Theories of failure

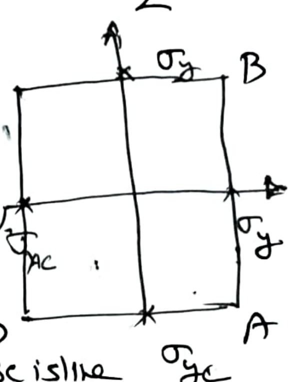
1) Maximum principal stress Theory. (Rankine's theory)

max principal stress theory. $\sigma_1 \le \sigma_y$ (ductile)

$$\sigma_1 \le \frac{\sigma_{yt}}{F.S}$$

For brittle material $\sigma_1 \le \dfrac{\sigma_u}{F.S.}$

graphical representation.



1) $\sigma_1 > \sigma_2$ (σ_1 is considered),
$\sigma_1 = +\sigma_{yt}$ ($\sigma_1^{+} =$ tensile),
AB is the line.

2) $\sigma_1 = -\sigma_{yc}$. (If σ_1 is compressive)
CD is line

3) $\sigma_2 > \sigma_2$ (σ_2 is considered)
$\sigma_2 = \sigma_{yt}$ (σ_2 is tensile) BC is line

4) $\sigma_2 = -\sigma_{yc}$ (σ_2 is compressive) AD is line

2) Max shear stress Theory. (Guest/Tresca's Theory)

$$\tau_{max} \le \tau_{yt}$$

$$\frac{\sigma_1 - \sigma_2}{2} \le \frac{\sigma_{yt}}{2}$$

$$\sigma_1 - \sigma_2 \le \sigma_{yt}$$

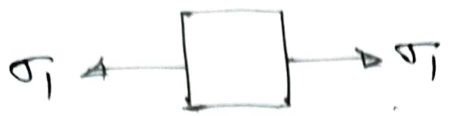
For safe design

$$\sigma_1 - \sigma_2 \le \frac{\sigma_{yt}}{F.S}$$

Failure ~~occurs~~ in material subjected to ~~when~~ biaxial or Triaxial stress occurs when max shear stress at any point in component = shear stress of standard specimen in tension test.

In Tension test

stresses in simple Te



σ_1 acts. $\sigma_2 = 0$

$$\tau_{max} = \frac{\sigma_1}{2}.$$

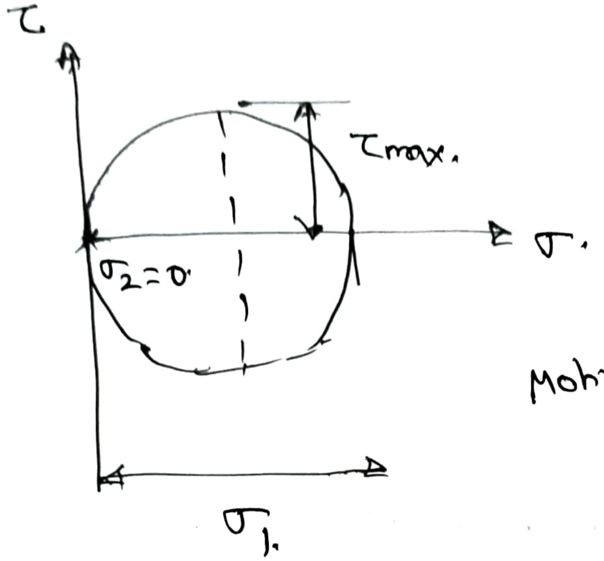
when specimen starts yielding

$\sigma_1 = \sigma_{yt}$

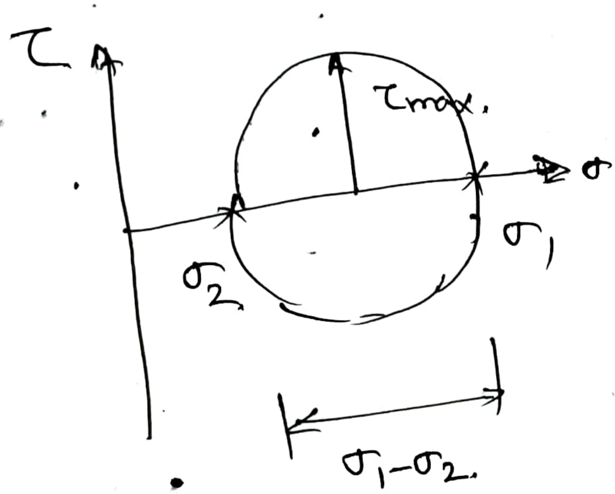
$$\tau_{max} = \frac{\sigma_{yt}}{2}$$

$\therefore$ max shear stress predict

$\tau_{max} = 0.5\,\sigma_{yt}$.



Mohr's circle for stresses.



$$\therefore \tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

$\tau_y = 0.5\,\sigma_{yt}$

$$\boxed{\text{actually. } \tau_y = \frac{\sigma_{yt}}{\sqrt{3}}}$$

If $\sigma_1, \sigma_2, \sigma_3$ are the three principal stresses at a point on the component.

Then shear stress on three different planes are.

$$\left.\begin{aligned} \tau_{12} &= \frac{\sigma_1-\sigma_2}{2} \\ \tau_{23} &= \frac{\sigma_2-\sigma_3}{2} \\ \tau_{31} &= \frac{\sigma_3-\sigma_1}{2} \end{aligned}\right\} \text{— (a)}$$

The largest of these stress is

$$\text{max of} \begin{bmatrix} \tau_{12} \\ \tau_{23} \\ \tau_{31} \end{bmatrix} = \tau_{max} = \frac{\sigma_{yt}}{2}$$

(ie) $\frac{\sigma_1-\sigma_2}{2} = \frac{\sigma_{yt}}{2}$ with safety $\frac{\sigma_1-\sigma_2}{2} = \frac{\sigma_{yt}}{F.S}$

(or) $\sigma_2-\sigma_3 = \frac{\sigma_{yt}}{F.S.}$

$\sigma_3-\sigma_1 = \frac{\sigma_{yt}}{F.S}$

used for finding dimension of components

neglecting F.S. — for compressive stress

refer eqn (a).

$\frac{\sigma_1-\sigma_2}{2} = \tau_{12}$

$\tau_{max} = \tau_{12}$

$\frac{\sigma_1-\sigma_2}{2} = \tau_{max} = \frac{\sigma_{yt}}{2}$

$\sigma_1-\sigma_2 = \sigma_{yt}$

$\sigma_2-\sigma_3 = \sigma_{yt}$

$\sigma_3-\sigma_1 = \sigma_{yt}$

$\sigma_1-\sigma_2 = -\sigma_{yc}$

$\sigma_2-\sigma_3 = -\sigma_{yc}$

$\sigma_3-\sigma_1 = -\sigma_{yc}$

The above equation can be written as

$\sigma_1 - \sigma_2 = \pm \sigma_{yt}$

$\sigma_2 - \sigma_3 = \pm \sigma_{yc} = \pm \sigma_{yt}$ [assuming $\sigma_{yt} = \sigma_{yc}$]

$\sigma_3 - \sigma_1 = \pm \sigma_{yt}$.

For uniaxial stress condition.

$\sigma_2 = 0$, $\sigma_3 = 0$. $\therefore \sigma_1 = \pm \sigma_{yt}$

For biaxial stress condition.

$\sigma_3 = 0$.

$\sigma_1 - \sigma_2 = \pm \sigma_{yt}$. — (applicable in 2nd & 4th quadrant)

$\left.\begin{array}{l} \sigma_2 = \pm \sigma_{yt} \\ \sigma_1 = \pm \sigma_{yt} \end{array}\right\}$ applicable in 1st & 3rd quadrant)

In ~~third~~ second quadrant.

σ_1 & σ_2 are of opposite signs

yielding will occur when

$\sigma_1 - \sigma_2 = \pm \sigma_{yt}$.

One stress is tensile, while other is compressive.

DC constructed such that

$\sigma_1 - \sigma_2 = - \sigma_{yt}$.

DC is constructed such that.

$\sigma_1 - \sigma_2 = - \sigma_{yt}$.

Intercept on x axis $\sigma_2 = 0$, $\sigma_1 = -\sigma_{yt}$.

on y axis $\sigma_1 = 0$, $\sigma_2 = \sigma_{yt}$.

similarly

In fourth quadrant FA is drawn, such that

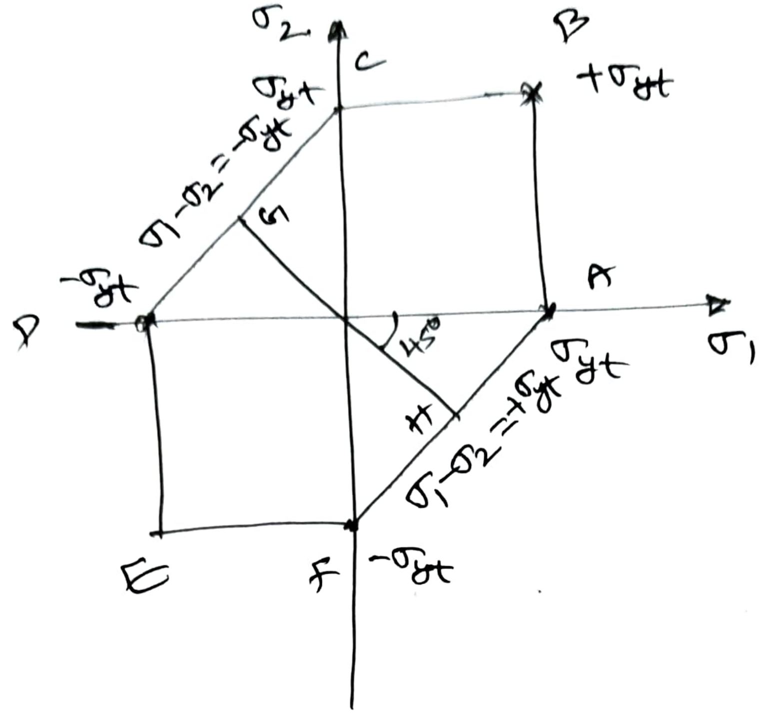
$\sigma_1 - \sigma_2 = + \sigma_{yt}$.

Intercept about x axis.

$\sigma_1 = \sigma_{yt}$, $\sigma_2 = 0$.

Intercept about y axis.

$\sigma_2 = -\sigma_{yt}$, $\sigma_1 = 0$



Boundary for max shear stress theory under Biaxial stress

In the first quadrant.

σ_1, σ_2 = positive.

yielding will depend on σ_1 or σ_2 whichever is greater in magnitude.

suppose $\sigma_1 > \sigma_2$.

$\sigma_1 = \sigma_{yt}$

AB is constructed such that $\sigma_1 = \sigma_{yt}$.

If $\sigma_2 > \sigma_1$.

$\sigma_2 = +\sigma_{yt}$.

Horizontal line. BC is constructed such that $\sigma_2 = \sigma_{yt}$

similarly in the third quadrant

If $\sigma_1 > \sigma_2$ $\quad \sigma_1 = -\sigma_{yt}$. (DE is constructed.

$\sigma_1 < \sigma_2$ $\quad \sigma_2 = -\sigma_{yt}$ (EF is constructed.

The complete region. ABCDEFA is a hexagon.

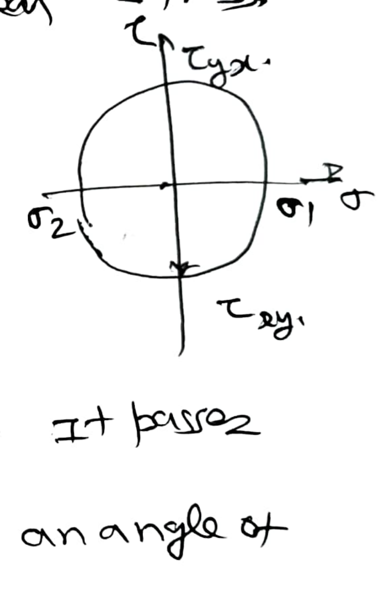
In case of biaxial stress,

If a point with coordinates (σ_1, σ_2) falls outside hexagon failure occurs.

If it falls inside, It is safe.

Shear diagonal.

line of pure shear (locus of all points) corresponding to pure shear stress



Here $\sigma_1 = -\sigma_2 = \tau_{12}$.

$$\frac{\sigma_1}{\sigma_2} = -1 = -\tan 45^\circ$$

GH is constructed such that. It passes through origin and makes an angle of -45° with x axis.

This line is called shear diagonal or line ~~wit~~ of pure shear.

It intersects Hexagon at two point G & H.

The pt of intersection of line (FA) and and GH is G.

For FA $\quad \sigma_1 - \sigma_2 = +\sigma_{yt}$.

GH $\quad \dfrac{\sigma_1}{\sigma_2} = -1$

<u>we know</u> for pure shear.

$$\sigma_1 = -\sigma_2 = \tau_{12} = \frac{\sigma_{yt}}{2}$$

$$\therefore \quad \sigma_1 - (-\sigma_2) = \cancel{1}\cancel{1/2}\,\sigma_{yt}$$

$$\sigma_1 = \cancel{1}\cancel{1/2}\,\frac{\sigma_{yt}}{2}$$

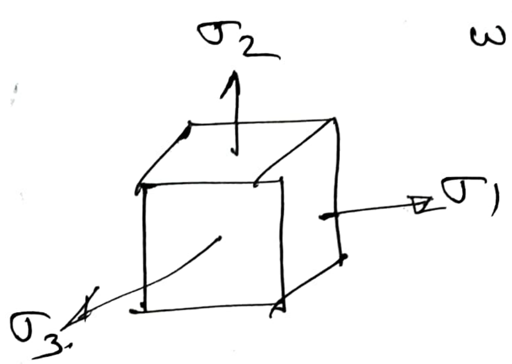
$$\tau_{12} = \frac{\sigma_{yt}}{2}$$

since it is at border line. (This is limiting value).

$$\boxed{\tau_{max} = \frac{\sigma_{yt}}{2}}$$

Maximum Distortion Energy theory

Failure of the mechanical component subjected to biaxial, (or) triaxial stresses occur when strain Energy / unit volume at any point in the component = strain energy of distortion / volume in the standard component of tension test when yielding starts.



For unit cube subjected to three principal stress $\sigma_1, \sigma_2, \sigma_3$.

Total strain Energy U of the cube is

$$U = \frac{1}{2}\sigma_1\epsilon_1 + \frac{1}{2}\sigma_2\epsilon_2 + \frac{1}{3}\sigma_3\epsilon_3 \quad \text{— (a)}$$

$\epsilon_1, \epsilon_2, \epsilon_3 \rightarrow$ strain in respective direction.

$$\epsilon_1 = \frac{1}{E}\left[\sigma_1 - \mu(\sigma_2 + \sigma_3)\right]$$

$$\epsilon_2 = \frac{1}{E}\left[\sigma_2 - \mu(\sigma_1 + \sigma_3)\right]$$

$$\epsilon_3 = \frac{1}{E}\left[\sigma_3 - \mu(\sigma_1 + \sigma_2)\right] \quad \text{— (b)}$$

~~For direction - 1~~

~~along - 1 = $\frac{\sigma_1}{E}$~~

~~- 2. =~~

sub (b) in (a)

$$U = \frac{1}{2E}\Big[\sigma_1^2 - \mu(\sigma_1\sigma_2 + \sigma_1\sigma_3)$$

$$+ \sigma_2^2 - \mu(\sigma_1\sigma_2 + \sigma_2\sigma_3)$$

$$+ \sigma_3^2 - \mu(\sigma_1\sigma_3 + \sigma_2\sigma_3)\Big]$$

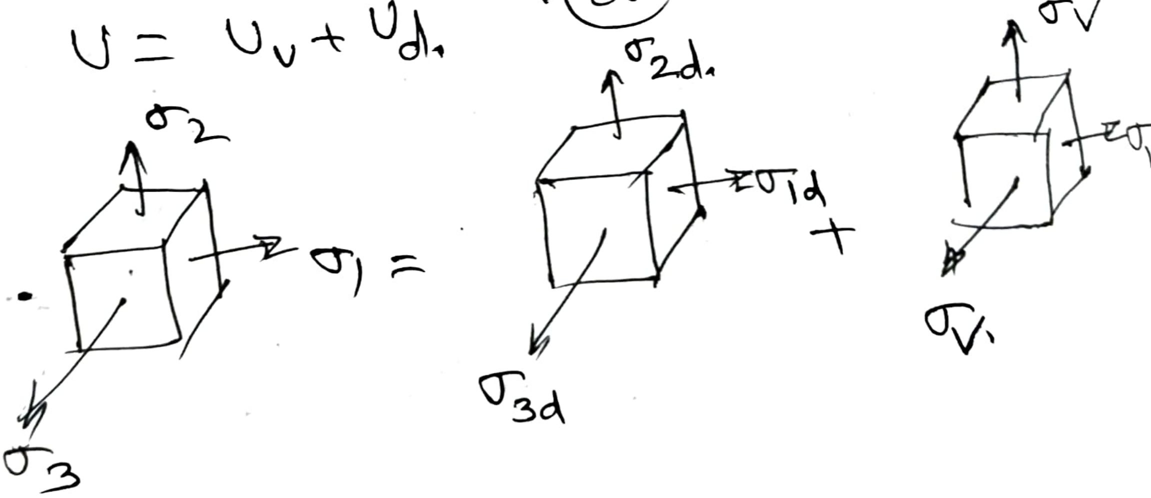
$$= \frac{1}{2E}\Big[\big[\sigma_1^2 + \sigma_2^2 + \sigma_3^2\big] - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)\Big] \quad (c)$$

The total strain Energy resolved into two component.

$U_v \rightarrow$ change of volume with no distortion

$U_d \rightarrow$ distortion of the element with no change in volume.

$$U = U_v + U_d \quad (d)$$



$$\left.\begin{aligned}\sigma_1 &= \sigma_{1d} + \sigma_v \\ \sigma_2 &= \sigma_{2d} + \sigma_v \\ \sigma_3 &= \sigma_{3d} + \sigma_v.\end{aligned}\right\} \quad (e)$$

$\sigma_{1d}, \sigma_{2d}, \sigma_{3d}$ } cause distortion of cube.

$\sigma_v \rightarrow$ results in volumetric change.

$\sigma_{1d}, \sigma_{2d}, \sigma_{3d}$ cause distortion of cube with.

no change in volume.

$\sigma_v =$ results in volume change.

$\because$ $\sigma_{1d}, \sigma_{2d}, \sigma_{3d}$ donot change the volume of the cube.

$$\epsilon_{1d} + \epsilon_{2d} + \epsilon_{3d} = 0. \quad - (f)$$

$$\left.\begin{aligned}\epsilon_{1d} &= \frac{1}{E}\left[\sigma_{1d} - \mu(\sigma_{2d} + \sigma_{3d})\right] \\ \epsilon_{2d} &= \frac{1}{E}\left[\sigma_{2d} - \mu(\sigma_{1d} + \sigma_{3d})\right] \\ \epsilon_{3d} &= \frac{1}{E}\left[\sigma_{3d} - \mu(\sigma_{1d} + \sigma_{2d})\right].\end{aligned}\right\} \quad (g)$$

sub (g) in (f)

$$\frac{1}{E}\left[(\sigma_{1d} + \sigma_{2d} + \sigma_{3d}) - 2\mu\left[\sigma_{1d} + \sigma_{2d} + \sigma_{3d}\right]\right] = 0$$

$$(1-2\mu)(\sigma_{1d} + \sigma_{2d} + \sigma_{3d}) = 0.$$

$$(1-2\mu) \neq 0.$$

$$\therefore \sigma_{1d} + \sigma_{2d} + \sigma_{3d} = 0. \quad - (h)$$

substitute (h) in (e)

$$(\sigma_1 - \sigma_v) + (\sigma_2 - \sigma_v) + (\sigma_3 - \sigma_v) = 0$$

$$\sigma_1 + \sigma_2 + \sigma_3 = 3\sigma_v.$$

$$\sigma_v = (\sigma_1 + \sigma_2 + \sigma_3)\frac{1}{3} \rightarrow (i)$$

like eqn (a)

$$U_v = \frac{1}{2}\sigma_v \epsilon_v + \frac{1}{2}\sigma_v \epsilon_v + \frac{1}{2}\sigma_v \epsilon_v.$$

$$U_v = \frac{3}{2}[\sigma_v \epsilon_v]. = 3\left(\frac{\sigma_v \epsilon_v}{2}\right) (j)$$

from (b)

$$\epsilon_v = \frac{1}{E}[\sigma_v - \mu(\sigma_v + \sigma_v)]$$

$$\epsilon_v = \frac{(1-2\mu)\sigma_v}{E} \quad (k)$$

~~sub j in i~~ from (j) and (k)

$$U_v = \cancel{\frac{3}{2}} \times \cancel{\frac{1}{3}} ($$

$$U_v = \frac{3}{2}\sigma_v \times \frac{(1-2\mu)\sigma_v}{E}$$

$$\boxed{U_v = \frac{3}{2E}(1-2\mu)\sigma_v^2} \quad (l)$$

From (1) and (l)

$$U_v = \frac{3}{2E}(1-2\mu).\ \frac{(\sigma_1+\sigma_2+\sigma_3)^2}{3^2}$$

$$U_v = \frac{1}{6E}(1-2\mu)(\sigma_1+\sigma_2+\sigma_3)^2 \quad \text{—(m)}$$

From (c) (d) and (m)

$$U_d = U - U_v$$

$$= \frac{1}{2E}\left[(\sigma_1^2+\sigma_2^2+\sigma_3^2) - 2\mu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_1\sigma_3)\right]$$

$$- \frac{1}{6E}(1-2\mu)(\sigma_1+\sigma_2+\sigma_3)^2$$

$$= \frac{1}{2E}\Big[\sigma_1^2+\sigma_2^2+\sigma_3^2 - 2\mu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_1\sigma_3)$$

$$- \frac{1}{3}(1-2\mu)(\sigma_1^2+\sigma_2^2+\sigma_3^2+2\sigma_1\sigma_2+2\sigma_1\sigma_3+2\sigma_2\sigma_3)$$

$$= \frac{1}{6E}\Big[3\sigma_1^2+3\sigma_2^2+3\sigma_3^2-6\mu\sigma_1\sigma_2-6\mu\sigma_2\sigma_3-6\mu\sigma_1\sigma_3$$

$$-\sigma_1^2-\sigma_2^2-\sigma_3^2-2\sigma_1\sigma_2-2\sigma_1\sigma_3-2\sigma_2\sigma_3$$

$$+2\mu(\sigma_1^2+\sigma_2^2+\sigma_3^2)+2\mu(2\sigma_1\sigma_2+2\sigma_1\sigma_3+2\sigma_2\sigma_3)$$

$$= \frac{1}{6E}\Big[\,2\sigma_1^2 + 2\sigma_2^2 + 2\sigma_3^2 - \cancel{2}\mu\sigma_1\sigma_2 - \cancel{2}\mu \ldots$$

$$- \cancel{2}\mu\,\sigma_1\sigma_3$$

$$- 2\sigma_1\sigma_2 - 2\sigma_1\sigma_3 - 2\sigma_2\sigma_3.$$

$$+ 2\mu\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)\Big]$$

$$= \frac{1}{6E}\Big[\,2\sigma_1^2 + 2\sigma_2^2 + 2\sigma_3^2 - 2\sigma_1\sigma_2 - 2\sigma_1\sigma_3 - 2\sigma_2\sigma_3\Big].$$

$$+ \frac{\mu}{6E}\Big[\,2\sigma_1^2 + 2\sigma_2^2 + 2\sigma_3^2 - 2\sigma_1\sigma_2 - 2\sigma_1\sigma_3 - 2\sigma_2\sigma_3\Big]$$

$$U_d = \left(\frac{1+\mu}{6E}\right)\Big[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\Big]. \quad \text{---} (n)$$

In simple Tension test $\sigma_1 = \sigma_y$.

$$\sigma_2 = \sigma_3 = 0$$

eqn (n) becomes

$$U_d = \frac{1+\mu}{3E}\left[\sigma_y^2\right]. \Rightarrow (o)'$$

From (n) and (o)'

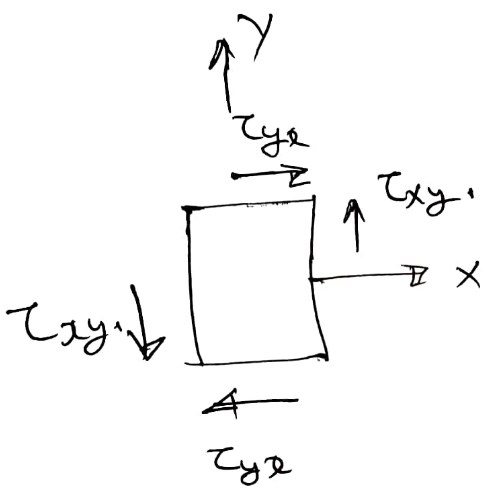
$$2\sigma_y^2 = (\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2$$

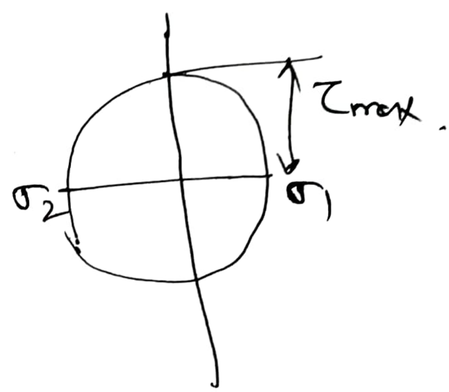
For Biaxial stress state. $\sigma_3 = 0.$

$$2\sigma_y^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2 + \sigma_2^2 + \sigma_3^2$$

$$2\sigma_y^2 = 2\sigma_1^2 + 2\sigma_2^2 - 2\sigma_1\sigma_2.$$

$$\boxed{\sigma_y^2 = \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2.} \quad (A)$$





For component subjected to pure shear stress and corresponds mohr circle is.

$\cancel{\sigma_1 = \sigma_y}$ $\qquad \sigma_1 - \sigma_2 = \tau_{xy},$

$\cancel{\sigma_2 = -\sigma_y}.$ $\qquad \tau_{xy} = \dfrac{\sigma_y}{\sqrt{3}}$

τ_{xy}

$$= \sigma_y \times 0.577.$$

$\cancel{0.577}$.

$$\tau_{xy} = \underline{\underline{0.5\,\sigma_y}}$$