3

Interpolation

3.1. INTRODUCTION

Interpolation means to find (approximate) values of a function $y = f(x)$ for an x between **different** x-values $x_0, x_1, ..., x_n$ at which the values of $f(x)$ are given.

So the given values $y_0 = f(x_0),\ y_1 = f(x_1), ..., y_n = f(x_n)$ may come from a "*mathematical function*" given by a formula (which we may wish to approximate by a simple function) or from an "*emperical function*" resulting from observations or experiments.

A standard idea in interpolation now is to find a polynomial $p_n(x)$ of degree n(or less) that assumes the given values ; thus

$$p_n(x_0) = y_0,\ p_n(x_1) = y_1, ..., p_n(x_n) = y_n \qquad ...(1)$$

We call this p_n an **interpolation polynomial** and $x_0, x_1, ... x_n$ the **nodes.** And if $f(x)$ is a mathematical function, we call p_n an **approximation** of f(or a polynomial approximation). We use p_n to get (approximate) values of f for x's between x_0 and x_n ("**interpolation**") or sometimes outside the given interval ("**extrapolation**"). Also p_n satisfying (1) for given data exists and is unique.

Linear interpolation is interpolation by the straight line through (x_0, y_0), (x_1, y_1). Fig. 3.1 illustrates this. The same idea has been used in general Lagrange interpolation polynomial of degree n(or less).

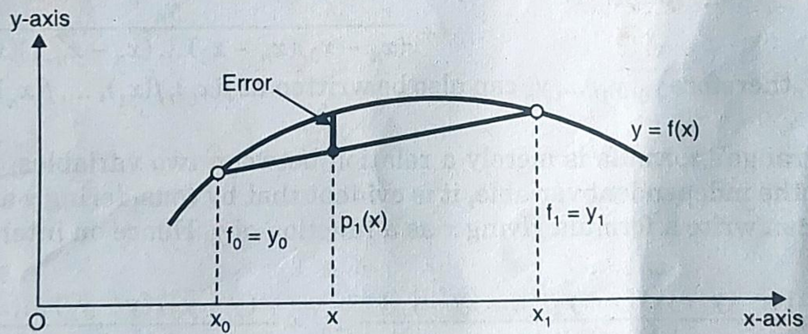

Fig. 3.1. Linear interpolation.

3.2. LAGRANGE'S INTERPOLATION FORMULA

Let $y = f(x)$ be a function which takes the values $(x_0, y_0), (x_1, y_1), ..., (x_n, y_n)$. Since there are $(n + 1)$ pairs of values of x and y, we can represent $f(x)$ by a polynomial in x of degree n. Let this polynomial be of the form.

$$y = f(x) = a_0(x - x_1)(x - x_2) ...... (x - x_n)$$
$$+ a_1(x - x_0)(x - x_2) ...... (x - x_n) + ... + a_n(x - x_0)(x - x_1) ...... (x - x_{n-1}) \qquad ...(1)$$

where $a_0, a_1, ..., a_n$ are the constants to be found.

Putting $\quad x = x_0$ and $y = y_0$ in (1), we get

$$y_0 = a_0(x_0 - x_1)(x_0 - x_2) \ldots\ldots (x_0 - x_n)$$

$$\therefore \quad a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2) \ldots\ldots (x_0 - x_n)}$$

Similarly putting $x = x_1$ and $y = y_1$ in (1), we get

$$y_1 = a_1(x_1 - x_0)(x_1 - x_2) \ldots\ldots (x_1 - x_n)$$

$$\therefore \quad a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \ldots\ldots (x_1 - x_n)}$$

Proceeding in the same manner, we find $a_2, a_3, \ldots\ldots, a_n$

Now substituting the values of $a_0, a_1, \ldots\ldots, a_n$ in (1), we get

$$y = f(x) = p_n(x) = \frac{(x - x_1)(x - x_2) \ldots\ldots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \ldots\ldots (x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2) \ldots\ldots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \ldots\ldots (x_1 - x_n)} y_1$$

$$+ \ldots\ldots + \frac{(x - x_0)(x - x_1) \ldots\ldots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \ldots\ldots (x_n - x_{n-1})} y_n \quad \ldots(2)$$

This is known as **Lagrange's interpolation formula for unequal intervals** and it gives $y = y_0, y_1, \ldots\ldots, y_n$ when $x = x_0, x_1, \ldots\ldots, x_n$ respectively. The values of the independent variable may or may not be equidistant.

The formula given by (2) can also be written in the following form :

$$\frac{y}{(x - x_0)(x - x_1) \ldots (x - x_n)} = \frac{y_0}{(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)(x - x_0)}$$

$$+ \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \ldots (x_1 - x_n)(x - x_1)} + \ldots$$

$$+ \frac{y_n}{(x_n - x_0)(x_n - x_1) \ldots (x_n - x_{n-1})(x - x_n)} \quad \ldots(3)$$

As $y = f(x)$, therefore $y_0, y_1, \ldots, y_n$ can also be written as $f(x_0), f(x_1), \ldots, f(x_n)$ respectively in (2) and (3).

Since Lagrange's formula is merely a relation between two variables, either of which may be taken as the independent variable, it is evident that by considering y as the independent variable we can write a formula giving x as a function of y. Hence on interchanging x and y in (2), we get

$$x = \frac{(y - y_1)(y - y_2) \ldots\ldots (y - y_n)}{(y_0 - y_1)(y_0 - y_2) \ldots\ldots (y_0 - y_n)} x_0 + \frac{(y - y_0)(y - y_2) \ldots\ldots (y - y_n)}{(y_1 - y_0)(y_1 - y_2) \ldots\ldots (y_1 - y_n)} x_1$$

$$+ \ldots\ldots\ldots\ldots + \frac{(y - y_0)(y - y_1) \ldots\ldots (y - y_{n-1})}{(y_n - y_0)(y_n - y_1) \ldots\ldots (y_n - y_{n-1})} x_n \quad \ldots(4)$$

This is Lagrange's formula for Inverse Interpolation. Therefore, the process of estimating the value of x for a value of y is called Inverse Interpolation.

WORKING RULES FOR SOLVING PROBLEMS

Step I. Denote the given function as $y = f(x)$.

Step II. From the given data or table, identify the values $y_0, y_1, y_2, \ldots, y_n$ and corresponding values $x_0, x_1, x_2, \ldots, x_n$.

Step III. Substitute the value of x at which $f(x)$ is to be evaluated and values identified in step II in Lagrange's formula,

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}y_1 + \dots$$

$$+ \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}y_n$$

to get the value of $f(x)$ at the specified point.

Example 1. *From the following table estimate f(3.2) using only five of the given values.*

x	0	1	2	3	4	5	6
$f(x)$	2	4	10	16	20	24	38

Sol. Taking the following five values from the table

$$x_0 = 1, \quad x_1 = 2, \quad x_2 = 3, \quad x_3 = 4, \quad x_4 = 5$$
$$y_0 = 4, \quad y_1 = 10, \quad y_2 = 16, \quad y_3 = 20, \quad y_4 = 24$$

Using Lagrange's Interpolation Formula, we get

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}y_3$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}y_4$$

Substituting values, we get

$$f(3.2) = \frac{(3.2-2)(3.2-3)(3.2-4)(3.2-5)}{(1-2)(1-3)(1-4)(1-5)}4 + \frac{(3.2-1)(3.2-3)(3.2-4)(3.2-5)}{(2-1)(2-3)(2-4)(2-5)}10$$

$$+ \frac{(3.2-1)(3.2-2)(3.2-4)(3.2-5)}{(3-1)(3-2)(3-4)(3-5)}16 + \frac{(3.2-1)(3.2-2)(3.2-3)(3.2-5)}{(4-1)(4-2)(4-3)(4-5)}20$$

$$+ \frac{(3.2-1)(3.2-2)(3.2-3)(3.2-4)}{(5-1)(5-2)(5-3)(5-4)}24$$

$$= \frac{(1.2)(0.2)(-0.8)(-1.8)}{(-1)(-2)(-3)(-4)}4 + \frac{(2.2)(0.2)(-0.8)(-1.8)}{(1)(-1)(-2)(-3)}10$$

$$+ \frac{(2.2)(1.2)(-0.8)(-1.8)}{(2)(1)(-1)(-2)}16 + \frac{(2.2)(1.2)(0.2)(-1.8)}{(3)(2)(1)(-1)}20$$

$$+ \frac{(2.2)(1.2)(0.2)(-0.8)}{(4)(3)(2)(1)}24$$

$$= 0.0576 - 1.056 + 15.2064 + 3.168 - 0.4224$$
$$= 16.9536.$$

Example 2. *For the following table of values :*

x	1	2	3	4
$f(x)$	1	8	27	64

Find f(3.5) using a Lagrange Interpolation with a quadratic interpolating polynomial.

Sol. For obtaining a quadratic polynomial we shall use only the following three values from the given table

$$x_1 = 2, x_2 = 3, x_3 = 4, y_1 = 8, y_2 = 27, y_3 = 64$$

Using Lagrange's Interpolation formula we can write

$$y = f(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} y_1 + \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} y_3$$

or
$$f(x) = \frac{(x-3)(x-4)}{(2-3)(2-4)} \times 8 + \frac{(x-2)(x-4)}{(3-2)(3-4)} \times 27 + \frac{(x-2)(x-3)}{(4-2)(4-3)} \times 64$$

$$= \frac{8(x^2 - 7x + 12)}{2} + \frac{27(x^2 - 6x + 8)}{-1} + \frac{64(x^2 - 5x + 6)}{2}$$

$$= 4(x^2 - 7x + 12) - 27(x^2 - 6x + 8) + 32(x^2 - 5x + 6)$$

$$= 4x^2 - 28x + 48 - 27x^2 + 162x - 216 + 32x^2 - 160x + 192$$

$$\therefore \quad f(x) = 9x^2 - 26x + 24$$

$$\therefore \quad f(3.5) = 9(3.5)^2 - 26(3.5) + 24$$

$$= 110.25 - 91 + 24 = 43.25.$$

Example 3. *Use Lagrange's Interpolation Formula to find the value of y when x = 10, if the following values of x and y are given :*

x	5	6	9	11
y	12	13	14	16

Sol. Here $x_0 = 5, x_1 = 6, x_2 = 9, x_3 = 11$ and $y_0 = 12, y_1 = 13, y_2 = 14, y_3 = 16$

According to Lagrange's Interpolation Formula for unequal intervals

$$f(x) = \frac{(x - x_1)(x - x_2)\dots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2)\dots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\dots(x_1 - x_n)} y_1 + \dots$$

$$\dots\dots + \frac{(x - x_0)(x - x_1)\dots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\dots(x_n - x_{n-1})} y_n \quad \dots(1)$$

Substituting the values of $x_1, x_2, \ldots$ and taking $x = 10$, we have

$$f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$= 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3} = 2 - 4.33 + 11.667 + 5.33$$

$$f(10) = 14.67.$$

Example 4. *The following table gives the viscosity of an oil as a function of temperature. Use Lagrange's formula to find viscosity of oil at a temperature of 140°.*

Temp(°)	*110*	*130*	*160*	*190*
Viscosity	*10.8*	*8.1*	*5.5*	*4.8*

Sol. Here $x_0 = 110, \quad x_1 = 130, \quad x_2 = 160, \quad x_3 = 190$

and $y_0 = 10.8, \quad y_1 = 8.1, \quad y_2 = 5.5, \quad y_3 = 4.8$

Putting $x = 140$ and substituting the above values in Lagrange's formula, we get

$$f(140) = \frac{(140-130)(140-160)(140-190)}{(110-130)(110-160)(110-190)} \times 10.8$$

$$+ \frac{(140-110)(140-160)(140-190)}{(130-110)(130-160)(130-190)} \times 8.1 + \frac{(140-110)(140-130)(140-190)}{(160-110)(160-130)(160-190)} \times 5.5$$

$$+ \frac{(140-110)(140-130)(140-160)}{(190-110)(190-130)(190-160)} \times 4.8$$

$$= \frac{(10)(-20)(-50)}{(-20)(-50)(-80)} \times 10.8 + \frac{(30)(-20)(-50)}{(20)(-30)(-60)} \times 8.1$$

$$+ \frac{(30)(10)(-50)}{(50)(30)(-30)} \times 5.5 + \frac{(30)(10)(-20)}{(80)(60)(30)} \times 4.8$$

$$= -1.35 + 6.75 + 1.8333 - 0.2000$$

$$f(140) = 7.03.$$

Example 5. *Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$. Find $\log_{10} 656$ using Lagrange's Formula.*

Sol. Here $x_0 = 654, \quad x_1 = 658, \quad x_2 = 659, \quad x_3 = 661$

and $y_0 = 2.8156, \quad y_1 = 2.8182, \quad y_2 = 2.8189, \quad y_3 = 2.8202$

Putting $x = 656$ and substituting the above values in Lagrange's Formula

$$f(656) = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \times 2.8156$$

$$+ \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} \times 2.8182$$

$$+ \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} \times 2.8189$$

$$+ \frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)} \times 2.8202$$

$$= \frac{(-2)(-3)(-5)}{(-4)(-5)(-7)} \times 2.8156 + \frac{2 \times (-3)(-5)}{4 \times (-1)(-3)} \times 2.8182 + \frac{2 \times (-2) \times (-5)}{5 \times 1 \times (-2)} \times 2.8189 + \frac{2 \times (-2) \times (-3)}{7 \times 3 \times 2} \times 2.8202$$

$$= 0.6033 + 7.0455 - 5.6378 + 0.8058 = 2.8168.$$

Example 6. *Given $y_0 = -12, y_1 = 0, y_3 = 6, y_4 = 12$. Find y_2 using Lagrange's Formula.*

Sol. Here $x_0 = 0,\quad x_1 = 1,\quad x_3 = 3,\quad x_4 = 4$

and $y_0 = -12,\quad y_1 = 0,\quad y_3 = 6,\quad y_4 = 12$

Putting $x = 2$ and substituting these values in Lagrange's formula, we get

$$f(2) = \frac{(2-1)(2-3)(2-4)}{(0-1)(0-3)(0-4)} \times (-12) + \frac{(2-0)(2-3)(2-4)}{(1-0)(1-3)(1-4)} \times 0 + \frac{(2-0)(2-1)(2-4)}{(3-0)(3-1)(3-4)} \times 6 + \frac{(2-0)(2-1)(2-3)}{(4-0)(4-1)(4-3)} \times 12$$

$$= \frac{(1)(-1)(-2)}{(-1)(-3)(-4)} \times (-12) + \frac{(2)(-1)(-2)}{(1)(-2)(-3)} \times 0 + \frac{(2)(1)(-2)}{(3)(2)(-1)} \times 6 + \frac{(2)(1)(-1)}{(4)(3)(1)} \times 12$$

$$= 2 + 0 + 4 - 2$$

$$f(2) = 4$$

$$\therefore\ y_2 = 4.$$

Example 7. *Use Lagrange's Formula to find the form of $f(x)$, given that*

x	0	2	3	6
$f(x)$	648	704	729	792

Sol. Here $x_0 = 0,\quad x_1 = 2,\quad x_2 = 3,\quad x_3 = 6$

and $y_0 = 648,\quad y_1 = 704,\quad y_2 = 729,\quad y_3 = 792$

Lagrange's Formula is

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} y_1 + \dots \dots\dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} y_n \quad \dots(1)$$

Substituting the table values in (1), we have

$$f(x) = \frac{(x-2)(x-3)(x-6)}{(0-2)(0-3)(0-6)} \times 648 + \frac{(x-0)(x-3)(x-6)}{(2-0)(2-3)(2-6)} \times 704 + \frac{(x-0)(x-2)(x-6)}{(3-0)(3-2)(3-6)} \times 729 + \frac{(x-0)(x-2)(x-3)}{(6-0)(6-2)(6-3)} \times 792$$

$$\Rightarrow \quad f(x) = \frac{(x-2)(x-3)(x-6)}{(-2)(-3)(-6)} \times 648 + \frac{(x)(x-3)(x-6)}{(2)(-1)(-4)} \times 704 + \frac{(x)(x-2)(x-6)}{(3)(1)(-3)} \times 729 + \frac{(x)(x-2)(x-3)}{(6)(4)(3)} \times 79$$

$$f(x) = -18\,[(x-2)(x-3)(x-6)] + 88\,[(x)(x-3)(x-6)] - 81\,[(x)(x-2)(x-6)] + 11\,[(x)(x-2)(x-$$

$$f(x) = -18\,[x^3 - 11x^2 + 36x - 36] + 88\,[x^3 - 9x^2 + 18x] - 81\,[x^3 - 8x^2 + 12x] + 11\,[x^3 - 5x^2 +$$

$\therefore \qquad f(x) = -x^2 + 30x + 648.$

Example 8. *By means of Lagrange's formula prove that*

(i) $y_1 = y_3 - 0.3[y_5 - y_{-3}] + 0.2[y_{-3} - y_{-5}]$

(ii) $y_0 = \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right].$

Sol. *(i)* y_{-5}, y_{-3}, y_3 and y_5 are given, therefore the values of the arguments are $-5, -3, 3, 5$, y_1 is to be obtained. By Lagrange's formula.

$$y_x = \frac{[x-(-3)](x-3)(x-5)}{[-5-(-3)](-5-3)(-5-5)} y_{-5} + \frac{[x-(-5)](x-3)(x-5)}{[-3-(-5)](-3-3)(-3-5)} y_{-3}$$

$$+ \frac{[x-(-5)](x-(-3))(x-5)}{[3-(-5)][3-(-3)](3-5)} y_3 + \frac{[x-(-5)][x-(-3)](x-3)}{[5-(-5)][5-(-3)](5-3)} y_5$$

Taking $x = 1$, we have

$$y_1 = \frac{(1+3)(1-3)(1-5)}{(-5+3)(-5-3)(-5-5)} y_{-5} + \frac{(1+5)(1-3)(1-5)}{(-3+5)(-3-3)(-3-5)} y_{-3}$$

$$+ \frac{(1+5)(1+3)(1-5)}{(3+5)(3+3)(3-5)} y_3 + \frac{(1+5)(1+3)(1-3)}{(5+5)(5+3)(5-3)} y_5$$

$$\Rightarrow \qquad y_1 = \frac{(4)(-2)(-4)}{(-2)(-8)(-10)} y_{-5} + \frac{(6)(-2)(-4)}{(2)(-6)(-8)} y_{-3} + \frac{(6)(4)(-4)}{(8)(6)(-2)} y_3 + \frac{(6)(4)(-2)}{(10)(8)(2)} y_5$$

$$= -\frac{y_{-5}}{5} + \frac{y_{-3}}{2} + y_3 - \frac{3}{10} y_5$$

$$= y_3 - 0.2y_{-5} + 0.5y_{-3} - 0.3y_5$$

$$= y_3 - 0.2y_{-5} + 0.2y_{-3} + 0.3y_{-3} - 0.3y_5$$

Hence $\quad y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5})$

(ii) y_{-3}, y_{-1}, y_1, and y_3 are given, y_0 is to be obtained. By Lagrange's formula

$$y_0 = \frac{(0+1)(0-1)(0-3)}{(-3+1)(-3-1)(-3-3)} y_{-3} + \frac{(0+3)(0-1)(0-3)}{(-1+3)(-1-1)(-1-3)} y_{-1}$$

$$+ \frac{(0+3)(0+1)(0-3)}{(1+3)(1+1)(1-3)} y_1 + \frac{(0+3)(0+1)(0-1)}{(3+3)(3+1)(3-1)} y_3$$

$$= -\frac{1}{16} y_{-3} + \frac{9}{16} y_{-1} + \frac{9}{16} y_1 - \frac{1}{16} y_3$$

$$= \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{16}[(y_3 - y_1) - (y_{-1} - y_{-3})]$$

$$\therefore \qquad y_0 = \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right].$$

Example 9. *Show that Lagrange's interpolation formula can be written in the form*

$$f(x) = \sum_{r=0}^{r=n} \frac{\phi(x_r)\, f(x_r)}{(x - x_r)\, \phi'(x_r)}$$

where $\phi(x) = (x - x_0)(x - x_1) \ldots (x - x_n)$ *and* $\phi'(x_r) = \frac{d}{dx}[\phi(x)]$ *at* $x = x_r$.

Sol. We know that Lagrange's interpolation formula is

$$y = f(x) = \frac{(x - x_1)(x - x_2) \ldots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2) \ldots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \ldots (x_1 - x_n)} f(x_1)$$
$$+ \ldots + \frac{(x - x_0)(x - x_1) \ldots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \ldots (x_n - x_{n-1})} f(x_n) \quad \ldots(1)$$

We are given that $\phi(x) = (x - x_0)(x - x_1) \ldots (x - x_n)$ ...(2)

Taking log on both sides of (2), we have

$$\log (\phi(x)) = \log (x - x_0) + \log (x - x_1) + \ldots + \log (x - x_n)$$

Differentiating w.r.t. x, we have

$$\frac{\phi'(x)}{\phi(x)} = \frac{1}{x - x_0} + \frac{1}{x - x_1} + \ldots + \frac{1}{x - x_n}$$

or
$$\phi'(x) = (x - x_1)(x - x_2) \ldots (x - x_n) + (x - x_0)(x - x_2) \ldots (x - x_n)$$
$$+ \ldots + (x - x_0)(x - x_1) \ldots (x - x_{r-1})(x - x_{r+1}) \ldots (x - x_n)$$
$$+ \ldots + (x - x_0)(x - x_1) \ldots (x - x_{n-1}) \quad \ldots(3)$$

$$\therefore \quad \phi'(x_r) = (x_r - x_0)(x_r - x_1) \ldots (x_r - x_{r-1})(x_r - x_{r+1}) \ldots (x_r - x_n) \quad \ldots(4)$$

R.H.S. of (4) does not contain $(x - x_r)$ factor. Since each term involving $(x - x_r)$ vanishes by putting $x = x_r$.

Using (2) and (4) in (1), we have

$$f(x) = \sum_{r=0}^{r=n} \frac{\phi(x)}{(x - x_r)\, \phi'(x_r)} f(x_r).$$

Uses of Lagrange's Formula

The two main uses of Lagrange's formula are :

(*i*) To find any value of a function when the given values of the independent variable are not equidistant.

(*ii*) To find the value of the independent variable corresponding to a given value of the function.

3.3. HERMITE INTERPOLATION FORMULA

The Hermite interpolating polynomial interpolates not only the function $f(x)$ but also its (certain order) derivatives at a given set of tabular points. The simple interpolating conditions are given in (1). We now give an explicit expression for the interpolating polynomial satisfying (1), that is

$$\left.\begin{aligned} H(x_i) &= f(x_i) \\ H'(x_i) &= f'(x_i), \quad i = 0, 1, \ldots, n. \end{aligned}\right\} \quad \ldots(1)$$

Since there are $2n + 2$ conditions to be satisfied, $H(x)$ must be a polynomial of degree $\leq 2n + 1$. The required polynomial may be written as

$$H(x) = \sum_{i=0}^{n} A_i(x)\, f(x_i) + \sum_{i=0}^{n} B_i(x)\, f'(x_i) \quad \ldots(2)$$