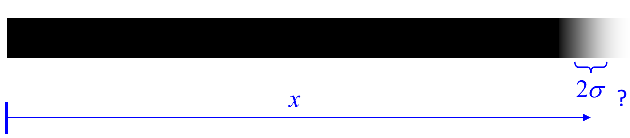
Computational Methods in EE

Uncertainty Analysis

Computational Methods in EE

What is Uncertainty?

How long is this object?



We write our measurement as $x \pm \sigma$

Since the measurement error is “blurred,” we are not entirely certain about the error. The uncertainty σ is a statistically derived quantity.

Notes

- This is a more realistic way to treat error because we do not need to know the true value.
- This is used to predict errors in computations.

How to Interpret σ

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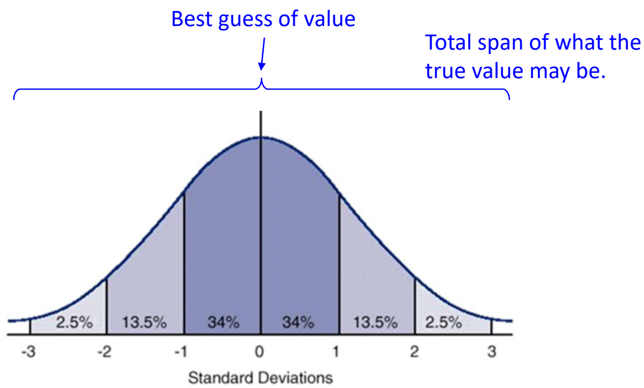

σ is one standard deviation.

There is a 68% chance that the error will be less than 1σ.
There is a 32% chance that the error will be greater than 1σ.

Uncertainly Analysis

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Suppose we have some number of parameters, each with an associated uncertainty.

$$x \pm \sigma_x \qquad y \pm \sigma_y \qquad z \pm \sigma_z$$

Now suppose we calculate a new quantity from these.

$$f(x, y, z)$$

What is the uncertainty σ_f of $f(x,y,z)$?

$$\sigma_f^2 = \left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2$$

We call this propagating uncertainty through a calculation.

Table of Uncertainty Calculations

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Function	Uncertainty
$f = ax$	$\sigma_f = a\sigma_x$
$f = ax \pm by$	$\sigma_f^2 = (a\sigma_x)^2 + (b\sigma_y)^2$
$f = xy$	$(\sigma_f/f)^2 = (\sigma_x/x)^2 + (\sigma_y/y)^2$
$f = x/y$	$(\sigma_f/f)^2 = (\sigma_x/x)^2 + (\sigma_y/y)^2$
$f = x^{\pm b}$	$\sigma_f/f = b\sigma_x/x$
$f = \ln(\pm bx)$	$\sigma_f = b\sigma_x/x$
$f = \log x$	$\sigma_f = b\sigma_x/(x \ln 10)$
$f = e^{\pm bx}$	$\sigma_f/f = b\sigma_x$
$f = a^{\pm bx}$	$\sigma_f/f = b\sigma_x \ln a$
$f = \sin x$	$\sigma_f = \sigma_x \cos x$
$f = \cos x$	$\sigma_f = \sigma_x \sin x$
$f = \tan x$	$\sigma_f = \sigma_x/\cos^2 x$
$f = \sin^{-1}(x)$	$\sigma_f^2 = \sigma_x^2/(1-x^2)$
$f = \cos^{-1}(x)$	$\sigma_f^2 = \sigma_x^2/(1-x^2)$
$f = \tan^{-1}(x)$	$\sigma_f = \sigma_x/(1+x^2)$

Uncertainty Through Multiple Calculations

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Suppose we wish to find the uncertainty σ_f when multiple calculations are involved.

$$f(a,b,c) = \ln(a+bc) \qquad \text{Given } \sigma_a,\ \sigma_b,\ \text{and } \sigma_c$$

	Value	Uncertainty
1	bc	$\left(\frac{\sigma_{bc}}{bc}\right)^2 = \left(\frac{\sigma_b}{b}\right)^2 + \left(\frac{\sigma_c}{c}\right)^2$
2	$a+bc$	$\sigma_{a+bc} = \sigma_a^2 + \sigma_{bc}^2$
3	$\ln(a+bc)$	$\sigma_{\ln(a+bc)} = \frac{\sigma_{a+bc}}{a+bc}$

Example 1

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What is the uncertainty of a decibel quantity?

$$10\log_{10}\left(x \pm \sigma_x\right)$$

Solution

$$\sigma_f^2 = \left(\sigma_x \frac{\partial f}{\partial x}\right)^2$$

$$f(x) = 10\log_{10}(x)$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[10\log_{10}(x)\right] = 10\left[\frac{\partial}{\partial x}\log_{10}(x)\right] = 10\frac{1}{x}\frac{1}{\ln 10} = \frac{10}{x\ln 10}$$

$$\sigma_f^2 = \left(\sigma_x \frac{10}{x\ln 10}\right)^2$$

$$\boxed{\sigma_f = 4.3429\frac{\sigma_x}{x}}$$

Example 2

Computational Methods in EE

The signal-to-noise ratio (SNR) of a system is 50±1.2. What is the SNR in decibels along with the uncertainty?

Solution

The SNR in decibels is

$$10\log_{10}(50) = 16.99 \text{ dB}$$

The uncertainty is calculated using the equation derived in the previous example.

$$\sigma_f = 4.3429\frac{\sigma_x}{x} = 4.3429\frac{1.2}{50} = 0.1$$

The final answer is

$$\boxed{\text{SNR} = 16.99 \pm 0.1 \text{ dB}}$$

Example 3

Computational Methods in EE

What is the uncertainty of the sum of two quantities?

$$x + y$$

Solution

$$\sigma_f^2 = \left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 \qquad f = x + y$$

$$\sigma_f^2 = \left[\sigma_x \frac{\partial}{\partial x}(x+y)\right]^2 + \left[\sigma_y \frac{\partial}{\partial y}(x+y)\right]^2 = \left[\sigma_x \cdot 1\right]^2 + \left[\sigma_y \cdot 1\right]^2$$

$$\boxed{\sigma_f = \sqrt{\sigma_x^2 + \sigma_y^2}}$$

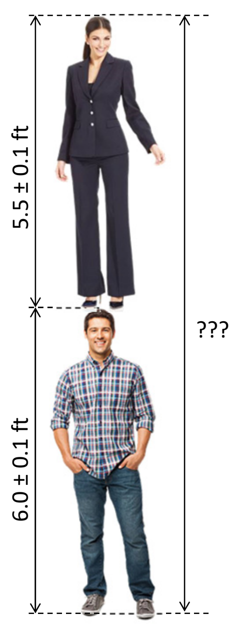
Example 4

Computational Methods in EE

The height of person 1 was measured to be 6.0 ± 0.1 ft.

The height of person 2 was measured to be 5.5±0.1 ft.

If person 2 stands on the head of person 1, what is the total height and the uncertainty of the total height?



Solution

Total height

$$h = h_1 + h_2$$
$$= (6.0\ \text{ft}) + (5.5\ \text{ft})$$
$$= 11.5\ \text{ft}$$

Uncertainty

$$\sigma_h = \sqrt{\sigma_1^2 + \sigma_2^2}$$
$$= \sqrt{(0.1\ \text{ft})^2 + (0.1\ \text{ft})^2}$$
$$= 0.14\ \text{ft}$$

Final Answer

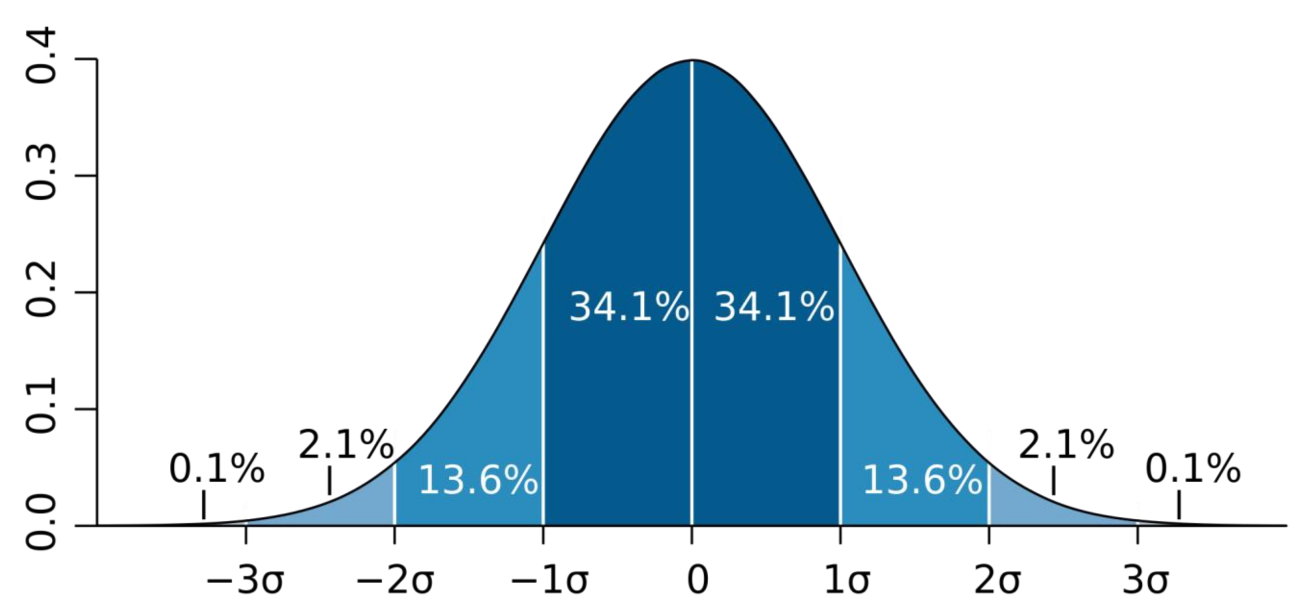
$$\boxed{h = 11.5 \pm 0.14\ \text{ft}}$$

Description of the Problem

Very often we do not know quantities exactly because there is uncertainty in the measurements. For example, the height of a person might be written as

$$h = 2.8 \pm 0.02 \text{ m}$$

In this case, the uncertainty in the measurement is 0.02 m. We interpret this as the standard deviation σ of a normal distribution.



Propagating Uncertainty

Suppose we wish to calculate some quantity that is a function of multiple variables, each having its own uncertainty.

$$f\left(x_1, x_2, \cdots, x_M\right)$$

What is the overall uncertainty for σ_f?

$$\sigma_f^2 = \left(\sigma_1 \frac{\partial f}{\partial x_1}\right)^2 + \left(\sigma_2 \frac{\partial f}{\partial x_2}\right)^2 + \cdots + \left(\sigma_M \frac{\partial f}{\partial x_M}\right)^2$$

Table of Equations

Function	Uncertainty
$f = ax$	$\sigma_f = a\sigma_x$
$f = ax \pm by$	$\sigma_f^2 = \left(a\sigma_x\right)^2 + \left(b\sigma_y\right)^2$
$f = xy$	$\left(\sigma_f / f\right)^2 = \left(\sigma_x / x\right)^2 + \left(\sigma_y / y\right)^2$
$f = x/y$	$\left(\sigma_f / f\right)^2 = \left(\sigma_x / x\right)^2 + \left(\sigma_y / y\right)^2$
$f = x^{\pm b}$	$\sigma_f / f = b\,\sigma_x / x$
$f = \ln\left(\pm bx\right)$	$\sigma_f = b\,\sigma_x / x$
$f = \log x$	$\sigma_f = b\,\sigma_x / \left(x \ln 10\right)$
$f = e^{\pm bx}$	$\sigma_f / f = b\sigma_x$
$f = a^{\pm bx}$	$\sigma_f / f = b\sigma_x \ln a$
$f = \sin x$	$\sigma_f = \sigma_x \cos x$
$f = \cos x$	$\sigma_f = \sigma_x \sin x$
$f = \tan x$	$\sigma_f = \sigma_x / \cos^2 x$
$f = \sin^{-1}\left(x\right)$	$\sigma_f^2 = \sigma_x^2 / \left(1 - x^2\right)$
$f = \cos^{-1}\left(x\right)$	$\sigma_f^2 = \sigma_x^2 / \left(1 - x^2\right)$
$f = \tan^{-1}\left(x\right)$	$\sigma_f = \sigma_x / \left(1 + x^2\right)$