2012 Leaving Cert Higher Level
Official Sample Paper 2

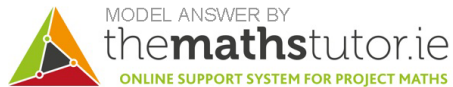| Section A	Concepts and Skills	150 marks

Question 1 (25 marks)

The events A and B are such that $P(A) = 0.7, P(B) = 0.5$ and $P(A \cap B) = 0.3$.
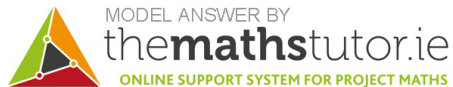
(a) Find $P(A \cup B)$.

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
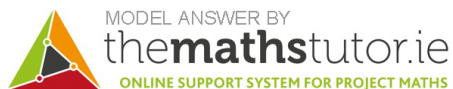&= 0.7 + 0.5 - 0.3 \\
&= 0.9
\end{aligned}
$$

(b) Find $P(A|B)$.

$$
\begin{aligned}
P(A|B) &= \frac{P(A \cap B)}{P(B)} \\
&= \frac{0.3}{0.5} \\
&= 0.6
\end{aligned}
$$

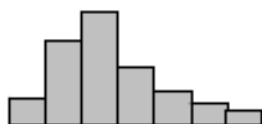**(c)** State whether A and B are independent events and justify your answer

If A and B are independent events then $P(A \cap B) = P(A)P(B)$.
Here, $P(A \cap B) = 0.3$ but $P(A)P(B) = (0.7)(0.9) = 0.63$.
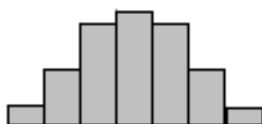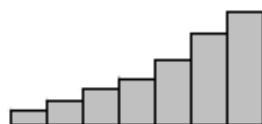So A and B are NOT independent events.

Question 2 (25 marks)

(a) Complete the table below indicating whether a statement is correct (✓) or incorrect with respect to each data set.
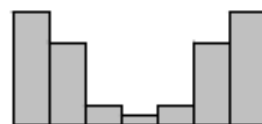

A   B   C   D

	A	B	C	D
The data are skewed to the left			✓	
The data are skewed to the right	✓			
The mean is equal to the median		✓		✓
The mean is greater than the median	✓			
There is a single mode	✓	✓	✓	

(b) Assume the four histograms are drawn on the same scale. State which of them has the largest standard deviation, and justify your answer.

Histogram D has the largest standard deviation.
Justification: The standard deviation measures how far the data are spread out from the mean. Since histogram D is symmetric, the mean is in the middle.
But we can also see that in histogram D, most of the data are spread out to the far right and left of the diagram, generally far from the mean.
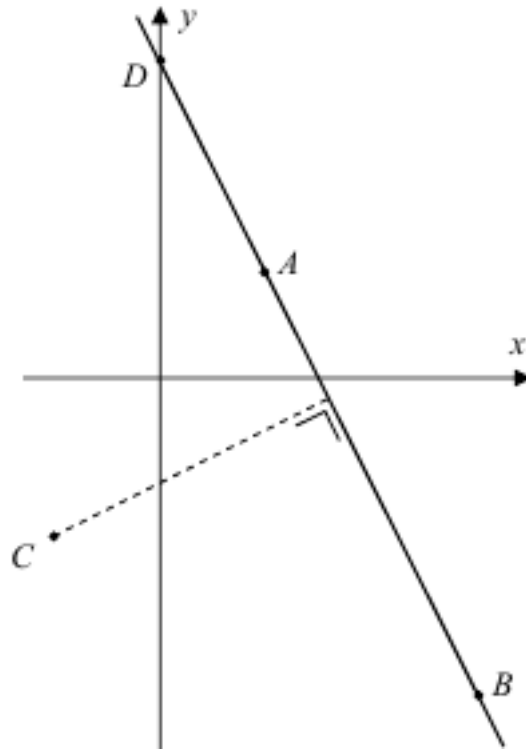Therefore, of the four histograms, D has the largest standard deviation.

Question 3

The co-ordinates of three points A, B, and C are:
$A(2,2), B(6,-6), C(-2,-3)$ (see diagram).

(a) Find the equation of AB.

Here $(x_1, y_1) = (2,2)$ and $(x_2, y_2) = (6,-6)$. First find the slope of AB

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6-2}{6-2} = \frac{-8}{4} = -2$$

Now use the equation of a line formula

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 2 &= -2(x - 2) \\
y - 2 &= -2x + 4 \\
2x + y - 6 &= 0
\end{aligned}
$$

(b) The line AB intersects the y-axis at D. Find the coordinates of D.

Use the equation of AB and set $x = 0$ to get

$$2(0) + y - 6 = 0$$

$$y - 6 = 0$$

which means $y = 6$ and so $D = (0,6)$.

(c) Find the perpendicular distance from C to AB.

Using the perpendicular distance formula:

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

From the equation of AB we have $a = 2, b = 1, c = -6$ and from the coordinates of C we have $x_1 = -2, y_1 = -3$. Then the perpendicular distance between C and AB is

$$\frac{|2(-2) + 1(-3) - 6|}{\sqrt{2^2 + 1^2}} = \frac{|-4 - 3 - 6|}{\sqrt{4 + 1}} = \frac{|-13|}{\sqrt{5}} = \frac{13}{\sqrt{5}}$$

(d) Hence, find the area of the triangle ADC.

If *AD* is the base of the triangle then the perpendicular height is the answer from part (**c**). The distance from *A* to *D* is

$$|AD| = \sqrt{(0-2)^2 + (6-2)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

So the area of *ADC* is

$$\frac{1}{2}\text{base} \times \text{perpendicular height} \quad = \quad \frac{1}{2}(2\sqrt{5})\left(\frac{13}{\sqrt{5}}\right)$$

$$= \quad 13 \text{ units squared}$$

Question 4 (25 marks)

(a) Write down the equation of the circle with centre $(-3, 2)$ and radius 4.

Let the centre of the circle $(h, k) = (-3, 2)$ and $r = 4$. So the equation of the circle is

$$
\begin{aligned}
(x - h)^2 + (y - k)^2 &= r^2 \\
(x + 3)^2 + (y - 2)^2 &= 4^2 \\
x^2 + 6x + 9 + y^2 - 4y + 4 &= 16 \\
x^2 + y^2 + 6x - 4y - 3 &= 0
\end{aligned}
$$

(b) A circle has equation $x^2 + y^2 - 2x + 4y - 15 = 0$. Find the values of m for which the line $mx + 2y - 7 = 0$ is a tangent line.

Re-write this equation as $x^2 + y^2 + 2(-1)x + 2(2)y - 15 = 0$ which matches the equation of a circle with $g = -1, f = 2, c = -15$. So this circle has centre $(-g, -f) = (1, -2)$ and radius $r = \sqrt{g^2 + f^2 - c} = \sqrt{1 + 4 + 15} = \sqrt{20}$.

For a line to be tangent to this circle, the perpendicular distance from that line to the centre $(1, -2)$ must be equal to the radius. The distance from the line $mx + 2y - 7 = 0$ to $(1, -2)$ is

$$
\frac{|m(1) + 2(-2) - 7|}{\sqrt{m^2 + 2^2}}
$$

This must be equal to the radius in order to be tangent which means

$$
\begin{aligned}
\frac{|m(1) + 2(-2) - 7|}{\sqrt{m^2 + 2^2}} &= \sqrt{20} \\
|m - 4 - 7| &= \sqrt{20}\sqrt{m^2 + 4} \\
|m - 11| &= \sqrt{20m^2 + 80} \\
(m - 11)^2 &= 20m^2 + 80 \\
m^2 - 22m + 121 &= 20m^2 + 80 \\
0 &= 19m^2 + 22m - 41
\end{aligned}
$$

We can solve this quadratic to get solutions $m = 1$ and $m = -\dfrac{41}{19}$
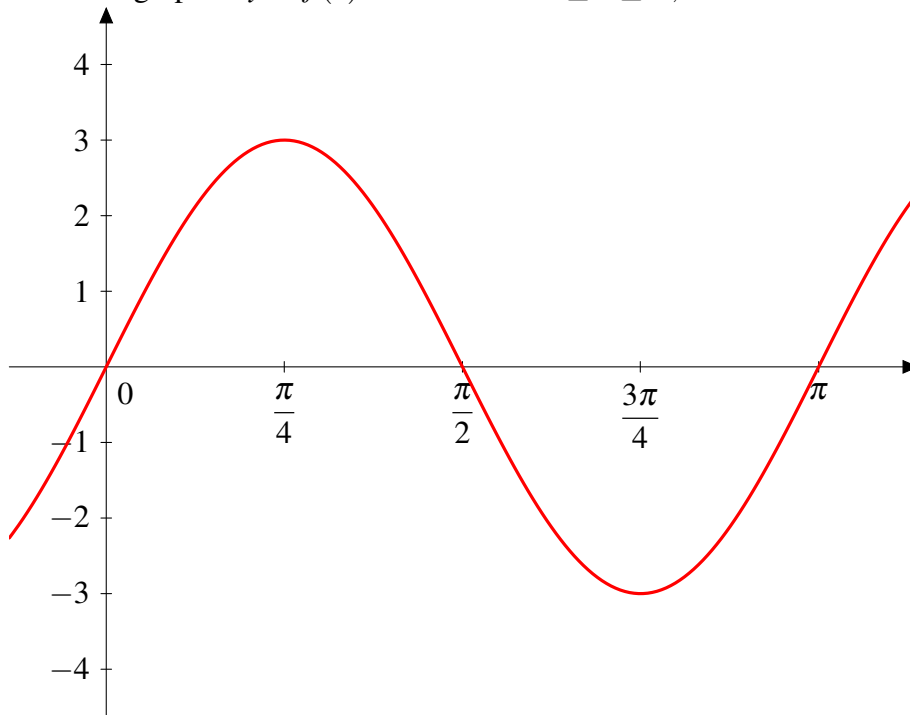
Question 5 (25 marks)

The function $f : x \mapsto 3\sin(2x)$ is defined for $x \in \mathbb{R}$.

(a) Complete the table below

x	0	$\dfrac{\pi}{4}$	$\dfrac{\pi}{2}$	$\dfrac{3\pi}{4}$	π
$2x$	0	$\dfrac{\pi}{2}$	π	$\dfrac{3\pi}{2}$	2π
$\sin(2x)$	0	1	0	-1	0
$3\sin(2x)$	0	3	0	-3	0

(b) Draw the graph of $y = f(x)$ in the domain $0 \le x \le \pi, \ x \in \mathbb{R}$.

(c) Write down the range and period of f.

$$\text{Range} = [-3, 3] \qquad\qquad \text{Period} = \pi$$

Question 6A (25 marks)

Explain with the aid of an example what is meant by *proof by contradiction*.

Note: you do not need to provide the full proof in your example. Give a sufficient outline to demonstrate how contradiction is used.

Explanation:

To prove a statement by contradiction, we assume that the statement is false and then prove that this assumption contradicts another statement that is known to be true.
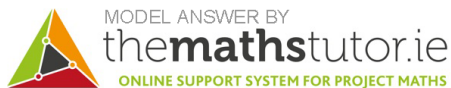
Example:

Consider the statement

The lines $y = x$ and $y = x + 1$ do not intersect.

We can prove this by contradiction as follows.
Assume that that the statement is false. So there is some point (a, b) that lies on both lines. Therefore $b = a$ since the point is on the line $y = x$ and $b = a + 1$ since the point is on the line $y = x + 1$.
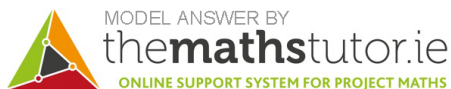Combining these equation, we get
$$a = a + 1$$
which implies that
$$0 = 1.$$

However this contradicts that fact that $0 \neq 1$. Therefore our original assumption is false. So we have proved that the lines $y = x$ and $y = x + 1$ do not intersect.
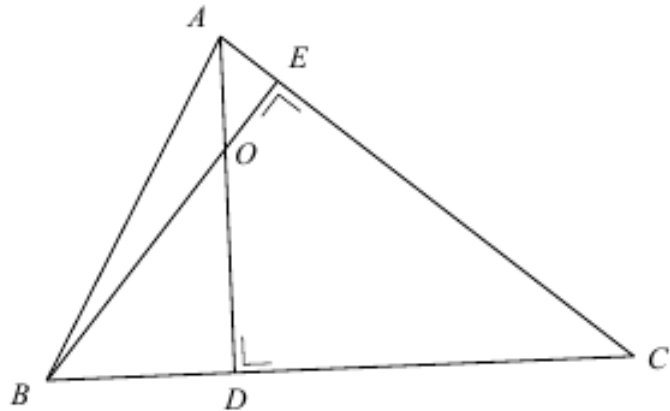
Question 6B (25 marks)

ABC is a triangle.

D is a point on *BC* such that $AD \perp BC$.
E is a point on *AC* such that $BE \perp AC$.

AD and *BE* intersect at 0.

Prove that $|\angle DOC| = |\angle DEC|$.

Consider the quadrilateral *DOEC*. We have

$$|\angle CDO| + |\angle OEC| = 90° + 90° = 180°.$$

Therefore *DOEC* is a cyclic quadrilateral (by the converse of Corollary 5).

Therefore $\angle DOC$ and $\angle DEC$ are angles standing on the same arc of a circle (the circumcircle of *DOEC*).

Therefore, by Theorem 19

$$|\angle DOC| = |\angle DEC|$$

as required.

Question 7   **(75 marks)**

(a) An economics students wants to find out whether the length of time people spend in education affects the income they earn. The student carries out a small study. Twelve adults are asked to state their annual income and the number of years they spent in full-time education. The data are given in the table below, and a partially completed scatter plot is given.

Years of education	Income /€1,000
11	28
12	30
13	35
13	43
14	55
15	38
16	45
16	38
17	55
17	60
17	30
19	58
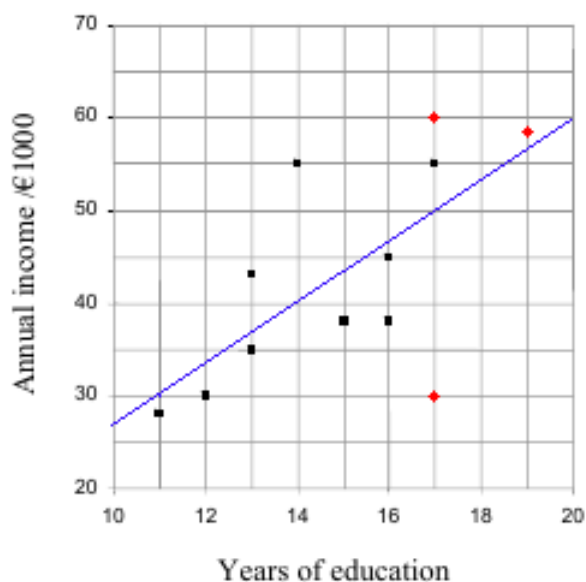
    (i) The last three rows of the data have not been included on the scatter plot. Insert them now.

See red points inserted in the scatter plot above.

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(ii) Calculate the correlation coefficient.

The correlation coefficient may be calculated using your electronic calculator. This is sufficient for exam purposes.

For information, it is calculated by hand as follows:

Let $x_1, x_2, \ldots, x_{12}$ denote the values for "Years" and let $y_1, y_2, \ldots y_{12}$ denote the values for "Income". Then, the correlation coefficient is defined by

$$\rho = \frac{n \sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} y_i \right)}{\sqrt{n \sum_{i=1}^{n} (x_i)^2 - \left(\sum_{i=1}^{n} x_i \right)^2} \sqrt{n \sum_{i=1}^{n} (y_i)^2 - \left(\sum_{i=1}^{12} y_i \right)^2}}$$

where n is the number of data points. Consider the following table:

x_i	y_i	$x_i y_i$	$(x_i)^2$	$(y_i)^2$
11	28	308	121	784
12	30	360	144	900
13	35	455	169	1,125
13	43	559	169	1,849
14	55	770	196	3,025
15	38	570	225	1,444
16	45	720	256	2,025
16	38	608	256	1,444
17	55	935	289	3,025
17	60	1.020	289	3,600
17	30	510	289	900
19	58	1.102	361	3,364
180	515	7,917	2,764	23,585
$\sum_{i=1}^{12} x_i$	$\sum_{i=1}^{12} y_i$	$\sum_{i=1}^{12} x_i y_i$	$\sum_{i=1}^{12} (x_i)^2$	$\sum_{i=1}^{12} (y_i)^2$

Thus, our correlation coefficient becomes

$$\rho = \frac{12(7917) - (180)(515)}{\sqrt{12(2,764) - (180)^2} \sqrt{12(23,585) - (515)^2}} = 0.62324$$

(iii) What can you conclude from the scatter plot and the correlation coefficient?

The above scatter plot implies that there is a rough linear relationship between years in education and income. We can see that the data is increasing roughly together, although there are some outliers which do not follow the same pattern as the rest of data.

This agrees with the correlation coefficient. A value close to 0 represents no relationship between the data, and a value close to 1 represents a strong positive linear relationship. Our value ($\rho = 0.623$) means that there is a positive linear relationship, but not a very strong one.

We can conclude that there is a moderate positive correlation between number of years in education and annual income i.e. as one increases, the other tends to also increase.
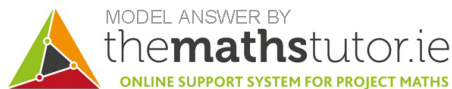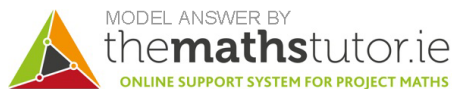
(iv) Add the line of best fit to the scatter plot above.
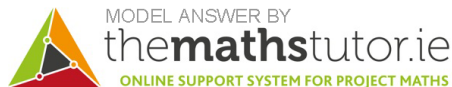
See blue line above.
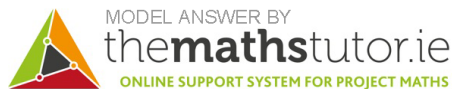
(v) Use the line of best fit to estimate the annual income of somebody who has spent 14 years in education.

€41,000

(vi) By taking suitable readings from your diagram, or otherwise, calculate the slope of the line of best fit.

Using the points $(14, 41)$ and $(17, 50)$ the slope of the line is $\dfrac{50 - 41}{17 - 14} = \dfrac{9}{3} = 3$.

(vii) Explain how to interpret the slope in this context.

The slope is the rise over the run. In this case, as we can write the slope of 3 as $\frac{3}{1}$, it implies that for every 1 extra year of education, there is a corresponding rise of 3 units of income. As the units of income are thousands of euro, then this implies a rise of 3000 euro in annual income per extra year of education.

(viii) The student collected the data using a telephone survey. Numbers were randomly chosen from the Dublin area telephone directory. The calls were made in the evenings, between 7 and 9pm. If there was no answer, or if the person who answered did not agree to participate, then another number was chosen at random.

List **three** possible problems regarding the sample and how it was collected that might make the result of the investigation unreliable. In each case, state clearly why the issue you mention could cause a problem.
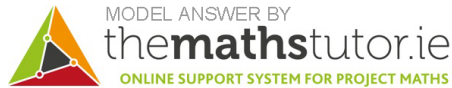
Problem 1:

Calls were made in the evening. This can affect the likelihood of getting responses from some demographics versus other demographics e.g. office workers vs shift workers, or employed people vs unemployed people, or parents of young children vs others. This may make the sample unrepresentative and could cause bias in the data.
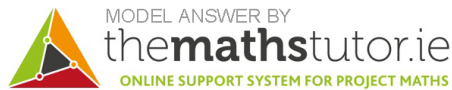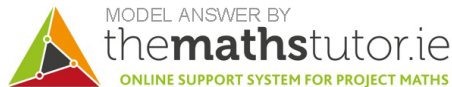
Problem 2:

The telephone directory shows landline numbers only. Many younger people do not have a landline. This may make the sample unrepresentative, causing bias in the data..

Problem 3:

If the survey is intended to represent the whole country, then using the Dublin telephone directory will mean that the data is unrepresentative, causing bias.

(b) The distribution of the hourly earnings of all employees in Ireland in October 2009 is shown in the diagram. It can be seen that the distribution is positively skewed.

The mean is €22.05.
The median is €17.82.
The standard deviation is €10.64
The lower quartile is €12.80
The upper quartile is €26.05
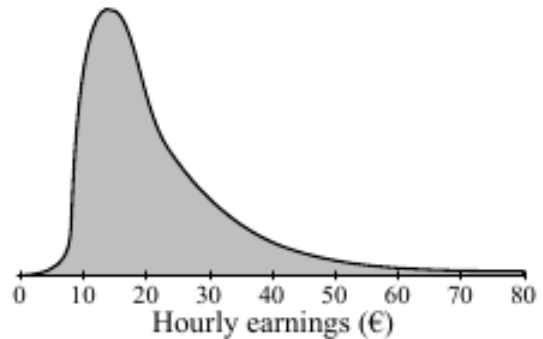
Hourly earnings (€)

(i) If six employees are selected at random from this population, what is the probability that exactly four of them had hourly earnings of more than €12.80?

Since €12.80 is the lower quartile, the probability of any <u>one</u> randomly selected person having hourly earnings of more than €12.80 is 0.75. If we repeat this selecting 6 times we are conducting a Bernouilli trial

$$P(r \text{ successes from } n \text{ trials}) = \binom{n}{r} p^r (1-p)^{n-r}$$

where p is the probability of a success. Let "success" be earning over €12.80. Then the probability of choosing 4 people who earn over €12.80 in 6 trials is

$$\binom{6}{4} 0.75^4 (0.25)^2$$
$$= (15)(0.3164)(0.0625)$$
$$= 0.296625$$

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(ii) In a computer simulation, random samples of size 200 are repeatedly selected from this population and the mean of each sample is recorded. A thousand such sample means are recorded.

Describe the expected distribution of these sample means. Your description should refer to the shape of this distribution and to its mean and standard deviation.

The Law of Large Numbers says that these sample means will be normally distributed with mean = 22.05 and standard deviation = $\dfrac{10.64}{\sqrt{200}} = 0.752$
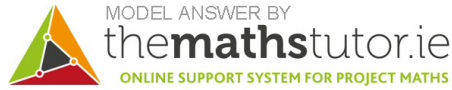
(iii) How many of the sample means would you expect to be greater than €23?

Let X be a normally distributed random variable with mean 22.05 and standard deviation 0.752

We will begin by finding the probability that $X \leq 23$. Since X is normally distributed with mean 22.05 and standard deviation 0.752, we will convert it into a Standard Normal Z distribution:

$$X \leq 23 \iff Z \leq \frac{23 - 22.05}{0.752} = 1.26$$

Note that the last figure is taken to two decimal places. From this calculation, we know that $P(X \leq 23) = P(Z \leq 1.26)$. From the z-tables, this probability works out to be 0.8762. This means that $P(X > 23) = 1 - 0.8762 = 0.1238$. A thousand sample means are calculated, so the expected number of sample means greater than €23 is $1000 \times 0.1238 \approx 124$.
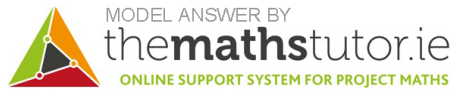
Question 8 (75 marks)

(a) Two surveyors want to find the height of an electricity pylon. There is a fence around the pylon that they cannot cross for safety reasons. The ground is inclined at an angle. They have a clinometer (for measuring angles of elevation) and a 100 metre tape measure. They have already used the clinometer to determine that the ground is inclined at $10°$ to the horizontal.

(i) Explain how they could find the height of the pylon.

Your answer should be illustrated on the diagram below. Show the points where you think they should take measurements, write down clearly what measurements they should take, and outline briefly how these can be used to find the height of the pylon.

10°
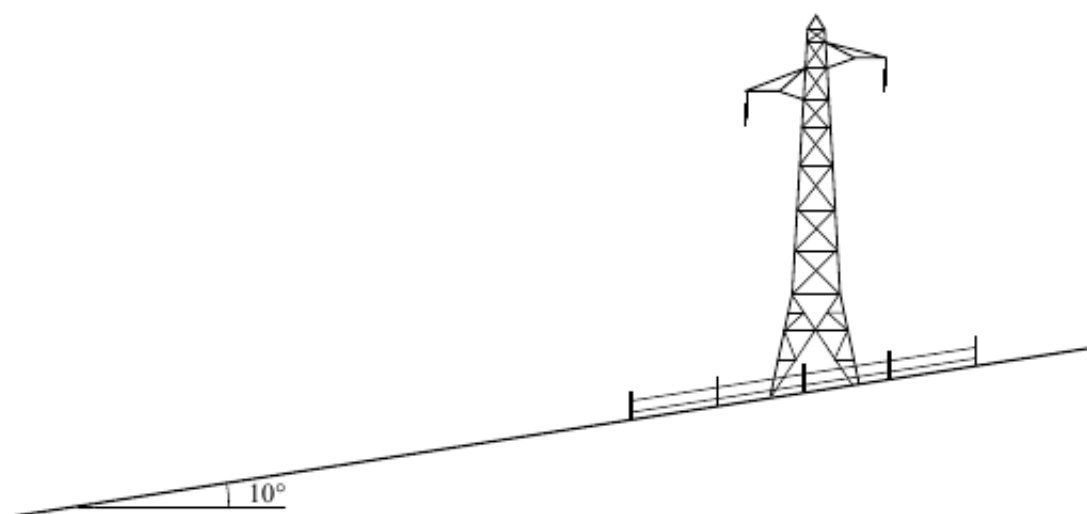
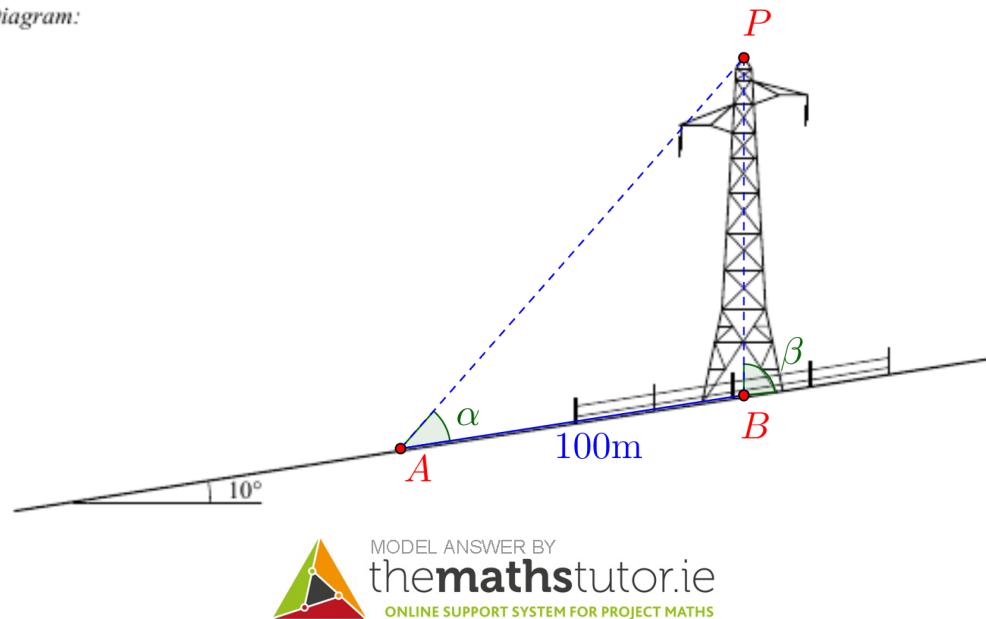*Measurements to be taken:*

Mark point *B* directly under the pylon.

Mark another point *A* 100 metres downhill from point *B*.

Use the clinometer to measure the angle of elevation at *A*.

We know the pylon makes a right-angle with the horizontal.

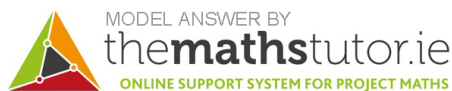Subtract 10° from each of these angles to find the measure of the angles α and β marked in green below.

Diagram:

Procedure used to find the height:

Firstly, the angle $\angle PBA$ is equal to $180° - \angle\beta$. Once this is calculated we know 2 angles inside the triangle $\triangle APB$ and can find the third because these must add up to $180°$. Now use the sine rule:

$$\frac{|BP|}{\sin\alpha} = \frac{|AB|}{\sin\angle APB}$$
$$\frac{|BP|}{\sin\alpha} = \frac{100}{\sin\angle APB}$$
$$|BP| = \frac{100\sin\alpha}{\sin\angle APB}$$

(ii) Write down the possible values for the measurements taken, and use them to show how to find the height of the pylon. (That is, find the height of the pylon using your

measurements, and showing your work.)

From the diagram, a protractor can be used to estimate $\alpha = 42°$ and we know that $\beta = (90 - 10) = 80°$.

Then $\angle PBA = 180° - 80° = 100°$. Now we get

$$
\begin{aligned}
\angle APB + \angle PBA + \alpha &= 180° \\
\angle APB + 100° + 42° &= 180° \\
\angle APB &= 180° - 100° - 42° \\
\angle APB &= 38°
\end{aligned}
$$
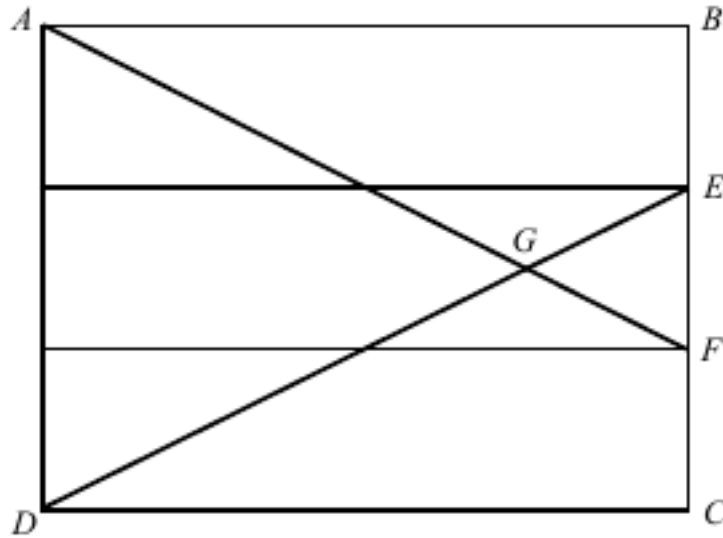
Finally, using the sine rule:

$$
\begin{aligned}
\frac{|BP|}{\sin 42°} &= \frac{100}{\sin 38°} \\
\frac{|BP|}{0.6691} &= \frac{100}{0.6157} \\
|BP| &= \frac{(100)(0.6691)}{0.6157} \\
|BP| &= 108.7 \text{ metres}
\end{aligned}
$$

(b) Anne is having a new front gate made and has decided on the design below.

The gate is 2 metres wide and 1.5 metres high. The horizontal bars are 0.5 metres apart.

(i) Calculate the common length of the bars $[AF]$ and $[DE]$, in metres, correct to three decimal places.

Use Pythagoras theorem on the triangle $\triangle AFB$:

$$\begin{aligned} |AF|^2 &= 2^2 + 1^2 \\ |AF|^2 &= 5 \end{aligned}$$

which means $|AF| = \sqrt{5} = 2.236$ metres correct to three decimal places.

(ii) In order to secure the bar $[AF]$ and $[DE]$ the manufacturer needs to know:

- The measure of the angle EGF, and
- the common distance $|AG| = |DG|$

Find these measures. Give the angle correct to the nearest degree and the length correct to three decimal places.

- First find the angle $\angle AFB$. Note that

$$\tan \angle AFB = \frac{opposite}{adjacent} = \frac{2}{1}$$

which means $\angle AFB = \tan^{-1}(2) = 63.43°$ to 2 decimal places.

Now focus on the triangle $\triangle GEF$.
The angle $\angle GFE = 63.43°$.
The angle $\angle GEF$ must also be $63.43°$ since the triangle $\triangle DCE$ is congruent to $\triangle AFB$.
Since the three angles in $\triangle GEF$ must add up to $180°$ we have

$$
\begin{aligned}
\angle GFE + \angle GEF + \angle EGF &= 180° \\
63.43° + 63.43° + \angle EGF &= 180° \\
\angle EGF &= 180° - 63.43° - 63.43° \\
\angle EGF &= 53.14° \\
\angle EGF &= 53° \text{ to the nearest degree}
\end{aligned}
$$

- Using the sin rule on $\triangle GEF$ we have

$$\frac{\sin 63}{|GE|} = \frac{\sin 53}{0.5}$$

which means

$$
\begin{aligned}
|GE| &= \frac{0.5 \sin 63}{\sin 53} \\
&= 0.558
\end{aligned}
$$

correct to three decimal places. Finally we have

$$
\begin{aligned}
|DG| + 0.558 &= 2.236 \\
|DG| &= 1.678 \text{ metres}
\end{aligned}
$$