Echoes of the past: A unified perspective on fading memory and echo states

Juan-Pablo Ortega*

Florian Rossmannek*

Abstract

Recurrent neural networks (RNNs) have become increasingly popular in information processing tasks involving time series and temporal data. A fundamental property of RNNs is their ability to create reliable input/output responses, often linked to how the network handles its memory of the information it processed. Various notions have been proposed to conceptualize the behavior of memory in RNNs, including steady states, echo states, state forgetting, input forgetting, and fading memory. Although these notions are often used interchangeably, their precise relationships remain unclear. This work aims to unify these notions in a common language, derive new implications and equivalences between them, and provide alternative proofs to some existing results. By clarifying the relationships between these concepts, this research contributes to a deeper understanding of RNNs and their temporal information processing capabilities.

1. Introduction

Recurrent neural networks (RNNs) have become increasingly popular as machine learning models in tasks involving time series and temporal data generated by dynamical systems. What makes RNNs so popular in applications is their efficient scalability and ease of training, owing to model designs such as reservoir computing (RC). Other machine learning paradigms such as large language models require vast amounts of training data and cost-intensive computational resources for the initial training as well as retraining sessions, which brings increasingly difficult challenges in its wake pertaining to energy consumption and cost of chatbot queries. [1,13]. In contrast, RC is designed so that only a small part of the model is being trained algorithmically, the main part of the model being chosen randomly [12, 19] or even realized by a physical system. The latter has sparked a particular interest in the physics and robotics community, which explore different systems that can function as these reservoirs. The first example of such a physical reservoir was a bucket of water, whose computational ability resides in translating input sound waves to ripples of waves on the surface of the water [5]. Since then many other physical reservoirs have been proposed from photonics and analog circuits to mechanical and biological bodies, to name only a few [26,30]. A particularly promising direction is the use of quantum systems as reservoirs for so-called quantum RC [6-8,25].

On the theoretical side, a fundamental property of RNNs is their ability to create reliable input/output responses. This ability is often linked to how the network handles its memory of the information it processed – or rather forgets its memory. Over the past decades, different notions have been brought forth to conceptualize the decay of its memory. Prominent keywords are steady states [2,4], echo states, state forgetting, input forgetting [12], and fading memory [2]. It is a long-standing folklore belief that these notions can be used interchangeably. Although some subtleties concerning this belief have been discovered in recent years [27, 29], a deeper understanding of the relationship between these various notions is lacking. It is difficult to analyze the precise relationships in part because the notions are not even defined using the same terminology. This gap will be filled in the present work: we unify all notions and present new results linking them.

Broadly speaking, an RNN processes a sequence of inputs $(\underline{u}_t)_t$ and produces a sequence of states $(\underline{x}_t)_t$, where at time t the current input \underline{u}_t and the previous state \underline{x}_{t-1} are transformed into the next state \underline{x}_t by the so-called state map f. In applications, it is a crucial part of the model design that the states are further transformed by a readout map to yield the final output. But adding a readout does not affect the underlying dynamics or its memory of the processed information and is not of relevance in this work. When an RNN ought to create a reliable response to a given input sequence, the state \underline{x}_t at time t is expected to depend to much greater extent on the most recent inputs $\underline{u}_t, \underline{u}_{t-1}, \ldots$ than on inputs $\underline{u}_s, s \ll t$, from the distant past. This naturally leads to the idea that the system is 'input forgetting' [10, 12]. The states, in this case, have been coined 'steady states' or 'steady state solution' in earlier works [2, 4] and 'echo states' in later works [12]. We caution that sometimes the terminology 'echo states' includes the requirement that the sequence of states is unique for each given input sequence.

The same idea as for input forgetting lies behind the notion of 'fading memory' but the mathematical formulation differs. Customarily, fading memory is defined as continuous dependence of the echo states on the input sequence [2,10]. On first inspection, it may not be clear how continuous dependence captures the idea of fading memory. The key to this definition is the choice of topology with respect to which continuity is required. Indeed, some topologies on the space of input sequences boast the property that two sequences are close if their recent entries are

^{*}Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore

close even if their distant past entries differ significantly. Since there are several topologies with this property, there are different notions of fading memory depending on the choice of topology. Although they are closely related to each other and sometimes overlap, various choices give rise to non-identical notions of fading memory [27,29]. The notions of input forgetting and fading memory are concerned with echo states of distinct input sequences that are similar to each other in the recent past. On the contrary, the notion of 'state forgetting' considers a single input sequence and two different initial states. If the next states produced by the state map become increasingly similar to each other as time evolves regardless of the dissimilarity of the initial states, then the system is state forgetting [10,12]. The initial states may not be part of the sequence forming the input's echo states, but some form of attraction is at play, where the next states obtained from the initial ones approach the echo states. Indeed, one of the overarching principles governing all these various notions is that the echo states form the dynamical attractor of the RNN to which the states converge [3, 12, 22, 29].

There had been a string of works that derived sufficient conditions for the existence of unique echo states, and it was observed that fading memory as well as various input and state forgetting properties held as well [9,10,23,28]. However, in those results, all these properties were the consequence of a single sufficient criterion (always some form of contractivity of the state map), and it had therefore not been unveiled whether the notions imply each other independently of the common sufficient criterion.

In this work, we unify the various notions surveyed above in a common language, in which it will become easier to see how they are linked to forward and pullback attraction of the echo states. We will derive new implications and equivalences between several notions, and along the way we will encounter alternative and shorter proofs to some of the classical results in the literature. The main notions and statements are discussed in Section 2, and the proofs together with additional details are presented in Section 2.6.

2. Echoes of the past

2.1 Mathematical framework

In the following, we fix the notation and the underlying mathematical objects that will be used throughout. We denote by \mathbb{Z} the set of integers, by \mathbb{N} the set of strictly positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{N}$. Subsets of topological spaces and products of topological spaces are endowed with the subspace topology and the product topology, respectively. Throughout, let \mathcal{U} be a Hausdorff space and $(\mathcal{X}, d_{\mathcal{X}})$ be a metric space. Sequences are denoted as underlined letters, e.g. $\underline{u} = (\underline{u}_t)_{t \in \mathbb{Z}_-} \in \mathcal{U}^{\mathbb{Z}_-}$. We denote the right-shift operator $(\underline{u}_t)_t \mapsto (\underline{u}_{t-1})_t$ on all left- and bi-infinite sequence spaces by the letter T and the left-shift operator $(\underline{u}_t)_t \mapsto (\underline{u}_{t+1})_t$ on bi-infinite se-

quence spaces by σ . Fix backwards shift-invariant subsets $\underline{\mathcal{U}}^- \subseteq \mathcal{U}^{\mathbb{Z}_-}$ and $\underline{\mathcal{X}}^- \subseteq \mathcal{X}^{\mathbb{Z}_-}$, that is, $T^{-1}(\underline{\mathcal{U}}^-) = \underline{\mathcal{U}}^-$ and likewise for $\underline{\mathcal{X}}^-$, and fix a shift-invariant subset $\underline{\mathcal{U}} \subseteq \mathcal{U}^{\mathbb{Z}}$ that is mapped surjectively onto $\underline{\mathcal{U}}^-$ by the truncation, that is, $\sigma(\underline{\mathcal{U}}) = \underline{\mathcal{U}}$ and $\tau(\underline{\mathcal{U}}) = \underline{\mathcal{U}}^-$, where $\tau \colon \underline{\mathcal{U}} \to \underline{\mathcal{U}}^-$, $\underline{u} \mapsto (\underline{u}_t)_{t \in \mathbb{Z}_-}$. In addition, we assume that $\underline{\mathcal{U}}$ contains all sequences of the form $(\dots, \underline{u}_{-1}^-, \underline{u}_0^-, \underline{u}_1, \underline{u}_2, \dots)$ with $\underline{u}^- \in \underline{\mathcal{U}}^-$ and $\underline{u} \in \underline{\mathcal{U}}$.

2.2 Echo states and fading memory

Consider the state-space system governed by a continuous state map $f: \mathcal{X} \times \mathcal{U} \to \mathcal{X}$. A **solution for the input** $\underline{u} \in \underline{\mathcal{U}}^-$ is an element $\underline{x} \in \underline{\mathcal{X}}^-$ that satisfies $\underline{x}_t = f(\underline{x}_{t-1}, \underline{u}_t)$ for all $t \in \mathbb{Z}_-$. We also call the pair $(\underline{x}, \underline{u})$ a **solution**. The set of all solutions will be denoted

$$\mathcal{S} = \{(\underline{x}, \underline{u}) \in \underline{\mathcal{X}}^- \times \underline{\mathcal{U}}^- : \underline{x}_t = f(\underline{x}_{t-1}, \underline{u}_t) \text{ for all } t \in \mathbb{Z}_-\}.$$

We say that the state-space system has the echo state **property** (ESP) if there exists a unique solution $\underline{x} \in \mathcal{X}$ for any given input $\underline{u} \in \underline{\mathcal{U}}^-$. In the literature, this definition of the ESP is sometimes called the unique solution property [21]. As mentioned in the introduction, in the presence of the ESP, we say that the state-space system has the fading memory property (FMP) if the map $\underline{\mathcal{U}}^- \to \underline{\mathcal{X}}^-$ that associates to any given input its unique solution is continuous. The FMP can be defined without presupposing the ESP [20, 29] but that generalization is not of relevance here. We discussed in the introduction that different choices of topologies on the sequence spaces give rise to different notions of FMP. In this work, we use the product topologies. The resulting FMP was termed product FMP in [27]. The FMP originally considered in [2] was based on the topology induced by a weighted sup-norm. Since the product topology is at least as coarse as the topology induced by a weighted sup-norm [9], the product FMP always implies the FMP in [2]. In that sense, the product FMP is the strongest notion of FMP encountered in the literature (topologies coarser than the product topology make coordinate projections discontinuous and are therefore never used). Restricting our attention to the strongest FMP is satisfactory for the results we present in this work. For detailed discussions and results about the different FMPs, we refer the reader to [10, 27, 29]. Lastly, we point out that compactness of $\underline{\mathcal{X}}^-$ or $\underline{\mathcal{U}}$ with respect to the product topology, which we encounter later as assumptions, is a weaker assumption than compactness with respect to a finer topology. Thus, by choosing the product topology, we get the strongest FMP and the weakest assumptions later.

2.3 State and input forgetting

Consider the family of functions $\psi_n \colon \mathcal{X} \times \underline{\mathcal{U}} \to \mathcal{X}, n \in \mathbb{N}$, defined recursively by $\psi_1(x,\underline{u}) = f(x,\underline{u}_1)$ and $\psi_n(x,\underline{u}) = f(\psi_{n-1}(x,\underline{u}),\underline{u}_n)$. A variation of this family of functions has appeared in [10] under the name 'reservoir flow'. We

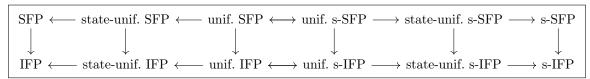


Figure 1: Relations between state and input forgetting properties.

will define state and input forgetting properties based on the relations

$$\lim \sup_{n \to \infty} d_{\mathcal{X}}(\psi_n(x, \underline{u}), \psi_n(x', \underline{u})) = 0$$
 (2.1)

and

$$\lim_{n \to \infty} \sup d_{\mathcal{X}}(\psi_n(x, T^n(\underline{u})), \psi_n(x', T^n(\underline{u}))) = 0, \qquad (2.2)$$

respectively. A state is called *reachable* if it appears as the time zero component of some solution. We denote the set of all reachable states by

$$\mathcal{R} = \{x \in \mathcal{X} : \text{there exists } (\underline{x}, \underline{u}) \in \mathcal{S} \text{ with } \underline{x}_0 = x\}.$$

Definition 2.1. We say that the state-space system has

- (i) ... the state forgetting property (SFP) if (2.1) holds for all $u \in \mathcal{U}$ and $x, x' \in \mathcal{X}$.
- (ii) ... the *input forgetting property (IFP)* if (2.1) holds for all $\underline{u} \in \underline{\mathcal{U}}$ and $x, x' \in \mathcal{R}$.
- (iii) ... the *shifted state forgetting property (s-SFP)* if (2.2) holds for all $\underline{u} \in \underline{\mathcal{U}}$ and $x, x' \in \mathcal{X}$.
- (iv) ... the *shifted input forgetting property (s-IFP)* if (2.2) holds for all $\underline{u} \in \underline{\mathcal{U}}$ and $x, x' \in \mathcal{R}$.

We add the attribute **state-uniform** (uniform) if the convergence of the limit superior is uniform in x, x' (in u, x, x').

From the given definition, it may not be immediately apparent why the IFP indeed conceptualizes 'input forgetting'. To see how the IFP is motivated, consider $x^1, x^2 \in \mathcal{R}$. Take solutions $(\underline{x}^1, \underline{u}^1), (\underline{x}^2, \underline{u}^2) \in \underline{\mathcal{X}}^- \times \underline{\mathcal{U}}^-$ with $\underline{x}^1_0 = x^1$ and $\underline{x}^2_0 = x^2$. Then, $\psi_n(x^i, \underline{u})$ is exactly the time zero component of a solution for the input $\underline{u}^{i,n} := (\dots, \underline{u}^i_{-1}, \underline{u}^i_0, \underline{u}_1, \dots, \underline{u}_n)$. The n most recent inputs in the sequences $\underline{u}^{1,n}$ and $\underline{u}^{2,n}$ are the same. Thus, although the IFP is defined through states $x^1, x^2 \in \mathcal{R}$, it really considers solutions of different input sequences whose difference lies further and further in the past as n grows.

Remark 2.2. The implications and equivalences depicted in the diagram in Figure 1 hold by definition.

2.4 Attraction

We hinted in the introduction that one of the overarching principles is that the echo states form the dynamical attractor of the RNN to which the states converge. To make this claim rigorous, let us specify the precise dynamical system. Despite being intrinsically non-autonomous, it was proposed in [29] to model the dynamics autonomously on an extended sequence space. To this end, consider the map $\varphi \colon \underline{\mathcal{X}}^- \times \underline{\mathcal{U}} \to \underline{\mathcal{X}}^- \times \underline{\mathcal{U}}$ given by $\varphi(\underline{x},\underline{u}) =$ $((\underline{x}, f(\underline{x}_0, \underline{u}_1)), \sigma(\underline{u}))$, whose iterates can be seen as extensions of the maps ψ_n . Concretely, $\psi_n \circ (p_0 \times \pi) = p_0 \circ \varphi^n$, where $p_0: \underline{\mathcal{X}}^- \times \underline{\mathcal{U}} \to \mathcal{X}$ and $\pi: \underline{\mathcal{X}}^- \times \underline{\mathcal{U}} \to \underline{\mathcal{U}}$ are the projections $p_0(\underline{x},\underline{u}) = \underline{x}_0$ and $\pi(\underline{x},\underline{u}) = \underline{u}$. This relation implies that $\psi_n(x,\underline{u})$ belongs to the p_0 -projection of the fiber $\pi^{-1}(\sigma^n(\underline{u}))$, and $\psi_n(x, T^n(\underline{u}))$ belongs to the p_0 projection of the fiber $\pi^{-1}(\underline{u})$. It has been shown in [29] that the global attractor $\mathcal{A} := \bigcap_{n \in \mathbb{N}_0} \varphi^n(\underline{\mathcal{X}}^- \times \underline{\mathcal{U}})$ of φ is exactly equal to $\tau^{-1}(\mathcal{S})$. This implies that if $x \in \mathcal{R}$, then $\psi_n(x,\underline{u})$ and $\psi_n(x,T^n(\underline{u}))$ belong to the p_0 -projection of the attractor \mathcal{A} . The fibers $\mathcal{A} \cap \pi^{-1}(\underline{u})$ of the attractor capture pullback attraction, and the fibers $\mathcal{A} \cap \pi^{-1}(\sigma^n(u))$ capture forward attraction. We now see that the SFP realizes forward attraction: if the SFP holds, then for all $u \in \mathcal{U}$ and $x \in \mathcal{X}$

$$\limsup_{n \to \infty} \operatorname{dist}(\psi_n(x, \underline{u}), \mathcal{A} \cap \pi^{-1}(\sigma^n(\underline{u})))$$

$$\leq \limsup_{n \to \infty} d_{\mathcal{X}}(\psi_n(x, \underline{u}), \psi_n(x', \underline{u})) = 0,$$

where x' is an arbitrary element in \mathcal{R} . Likewise, the s-SFP realizes pullback attraction:

$$\limsup_{n \to \infty} \operatorname{dist}(\psi_n(x, T^n(\underline{u})), \mathcal{A} \cap \pi^{-1}(\underline{u}))$$

$$\leq \limsup_{n \to \infty} d_{\mathcal{X}}(\psi_n(x, T^n(\underline{u})), \psi_n(x', T^n(\underline{u}))) = 0.$$

Remark 2.3. If the inputs are known to be generated by a deterministic dynamical system $\phi \colon \mathcal{M} \to \mathcal{M}$, transformed by an observation function $\omega \colon \mathcal{M} \to \mathcal{U}$, then instead of the maps ψ_n defined on $\mathcal{X} \times \underline{\mathcal{U}}$ one would consider the alternative maps $\hat{\psi}_n \colon \mathcal{X} \times \mathcal{M} \to \mathcal{M}$ defined by $\hat{\psi}_1(x,p) = f(x,\omega \circ \phi(p))$ and $\hat{\psi}_n(x,p) = f(\hat{\psi}_{n-1}(x,p),\omega \circ \phi^n(p))$. However, the (shifted) SFP and IFP defined through these maps $\hat{\psi}_n$ are equivalent to the (shifted) SFP and IFP defined through ψ_n with $\underline{\mathcal{U}} = \{(\omega(\phi^t(p)))_{t \in \mathbb{Z}} \colon p \in \mathcal{M}\}$. Our setup working with ψ_n on $\mathcal{X} \times \underline{\mathcal{U}}$ covers the most general case of possible inputs.

2.5 Main result

In our main result, stated below, we extend the diagram in Theorem 2.2 with additional implication arrows under compactness assumptions. It generalizes [12, Proposition 1], [21, Theorem 1], and [18, Remark 1], and provides alternative and shorter proofs.

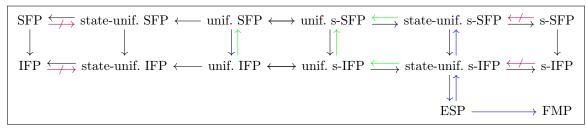


Figure 2: Relations between state and input forgetting properties in Theorem 2.4. The blue implication arrows hold if $\underline{\mathcal{X}}^-$ is compact; the green implication arrows hold if $\underline{\mathcal{X}}^-$ and $\underline{\mathcal{U}}$ are compact; the purple implication arrows do not hold in general.

Theorem 2.4. Consider the diagram depicted in Figure 2. If $\underline{\mathcal{X}}^-$ is compact, then the implication arrows indicated in blue are valid; and if $\underline{\mathcal{X}}^-$ and $\underline{\mathcal{U}}$ are compact, then the implication arrows indicated in green are valid. Furthermore, we will see counterexamples to the implication arrows indicated in purple.

Remark 2.5. We point out that $\underline{\mathcal{X}}^-$ is compact if and only if \mathcal{X} is compact and $\underline{\mathcal{X}}^-$ is a closed subset of $\mathcal{X}^{\mathbb{Z}_-}$. Likewise, $\underline{\mathcal{U}}$ is compact if and only if \mathcal{U} is compact and $\underline{\mathcal{U}}$ is a closed subset of $\mathcal{U}^{\mathbb{Z}}$.

2.6 Technical details

Let us prove Theorem 2.4. That the ESP implies the FMP if \mathcal{X}^- is compact has been proved in [20,29]. In fact, we mentioned in Section 2.2 that fading memory can also be defined without presupposing the ESP, which involves generalizing continuity of the solution map to a set-valued solution map. In this case, the reverse implication – that the FMP implies the ESP – holds as soon as there is some input for which there exists exactly one solution [20, 29]. All remaining implications in Theorem 2.4 follow from the next two lemmas.

Lemma 2.6. Suppose the state-space system has the state-uniform s-IFP. Then, there is at most one solution for each input. In particular, if $\underline{\mathcal{X}}^-$ is compact, then the state-space system has the ESP.

Proof. Suppose $\underline{x}, \underline{x}' \in \underline{\mathcal{X}}^-$ are two solutions for a given input $\underline{u}^- \in \underline{\mathcal{U}}^-$. Fix $t \in \mathbb{Z}_-$. Take any $\underline{u} \in \tau^{-1}(\underline{u}^-)$ and note that $\psi_n(\underline{x}_{t-n}, T^{n-t}(\underline{u})) = \underline{x}_t$. By the s-IFP applied to $T^{-t}(\underline{u})$,

$$d_{\mathcal{X}}(\underline{x}_t, \underline{x}_t') = d_{\mathcal{X}}(\psi_n(\underline{x}_{t-n}, T^{n-t}(\underline{u})), \psi_n(\underline{x}_{t-n}', T^{n-t}(\underline{u})))$$

converges to zero since the convergence is state-uniform. Thus, $\underline{x} = \underline{x}'$. If $\underline{\mathcal{X}}^-$ is compact, then the fibers of the attractor \mathcal{A} are non-empty by a standard topology argument. Indeed, each $\mathcal{A} \cap \pi^{-1}(\underline{u})$ becomes a nested intersection of the non-empty compact sets $\varphi^n(\underline{\mathcal{X}}^- \times \pi^{-1}(T^n(\underline{u})))$. The existence of at least one solution for each input is equivalent to \mathcal{A} having non-empty fibers.

Lemma 2.7. Suppose the state-space system has the ESP.

- If X⁻ is compact, then the state-space system has the state-uniform s-SFP.
- (ii) If $\underline{\mathcal{U}}$ and $\underline{\mathcal{X}}^-$ are compact, then the state-space system has the uniform s-SFP.

Proof. (ii) Suppose for contradiction the uniform s-SFP does not hold, that is, there exists an $\varepsilon > 0$, a strictly increasing sequence $(n_k)_k \subseteq \mathbb{N}$, and sequences $(\underline{u}^k)_k \subseteq \underline{\mathcal{U}}$, $(x_k)_k, (x_k')_k \subseteq \mathcal{X}$ with

$$d_{\mathcal{X}}(\psi_{n_k}(x_k, T^{n_k}(\underline{u}^k)), \psi_{n_k}(x_k', T^{n_k}(\underline{u}^k))) \ge \varepsilon.$$

Take $\underline{x}^k, \hat{\underline{x}}^k \in \underline{\mathcal{X}}^-$ with $\underline{x}_0^k = x_k$ and $\hat{\underline{x}}_0^k = x_k'$. By compactness, $\varphi^{n_k}(\underline{x}^k, T^{n_k}(\underline{u}^k))$ and $\varphi^{n_k}(\hat{\underline{x}}^k, T^{n_k}(\underline{u}^k))$ have some accumulation points $(\underline{x}, \underline{u})$ and $(\underline{x}', \underline{u}')$ satisfying $\underline{u} = \underline{u}'$. Since these accumulation points necessarily belong to \mathcal{A} , the ESP implies $\underline{x} = \underline{x}'$. However, this contradicts $d_{\mathcal{X}}(p_0(\underline{x}), p_0(\underline{x}')) \geq \varepsilon$, which follows from the relation $\psi_n \circ (p_0 \circ \pi) = p_0 \circ \varphi^n$.

(i) The same argument as for (ii) with $\underline{u}^k = \underline{u}$ not depending on k yields the state-uniform s-SFP since then $\varphi^{n_k}(\underline{x}^k, T^{n_k}(\underline{u})) \in \underline{\mathcal{X}}^- \times \{\underline{u}\}$ still belongs to a compact space and any accumulation point thereof must belong to $A \cap \pi^{-1}(\underline{u})$.

The following example shows that the implication arrows in Theorem 2.4 indicated in purple do not hold in general, even if the underlying spaces are compact and connected.

Example 2.8. Consider the homeomorphism $x\mapsto x^2$ of the unit interval [0,1]. Since the endpoints of the interval are fixed points, this map induces a homeomorphism $g\colon S^1\to S^1$ of the unit circle. Note that $g^n(x)$ converges to 0 as $n\to\infty$ for all $x\in S^1$. Thus, the state-space system governed by $f\colon S^1\times \mathcal{U}\to S^1,\ (x,u)\mapsto g(x)$ has the SFP. On the other hand, the SFP is not state-uniform. Every state is reachable since g is a homeomorphism, that is, $\mathcal{R}=S^1$. This implies that the ESP does not hold and that the IFP coincides with the SFP. Lastly, if we take $\underline{\mathcal{U}}$ to contain only constant sequences, then the s-SFP and s-IFP also coincide with the SFP. We remark that the state map can be perturbed to no longer be input-independent but retain all the other properties of f.

Remark 2.9. If the inputs are observations of a hidden dynamical system as in Theorem 2.3, then the maps

 $\psi_n(x,T^n(\underline{u}))$ recover the 'echo state family' considered in [11]. Therein, it was shown that this echo state family converges under a contractivity condition on the state map, which guarantees the ESP. Since [11] assumes compactness, it follows from our results that the contractivity condition is not necessary. Indeed, as soon as the ESP holds, the s-SFP implies convergence of the echo state family.

3. Unifying notions

To show that the newly presented definitions of state and input forgetting unify previous notions in the literature, we discuss the relevant definitions found in [2, 10, 12, 21].

3.1 Contracting and forgetting à la Jaeger

We point out that reachable states are what Jaeger [12] called 'end-compatible'. More specifically, a state $x \in \mathcal{X}$ is end-compatible with an input sequence $\underline{u} \in \underline{\mathcal{U}}^-$ if there exists a solution $\underline{x} \in \underline{\mathcal{X}}^-$ for the input \underline{u} with $\underline{x}_0 = x$. The notions of state contracting, state forgetting, and input forgetting in [12] are formulated with left-and right-infinite input sequences. Although the maps ψ_n formally take bi-infinite sequences as input, $\psi_n(x,\underline{u})$ depends only on $\underline{u}_1,\ldots,\underline{u}_n$ and $\psi_n(x,T^n(\underline{u}))$ depends only on $\underline{u}_{-n+1},\ldots,\underline{u}_0$. This allows us to express Jaeger's definitions of state contracting and state forgetting with the maps ψ_n . To state his definition of input forgetting, let $\gamma^n \colon \underline{\mathcal{U}}^- \times \underline{\mathcal{U}}^- \to \underline{\mathcal{U}}^-$, $n \in \mathbb{N}$, be the maps $\gamma^n(\underline{u}',\underline{u}) = (\ldots,\underline{u}'_{-1},\underline{u}'_0,\underline{u}_{-n+1},\ldots,\underline{u}_0)$. Now, in the language of [12], the system is

- state contracting if for every $\underline{u} \in \underline{\mathcal{U}}$ there exists some null-sequence $(\delta_n)_{n \in \mathbb{N}} \subseteq (0,1)$ such that for all $n \in \mathbb{N}$ and all states $x, x' \in \mathcal{X}$ it holds that $d_{\mathcal{X}}(\psi_n(x,\underline{u}),\psi_n(x',\underline{u})) < \delta_n$. If the null-sequence does not depend on \underline{u} , then the system is uniformly state contracting. Unpacking the definition of the limit superior in (2.1), it is clear that this notion of state contracting is precisely the state-uniform SFP, and uniformly state contracting is exactly the uniform SFP.
- state forgetting if for every $\underline{u} \in \underline{\mathcal{U}}$ there exists some null-sequence $(\delta_n)_{n \in \mathbb{N}} \subseteq (0,1)$ such that for all $n \in \mathbb{N}$ and all states $x, x' \in \mathcal{X}$ it holds that $d_{\mathcal{X}}(\psi_n(x, T^n(\underline{u})), \psi_n(x', T^n(\underline{u}))) < \delta_n$. As before, unpacking the definition of the limit superior in (2.2), it is clear that this notion of state forgetting is precisely the state-uniform s-SFP.
- input forgetting if for every $\underline{u}^0 \in \underline{\mathcal{U}}$ there exists some null-sequence $(\delta_n)_{n \in \mathbb{N}} \subseteq (0,1)$ such that for all $n \in \mathbb{N}$, all $\underline{u},\underline{u}' \in \underline{\mathcal{U}}^-$, and all states $x,x' \in \mathcal{X}$ that are end-compatible with $\gamma^n(\underline{u},\tau(\underline{u}^0))$ and $\gamma^n(\underline{u}',\tau(\underline{u}^0))$, respectively, it holds that $d_{\mathcal{X}}(x,x') < \delta_n$. Note that x being end-compatible with $\gamma^n(\underline{u},\tau(\underline{u}^0))$ is equivalent to the existence of a solution \underline{x} for the input \underline{u} such

that $x = \psi_n(\underline{x}_0, T^n(\underline{u}^0))$. In particular, $\underline{x}_0 \in \mathcal{R}$ is reachable. Thus, being input forgetting is equivalent to: for every $\underline{u}^0 \in \underline{\mathcal{U}}$ there exists some null-sequence $(\delta_n)_{n \in \mathbb{N}} \subseteq (0, 1)$ such that for all $n \in \mathbb{N}$ and all $x, x' \in \mathcal{R}$ it holds that $d_{\mathcal{X}}(\psi_n(x_0, T^n(\underline{u}^0)), \psi_n(x'_0, T^n(\underline{u}^0))) < \delta_n$. This shows that this notion of input forgetting is precisely the state-uniform s-IFP.

3.2 Forgetting à la Grigoryeva and Ortega

The (uniform) state forgetting property introduced by Grigoryeva and Ortega in [10] is exactly the (uniform) SFP we introduced in Section 2.3. Their variant of the input forgetting property is formulated with the *induced func*tional, which, assuming the state-space system has the ESP, is the unique map $\mathcal{H}: \underline{\mathcal{U}}^- \to \mathcal{X}$ that assigns to every input sequence the time zero component of its unique solution, that is, $\mathcal{H}(\underline{u}) = \underline{x}_0$ for all $(\underline{x}, \underline{u}) \in \mathcal{S}$. In [10], the statespace system is said to have the (uniform) input forgetting property if for all $\underline{u}, \underline{u}' \in \underline{\mathcal{U}}$ with $\underline{u}_t = \underline{u}'_t, t \in \mathbb{N}$, (uniformly in $\underline{u}, \underline{u}'$ $\lim_{n\to\infty} d_{\mathcal{X}}(\mathcal{H} \circ \tau \circ \sigma^n(\underline{u}), \mathcal{H} \circ \tau \circ \sigma^n(\underline{u}')) = 0$. Note that $\mathcal{H} \circ \tau \circ \sigma^n(\underline{u}) = \psi_n(\mathcal{H}(\tau(\underline{u})),\underline{u})$ and $\mathcal{H} \circ \tau \circ \sigma^n(\underline{u}') =$ $\psi_n(\mathcal{H}(\tau(\underline{u}')),\underline{u})$. The state $x' := \mathcal{H}(\tau(\underline{u}'))$ is reachable. Thus, the (uniform) input forgetting property in the sense of [10] is equivalent to: for all $\underline{u} \in \underline{\mathcal{U}}$ and all $x' \in \mathcal{R}$ (uniformly in \underline{u}, x' $\lim_{n \to \infty} d_{\mathcal{X}}(\psi_n(\mathcal{H}(\tau(\underline{u})), \underline{u}), \psi_n(x', \underline{u})) = 0.$ By the triangle inequality, this is further equivalent to: for all $\underline{u} \in \underline{\mathcal{U}}$ and all $x, x' \in \mathcal{R}$ (uniformly in \underline{u}, x, x') $\lim_{n\to\infty} d_{\mathcal{X}}(\psi_n(x,\underline{u}),\psi_n(x',\underline{u}))=0$. This is precisely the (uniform) IFP we introduced in Section 2.3.

3.3 Attraction à la Manjunath

Attractors have long played a central role in dynamical systems theory and come in many flavors [24]. Here, we review the choice of definition made in Manjunath's work [21], which discussed links between echo states, fading memory, and attractors, and is highly relevant to our discussion. In [21], the state-space system is said to have the uniform attracting property if for every $\underline{u} \in \mathcal{U}$ there exists a solution $\underline{x}(\underline{u}) \in \mathcal{X}^-$ for the input $\tau(\underline{u})$ such that

$$\lim_{n \to \infty} \sup_{t \in \mathbb{Z}_{-}} \sup_{\underline{u} \in \underline{\mathcal{U}}} \sup_{x \in \mathcal{X}} d_{\mathcal{X}} \left(\psi_n(x, T^{n-t}(\underline{u})), \underline{x}_t(\underline{u}) \right) = 0.$$

Note that if the uniform attracting property holds, then the ESP must hold and $\underline{x}_t(\underline{u}) = \mathcal{H} \circ \tau \circ T^{-t}(\underline{u}) = \psi_n(\mathcal{H} \circ \tau \circ T^{n-t}(\underline{u}), T^{n-t}(\underline{u}))$. Since $\underline{\mathcal{U}}$ is shift-invariant, the supremum over $\underline{u} \in \underline{\mathcal{U}}$ can be substituted by the supremum over $T^{n-t}(\underline{u}) \in \underline{\mathcal{U}}$. Thus, the uniform attracting property is equivalent to having the ESP and

$$\lim_{n\to\infty} \sup_{\underline{u}\in\underline{\mathcal{U}}} \sup_{x\in\mathcal{X}} d_{\mathcal{X}} \left(\psi_n(x,\underline{u}), \psi_n(\mathcal{H} \circ \tau(\underline{u}),\underline{u}) \right) = 0.$$

From this it is clear that the ESP and uniform SFP together imply the uniform attracting property, and the reverse implication also holds by the triangle inequality.

3.4 Steady states à la Boyd, Chua, and Green

The notion of steady-states or steady-state solutions dates back to the classical works in linear filtering and control [14, 17] but the precise definitions vary between different sources and earlier works preferred the terminology of asymptotic stability [15, 16]. Here, we use the definition from the work of Chua and Green [4] and of Boyd and Chua [2] due to the high impact that they had on the field, partly because they were the first to establish a formal link to fading memory. Their definitions are stated for systems in continuous time but are easily translated to a discrete time setting. It should be noted that [2,4] only consider bounded solutions. Thus, to recover their definitions exactly we assume \mathcal{X} to be bounded in this subsection. Steady states are defined through asymptotic convergence of solutions in forward time. When the state equation is considered in forward time only, solutions are uniquely determined by the maps ψ_n once the initial condition at time zero is fixed. Now, the state-space system is said to have the unique steady-state property if for all $\underline{u} \in \underline{\mathcal{U}}$ and all reachable states $x, x' \in \mathcal{R}$ it holds that $\lim_{n\to\infty} d_{\mathcal{X}}(\psi_n(x,\underline{u}),\psi_n(x',\underline{u}))=0$. This is precisely the IFP.

3.5 Further equivalences

In a similar spirit to the previous identifications, the next lemma highlights equivalent characterizations of the (shifted) SFP and IFP in the presence of the ESP.

Lemma 3.1. Suppose the state-space system has the ESP, and let $\mathcal{H}: \underline{\mathcal{U}}^- \to \mathcal{X}$ be the induced functional that satisfies $\mathcal{H}(\underline{u}) = \underline{x}_0$ for all solutions $(\underline{x},\underline{u}) \in \underline{\mathcal{X}}^- \times \underline{\mathcal{U}}^-$. Then, the state-space system has

(i) the (uniform) IFP if and only if for all $\underline{u}, \underline{u}' \in \underline{\mathcal{U}}$ with $u_t = u'_t$), $t \in \mathbb{N}$, (uniformly in u, u')

$$\lim\sup_{n\to\infty} d_{\mathcal{X}}(\mathcal{H}\circ\tau\circ\sigma^n(\underline{u}),\mathcal{H}\circ\tau\circ\sigma^n(\underline{u}'))=0.$$

(ii) the (state-uniform) [uniform] s-IFP if and only if for all $\underline{u},\underline{u}',\underline{u}'' \in \underline{\mathcal{U}}^-$ (uniformly in $\underline{u}',\underline{u}''$) [uniformly in $\underline{u},\underline{u}',\underline{u}''$]

$$\limsup_{n\to\infty} d_{\mathcal{X}}(\mathcal{H}\circ\gamma^n(\underline{u}',\underline{u}),\mathcal{H}\circ\gamma^n(\underline{u}'',\underline{u}))=0.$$

(iii) the (state-uniform) [uniform] SFP if and only if for all $\underline{u} \in \underline{\mathcal{U}}$ and $x \in \mathcal{X}$ (uniformly in x) [uniformly in \underline{u}, x]

$$\limsup_{n\to\infty} d_{\mathcal{X}}(\psi_n(x,\underline{u}),\mathcal{H}\circ\tau\circ\sigma^n(\underline{u}))=0.$$

(iv) the state-uniform (uniform) s-SFP if and only if for all $\underline{u} \in \underline{\mathcal{U}}$ and $x \in \mathcal{X}$ uniformly in x (uniformly in \underline{u}, x)

$$\lim \sup_{n \to \infty} d_{\mathcal{X}}(\psi_n(x, T^n(\underline{u})), \mathcal{H} \circ \tau(\underline{u})) = 0.$$

Proof. (i) This was shown in Section 3.2.

- (ii) Since $\mathcal{H} \circ \gamma^n(\underline{u}',\underline{u})$ is the unique state that is end-compatible with $\gamma^n(\underline{u}',\underline{u})$, this item is shown as in the last bullet point in Section 3.1.
- (iii) One direction of the equivalence follows from the triangle inequality and other one from the fact that $\mathcal{H} \circ \tau \circ \sigma^n(\underline{u}) = \psi_n(\mathcal{H} \circ \tau(\underline{u}), \underline{u}).$
- (iv) As for (iii), one direction of the equivalence follows from the triangle inequality and other one from the fact that $\mathcal{H} \circ \tau(\underline{u}) = \psi_n(\underline{x}_{-n}, T^n(\underline{u}))$. However, note that the equivalence does not remain valid without the attribute 'state-uniform' because the argument \underline{x}_{-n} of ψ_n depends on n.

Using this characterization, we present one final result. Suppose the ESP holds. Theorem 2.7 guarantees the state-uniform s-SFP if $\underline{\mathcal{X}}^-$ is compact. Since the FMP is also implied by compactness of $\underline{\mathcal{X}}^-$ in the presence of the ESP, one can use the FMP instead as a weaker assumption. In this case, we still get the state-uniform s-IFP. Here, it is crucial that we used the product topology to define the FMP.

Proposition 3.2. Suppose \mathcal{U} is metrizable and the state-space system has the ESP and the FMP. Then, the state-space system has the state-uniform s-IFP.

Proof. Let $d_{\mathcal{U}}$ be a metric that metrizes the topology on \mathcal{U} . Then, the product topology on $\underline{\mathcal{U}}^-$ is metrized by $d_{\underline{\mathcal{U}}^-}(\underline{u}',\underline{u}) = \sup_{t \in \mathbb{Z}_-} 2^t \min\{1, d_{\mathcal{U}}(\underline{u}'_t,\underline{u}_t)\}$. In particular, $d_{\underline{\mathcal{U}}^-}(\gamma^n(\underline{u}',\underline{u}),\underline{u}) \leq 2^{-(n+1)}$ for any $\underline{u},\underline{u}' \in \underline{\mathcal{U}}^-$ and $n \in \mathbb{N}$. Now, given any $\underline{u} \in \underline{\mathcal{U}}^-$ and $\varepsilon > 0$, the FMP yields some $\delta > 0$ such that $d_{\mathcal{X}}(\mathcal{H}(\underline{u}'),\mathcal{H}(\underline{u})) < \varepsilon/2$ for all $\underline{u}' \in \underline{\mathcal{U}}^-$ that satisfy $d_{\underline{\mathcal{U}}^-}(\underline{u}',\underline{u}) < \delta$. Take $N \in \mathbb{N}$ so that $2^{-(N+1)} < \delta$. Then, $d_{\mathcal{X}}(\mathcal{H} \circ \gamma^n(\underline{u}',\underline{u}),\mathcal{H}(\underline{u})) < \varepsilon/2$ for any $\underline{u}' \in \underline{\mathcal{U}}^-$ and $n \geq N$. Theorem 3.1.(ii) and the triangle inequality yield the state-uniform s-IFP.

4. Conclusion

This work has provided a unified framework for understanding the various notions of memory in RNNs, including steady states, echo states, state forgetting, input forgetting, and fading memory. By clarifying the relationships between these concepts, we have shed light on the intricate dynamics of RNNs and their memory capacities. Our results have established new implications and equivalences between these notions, providing a deeper understanding of how RNNs create reliable input/output responses, and have provided shorter proofs for previous related results. Future research can build upon this foundation to further explore the properties and limitations of RNNs, ultimately leading to more efficient and effective models for a wide range of applications.

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