Thus, both $x = \phi_1(x)$ and $x = \phi_3(x)$, give us convergent sequences of iteration, we shall take $x = \phi_1(x) = \frac{1}{x^2+1}$ for more rapid convergence than $x = \phi_3(x) = (1-x)^{1/3}$. We take $x_0 = 0.5$ and $x = \phi_1(x) = \frac{1}{x^2+1}$.

n	x_n	$\phi_1(x_n)$
0	0.5	0.8
1	0.8	0.61
2	0.61	0.73
3	0.73	0.65
4	0.65	0.70
5	0.70	0.67
6	0.67	0.69
7	0.69	0.677
8	0.677	0.6857
9	0.6857	0.6802
10	0.6802	0.6836
check 11	0.6836	0.6815

$\therefore$ **0.68** is the root of the equation, correct up to two decimal places.

► EXAMPLE 6.6.6 *Find, by the method of iteration, the root of* $2x - \log_{10} x - 7 = 0$, *correct to four decimal places, which between* 3 *and* 4.

SOLUTION: Let $f(x) = 2x - \log_{10} x - 7$.
Here $f(3) = -1.47 < 0$ and $f(4) = 0.39 > 0$.
Thus, one root of the equation between 3 and 4. We re-write the equation as

$$x = \frac{1}{2}[\log_{10} x + 7] = \phi(x).$$

$\therefore\ |\phi'(x)| = \frac{1}{2x} \cdot \frac{1}{\log_e 10} < 1$ in $(3, 4)$,
as Max $[|\phi'(3)|, |\phi'(4)|] = \text{Max}\ [0.07, 0.05] = 0.07 < 1$. We take $x_0 = 3$, and $\phi(x) = \frac{1}{2}[\log_{10} x + 7]$.

n	x_n	$\phi(x_n)$
0	3.0	3.7
1	3.7	3.78
2	3.78	3.7887
3	3.7887	3.7892
4	3.7892	3.78927
5	3.78927	3.78928
check 6	3.78928	3.789278

$\therefore$ **3.7893** is the root of the equation, correct up to four decimal places.

6.7 Newton-Raphson Method

This is also an iterative method and is used to find isolated roots of an equation $f(x) = 0$. The object of this method is to correct the approximate root x_0 (say) successively to its exact value α. Initially, a crude approximation small interval $[a_0, b_0]$ is found out in which only one root α (say) of $f(x) = 0$ lies.

Let $x = x_0 (a_0 \leq x_0 \leq b_0)$ is an approximation of the root α of the equation $f(x) = 0$. Let h be a small correction on x_0, then $x_1 = x_0 + h$ is the correct root.

$\therefore f(x_1) = 0 \Rightarrow f(x_0 + h) = 0.$

Therefore, by Taylor series expansion, we get,

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0.$$

As h is small, neglecting the second and higher power of h, we get,

$$h = -\frac{f(x_0)}{f'(x_0)} \tag{6.7.1}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}. \tag{6.7.2}$$

Further, if h_1 be the correction on x_1, then $x_2 = x_1 + h_1$ is the correct root of $f(x) = 0$.

$$\therefore f(x_2) = f(x_1 + h_1) = 0.$$

Thus,

$$f(x_1) + h_1 f'(x_1) + \frac{h_1}{2!} f''(x_1) + \cdots = 0.$$

Neglecting the second and higher power of h_1, we get,

$$h_1 = -\frac{f(x_1)}{f'(x_1)}.$$

$$\therefore x_2 = x_1 + h_1 = x_1 - \frac{f(x_1)}{f'(x_1)}. \tag{6.7.3}$$

Proceeding in this way, we get the $(n+1)$th corrected root as,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{6.7.4}$$

The formula (6.7.4) generates a sequence of successive corrections on an approximate root x_0 to get the correct root α of $f(x) = 0$, provided the sequence is convergent. The formula (6.7.4) is known as the iteration formula for Newton-Raphson Method. The number of iterations required depends upon the desired degree of accuracy of the root.

6.7.1 Convergence of Newton-Raphson Method

Comparing with the iteration method, we may assume the iteration function as

$$\phi(x) = x - \frac{f(x)}{f'(x)}.$$

Thus, the above sequence will be convergent, if and only if,

$$|\phi'(x)| = \left|1 - \frac{\{f'(x)\}^2 - f(x)f''(x)}{\{f'(x)\}^2}\right| < 1, \text{ i.e., } \left|\frac{f(x)f''(x)}{\{f'(x)\}^2}\right| < 1$$

$$\therefore |\{f'(x)\}^2| > |f(x) \cdot f''(x)|. \tag{6.7.5}$$

6.7.2 Estimation of Error in Newton-Raphson Method

If x_{n+1} be the $(n+1)$th approximation of a root α of $f(x) = 0$ and ε_{n+1} be the corresponding error, we have

$$\alpha - x_{n+1} = \varepsilon_{n+1} \tag{6.7.6}$$

$$\begin{aligned}
\text{and} \quad x_{n+1} &= x_n + h_n \\
\text{or } \alpha - x_{n+1} &= \alpha - x_n - h_n \\
\text{or} \quad \varepsilon_{n+1} &= \varepsilon_n - h_n \quad \text{[by (6.7.6)]} \\
\text{or} \quad \varepsilon_n &= \varepsilon_{n+1} + h_n
\end{aligned} \tag{6.7.7}$$

where

$$h_n = -\frac{f(x_n)}{f'(x_n)} \quad \text{or,} \quad f(x_n) + h_n f'(x_n) = 0. \tag{6.7.8}$$

Again we have

$$\alpha = x_n + \varepsilon_n$$

$$\text{or,} \quad f(\alpha) = 0 = f(x_n + \varepsilon_n) = f(x_n) + \varepsilon_n f'(x_n) + \frac{\varepsilon_n^2}{2!} f''(\xi_n);\ (x_n < \xi_n < \alpha). \tag{6.7.9}$$

Subtracting (6.7.8) from (6.7.9), we get,

$$(\varepsilon_n - h_n) f'(x_n) + \frac{\varepsilon_n^2}{2!} f''(\xi_n) = 0, \text{ or, } \varepsilon_{n+1} f'(x_n) + \frac{\varepsilon_n^2}{2!} f''(\xi_n) = 0$$

$$\text{or,} \quad \left| \frac{\varepsilon_{n+1}}{\varepsilon_n^2} \right| = \frac{1}{2} \left| \frac{f''(\xi_n)}{f'(x_n)} \right|.$$

If the iteration converges, i.e., $x_n, \xi_n \to \alpha$, as $n \to \infty$

$$\lim_{n\to\infty} \left| \frac{\varepsilon_{n+1}}{\varepsilon_n^2} \right| = \frac{1}{2} \left| \frac{f''(\alpha)}{f'(\alpha)} \right| \quad \text{or,} \quad \lim_{n\to\infty} \left| \frac{\alpha - x_{n+1}}{(\alpha - x_n)^2} \right| = \frac{1}{2} \left| \frac{f''(\alpha)}{f'(\alpha)} \right|. \tag{6.7.10}$$

Thus, it is clear that Newton-Raphson iteration method is a second order iterative process. Therefore, this iterative process converges vary rapidly with constant asymptotic error equal to $\frac{1}{2} \left| \frac{f''(\alpha)}{f'(\alpha)} \right|$.

6.7.3 Geometrical Interpretation of Newton-Raphson Method

We draw the graph [Fig. 6.7] of the curve $y = f(x)$ w.r.t. ox and oy as axes. Let the tangent at $P_0[x_0, f(x_0)]$ meet the x-axis, A_1, where $oA_1 = x_1$ and the tangent at $P_1[x_1, f(x_1)]$ meet the x-axis at A_2, where $oA_2 = x_2$, etc.

Thus,

$$\begin{aligned}
P_0A_0 &= A_1A_0 \tan \angle P_0A_1A_0 = A_1A_0 f'(x_0) \\
\text{or } f(x_0) &= (x_0 - x_1) f'(x_0) \\
\text{or} \quad x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)}.
\end{aligned}$$

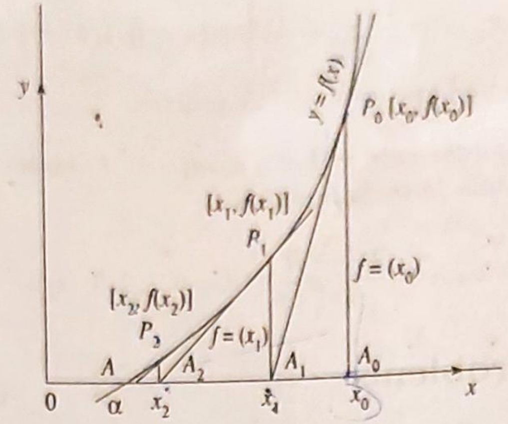

Fig. 6.7

Thus, it is clear that the successive approximations of the root, i.e., $x_1, x_2, x_3, \cdots, x_{n+1}$ are obtained respectively by the points at which the tangents at $x_0, x_1, x_2, \cdots, x_n$ to the curve $y = f(x)$ meet the x-axis.

☐ REMARK 6.7.1 *If ε_n be a tolerable error, we should terminate the iteration when*

$$|x_{n+1} - x_n| \leq \varepsilon_n.$$

☐ REMARK 6.7.2 *Newton-Raphson Method fails when, $f'(x) = 0$ or very small in the neighbourhood of the root.*

☐ REMARK 6.7.3 *If the initial approximation is very close to the root, then the convergence in Newton-Raphson Method is faster than the iteration method.*

☐ REMARK 6.7.4 *The initial approximation must be taken very close to the root, otherwise the iterations may diverge.*

6.8 Newton's Iterative Formula for Finding qth root of a Positive Real Number R

Let $x = \sqrt[q]{R} \therefore x^q - R = 0$.

Using Newton-Raphson Method to $f(x) \equiv x^q - R = 0$, we have the iteration formula as:

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^q - R}{q x_n^{q-1}} \\ &= \frac{(q-1)x_n^q + R}{q x_n^{q-1}} \quad (n = 0, 1, 2, \cdots). \end{aligned} \tag{6.8.1}$$

Number of iterations required depends upon the desired degree of accuracy.

Corollary 6.8.1 *To find the square root of a positive real number R, we substitute $q = 2$ in (6.8.1) and then the iteration formula becomes*

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{R}{x_n}\right] \quad (n = 0, 1, 2, \cdots). \tag{6.8.2}$$

This iterative procedure known as Hero's process.

Corollary 6.8.2 *To find the cube root of a positive real number R, we substitute $q = 3$ in (6.8.1) and the iteration formula transfers to*

$$x_{n+1} = \frac{2x_n^3 + R}{3x_n^2} \quad (n = 0, 1, 2, 3, \cdots). \tag{6.8.3}$$

6.9 Solved Problems

► EXAMPLE 6.9.1 *Find the root of $x^3 - 8x - 4 = 0$, which between 3 and 4, by Newton-Raphson Method, correct to four decimal places.* **[VH '89]**

SOLUTION: Let $f(x) = x^3 - 8x - 4$. $\therefore f(3) = -1 < 0$ and $f(4) = 28 > 0$

Thus, $f(x) = 0$, has a root between 3 and 4.

$$\therefore f'(x) = 3x^2 - 8 \quad \text{and} \quad f'(3) = 19.$$

We take, $x_0 = 3$ and the successive approximations are computed in the table as follows:

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n + hn$
0	3	−1.0	19.0	0.05	3.05
1	3.05	−0.027	19.9075	0.0014	3.0514
2	3.0514	0.000513	19.93312	−0.0000257	3.051374
check 3	3.051374	−0.000005	19.932649	0.00000025	3.0513742

Thus, **3.0514** is the root of the given equation, correct up to four decimal places.

► EXAMPLE 6.9.2 *Find a positive root of $x^2 + 2x - 2 = 0$, by Newton-Raphson Method, correct to two significant figures.* **[CH '81], [VH '90]**

SOLUTION: Let $f(x) = x^2 + 2x - 2$. $\therefore f(0) = -2, f(0.5) = -0.75$, $f(0.7) = -0.11, f(0.8) = 0.24$. Here $f'(x) = 2x + 2$ and $f'(0.7) = 3.4$.

Thus, $f(x) = 0$ has a root between 0.7 and 0.8.

We take, $x_0 = 0.7$. Now we compute the successive approximations in a table as follows:

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n + h_n$
0	0.7	−0.11	3.4	0.03	0.73
1	0.73	−0.0071	3.46	0.00205	0.73205
check 2	0.73205	−0.0000028	3.4641	0.0000008	0.732051

$\therefore$ **0.73** is the root of the equation, correct up to two significant figures.

► EXAMPLE 6.9.3 *Find a positive root of $x + \ln x - 2 = 0$, by Newton-Raphson Method, correct to six significant figures.*

$(1,2)\times$
$(0,1)$✓

SOLUTION: Let $f(x) = x + \ln x - 2$.
$\therefore f(1) = -1, f(1.5) = -0.09, f(2) = 0.69$.
$\therefore f(x) = 0$ has a root between 1.5 and 2.0.
Now $f'(x) = 1 + \frac{1}{x}$ and $f'(1.5) = 1.67$.
Taking $x_0 = 1.5$, the successive approximations are computed in the table as follows:

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n + h_n$
0	1.5	−0.09	1.67	0.054	1.554
1	1.554	−0.00517	1.6435	0.003146	1.557146
2	1.557146	0.0000007	1.6422	−0.0000004	1.5571453
3	1.5571453	−0.0000005	1.6422	0.0000003	1.5571456
check 4	1.5571456	0.0000000	1.6422	0.0000000	1.5571456

$\therefore$ **1.55714** is the root of the equation, correct to six significant figures.

► EXAMPLE 6.9.4 *Find by Newton-Raphson Method the real root of*

$$3x - \cos x - 1 = 0.$$ [CH '87]

SOLUTION: Let $f(x) = 3x - \cos x - 1 \therefore f(0) = -2, f(0.5) = -0.37, f(0.7) = 0.34$.
Thus, one real root of $f(x) = 0$ between 0.5 and 0.7.
Now $f'(x) = 3 + \sin x$ and $f'(0.5) = 3.48$.
Taking $x_0 = 0.5$, the successive approximations of the root are computed in the following table.

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n + h_n$
0	0.5	−0.37	3.48	0.1063	0.6063
1	0.6063	−0.00286	3.56983	0.000801	0.607101
2	0.607101	−0.00000231	3.570489	0.00000064	0.60710164
check 3	0.6071016	−0.00000017	3.570489	0.00000003	0.60710163

$\therefore$ **0.60710** is the root of $f(x) = 0$, correct up to five decimal places.

► EXAMPLE 6.9.5 *Find a real root of $x^x + x - 4 = 0$, by Newton-Raphson Method, correct to six decimal places.*

SOLUTION: Let $f(x) = x^x + x - 4$ and $f'(x) = x^x(1 + \ln x) + 1$
$\therefore f(1) = -2, f(1.5) = -0.66, f(1.6) = -0.27, f(1.7) = 0.16$
Thus, $f(x) = 0$ has a root between 1.6 and 1.7.
$\therefore f'(1.6) = 4.12$.
Taking $x_0 = 1.6$, the successive iterations are computed in the following table:

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x)}{f'(x)}$	$x_{n+1} = x_n + h_n$
0	1.6	−0.27	4.12	0.066	1.666
1	1.666	0.0065	4.5352	−0.00143	1.66457
2	1.66457	0.0000318	4.525536	−0.000007	1.664563
3	1.664563	0.0000002	4.5254887	−0.00000004	1.664563

Thus, **1.664563** is a root of $f(x) = 0$, correct up to six decimal places.

► EXAMPLE 6.9.6 *Find a positive root of $10^x + x - 4 = 0$, by Newton-Raphson Method, correct to six significant figures.*

SOLUTION: Let $f(x) = 10^x + x - 4$. $\therefore f'(x) = 10^x \times \ln 10 + 1$
$\therefore f(0) = -3, f(0.5) = -0.34, f(0.6) = 0.58$. Therefore, one root of $f(x) = 0$ between 0.5 and 0.6.

Now $f'(0.5) = 8.28$.

Taking $x_0 = 0.5$, we compute the successive approximations as follows:

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x)}{f'(x)}$	$x_{n+1} = x_n + h_n$
0	0.5	−0.34	8.28	0.04	0.54
1	0.54	0.007369	8.9839	−0.00082	0.53918
2	0.53918	0.0000079	8.968851	−0.00000088	0.5391792
3	0.5391792	−0.0000007	8.9688360	0.00000008	0.5391792

Thus, **0.539179** is root of $f(x) = 0$, correct to six significant figures.

► EXAMPLE 6.9.7 *Compute the positive root of $x^3 - x - 0.1 = 0$, by Newton-Raphson Method, correct to six significant figures.*

SOLUTION: Let $f(x) = x^3 - x - 0.1$ $\therefore f'(x) = 3x^2 - 1$.

Now $f(0) = -0.1, f(1) = -0.1, f(1.5) = 4.78, f(1.1) = 0.131$.

Thus, one root of $f(x) = 0$ between 1.0 and 1.1.

And $f'(1.1) = 2.63$.

Now taking $x_0 = 1.0$, we compute the successive iterations as follows:

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n + h_n$
0	1.0	−0.1	2.0	0.05	1.05
1	1.05	0.0076	2.3075	−0.00329	1.04671
2	1.04671	0.000067	2.28680	−0.000029	1.046681
3	1.046681	0.0000011	2.286623	−0.00000048	1.046681

$\therefore$ **1.046681** is a root of $f(x) = 0$, correct up to six significant figures.

► EXAMPLE 6.9.8 *Use Hero's process to compute* (i) $\sqrt{27}$ *and* (ii) $\sqrt{8}$, *correct up to six decimal places.*

SOLUTION: (i) Let $x = \sqrt{27}$ and $f(x) \equiv x^2 - 27 = 0$.

Substituting $R = 27$ in (6.8.2), we get the iteration formula as

$$x_{n+1} = \frac{x_n^2 + 27}{2x_n}.$$

Taking $x_0 = 5.0$, we have

n	x_n	x_{n+1}
0	5.0	5.2
1	5.2	5.196
2	5.196	5.196154
3	5.196154	5.196154

$\therefore$ $\sqrt{27}$ **= 5.196154**, correct up to six decimal places.

(ii) Let $x = \sqrt{8}$ and $f(x) \equiv x^2 - 8 = 0$.
Substituting $R = 8$ in (6.8.2), we get the iteration formula as

$$x_{n+1} = \frac{x_n^2 + 8}{2x_n}.$$

Taking $x_0 = 2.5$, we have the successive approximations as follows:

n	x_n	x_{n+1}
0	2.5	2.85
1	2.85	2.8285
2	2.8285	2.8284271
3	2.8284271	2.8284271

$\therefore$ $\sqrt{8} = \mathbf{2.828427}$, correct up to six decimal places.

EXAMPLE 6.9.9 *From the equation $x^5 - a = 0$, deduce Newtonian iterative procedure.*

$$x_{n+1} = \frac{1}{5}\left[4x_n + \frac{a}{x_n^4}\right]$$

for $\sqrt[5]{a}$. Use this method to find $\sqrt[5]{3}$. **[VH '88]**

SOLUTION: Let $x = \sqrt[5]{a}$. $\therefore x^5 = a$.
Using Newton-Raphson Method to $f(x) \equiv x^5 - a = 0$, we have the iteration formula as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^5 - a}{5x_n^4} = \frac{1}{5}\left[4x_n^5 + \frac{a}{x_n^4}\right].$$

To compute $\sqrt[5]{3}$, putting $a = 3$ in the above iteration formula, we have

$$x_{n+1} = \frac{4x_n^5 + 3}{5x_n^4}.$$

Taking $x_0 = 1$, we have successive approximations as follows:

n	x_n	x_{n+1}
0	1	1.4
1	1.4	1.276
2	1.276	1.2471
3	1.2471	1.2457
4	1.2457	1.2457301

$\therefore$ $\sqrt[5]{3} = \mathbf{1.2457}$, correct up to four decimal places.

EXAMPLE 6.9.10 *For finding the square root of 'a' ($a > 0$) derive the iteration formula*

$$x_{n+1} = \tfrac{1}{2}\left(x_n + \tfrac{a}{x_n}\right), \quad n = 0, 1, 2, 3, \cdots,$$

where $x_0 (> 0)$ is any initial approximation and x_n is the nth approxim... this formula to find $\sqrt{2}$ correct up to three significant figures.

SOLUTION: Let $x = \sqrt{a}$. $\therefore x^2 - a = 0$.

Using Newton-Raphson Method to $f(x) \equiv x^2 - a = 0$, we get the iteration formula as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}\left[\frac{x_n^2 + a}{x_n}\right]$$
$$= \frac{1}{2}\left[x_n + \frac{a}{x_n}\right], n = 0, 1, 2, \cdots.$$

To compute $\sqrt{2}$, putting $a = 2$, in the above formula, we get the interation formula as:

$$x_{n+1} = \frac{1}{2}\left[\frac{x_n^2 + 2}{x_n}\right].$$

Now taking $x_0 = 1$, we have successive approximations as follows:

n	x_n	x_{n+1}
0	1	1.5
1	1.5	1.42
2	1.42	1.414
3	1.414	1.4142

$\therefore$ **$\sqrt{2} = 1.41$,** correct up to three significant figures.

6.10 Regula-Falsi Method or Method of False Position

In this method we first find a sufficiently small interval $[a_0, b_0]$, such that $f(a_0)f(b_0) < 0$, by tabulation or graphical method, and which contains only one root α (say) of $f(x) = 0$, i.e., $f'(x)$ maintains same sign in $[a_0, b_0]$.

This method is based on the assumption that the graph of $y = f(x)$ in small interval $[a_0, b_0]$ can be represented by the chord joining $(a_0, f(a_0))$ and $(b_0, f(b_0))$. Then, the point $x = a_1 = a_0 + h_0$ at which the chord meets the x-axis gives us an approxe value of the root α of the equation $f(x) = 0$. Thus, we obtain two intervals [and $[x_1, b_0]$, one of which must contain the root α, depending upon the cond $f(a_0)f(x_1) < 0$ or $f(x_1) \cdot f(b_0) < 0$. If $f(x_1)f(b_0) < 0$, then α lies in the in $[x_1, b_0]$ which we re-name as $[a_1, b_1]$. Again we consider that the graph of $y = f$ $[a_1, b_1]$ as the chord joining $(a_1, f(a_1))$ and $(b_1, f(b_1))$, thus, the point of inters of the chord with the x-axis (say) $x_2 = a_1 + h_1$ gives us an approximate value root α and x_2 is called the *second approximation* of the root α.

Proceeding in this way, we shall get a sequence $\{x_1, x_2, x_3, \cdots, x_n\}$. each m of which is the successive aproximation of the exact root α of the equation $f(x)$

Now we are going to establish an iteration formula which may generate a seq uccessive approximations of an exact root α of the equation $f(x) = 0$, geometri e adjoining Fig. 6.8, we assume that one real root α of $f(x) = 0$ lies i $[a_n, b_n]$ and $f(a_n) < 0$ and $f(b_n) > 0$. Let PRQ be the graph of $y =$

$\therefore$ **$\sqrt{27} = 5.196154$,** co the x-axis at R.