Reading report of *Principles of Mathematical Analysis* -Part I

002 gxl

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Chapter 1

The Real and Complex Number Systems

1.1 Notes

The chapter 1 mainly talks about The real number system,**R**.

1.1.1 Why we need R

The rational number system is inadequate for many purposes , both as a field (Dedekind principle) and as an ordered set (the least-upper-bound property). And we can proof that **R** is perfectly matched.

1.1.2 Basic property of R

Theorems there seems to *OBVIOUS* to proof , but some of them are interesting.

Propositon 1. *The following statements are true in every ordered field.*

(a) *If $x > 0$ then $-x < 0$,and vice versa.*

Proof*(a)*
If $x > 0$ then $0 = -x + x > -x + 0$,so that $-x < 0$. If $x < 0$ then $0 = -x + x < -x + 0$,so that $-x > 0$. This proves (a).

Theorem 1. *If $x \in \mathbf{R}, y \in \mathbf{R}$,and $x > 0$, and there is a positive integer n such that*

$$nx > y$$

Proof *(Tool:the least-upper-bound property)*
Let $A = \{nx | n \in \mathbf{N}^*\}$. *If Theorem were false,then* y *would be an upper bound of* A. *Put* $\alpha = supA$. *Since* $x > 0$,*then* $\alpha - x$ *is not an upper bound of* A. *Hence* $\alpha - x < mx$ *for some* $mx \in A$. *But then* $\alpha < (m+1)x \in A$, *which is impossible.*

1.1.3 The fields that contains R as a subfield

1.2 Exercises

Proof 1. *If* $r + x$ *were rational ,* $x = (r + x) - r$ *is also rational,which is contradictory. The similar argument holds for* rx *.*

Proof 2. *If there existed a rational number* $x = \frac{m}{n}$, $gcd(m, n) = 1$. *and* $x^2 = 12$.*Then we can get* $m^2 = 12n^2 = 3 \times 4n^2$. *Hence* m *is divided by* 3,*and* m^2 *is divided by* 9. *Thus* q *is also divided by* 3,*which is contradictory.*

Proof 3. *For (a) , if* $x \neq 0$ *and* $xy = xz$ *, then*

$$\begin{aligned} y = 1 \times y = \frac{x}{x} \times y = \frac{xy}{x} \\ = \frac{xz}{x} = \frac{x}{x} \times z = z \end{aligned}$$

The similar argument holds for (b)(c)(d) .

Proof 4. *Assuming one element* $x \in E$ *, it follows from the definition that*

$$\alpha \leq x \leq \beta$$

Proof 5. *For each* $x \in A$, $supA \geq x$, *thus for each* $-x \in B$, $-supA \leq x$, *which proves* $-supA = infB$.

Proof 6. *(a) Let* $\beta = (b^m)^{\frac{1}{n}}$,*then*

$$\beta^{nq} = b^{mq} = b^{np}$$

$$\text{Thus } \beta^q = b^p \ , \ \beta = (b^p)^{\frac{1}{q}}$$

(b)Assuming $r = \frac{m}{n}$ *and* $s = \frac{p}{q}$ *,then*

$$b^{r+s} = b^{\frac{mq+np}{nq}} = (b^{mq})^{\frac{1}{nq}} (b^{np})^{\frac{1}{nq}} = b^r b^s$$

(c)For each $x \leq r$, $b^x \leq b^r$
Hence for each $b^x \in B(r)$, $b^r \geq b^x$ *, which proves*

$$b^r = supB(r)$$

(d)$\forall r, s \in \mathbf{Q}$, *and* $r \leq x, s \leq y$

$$b^r b^s = b^{r+s} \leq sup\ B(x + y) = b^{x+y}$$

(e)

Proof 7. *(a)*$b^n - 1 = (b - 1) \sum_{i=1}^{n} b^i \geq n(b - 1)$.
(b)Just let b in (a) become $b^{\frac{1}{n}}$.
(c)Use (b)

(e)To see this , apply part(c) with $t = \frac{b^w}{y}$.
(f)Obviously $b^x \leq y$. *If* $b^x < y$*, according to (d),*$b^{x+\frac{1}{n}} < y$ *for sufficiently large* n. *Then* $x + \frac{1}{n} \in A$*,which is contradictory.Hence* $b^x = y$.
(g)Assume that $x > y$,

$$b^x = b^y b^{x-y} > b^y$$

.

Proof 8. *Assuming* $\mathbf{C}$ *were a ordered field,if* $i > 0$ *,then* $i^2 = -1 > 0$*,which is contradictory.*

Proof 9. *(i)In this definition , it is obvious that if* $x \in \mathbf{C}$ *and* $y \in \mathbf{C}$*,then one and only one of the statements*

$$x < y, \qquad x = y, \qquad y < x$$

is true.
(ii)If $x, y, z \in \mathbf{C}$ *and assuming* $x < y$, $y < z$ *,then*

$$\begin{aligned}
&1.Re(x) < Re(y) < Re(z) && \Rightarrow x < z \\
&2.Re(x) = Re(y) < Re(z) \quad or \quad Re(x) < Re(y) = Re(z) && \Rightarrow x < z \\
&3.Re(x) = Re(y) = Re(z) \quad and \quad Im(x) < Im(y) < Im(z) && \Rightarrow x < z
\end{aligned}$$

which proves $x < z$.

Proof 10.

$$\begin{aligned}
z^2 &= a^2 - b^2 + 2abi \\
&= u + \sqrt{|w|^2 - u^2} i \\
&= u + |v| i
\end{aligned}$$

$$\begin{aligned}
\overline{z}^2 &= a^2 - b^2 - 2abi \\
&= u - \sqrt{|w|^2 - u^2} i \\
&= u - |v| i
\end{aligned}$$

Proof 11. *Let* $w_\theta = cos\theta + isin\theta$ *, and* $0 \leq \theta \leq 2\pi$. *It is clear that* $|w\theta| = 1$ *and that* $\forall \theta_1, \theta_2$ *, if* $\theta_1 \neq \theta_2$ *, then* $w_{\theta_1} \neq w_{\theta_2}$.
1.If $|z| = 0$ *, then* $r = 0$. w *is obviously not unique. 2.If* $|z| \neq 0$ *, let* $tan\theta = \frac{Im(z)}{Re(z)}$ *and* $z = |z| w\theta$*.They are both unique.*

Proof 12. *We give a proof by induction . The inequality holds obviously for* $n = 2$. *Assume that the inequality holds for* $n = m - 1$ *, for* $n = m$ *,*

$$|z_1 + \cdots + z_{n-1} + z_n| \leq |z_1 + \cdots + z_{n-1}| + |z_n| \leq |z_1| + \cdots + |z_{n-1}| + |z_n|$$

Proof 13. *Let* $|x| > |y|$ *, then the propositon equals to*

$$|x| \leq |x - y| + |y|$$

According to Proof 12,it is clear to prove.

Proof 14.

$$\begin{aligned}|1+z|^2+|1-z|^2 &= (1+z)(1+\overline{z})+(1-z)(1-\overline{z}) \\ &= 2+z+\overline{z}+2-z-\overline{z} \\ &= 4\end{aligned}$$

Proof 15. *The equality holds when* $(a_1, a_2, \cdots, a_n)$ *and* $(b_1, b_2, \cdots, b_n)$ *are proportional .*

Proof 16. *Let x(0,0,...,0),y(d,0,...,0).*
(a)every z like $(\frac{d}{2}, z_2, \ldots, z_k)$*,* $z_2^2+\cdots+z_k^2 = r^2-\frac{r^2}{4}$*.*
(b)The only z is $(\frac{d}{2}, 0, \ldots, 0)$*.*
(c)$\forall z \in \mathbf{R}^k, |z-x|+|z-y| \geq |x-y| > 2r$*.*

Proof 17.

$$\begin{aligned}|x+y|^2+|x-y|^2 &= \sum_{i=1}^{n}[(x_i+y_i)^2+(x_i-y_i)^2] \\ &= \sum_{i=1}^{n}(x_i^2+y_i^2) \\ &= 2|x|^2+2|y|^2\end{aligned}$$

Geometrical interpretation : In a parallelogram , the square sum of a pair of adjacent sides equals to half the square sum of two diagonals.

Proof 18. *This is certainly false when k=1.*
If $k=2$ *,* $x \cdot y = 0 \Rightarrow x_1x_2 + y_1y_2 = 0$*.*
Let $y_1 = 1$ *, and* $y_2 = x_1x_2$*.Obviously* $y \neq 0$*.*
The similar argument holds if $k > 2$*.*