$\therefore$ Absolute error $E_a = E_r \times u$.

Thus the relative error of the product of n approximate numbers is less than or equal to the sum of the relative errors of all the approximate numbers.

(d) *Division:* Let $u = \frac{u_1}{u_2}$, where u_1 and u_2 are two approximate numbers. Then

$$\ln u = \ln u_1 - \ln u_2$$

$$\text{or} \quad \frac{\Delta u}{u} = \frac{\Delta u_1}{u_1} - \frac{\Delta u_2}{u_2}$$

$$\therefore \quad E_r = \left|\frac{\Delta u}{u}\right| = \left|\frac{\Delta u_1}{u_1} - \frac{\Delta u_2}{u_2}\right| \leq \left|\frac{\Delta u_1}{u_1}\right| + \left|\frac{\Delta u_2}{u_2}\right|. \tag{1.8.4}$$

Thus the relative error of a quotient of two approximate numbers is less than or equal to the sum of the relative errors of the numbers.

□ REMARK 1.8.1 *In modern arithmetical computation, the following rules should be followed.*

(a) *For addition:*

(i) The numbers which are least number of correct significant figures should be kept unchanged.

(ii) The rest numbers should be rounded off to one or two decimal places relative to those which are least number of correct significant figures.

(iii) Then the numbers so obtained be added and then rounded off the added number by retaining as many significant figures as there are in the last accurate number among the given numbers.

▶ EXAMPLE 1.8.1 *Find the sum of the approximate numbers:*

$$245.72,\ 258.8,\ 17.5,\ 0.0087,\ 0.00031,\ 41.5.$$

The least accurate numbers are 258.8 and 41.5.

Rounding off the remaining numbers to two decimal places, we have

$$245.72 + 258.8 + 17.5 + 0.01 + 0.00 + 41.5 = 563.53 \simeq \mathbf{563.5}.$$

(b) *For Subtraction:* When the subtraction of two approximate numbers is required, both the numbers should be rounded off to same decimal places at first, then after the numbers should be subtracted.

▶ EXAMPLE 1.8.2 *Subtract the approximate numbers* 57.4662 *from* 786.85.

We first round off the number 57.4662 correct to two places of decimal.

$\therefore$ $786.85 - 57.47 = \mathbf{729.38}.$

(c) *For Multiplication:* In this case the rules are same as was followed for addition.

▶ EXAMPLE 1.8.3 *Find the product of* 171.72, 51.027, 0.0085

$$171.72 \times 51.03 \times 0.01 = 87.628716 \simeq \mathbf{87.63}.$$

1.9 Solved Problems

▶ EXAMPLE 1.9.1 *Round-off the following numbers correct up to 4-decimal places:* (a) 56.243827, (b) 0.235082, (c) 0.560012, (d) 40.35856, (e) 2.789654, (f) 0.005789, (g) 0.003156.

SOLUTION: (a) **56.2438**, (b) **0.2351**, (c) **0.5600**, (d) **40.3586**, (e) **2.7896**, (f) **0.0058**, (g) **0.0032**.

▶ EXAMPLE 1.9.2 *Round-off the following numbers correct to four significant figures:* (a) 5.2056, (b) 0.24062, (c) 0.0055672, (d) 87.26846, (e) 1.03578, (f) 546.9423, (g) 40.3585, (h) 0.98736, (i) 0.0056812.

SOLUTION: (a) **5.206**, (b) **0.2406**, (c) **0.005567**, (d) **87.27**, (e) **1.036**, (f) **546.9**, (g) **40.36**, (h) **0.9874**, (i) **0.005681**.

▶ EXAMPLE 1.9.3 *Find the sum of the following approximate numbers, if the numbers are correct to the last digit:* 2.56, 4.56273, 1.253, 1.05342. **[CP '97]**

SOLUTION: Here the least correct number is 2.56. Rounding-off the remaining number up to 4 decimal places (more than two decimal places relative to 2.56), we get,

$$2.56 + 4.5627 + 1.2530 + 1.0534 = 9.4291 \simeq \mathbf{9.43}.$$

▶ EXAMPLE 1.9.4 *Write down the approximate representation of $\frac{2}{3}$ correct to four significant figures and then find:* (a) *Absolute error,* (b) *relative error,* (c) *relative percentage error.* **[CH '93, CP '97]**

SOLUTION: $\frac{2}{3} = 0.6667$.

(a) Absolute Error, $E_a = |V_T - V_A| = |\frac{2}{3} - 0.6667| = \mathbf{0.000033}$

(b) Relative Error, $E_r = \frac{E_a}{V_T} = \frac{0.000033}{2/3} = 0.0000495 \simeq \mathbf{0.00005}$

(c) Relative percentage Error, $E_p = E_r \times 100 = \mathbf{0.005\%}$.

▷ NOTE 1.9.1 *From equation* (1.5.1), *we have* $E_a < \frac{1}{2} \cdot 10^{-m} = \frac{1}{2} \cdot 10^{-4} = 0.00005$.

$\therefore E_a = 0.000033 < 0.00005$.
From equation (1.6.1)

$$E_r < \frac{1}{k \times 10^{n-1}}.$$

Here $k = 6$, $n = 4$

$$\therefore E_r < \frac{1}{6 \times 10^3} = \frac{0.001}{6} = 0.00166 \simeq 0.0017$$
$$\therefore E_r = 0.00005 < 0.0017.$$

▶ EXAMPLE 1.9.5 *Write down the approximate value of $\frac{\pi}{4}$ correct to four significant figures and then find:* (a) *Absolute errors,* (b) *Relative error and* (c) *Relative percentage error.*

SOLUTION: $\frac{\pi}{4} = \frac{22}{28} \simeq 0.7854$.

(a) The absolute error: $E_a = \left|\frac{11}{14} - 0.7854\right| = \mathbf{0.00031}$.

(b) The relative error: $E_r = \frac{E_a}{V_T} = \frac{0.00031}{11/14} = 0.00039 \simeq \mathbf{0.0004}$.

(c) The relative percentage error: $E_p = E_r \times 100 = 0.0004 \times 100 = \mathbf{0.04\%}$.

EXAMPLE 1.9.6 *Find the number of significant figures in*
(a) $V_A = 11.2461$ *given its absolute error as* 0.25×10^{-2};
(b) $V_T = 1.5923$ *given its relative error as* 0.1×10^{-3};
(c) $V_A = 0.3941$ *given its absolute error as* 0.25×10^{-2}. [CH '84]

SOLUTION:

(a) Here the absolute error is $0.25 \times 10^{-2} = \frac{1}{4} \times 10^{-2} < \frac{1}{2} \times 10^{-2}$.
Thus by (1.5.1), V_A is correct to two decimal places.
$\therefore$ V_A is correct to *four* significant figures.

(b) Absolute error $= E_a = V_T \times E_r = 1.5923 \times 0.1 \times 10^{-3}$
$= 0.15923 \times 10^{-3} \simeq 0.2 \times 10^{-3}$
$= 0.02 \times 10^{-2} < 0.05 \times 10^{-2} = \frac{1}{2} \times 10^{-2}$.
Thus by (1.5.1), the number is correct to two decimal places.
Thus the number of significant figures is *three*.

(c) Here the absolute error is $E_a = 0.25 \times 10^{-2} = \frac{1}{4} \times 10^{-2} < \frac{1}{2} \cdot 10^{-2}$.
Thus by (1.5.1) the number is correct to two decimal places.
Hence the number of significant figures is *two*.

EXAMPLE 1.9.7 *Find the number of significant figures in* V_A *w.r.t.* V_T *where* $V_A = 0.05411, V_T = 0.05418$.

SOLUTION: Here

$$\begin{aligned} E_a &= |V_T - V_A| = |0.05418 - 0.05411| = 0.00007 \\ &= 0.7 \times 10^{-4} = 0.07 \times 10^{-3} < 0.5 \times 10^{-3} = \tfrac{1}{2} \times 10^{-3}. \end{aligned}$$

Thus, by (1.5.1), V_A is correct to 3-decimal places.
So V_A is a number correct to *two* significant figures.

EXAMPLE 1.9.8 *Find the relative error in the computation of* $x - y$ *for* $x = 12.05$ *and* $y = 8.02$ *having absolute errors* $\Delta x = 0.005$ *and* $\Delta y = 0.001$.

SOLUTION: As the absolute error in $x = 0.005$, thus the relative error in

$$x = \frac{0.005}{12.05} = 0.000415.$$

Similarly the relative error in

$$y = \frac{0.001}{8.02} = 0.000125.$$

$\therefore$ The relative error in

$$x - y = 0.000415 - 0.000125 = \mathbf{0.00029}.$$

EXAMPLE 1.9.9 *If $\Delta x = 0.005$ and $\Delta y = 0.001$ be the absolute errors in $x = 2.11$ and $y = 4.15$, find the relative error in computation of $x + y$.*

SOLUTION: Here $x + y = 2.11 + 4.15 = 6.26$, this result has an absolute error

$$\Delta x + \Delta y = 0.005 + 0.001 = 0.006.$$

$\therefore$ The relative error in

$$x + y = \frac{0.006}{6.26} = 0.000958 \simeq \mathbf{0.001}.$$

EXAMPLE 1.9.10 *If $y = 4x^6 - 5x$, find the percentage error in y at $x = 1$, if the error in $x = 0.04$.*

SOLUTION: We have

$$\Delta y \simeq \frac{dy}{dx} \cdot \Delta x = (24x^5 - 5) \times 0.04.$$

Thus at $x = 1$, the absolute error in computation of y, $\Delta y = (24.1^5 - 5) \times 0.04 = 0.76$. And y for $x = 1, y = -1$.

Thus percentage error in

$$y = \left|\frac{\Delta y}{y}\right| \times 100 = \frac{0.76}{1} \times 100 = \mathbf{76\%}.$$

EXAMPLE 1.9.11 *If $f(x) = 4\cos x - 6x$, find the relative percentage error in $f(x)$ for $x = 0$, if the error in $x = 0.005$.*

SOLUTION: The error in computation of $f(x)$ is given by

$$\Delta f(x) = \frac{df}{dx} \cdot \Delta x = [-4\sin x - 6] \times \Delta x.$$

At $x = 0$, the absolute error in computation $f(x)$,

$$\Delta f(x) = [-6] \times 0.005 = -0.03 \text{ and } f(0) = 4.$$

$\therefore$ Relative percentage error is

$$\frac{|\Delta f(x)|}{|f(0)|} \times 100 = \frac{0.03}{4} \times 100 = \mathbf{0.75\%}.$$

► EXAMPLE 1.9.12 *N is a function of five different measurable quantities u, v, w, x, y and is given by*

$$N = \frac{u^p v^q w^r}{x^s y^t}$$

find an upper limit to the relative error in the measure of N. [CH '87]

SOLUTION: [This problem may be considered as direct application of equation (1.8.3)]
Here

$$N = \frac{u^p v^q w^r}{x^s y^t}.$$

Taking $\log_e$ on both sides, we have

$$\ln N = p \ln u + q \ln v + r \ln w - s \ln x - t \ln y$$

$$\therefore \frac{\Delta N}{N} = p \cdot \frac{\Delta u}{u} + q \cdot \frac{\Delta v}{v} + r \cdot \frac{\Delta w}{w} - s \cdot \frac{\Delta x}{x} - t \cdot \frac{\Delta y}{y}.$$

Thus the relative error in N is

$$E_r = \left|\frac{\Delta N}{N}\right| = \left|p \cdot \frac{\Delta u}{u} + q \cdot \frac{\Delta v}{v} + r \cdot \frac{\Delta w}{w} - s \cdot \frac{\Delta x}{x} - t \cdot \frac{\Delta y}{y}\right|$$

$$\leq p \cdot \left|\frac{\Delta u}{u}\right| + q \cdot \left|\frac{\Delta v}{v}\right| + r \cdot \left|\frac{\Delta w}{w}\right| + s \cdot \left|\frac{\Delta x}{x}\right| + t \cdot \left|\frac{\Delta y}{y}\right|.$$

Thus the upper limit of the relative error in N is

$$p\left|\frac{\Delta u}{u}\right| + q\left|\frac{\Delta v}{v}\right| + r\left|\frac{\Delta w}{w}\right| + s\left|\frac{\Delta x}{x}\right| + t\left|\frac{\Delta y}{y}\right|.$$

1.10 Exercises

1. Round-off the following numbers correct up to 4-significant figures:
   a) 2.56235; b) 0.34026; c) 4.50089; d) 0.000243468;
   e) 2456.782; f) 1.3456; g) 2.5555; h) 8.0008.

   [*Ans:* (a) 2.562; (b) 0.3403; (c) 4.501; (d) 0.0002435; (e) 2457; (f) 1.346; (g) 2.226; (h) 8.001]

2. Round-off the following numbers correct up to 3 decimal places:
   a) 2.46289; b) 0.46999; c) 0.0035869;
   d) 0.0015; e) 1.46294; f) 1.35008.

   [*Ans:* (a) 2.463; (b) 0.470; (c) 0.004; (d) 0.002; (e) 1.463; (f) 1.350]

3. The following numbers are correct to the last digit. Find the sum:
   (a) 1.3526, 2.00462, 1.532, 28.201, 31.0012;
   (b) 0.532, 7.46571, 1.501, 3.62102;
   (c) 5.2146, 20.12, 11.2356, 1.8948.

   [*Ans:* (a) 64.091; (b) 13.120; (c) 38.46]

4. If V_T, V_A, E_a, E_r and E_p, are the true value, approximate value, absolute error, relative error and relative percentage error respectively.

(a) Given $V_T = 2.546282, V_A = 2.5463$. Find E_a, E_r and E_p.

(b) Given $E_a = 0.2 \times 10^{-3}, E_r = 0.32 \times 10^{-5}$. Find V_T.

(c) Given $V_T = 2.73456, E_r = 0.2 \times 10^{-2}$. Find E_a, E_p.

[*Ans:* (a) $E_a = 1.8 \times 10^{-5}, E_r = 7.07 \times 10^{-6}, E_p = 7.07 \times 10^{-4}\%$; (b) 62.5; (c) $E_a = 0.00547, E_p = 0.2\%$]

5. Find the relative error in computation of $x+y$ for $x = 11.75$ and $y = 7.23$ having absolute errors $\Delta x = 0.002$ and $\Delta y = 0.005$ respectively.

[*Ans:* 0.00037]

6. Find the relative error in computation of $x - y$ for $x = 9.05$ and $y = 6.56$ have absolute errors $\Delta x = 0.001$ and $\Delta y = 0.003$ respectively.

[*Ans:* 0.00034]

7. If $\frac{5}{6}$ be represented approximately by 0.8333, find a) relative error and b) percentage error.

[*Ans:* $E_r = 0.00004, E_p = 0.004\%$]

8. Find the relative percentage error in approximate representation of $\frac{4}{3}$ by 1.33.

[*Ans:* $E_p = 0.25\%$]

9. Determine the number of correct digits in the number x, given its absolute error E_a.

(a) $x = 0.3681,\ E_a = 0.23 \times 10^{-2}$;

(b) $x = 841.256,\ E_a = 0.1$;

(c) $x = -33.783,\ E_a = 0.3 \times 10^{-2}$.

[*Ans:* (a) 2; (b) 3; (c) 4]

10. Determine the number of correct digits in the number x given its relative error E_r.

(a) $x = 0.4785, E_r = 0.2 \times 10^{-2}$;

(b) $x = 386.4, E_r = 0.3$;

(c) $x = 86.34, E_r = 0.1$.

[*Ans:* (a) 2; (b) 1; (c) 0]

11. Find the number of significant figures in $V_A = 1.8921$, given its relative error as 0.1×10^{-2}. [CH '84]

[*Ans:* 3]