Problem Set 5

1. 1. $X \sim \dfrac{\lambda^x e^{-\lambda}}{x!}$

$$\text{MLE} = \arg\max_{\lambda} \log\left(\frac{\lambda^x e^{-\lambda}}{x!}\right)$$

log likelihood :

$$\ln P(D/\lambda) = \sum_{i=1}^{n} x^{(i)} \log \lambda - \lambda - \log(x^{(i)}!)$$

$$= \log \lambda \sum_{i=1}^{n} x^{(i)} - n\lambda - \sum_{i=1}^{n} \log(x^{(i)}!)$$

$$\frac{\partial}{\partial \lambda} \ln P(D/\lambda) = \frac{1}{\lambda} \sum_{i=1}^{n} x^{(i)} - n = 0$$

$$\lambda' = \frac{1}{n} \sum_{i=1}^{n} x^{(i)}$$

$$\boxed{\lambda' = \bar{X}}$$

2. $r(\lambda) = \frac{1}{5} \max\{-\lambda/10 + 1, 0\}$

$$P(D/\lambda) = \prod_{i=1}^{n} \frac{1}{5} \max\left\{\frac{-\lambda}{10} + 1, 0\right\} \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \begin{cases} \prod_{i=1}^{n} \frac{1}{5} \cancel{\max} \left(1 + \left(\frac{-\lambda}{10}\right)\right)\left(\frac{\lambda^x e^{-\lambda}}{x!}\right) & \lambda < 10 \\ 0 & \lambda > 10 \end{cases}$$

Assuming $0 < \lambda < 10$

$L =$

$\ln P(D/\lambda) = m \log \frac{1}{5} + n \log\left(1 - \frac{\lambda}{10}\right)$

$+ \log \lambda \sum_{i=1}^{n} x_i - n\lambda - \sum_{i=1}^{n} \log x_i!$

$\frac{\partial L}{\partial \lambda}_{(0<\lambda<10)} = \frac{n}{\left(1 - \frac{\lambda}{10}\right)}\left(\frac{-1}{10}\right) + \frac{1}{\lambda}\sum_{i=1}^{n} x_i - n = 0$

$\Rightarrow \frac{n}{\lambda - 10} + \frac{1}{\lambda}\sum_{i=1}^{n} x = n$

$n\lambda + (\lambda - 10)\sum_{i=1}^{n} x_i = \lambda(\lambda - 10)n$

$\lambda^2 - 11\lambda n - \lambda x - 10x = 0 \quad \text{——} \quad (1)$

Solving for λ

$$\lambda = \frac{11n + \frac{\sum_{i=1}^{n} x_i}{n} \pm \sqrt{\left(11n + \frac{\sum_{i=1}^{n} x_i}{n}\right)^2 - 40n\frac{\sum_{i=1}^{n} x_i}{n}}}{2n} \quad \text{——} \quad (2)$$

Let $z = \frac{\sum_{i=1}^{n} x_i}{n}$.

Taking second derivative of (1)

$\frac{\partial^2 L}{\partial \lambda^2} = 2\lambda n - (11n + z)$

Substituting the roots from (2)

$$\cancel{\lambda}\left(\frac{\cancel{11n+z} \pm \sqrt{(11n+z)^2+40nz}}{\cancel{2n}}\right)\cancel{n} - (\cancel{11n+z})$$

for $\lambda = \dfrac{11n+z-\sqrt{(11n+z)^2+40nz}}{2n}$ we get

$\dfrac{\partial L^2}{\partial \lambda^2} < 0 \Rightarrow$ Above λ is the Maximum A priori

1.3

→ $\frac{1}{5}\max\{-\frac{\lambda}{10}+1, 0\}$ is not a parferable prior as it has a probability of 0 for all values of $\lambda > 10$

→ Also the prior is not differentiable every where which makes calcuting MAP difficult

→ We can choose Gamma distribution as a prior as it is the conjugate prior for Poisson distribution.

2   1. Lognormal distribution

$$X \sim \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$$

Loglikelihood:

$$L(\mu,\sigma^2/x) = \log\left(\prod_{i=1}^{n}\left(\frac{1}{x_i\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln x_i - \mu)^2}{2\sigma^2}}\right)\right)$$

$$= \log\left((2\pi\sigma^2)^{-n/2} \prod_{i=1}^{n} \frac{1}{x_i} \exp\left(\frac{\sum_{i=1}^{n}(\ln(x_i) - \mu)^2}{2\sigma^2}\right)\right)$$

$$= -\frac{n}{2}\ln(2\pi\sigma^2) - \sum_{i=1}^{n}\ln x_i - \frac{\sum_{i=1}^{n}(\ln(x_i) - \mu)^2}{2\sigma^2}$$

$$= -\frac{n}{2}\ln(2\pi\sigma^2) - \sum_{i=1}^{n}\ln(x_i) - \frac{\sum_{i=1}^{n}\ln(x_i)^2}{2\sigma^2}$$

$$+ 2\frac{\sum_{i=1}^{n}\ln(x_i)\mu}{2\sigma^2} - \frac{\sum_{i=1}^{n}\mu^2}{2\sigma^2}$$

$$l(\mu,\sigma^2/x) = -\frac{n}{2}\ln(2\pi\sigma^2) - \sum_{i=1}^{n}\ln(x_i) - \frac{\sum_{i=1}^{n}\ln(x_i)^2}{2\sigma^2}$$

$$+ \frac{\sum_{i=1}^{n}\ln(x_i)\mu}{\sigma^2} - \frac{n\mu^2}{2\sigma^2}$$

2. MLE for μ

$$\frac{\partial L}{\partial \mu} = 0$$

$$\sum_{i=1}^{n} \frac{\ln(x_i)}{\sigma^2} - \frac{2n\mu}{2\sigma^2} = 0$$

$$\frac{n\mu}{\cancel{\sigma^2}} = \frac{\sum_{i=1}^{n} \ln(x_i)}{\cancel{\sigma^2}}$$

$$\mu_{MLE} = \frac{1}{n} \sum_{i=1}^{n} \ln(x_i)$$

MLE for σ^2

$$\frac{\partial L}{\partial \sigma^2} = 0$$

$$-\frac{n}{2}\frac{1}{\sigma^2} - \frac{\sum_{i=1}^{n} (\ln(x_i) - \mu_{MLE})^2}{-2(\sigma^2)^2} = 0$$

$$\frac{n}{\cancel{2}\cancel{\sigma^2}} = \frac{\sum_{i=1}^{n} (\ln x_i - \mu_{MLE})^2}{\cancel{2}(\sigma^2)^{\cancel{2}}}$$

$$\sigma^2_{MLE} = \frac{\sum_{i=1}^{n} (\ln x_i - \mu_{MLE})^2}{n}$$

3. Selecting n iid samples $x_1 \cdots x_n$

Sample mean $= \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$$E[\bar{x}] = E[\bar{x}_1 + \bar{x}_2 + \bar{x}_3 \cdots \bar{x}_n]$$

$$= E[\bar{x}_1] + E[\bar{x}_2] \cdots E[\bar{x}_n]$$

$$= \mu$$

μ is unbiased.

Sample variance s^2

let $Y = \ln(X)$

$$E[s^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2\right]$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i-\mu)^2 - \frac{2}{n}(\bar{Y}-\mu)\sum_{i=1}^{n}(Y_i-\mu) + \frac{1}{n}(\bar{Y}-\mu)^2\sum_{i=1}^{n}\right]$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i-\mu)^2 - \frac{2}{n}(\bar{Y}-\mu)\sum_{i=1}^{n}(Y_i-\mu) + (\bar{Y}-\mu)^2\right]$$

$$\bar{Y} - \mu = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \mu)$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(y_i-\mu)^2 - \frac{2}{n}(\bar{y}-\mu)n(\bar{y}-\mu) + (\bar{y}-\mu)^2\right]$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(y_i-\mu)^2 - (\bar{y}-\mu)^2\right]$$

$$= \sigma^2 - E\left[(\bar{y}-\mu)^2\right]$$

$$= \sigma^2 - \frac{1}{n}\sigma^2 < \sigma^2$$

Similar to Gaussian distribution log normal distribution has variance is biased.

4. Gaussian distribution is not a good choice of prior distribution over μ as it is not the conjugate prior of log normal distribution