P_r(E_1 \cup E_2) = P_r(E_1) + P_r(E_2)$$

$$P_r(E_1 \cup E_2) = \frac{3}{6} + \frac{1}{6}$$

$$P_r(E_1 \cup E_2) = \frac{4}{6}$$

$$P_r(E_1 \cup E_2) = \frac{2}{3}$$

Example 3-2

If A, B, and C are mutually exclusive events, is it possible for P(A) = 0.3, P(B) = 0.4, and P(C) = 0.5? Why or why not?

Solution

No, it is not possible. Below are the reasons:

$Pr(A \cup B \cup C) = Pr(A) + Pr(B) + Pr(C)$ if A, B, and C are mutually exclusive events.

and

$Pr(A \cup B \cup C) = 0.3 + 0.4 + 0.5$

$Pr(A \cup B \cup C) = 1.2 > 1.$

But $Pr(A \cup B \cup C)$ cannot be greater than 1, so it's not possible for P(A) = 0.3, P(B) = 0.4, and P(C) = 0.5 when A, B, and C are mutually exclusive.

Example 4-2

The probability that a student passes STA 101 test is $\frac{2}{3}$, and the probability that the student passes STA 103 test is $\frac{4}{9}$. If the probability of passing both courses is $\frac{7}{9}$; what is the probability that he will pass STA 101 or STA 103 test?

Solution

Let $E_1 = event\ of\ passing\ STA\ 101\ test$

$E_2 = event\ of\ passing\ STA\ 103\ test$

$$P_r(E_1\ or\ E_2\ ) = P_r(E_1) + P_r(E_2) - P_r(E_1 \cap E_2)$$

$$P_r(E_1\ or\ E_1\ ) = \frac{2}{3} + \frac{4}{9} - \frac{7}{9}$$

$$P_r(E_1\ or\ E_1\ ) = \frac{6 + 4 - 7}{9}$$

$$P_r(E_1\ or\ E_1\ ) = \frac{3}{9}$$

$$P_r(E_1\ or\ E_1\ ) = \frac{1}{3}$$

Exercise

The probability that a student passes STA 101 test is $\frac{2}{3}$, and the probability that the student passes STA 103 test is $\frac{4}{9}$. If the probability of passing at least one course is $\frac{7}{9}$; what is the probability that he will pass STA 101 or STA 103 test but not both?

Example 5-2

On New Year's Eve, the probability of a person driving while intoxicated is 0.32, the probability of a person having a driving accident is 0.09, and the probability of a person having a driving accident while intoxicated is 0.06. What is the probability of a person driving while intoxicated or having a driving accident?

Solution

Let $E_1 = driving\ while\ intoxicated$

$E_2 = having\ driving\ accident$

$P_r(E_1) = 0.32$, $P_r(E_2) = 0.09$ and $P_r(E_1 \cap E_2) = 0.06$

$P_r(E_1\ or\ E_1\ ) = P_r(E_1) + P_r(E_2) - P_r(E_1 \cap E_1)$

$P_r(E_1\ or\ E_1\ ) = 0.32 + 0.09 - 0.06$

$P_r(E_1\ or\ E_1\ ) = 0.35$

LESSON THREE

3.0 Multiplication Rule

3.1 Multiplication Rule 1 [Independent Events]

When two events $E_1\ and\ E_2$ are independent, the probability of both occurring is

$P_r(E_1\ and\ E_1\ ) = P_r(E_1) \times P_r(E_2)$

$P_r(E_1\ \cap\ E_1\ ) = P_r(E_1) \times P_r(E_2)$

For three eventsE_1, E_2 and E_3,

$P_r(E_1\ \cap\ E_1 \cap\ E_3) = P_r(E_1) \times P_r(E_2) \times P_r(E_3)$

3.1.1 Multiplication Rule 2 [Dependent Events]

When two events $E_1\ and\ E_2$ are dependent, the probability of both occurring is

$P_r(E_1\ \cap\ E_1\ ) = P_r(E_1) \times P_r(E_2|E_1)$

Example 1-3

A coin is flipped and a die is rolled. Find the probability of getting a head on the coin and a 4 on the die.

Solution

$S = \{H1\ H2\ H3\ H4\ H5\ H6\ T1\ T2\ T3\ T4\ T5\ T6\}$

$P_r(a\ head\ and\ a\ 4) = P_r(H) \times P_r(4)$ $\qquad [both\ events\ are\ independent]$

$P_r(a\ head\ and\ a\ 4) = \frac{1}{2} \times \frac{1}{6}$

$P_r(a\ head\ and\ a\ 4) = \frac{1}{12}$

Example 2-3

A man owns a house in town and a cottage in the country. In any one year the probability of the house being burgled is 0.01 and the probability of the cottage being burgled is 0.05. In any one year what is the probability that:

(a) both will be burgled? (b) one or the other (but not both) will be burgled?

Solution:

Let $H = \{house\ is\ burgled\}$ $\quad C = \{cottage\ is\ burgled\}$

a. $P(H \cap C) = P(H) \cdot P(C)$ [Since events are independent]

$= (0.01) \times (0.05)$

$= 0.0005$

b. $P\big(One\ or\ the\ other(but\ not\ both)\big) = P(Only\ house\ is\ burgled)\ or\ (Only\ cottage\ is\ burgled)$

$= P(H \cap C') \cup P(H' \cap C)$

$= P(H \cap C') + P(H' \cap C)$

$= \big(0.01 \times (1 - 0.05)\big) + \big((1 - 0.01) \times 0.05\big)$

$= (0.01 \times 0.95) + 0.99 \times 0.05$

$= 0.059$

Example 3-3 [University Scholarship]

At FUL, there were 5 University scholars reported in 2009, 16 in 2010, and 32 in 2012. If a researcher wishes to select at random two scholars to prepare for scholarship in 2013, find the probability that both will be selected from 2009.

Solution

In this case, the events are dependent since the researcher wishes to select two distinct scholars. Hence the first scholar is selected and not replaced.

Let $S = Scholar$

$$P_r(S_1 and\ S_2) = P_r(S_1) \times P_r(S_2|S_1)$$

$$= \frac{5}{53} \times \frac{4}{52}$$

$$= 20/2862$$

$$= 0.007$$

Example 4-3 [Homeowner's and Automobile Insurance]

World Wide Insurance Company found that 53% of the residents of a city had homeowner's insurance (H) with the company. Of these clients, 27% also had automobile insurance (A) with the company. If a resident is selected at random, find the probability that the resident has both homeowner's and automobile insurance with World Wide Insurance Company.

Solution

$$P(H\ and\ A) = P(H) \times P(A|H) = (0.53) \times (0.27) = 0.1431$$

Example 5-3

Suppose that we have a fuse box containing 20 fuses of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first fuse. What is the probability that both fuses are defective?

Solution

Total number of fuses in the box $= 20$

Let $E_1 = First\ defective\ fuse$

$E_2 = Second\ defective\ fuses$

$P_r(E_1) = \frac{5}{20}$

$P_r(E_2) = \frac{4}{19}$

$$P_r(E_1 \cap E_2) = P_r(E_1) \cap P_r(E_2|E_1)$$
$$= \frac{5}{20} \times \frac{4}{19}$$
$$= \frac{1}{19}$$
$$= 0.05$$

Example 6-3 [Winning a Door Prize]

At a gathering consisting of 10 men and 20 women, two door prizes are awarded. The winning ticket is not replaced. Find the probability that

a. both prizes are won by men.
b. both prizes are won by women
c. a man wins one and a woman wins one.
d. Would you consider this event in a, b, and c likely or unlikely to occur?

Solution

$M = Men, \quad n(M) = 10$

$W = Women, \ n(W) = 20$

$n(S) = 2, \ P_r(M) = \frac{10}{30}, \ P_r(W) = \frac{20}{30}$

a. $P_r(both\ Men) = P_r(M) \times P_r(M|M)$
$$= \frac{10}{30} \times \frac{9}{29}$$
$$= \frac{3}{29}$$
$= 0.10$ [The event is unlikely]

b. $P_r(both\ Women) = P_r(W) \times P_r(W|W)$
$$= \frac{20}{30} \times \frac{19}{29}$$
$$= \frac{38}{87}$$
$= 0.44$ [The event is more likely]

c. $P_r(A\ man\ and\ a\ woman\ wins\ one\ each) = P_r(M \cap W|M) + P_r(W \cap M|W)$
$$= \left(\frac{10}{30} \times \frac{20}{29}\right) + \left(\frac{20}{30} \times \frac{10}{29}\right)$$
$$= \frac{200}{870}$$
$= 0.23$ [The event is unlikely]

Exercise 1-3

Two boys are chosen at random from a class consisting of 18 boys and 12 girls. What is the probability that the two students selected are

i. both boys
ii. both girls
iii. of the same sex
iv. a boy and a girl

3.2 Sampling with and without Replacement

In the theory of Probability and Statistics, counting processes involve selection (combination) and arrangement (permutations) respectively. Sampling with replacement as the name implies occurs when object is selected and then replaced before the next is selected. It is synonymous to permutation because it involves repetition. On the other hand, sampling without replacement occurs when object is selected but it is not replaced before the next is selected. It is synonymous to combination. It is such that after the first selection, the total number of items for the next selection is the initial total minus one. In computing probability values of events using these schemes, the denomination remains the same, with replacement, while it reduces with each selection, without replacement.

Example 7-3

A bag contains 7 W, 3 R, and 5 B balls. Three are drawn without replacement. Find the Probability that:

a. No ball is Red
b. Exactly one is Red
c. At least one is Red
d. All are of the same colour
e. No two are of the same colour

Solution

$\boldsymbol{S = 7W, 5W, 3R, \ n(S) = 15}$

a. Let $E_1 = Red\ balls, \quad E_2 = Not\ Red\ balls$

$n(E_1) = 3,$

$n(E_2) = 7W + 5B = 12$

$$P_r(No\ ball\ is\ Red) = \frac{12}{15} \times \frac{11}{14} \times \frac{10}{13}$$
$$= \frac{132}{273}$$
$$= \frac{44}{91}$$
$$= 0.4835$$

b. $$P_r(Exactly\ one\ is\ Red) = P_r(E_1 \cap E_2 \cap E_2) \cup P_r(E_2 \cap E_1 \cap E_2) \cup P_r(E_2 \cap E_2 \cap E_1)$$
$$= \left(\frac{3}{15} \times \frac{12}{14} \times \frac{11}{13}\right) + \left(\frac{12}{15} \times \frac{3}{14} \times \frac{2}{13}\right) + \left(\frac{12}{15} \times \frac{11}{14} \times \frac{3}{13}\right)$$
$$= 0.4352$$

c. $$P_r(At\ least\ one\ is\ Red) = 1 - P_r(no\ Red)$$
$$= 1 - 0.4352$$
$$= 0.5165$$

d. $$P_r(The\ same\ colour\ ) = P_r(WWW) \cup P_r(RRR) \cup P_r(BBB)$$
$$= \left(\frac{7}{15} \times \frac{6}{14} \times \frac{5}{13}\right) + \left(\frac{3}{15} \times \frac{2}{14} \times \frac{1}{13}\right) + \left(\frac{5}{15} \times \frac{4}{14} \times \frac{3}{13}\right)$$
$$= \left(\frac{210}{2730}\right) + \left(\frac{3}{2730}\right) + \left(\frac{60}{2730}\right)$$
$$= \frac{273}{2730}$$
$$= 0.1$$

e. $$P_r(Different\ colours) = P_r(WBR) \cup P_r(WRB) \cup P_r(BRW) \cup P_r(BWR) \cup P_r(RWB) \cup P_r(RBW)$$

$$= \left(\frac{7}{15} \times \frac{5}{14} \times \frac{3}{13}\right) \times 6$$
$$= 0.231$$

LESSON FOUR

4.0 Conditional Probability

The conditional probability of an event E_2 in relationship to an event E_1 was defined as the probability that event E_2 occurs after event E_1 has already occurred. The conditional probability of an event can be found by dividing both sides of the equation for multiplication rule 2 by $P_r(E_1)$, as shown next:

$$\frac{P_r(E_1 and\ E_2)}{P_r(E_1)} = \frac{P_r(E_1) \times P_r(E_2|E_1)}{P_r(E_1)}$$
$$P_r(E_2|E_1) = \frac{P_r(E_1 and\ E_2)}{P_r(E_1)}$$

Interpretation:

The probability that the second event E_2 occurs given that the first event E_1 has occurred can be found by dividing the probability that both events occurred by the probability that the first event has occurred.

Example 1-4

A FUL student is enrolled in a course in Geology (G) and a course in Statistics (S). The probabilities that the student will pass Geology, Statistics, or both subjects are, respectively, $P(G) = 0.8$, $P(S) = 0.7$, and $P(G \cap M) = 0.56$.

(a) What is the probability that the student will pass Geology given that the student passes Statistics?

(b) Are the events G and S independent?

Solution

a. We need to find $P_r(G|S) = \frac{P_r(G\ and\ S)}{P_r(S)}$
$$= \frac{0.56}{0.7}$$
$$= 0.8$$

b. Since $P_r(G|S) = P_r(G) = 0.8$ [Events G and S are independent]

Example 2-4

A maintenance firm has gathered the following information regarding the failure mechanisms for air conditioning systems:

Evidence of electrical failure	Evidence of Gas Leak		
		Yes	No
	Yes	55	17

	No	32	3

The units without evidence of gas leaks or electrical failure showed other types of failure. If this is a representative sample of AC failure, find the probability

(a) That failure involves a gas leak

(b) That there is evidence of electrical failure given that there was a gas leak

(c) That there is evidence of a gas leak given that there is evidence of electrical failure

Solution

a) $P(gas\ leak) = (55 + 32)/107 = 0.813$

b) $P(electric\ failure|gas\ leak) = (55/107)/(87/102) = 0.632$

c) $P(gas\ leak|\ electric\ failure) = (55/107)/(72/107) = 0.764$

Example 3-4

In a sample of 100 college students, 60 said they own a car, 30 said they own a stereo, and 10 said they own both a car and a stereo. What is the probability of a student's having a stereo given the student has a car?

Solution

$C = A\ student\ owns\ a\ car,\ \ P_r(C) = \frac{60}{100}$

$S = A\ student\ owns\ a\ stereo, P_r(S) = \frac{30}{100}$

$$P_r(Cand\ S) = \frac{10}{100}$$

$$P_r(S|C) = \frac{P_r(C \cap S)}{P_r(C)}$$

$$P_r(S|C) = \frac{10}{100} \div \frac{60}{100}$$

$$P_r(S|C) = \frac{10}{100} \times \frac{100}{60}$$

$$P_r(S|C) = \frac{1}{6}$$

4.1 Total Probability Rule

4.1.1 Total Probability Rule for Two Events

For any two events A and B such that A and A' are two distict partitions of the sample space and $P_r(A_i) > 0$ for all A_i, the total probability rule is given as

$$P_r(B) = P_r(B \cap A) + P_r(B \cap A')$$

$$P_r(B) = P_r(B|A)P_r(A) + P_r(B|A')P_r(A')$$

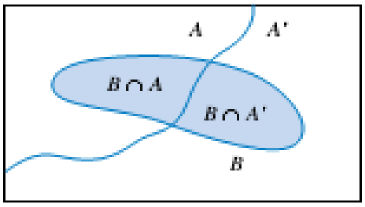


Figure 1-4: Partitioning An event into two mutually exclusive subsets

One of the uses of the rule is to compute the probability of various events B for which the conditional probabilities $P_r(B|A_i)$ are known or easy to derive.

4.1.2 Total Probability Rule for More than Two Events

Let $E_1, E_2, \cdots, E_n$ are n mutually exclusive and exhaustive events which are partitions of B in the sample. Then

$P_r(B) = P_r(B \cap E_1) + P_r(B \cap E_2) + \cdots + P_r(B \cap E_n)$

$P_r(B) = P_r(B|E_1)P_r(E_1) + P_r(B|E_2)P_r(E_2) + \cdots + P_r(B|E_n)P_r(E_n)$

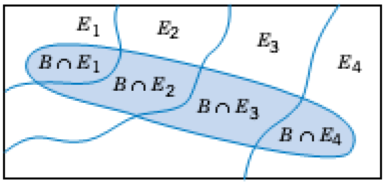


Figure 2-4: Partitioning an event into several mutually exclusive subsets

Example 4-4:

Suppose that $P_r(A|B) = 0.2,\ P_r(A|B') = 0.3\ and\ P_r(B) = 0.8$, what is $P_r(A)$?

Solution:

$P_r(A) = P_r(A \cap B) + P_r(A \cap B')$

$P_r(A) = P_r(A|B)P_r(B) + P_r(A|B')P_r(B')$

$P_r(A) = 0.2 \times 0.8 + 0.3 \times 0.2$

$P_r(A) = 0.16 + 0.6$

$P_r(A) = 0.22$

Example 5-4:

The probability is 1% that an electrical connector that is kept dry fails during the warranty period of a portable connector. If the connector is ever wet, the probability of a failure during the warranty period is 5%. If 90% of the connectors are kept dry and 10% are wet, what proportion of connectors fail during the warranty period?

Solution

Let $A_1 = Dry\ connector\ fails\ during\ warranty\ period,\ P_r(A_1) = 0.01$

$A_2 = Wet\ connector\ fails\ during\ warranty\ period, \qquad P_r(A_2) = 0.05$
$B = connector\ fails\ during\ warranty\ period, \qquad P_r(B) = ?$
$P_r(B|A_1) = 0.9\ and\ P_r(B|A_2) = 0.1$
$P_r(B) = P_r(B \cap A_1) \cup P_r(B \cap A_2)$
$P_r(B) = P_r(B|A_1)P_r(A_1) + P_r(B|A_2)P_r(A_2)$
$P_r(B) = 0.9 \times 0.01 + 0.1 \times 0.05$
$P_r(B) = 0.014$

Example 6-4:

Suppose 2% of cotton fabric rolls and 3% of nylon fabric rolls contain flaws. Of the rolls used by a manufacturer, 70% are cotton and 30% are nylon. What is the probability that a randomly selected roll used by the manufacturer contains flaws?

Solution:

Let $A_1 = Fabric\ rolls\ contain\ flaws, \ P_r(A_1) = 0.02$
$A_2 = Nylon\ rolls\ contain\ flaws, \qquad P_r(A_2) = 0.03$
$B = Selected\ roll\ contains\ flaw, \qquad P_r(B) = ?$
$P_r(B|A_1) = 0.7\ and\ P_r(B|A_2) = 0.3$
$P_r(B) = P_r(B \cap A_1) \cup P_r(B \cap A_2)$
$P_r(B) = P_r(B|A_1)P_r(A_1) + P_r(B|A_2)P_r(A_2)$
$P_r(B) = 0.7 \times 0.02 + 0.3 \times 0.03$
$P_r(B) = 0.023$

Example 7-4:

Samples of laboratory glass are in small, light packaging or heavy, large packaging. Suppose that 2 and 1% of the sample shipped in small and large packages, respectively, break during transit. If 60% of the samples are shipped in large packages and 40% are shipped in small packages, what proportion of samples break during shipment?

Solution

Let $A_1 = Glass\ samples\ shipped\ in\ small\ packaging\ break\ in\ transit, \ P_r(A_1) = 0.02$
$A_2 = Glass\ samples\ in\ large\ packaging\ break\ in\ transit, \qquad P_r(A_2) = 0.01$
$B = Proportion\ of\ glass\ samples\ breakages\ during\ shipment, \qquad P_r(B) = ?$
$P_r(B|A_1) = 0.4\ and\ P_r(B|A_2) = 0.6$
$P_r(B) = P_r(B \cap A_1) \cup P_r(B \cap A_2)$
$P_r(B) = P_r(B|A_1)P_r(A_1) + P_r(B|A_2)P_r(A_2)$
$P_r(B) = 0.4 \times 0.02 + 0.7 \times 0.01$
$P_r(B) = 0.015$

4.2 Bayes' Rule

Bayes' Rule

Let $E_1, E_2, \cdots, E_n$ be disjoint events that form a partition of the sample space, and assume that $P(E_i) > 0$, for all i. Then, for any event B such that $P(B) > 0$, we have

$$P_r(E_i|B) = \frac{P_r(E_i) \times P_r(B|E_i)}{P_r(B)}$$

$$P_r(E_i|B) = \frac{P_r(E_i) \times P_r(B|E_i)}{P_r(E_1) \times P_r(B|E_i) + \cdots + P_r(E_n) \times P_r(B|E_n)}$$

Example 8-4: [The False-Positive Puzzle]

A test for a certain rare disease is assumed to be correct 95% of the time: if a person has the disease, the test results are positive with probability 0.95, and if the person does not have the disease, the test results are negative with probability 0.95. A random person drawn from a certain population has probability 0.001 of having the disease. Given that the person just tested positive, what is the probability of having the disease?

Solution

If A is the event that the person has the disease, and B is the event that the test results are positive, the desired probability $P_r(A|B)$, is

$$P_r(A|B) = \frac{P_r(A)P_r(B|A)}{P_r(A)P_r(B|A) + P_r(A')P_r(B|A')}$$

$$P_r(A|B) = \frac{0.001 \times 0.95}{0.001 \times 0.95 + 0.999 \times 0.05}$$

$$P_r(A|B) = 0.0187$$

Example 9-4:

Customers are used to evaluate preliminary product designs. In THE PAST, 95% of highly successful products received good reviews, 60% of moderately successful products

Received good reviews, and 10% of poor products received good reviews. In addition, 40% of products have been highly successful, 35% have been moderately successful, and 25% have been poor products.

a. What is the probability that a product attains a good review?
b. If a new design attains a good review, what is the probability that it will be a successful product?
c. If a product does not attain a good review, what is the probability that it will be highly successful?

Solution:

Let G denote a product that received a good review. Let H, M, and P denote products that were high, moderate, and poor performers, respectively.

a. $P_r(G) = P_r(G|H)P_r(H) + P_r(G|M)P_r(M) + P_r(G|P)P_r(P)$

$P_r(G) = 0.95(0.40) + 0.60(0.35) + 0.10(0.25)$

$P_r(G) = 0.615$

b. Using the results from part a.,

$$P_r(H|G) = \frac{P_r(G|H)P_r(G|H)}{P_r(G)}$$

$$P_r(H|G) = \frac{0.95(0.40)}{0.615}$$

$$P_r(H|G) = 0.618$$

c. $P_r(H|G') = \frac{P_r(G'|H)P_r(H)}{P_r(G')}$

$$P_r(H|G) = \frac{0.05(0.40)}{1 - 0.615}$$
$$P_r(H|G) = 0.052$$

Example 10-4:
Suppose that $P_r(A|B^{'}) = 0.7$, $P_r(A) = 0.5$ and $P_r(B^{'}) = 0.2.$ $Determine\ P_r(B|A)$.

Solution
$$P_r(B|A) = \frac{P_r(B) \times P_r(A|B)}{P_r(B) \times P_r(A|B) + P_r(B') \times P_r(A|B^{'})}$$
$$P_r(B|A) = \frac{0.2 \times 0.7}{0.2 \times 0.7 + 0.8 \times 0.3}$$
$$P_r(B|A) = \frac{0.14}{0.14 + 0.24}$$
$$P_r(B|A) = 0.3684$$
$$P_r(B|A) = 0.4$$

LESSON 5
Factorial Arithmetic, Permutation and Combination

5.0 Factorial Arithmetic

In 1808, Christian Kramp first used a counting product called the Factorial notation. Factorials are denoted by the exclamation mark (!). The number of different arrangements of n different objects is denoted by n!. Hence

$n! = n \times (n-1) \times (n-2) \times, \cdots, \times 4 \times 3 \times 2 \times 1$

0! = 0

1! = 1

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

$5! = 5 \times 4! = 120$

$4! = 4 \times 3! = 24$
$3! = 3 \times 2! = 6$
$2! = 2 \times 1 = 2$ etc.

5.1 Permutation

Permutation simply means arrangement in mathematical terms. It is an arrangement of n objects in a specified order. Therefore, the arrangement of n objects in a specific order using r objects at a time is called a permutation of n objects taking r objects at a time. It is written as $n_{P_r} = \frac{n!}{(n-r)!}$.

$$5_{P_3} = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\times4\times3\times2!}{2!} = 5 \times 4 \times 3 = 60$$

$$5_{P_5} = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{1} = 120$$

$$20_{P_4} = \frac{20!}{(20-4)!} = \frac{20!}{16!} = \frac{20 \times 19 \times 18 \times 17 \times 16!}{16!} = 20 \times 19 \times 18 \times 17 = 116280$$

Example1-5: [Arrangement of graduands at a convocation ceremony]

Suppose you to arrange graduands from the 6 colleges in FUL for collection of certificates at a convocation ceremony taking only 2 colleges at a time. Determine the number of possible ways.

Solution

Any college can be selected first in 6 ways, any other college can be selected in the remaining (6-1) ways. The required number of ways $= 6 \cdot 5 = 30 ways$

Alternatively,

$n = 6 \; and \; r = 2$

$$6_{P_2} = \frac{6!}{(6-2)!} = \frac{6!}{4!} = \frac{6\times5\times4!}{4!} = 30 ways$$

Example 2-5:[Arranging answerss in a test paper]

A test paper has six questions but four are to be answered. In how many ways can the answers be arranged?

Solution

First question can be answered in 6 different ways, second can be answered in 5 different ways, third can be answered in 4 different ways and the fourth can be answered in 3 different ways. The required number of ways is therefore $6 \cdot 5 \cdot 4 \cdot 3 = 360 ways$

Alternatively,

$$6_{P_4} = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6\times5\times4\times3\times2!}{2!} = 30 ways$$

Example 3-5: [Manufacturing Tests]

An inspector must select 3 tests to perform in a certain order on a manufactured part. He has a choice of 7 tests. How many ways can he perform 3 different tests? 210

Solution

$$7_{P_3} = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7 \times 6 \times 5 \times 4!}{4!} = 210 \; ways$$

5.1.1 Permutation (arrangement) of identical objects

The number of ways of arranging n objects taking r at a time of which $r_1, r_2, \cdots, r_n$ are identical is $\frac{n!}{r_1! \times r_2!, \cdots r_{n!}}$

Example 5-5

In how many ways can letters of the word MATHEMATICS be arranged?

Solution

There are 11 letters of which there 2As, 2Ms and 2Ts. Hence the required number of ways is $\frac{11!}{2! \times 2! \times 2!} = 6652800 ways$

5.1.2 Conditional Permutation

This involves arrangements of n objects taking r of them a time following some restrictions on the order (or manner) of the arrangement. For instance we may be interested in the number of ways in which the letters of the word SHALLOW can be arranged if

i. the two Ls must not come together
ii. the two Ls must always come together

Solution

By removing the two Ls, the remaining letter (SHAOW) can be arranged in 5! Ways.

i. If the two Ls must not come together, the first L can occupy any of 6 places below
   •S•H•A•O•W•
   When this is done, there are 5 places for the second L not next to the first. Hence, the required number of arrangements with separate Ls 5! •6•5 supposing the Ls are distinguishable. But here, the two Ls are nit distinguishable. Hence the number of arrangements is $\frac{5! \bullet 6 \bullet 5}{2} = 5! \times 15 = 1800\ ways$
ii. The two Ls are identical here. S we take them as one object. There are 6 places for each of the 5! Arrangements of the word SHAOW. Hence, the required number of ways $= 5! \times 6 = 6! = 720\ ways$

Example 6-5

Find the number of arrangements using all the letters of the word PERCENTAGE if Es must be placed next to each other.

Solution

By removing the Es, the word PERCENTAGE becomes PRCNTAG and it can be arranged in 7! Ways. The letter Es are taken as one (since they must be placed side by side) and can occupy the 8 places in •P•R•C•N•T•A•G•

Hence, the required number of arrangements $7! \times 8 = 8! = 40320\ arrangements$

5.1.3 Permutation in ring (or cyclic permutation)

This involves arrangement of n objects in a cycle, in a thread (such as beads in thread) or in a round table (such as people in a round table meeting or conference) such one of them must be taken as a start point or reference the leaving the remaining (n-1)! from arrangement from either directions. It should be noted that if the objects can move about or rotate in their positions, the number arrangements should be $\frac{(n-1)!}{2}$.

Eample 7-5

a. In how many ways can six people take places at a round table?

b. How many ways are there if two people must sit next to each other?
c. How many ways are there if two people must not sit next to each other?

Solution

a. Here, one of them will take a place (no matter where). So, there are 5 choices for the second person, 4 choices for the third person, 3 choices for the fourth person, 2 choices for the fifth person and 1 choice for the sixth. The number of ways
= 5•4•3•2•1 = 120
b. For two people to sit next to each other at the table, there are 2 possible ways: the second can sit to the right or left of the first. The third person then has a choice of 4 places, the fourth person has a choice of 3 places, the fifth person has a choice of 2 places and the 6 person has a choice of 1 place. Hence the number of ways
= 2•4•3•2•1 = 48
c. Two methods are possible here,
Method1
Number of ways of sitting 6 people on round table = 120

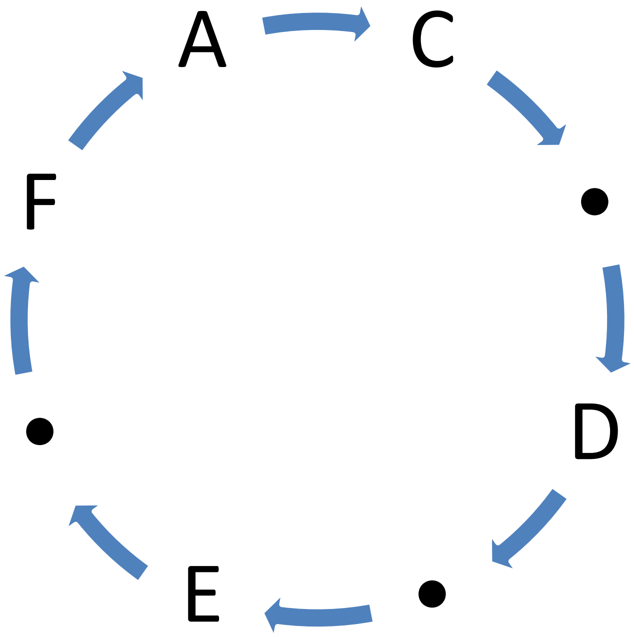


Figure 1-5: Permutation at a round table

Number of ways of sitting 6 people, if two people must sit next to each other = 48. Hence number of ways of sitting 6 people at a round table if 2 people must not sit next to each other = 120 -48
= 72

Method 2:

Refer to figure 1-5 above. Let one of the two people (say A) take a position, leaving behind the other person for the moment. The rest people (C,D,E,F) can take their places in 4! Ways. By now B has 3 choices so as not to sit next to A. Therefore, the required number of ways $= 24 \times 3 = 72\ ways$

Exercise 1-5

1. In how many ways can four men and two women be seated at a round table if the women do not sit next to each other.

2. Application: How many ways can 5 people sit on a park bench if the bench can only seat 3 people?
3. A bookshelf has space for exactly 5 books. How many different ways can 5 books be arranged on this bookshelf?
4. You are moderating a debate of gubernatorial candidates. How many different ways can you seat the panelists in a row? Call them Rasheedat, Abduljaleel, Melody, Ayinla, and Sophiat.
5. A wine taster claims that she can distinguish four vintages or a particular Cabernet. What is the probability that she can do this by merely guessing (she is confronted with 4 unlabeled glasses)? (hint: without replacement)
6.

5.2 Combination

The term combination simply means selection. A selection of distinct objects without regard to order is called a combination. The number of ways that r objects can be selected from
n objects without regard to order, is called combination rule given as

$$n_{C_r} = \frac{n!}{(n-1)! \times r!} \quad for\ n \geq r$$

Where

$$n_{C_r} = n_{C_{(n-r)}}$$

Poof:

Recall that

$$n_{C_r} = \frac{n!}{(n-1)! \times r!}$$

Hence,

$$n_{C_{(n-r)}} = \frac{n!}{\big(n-(n-r)\big)! \times (n-r)!}$$

$$n_{C_{(n-r)}} = \frac{n!}{(n-n+r)! \times (n-r)!}$$

$$n_{C_{(n-r)}} = \frac{n!}{r! \times (n-r)!} \equiv n_{C_r}$$

Example 8-5:

Sample Space	Number of Ways
$1P, 3H$	$3C_1 \bullet 5C_3 = 3 \bullet 10 = 30$
$2P, 2H$	$3C_2 \bullet 5C_2 = 3 \bullet 10 = 30$
$3P, 1H$	$3C_3 \bullet 5C_1 = 1 \bullet 5 = 5$
Total	$30 + 30 + 5 = 65$