Error Analysis

Rounding-Off Error

↳ 24.564986 ⟶ 24.5650

Absolute Error

↳ Difference between true value and approximate value

Relative Error

↳ Ratio of absolute error and true ~~error~~ value

Relative Percentage Error

↳ Relative Error × 100

Q. Write down the approximate representation of $\frac{2}{3}$ correct to 4th significant ~~significa~~ figure and then find (i) absolute error (ii) Relative error (iii) Relative % error

→ Approximate value: 0.6667

Absolute error = True value − Approximate value

$$= \left|\frac{2}{3} - 0.6667\right| = +3.333333333 \times 10^{-5}$$

$$\text{Relative Error} = \frac{\text{Absolute Error}}{\text{True Value}} = +5 \times 10^{-5}$$

$$\text{Relative \% Error} = 5 \times 10^{-5} \times 100$$

$$= 5 \times 10^{-3} = 0.005$$

Q. write down the approx value of $\pi/4$ and find the

(i) absolute error (ii) relative error (iii) relative % error

→ Approximate value = ~~1.83~~ 0.7854

Absolute error $= \left| -1.83660255 \times 10^{-6} \right| = 1.83660255 \times 10^{-6}$

Relative error $= +\,2.338434995 \times 10^{-6}$

Relative % error $= +\,2.338434995 \times 10^{-4}$

(#) Method of Bisection

↳ Let $f(x)$ be a continuous function in $[a,b]$ such that $f(a)f(b) < 0$, then one of the roots will lie between a and b

Q. Find the positive roots of the eq^n $x^3 - 3x + 1.06 = 0$ by method of bisection by upto 3 decimal places

→ $f(x) = x^3 - 3x + 1.06$

$f(0) = 1.06 > 0$

$f(1) = -0.94 < 0$

$f(2) = 3.06 > 0$

Hence, we can say that two of the roots lie between $x = (0,1)$ and $x = (1,2)$ respectively

For $x = (0,1)$

a_n (+ve)	b_n (−ve)	$x_{n+1} = \frac{a_n + b_n}{2}$	$f(x_{n+1})$
0	1	0.5	−0.315
0	0.5	0.25	0.325
0.25	0.5	0.375	~~0.0~~ −0.0122
0.25	0.375	0.3125	0.1530
0.3125	0.375	0.34375	0.069
0.34375	0.375	0.359375	0.029 ~~4.64×10^{-13}~~
0.359	0.375	0.367	0.008
0.367	0.375	0.371	−0.002
0.367	0.371	0.369	0.003
0.369	0.371	0.370	0.0006

a_n (+ve)	b_n (−ve)	$x_{n+1} = \frac{a_n + b_n}{2}$	$f(x_{n+1})$
0.370	0.371	0.3705	+0.0006
0.370	0.3705	0.37025	−0.000006
0.37025	0.3705	0.370375	−0.0003
0.37025	0.370375	0.370312	

Hence, the value of root is 0.3703

Q. Solve the equation $x^3 - 9x + 1 = 0$ which is lying between 2 and 3 correct to three significant figure

→ $f(x) = x^3 - 9x + 1$

a_n (−ve)	b_n (+ve)	$x_{n+1} = \frac{a_n + b_n}{2}$	$f(x_{n+1})$
2	3	1.5	−9.125
1.5	3	2.25	−7.8593
2.25	3	2.625	−4.5371
2.625	3	2.8125 ~~2.84375~~	−2.065185547
2.8125	3	2.90625	−0.6092
2.90625		2.953125	0.175922
2.90625	2.953125	2.92968	~~5.635×10^{-17}~~ −0.22160
2.92968	2.953125 ~~2.92968~~	2.9414025	−0.02405
2.9414025	2.953125	2.94726375	0.0756
2.9414025	2.94726375	2.944333125	0.02571
2.9414025	2.944333125	2.942867813	0.008
2.9414025	2.942867813	2.942135157	−0.0116
2.942135157	2.942867813	2.942501485	

09/01/2023

Newton Rapson Method

Q. Find the root of $x^3 - 8x - 4 = 0$ which is between 3 and 4 by Newton Rapson Method correct upto 3 decimal places

$\Rightarrow$ $f(x) = x^3 - 8x - 4$

$f(3) = -1$

$f(4) = 28$

$f'(x) = 3x^2 - 8$

$f'(3) = 19$

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n + h_n$
0	3	-1	19	0.05	3.05
1	3.05	-0.027	19.9075	0.0014	3.0514
2	3.0514	0.000513	19.9331	-0.0000257	3.051374
3	3.051374	-0.000005	19.932649	0.00000025	3.0513742

Hence, the value of the root is 3.051

Q. Find the positive root of $x^2 + 2x - 2 = 0$ by Newton Rapson Method correct upto 2 decimal figure

$\rightarrow$ $f(0) = -2$ $\quad$ $f(x) = x^2 + 2x - 2$

$f(1) = 1$ $\quad$ $f'(x) = 2x + 2$

Hence, root is between 0 and 1

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = \frac{-f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n + h_n$
0	0	-2	2	1	1
1	1	1	3	$-1/3$	$2/3$
2	0.6667	-0.2221	3.3334	0.0666	0.7333
3	0.7333	0.00432	3.4666	-0.001246	0.732054
4	0.732054	0.0000105[illegible]	3.464108	0.000003057	0.732057

Regular Falsi Method

(H) Regular Falsi Method

Q. Compute the roots of eq^n $2x - \log_{10} x - 7$ by Regular Falsi method, which lie between 3 and 4 upto 3 decimal places

→

n	a_n(-ve)	b_n(+ve)	$f(a_n)$	$f(b_n)$	$h_n = \frac{\lvert f(a_n)\rvert \lvert b_n - a_n\rvert}{\lvert f(a_n)\rvert + \lvert f(b_n)\rvert}$	$x_{n+1} = a_n + h_n$	$f(x_{n+1})$
0	3	4	-1.48	0.40	0.79	3.79	0.0014
1	3	3.79	-1.48	0.0014	0.789	3.789	-0.00052
2	3.789	3.79	-0.00052	0.0014	0.000271	3.789271	-0.00000014
3	3.789271	3.79	-0.00000014	0.0014	0.0000007	3.7892717	-

Hence, value of root = 3.789

Q. Find the root of the equation $3x - \cos x - 1 = 0$ by Regular falsi method correct to four significant figure.

→ $f(0) = -2$

$f(1) = 1.0001$

Hence root lies between 0 and 1

n	a_n(-ve)	b_n(+ve)	$f(a_n)$	$f(b_n)$	$h_n = \frac{\lvert f(a_n)\rvert \lvert b_n - a_n\rvert}{\lvert f(a_n)\rvert + \lvert f(b_n)\rvert}$	$x_{n+1} = a_n + h_n$	$f(x_{n+1})$
0	0	1	-2	1	+0.6667	0.6667	1.6×10^{-4}
1	0	0.6667	-2	1.6×10^{-4}	0.66664	0.6664	-1.23×10^{-5}
2	0.6664	0.6667	-1.23×10^{-5}	1.6×10^{-4}	-2.5012×10^{-5}	0.666374	

13/01 class missed

(A) Langrange's Interpolation Formula

$$\hookrightarrow y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}\,y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}\,y_1 + \cdots +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}\,y_n$$

Q. Find the value of f(3.2) by Langrange's Interpolation method

x	0	1	2	3	4	5	6
$f(x)$	2	4	10	16	20	24	38

$\rightarrow$ $\cancel{y = f(x) = (x-1)(x-2)(x-3)\cdots(x-6)\ 2}$

$$\cancel{y = f(x) = \frac{(x-2)(x-3)(x-4)(x-5)(x-6)}{(1-2)}}$$

$$\rightarrow \cancel{y = f(3.2) = \frac{(3.2-1)(3.2-2)(3.2-3)(3.2-4)(3.2-5)(3.2-6)}{(2-1)(2-2)}}$$

$$\rightarrow y = f(3.2) = \frac{(3.2-2)(3.2-3)(3.2-4)(3.2-5)(3.2-6)}{(1-2)(1-3)(1-4)(1-5)(1-6)}(4)$$

$$+ \frac{(3.2-1)(3.2-3)(3.2-4)(3.2-5)(3.2-6)}{(2-1)(2-3)(2-4)(2-5)(2-6)}(10)$$

$$+ \cdots +$$

$$+ \frac{(3.2-1)(3.2-2)(3.2-3)(3.2-4)(3.2-5)}{(6-1)(6-2)(6-3)(6-4)(6-5)}(38)$$

$= 0.032256 - 0.7392 + 17.7408 + 4.4352 - 1.18272 + 0.240768 = 20.527104$

17/01/2023

Hermite Interpolation Formula

$\rightarrow H(x_i) = f(x_i)$

$H'(x_i) = f'(x_i), \ (i = 0, 1, 2, \dots, n)$

$$H(x) = \sum_{i=0}^{n} A_i(x) f(x_i) + \sum_{i=0}^{n} B_i(x) f'(x_i)$$

$$A_i(x) = [1 - 2(x - x_i)\, l_i'(x_i)]\, l_i^2(x)$$

$$B_i(x) = (x - x_i)\, l_i^2(x)$$

$$l_i(x) = \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

Q. Given the following value of x, $f(x)$ and $f'(x)$, find the value of $f(-0.5)$ and $f(0.5)$ by Hermite Interpolation method

x	$f(x)$	$f'(x)$
-1	1	-5
0	1	1
1	3	7

$\rightarrow$ $\cancel{l_0(x)}$ $l_0 = \dfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \dfrac{(x-0)(x-1)}{(-1-0)(-1-1)} = \dfrac{x^2-x}{2}$

$l_0' = \dfrac{2x-1}{2} \Rightarrow l_0'(-1) = \dfrac{-3}{2}$

$$A_0(-1) = \left[1 - 2(0.5+1)\cdot\left(\frac{-3}{2}\right)\right]\left[\frac{0.5^2-0.5}{2}\right]^2$$

$$= [1+4.5]\left[\frac{0 - 0.25}{2}\right]^2$$

$$= (5.5)\left(\frac{1}{64}\right) = \frac{11}{128}$$

$l_1(x) = 0$

$l_2(x) = \frac{3}{2}$

And

$A_0(x) = \frac{1}{4}(3x^5 - 2x^4 - 5x^3 + 4x^2)$

$A_1(x) = x^4 - 2x^2 + 1$

$A_2(x) = \frac{1}{4}(-3x^5 - 2x^4 + 5x^3 + 4x^2)$

$B_0(x) = \frac{1}{4}(x^5 - x^4 - x^3 + x^2)$

$B_1(x) = x^5 - 2x^3 + x$

$B_2(x) = \frac{1}{4}(x^5 + x^4 - x^3 - x^2)$

Hence, $H(x) = A_0 f(0) + A_1 f(1) + A_2 f(2) +$
$B_0 f'(0) + B_1 f'(1) + B_2 f'(2)$
$= 2x^4 - x^2 + x + 1$

At $x = -0.5$, $H(-0.5) = -33/64$

At $x = 0.5$, $H(0.5) = 97/64$

20/01/2023

Numerical Differentiation

(1) Newton Forward and Backward formulae

↳ Newton Forward formulae

$$f'(x_0) = \frac{1}{h}\left[\Delta f(x_0) - \frac{1}{2}\Delta^2 f(x_0) + \frac{1}{3}\Delta^3 f(x_0) - \dots\right]$$

$$f''(x_0) = \frac{1}{h^2}\left[\Delta^2 f(x_0) - \Delta^3 f(x_0) + \frac{11}{12}\Delta^4 f(x_0) - \frac{5}{6}\Delta^5 f(x_0) \dots\right]$$

$$f'''(x_0) = \frac{1}{h^3}\left[\Delta^3 f(x_0) - \frac{3}{2}\Delta^4 f(x_0) + \frac{7}{4}\Delta^5 f(x_0) + \dots\right]$$

where $h \rightarrow$ interval

Newton Backward formulae

$$f'(x_n) = \frac{1}{h}\left[\Delta f(x_{n-1}) + \frac{1}{2}\Delta^2 f(x_{n-2}) + \frac{1}{3}\Delta^3 f(x_{n-3}) + \dots\right]$$

$$f''(x_n) = \frac{1}{h^2}\left[\Delta^2 f(x_{n-2}) + \Delta^3 f(x_{n-3}) + \frac{11}{12}\Delta^4 f(x_{n-1}) - \frac{5}{6}\Delta^5 f(x_{n-1})\right]$$

$$f'''(x_n) = \frac{1}{h^3}\left[\Delta^3 f(x_{n-3}) + \frac{3}{2}\Delta^4 f(x_{n-4}) + \frac{7}{4}\Delta^5 f(x_{n-5}) + \dots\right]$$

Q. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x=1$ and 6 for the function $y=f(x)$

x	y	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
1	2.7183	0.6027	0.1315	0.0321	0.0031	0.0058
2	3.3216	0.7342	0.1636	0.0352	0.0089	
3	4.0552	0.8978	0.1988	0.0441		
4	4.9530	1.0966	0.2429			
5	6.0496	1.3395				
6	7.3891					

For $x = 1$

$$f'(x) = \frac{1}{1}\left[0.6027 - \frac{1}{2}(0.2315) + \frac{1}{3}(0.0321) - \frac{1}{4}\times(0.0031) + \frac{1}{5}(0.0058)\right]$$

$$= 0.548035$$

$$f''(x) = \frac{1}{1^2}\left[0.1315 - 0.0321 + \frac{11}{12}(0.0031) - \frac{5}{6}(0.0058)\right]$$

~~For $x = 2$~~

$$= 0.0974083$$

For $x = 6$

$$f'(x) = \frac{1}{1}\left[(1.3395) - \frac{1}{2}(0.2429) + \frac{1}{3}(0.0441) - \frac{1}{4}(0.0089) + \frac{1}{5}(0.0058)\right]$$

$$= 1.231685$$

$$f''(x) = \frac{1}{1^2}\left[0.2429 - 0.0441 + \frac{11}{12}(0.0089) - \frac{5}{6}(0.0058)\right]$$

$$= 0.202125$$

23/01/2023

Gauss Elimination Method

⊕ Elimination Method

↳ $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_3 + a_{33}x_3 = b_3$

→ $a_{11}x_1 + a_{12}x_2 + a_{13} = b_1$

→ $\left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right)x_2 + \left(a_{23} - \frac{a_{13}a_{21}}{a_{11}}\right)x_3 = \left(b_2 - \frac{b_1 a_{21}}{a_{11}}\right)$

→ $\left(a_{32} - \frac{a_{12}a_{31}}{a_{11}}\right)x_2 + \left(a_{33} - \frac{a_{13}a_{31}}{a_{11}}\right)x_3 = \left(b_3 - \frac{b_1 a_{31}}{a_{11}}\right)$

Q. Solve the equation $2x_1 + 3x_2 + x_3 = 9$, ~~and~~

$x_1 + 2x_2 + 3x_3 = 6$

$3x_1 + x_2 + 2x_3 = 8$ upto 3 significant figure

⇒ $a_{11} = 2$ | $a_{12} = 3$ | $a_{13} = 1$ | $b_1 = 9$

$a_{21} = 1$ | $a_{22} = 2$ | $a_{23} = 3$ | $b_2 = 6$

$a_{31} = 3$ | $a_{32} = 1$ | $a_{33} = 2$ | $b_3 = 8$

$\left(2 - \frac{(3)(1)}{2}\right)x_2 + \left(3 - \frac{(1)(1)}{2}\right)x_3 = \left(6 - \frac{9(1)}{2}\right)$

⇒ $\frac{x_2}{2} + \frac{5x_3}{2} = \frac{3}{2}$ ⇒ $x_2 + 5x_3 = 3$

$\left(1 - \frac{3(3)}{2}\right)x_2 + \left(2 - \frac{(1)(3)}{2}\right)x_3 = 8 - \frac{9(3)}{2}$

⇒ $-\frac{7x_2}{2} + \frac{x_3}{2} = -5.5$ ⇒ $7x_2 - x_3 = 11$

→ $x_2 = \frac{-203 \times 2}{36 \times 7} = 1.61$

$x_3 = \frac{5}{18} = 0.278$

⇒ $x_1 = \frac{9 - x_3 - 3x_2}{2} = 1.946$

24/01/2023

Gauss-Siedal Method

$$x_1 + x_2 + 4x_3 = 9 \quad \text{—— (iii)}$$
$$8x_1 - 3x_2 + 2x_3 = 20 \quad \text{—— (i)}$$
$$4x_1 + 11x_2 - x_3 = 33 \quad \text{—— (ii)}$$

$$x_1 = \frac{20 + 3x_2 - 2x_3}{8} \quad \text{—— (iv)}$$

$$x_2 = \frac{33 - 4x_1 + x_3}{11} \quad \text{—— (v)}$$

$$x_3 = \frac{9 - x_1 - x_2}{4} \quad \text{—— (vi)}$$

From (iv), putting $x_2 = x_3 = 0$, $x_1 = 2.5$

Putting $x_1 = 2.5$ and $x_3 = 0$, we get $x_2 = 2.0909$

Putting $x_1 = 2.5$ and $x_2 = 2.0909$, we get $x_3 = 1.1023$

~~Finally putting~~

Now, putting $x_2 = 2.0909$ and $x_3 = 1.1023$ in (iv), $x_1 = 3.0085$

Putting $x_1 = 3.0085$ and $x_3 = 1.1023$ in (v), we get $x_2 = 2.0062$

Putting $x_2 = 2.0062$ and $x_1 = 3.0085$, we get $x_3 = 0.9963$

Now, $x_2 = 2.0062$ and $x_3 = 0.9963$ in (iv), $x_1 = 3.0032 \approx 3.00$

$x_1 = 3.0032$ and $x_3 = 0.9963$ in (v), $x_2 = 1.9985 \approx 2.00$

$x_1 = 3.0032$ and $x_2 = 1.9985$ in (vi), $x_3 = 0.9963 \approx 1.00$

Hence, $x_1 \rightarrow 3.00$

$x_2 \rightarrow 2.00$

$x_3 \rightarrow 1.00$