February 3, 2021 12:42 PM

Set $\frac{\theta}{\theta+\alpha} = \pi$, $1-\pi = \frac{\alpha}{\alpha+\theta}$

$$f_\Lambda(x) = \frac{1}{\theta^2}(\theta\pi + (1-\pi)x) e^{-\frac{x}{\theta}}$$

$$M_\Lambda(t) = \frac{1-\pi\theta t}{(1-\theta t)^2}$$

$$M_\Lambda(-(j+x)) = \frac{1+\pi\theta(j+x)}{(1+\theta(j+x))^2}$$

$$p(X=x) = \binom{m}{x}\sum_{j=0}^{m-x}\binom{m-x}{j}(-1)^j \frac{1+(j+x)\pi\theta}{(1+(j+x)\theta)^2}$$

$$\binom{m}{x}(e^{-\lambda})^x(1-e^{-\lambda})^{m-x}$$

$$= \binom{m}{x} e^{-\lambda x}\sum_{j=0}^{m-x}(e^{-\lambda})^j$$

$$= \binom{m}{x}\sum_{j=0}^{m-x}(-1)^j e^{-\lambda(x+j)}$$

$$X|\Lambda \sim bin(m, e^{-\Lambda})$$

$$\binom{m}{x}(e^{-\Lambda})^x(1-e^{-\Lambda})^{m-x}$$

$$= \binom{m}{x}\sum_{j=0}^{m-x}\binom{m-x}{j} e^{-\lambda x}(-e^{-\lambda})^j$$

$$= \binom{m}{x}\sum_{j=0}^{m-x}\binom{m-x}{j} e^{-\Lambda(x+j)}$$

$$EX = E\,E(X|\Lambda)$$
$$= E(m e^{-\Lambda}) = m\,E e^{-\Lambda}$$
$$= m\,M_\Lambda(-1) = m\frac{1+\pi\theta}{(1+\theta)^2}$$

$$M_\Lambda(t) = \pi(1-\theta t)^{-1} + (1-\pi)(1-\theta t)^{-2}$$
$$= \frac{\pi(1-\theta t)}{(1-\theta t)^2} + \frac{1-\pi}{(1-\theta t)^2} = \frac{\pi - \pi\theta t + 1 - \pi}{(1-\theta t)} = \frac{1-\pi\theta t}{(1-\theta t)^2}$$

$$V(X) = E(V(X|\Lambda)) + V(E(X|\Lambda))$$
$$= E(m e^{-\Lambda}(1-e^{-\Lambda})) + V(m e^{-\Lambda})$$
$$= m[M_\Lambda(-1) - M_\Lambda(-2)] + m^2(M_\Lambda(-2) - [M_\Lambda(-1)]^2)$$
$$= m\left[\frac{1+\pi\theta}{(1+\theta)^2} - \frac{1+2\pi\theta}{(1+2\theta)^2}\right] + m^2\left(\frac{1+2\pi\theta}{(1+2\theta)^2} - \frac{(1+\pi\theta)^2}{(1+\theta)^4}\right)$$

Feb. 9, 2021.

The estimation of parameters for Lindley model with two-parameters.

$W \sim$ Lindley(θ, α) with density function

$$f_W(x) = \frac{\theta^2}{\alpha+\theta}(1+\alpha x) e^{-\theta x}, \quad x>0, \quad \theta>, \quad \alpha+\theta>0$$

Now, By defining $w = \frac{\theta}{\alpha+\theta}$, $(1-w) = \frac{\alpha}{\alpha+\theta}$, $\beta = \frac{1}{\theta}$,

we have $\theta = \frac{1}{\beta}$, $\alpha = \frac{1}{\beta}\frac{1-\pi}{\pi} = \frac{1}{\beta\pi} - \frac{1}{\beta}$

In this Case,

$$f_W(x) = \frac{1}{\beta^2}(\beta\pi + (1-\pi)x) e^{-\frac{x}{\beta}}$$
$$= \pi\frac{1}{\beta}e^{-\frac{x}{\beta}} + (1-\pi)\frac{x}{\beta^2}e^{-\frac{x}{\beta}}$$
$$= \pi f_1(x) + (1-\pi) f_2(x) = \pi f_1(x|\beta) + (1-\pi) f_2(x|\beta)$$

where, $f_1(x|\beta)$ is the pdf of ~~expon~~ Gamma$(1, \beta)$ (exponential (β))

$f_2(x|\beta)$ is the pdf of Gamma$(2, \beta)$

where, $f_1(a|\beta)$ is the pdf of ~~expon~~ Gamma $(1, \beta)$ (exponential (β))

$f_2(a|\beta)$ is the pdf of Gamma $(2, \beta)$

In this sense, the Lindley distribution is the mixture of two gamma distributions. Further, Let $W \sim$ Lindley(θ, α) (Lindley (π, β))

$X \sim$ gamma$(1, \beta)$. $Y \sim$ gamma$(2, \beta)$ $Z \sim$ Bernoulli (π).

Then the random variable W can be stochastically represented as πX $(1-\pi) Y$

$$W = X^Z Y^{1-Z}.$$

It also means that
$$\underline{W} = \begin{cases} X & \text{if } \underline{Z=1} \\ Y & \text{if } Z = 0 \end{cases}$$

The latent variables are taken to be random indicator variables that specify to which mixture component each observation belongs.

Therefore, for a given w_i, $i=1, 2, \dots n$, there is an associated latent variable z_i and the distribution function of y_i can be written as
$$z_i = \begin{cases} 1 & \pi \\ 0 & 1-\pi \end{cases}$$

$$f_W(w_i | z_i, \beta) = \left[f_X(w_i|\beta)\right]^{z_i} \left[f_Y(w_i|\beta)\right]^{1-z_i}$$

where $z_i = 0, 1$ is a Bernoulli random variable with $P(z_i = 1) = \pi$. π

Let $p(z_i|\pi)$ represent the probability mass function. Then.

$$f_W(w_i | z_i, \beta) \pi(z_i | \pi) = \{\pi f_1(w_i|\beta)\}^{z_i} \{(1-\pi) f_2(w_i|\beta)\}^{1-z_i}$$
$$\propto \pi^{z_i} (1-\pi)^{1-z_i}$$

From the above expression, the full conditional distribution of z_i is given by

$$\boxed{z_i | \pi, \beta, y_i \sim \text{Bernoulli}(\pi_i^*)}$$

where
$$\pi_i^* = \frac{\pi f_1(w_i|\beta)}{\pi f_1(w_i|\beta) + (1-\pi) f_2(w_i|\beta)} = \frac{\pi \beta}{\pi\beta + (1-\pi) w_i}$$

By ~~and~~ introducing the latent variables z_i, the likelihood function for complete data $\{(z_i, w_i), i=1, 2, \dots n\}$ is

$$L(\pi, \beta | w, z_i) = \prod_{i=1}^{n} f_W(w_i | z_i \beta) \pi(z_i|\pi)$$
$$= \prod_{i=1}^{n} \left[\pi f_1(w_i|\beta)\right]^{z_i} \left[(1-\pi) f_2(w_i|\beta)\right]^{1-z_i}$$
$$= \prod_{i=1}^{n} \left(\pi \frac{1}{\beta} e^{-\frac{w_i}{\beta}}\right)^{z_i} \left[(1-\pi) \frac{w_i}{\beta^2} e^{-\frac{w_i}{\beta}}\right]^{1-z_i}$$
$$= \prod_{i=1}^{n} \pi^{z_i} (1-\pi)^{1-z_i} \beta^{-z_i} e^{-\frac{w_i z_i}{\beta}} \beta^{-2(1-z_i)} w_i^{(1-z_i)} e^{-\frac{w_i(1-z_i)}{\beta}}$$
$$= \pi^{\sum z_i} (1-\pi)^{n-\sum z_i} \beta^{-\sum z_i} e^{-\frac{\sum w_i}{\beta}} \beta^{-2n+2\sum z_i} \prod_{i}^{n} w_i^{(1-z_i)}$$

The log-likelihood function for complete data is

$$\ell(\pi, \beta | w, z_i) = \left(\sum_{i=1}^{n} z_i\right) \log \pi + \left(n - \sum_{i=1}^{n} z_i\right) \log(1-\pi) - \left(2n - \sum z_i\right) \log \beta + \sum_{i=1}^{n} (1-z_i) \log w_i - \frac{\sum w_i}{\beta}$$

$$+ \sum_{i=1}^{n}(1-z_i)\log w_i - \frac{\sum w_i}{\beta}$$

Now

$$\frac{\partial \ell}{\partial \pi} = \frac{\sum_{i=1}^{n} z_i}{\pi} - \frac{(n-\sum z_i)}{1-\pi}, \quad (1)$$

$$\frac{\partial \ell}{\partial \beta} = -\frac{2n-\sum z_i}{\beta} + \frac{1}{\beta^2}\sum w_i \quad (2)$$

From (1) (2) we have

$$\hat{\pi} = \frac{1}{n}\sum_{i=1}^{n} z_i,$$

$$\hat{\beta} = \frac{\sum_{i=1}^{n} w_i}{2n - \sum z_i}$$

$\{z_i \quad \lambda_i \quad x_i\}$

Since

$$E(z_i \mid \pi, \beta, w_i) = \pi_i^* = \frac{\pi\beta}{\pi\beta + (1-\pi)w_i}$$

$\lambda.\ F(\lambda|x)$

By using EM algorithm, let $\hat{\pi}^{(t)}$ and $\hat{\beta}^{(t)}$ are the estimates of β and π at t^{th} iterative step. Then we have the following algorithm for computing the MLE $\hat{\beta}, \hat{\pi}$ of parameters π, β

$$\hat{\pi}^{(t+1)} = \frac{1}{n}\sum_{i=1}^{n}\frac{\hat{\pi}^{(t)}\hat{\beta}^{(t)}}{\hat{\pi}^{(t)}\hat{\beta}^{(t)} + (1-\hat{\pi}^{(t)})w_i} \quad (3)$$

and

$$\hat{\beta}^{(t+1)} = \frac{\sum_{i=1}^{n} w_i}{2n - \sum_{i=1}^{n}\frac{\hat{\pi}^{(t)}\hat{\beta}^{(t)}}{\hat{\pi}^{(t)}\hat{\beta}^{(t)} + (1-\hat{\pi}^{(t)})w_i}} \quad (4)$$

Note that for the reparametrized Lindley (π, β), $0<\pi<1$ and $\beta>0$ we may choose the ~~that~~ initial value of π equal to $\frac{1}{2}$ and the initial value of β equal to $\frac{2}{3n}\sum w_i$ because for $\pi = \frac{1}{2}$

$$E w = \frac{1}{2}\beta + (1-\tfrac{1}{2})2\beta = \frac{3\beta}{2} \approx \frac{1}{n}\sum w_i \Rightarrow \beta \approx \frac{2}{3n}\sum w_i = \frac{2}{3}\bar{w}.$$

Generate the random sample from Lindley (π, β).

$w = (w_1, \dots w_n)$

$\pi_0 = 0.5$

$\beta_0 = \frac{2}{3n}\sum w_i$

for (t :

$$\pi\beta + (1-\pi)2\beta$$

$$\frac{\beta}{2} + \frac{1}{2}2\beta = \frac{3}{2}\beta$$

Feb. 14, 2021

$\Lambda \sim$ Lindley (π, β) (or Lindley (θ, α)) with $\pi = \frac{\theta}{\theta+\alpha}$, $\beta = \frac{1}{\alpha}$

$0<\pi<1$. $\left(\begin{array}{l} \theta>0 \\ \alpha>0 \end{array} \right)$

$X \sim$ Lindley - Binomial (m, π, β).

$X \mid \Lambda \sim$ Binomial $(m, e^{-\Lambda})$

$$P(X=x) = \binom{m}{x}\sum_{j=0}^{m-x}\binom{m-x}{j}(-1)^j \frac{1+(x+j)\pi\beta}{[1+(x+j)\beta]^2}$$

Now $P(X=x, \Lambda=\lambda)$

$$= \binom{m}{x}\sum_{j=0}^{m-x}\binom{m-x}{j}(-1)^j e^{-(x+j)\lambda}\left[\frac{\pi}{\beta}e^{-\frac{\lambda}{\beta}} + \frac{(1-\pi)\lambda}{\beta^2}e^{-\frac{\lambda}{\beta}}\right]$$

Therefore,

$$E(\Lambda|X) = \frac{\int_0^\infty \lambda P(X=x, \Lambda=\lambda)\,d\lambda}{P(X=x)}$$

$E\,\Lambda|x$

Now, $\int_0^\infty \lambda P(X=x, \Lambda=\lambda)\,d\lambda$

$$= \binom{m}{x}\sum_{j=0}^{m-x}\binom{m-x}{j}(-1)^j \int_0^\infty \lambda e^{-(x+j)\lambda}\left[\frac{\pi}{\beta}e^{-\frac{\lambda}{\beta}} + \frac{(1-\pi)\lambda}{\beta^2}e^{-\frac{\lambda}{\beta}}\right]d\lambda$$

$$= \binom{m}{x}\sum_{j=0}^{m-x}\binom{m-x}{j}(-1)^j \left[\int_0^\infty \frac{\lambda\pi}{\beta}e^{-\lambda\left(\frac{1}{\beta}+(x+j)\right)}d\lambda + \int_0^\infty \frac{(1-\pi)\lambda^2}{\beta^2}e^{-\lambda\left(\frac{1}{\beta}+x+j\right)}d\lambda\right]$$

$$= \binom{m}{x}\sum_{j=0}^{m-x}\binom{m-x}{j}(-1)^j\left[\frac{\pi\beta}{[1+(x+j)\beta]^2} + \frac{2(1-\pi)\beta}{[1+(x+j)\beta]^3}\right]$$

$$= \binom{m}{x}\sum_{j=0}^{m-x}\binom{m-x}{j}(-1)^j \frac{2\beta-\pi\beta+(x+j)\pi\beta^2}{[1+(x+j)\beta]^3}$$

Therefore

$$E[\Lambda|X] = \frac{\binom{m}{x}\sum_{j=0}^{m-x}\binom{m-x}{j}(-1)^j\frac{2\beta-\pi\beta+(x+j)\pi\beta^2}{[1+(x+j)\beta]^3}}{\binom{m}{x}\sum_{j=0}^{m-x}\binom{m-x}{j}(-1)^j\frac{1+(x+j)\pi\beta}{[1+(x+j)\beta]^2}}$$

$$= \frac{\beta\sum_{j=0}^{m-x}\binom{m-x}{j}(-1)^j\frac{2-\pi+(x+j)\pi\beta}{[1+(x+j)\beta]^3}}{\sum_{j=0}^{m-x}\binom{m-x}{j}(-1)^j\frac{1+(x+j)\pi\beta}{[1+(x+j)\beta]^2}}$$

f $L(\pi,\beta \mid \lambda_i, \delta_i, x_i)$

$$= \prod_{i=1}^{n}\binom{m}{x_i}(e^{-\lambda_i})^{x_i}(1-e^{-\lambda_i})^{m-x_i}\cdot\pi^{\delta_i}(1-\pi)^{1-\delta_i}\left(\frac{1}{\beta}e^{-\frac{\lambda_i}{\beta}}\right)^{\delta_i}\left(\frac{\lambda_i}{\beta^2}e^{-\frac{\lambda_i}{\beta}}\right)^{1-\delta_i}$$

$$= \prod_{i=1}^{n}\binom{m}{x_i}\; e^{-\lambda_i x_i}(1-e^{-\lambda_i})^{m-x_i}\;\pi^{\sum\delta_i}(1-\pi)^{n-\sum\delta}\,\beta^{-\sum\delta_i}\,e^{-\frac{\sum\delta_i\lambda_i}{\beta}}\;\beta^{-2(n-\sum\delta)}\prod(\lambda_i^{1-\delta_i})\;e^{-\frac{\sum(1-\delta)\lambda_i}{\beta}}$$

$$\ell(\quad) = (\quad) + \sum -\lambda_i x_i + \sum \ln(1-e^{-\lambda_i})^{m}$$

$$\propto \pi^{\sum\delta_i}(1-\pi)^{n-\sum\delta_i}\;\beta^{-2n+\sum\delta_i}\;e^{-\frac{\sum\lambda_i}{\beta}}$$

$\propto \pi^{\sum \delta_i} (1-\pi)^{n-\sum \delta_i} \beta^{-2n+\sum \delta_i} \cdot e^{-\frac{\sum \lambda_i}{\beta}}$

$E(\lambda \mid X_i) = \lambda_i \qquad \delta_i = \frac{\pi \beta}{\pi \beta + (1-\pi) \lambda_i}$

$$\begin{cases} X \sim \text{Lindley}(0, \alpha) \\ X \sim \text{bin}(m, 1-e^{-\lambda}) \end{cases}$$