

## Dimension

Let  $V$  be a vector space over a field  $F$ . Let  $U \subseteq V$  be finite. Let  $U$  be a basis of  $U$ . Define a function  $\dim \in \mathbb{N}$ , such that  $\dim U := |U|$ .

$$1. 1 \in S$$

$$1. 1 := \max \{ |U| \} = n$$

$$(1, a), (2, b)$$



## Basis

Let  $V$  be a vector space over a field  $F$ . Let  $U \subseteq V$  be a finite dimension. If  $U$  is lin. incl. and  $\text{span } U = V$ , then  $U$  is a basis of  $V$ .

## Isomorphism

...

## Subspaces

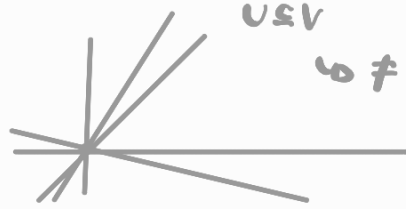
Let  $V$  be a vector space over  $F$ . Let  $U$  be a non-empty set. If  $0 \in U$  and  $\forall \alpha \in F$  and  $\forall v \in U, \alpha \cdot v \in U$  and  $\forall u, v \in U, u+v \in U$ , then  $U$  is a subspace of  $V$ .

$$0 \in U$$

$$\forall \alpha, v \in U \quad \alpha \cdot v \in U$$

$$\forall u, v \in U \quad u+v \in U$$

 If  $U$  is a plane or a line that passes through the origin ( $0 \in U$ ), then  $\bigcap_{U \subseteq V} U = 0$ ?



$\neq 0$ , if  $U_1 \subseteq V$  and  $U_2 \subseteq V$  are equal, but then they are the same set.

**Theorem:** The intersection of any collection of subspaces is a subspace.

Let  $V$  be a vector space over  $F$ . If  $\{U_\nu\}$  is a set of subspaces of  $V$ , then  $\bigcap_{\nu \in I} U_\nu$  is a subspace.

• The intersection reduces the "dimension"?

\*

1.  $0 \in \bigcap U_\nu$  ✓ trivial

2.  $\forall \alpha \in F, \forall v \in \bigcap U_\nu, \alpha \cdot v \in \bigcap U_\nu$  ?

3.  $\forall u, v \in \bigcap U_\nu, u+v \in \bigcap U_\nu$  ?

for any vector  $v$  if  $v \in U_1$  and  $v \in U_2$  by the definition,  $\forall \alpha \in F, \alpha \cdot v \in U_1$  as well as  $\alpha \cdot v \in U_2$

If  $\bigcap U_\nu = \{0\}$ , it is also a subspace

\* **Theorem:** A subspace  $M$  in an  $n$ -dimensional vector space  $V$  is a vector space of dimension  $\leq n$ .

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**Theorem:** Let  $V$  be a vector space over a field  $F$ . Let  $B$  and  $B'$  be bases of  $V$ . Then  $|B| = |B'|$

$B$  and  $B'$  linearly independent and span  $V$

$$\text{span}(B) = \left\{ \sum_{v \in B} \alpha(v) v : \alpha \in F^B \right\} = V$$

$$\text{span}(B') = \left\{ \sum_{v \in B'} \beta(v) v : \beta \in F^{B'} \right\}$$

$$|B| \neq |B'|, \text{ w.l.o.g. } B, B' \subseteq V$$

$$|B| < |B'| \Rightarrow \exists v \in B', v \notin B \Rightarrow \text{since } B' \text{ is lin. ind. } v \notin \text{span}(B), \text{ a contradiction}$$

Let  $B$  and  $B'$  be two bases of a vector space  $V$  over a field  $F$ .

$$B, B' \subseteq V$$

$B$  and  $B'$  are lin. ind.

$B$  span  $V$  and  $B'$  span  $V$

Suppose, by contradiction,  $|B| \neq |B'|$

Without loss of generality, if  $|B| < |B'|$ . Then, at least one element more than  $B$   $\exists v \in B'$  and  $v \notin B$

Since  $B'$  is lin. ind.,  $v \notin \text{span}(B)$

However  $v \in V$  and  $B$  span  $V$ , then  $v \in \text{span } B$ , a contradiction.

Hence,  $|B| = |B'|$

Let  $V$  be a vector space over a field  $F$ .

Let  $\mathcal{B} := \{B \subseteq V : B \text{ is a basis of } V\}$ , then  $|\{ |B| : B \in \mathcal{B} \}| = 1$

Let  $B$  and  $B'$  be two basis of  $V$ .