Solution - HW 1 - Logic Review

CS 536: Science of Programming, Fall 2019

9/30 p.2

1. (Full parenthesization)

1a. Yes; $((((p \wedge (\neg r)) \wedge s) \to (((\neg q) \vee r) \to (\neg p))) \leftrightarrow ((\neg s) \to t))$

1b. No, the two are $\not\equiv$. It's easier to see this from the full parenthesization, $(\exists\, \mathtt{m} \,.\, ((0 \le \mathtt{m} < \mathtt{n}) \wedge (\forall\, \mathtt{j} \,.\, ((0 \le \mathtt{j} < \mathtt{m}) \to (\mathtt{b[0]} \le \mathtt{b[j]} \le \mathtt{b[m]})))))$. The parens in $(\exists\, \mathtt{m} \,.\, 0 \le \mathtt{m} < \mathtt{n}) \wedge \ldots$ aren't compatible.

2. (Minimal parenthesization) All that was required was the final proposition or predicate, but I'm going to show the reasoning in detail, hoping it might help some people. **Notation**: Below I'm writing "op_1 over op_2" to mean that op_1 has higher precedence (= binds more strongly than) op_2. E.g., $\neg$ over $\wedge$.

2a. Subscripting the parentheses of $((\neg(p \vee q) \wedge r) \to (((\neg q) \vee r) \to ((p \vee (\neg r)) \vee (q \wedge s))))$ gives us

$$(_1\, (_2\, \neg (_3\, p \vee q)_3 \wedge r)_2 \to (_4 (_5 (_6\, \neg q)_6 \vee r)_5 \to (_7 (_8\, p \vee (_9\, \neg r)_9)_8 \vee (_{10}\, q \wedge s)_{10})_7)_4)_1$$

I'll work from the inside out, in phases. Of the most deeply embedded parentheses, pairs 6 ($\neg$ over $\wedge$), 9 ($\neg$ over $\vee$), and 10 ($\wedge$ over $\vee$) are redundant but pair 3 is necessary ($\neg$ over $\vee$). Removing pairs 6, 9 and 10 gives us

$$(_1\, (_2\, \neg (_3\, p \vee q)_3 \wedge r)_2 \to (_4 (_5\, \neg q \vee r)_5 \to (_7 (_8\, p \vee \neg r)_8 \vee q \wedge s)_7)_4)_1$$

Now, we can remove the redundant pairs 2 ($\vee$ over $\to$), 5 ($\vee$ over $\to$), and 8 ($\vee$ is associative):

$$(_1\, \neg (_3\, p \vee q)_3 \wedge r \to (_4\, \neg q \vee r \to (_7\, p \vee \neg r \vee q \wedge s)_7)_4)_1$$

The remaining pairs are redundant: 7 ($\vee$ over $\to$), 4 ($\to$ is right-associative), and 1 (the outermost pair). We can drop the subscript 3 and get the final answer,

$$\neg(p \vee q) \wedge r \to \neg q \vee r \to p \vee \neg r \vee q \wedge s.$$

2b. $(\exists\, \mathtt{i} \,.\, (((0 \le \mathtt{i}) \wedge (\mathtt{i} < \mathtt{m})) \wedge (\forall\, \mathtt{j} \,.\, (((\mathtt{m} \le \mathtt{j}) \wedge (\mathtt{j} < \mathtt{n})) \to (\mathtt{b[i]} = \mathtt{b[j]})))))$. Let me try not subscripting this time. The tests $\le$, $<$, and $=$ are stronger than $\wedge$, $\vee$, and $\to$ (also $\neg$ and $\leftrightarrow$ for that matter), so we can delete the parentheses around them and get

$$(\exists\, \mathtt{i}.((0 \le \mathtt{i} \wedge \mathtt{i} < \mathtt{m}) \wedge (\forall\, \mathtt{j}.((\mathtt{m} \le \mathtt{j} \wedge \mathtt{j} < \mathtt{n}) \to \mathtt{b[i]} = \mathtt{b[j]}))))$$

The body of $(\forall\, \mathtt{j}$ ends at its matching), so we can drop the parentheses around the body: $\forall \mathtt{j}.(\ldots)$. In that body, $\wedge$ is stronger than $\to$, so altogether we can simplify $(\forall\, \mathtt{j} \ldots)$:

$$(\exists\, \mathtt{i}.((0 \le \mathtt{i} \wedge \mathtt{i} < \mathtt{m}) \wedge (\forall\, \mathtt{j}.\mathtt{m} \le \mathtt{j} \wedge \mathtt{j} < \mathtt{n} \to \mathtt{b[i]} = \mathtt{b[j]})))$$

Similarly, the body of $(\exists\, \mathtt{i}$ ends at the matching).

$$(\exists\, \mathtt{i}.(0 \le \mathtt{i} \wedge \mathtt{i} < \mathtt{m}) \wedge (\forall\, \mathtt{j}.\mathtt{m} \le \mathtt{j} \wedge \mathtt{j} < \mathtt{n} \to \mathtt{b[i]} = \mathtt{b[j]}))$$

Finally, since $\wedge$ is associative, we can remove the parentheses in $(\ldots \wedge \ldots (\ldots))$ and also delete the outermost parentheses to get the final answer

$$\exists\, \mathtt{i}.0 \le \mathtt{i} \wedge \mathtt{i} < \mathtt{m} \wedge \forall\, \mathtt{j}.\mathtt{m} \le \mathtt{j} \wedge \mathtt{j} < \mathtt{n} \to \mathtt{b[i]} = \mathtt{b[j]}$$

(Note: In real life, we'd abbreviate $0 \le i \wedge i < m$ to $0 \le i < m$ and similarly for j to get

$\exists\, i.\, 0 \le i < m \wedge \forall\, j.\, m \le j < n \rightarrow b[i] = b[j]$

We could even use bounded quantifiers and get $\exists\, 0 \le i < m.\, \forall\, m \le j < n.\, b[i] = b[j]$. We'd have to rely on the context to know that the quantifiers are over i and j, not m or n.)

2c. $(\forall\, x.\, ((\exists\, y.(p \rightarrow q)) \rightarrow (\forall\, z.(q \vee (r \wedge s)))))$ minimizes to $\forall\, x.(\exists\, y.p \rightarrow q) \rightarrow \forall\, z.q \vee r \wedge s$. Here's a brief explanation of the highlights: The parentheses of $(\exists y \ldots)$ are necessary to keep the body $p \rightarrow q$ from becoming $p \rightarrow q \rightarrow (\forall\, z \ldots)$. The parentheses of $(\forall z \ldots)$ are redundant because the right parentheses is at the end of the predicate.

Note $\forall x.(\exists y.p \rightarrow q) \rightarrow (\forall z.q \vee r \wedge s)$ is not minimal but has a pleasant symmetry. It's what you might write in real life.

3. (Syntactic equality)

3a. They are $\not\equiv$: We have $p \wedge q \vee \neg r \rightarrow \neg p \rightarrow q \equiv ((p \wedge q) \vee \neg r) \rightarrow (\neg p \rightarrow q)$, but $((p \wedge q) \vee ((\neg r \rightarrow ((\neg p) \rightarrow q)))) \equiv p \wedge q \vee (\neg r \rightarrow (\neg p \rightarrow q))$

3b. They are $\not\equiv$. $\forall\, x\,.\, p \rightarrow \exists\, y\,.\, q \rightarrow r \equiv \forall\, x.(p \rightarrow \exists\, y.(q \rightarrow r))$, but $(\forall\, x\,.\, p) \rightarrow (\exists\, y\,.\, q) \rightarrow r \equiv ((\forall\, x.\, p) \rightarrow ((\exists\, y.\, q) \rightarrow r))$

4. (Tautology, contradiction, or contingency)

4a. $(p \rightarrow (q \rightarrow r)) \leftrightarrow ((p \rightarrow q) \rightarrow r)$ is a contingency. When p is T and q and r are F, this becomes T $\leftrightarrow$ T which is T, but when q is T and p and r are F, this becomes T $\leftrightarrow$ F, which is F.

4b. $(\forall\, x \in \mathbb{Z}.\forall\, y \in \mathbb{Z}.\, f(x, y) > 0) \rightarrow (\exists\, x \in \mathbb{Z}.\exists\, y \in \mathbb{Z}.\, f(x, y) > 0)$ is a tautology. If $f(x, y) > 0$ for all integers x and y, then any two integers prove the existentials: Since (for example) $f(0, 0) > 0$, there exist x and y with $f(x, y) > 0$.

5. ($\rightarrow$ versus $\leftarrow$)

5a. "p is sufficient for q" $\Leftrightarrow p \rightarrow q$.

5b. "p only if q" $\Leftrightarrow p \rightarrow q$ [9/30] (Also, another way to think about this is that "p only if q" is like "¬ q → ¬ p", which is the contrapositive of (and logically equivalent to) p → q)

6. ($\equiv$ vs $=$)

6a. For $e_1 \neq e_2$ implying $e_1 \not\equiv e_2$ (which it does), it might be easier to see the contrapositive, ($e_1 \equiv e_2$ implies $e_1 = e_2$): If $e_1 \equiv e_2$ then syntactically, they differ only in redundant parentheses, so they must have equal values.

6b. $e_1 = e_2$ does not imply $e_1 \equiv e_2$. Easy examples: (1) $2 + 2 = 4$ but $2 + 2 \not\equiv 4$; (2) $p \wedge q =$ (actually, $\Leftrightarrow$) $q \wedge p$, but we're taking $p \wedge q \not\equiv q \wedge p$.

7. (Prove a tautology) [It's okay to omit the "$\vee$ associative and commutative" step]

$p \wedge \neg(q \wedge r) \to q \wedge r \to \neg p$	
$\Leftrightarrow p \wedge \neg(q \wedge r) \to (\neg(q \wedge r) \vee \neg p)$	Defn $\to$
$\Leftrightarrow \neg(p \wedge \neg(q \wedge r)) \vee (\neg(q \wedge r) \vee \neg p)$	Defn $\to$
$\Leftrightarrow (\neg p \vee (q \wedge r)) \vee (\neg(q \wedge r) \vee \neg p)$	DeMorgan's law (on $\neg(\ldots \wedge \ldots)$) and $\neg\neg$
($\Leftrightarrow (q \wedge r) \vee \neg(q \wedge r) \vee \neg p \vee \neg p$	$\vee$ associative and commutative)
$\Leftrightarrow \mathrm{T} \vee \neg p \vee \neg p$	Excluded middle
$\Leftrightarrow \mathrm{T}$	Domination

8. (Remove $\neg$)

$\neg(\forall x . (\exists y . x \le y) \vee \exists z . x \ge z)$	
$\Leftrightarrow \exists x . \neg((\exists y . x \le y) \vee \exists z . x \ge z)$	DeMorgan's Law ($\neg\forall \Leftrightarrow \exists\neg$)
$\Leftrightarrow \exists x . \neg(\exists y . x \le y) \wedge \neg\exists z . x \ge z$	DeMorgan's Law ($\neg \vee \Leftrightarrow \neg \wedge \neg$)
$\Leftrightarrow \exists x . (\forall y . x > y) \wedge \neg\exists z . x \ge z$	DeMorgan's Law ($\neg\exists \Leftrightarrow \forall\neg$) and $\neg$ of $\le$
$\Leftrightarrow \exists x . (\forall y . x > y) \wedge \forall z . x < z$	DeMorgan's Law ($\neg\exists \Leftrightarrow \forall\neg$) and $\neg$ of $\ge$

9. $GT(b, x, m, k) \equiv \forall i . m \le i < m+k \to x > b[i]$ is one solution. $\forall j . 0 \le i < k \to x > b[m+j]$ is another.

10. (State well-formed, proper, and satisfies a predicate?)

a. $\{x = \textit{ten}, y = \textit{eight plus one}\} \vDash x+y > 0$. The state is well-formed (it $= \{x = 10, y = 9\}$), and $\vDash x+y > 0$.

b. $\{c = \alpha, d = 2\alpha, e = 3\alpha\}$ and $d/c+(0*z) \le w$: The state is well-formed but improper — it doesn't have a binding for z. (Even though the value of z wouldn't matter.) Note the binding for e isn't relevant, since it doesn't appear in the expression.