Set of words

- Let
- be
- $\exists$
- $\forall$
- $=$
- $\neq$
- and
- or
- not
- if... then
- iff
- Define
- s.t
- since
- Thus / Therefore

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Linear dependency and independency

Let $\{v_1, \dots, v_n\}$ be a set of vectors in a vector vector V. We say if there exist $\alpha_1, \dots, \alpha_n \in F$, such that for every vector v in V, $\{v_1, \dots, v_n\}$ spans V.

For every v in V, if ..., such that $v = \alpha_1 v_1 + \dots + \alpha_n v_n$

↳ Let V be a vector space over a field F. Let $U \subseteq V$ be finite. Call U spanning if $\forall v \in V, \exists \alpha \in F^U$ s.t. $v = \sum_{u \in U} \alpha(u) \cdot u$

↳ Call U linearly independent if $\{\alpha \in F^U : \sum_{u \in U} \alpha(u) \cdot u = 0\} = \{0\}$

↳ Call U linearly dependent if $\exists \alpha \in F^U$, s.t. $\alpha \neq 0$ and $\sum_{u \in U} \alpha(u)\, u = 0$

Let V be a vector space and let $U \subseteq V$ be finite. Then U is L.D. iff $\exists v \in U$ s.t. $v \in \operatorname{span}(U \setminus \{v\})$

($\Rightarrow$) If U is L.D. then $\exists v \in U$ s.t. $v \in \operatorname{span}(U \setminus \{v\})$

$(1,1) = \{\{1\}, \{1,1\}\}$

$(a,b) = \{a, \{a,b\}\}$

($\Leftarrow$) $\exists v \in U$, $v \in \operatorname{span}(U \setminus \{v\})$, then U is L.D.

$\{v_1, \dots, v_n\}$

F^U

$\underline{\beta(x)} = -\alpha(x)$

$x \in U \setminus \{v\}$

$\exists \alpha \in F^{U \setminus \{v\}}$, $v = \sum_{x \in U \setminus \{v\}} \alpha(x)\, x$

$\beta(v) = 1$

↳

$\exists \beta \in F^U$, s.t. $\beta \neq 0$ and $\sum_{u \in U} \beta(u)\, u = 0$

$\mathbb{R}^n$ $\quad \mathbb{R}^{[n]}$

$\downarrow$

$\mathbb{R}^{\{1, \dots, n\}}$

$v - \sum_{x \in U \setminus \{v\}} \alpha(x)\, x = 0$

$(\Leftarrow)$ $\exists v \in U, v \in \operatorname{span}(U \setminus \{v\})$

$\exists \alpha \in F^{U \setminus \{v\}}, v = \sum_{x \in U \setminus \{v\}} \alpha(x) x$

Define $\beta \in F^U$ s.t. $\forall x \in U \setminus \{v\}, \beta(x) := -\alpha(x)$ and $\beta(v) = 1$

Since $\beta(v) = 1$, $\beta \neq 0$

Since $v = \sum_{x \in U \setminus \{v\}} \alpha(x) x$, $v + \sum_{x \in U \setminus \{v\}} -\alpha(x) x = 0$

Thus $\sum_{u \in U} \beta(u) u = 0$

Therefore, U is L.D.

~~$(\Rightarrow)$ If U is L.D. then $\exists v \in U$ s.t. $v \in \operatorname{span}(U \setminus \{v\})$~~

$(\Rightarrow)$ $\exists \alpha \in F^U$, s.t. $\alpha \neq 0$ and $\sum_{u \in U} \alpha(u) u = 0$

Since $\alpha \neq 0$, then $\exists v \in U$ $\alpha(v) \neq 0$

Then, $\alpha(v) v + \sum_{u \in U \setminus \{v\}} \alpha(u) u = 0$ (I)

$\forall x \in U \setminus \{v\}$, define $\beta(x) := -\dfrac{\alpha(x)}{\alpha(v)}$

(I) implies $v = \sum_{u \in U \setminus \{v\}} \beta(u) u$

Thus, $v \in \operatorname{span}(U \setminus \{v\})$

Let V be a vector space and let $U \subseteq V$ be finite. Define $\operatorname{span}(U) :=$

$$\left\{ \sum_{v \in U} \alpha(v) v : \alpha \in F^U \right\}$$

$\alpha = 0 \iff \forall v \in U \ \ \alpha(v) = 0$

$\alpha \neq 0 \iff \exists v \in U \ \ \alpha(v) \neq 0$

$\neg(\alpha = 0) \iff \neg(\forall v \in U \ \ \alpha(v) = 0)$

$\exists v \in U \ \neg(\alpha(v) = 0)$

1. $\{1, 2, 3\}$

2. $\{x \in X : x \text{ satisfies} \dots\}$
   X is already defined

$\times$ $\{x : x \text{ is a set}\}$

3. $\{f(x) : x \in X\}$
   $f \in A^X$, $f : \alpha \in F^U \longmapsto \sum_{v \in U} \alpha(v) v$

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Let V be a vector space and let $U \subseteq V$ be finite. Define $\text{span}(U) :=$ $\{ \sum_{v \in U} \alpha(v) v : \alpha \in F^U \}$. Call U spanning if $\text{span}(U) = V$

Let V be a vector space and let $U \subseteq V$ be finite. Call U a basis if U is spanning and linearly independent.

* every V has a basis

Theorem: Let V be a vector space over a field F. Let B and B' be bases of V. Then $|B| = |B'|$

adapt Holmes theorem.

Corollary: Let V be a vector space over a field F.

prove it

Let $\beta := \{ B \subseteq V : B \text{ is a basis of } V \}$

$\{ B \in \mathcal{P}(V) : B \text{ is a basis of } V \}$

Then $|\{ |B| : B \in \beta \}| = 1$

Definition Let V be a vector space over a field F.

Let $\beta := \{ B \subseteq V : B \text{ is a basis of } V \}$

By corollary, $|\{ |B| : B \in \beta \}| = 1$

Define $\dim(V)$ to be the unique element of $\{ |B| : B \in \beta \}$

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Let U and V be vector spaces over a field F.

A linear transformation is a function $T: V \longrightarrow U$ such that

$\forall x,y \in V$, $T(x+y) = T(x) + T(y)$

$\forall \alpha \in F$ $\forall x \in V$, $T(\alpha x) = \alpha T(x)$

Let U and V be vector spaces over a field F.

A linear transformation is a function $T: V \longrightarrow U$ such that

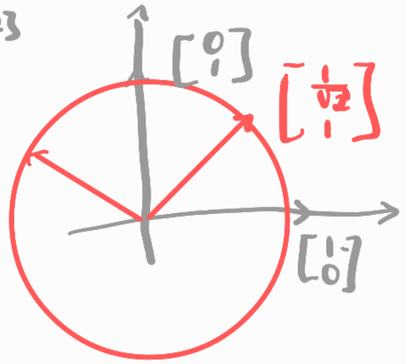
$\forall x,y \in V$ $\forall \alpha, \beta \in F$ $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

$V = \mathbb{R}^{[2]}$

$U = \mathbb{R}^{[2]}$

$$R_{2\times 2} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta = \frac{\pi}{4}$$

$T(x) := R.x, \forall x \in \mathbb{R}^{[2]}$



$T(x+y) = T(x) + T(y)$

Let $x, y \in \mathbb{R}^{[2]}$

$$\begin{aligned} T(x+y) &= R(x+y) \\ &= Rx + Ry \\ &= T(x) + T(y) \end{aligned}$$

Let $x \in \mathbb{R}^{[2]}$

Let $\alpha \in F$

$$\begin{aligned} T(\alpha x) &= R(\alpha x) \\ &= \alpha Rx \\ &= \alpha T(x) \end{aligned}$$

$$R_{2\times 2} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\forall \theta \in \mathbb{R}, \quad R_\theta := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R : \mathbb{R} \longrightarrow \mathbb{R}^{[2]\times[2]}$$

$$R_\theta R_\psi = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}$$

$-(\cos\theta \sin\psi$

$$= \begin{bmatrix} \cos\theta\cos\psi - \sin\theta\sin\psi & \cos\theta(-\sin\psi) - \sin\theta\cos\psi \\ \sin\theta\cos\psi + \cos\theta\sin\psi & -\sin\theta\sin\psi + \cos\theta\cos\psi \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\psi) & -\sin(\theta+\psi) \\ \sin(\theta+\psi) & \cos(\theta+\psi) \end{bmatrix}$$

$$= R_{\theta+\psi}$$

Let U, V, W be vector spaces over a field F

Let $T : U \longrightarrow V$ be a linear transformation

Let $R : V \longrightarrow W$ be a linear transformation

Then $R \circ T$ (after) is a linear transformation

Let $x, y \in V$. Let $\alpha, \beta \in F$

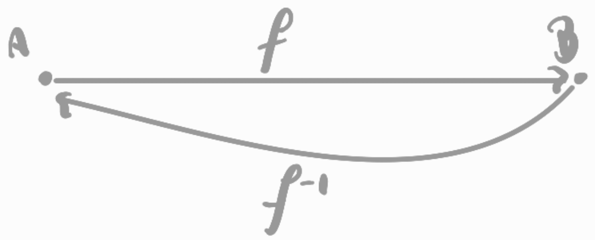
$x \in U$, $y \in V$

finish it

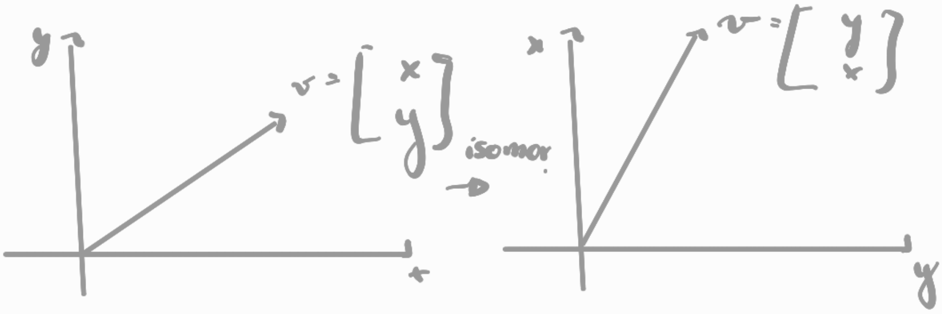
$$(R\circ T)(\alpha x + \beta y) = R(T(\alpha x + \beta y))$$
$$= R(\alpha Tx + \beta Tx)$$
$$= R(\alpha Tx) + R(\beta Tx)$$
$$= \alpha R(Tx) + \beta R(Tx)$$
$$= \alpha(R\circ T)(x) + \beta(R\circ T)(y)$$

Want $\forall x,y \in V \ \forall \alpha,\beta \in F \ (R \circ T)(\alpha x + \beta y) = \alpha (R\circ T)(x) + \beta (R \circ T)(y)$

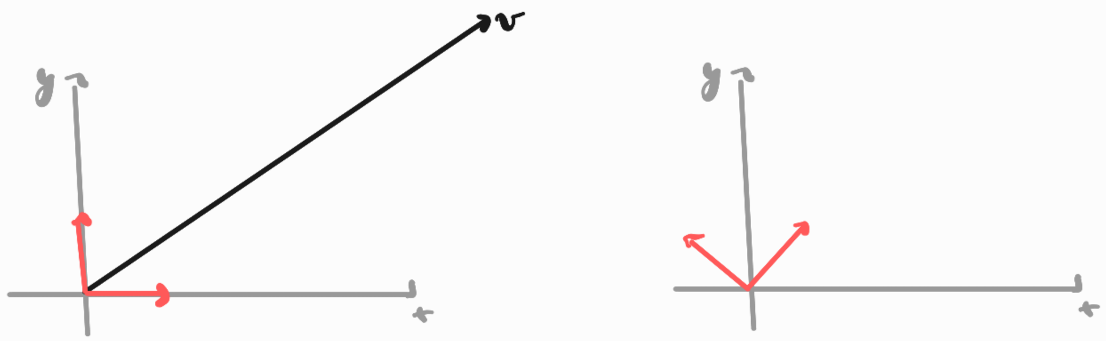
A homomorphism (linear function) f is an isomorphism if f is bijective and f^{-1} is a homomorphism (linear function)

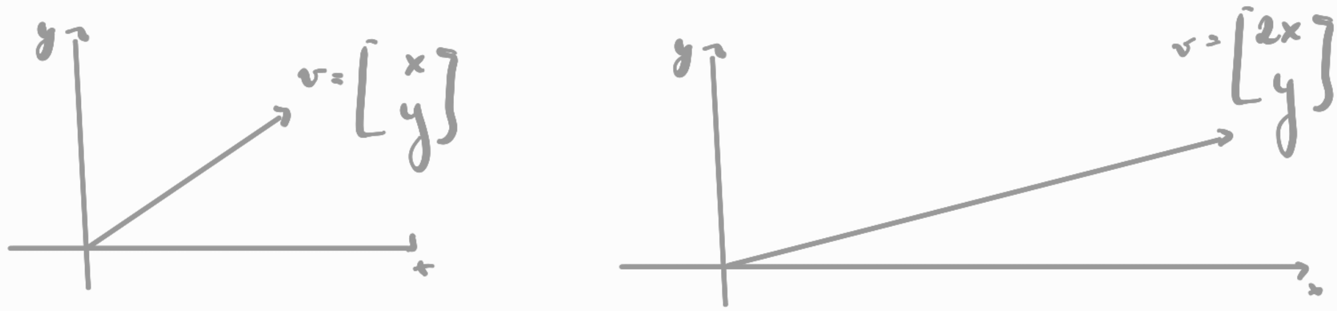


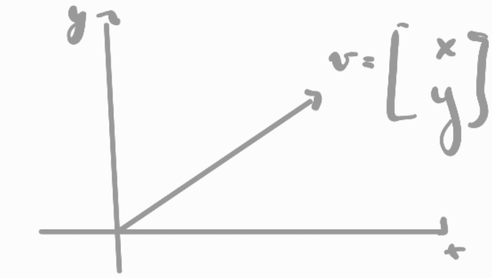
$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

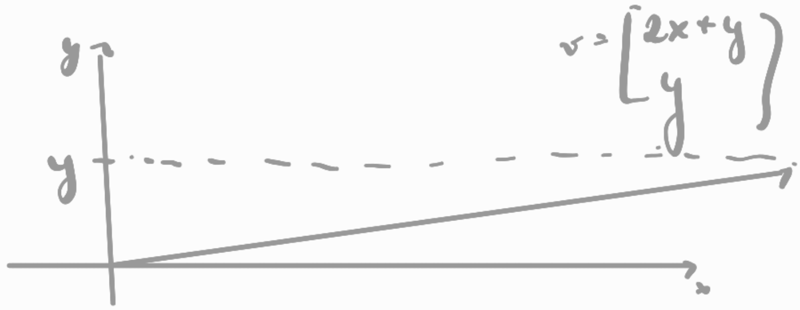


Exercise: Let V, W be vector spaces over a field F. Let $L: V \to W$ be a linear bijection. Then L^{-1} is linear.

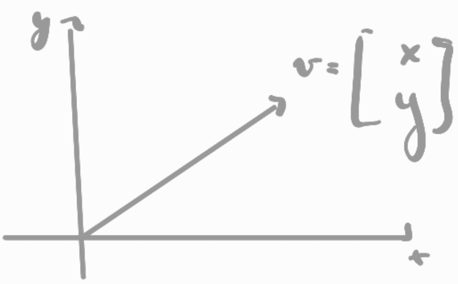




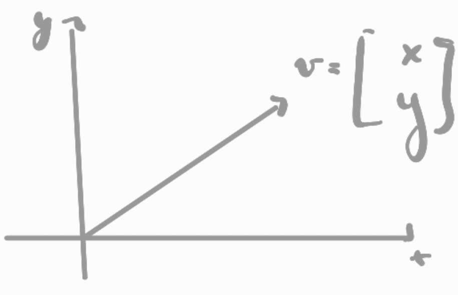

y
x
v = [x y]

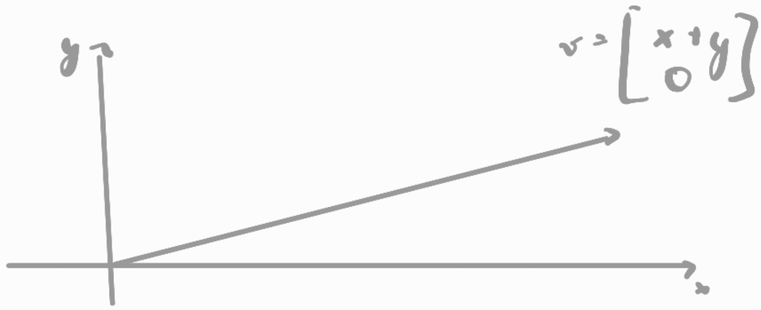

y
x
y
v = [2x+y y]

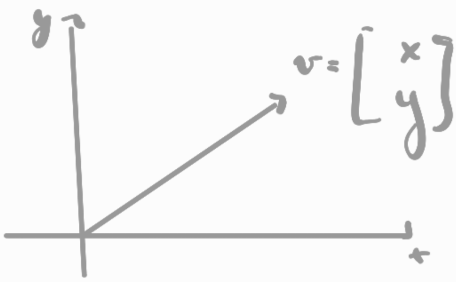
not isomor.

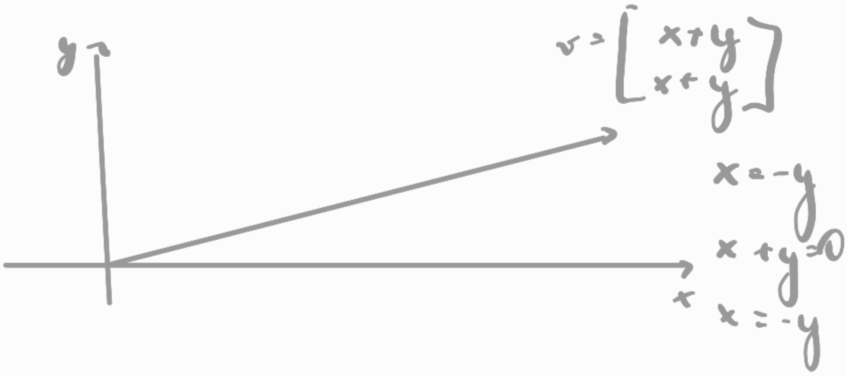

y
x
v = [x y]


y
x
v = [x 0]


y
x
v = [x y]


y
x
v = [x+y 0]


y
x
v = [x y]


y
x
v = [x+y x+y]
x = -y
x + y = 0
x = -y