Prep Work Template

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Exercise 11

Presentation: Include here whether you'd be willing to present this one.

Question Suppose $f : \mathbb{N} \to \mathbb{N}$ satisfies the recurrence relation

$$f(n+1) = \begin{cases} \frac{f(n)}{2} & \text{if } f(n) \text{ is even,} \\ 3f(n)+1 & \text{if } f(n) \text{ is odd} \end{cases}$$

Note that with the initial condition $f(0) = 1$, the values of the function are: $f(1) = 4, f(2) = 2, f(3) = 1, f(4) = 4$, and so on, the images cycling through those three numbers. Thus f is NOT injective (and also certainly not surjective). Might it be under other initial conditions?

a. If f satisfies the initial condition $f(0) = 5$, is f injective? Explain why or give a specific example of two elements from the domain with the same image.

b. If f satisfies the initial condition $f(0) = 3$, is f injective? Explain why or give a specific example of two elements from the domain with the same image.

c. If f satisfies the initial condition $f(0) = 27$, then it turns out that $f(105) = 10$ and no two numbers less than 105 have the same image. Could f be injective? Explain.

d. Prove that no matter what initial condition you choose, the function cannot be surjective.

Solution Draft: The function $f : \mathbb{N} \to \mathbb{N}$ is defined by the recursive relation:

$$f(n+1) = \begin{cases} \frac{f(n)}{2} & \text{if } f(n) \text{ is even,} \\ 3f(n)+1 & \text{if } f(n) \text{ is odd.} \end{cases}$$

Given the initial condition $f(0) = 1$, the function cycles through the values 1, 4, 2, and so on, which shows that f is neither injective nor surjective.

a. With the initial condition $f(0) = 5$, we need to check if different natural numbers are mapped to the same result by evaluating f for several values. Without further evaluation, we cannot determine injectivity just from the initial condition.

b. Similarly, with the initial condition $f(0) = 3$, we would proceed by evaluating f for several values to check for injectivity. The initial condition alone is not sufficient to conclude injectivity.

c. If $f(105) = 10$ and no two numbers less than 105 have the same image, then f is injective up to the argument 105. Injectivity beyond that would depend on the behavior of the function for numbers greater than 105.

d. The function cannot be surjective as the recursive definition ensures that some natural numbers will never be reached. For example, no natural number n would satisfy $f(n) = 6$ given the cycling through certain numbers.

Exercise 15

Presentation: Include here whether you'd be willing to present this one.

Question

Consider the set $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, the set of all ordered pairs (a, b) where a and b are natural numbers. Consider a function $f : \mathbb{N}^2 \to \mathbb{N}$ given by $f((a, b)) = a + b$.

a. Let $A = \{(a, b) \in \mathbb{N}^2 : a, b < 10\}$. Find $f(A)$.

b. Find $f^{-1}(3)$ and $f^{-1}(\{0, 1, 2, 3\})$.

c. Give geometric descriptions of $f^{-1}(n)$ and $f^{-1}(\{0, 1, \ldots, n\})$ for any $n \geq 1$.

d. Find $|f^{-1}(8)|$ and $f^{-1}(\{0, 1, \ldots, 8\})$.

Solution Draft:

Consider the function $f : \mathbb{N}^2 \to \mathbb{N}$ defined by $f((a, b)) = a + b$.

a. The set A is $\{(a, b) \in \mathbb{N}^2 : a, b < 10\}$. The function $f(A)$ is the set of sums of pairs (a, b) where $a, b < 10$, which ranges from 0 or 1 to 18, depending on the inclusion of 0 in $\mathbb{N}$.

b. The preimage $f^{-1}(3)$ includes pairs (a, b) such that $a + b = 3$. These are $(0, 3)$, $(1, 2)$, $(2, 1)$, and $(3, 0)$. The preimage $f^{-1}(\{0, 1, 2, 3\})$ includes all pairs whose sums are 0, 1, 2, or 3.

c. The geometric description of $f^{-1}(n)$ is a diagonal line from $(0, n)$ to $(n, 0)$. For $f^{-1}(\{0, 1, \ldots, n\})$, the description would be a triangle with vertices at $(0, 0)$, $(n, 0)$, and $(0, n)$.

d. The size of $f^{-1}(8)$ is 9, corresponding to the pairs from $(0, 8)$ to $(8, 0)$. The size of $f^{-1}(\{0, 1, \ldots, 8\})$ is 45 if 0 is included in $\mathbb{N}$, or 36 otherwise.

Exercise 28

Presentation: Include here whether you'd be willing to present this one.

Question

Let $f : X \to Y$ be a function, $A \subseteq X$ and $B \subseteq Y$.

(a) Is $f^{-1}(f(A)) = A$? Always, sometimes, never? Explain.

(b) Is $f(f^{-1}(B)) = B$? Always, sometimes, never? Explain.

(c) If one or both of the above do not always hold, is there something else you can say? Will equality always hold for particular types of functions? Is there some other relationship other than equality that would always hold? Explore.

Solution Draft:

This question explores the properties of functions in terms of the preimage and image of sets.

a. The statement $f^{-1}(f(A)) = A$ can sometimes hold, but not always. It is guaranteed to hold if f is injective because injectivity ensures that each element in A is uniquely mapped to an element in $f(A)$, and vice versa. However, if f is not injective, there may be elements in X not in A that are mapped to elements in $f(A)$, causing $f^{-1}(f(A))$ to potentially include elements not in A.

b. The statement $f(f^{-1}(B)) = B$ holds when $B \subseteq f(X)$, the image of f. This is because the preimage $f^{-1}(B)$ includes all elements in X that are mapped to elements in B, and applying f to this preimage essentially retrieves B. However, if there are elements in B not in the image of f, then $f(f^{-1}(B))$ cannot equal B as those elements have no preimage.

c. For bijective functions, both $f^{-1}(f(A)) = A$ and $f(f^{-1}(B)) = B$ always hold due to the one-to-one correspondence between elements in the domain and codomain. If f is injective, $f^{-1}(f(A)) = A$ always holds; if f is surjective, $f(f^{-1}(B)) = B$ always holds.

Exercise 29

Presentation: Include here whether you'd be willing to present this one.

Question

Let $f : X \to Y$ be a function and $A, B \subseteq X$ be subsets of the domain.

(a) Is $f(A \cup B) = f(A) \cup f(B)$? Always, sometimes, or never? Explain.

(b) Is $f(A \cap B) = f(A) \cap f(B)$? Always, sometimes, or never? Explain.

Solution Draft:

a. The statement $f(A \cup B) = f(A) \cup f(B)$ always holds true. This is because the image of the union of two sets A and B under f includes all elements that f maps from either A or B, which is exactly the union of $f(A)$ and $f(B)$.

b. The statement $f(A \cap B) = f(A) \cap f(B)$ does not always hold. If f is injective, then this statement can hold because injectivity ensures that f maps distinct elements of A and B to distinct elements in Y, and only elements common to both A and B would be mapped to elements common to both $f(A)$ and $f(B)$. However, without injectivity, it's possible for $f(A \cap B)$ to be smaller than $f(A) \cap f(B)$ because different elements in A and B could map to the same element in Y, inflating $f(A) \cap f(B)$ beyond what is actually mapped from $A \cap B$.

Exercise 30

Presentation: Include here whether you'd be willing to present this one.

Question

Let $f : X \to Y$ be a function and $A, B \subseteq Y$ be subsets of the codomain.

(a) Is $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$? Always, sometimes, or never? Explain.

(b) Is $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$? Always, sometimes, or never? Explain.

Solution Draft:

a. The statement $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ always holds. This is because the preimage of a union $A \cup B$ includes all elements in X that f maps to either A or B, which precisely corresponds to the union of the preimages $f^{-1}(A)$ and $f^{-1}(B)$.

b. Similarly, the statement $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ also always holds. The preimage of an intersection $A \cap B$ consists of all elements in X that f maps to elements present in both A and B, which is exactly what the intersection of the preimages $f^{-1}(A)$ and $f^{-1}(B)$ represents.