Reading report of
Principles of Mathematical Analysis -Part II

002 gxl

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BASIC TOPOLOGY

Notes

2.8 Theorem

Every infinite subset of a countable set **A** is countable .
The theorem shows that countable sets represent the ”smallest” infinity.
Also,this theorem can be used **to show one infinite set is a countable set** , if we can find this set is equivalent to one subset of a countable set **A** .

2.12 Theorem

Why the author use the word ”Hence” ? I can't understand it .

*Typical countable sets

$\mathbb{N}$ $\mathbb{Q}$ $\bigcup_{\alpha\in\mathbf{A}}\mathbf{B}_\alpha$, when $\mathbf{A},\mathbf{B}_\alpha,\forall\alpha\in\mathbf{A}$ are at most countable .
set of all **algebraic numbers**.

Cantor's diagonal process

In this text , this process is used to proof **2.14 Theorem** , and it can also be used to proof a typical theorem:
Theorem Let $\mathbf{A}=\{x|x\in[0,1]\}$, and **A** is uncountable.

One more thing about 2.14 Theorem

In order to proof **A** is uncountable , the author proofs that every subset of **A** is a proper subset of **A** . It should be seperated from **2.6 Remark** (**A** is infinite if **A** is equivalent to one of its proper subsets)

Perfect

2.18 Definition (h)
E is *perfect* if **E** is closed and if every point of **E** is a limit point of **E**.
We say $\mathbb{Q}$ is **not** perfect due to *Dedekind principle*.

Dense

2.18 Definition (j)
E is *dense* in **X** if every point of **X** is a limit point of **E**, or a point of **E** .
We say $\mathbb{Q}$ is dense in **R**, that means :
$\forall x \in \mathbb{R}, \forall \epsilon > 0, \exists y \in \mathbb{Q}, d(x, y) < \epsilon$.

2.23 Theorem

Main reason: $x \in \mathbf{E} \Longleftrightarrow x \notin \mathbf{E}^c$.
It's important that in some conditon it is hard to proof a set is closed , then we can turn to proof its complement is open.

Compact sets

At the quiz of Unit 2 we have seen this definition . A set is compact only if every open cover of it contains a finite subcover.
A compact has many fine property:

2.33 Theorem shows compactness behaves well because it does not rely to the metric it is relative.
2.34 2.35 Theorem shows compactness has deep relation to closeness.
2.36 2.37 Theorem links the compactness with limit.
By using **2.40 Theorem** , we can put every bounded set E in R^k into a compact set (a $k - cell\ I$) and use powerful property above.
2.41 Theorem point out the connection between property and definition.

All in all , compact sets is a powerful tool for us.

2.42 Theorem(Weierstrass)

We have known a special case of this theorem when we learn **limit of sequence** , which says a bounded sequence always has a convergent subsequence .

Exersise

1.

Proof. Let A be a empty set.If ∃ B , and A is not a subset of B. Then there must be a x ,which $x \in A$ but $x \notin B$. However ,there is no element in A. Then no x will be found. $\square$

2.

Proof. $\square$

3.

Proof. From 2. ,we know the set of all algebraic numbers is countable. But $\mathbb{R}$ is uncountable. $\square$

4.

No . If so , $\mathbb{R}$ were countable.

5.

Let $A_k = \{k + \frac{1}{n+1} \mid n \in \mathbb{N}^*\}$. $B = \bigcup_{i=1}^{3} A_i$. It obvious that $\forall x \in B, 0 \leq x \leq 4$. So B is the set we seek for.

6.

(1). $\forall x_0 \notin E'$,then $\exists r > 0, \forall x \in (x_0 - r, x_0 + r), x \notin E$. Now we will proof $x \notin E'$, either. There must exists $\epsilon > 0$, then $x_0 - r < x - \epsilon < x_0 + r$. So x can not be a limit point ,that means $x \notin E'$. $\square$

(2). Assuming $x_0 \notin E'$ but x_0 is a limit point of $\overline{E}$. But we have shown that if x_0 is not a limit point of E , it will not be a limit point of E' ,too. So $\exists r > 0, \forall x \in (x_0 - r, x_0 + r), x \notin E \textit{ and } x \notin E'$, which means $x \notin E \cup E'$. $\square$

Surely E and E' do not always have the same limit points.Just consider the set in Exersise-5.

7.

(a). When $n = 1$,this theorem is obvious.
Let $n = 2$, and what we can proof that :

$$A_1' \cup A_2' = B_n'$$

$$\begin{aligned} x \in B_n' &\iff \forall \delta > 0 \ , \ \exists y \in A_1 \cup A_2 \ , \ y \in (x - \delta, x + \delta) \\ &\iff \forall \delta > 0 \ , \ \exists y \in A_1 \textit{ or } y \in A_2 \ , \ y \in (x - \delta, x + \delta) \\ &\iff x \in A_1' \cup A_2' \end{aligned}$$

$\square$

(b). Let us proof:

$$\bigcup_{i=1}^{\infty} A_i' \subset B'$$

$\forall x_0 \in \bigcup_{i=1}^{\infty} A_i'$, $\forall \epsilon > 0$, $\exists x \in (x_0 - \epsilon, x_0 + \epsilon)$, $x \in A_i$ for some i. Hence $x_0 \in (\bigcup_i A_i)'$. Then $x \in B'$. $\square$

This is a example to indicate this inclusion can be proper.
$A_i = \{\frac{1}{i}(1 + \frac{1}{n}) \mid n \in \mathbb{N}^*\}$.

Then $A_i' = \{\frac{1}{i}\}$. $\bigcup_{i=1}^{\infty} A_i' = \{\frac{1}{n} \mid n \in \mathbb{N}^*\}$.

But $B = \{0\} \cup \bigcup_{i=1}^{\infty} A_i'$.

8.

(1). Surely . As its definition says ,
$\forall x_0 \in E$, $\exists \epsilon > 0$, $\forall x \in (x_0 - \epsilon, x_0 + \epsilon)$, $x \in E$.
Then x is a limit point of E. $\square$

(2). No. Just consider a finite set . It has no limit point but is closed. $\square$

9.

(a). Consider one element in E^o called x.
$\forall x_0 \in E$, $\exists \epsilon > 0$, $\forall y \in (x - \epsilon, x + \epsilon)$, $y \in E$. Then $\forall x' \in (x - \frac{\epsilon}{3}, x + \frac{\epsilon}{3})$, there exists $\delta = \frac{\epsilon}{3}$, then $\forall y \in (x - \delta, x + \delta)$, $y \in E$.
So we have found a neighbourhood of x_0. $\square$

(b). According to the definition? $\square$

(c). If there is $x \in G$ but $x \notin E^o$, then x is not a interior point .
Then G is not open. This is contradictory. $\square$

(d). $x \notin E^o$ equals to $x \notin E$ $(x \in E^c)$ or every neighbourhood of x has a $y \notin E$ $(y \in E^c)$. $\square$

(e). No.Let $E = (-\infty, 0) \cup (0, \infty)$, and $\overline{E} = \mathbb{R}$. Hence $0 \notin E^o$ but $0 \in \overline{E}^o$. $\square$

(f). $\square$

10.

Proof. This is a metric space.
(1)$d(p,p) = 0$, and $d(p,q) = 1 > 0$.
(2)If $p \neq q$ then $d(p,q) = d(q,p) = 1$.
(3)If $p \neq q$, $d(p,q) = 1$, $d(p,r) + d(r,q) \geq 1$. □

open subset: every subset
closed subset: empty subset?
compact subset: empty subset?

11.

1). This is not a metric.
(3)If $p \neq q$, $d(p,q) = (p-q)^2$, $d(p,r) + d(r,q) = (p-r)^2 + (q-r)^2$.Then we should proof

$$(p-q)^2 = p^2 + q^2 - 2pq \leq p^2 + q^2 + 2r^2 - 2r(p+q) = (p-r)^2 + (q-r)^2$$
$$0 \leq (r-q)(r-p)$$

But let $r > min\{p,q\}$ and $r < min\{p,q\}$. □

2). This is a metric.
(1)$d(p,p) = 0$, and $d(p,q) = \sqrt{|p-q|} > 0$.
(2)If $p \neq q$ then $d(p,q) = d(q,p) = \sqrt{|p-q|}$.
(3)If $p \neq q$, $d(p,q) = \sqrt{|p-q|}$, $d(p,r) + d(r,q) = \sqrt{|p-r|} + \sqrt{|q-r|}$.
Now let's proof :

$$\sqrt{|p-q|} \leq \sqrt{|p-r|} + \sqrt{|q-r|}$$
$$|p-q| \leq |p-r| + |q-r| + \sqrt{|p-r||q-r|}$$

Since $|p-q| \leq |p-r| + |q-r|$ and $\sqrt{|p-r||q-r|} \geq 0$, this inequaiton is true. □

3). This is a metric.
(1)$d(p,p) = 0$, and $d(p,q) = |p^2 - q^2| > 0$.
(2)If $p \neq q$ then $d(p,q) = d(q,p) = |p^2 - q^2|$.
(3)If $p \neq q$, $d(p,q) = |p^2 - q^2|$, $d(p,r) + d(r,q) = |p^2 - r^2| + |r^2 - q^2|$.
Since $|p^2 - q^2| \leq |p^2 - r^2| + |r^2 - q^2|$, $d(p,q) \leq d(p,r) + d(r,q)$ is true. □

4). This is not a metric.
(2)$d(p,q) = |p-2q|$ but $d(q,p) = |q-2p|$.Let $q = 0$, $p = 1$, and $d(p,q) \neq d(q,p)$
. □

5). This is a metric.
(1)$d(p,p)=0$, and $d(p,q)=\frac{|p-q|}{1+|p-q|}>0$.
(2)If $p\neq q$ then $d(p,q)=d(q,p)=\frac{|p-q|}{1+|p-q|}$.
(3)If $p\neq q$, $d(p,q)=\frac{|p-q|}{1+|p-q|}$, $d(p,r)+d(r,q)=\frac{|p-r|}{1+|p-r|}+\frac{|r-q|}{1+|r-q|}$.

$$\frac{|p-q|}{1+|p-q|}\leq\frac{|p-r|}{1+|p-r|}+\frac{|r-q|}{1+|r-q|}$$
$$\frac{1}{1+|p-r|}+\frac{1}{1+|r-q|}\leq 1+\frac{1}{1+|p-q|}\leq 1+\frac{1}{1+|p-r|+|r-q|}$$
$$\frac{2+|p-r|+|r-q|}{1+|p-r|+|r-q|+|p-r||r-q|}\leq\frac{2+|p-r|+|r-q|}{1+|p-r|+|r-q|}$$

□

12.

(What's *directly from the definiton* ?)

Proof. Let $A=[0,1]$ and A is compact . It is clear that $K\subset A$. And K is closed because its only limit point $0\in K$. So K is compact. □

13.

Proof. Let $K_i=\{i\}\cup\{i+\frac{1}{n}\mid n\in\mathbb{N}^*\}$. It has only limit point i . Now let $K=\bigcup_{i\in\mathbb{Z}}K_i$, and proof K is compact. Hence K is the compact set we are searching for. □

14.

Proof. Of course . Just think a subset of $\mathbb{R}^1$, $(0,1)$. It has a open cover $\mathcal{A}$, $\mathcal{A}=\bigcup_{i=1}^{\infty}(0,1-\frac{1}{i})$.
It is clear that $\mathcal{A}$ does not have a finity subset. □

15.

Proof. □

16.

Proof. □

17.

Countable. No.If we consider 4 as 0 and 7 and 1 in binary system , thus we connect each $x \in E$ with $y \in [0, 1]$.
Hence $[0, 1]$ is uncountable in R^1 , E is uncountable. □

Dense. No . Just think about 0 or 1. □

compact. □

perfect. □

18.

Proof. Yes . Just consider $\mathbb{R}/\mathbb{Q}$. □

19.

(a). Hence A and B are closed , then $\overline{A} = A$ and $\overline{B} = B$. If they are not separated , there must be a $x \in \overline{A} \cup B$ or $x \in \overline{B} \cup A$. In other words , $x \in A \cup B$.
Then A and B is adjoint , in contradiction to our suppose. □

(b). Assuming there are some $x \in A' \cup B$, then for every neighbourhood of x , there exists a $y \in A$ and $\in B$, in contradiction to our suppose. □

(c). Using (b). □

(d). □

20.

closure. □

interior. No . Let $A = \{(a_0, b) \ |b \in \mathbb{R}\}$ and $B = \{(a_0, b) \ |a \in \mathbb{R}\}$. A and B are subsets of $\mathbb{R}^2$ and it is clear $A \cup B$ is connected. Then $(A \cup B)^o$ is empty , thus it is not connected . □

21.

Proof. □

22.

Proof. Consider the set A whose points have only rational coordinates . $\mathbb{Q}$ is countable and dense , hence A is countable and dense .
$A \subset R^k$ so R^k is separable . □

23.

Proof. From **22**. we have known every separable metric space has a countable dense subset . Take all neighbourhoods with rational radius and center in this countable dense subset .
We called this collection of open subsets $V_{\alpha,r}$. $V_{\alpha,r}$ is the neighbourhood whose radius is r and center is α .
For every open subset E, due to its openness and the dense of $\mathbb{Q}$, this subset if the union of subcollection of $V_{\alpha,r}$, which $\alpha \in E$ and $(\alpha - r, \alpha + r) \subset X$. □

24.

Proof. □

25.

Proof. □

26.

Proof. □

27.

Proof. □

28.

Proof. □

29.

Proof. □

30.

I'm so vegetable.