Prep Work 12

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Presentation:

Exercise 5 3.2

Prove that for all integers n, it is the case that n is even if and only if $3n$ is even. That is, prove both implications: if n is even, then $3n$ is even, and if $3n$ is even, then n is even.

Solution Draft:

If n is even, then $3n$ is even.
Assuming n is even. Then, there exists an integer k such that $n = 2k$. Consider the product $3n$:

$$\begin{aligned} 3n &= 3(2k) \\ &= 6k \\ &= 2(3k) \end{aligned}$$

Since $3k$ is an integer, $6k$ is even, proving the first part.

If $3n$ is even, then n is even.
Assuming $3n$ is even. Then, there exists an integer m such that $3n = 2m$. To prove n is even by contradiction, we assume n is odd, so $n = 2k + 1$ for some k. Then,

$$\begin{aligned} 3n &= 3(2k + 1) \\ &= 6k + 3 \\ &= 2(3k) + 3 \end{aligned}$$

This is not divisible by 2, a contradiction. Therefore, n must be even.

Exercise 6 3.2

Prove that $\sqrt{3}$ is irrational.

Solution Draft:

To begin, we will assume that $\sqrt{3}$ is rational, meaning it can be expressed as a fraction $\frac{a}{b}$, where a and b are integers with no common factor other than 1, and $b \neq 0$.

Given this assumption, we have:

$$\sqrt{3} = \frac{a}{b}$$

Squaring both sides yields:

$$3 = \frac{a^2}{b^2}$$

Rearranging gives:

$$a^2 = 3b^2$$

This implies that a^2 is a multiple of 3. For a^2 to be a multiple of 3, a itself must also be a multiple of 3 as the square of a non-multiple of 3 cannot be a multiple of 3. Let us denote a as $3k$, where k is an integer.

Substituting $3k$ for a in the equation $a^2 = 3b^2$ we get:

$$(3k)^2 = 3b^2$$
$$9k^2 = 3b^2$$
$$b^2 = 3k^2$$

This shows that b^2 is also a multiple of 3, and hence b must also be a multiple of 3.

However, since both a and b are multiples of 3, they share a common factor greater than 1. This contradicts our initial assertion that a and b have no common factor other than 1. Therefore, our assumption that $\sqrt{3}$ is rational must be false.

Exercise 14 3.2

Prove that there are no integer solutions to the equation $x^2 = 4y + 3$.

Solution Draft:

To start, we assume integers x and y exist and complete the equation:

$$x^2 = 4y + 3$$

On the right side, $4y + 3$, indicates that for any integer y, $4y$ is divisible by 4, and adding 3 to it makes $4y + 3$ equal to one more than a multiple of 4. Thus, dividing $4y + 3$ by 4 leaves a remainder of 3.

On the left side, the remainder when an integer squared is divided by 4 can only be 0 or 1. This is because: - If x is even, then $x^2 = (2k)^2 = 4k^2$; divisible by 4. - If x is odd (say $x = 2k + 1$), this leaves a remainder of 1 when divided by 4.

$$\begin{aligned} x^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 4(k^2 + k) + 1 \end{aligned}$$

Therefore, a squared integer can never have a remainder of 3 after divided by 4. This contradicts the original statement. So, there are no integer solutions to the equation $x^2 = 4y + 3$.