SOLUTION: Let $x = \sqrt{a}$. $\therefore x^2 - a = 0$.

Using Newton-Raphson Method to $f(x) \equiv x^2 - a = 0$, we get the iteration formula as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}\left[\frac{x_n^2 + a}{x_n}\right]$$
$$= \frac{1}{2}\left[x_n + \frac{a}{x_n}\right], n = 0, 1, 2, \cdots.$$

To compute $\sqrt{2}$, putting $a = 2$, in the above formula, we get the interation formula as:

$$x_{n+1} = \frac{1}{2}\left[\frac{x_n^2 + 2}{x_n}\right].$$

Now taking $x_0 = 1$, we have successive approximations as follows:

n	x_n	x_{n+1}
0	1	1.5
1	1.5	1.42
2	1.42	1.414
3	1.414	1.4142

$\therefore$ **$\sqrt{2} = 1.41$,** correct up to three significant figures.

6.10 Regula-Falsi Method or Method of False Position

In this method we first find a sufficiently small interval $[a_0, b_0]$, such that $f(a_0)f(b_0) < 0$, by tabulation or graphical method, and which contains only one root α (say) of $f(x) = 0$, i.e., $f'(x)$ maintains same sign in $[a_0, b_0]$.

This method is based on the assumption that the graph of $y = f(x)$ in small interval $[a_0, b_0]$ can be represented by the chord joining $(a_0, f(a_0))$ and $(b_0, f(b_0))$. The point $x = a_1 = a_0 + h_0$ at which the chord meets the x-axis gives us an ap value of the root α of the equation $f(x) = 0$. Thus, we obtain two interv and $[x_1, b_0]$, one of which must contain the root α, depending upon the $f(a_0)f(x_1) < 0$ or $f(x_1) \cdot f(b_0) < 0$. If $f(x_1)f(b_0) < 0$, then α lies in t $[x_1, b_0]$ which we re-name as $[a_1, b_1]$. Again we consider that the graph of $[a_1, b_1]$ as the chord joining $(a_1, f(a_1))$ and $(b_1, f(b_1))$, thus, the point of i of the chord with the x-axis (say) $x_2 = a_1 + h_1$ gives us an approximate v root α and x_2 is called the *second approximation* of the root α.

Proceeding in this way, we shall get a sequence $\{x_1, x_2, x_3, \cdots, x_n\}$, eac of which is the successive aproximation of the exact root α of the equation f

Now we are going to establish an iteration formula which may generate a successive approximations of an exact root α of the equation $f(x) = 0$, geon

[illegible] adjoining Fig. 6.8, we assume that one real root α of $f(x) = 0$ l [illegible] $[a_n, b_n]$ and $f(a_n) < 0$ and $f(b_n) > 0$. Let PRQ be the graph of the x-axis at R.

$\therefore$ **$\sqrt{27} = 5.196154$,** correct

Thus, $x = 0R(= \alpha)$ gives the exact value of the root α. ... onsider curve PRQ as the chord PQ, in the small interval $[a_n, b_n]$, which intersect ... axis at C, then $OC = x_{n+1} = a_n + h_n$ approximates the root α of the equation $f(x$ ...

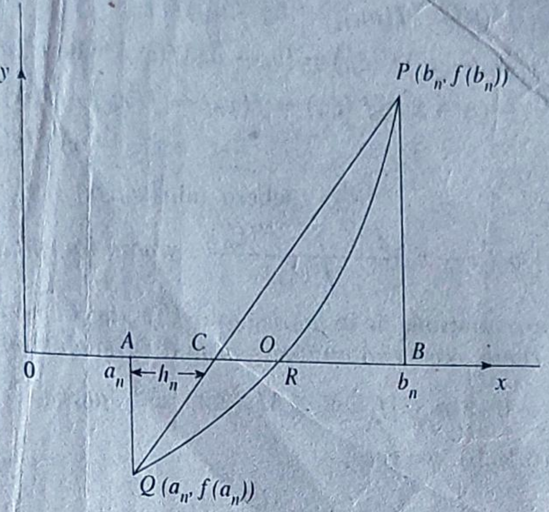

Fig. 6.8

Now from similar triangles AQC and CBP, we get,

$$\frac{AC}{AQ} = \frac{CB}{BP} \text{ or } AC = \frac{AQ}{BP} \cdot CB = \frac{|f(a_n)|}{|f(b_n)|}(AB - AC)$$

$$\text{or, } AC\left[1 + \frac{|f(a_n)|}{|f(b_n)|}\right] = \frac{|f(a_n)|}{|f(b_n)|} \cdot AB = \frac{|f(a_n)|}{|f(b_n)|}(b_n - a_n)$$

$$\therefore\ AC = h_n = \frac{|f(a_n)|}{|f(a_n)| + |f(b_n)|}(b_n - a_n).$$

Thus,

$$x_{n+1} = a_n + h_n = a_n + \frac{|f(a_n)|}{|f(a_n)| + |f(b_n)|} \cdot (b_n - a_n). \tag{6.10.1}$$

The above formula (6.10.1) is known as the iteration formula for Regula-Falsi Method.

☐ REMARK 6.10.1 *As in the bisection method, the sequence of successive approximations obtained by formula (6.10.1) is invariably convergent.*

6.10.1 Convergence of Regula-Falsi Method

As $f(a_n) \cdot f(b_n) < 0$, considering the proper sign of $f(a_n)$ and $f(b_n)$, we can write the equation (6.10.1) as in the following relations

$$x_{n+1} = a_n - \frac{b_n - a_n}{f(b_n) - f(a_n)} f(a_n) \quad \text{or} \quad x_{n+1} = b_n - \frac{b_n - a_n}{f(b_n) - f(a_n)} \cdot f(b_n). \tag{6.10.2}$$

Since $x_n = a_n$ or b_n ... have for both the relations (6.10.2),

$$x_{n+1} \quad - \frac{b_n - a_n}{f(b_n) - f(a_n)} f(x_n)$$

or $\quad - x_{n+1})[f(b_n) - f(a_n)] = (b_n - a_n) f(x_n)$

$(x_n - x_{n+1})(b_n - a_n) f'(\xi_n) = (b_n - a_n) f(x_n)$ where $(a_n < \xi_n < b_n)$

or, $[(\alpha - x_{n+1}) - (\alpha - x_n)] f'(\xi_n) = f(x_n) = f(x_n) - f(\alpha) \quad [\because \; f(\alpha) = 0]$

$$= (x_n - \alpha) f'(\xi'_n)$$

where $\min\{x_n, \alpha\} < \xi'_n < \max\{x_n, \alpha\}$

or, $(\alpha - x_{n+1}) = (\alpha - x_n) \dfrac{f'(\xi_n) - f'(\xi'_n)}{f'(\xi_n)}$ where $(a_0 < \xi_n, \xi'_n < b_0)$. (6.10.3)

Since, all the approximations lie in $[a_0, b_0]$ and $f'(x)$ is continuous and has the same sign in $[a_0, b_0]$, then there exist two numbers m and M such that

$$0 < m \leq |f'(x)| \leq M \quad \text{for} \quad x \in [a_0, b_0].$$

Therefore, from (6.10.3), we get

$$|\alpha - x_{n+1}| \leq \frac{M - m}{m} |\alpha - x_n|.$$

Now putting $n-1, n-2, n-3, \cdots, 2, 1, 0$ for n suceessively and multiplying all the $(n+1)$ relations we get

$$\varepsilon_{n+1} = |\alpha - x_{n+1}| \leq \left[\frac{M - m}{m}\right]^{n+1} \cdot |\alpha - x_0|. \tag{6.10.4}$$

If we choose the interval $[a_0, b_0]$ so small, such that $M < 2m$. then $0 < \frac{M-m}{m} < 1$, therefore,

$$\lim_{n \to \infty} \varepsilon_{n+1} = \lim_{n \to \infty} |\alpha - x_{n+1}| < \lim_{n \to \infty} \left(\frac{M - m}{m}\right)^{n+1} |\alpha - x_0| = 0$$

$$\therefore \quad \lim_{n \to \infty} x_{n+1} = \alpha.$$

☐ REMARK 6.10.2 *Thus, for the convergence of Regula-Falsi Method, the interval* $[a_0, b_0]$ *must be very small.*

6.11 Computation Scheme

1. Find an interval $[a_0, b_0]$, where $f(a_0) \cdot f(b_0) < 0$ and $f'(x)$ maintains the same sign.

2. Write n (number of iteration),

$$a_n, \; b_n, \; f(a_n); \; f(b_n), \; h_n = \frac{|f(a_n)|(b_n - a_n)}{|f(a_n)| + |f(b_n)|}, \; x_{n+1} = a_n + h_n \text{ and } f(x_n \ldots)$$

in a row.

3. Insert, $+ve$ or $-ve$ sign with a_n, as $a_n(+)$ or $a_n(-)$ according as $f(a_0) \ldots 0$ or $f(a_0) < 0$ and $-ve$ or $+ve$ sign with b_n as $b_n(-)$ or $b_n(+)$ according as $f(\ldots) < 0$ or $f(b_0) > 0$.

4. In the $(r+1)$th iteration, write x_{r+1}, in the column of $a_n(+)$ if $f(x_{r+1}) > 0$ and keep b_r fixed in the column $b_n(-)$. Otherwise, write x_{r+1} in the column of $b_n(-)$ if $f(x_{r+1}) < 0$ and keep a_r fixed in the column of $a_n(+)$.

Scheme 6.3

When $f(a_0) > 0$ and $f(b_0) < 0$.

n	$a_n(+)$	$b_n(-)$	$f(a_n)$	$f(b_n)$	$h_n = \frac{\lvert f(a_n)\rvert(b_n-a_n)}{\lvert f(a_n)\rvert+\lvert f(b_n)\rvert}$	$x_{n+1} = a_n + h_n$	$f(x_{n+1})$
0	a_0	b_0	$f(a_0)$	$f(b_0)$	$h_0 = \frac{\lvert f(a_0)\rvert(b_0-a_0)}{\lvert f(a_0)\rvert+\lvert f(b_0)\rvert}$	$x_1 = a_0 + h_0$	$f(x_1) > 0$
1	$a_1(=x_1)$	$b_1(=b_0)$	$f(a_1)$	$f(b_1)$	$h_1 = \frac{\lvert f(a_1)\rvert(b_1-a_1)}{\lvert f(a_1)\rvert+\lvert f(b_1)\rvert}$	$x_2 = a_1 + h_1$	$f(x_2) < 0$
2	$a_2(=a_1)$	$b_2(=x_2)$	$f(a_2)$	$f(b_2)$	$h_2 = \frac{\lvert f(a_2)\rvert(b_2-a_2)}{\lvert f(a_2)+\lvert f(b_2)\rvert}$	$x_3 = a_2 + h_2$	$f(x_3) > 0$
3	$a_3(=x_3)$	$b_3(=b_2)$	$f(a_3)$	$f(b_3)$	$h_3 = \frac{\lvert f(a_3)\rvert(b_3-a_3)}{\lvert f(a_3)\rvert+\lvert f(b_3)\rvert}$	$x_4 = a_3 + h_3$	$f(x_4) > 0$
4	$a_4(=x_4)$	$b_4(=b_3)$	$f(a_4)$	$f(b_4)$	$h_4 = \frac{\lvert f(a_4)\rvert(b_4-a_4)}{\lvert f(a_4)\rvert+\lvert f(b_4)\rvert}$	$x_5 = a_4 + h_4$	$f(x_5) > 0$

and so on.

Scheme 6.4

When $f(a_0) < 0$ and $f(b_0) > 0$.

n	$a_n(-)$	$b_n(+)$	$f(a_n)$	$f(b_n)$	$h_n = \frac{\lvert f(a_n)\rvert(b_n-a_n)}{\lvert f(a_n)\rvert+\lvert f(b_n)\rvert}$	$x_{n+1} = a_n + h_n$	$f(x_{n+1})$
0	a_0	b_0	$f(a_0)$	$f(b_0)$	$h_0 = \frac{\lvert f(a_0)\rvert(b_0-a_0)}{\lvert f(a_0)\rvert+\lvert f(b_0)\rvert}$	$x_1 = a_0 + h_0$	$f(x_1) > 0$
1	$a_1(=a_0)$	$b_1(=x_1)$	$f(a_1)$	$f(b_1)$	$h_1 = \frac{\lvert f(a_1)\rvert(b_1-a_1)}{\lvert f(a_1)\rvert+\lvert f(b_1)\rvert}$	$x_2 = a_1 + h_1$	$f(x_2) > 0$
2	$a_2(=a_1)$	$b_2(=x_2)$	$f(a_2)$	$f(b_2)$	$h_2 = \frac{\lvert f(a_2)\rvert(b_2-a_2)}{\lvert f(a_2)\rvert+\lvert f(b_2)\rvert}$	$x_3 = a_2 + h_2$	$f(x_3) < 0$
3	$a_3(=x_3)$	$b_3(=b_2)$	$f(a_3)$	$f(b_3)$	$h_3 = \frac{\lvert f(a_3)\rvert(b_3-a_3)}{\lvert f(a_3)\rvert+\lvert f(b_3)\rvert}$	$x_4 = a_3 + h_3$	$f(x_4) < 0$
4	$a_4(=x_4)$	$b_4(=b_3)$	$f(a_4)$	$f(b_4)$	$h_4 = \frac{\lvert f(a_4)\rvert(b_4-a_4)}{\lvert f(a_4)\rvert+\lvert f(b_4)\rvert}$	$x_5 = a_4 + h_4$	$f(x_5) > 0$
5	$a_5(=a_4)$	$b_5(=x_5)$	$f(a_5)$	$f(b_5)$	$h_5 = \frac{\lvert f(a_5)\rvert(b_5-a_5)}{\lvert f(a_5)\rvert+\lvert f(b_5)\rvert}$	$x_6 = x_5 + h_5$	$f(x_6) > 0$

and so on.

6.12 Solved Problems

▶ EXAMPLE 6.12.1 *Compute the root of the equation $2x - \log_{10} x - 7 = 0$, by Regula-Falsi Method, which between 3 and 4, correct to three decimal places.*

SOLUTION: Let $f(x) = 2x - \log_{10} x - 7$.
Here $f(3) = -1.48$, $f(4) = 0.40$. Therefore, one root of $f(x) = 0$ between 3 and 4.
Now we compute the successive approximations of the root as under:

n	$a_n(-)$	$b_n(+)$	$f(a_n)$	$f(b_n)$	h_n^*	x_{n+1}^{**}	$f(x_{n+1})$
0	3.0	4.0	−1.48	0.40	0.79	3.79	$0.0014 > 0$
1	3.0	3.79	−1.48	0.0014	0.789	3.789	$-0.00052 < 0$
2	3.789	3.79	−0.00052	0.0014	0.000271	3.789271	$-0.0000014 < 0$
check 3	3.789271	3.79	−0.0000014	0.0014	0.0000007	3.7892717	$-0.0000012 < 0$

$* \; h_n = \frac{|f(a_n)|(b_n - a_n)}{|f(a_n)| + |f(b_n)|}$; $** \; x_{n+1} = a_n + h_n$

Thus, **3.789** is a root of $f(x) = 0$, correct to three decimal places. In iteration number 2, we can conclude that **3.79** is a root, correct up to two decimal places.

► EXAMPLE 6.12.2 *Find a root of the equation $3x - \cos x - 1 = 0$, by Regula-Falsi Method, correct to four significant figures.* [CH '87]

$a_0 = 0$ / $b_0 = 1$

SOLUTION: Let $f(x) = 3x - \cos x - 1$ $\therefore$ $f(0) = -1$ and $f(1) = 1.46$.
Thus, $f(x) = 0$ has a root between 0 and 1. Now we compute the root as follows:

n	$a_n(-)$	$b_n(+)$	$f(a_n)$	$f(b_n)$	h_n^*	x_{n+1}^{**}	$f(x_{n+1})$
0	0.0	1.0	−1	1.46	0.41	0.41	$-0.67 < 0$
1	0.41	1.0	−0.67	1.46	0.18	0.59	$-0.061 < 0$
2	0.59	1.0	−0.061	1.46	0.0164	0.6064	$-0.0025 < 0$
3	0.6064	1.0	−0.0025	1.46	0.00067	0.60707	$-0.000113 < 0$
4	0.60707	1.0	−0.000113	1.46	0.0000304	0.6071004	$-0.0000045 < 0$
check 5	0.6071004	1.0	−0.0000045	1.46	0.0000012	0.6071016	$-0.00000017 < 0$

$* \; h_n = \frac{|f(a_n)|(b_n - a_n)}{|f(a_n)| + |f(b_n)|}$; $** \; x_{n+1} = a_n + h_n$.

Thus, **0.6071** is a root of $f(x) = 0$, correct up to four significant figures.

► EXAMPLE 6.12.3 *Using Regula-Falsi Method, find a root of $x^3 + 2x - 2 = 0$, correct up to three significant figures.*

SOLUTION: Let $f(x) = x^3 + 2x - 2$. Here $f(0) = -2$, $f(1) = 1$.
Thus, $f(x) = 0$ has a root between 0 and 1. Now we compute the successive approximations of the roots as follows:

n	$a_n(-)$	$b_n(+)$	$f(a_n)$	$f(b_n)$	h_m^*	x_{n+1}^{**}	$f(x_{n+1})$
0	0.0	1.0	−2.0	1.0	0.67	0.67	$-0.36 < 0$
1	0.67	1.0	−0.36	1.0	0.087	0.757	$-0.052 < 0$
2	0.757	1.0	−0.052	1.0	0.012	0.769	$-0.00724 < 0$
3	0.769	1.0	−0.00724	1.0	0.00166	0.77066	$-0.00097 < 0$
check 4	0.77066	1.0	−0.00097	1.0	0.000222	0.770882	$-0.00013 < 0$

$* \; h_m = \frac{|f(a_n)|(b_n - a_n)}{|f(a_n)| + |f(b_n)|}$; $** \; x_{n+1} = a_n + h_n$.

Thus, **0.771** is a root of $f(x) = 0$, correct up to three significant figures.

► EXAMPLE 6.12.4 *Compute a root of $x \ln x = 1$, by Regula-Falsi Method, correct to three decimal places.*

SOLUTION: Let $f(x) = x \ln x - 1$. Here $f(1) = -1$ and $f(2) = 0.39$.
Thus, $f(x) = 0$ has a root between 1 and 2. The iterations table is given below:

n	$a_n(-)$	$b_n(+)$	$f(a_n)$	$f(b_n)$	h_n^*	x_{n+1}^{**}	$f(x_{n+1})$
0	1.0	2.0	−1.0	0.39	0.72	1.72	$-0.067 < 0$
1	1.72	2.0	−0.067	0.39	0.0411	1.7611	$-0.00333 < 0$
2	1.7611	2.0	−0.00333	0.39	0.002022	1.763122	$-0.000158 < 0$
check 3	1.763122	2.0	−0.000158	0.39	0.000096	1.763218	$-0.0000075 < 0$

$* \ h_n = \frac{|f(a_n)|(b_n - a_n)}{|f(a_n)| + |f(b_n)|}$; $** \ x_{n+1} = a_n + b_n$.

Thus, **1.763** is root of $f(x) = 0$, correct up to three decimal places.

► EXAMPLE 6.12.5 *Find a root of the equation $\sin x + \cos x = 1$, by Regula-Falsi Method, correct to four significant figures.*

SOLUTION: Let $f(x) = \sin x + \cos x - 1$. Here $f(0) = 0, f(0.5) = 0.36, f(1.0) = 0.38, f(1.5) = 0.07, f(2.0) = -0.51$. Thus, $f(x) = 0$ has a root between 1.5 and 2.0.
The computational table is given below:

n	$a_n(+)$	$b_n(-)$	$f(a_n)$	$f(b_n)$	h_n^*	x_{n+1}^{**}	$f(x_{n+1})$
0	1.5	2.0	0.07	−0.51	0.06	1.56	$0.0107 > 0$
1	1.56	2.0	0.0107	−0.51	0.00904	1.56904	$0.00175 > 0$
2	1.56904	2.0	0.00175	−0.51	0.00147	1.57051	$0.000286 > 0$
3	1.57051	2.0	0.000286	−0.51	0.000241	1.570751	$0.000045 > 0$
check 4	1.570751	2.0	0.000045	−0.51	0.000038	1.570789	$0.0000073 > 0$

$* \ h_n = \frac{|f(a_n)|(b_n - a_n)}{|f(a_n)| + |f(b_n)|}$; $** \ x_{n+1} = a_n + h_n$.

Thus, **1.571** is a root of $f(x) = 0$, correct up to four significant figures.

6.13 Graeffe's Root Squaring Method for Polynomial Equation

The Graeffe's root squaring method is a special method to find all the roots (real numerically unequal or equal and complex) at a time of an algebraic or polynomial equation. The advantage of this method is that it does not require any prior information about the roots as in the case of an iteration method.

In this method, we find the equation where roots are the squares of the roots of the given equation. Repeating this process to a sufficient number of times k (say) finally, we shall obtain an equation whose roots are the mth ($m = 2^k, k = 1, 2, 3, \cdots$) power of the roots of the original equation. If m is sufficiently large, the roots of the final equation are widely separated such that, they can be directly calculated in terms of the coefficients of the equation.