Computer Vision

CS 512

Homework 0

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Homework 1

A.

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad C = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

1. $$2A - B = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$

2. $$\|A\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

$$\cos\theta = \frac{1}{\sqrt{14}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) = 74.4986$$

3. Magnitude of $A = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

$$\hat{A} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$$

4. Magnitude of $A = \sqrt{14}$

$$\cos\alpha = \frac{x}{|A|} = \frac{1}{\sqrt{14}}$$

$$\cos\beta = \frac{y}{|A|} = \frac{2}{\sqrt{14}}$$

$$\cos\gamma = \frac{z}{|A|} = \frac{3}{\sqrt{14}}$$

5. $$A.B = \sum_{i=1}^{n} A_i B_i = (1\times4) + (2\times5) + (3\times6)$$

$$= 4 + 10 + 18$$

$$= 32$$

$$B.A = \sum_{i=1}^{n} B_i A_i = (4\times1) + (5\times2) + (6\times3)$$

$$= 4 + 10 + 18$$

$$= 32$$

6. $A \cdot B = 32$

$\|A\| = \sqrt{14}$

$\|B\| = \sqrt{4^2+5^2+6^2} = \sqrt{77}$

$\cos\theta = \frac{A \cdot B}{\|A\| \cdot \|B\|} = \frac{32}{\sqrt{14} \cdot \sqrt{77}} = 0.97463$

$\theta = 12.9$

7. Let the perpendicular vector ~~&~~ be $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$A \cdot X = 1x + 2y + 3z = 0$

Assuming $x=1$, $y=1$, calculate for 'z'

$1 + 2 + 3z = 0$

$3z = -3$

$\therefore z = -1$

~~Res~~ Therefore $X = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

8. $A \times B = \begin{bmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{bmatrix}$

$= \begin{bmatrix} 2 \times 6 - 3 \times 5 \\ 3 \times 4 - 1 \times 6 \\ 1 \times 5 - 2 \times 4 \end{bmatrix}$

$= \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$

$B \times A = \begin{bmatrix} B_y A_z - B_z A_y \\ B_z A_x - B_x A_z \\ B_x A_y - B_y A_x \end{bmatrix}$

$= \begin{bmatrix} 5 \times 3 - 6 \times 2 \\ 6 \times 1 - 4 \times 3 \\ 4 \times 2 - 5 \times 1 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$

9.

9. A vector which is perpendicular to both A and B is $A \times B$

$$\therefore A \times B = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

10. $A^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

$$A^T B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 32 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

$$AB^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$$

B.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix}$$

1. $$2A - B = 2\begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 8 & -4 & 6 \\ 0 & 10 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 5 \\ 6 & -5 & 10 \\ -3 & 12 & -3 \end{bmatrix}$$

2. $$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+9 & 2+2-6 & 1-8+3 \\ 4-4+9 & 8-2-6 & 4+8+3 \\ 0+10-3 & 0+5+2 & 0+-20-1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+8+0 & 2-4+5 & 3+6-1 \\ 2+4+0 & 4-2-20 & 6+3+4 \\ 3-8+0 & 6+4+5 & 9-6-1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 3 & 8 \\ 6 & -18 & 13 \\ -5 & 15 & 2 \end{bmatrix}$$

3. $AB = \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}$

$$(AB)^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 & 0 \\ 2 & -2 & 5 \\ 3 & 3 & -1 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 2 & -2 & 5 \\ 3 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+9 & 4-4+9 & 0+10-3 \\ 2+2-6 & 8-2-6 & 0+5+2 \\ 1-8+3 & 4+8+3 & 0-20-1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$$

4. $|A| = 1(2-15) - 2(-4-0) + 3(20-0)$

$= -13 + 8 + 60$

$= 55$

$|C| = 1(15-6) - 2(12+6) + 3(4+5)$

$= 9 - 36 + 27$

$= 0$

5. ~~Two vect~~ A matrix is orthogonal if

$$AA^T = I$$

For A,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 2 & -2 & 5 \\ 3 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+9 & 4-4+9 & 0+10-3 \\ 4-4+9 & 16+4+9 & 0-10-3 \\ 0+10-3 & 0-10-3 & 0+25+1 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 9 & 7 \\ 9 & 29 & -13 \\ 7 & -13 & 26 \end{bmatrix}$$

For B,

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+1 & 2+2-4 & 3-4+1 \\ 2+2-4 & 4+1+16 & 6-2-4 \\ 3-4+1 & 6-2-4 & 9+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

For C,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & -1 \\ 2 & 5 & 1 \\ 3 & 6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+9 & 4+10+18 & -1+2+9 \\ 4+10+18 & 16+25+36 & -4+5+18 \\ -1+2+9 & -4+5+18 & 1+1+9 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 32 & 10 \\ 32 & 77 & 19 \\ 10 & 19 & 11 \end{bmatrix}$$

~~None of the matrices A, B, or C have~~

Only B has row vectors that form an orthogonal set.

6. $$A^{-1} = \begin{bmatrix} 2-8 & -4-0 & 20-0 \\ -2-8 & -1-0 & 5-0 \\ 6+6 & 3-12 & +2-8 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -4 & 20 \\ -10 & -1 & 5 \\ 6 & -9 & -10 \end{bmatrix}$$

6. cofactor of A $= \begin{bmatrix} 2-15 & -(-4-0) & 20+0 \\ -(-2-15) & -1-0 & -(5-0) \\ 6+6 & -(3+12) & -2-8 \end{bmatrix}$

$= \begin{bmatrix} -13 & 4 & 20 \\ 17 & -1 & -5 \\ 12 & -9 & -10 \end{bmatrix}$

$(\text{cofactor of } A)^T = \begin{bmatrix} -13 & 17 & 12 \\ 4 & -1 & -9 \\ 20 & -5 & -10 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \begin{bmatrix} -13 & 17 & 12 \\ 4 & -1 & -9 \\ 20 & -5 & -10 \end{bmatrix}$

$= \begin{bmatrix} -13/55 & 17/55 & 12/55 \\ 4/55 & -1/55 & -9/55 \\ 20/55 & -5/55 & -10/55 \end{bmatrix}$

$$\text{cofactor of } B = \begin{bmatrix} 1+8 & -(2+12) & -4-3 \\ -(2+2) & 1-3 & -(-2-6) \\ (-8-1) & -(-4-2) & 1-4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -14 & -7 \\ -4 & -2 & 8 \\ -9 & 6 & -3 \end{bmatrix}$$

$$(\text{cofactor of } B)^T = \begin{bmatrix} 9 & -4 & -9 \\ -14 & -2 & 6 \\ -7 & 8 & -3 \end{bmatrix}$$

$$|B| = 1(1-8) - 2(2+12) + 1(-4-3)$$

$$= -7 - 28 - 7$$

$$= -42$$

$$B^{-1} = \frac{1}{|B|} \begin{bmatrix} 9 & -4 & -9 \\ -14 & -2 & 6 \\ -7 & 8 & -3 \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} 9/26 & 4/26 & +9/26 \\ 14/26 & 2/26 & -6/26 \\ 7/26 & -8/26 & 3/26 \end{bmatrix}}$$

$$= \begin{bmatrix} -9/42 & 4/42 & 9/42 \\ 14/42 & 2/42 & -6/42 \\ 7/42 & -8/42 & 3/42 \end{bmatrix}$$

C. $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$

1. Let λ be the eigenvalue and x be the corresponding eigenvector

$$Ax = \lambda x$$
$$Ax - \lambda x = 0$$
$$(A - \lambda I)x = 0$$

To solve for λ,

$$|(A - \lambda I)| = 0$$

$$\left| \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix} \right| = 0$$

$$(1-\lambda)(2-\lambda) - (2\times 3) = 0$$
$$2 - \lambda - 2\lambda + \lambda^2 - 6 = 0$$
$$\lambda^2 - 3\lambda - 4 = 0$$
$$(\lambda - 4)(\lambda + 1) = 0$$

$\boxed{\lambda = 4}$ or $\boxed{\lambda = -1}$

For $\lambda = 4$

$(A - \lambda I)x = 0$

$$\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-4 & 2 \\ 3 & 2-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right]$$

$R_1 + R_2 \to R_2$

$$\begin{array}{cc} x_1 & x_2 \end{array}$$
$$\left[\begin{array}{cc|c} -3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$-3x_1 + 2x_2 = 0$

$3x_1 = 2x_2$

Let $x_1 = 2$,

$6 = 2x_2$

$\therefore x_2 = 3$

For $\lambda = 4$

eigenvector $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

For $\lambda = -1$

$(A - \lambda I)x = 0$

$$\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

$R_1 - 2/3 R_2 \rightarrow R_2$

$$\begin{array}{cc} x_1 & x_2 \end{array}$$
$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$2x_1 + 2x_2 = 0$

$x_1 = -x_2$

Let $x_1 = 1$

$\therefore x_2 = -1$

For $\lambda = -1$, eigenvector $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2. $V = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$

$|V| = -2-3 = -5$

$V^{-1} = \frac{1}{|V|} \begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix}$

$= \frac{1}{-5} \begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{bmatrix}$

$V^{-1} A V = V^{-1} (AV)$

$\cancel{= V^{-1} \left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{bmatrix} \right)}$

$\cancel{V^{-1}\ 7/5}$

$= V^{-1} \left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \right)$

$= V^{-1} \begin{bmatrix} 2+6 & 1-2 \\ 6+6 & 3-2 \end{bmatrix}$

$$= \begin{bmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{bmatrix} \begin{bmatrix} 8 & -1 \\ 12 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8/5 + 12/5 & -1/5 + 1/5 \\ 24/5 - 24/5 & -3/5 - 2/5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

3. Dot product of eigenvectors of A

$= (1 \times 2) + (-1 \times 3)$

$= 2 - 3$

$= -1$

4. Let λ be the eigenvalues and x be the eigenvectors of B.

$Bx = \lambda x$

$Bx - \lambda x = 0$

$(B - \lambda I)x = 0$

To solve for λ

$$|B - \lambda I| = 0$$

$$\left| \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{bmatrix} \right| = 0$$

$$(2-\lambda)(5-\lambda) - (-2 \times -2) = 0$$
$$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$
$$(\lambda - 6)(\lambda - 1) = 0$$

$$\boxed{\lambda = 6} \text{ or } \boxed{\lambda = 1}$$

For $\lambda = 6$,

$$(B - \lambda I)x = 0$$

$$\begin{bmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -4 & -2 & 0 \\ -2 & -1 & 0 \end{array}\right]$$

$R_1 - 2R_2 \rightarrow R_2$

$$\begin{array}{cc} x_1 & x_2 \\ \multicolumn{2}{c}{\left[\begin{array}{cc|c} -4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]} \end{array}$$

$$-4x_1 - 2x_2 = 0$$

$$\cancel{4x_1 = -2x_2}$$

$$2x_1 = -x_2$$

Let $x_1 = 1$

$\therefore x_2 = -2$

For $\lambda = 6$, eigenvector, $x = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

For $\lambda = 1$,

$$(B - \lambda I)x = 0$$

$$\begin{bmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 4 & 0 \end{array}\right]$$

$2R_1 + R_2 \rightarrow R_2$

$$\begin{array}{cc} x_1 & x_2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 - 2x_2 = 0$

$x_1 = 2x_2$

Let $x_1 = 2$

$\therefore x_2 = 1$

For $\lambda = 1$, eigenvector $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Dot product of eigenvectors of B

$= (2 \times 1) + (1 \times -2)$

$= 2 - 2$

$= 0$

5. since the dot product is 0 the eigenvectors of B are perpendicular to each other.

D.

$$f(x) = x^2 + 3 \qquad g(x, y) = x^2 + y^2$$

1. Let h be change in x

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{(x+h)^2 + 3 - (x^2 + 3)}{h}$$

$$= \frac{\cancel{x^2} + h^2 + 2xh + \cancel{3} - \cancel{x^2} - \cancel{3}}{h}$$

$$= h + 2x$$

When h approaches 0 we get 2x

$$\therefore f'(x) = 2x$$

$$f''(x) = \frac{f'(x+h) - f'(x)}{h}$$

$$= \frac{2(x+h) - 2x}{h}$$

$$= \frac{\cancel{2x} + 2h - \cancel{2x}}{h}$$

$$= 2$$

$$\therefore f''(x) = 2$$

2. $\frac{\partial g}{\partial x} = 2x + 0 = 2x$

$\frac{\partial g}{\partial y} = 0 + 2y = 2y$

3. $\nabla g(x,y) = \begin{bmatrix} \partial g/\partial x \\ \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

4. The probability density function of a ~~univest~~ univariate Gaussian distribution is

$$f(x) = \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}}$$

where μ = location parameter

σ = scale parameter