Gaussian Elimination

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Introduction

A **system of equations** (linear) is a group of (linear) equations with various unknown factors. Generally speaking, the unknown factors appear in various equations.

> What an equation with various unknown factors does is relates them amongst each other.

Solving a system consists in finding the value for the unknown factors in a way that verifies **all** the equations that make up the system.

- If there is a single solution (one value for each unknown factor) we will say that the system is **Consistent Independent System (CIS)**.
- If there are various solutions (the system has infinitely many solutions), we say that the system is a **Consistent Dependent System (CDS).** .
- If there is no solution, and this will happen if there are two or more equations that can't be verified at the same time, we say it's an **Inconsistent System (IS)**. For example, the following system of equations

$$\begin{cases} y & = & 0 \\ 2x + y & = & 0 \\ 2x + y & = & 2 \end{cases}$$

  is inconsistent because of we obtain the solution $x = 0$ from the second equation and, from the third, $x = 1$.

In this section we are going to solve systems using the **Gaussian Elimination** method, which consists in simply doing elemental operations in row or column of the augmented matrix to obtain its **echelon form** or its **reduced echelon form** (Gauss-Jordan).

Resolution Method

1. We apply the **Gauss-Jordan Elimination** method: we obtain the **reduced row echelon form** from the augmented matrix of the equation system by performing elemental operations in rows (or columns).

2. Once we have the matrix, we apply the **Rouché-Capelli theorem** to determine the type of system and to obtain the solution(s), that are as:

   Let $A \cdot X = B$ be a system of m linear equations with n unknown factors, m and n being natural numbers (not zero):

   - $AX = B$ is **consistent** if, and only if,

   $$rank(A) = rank(A|B)$$

   - $AX = B$ is **consistent independent** if, and only if,

   $$rank(A) = n = rank(A|B)$$

Note: The elemental operations in rows or columns allow us to obtain equivalent systems to the initial one, but with a form that simplifies obtaining the solutions (if there are). Also, there are quicker tools to work out the solutions in the CIS, like Cramer's rule.

Solved Systems

System 1

$$\begin{aligned} 5x + 2y &= 3 \\ -3x + 3y &= 15 \end{aligned}$$

Show Solution

The augmented matrix of the system is

$$\left(\begin{array}{cc|c} 5 & 2 & 3 \\ -3 & 3 & 15 \end{array}\right)$$

of the same dimension as the system (2x3). The vertical line that separates the matrix coefficients from the vector of the independent terms.

We perform elemental operations in the rows to obtain the reduced row echelon form:

We multiply the first row by *1/5* and the second by *1/3*

$$\left(\begin{array}{cc|c} 1 & 2/5 & 3/5 \\ -1 & 1 & 5 \end{array}\right)$$

We add the second row with the first

$$\left(\begin{array}{cc|c} 1 & 2/5 & 3/5 \\ 0 & 7/5 & 28/5 \end{array}\right)$$

We multiply the second row by *5/7*

$$\left(\begin{array}{cc|c} 1 & 2/5 & 3/5 \\ 0 & 1 & 4 \end{array}\right)$$

We add the first row with the second one multiplied by *-2/5*

$$\left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \end{array}\right)$$

This last equivalent matrix is in the reduced row echelon form and it allows us to quickly see the rank of the coefficient matrix and the augmented one.

We calculate the ranks:

$$rango\left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \end{array}\right) = 2$$

$$rango\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

By the Rouché-Capelli theorem, the system is consistent Independent. The matrix we have obtained represents the system

$$\begin{aligned} x &= -1 \\ y &= 4 \end{aligned}$$

which is the solution to the initial system.

System 2

$$\begin{aligned} 3x \quad - y &= 2 \\ -6x + 2y &= -4 \end{aligned}$$

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System 3

$$\begin{aligned} -5x \quad + y &= 0 \\ x - \frac{1}{5}y &= -3 \end{aligned}$$

Show Solution

System 4

$$\begin{aligned} 5x + 2y \quad\;\; &= 2 \\ 2x + y - z &= 0 \\ 2x + 3y - z &= 3 \end{aligned}$$

Show Solution

The augmented matrix of the system is

$$\left(\begin{array}{ccc|c} 5 & 2 & 0 & 2 \\ 2 & 1 & -1 & 0 \\ 2 & 3 & -1 & 3 \end{array}\right)$$

(dimension 3x4).

We perform elemental operations in the rows to obtain the reduced row echelon form

We multiply the first row by *1/5*

$$\left(\begin{array}{ccc|c} 1 & 2/5 & 0 & 2/5 \\ 2 & 1 & -1 & 0 \\ 2 & 3 & -1 & 3 \end{array}\right)$$

We add the second and third rows with the first one multiplied by *-2*

$$\left(\begin{array}{ccc|c} 1 & 2/5 & 0 & 2/5 \\ 0 & 1/5 & -1 & -4/5 \\ 0 & 11/5 & -1 & 11/5 \end{array}\right)$$

We multiply the second and third rows by *5*

$$\left(\begin{array}{ccc|c} 1 & 2/5 & 0 & 2/5 \\ 0 & 1 & -5 & -4 \\ 0 & 11 & -5 & 11 \end{array}\right)$$

We add the second row with the third one multiplied by *-1*

$$\left(\begin{array}{ccc|c} 1 & 2/5 & 0 & 2/5 \\ 0 & -10 & 0 & -15 \\ 0 & 11 & -5 & 11 \end{array}\right)$$

We multiply the second row by *-1/10* and the third one by *1/11*

$$\left(\begin{array}{ccc|c} 1 & 2/5 & 0 & 2/5 \\ 0 & 1 & 0 & 3/2 \\ 0 & 1 & -5/11 & 1 \end{array}\right)$$

We add the first row with the second one multiplied by *-2/5* and the third with the second one multiplied by *-1*

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1/5 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & -5/11 & -1/2 \end{array}\right)$$

We multiply the third row by *-11/5*

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1/5 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 11/10 \end{array}\right)$$

This last equivalent matrix is in the reduced row echelon form (we know it because it is the identity matrix). By having the identity matrix, we know that it is a consistent independent system and we can obtain the single solution.

We calculate the ranks

$$rango\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1/5 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 11/10 \end{array}\right) = 3$$

$$rango\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3$$

By the Rouché-Capelli theorem, the system is consistent Independent. The matrix we have obtained represents the system

$$x = -1/5$$
$$y = 3/2$$
$$z = 11/10$$

which is the solution to the system.

System 5

$$\begin{aligned} 2x \quad - y + 3z &= 5 \\ 2x + 2y + 3z &= 7 \\ -2x + 3y \qquad &= -3 \end{aligned}$$

Show Solution

System 6

$$\begin{aligned} x + 2y + 3z &= 1 \\ -3x - 2y \quad - z &= 2 \\ 4x + 4y + 4z &= 3 \end{aligned}$$

Show Solution

System 7

$$\begin{aligned} 3x - y + 7z &= 1 \\ 5x \qquad + z &= 2 \end{aligned}$$

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System 8

$$\begin{aligned} x \ - y + 5z &= \sqrt{2} \\ \sqrt{5}x \qquad + z &= \sqrt{3} \\ \frac{2}{5}x + 3y + 2z &= \frac{5}{2} \end{aligned}$$

Show Solution

System 9

$$\begin{aligned} x + 2y - 3z - t &= 0 \\ -3y + 2z + 6t &= -8 \\ -3x - y + 3z + t &= 0 \\ 2x + 3y + 2z - t &= -8 \end{aligned}$$

Show Solution

System 10

$$\begin{aligned} (1+i)x - iy &= 3i \\ 2x + iy &= -i \end{aligned} \qquad i = \sqrt{-1} \in \mathbb{C}$$

Show Solution
