Table: $T_2^{(\gamma)}(4.9 < \gamma < 5.0)$

x	:	4.90	4.91	4.92	4.93	4.94	4.95	4.96	4.97	4.98	4.99	5.00
$f(x)$	:	-6.48	-5.79	$\cdots$	$\cdots$	$\cdots$	-2.89	$\cdots$	$\cdots$	-0.59	0.20	$\cdots$

Here $f(4.98) \cdot f(4.99) < 0$, $\therefore \gamma$ between 4.98 and 4.99. Again we compute $f(x)$ for successive values of x starting with 4.98 and with step length 0.001.

Table: $T_3^{(\gamma)}(4.98 < \gamma < 4.99)$

x	:	4.980	4.981	4.982	4.983	4.984	4.985	4.986	4.987	4.988	4.989
$f(x)$	:	-0.59	$\cdots$	$\cdots$	$\cdots$	$\cdots$	-0.201	$\cdots$	-0.042	0.037	$\cdots$

As $f(4.987) \times f(4.988) < 0, \gamma$ lies within the interval $(4.987, 4.988)$. Thus, $\gamma = 4.99$ is a root of $f(x) = 0$, up to two decimal places.

▷ NOTE 6.2.1 *As the method is laborious and we are only interested in finding an expected sufficiently small interval in which $f(x)$ changes sign, we use a short-cut approach on and from table $T_3^{(\alpha)}$. In table $T_2^{(\alpha)}$, it is confirmed that only one root α lies in $(1.76, 1.77)$. For table $T_3^{(\alpha)}$, we first write $f(1.760) = -0.07 < 0$ from table $T_2^{(\alpha)}$, then $f(1.765) = -0.03 < 0$ and $f(1.769) = 0.005 > 0$ are computed, shows the change of sign. Now we calculate $f(1.768) = -0.003$ which indicates that the root α lies in the interval $(1.768, 1.769)$. So our purpose is reached and there is no need for computation $f(x)$ for other arguments $x < 1.768$. This technique may be used in any interval where we confirm that only one root lies within the said interval.*

6.3 Method of Bisection

It is an iterative method and is based on a well known theorem which states that if $f(x)$ be a continuous function in a closed interval $[a, b]$ and $f(a)f(b) < 0$, then there exists at least one real root of the equation $f(x) = 0$, between a and b. If further $f'(x)$ exists and $f'(x)$ maintains same sign in $[a, b]$, i.e., $f(x)$ strictly monotonic, then there is only one real root of $f(x) = 0$, in $[a, b]$. The method of Bisection is nothing but a repeated applications of the above theorem.

We shall determine a sufficiently small interval $[a_0, b_0]$, by Graphical or Tabulation method, in which $f(a_0)f(b_0) < 0$ and $f'(x)$ maintains same sign in $[a_0, b_0]$, so that there is only one real root of $f(x) = 0$. Now we shall find a sequence $\{x_n\}$, each member of which is a successive better approximation of a root say, α of $f(x) = 0$, in $[a_0, b_0]$ as follows. Let the interval $[a_0, b_0]$ be divided in two equal parts by x_1, i.e., $x_1 = \frac{a_0+b_0}{2}$ and $f(x_1)$ is calculated. If $f(x_1) = 0$, then x_1 is an exact root of $f(x) = 0$. If $f(x_1) \neq 0$, then either $f(a_0)f(x_1) < 0$ or $f(x_1)f(b_0) < 0$. If $f(a_0)f(x_1) < 0$, then the root α lies in $[a_0, x_1]$, otherwise α lies in $[x_1, b_0]$. For convenience we assume that α lies in $[x_1, b_0]$ and we re-name the interval as $[a_1, b_1]$ so that $b_1 - a_1 = \frac{1}{2}(b_0 - a_0)$. Now we take $x_2 = \frac{a_1+b_1}{2}$ and $f(x_2)$ is computed, then either $f(a_1)f(x_2) < 0$ or $f(x_2)f(b_1) < 0$, provided $f(x_2) = 0$ where x_2 is the exact root of $f(x) = 0$. We assume here that $f(a_1)f(x_2) < 0$, then the root α of $f(x) = 0$ lies in $[a_1, x_2]$ and we call it as $[a_2, b_2]$, where $b_2 - a_2 = \frac{1}{2}(b_1 - a_1) = \frac{1}{2^2}(b_0 - a_0)$. Proceeding in this manner, we find $x_{n+1} = \frac{a_n+b_n}{2}$ which is the $(n+1)$th approximation of the root α of $f(x) = 0$ and lies in the interval $[a_n, b_n]$ where

$b_n - a_n = \frac{1}{2^n}(b_0 - a_0)$ and $a_0 \leq a_n < b_n \leq b_0$ for all n. If ε_{n+1} be the error in approximating α by x_{n+1} then $\varepsilon_{n+1} = |\alpha - x_{n+1}| < b_n - a_n < \frac{b_0-a_0}{2^n} \to 0$, as $n \to \infty$. Thus, this iterative process surely converges. To get a root of $f(x) = 0$ correct to p significant figures, we are to go up to qth iteration so that x_q and x_{q+1} have the same p significant figures.

6.3.1 Computation Scheme

1. Find an interval $[a_0, b_0]$ where $f(a_0)f(b_0) < 0$ and $f'(x)$ maintains same sign.

2. Write, n (number of iteration), a_n, b_n, x_{n+1} $(= \frac{a_n+b_n}{2})$ and $f(x_{n+1})$ horizontally.

3. Insert $+ve$ or $-ve$ sign with a_n, as $a_n(+ve)$ or $a_n(-ve)$ according as $f(a_0) > 0$ or $f(a_0) < 0$ and $-ve$ or $+ve$ sign with b_n, as $b_n(-ve)$ or $b_n(+ve)$ according as $f(b_0) < 0$ or $f(b_0) > 0$.

4. In $(r+1)$th iteration, write x_{r+1} $(= \frac{a_r+b_r}{2})$ in the column of $a_n(+ve)$, if $f(x_{r+1}) > 0$ keeping b_r fixed in the column of $b_n(-ve)$. Otherwise, write x_{r+1} $(= \frac{a_r+b_r}{2})$ in the column of $b_n(-ve)$, if $f(x_{r+1}) < 0$, keeping a_r fixed in the column of $a_n(+ve)$.

Scheme 6.1

When $f(a_0) > 0$ and $f(b_0) < 0$

n	$a_n(+ve)$	$b_n(-ve)$	x_{n+1} $(= \frac{a_n+b_n}{2})$	$f(x_{n+1})$
0	a_0	b_0	x_1 $(= \frac{a_0+b_0}{2})$	$f(x_1) > 0$
1	$x_1(= a_1)$	$b_0(= b_1)$	x_2 $(= \frac{a_1+b_1}{2})$	$f(x_2) > 0$
2	$x_2(= a_2)$	$b_0(= b_2)$	x_3 $(= \frac{a_2+b_2}{2})$	$f(x_3) < 0$
3	$a_2(= a_3)$	$x_3(= b_3)$	x_4 $(= \frac{a_3+b_3}{2})$	$f(x_4) < 0$
4	$a_2(= a_4)$	$x_4(= b_4)$	x_5 $(= \frac{a_4+b_4}{2})$	$f(x_5) > 0$
5	$x_5(= a_5)$	$b_4(= b_5)$	x_6 $(= \frac{a_5+b_5}{2})$	$f(x_6) < 0$
6	$a_5(= a_6)$	$x_6(= b_6)$	x_7 $(= \frac{a_6+b_6}{2})$	$f(x_7) < 0$
7	$a_6(= a_7)$	$x_7(= b_7)$	x_8 $(= \frac{a_7+b_7}{2})$	
	and so on.			

Scheme 6.2

When $f(a_0) < 0$ and $f(b_0) > 0$

n	$a_n(-ve)$	$b_n(+ve)$	$x_{n+1}\ (= \frac{a_n+b_n}{2})$	$f(x_{n+1})$
0	a_0	b_0	$x_1\ (= \frac{a_0+b_0}{2})$	$f(x_1) > 0$
1	$a_0(= a_1)$	$x_1(= b_1)$	$x_2\ (= \frac{a_1+b_1}{2})$	$f(x_2) > 0$
2	$a_0(= a_2)$	$x_2(= b_2)$	$x_3\ (= \frac{a_2+b_2}{2})$	$f(x_3) > 0$
3	$a_0(= a_3)$	$x_3(= b_3)$	$x_4\ (= \frac{a_3+b_3}{2})$	$f(x_4) < 0$
4	$x_4(= a_4)$	$b_3(= b_4)$	$x_5\ (= \frac{a_4+b_4}{2})$	$f(x_5) < 0$
5	$x_5(= a_5)$	$b_3(= b_5)$	$x_6\ (= \frac{a_5+b_5}{2})$	$f(x_6) > 0$
6	$a_5(= a_6)$	$x_6(= b_6)$	$x_7\ (= \frac{a_6+b_6}{2})$	$f(x_7) < 0$
7	$x_7(= a_7)$	$b_6(= b_7)$	$x_8\ (= \frac{a_7+b_7}{2})$	
	and so on.			

(0, 1)
(0, .5)
f(a) f(b) < 0

6.4 Solved Problems

► EXAMPLE 6.4.1 *Find the positive roots of the equation $x^3 - 3x + 1.06 = 0$, by method of bisection, correct to three decimal places.*

SOLUTION: Let $f(x) = x^3 - 3x + 1.06$. By Descartes' rule of signs, the maximum number of positive roots is two.

Now $f(0) = 1.06 > 0$, $f(1) = -0.94 < 0$, $f(2) = 3.06 > 0$. Thus, one positive root α lies in $(0, 1)$ and other β lies in $(1, 2)$.

(i) Computation of $\alpha (0 < \alpha < 1)$

$a_0 = 0$
$b_0 = 1$

	n	$a_n(+ve)$	$b_n(-ve)$	$x_{n+1}\ (= \frac{a_n+b_n}{2})$	$f(x_{n+1})$
	0	0	1	0.5	−0.32
	1	0	0.5	0.25	0.33
	2	0.25	0.5	0.375	−0.012
	3	0.25	0.375	0.312	0.154
	4	0.312	0.375	0.343	0.071
	5	0.343	0.375	0.359	0.029
	6	0.359	0.375	0.367	0.008
	7	0.367	0.375	0.371	−0.002
	8	0.367	0.371	0.369	0.003
	9	0.369	0.371	0.370	0.0006
	10	0.370	0.371	0.3705	−0.0006
	11	0.370	0.3705	0.37025	0.000006
	12	0.37025	0.3705	0.370375	−0.0003
check	13	0.37025	0.370375	0.370312	

$\therefore\ \boldsymbol{\alpha = 0.370}$, correct to three decimal places.

(ii) Computation of $\beta(1 < \beta < 2)$. Here $f(1) = -0.94$ and $f(2) = 3.06$

	n	$a_n(-ve)$	$b_n(+ve)$	$x_{n+1}\left(=\frac{a_n+b_n}{2}\right)$	$f(x_{n+1})$
	0	1	2	1.5	−0.06
	1	1.5	2	1.75	1.17
	2	1.5	1.75	1.62	0.45
	3	1.5	1.62	1.56	0.17
	4	1.5	1.56	1.53	0.05
	5	1.5	1.53	1.515	−0.007
	6	1.515	1.53	1.5225	0.0217
	7	1.515	1.5225	1.5188	0.0067
	8	1.515	1.5188	1.5169	−0.0003
	9	1.5169	1.5188	1.51785	0.0034
	10	1.5169	1.51785	1.517375	0.0015
	11	1.5169	1.517375	1.517138	0.0006
check	12	1.5169	1.517138	1.517019	0.0001

$\therefore$ $\beta = \mathbf{1.517}$, correct to three decimal places. Here a_n, b_n and x_{n+1} are equal up to three decimal places at the 11th step.

► EXAMPLE 6.4.2 *Solve the equation $x^3 - 9x + 1 = 0$ for the root lying between 2 and 3, correct to 3-significant figures.* [CH '87]

SOLUTION: Let $f(x) = x^3 - 9x + 1$, Here $f(2) = -9, f(3) = 1$.
$\therefore f(2) \cdot f(3) < 0$

	n	$a_n(-ve)$	$b_n(+ve)$	$x_{n+1}\left(=\frac{a_n+b_n}{2}\right)$	$f(x_{n+1})$
	0	2	3	2.5	−5.8
	1	2.5	3	2.75	−2.9
	2	2.75	3	2.88	−1.03
	3	2.88	3	2.94	−0.05
	4	2.94	3	2.97	0.47
	5	2.94	2.97	2.955	0.21
	6	2.94	2.955	2.9475	0.08
	7	2.94	2.9475	2.9438	0.017
	8	2.94	2.9438	2.9419	−0.016
check	9	2.9419	2.9438	2.9428	−0.003

In the 8th step, a_n, b_n and x_{n+1} are equal up to three significant figures, $\therefore$ **2.94** is the root, up to three significant figures.

► EXAMPLE 6.4.3 *Compute one positive root of $2x - 3\sin x - 5 = 0$, by the bisection method, correct to three significant figures.*

SOLUTION: Let $f(x) = 2x - 3\sin x - 5$. Here $f(0) = -5$, $f(1) = -5.5$, $f(2) = -3.7$, $f(3) = 0.57$, $\therefore f(2) \cdot f(3) < 0$. Thus, only one root lies between 2 and 3, since $f'(x) = 2 - 3\cos x > 0$ for $x \in [2, 3]$.

n	$a_n(-ve)$	$b_n(+ve)$	$x_{n+1}\left(=\frac{a_n+b_n}{2}\right)$	$f(x_{n+1})$
0	2.0	3.0	2.5	−1.79
1	2.5	3.0	2.75	−0.64
2	2.75	3.0	2.875	−0.04
3	2.875	3.0	2.938	0.27
4	2.875	2.938	2.906	0.11
5	2.875	2.906	2.8905	0.036
6	2.875	2.8905	2.8828	−0.0021
7	2.8828	2.8905	2.8866	0.0165
8	2.8828	2.8866	2.8847	0.0072
9	2.8828	2.8847	2.8838	0.0028
check 10	2.8828	2.8838	2.8833	0.0003

In 9th step, a_n, b_n and x_{n+1} are equal up to three significant figures.
$\therefore$ **2.88** is the root, correct to three significant figures.

▶ EXAMPLE 6.4.4 *Find one root of $10^x + \sin x + 2x = 0$, by the bisection method, up to three significant figures.*

SOLUTION: Let $f(x) = 10^x + \sin x + 2x$, $f(0) = 1$, $f(1) = 12.8$, $f(-1) = -2.74$, and $f(-1) \cdot f(0) < 0$. Also, $f'(x) = 10^x \log_e x + \cos x + 2 > 0$ for $x \in [-1, 0]$.
$\therefore$ Only one root between −1 and 0.

n	$a_n(-ve)$	$b_n(+ve)$	$x_{n+1}\left(=\frac{a_n+b_n}{2}\right)$	$f(x_{n+1})$
0	−1.0	0.0	−0.5	−1.16
1	−0.5	0.0	−0.25	−0.18
2	−0.25	0.0	−0.125	0.37
3	−0.25	−0.125	−0.1875	0.088
4	−0.25	−0.1875	−0.2187	−0.050
5	−0.2187	−0.1875	−0.2031	0.018
6	−0.2187	−0.2031	−0.2109	−0.016
7	−0.2109	−0.2031	−0.2070	0.001
8	−0.2109	−0.2070	−0.20895	−0.007
9	−0.20895	−0.2070	−0.20798	−0.003
10	−0.20798	−0.2070	−0.20749	−0.0008
11	−0.20749	−0.2070	−0.20724	0.0003
check 12	−0.20749	−0.20724	−0.20736	−0.0002

Here in the 11th stage, a_n, b_n and x_{n+1} are same up to three significant figures. Thus, **−0.207** is a root of $10^x + \sin x + 2x = 0$, correct to three significant figures.

▶ EXAMPLE 6.4.5 *Compute a real root of $x^3 - 1.1x^2 + 4x - 4.4 = 0$, correct to two significant figures, by method of bisection.*

SOLUTION: Let $f(x) = x^3 - 1.1x^2 + 4x - 4.4$. Here $f(0) = -4.4$, $f(1) = -0.6$, $f(2) = 16 > 0$. $\therefore$ $f(1) \cdot f(2) < 0$. Thus, only one root between 1 and 2, since $f'(x) = 3x^2 - 2.2x + 4 > 0$ for $x \in [1, 2]$.

n	$a_n(-ve)$	$b_n(+ve)$	$x_{n+1}\ (= \frac{a_n+b_n}{2})$	$f(x_{n+1})$
0	1	2	1.5	2.5
1	1	1.5	1.25	0.83
2	1	1.25	1.125	0.13
3	1	1.125	1.062	−0.195
4	1.062	1.125	1.094	−0.031
5	1.094	1.125	1.1095	0.049
6	1.094	1.1095	1.1018	0.0093
check 7	1.094	1.1018	1.0979	−0.0109

In the 6th step, a_n, b_n and x_{n+1} are equal up to two significant figures.
$\therefore$ **1.1** is a root of the equation, correct to two significant figures.

► EXAMPLE 6.4.6 *Compute one root of $x + \ln x - 2 = 0$ correct to two decimal places with lies between* 1 *and* 2.

SOLUTION: Let $f(x) = x + \ln x - 2 = 0$. Here $f(1) = -1$ and $f(2) = 0.69$.
$\therefore$ Only one root between 1 and 2, since $f'(x) = 1 + \frac{1}{x} > 0$ for $x \in [1, 2]$.

n	$a_n(-ve)$	$b_n(+ve)$	$x_{n+1}\ (= \frac{a_n+b_n}{2})$	$f(x_{n+1})$
0	1.0	2.0	1.5	−0.09
1	1.5	2.0	1.75	0.31
2	1.5	1.75	1.625	0.11
3	1.5	1.625	1.562	0.008
4	1.5	1.562	1.531	−0.043
5	1.531	1.562	1.5465	−0.017
6	1.5465	1.562	1.5542	−0.0048
7	1.5542	1.562	1.5581	0.0016
8	1.5542	1.5581	1.5562	−0.0015
9	1.5562	1.5581	1.55715	0.000007
check 10	1.5562	1.55715	1.55668	−0.00076

In the 9th step, a_n, b_n and x_{n+1} are equal to two decimal places.
$\therefore$ **1.56** is a root of the equation, correct to two decimal places.

► EXAMPLE 6.4.7 *Compute one root of $\sin x = 10(x - 1)$, correct to three significant figures.*

SOLUTION: Let $f(x) = \sin x - 10x + 10$. Here $f(1) = 0.84$, $f(1.5) = -4.002$, $f(1.1) = -0.108$. Thus, only one root of $f(x) = 0$ lies in $[1, 1.1]$, since $f'(x) = \cos x - 10 < 0$ for $x \in [1, 1.1]$.

n	$a_n(-ve)$	$b_n(+vc)$	$x_{n+1}\ (=\frac{a_n+b_n}{2})$	$f(x_{n+1})$
0	1.0	1.1	1.05	0.37
1	1.05	1.1	1.08	0.082
2	1.08	1.1	1.09	−0.013
3	1.08	1.09	1.085	0.034
4	1.085	1.09	1.0875	0.0104
5	1.0875	1.09	1.08875	−0.00145
check 6	1.0875	1.08875	1.08812	0.0046

In the 5th step, a_n, b_n and x_{n+1} are equal up to three significant figures. $\therefore$ $\boldsymbol{x = 1.09}$ is a root of $f(x) = 0$, correct to three significant figures.

▶ EXAMPLE 6.4.8 *Compute one root of $e^x - 3x = 0$, correct to two decimal places which between 1 and 2.*

SOLUTION: Let $f(x) = e^x - 3x$. Here $f(1) = -0.28$, $f(1.5) = -0.02$, $f(1.6) = 0.15$. Thus, only one root of $f(x) = 0$ between 1.5 and 1.6, since $f'(x) = e^x - 3 > 0$ for $x \in [1.5,\ 1.6]$.

n	$a_n(-ve)$	$b_n(+ve)$	$x_{n+1}\ (=\frac{a_n+b_n}{2})$	$f(x_{n+1})$
0	1.5	1.6	1.55	0.06
1	1.5	1.55	1.525	0.02
2	1.5	1.525	1.5125	0.00056
3	1.5	1.5125	1.5062	−0.00904
4	1.5062	1.5125	1.50935	−0.00426
check 5	1.50935	1.5125	1.51092	−0.00184

In the 4th step, $a_n. b_n$ and x_{n+1} are equal up to two decimal places. Thus, $\boldsymbol{x = 1.51}$ is a root of $f(x) = 0$, correct up to two decimal places.

▶ EXAMPLE 6.4.9 *Compute the root of* $\log x = \cos x$, *correct to two decimal places, which between 1 and 2.*

SOLUTION: Let $f(x) = \log_e x - \cos x$. Here $f(1) = -0.54$, $f(1.5) = 0.33$, $f(1.3) = -0.005$, $f(1.4) = 0.166$. Thus, one root of $f(x) = 0$ between 1.3 and 1.4.

n	$a_n(-ve)$	$b_n(+ve)$	$x_{n+1}\ (=\frac{a_n+b_n}{2})$	$f(x_{n+1})$
0	1.3	1.4	1.35	0.08
1	1.3	1.35	1.325	0.038
2	1.3	1.325	1.3125	0.0165
3	1.3	1.3125	1.3062	0.0056
4	1.3	1.3062	1.3031	0.00024
5	1.3	1.3031	1.3016	−0.0024
check 6	1.3016	1.3031	1.3024	−0.00098

In the 5th step, a_n, b_n and x_{n+1} are equal up to two decimal places. Thus, $\boldsymbol{x = 1.30}$ is root of $f(x) = 0$, correct up to two decimal places.

▶ EXAMPLE 6.4.10 *Find the root of* $\tan x + x = 0$, *up to two decimal places, which between 2 and 2.5.*

SOLUTION: Let $f(x) = \tan x + x$. Here $f(2) = -0.18$, $f(2.2) = 0.82$, $f(2.1) = 0.39$. Thus, the root between 2.0 and 2.1.

n	$a_n(-ve)$	$b_n(+ve)$	$x_{n+1}\left(=\frac{a_n+b_n}{2}\right)$	$f(x_{n+1})$
0	2.0	2.1	2.05	0.12
1	2.0	2.05	2.025	−0.023
2	2.025	2.05	2.0375	0.053
3	2.025	2.0375	2.03125	0.0152
4	2.025	2.03125	2.02812	−0.0039
5	2.02812	2.03125	2.02968	0.0056
check 6	2.02813	2.02968	2.02890	0.00087

In the 5th step, a_n, b_n and x_{n+1} are equal up to two decimal places.
$\therefore$ $\boldsymbol{x = 2.03}$ is a root of $f(x) = 0$, correct up to two decimal places.

▶ EXAMPLE 6.4.11 *Find a root of the equation* $x^x + 2x - 6 = 0$, *by method of bisection, correct to two decimal places.*

SOLUTION: Let $f(x) = x^x + 2x - 6$. Here $f(1) = -3$ and $f(2) = 2$. Thus, only one root of $f(x)$ between 1 and 2. Since $f'(x) = x^x(1 + \log_e x) + 2 > 0$ for $x \in [1, 2]$.

n	$a_n(-ve)$	$b_n(+ve)$	$x_{n+1}\left(=\frac{a_n+b_n}{2}\right)$	$f(x_{n+1})$
0	1.0	2.0	1.5	−1.16
1	1.5	2.0	1.75	0.16
2	1.5	1.75	1.625	−0.55
3	1.625	1.75	1.6875	−0.207
4	1.6875	1.75	1.71875	−0.026
5	1.71875	1.75	1.73438	0.067
6	1.71875	1.73438	1.72656	0.0206
7	1.71875	1.72656	1.72266	−0.0026
8	1.72266	1.72656	1.72461	0.00898
9	1.72266	1.72461	1.723635	0.00318
check 10	1.72266	1.72335	1.723148	0.00028

In the 9th step, a_n, b_n and x_{n+1} are equal up to two decimal places. Thus, $\boldsymbol{x = 1.72}$ is a root, correct up to two decimal places.

6.5 Method of Iteration or Fixed-Point Iteration

This iteration method is based on the principle of finding a sequence $\{x_n\}$ each element of which successively approximates a real root α of the equation $f(x) = 0$, in $[a, b]$.

We re-write $f(x) = 0$ as:

$$x = \phi(x). \qquad (6.5.1)$$