

# Math 170E – Intro to Probability

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This is math 170E taught by Professor Nguyen. The formal name of the class is **Introduction to Probability and Statistics 1: Probability**. The textbook used for the class is *Probability & Statistical Interference* 10<sup>th</sup> by *Hogg, Tanis*. We meet weekly on MWF from 10:00 – 10:50 and on Tue at the same time frame for discussion with our TA, Jason Snyder. You can also find other lecture notes at [ductuanvu.wordpress.com/notes/](https://ductuanvu.wordpress.com/notes/). Let me know through my [email](#) if you notice something mathematically wrong/concerning. Thank you!

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## §1.1 Properties of Probability

**Definition 1.1** — Consider the outcome of a random experiment, e.g. flipping a coin. The collection of all such outcomes, denoted by  $\underbrace{S}_{\omega \text{ in other advanced prob. textbook}}$ , is called the outcome space.

- A subset  $A \subseteq S$  is called an event.
- If  $A_1, A_2, \dots \subseteq S$  satisfy  $A_i \cap A_j = \emptyset, i \neq j$  then they are called “disjoint” (mutually exclusive)
- If  $A_1, A_2, \dots, A_n \subseteq S$  satisfy  $\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n = S$ . Then  $\{A_i\}_{i=1 \dots n}$  are called exhaustive (fully comprehensive).

**Example 1.2** 1. Flip two coins in order. Denote  $H$  = head,  $T$  = tail.

$$S = \{HH, HT, TH, TT\}$$

$$A = \{HH\} = \{\text{both coins are head}\}$$

$A \subseteq S$  is an event.

$$B = \{HT, TH\}$$

$B \subseteq S$  is another event.

$A \cap B = \emptyset$ , they are disjoint.

2. Flip 2 coins at once.

$$S = \{HH, HT, TT\}$$

$$A = \{\text{one head, one tail}\}$$

$$A = \{HT\}, \text{ is an event.}$$

Probability – A heuristic intro:

Consider an experiment and repeat  $n$  times. Let  $N(A)$  = number of times  $A$  occurs. The ratio  $\frac{N(A)}{n}$  is called the relative frequency of  $A$  in  $n$  repetitions of the experiment.

$$0 \leq \frac{N(A)}{n} \leq 1$$

As  $n \rightarrow \infty$ ,

$$\frac{N(A)}{n} \rightarrow p \in [0, 1]$$

This  $p$  is called the prob. that event  $A$  occurs.

**Example 1.3**

(a) Flip a coin

$$S = \{H, T\}$$

$$A = \{H\}$$

What is  $P(A)$ ?

(b) Sometimes, we can also assign prob. based on the nature of the event Pick a random point in the unit circle.

$$A = \{\text{chosen point} \in 1^{\text{st}} \text{quadrant}\}$$

$$P(A) = \frac{\text{Area of first quadrant}}{\text{Area of unit circle}} = \frac{1}{4}$$

(c) Pick a number randomly from  $\{0, 1, \dots, 9\}$ ,  $B = \{2 \text{ is picked}\}$ 

$$P(B) = \frac{1}{10}$$

Table 1: From example 1.3 (a)

$n$	$N(A)$	$\frac{N(A)}{n}$
50	37	.74
500	333	.66

It is safe to assign  $P(A) = 0.66$ **Definition 1.4** — Given an outcome space  $S$ , the probability of an event  $A \subseteq S$ , is a number satisfying:

1.  $P(A) \geq 0$
2.  $P(S) = 1$
3.  $A_1, \dots, A_n \subseteq S$  are disjoint events, i.e.  $A_i \cap A_j = \emptyset, i \neq j$ , then

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i) = P(A_1) + \dots + P(A_n)$$

More generally, if  $A_1, \dots, A_n, \dots \subseteq S$  are disjoint events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

**Theorem 1.5** 1. Denote  $A'$  to be the complement of  $A$  in  $S$ , i.e.

$$A' \cup A = S$$

$$A' \cap A = \emptyset$$

Then

$$P(A') = 1 - P(A)$$

$$2. P(\emptyset) = 0$$

$$3. \text{ If } A \leq B \text{ then } P(A) \leq P(B)$$

$$4. P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$5. P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Note: The pattern here is add the prob. of odd event(s) and subtract the prob. of even events. (for prop (4) and (5) of theorem 1.5).

*Proof.*

$$P(A') = 1 - P(A)$$

Since  $A' \cap A = \emptyset$  (by def of  $A'$ ). By property (c),

$$P(\underbrace{A' \cup A}_S) = P(A') + P(A)$$

$$\underbrace{P(S)}_{1 \text{ (by prop.(b))}} = P(A') + P(A)$$

Thus,

$$P(A') = 1 - P(A)$$

## §2 | Lec 2: Oct 5, 2020

Cont'd of Lec 1

(2)

$$\begin{aligned} P(\emptyset) &= 1 - P(S) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

(3)

$$P(A) \leq P(B)$$

$B \setminus A$  is the set s.t.

$$A \cup (B \setminus A) = B$$

$$A \cap (B \setminus A) = \emptyset$$

something here

implying

$$P(A) \leq P(B)$$

(4)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(5) Textbook Section 1.1. □

**Definition 2.1** — Suppose  $S = \{e_1, \dots, e_m\}$  where each  $e_i$  is a possible outcome. Denote  $n(s)$  = number of outcomes =  $m$ . If each  $e_i$  has the same prob. of occurring, then they are called equally likely. In particular,

$$P(e_i) = \frac{1}{n(s)} = \frac{1}{m}$$

Moreover, if  $A \subseteq S$  is an event s.t.  $n(A) = k$ . Then,

$$P(A) = \frac{n(A)}{n(s)} = \frac{k}{m}$$

### Example 2.2

Draw one card from a deck of 52 cards.

$$P(\text{each card is drawn}) = \frac{1}{52}$$

$A = \{\text{a king is drawn}\}$ , so  $n(A) = 4$ . Thus,

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52}$$

## §2.1 Method of Enumeration

### Multiplication Principle:

Suppose an experiment  $E_1$  has  $n_1$  outcomes

- For each outcome from  $E_1$ , a 2<sup>nd</sup> experiment  $E_2$  has  $n_2$  outcomes. Then the composite  $E_1 E_2$  has  $n_1 \cdot n_2$  outcomes.

### Permutation of size n:

**Definition 2.3** — Suppose there are  $n$  positions to be filled by  $n$  persons. One such arrangement is called a permutation of size  $n$ .

FACT: the total number of different such arrangements is given by “ $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$ ”

*Proof.* •  $E_1 =$  fill the 1<sup>st</sup> position from  $n$  persons  $\implies n$  outcomes for  $E_1$ .

- $E_2$  = fill the 2<sup>nd</sup> pos. from  $n - 1$  persons left  $\implies n - 1$  outcomes for  $E_2$
- $\vdots$
- $E_n$  = fill the  $n^{\text{th}}$  pos. from 1 person left  $\implies 1$  outcome for  $E_n$
- One arrangement =  $E_1 E_2 \dots E_n$

Thus, total number of arrangements is  $n!$ . □

### Permutation/Combination of $n$ objects taken $k$ :

**Definition 2.4** — Given  $k \leq n$  and suppose there are  $n$  objects. If  $k$  objects are taken from  $n$  **with/without** order, then such a selection is called **permutation/combination** of size  $n$  taken  $k$ .

Note: “Permutation of size  $n$ ” = “permutation of size  $n$  taken  $n$ ”.

**Fact 2.1.** 1. The total number of permutation  $n$  taken  $k$  (order is important here) is denoted by  ${}^n P_k$  is given by

$${}^n P_k = \frac{n!}{(n - k)!}$$

2. The total numbers of combination of  $n$  taken  $k$ , denoted by  ${}^n C_k$  or  $\binom{n}{k}$  is given by

$${}^n C_k = \binom{n}{k} = \frac{n!}{(n - k)!k!}$$

*Proof.*  $E_1$  = fill 1<sup>st</sup> pos. from  $n$   $\implies n$  for  $E_1$

$\vdots$

$E_k$  = fill  $k^{\text{th}}$  pos. from  $n - k + 1$  persons left. Thus,

$$\text{perm}k = n \cdot \dots \cdot (n - k + 1)$$

(2) Combination of  $n$  taken  $k$  :

Start with  ${}^n P_k$  as follow:

- $E_1$  = take  $k$  from  $n$  at once, outcome =  ${}^n C_k = \binom{n}{k}$
- $E_2$  = permute  $k$ , outcomes =  $k!$ . Thus,

$${}^n P_k = \binom{n}{k} \cdot k!$$

implying

$$\binom{n}{k} = \frac{{}^n P_k}{k!} = \frac{n!}{(n - k)!k!}$$

□

Practice 1: [https://ccle.ucla.edu/pluginfile.php/3766550/mod\\_resource/content/1/Practice%201.pdf](https://ccle.ucla.edu/pluginfile.php/3766550/mod_resource/content/1/Practice%201.pdf)

1. Consider  $S = \{1, \dots, 8\}$

a)

- $E_1$  = filling 1<sup>st</sup> pos  $\implies$  8 choices.
- Same for  $E_2 \implies$  8 choices.
- Likewise,  $E_3$  has 8 choices.

Thus, the number of 3 digit numbers can be formed is  $8^3$

b) “3 distinct digit numbers” = “permutation of size 8 taken 3”

Thus, total such numbers is  ${}_8P_3 = \frac{8!}{5!} = 8 \cdot 7 \cdot 6$

c) Considering subset where order is not taken into account  
Combination of size 8 taken 3. Thus, the answer is

$$\binom{8}{3} = \frac{8!}{3!5!}$$

d) 3 digit numbers and divisible by 5

- $E_1$  = choose 5 for the 3<sup>rd</sup> pos, so 1 choice.
- $E_2$  = 8 choices
- $E_3$  = 8 choices

Thus, the total of choices is  $8 \cdot 8 = 64$ .

e) 4 element subsets of  $S$  that has one even digit.

- $E_1$  = choose one even digit from  $S$ , so 4 choices (2,4,6,8).
- $E_2$  = choose 3 digits from  $\{1, 3, 5, 7\}$  without order, so  $\binom{4}{3}$

Thus, total =  $E_1 \cdot E_2 = 4 \cdot \binom{4}{3}$ .

e') What if “at least one even digit” instead of “exactly one even”?

1. Total = exactly “one even” + “two even” + “three even” + “four even”
2. Total = “4-element subset” - “4-element subset with no even digit”

## §3 | Lec 3: Oct 7, 2020

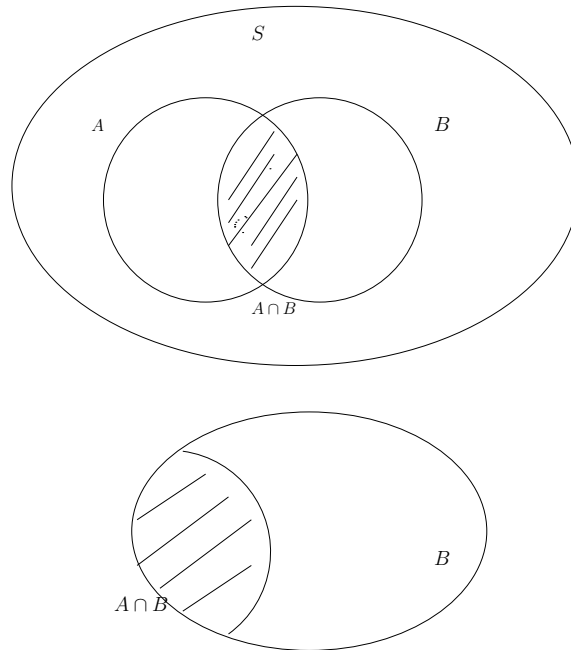
### §3.1 Conditional Probability

**Definition 3.1** — Let  $A, B \subseteq S$  be two events. The conditional prob. of  $A$ , given that  $B$  has occurred with  $P(B) > 0$ , is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

A heuristic explanation:  $A \cap B$ : “the portion in  $B$  that  $A$  occurs”

$$P(A|B) = \frac{\text{“area of A in B”}}{\text{“area of B”}}$$

**Example 3.2**

Suppose my family has two kids. Given that there is at least a boy, what is the prob. my family has two boys?

$$S = \{bb, bg, gb, gg\}$$

Now, let  $B = \{\text{at least a boy}\}$ . So we only look at the first three outcomes from  $S$  ( $B$ ). Define  $A = \{\text{two boys}\}$

$$A \cap B = \{bb\}$$

Note  $A = A \cap B$  since  $A \subseteq B$ . Thus,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Note: We can also consider the alternative outcome space without order as follows

$$S = \left\{ (b, b) - \frac{1}{4}, (b, g) - \frac{1}{2}, (g, g) - \frac{1}{4} \right\}$$

**Fact 3.1.**  $P(A|B)$  satisfies basic properties of probability:

- $P(A|B) \geq 0$
- $P(B|B) = 1$

Moreover, if  $B \leq C$  then

$$P(C|B) = 1$$

- If  $A_1, \dots, A_n, \dots$  are disjoint events,

$$P\left(\bigcup_{k=1}^{\infty} A_k | B\right) = \sum_{k=1}^{\infty} P(A_k | B)$$



*Proof.* (a)  $P(A|B) = \frac{P(A \cap B)}{P(B)} \geq 0$

(b)  $P(B|B) = \frac{P(B \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$   
 If  $B \subseteq C$  then  $B \cap C = B$

$$P(C|B) = \frac{P(B \cap C)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

$B \subseteq C$  means “if B occurs then C must occur”.

(c)  $P(\bigcup_{k=1}^{\infty} A_k|B) = \frac{P(\bigcup_{k=1}^{\infty} A_k \cap B)}{P(B)}$ . By distributive law,

$$\begin{aligned} &= \frac{P(\bigcup_{k=1}^{\infty} (A_k \cap B))}{P(B)} \\ &= \frac{\sum_{k=1}^{\infty} P(A_k \cap B)}{P(B)} \\ &= \sum_{k=1}^{\infty} P(A_k|B) \end{aligned}$$

□

\*INSERT: PRACTICE 1 #3 here\*

**Theorem 3.3** 1.  $P(A \cap B) = P(A|B) \cdot P(B)$  given that  $P(B) > 0$   
 2.  $P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$  given  $P(A), P(A \cap B) > 0$ .

*Proof.* 1. By defn of cond. prob.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

implying

$$P(B)P(A|B) = P(A \cap B)$$

2.  $P(A \cap B \cap C) = P(C \cap (A \cap B))$ . By part 1,

$$\begin{aligned} &= P(C|A \cap B)P(A \cap B)P(A \cap B) \\ &= P(C|A \cap B)P(B|A)P(A) \end{aligned}$$

□

**Practice 3.1.** The url: [https://ccle.ucla.edu/pluginfile.php/3776692/mod\\_resource/content/0/Practice%202.pdf](https://ccle.ucla.edu/pluginfile.php/3776692/mod_resource/content/0/Practice%202.pdf)

\*INSERT: Look at the online notes\*

## §4 | Lec 4: Oct 9, 2020

Cont'd (Practice)

3)

$$A = \{\text{spade}\} \quad B = \{\text{heart}\} \quad C = \{\text{diamond}\} \quad D = \{\text{club}\}$$

$P = (A \cap B \cap C \cap D = ?$  So,

$$= P(A)P(B|A)P(C|A \cap B)P(D|A \cap B \cap C)$$

(from problem 2 in practice 2)

- $P(A) = \frac{13}{52}$
- $P(B|A) =$ , now restricted to outcome space {51 cards including 13 hearts}  $B|A = \{\text{dealing a heart}\}$ . Thus,

$$P(B|A) = \frac{13}{51}$$

- Similarly,

$$P(C|A \cap B) = \frac{13}{50}$$

(13 diamond from 50 cards left)

- $P(D|A \cap B \cap C) = \frac{13}{49}$  (13 clubs from 49 cards left).

Hence,

$$P(A \cap B \cap C \cap D) = \frac{13}{52} \frac{13}{51} \frac{13}{50} \frac{13}{49}$$

### §4.1 Independent Events

**Example 4.1**

Flip a fair coin twice

$$S = \{ HH, HT, TH, TT \}$$

$$A = \{ 1^{\text{st}}H \}$$

$$B = \{ 2^{\text{nd}}T \}$$

$$C = \{ TT \}$$

$C \subseteq B$  “2 tails”  $\implies$  “2nd is T”. i.e., if C occurs then B must have occurred. Thus,

$$P(B|C) = 1$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{\frac{1}{4}}{\frac{1}{2}}$$

$$= \frac{1}{2}$$

$$P(A) = \frac{1}{2}$$

Thus,  $P(A|B) = P(A)$ , i.e., B occurring does not impact the occurrence of A.

Note also that

$$\frac{P(A \cap B)}{P(B)} = P(A|B) = P(A)$$

implying

$$P(A \cap B) = P(A)P(B)$$

**Definition 4.2** — Given two events  $A, B$  which are called independent iff

$$P(A \cap B) = P(A)P(B)$$

**Theorem 4.3**

The following are equivalent

- $A, B$  are independent
- $P(A|B) = P(A)$ , provided  $P(B) > 0$
- $P(B|A) = P(B)$ , provided  $P(A) > 0$

*Proof.* Left as an exercise. □

**Theorem 4.4** 1. If  $P(A) = 0$  then  $A$  is independent with any event.

2. If  $A$  and  $B$  are independent then so are the following pairs:

$$A, B' \quad A', B \quad A', B'$$

*Proof.* 1. Let  $B$  an arbitrary event, we need to show  $P(A \cap B) = P(A)P(B)$ . Since  $P(A) = 0$ ,  $P(A)P(B) = 0$ .

$$A \cap B \subseteq A$$

imply

$$0 \leq P(A \cap B) \leq P(A) = 0$$

thus  $P(A \cap B) = 0$ .

2. Textbook(section 1.5)

□

**Practice 4.1.** Practice 2 – Problem 4:

Let's consider  $C$  and  $D$  first

$$\begin{aligned} D &= \{ \text{sum of two rolls} = 12 \} \\ &= \{(6, 6)\} \end{aligned}$$

Thus,  $D \subseteq C = \{\text{first roll is 6}\}$ . Hence,  $C$  and  $D$  are dependent.  
A v.s. B

$$\begin{aligned} P(A) &= \frac{5}{6} \\ B &= \{ \text{sum is even} \} \\ &= \{ \text{first and second roll are even} \} \cup \{ \text{first and second roll are odd} \} \\ P(B) &= P(\text{first even})P(\text{second even}) + P(\text{first odd})P(\text{second odd}) \\ &= \frac{3}{6} \frac{3}{6} + \frac{3}{6} \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$

Now, consider  $A \cap B = \{1^{\text{st}} \neq 3, \text{sum is even}\}$ . So,

$$\begin{aligned} A \cap B &= \{1^{\text{st}} \neq 3, 1^{\text{st}} \text{ odd}, 2^{\text{nd}} \text{ odd}\} \cup \{1^{\text{st}} \neq 3, 1^{\text{st}} \text{ even}, 2^{\text{nd}} \text{ even}\} \\ P(A \cap B) &= P(1^{\text{st}} \neq 3, 1^{\text{st}} \text{ odd})P(2^{\text{nd}} \text{ odd}) + P(1^{\text{st}} \neq 3, 1^{\text{st}} \text{ even})P(2^{\text{nd}} \text{ even}) \\ &= \frac{2}{6} \frac{3}{6} + \frac{3}{6} \frac{3}{6} \\ &= \frac{5}{12} \end{aligned}$$

Since  $P(A \cap B) = \frac{5}{12} = \frac{5}{6} \frac{1}{2} = P(A)P(B)$ ,  $A$  and  $B$  are independent.

## §5 | Lec 5: Oct 12, 2020

### §5.1 Independent Events (cont'd)

**Definition 5.1** —  $A, B, C$  are called “mutually independent” if followings hold:

- pairwise independent

$$P(A \cap B) = P(A)P(B) \quad P(B \cap C) = P(B)P(C) \quad P(A \cap C) = P(A)P(C)$$

- “triple” wise independent, i.e.,

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

*Note:* analogous defn holds for  $A_1, \dots, A_n, \dots$  in which any pairs, triple, quadruple and so on must satisfy the similarly multiplication rules. Usually, the term “mutually” is dropped but it is understood that “independence” means “mutually independence”.

**Remark 5.2.** In general, pairwise independence does not imply triple-wise independence.

**Practice 5.1.** 2 – Problem 5:

$$A = \{1, 2\}, \quad B = \{1, 3\}, \quad C = \{1, 4\}$$

$$P(A) = \frac{2}{4} = P(B) = P(C)$$

$$A \cap B = \{1\} = B \cap C = A \cap C$$

$$P(A \cap B) = P(B \cap C) = P(C \cap A) = \frac{1}{4}$$

Thus,

$$P(A \cap B) = \frac{1}{4} = P(A)P(B)$$

Same for  $B, C$  and  $A, C$  – so pairwise independent.

Triple:

$$A \cap B \cap C = \{1\}$$

$P(A \cap B \cap C) = \frac{1}{4}$ ; on the other hand,  $P(A)P(B)P(C) = \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{1}{8}$ . They are not equal! Therefore,  $A, B, C$  are not mutually independent.

### §5.2 Bayes's Theorem

**Definition 5.3** — The events  $B_1, \dots, B_n$  ( $n$  may be finite or  $\infty$ ) are called a partition of the outcome space  $S$  if followings hold

- disjoint:  $B_i \cap B_k = \emptyset, i \neq k$
- exhausted:  $\bigcup_n B_i = S$

then,

$$P(B_1) + \dots + P(B_n) = P(S) = 1$$

**Theorem 5.4** (Law of total Probability)

Suppose  $B_1, \dots, B_n$  is a partition of  $S$  with  $P(B_i) > 0$  for  $i = 1, \dots, n$ . If  $A$  is an event in  $S$ , then

$$P(A) = \sum_{i=1}^n P(A|B_i)P(B_i)$$

where  $P(B_i)$  is called the prior probability.

*Proof.* (sketch)

$$\begin{aligned} P(A) &= P\left(\bigcup_{i=1}^n (A \cap B_i)\right) \\ &= \sum_{i=1}^n P(A \cap B_i) \\ &= \sum_{i=1}^n P(A|B_i)P(B_i) \end{aligned} \quad \square$$

**Practice 5.2.** 3 – problem 1:

$$\begin{aligned} P(I) &= .35 \\ P(II) &= .25 \\ P(III) &= .4 \end{aligned}$$

$A = \{ \text{a spring is defective} \}$ ,  $P(A) = ?$  We know

$$\begin{aligned} P(A|I) &= .02 \\ P(A|II) &= .01 \\ P(A|III) &= .03 \end{aligned}$$

By law of total prob:

$$\begin{aligned} P(A) &= P(A|I)P(I) + P(A|II)P(II) + P(A|III)P(III) \\ &= 0.0215 \end{aligned}$$

**Theorem 5.5** (Bayes's Theorem)

Suppose  $\{B_i\}_{i=1, \dots, n}$  is a partition of  $S$  with  $P(B_i) > 0$ . If  $A$  with  $P(A) > 0$ , then for all  $i = 1, \dots, n$

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{k=1}^n P(A|B_k)P(B_k)}$$

where  $P(B_i|A)$  is called posterior probability.

*Proof.*

$$\begin{aligned}
 P(B_i|A) &= \frac{P(B_i \cap A)}{P(A)} \\
 &= \frac{P(A \cap B_i)}{P(A)} \\
 &= \frac{P(A|B_i)P(B_i)}{P(A)} \\
 &= \frac{P(A|B_i)P(B_i)}{P(A|B_1)P(B_1) + \dots + P(A|B_n)P(B_n)} \quad \square
 \end{aligned}$$

**Practice 5.3.** 3 – problem 2:  $A = \{ \text{person has disease} \}$ ,  $P(A) = .005$ .

$$+ = \{ \text{test } + \}$$

$$- = \{ \text{test } - \}$$

$$P(+|A) = .99$$

$$P(\underbrace{+|A'}_{\text{false positive}}) = .03$$

$$P(A|+) = ?$$

By Bayes's Theorem:

$$\begin{aligned}
 P(A|+) &= \frac{P(+|A)P(A)}{P(+|A)P(A) + P(+|A')P(A')} \\
 &= \frac{(.99)(.005)}{(.99)(.005) + (.03)(.995)}
 \end{aligned}$$

$\{A, A'\}$  is a partition of  $S$ .

## §6 | Dis 1: Oct 6, 2020

### §6.1 Set Theory

**Definition 6.1** — A set is a collection of items.

**Example 6.2**

$$T = \{1, 2, 3, \text{red}, \text{blue}\}$$

$$S = \{1, 3, \text{red}\}$$

$$R = \{1, 2, 4\}$$

$$S \subseteq T$$

$$S' = S^c = \{2, \text{blue}\}$$

$$R \not\subseteq T$$

$$3 \underbrace{\in}_{\text{is an element of}} T$$

$$\{3\} \subseteq T$$

**Example 6.3**

$$A = \{1, 3, 7\} \quad A \cup B = \{1, 2, 3, 4, 7\}$$

$$B = \{2, 3, 4\} \quad A \cap B = \{3\}$$

$$A \setminus B = \{1, 7\} \quad B \setminus A = \{2, 4\}$$

De Morgan Laws:

$$(A \cup B)' = A' \cap B'$$

$$(A_1 \cup A_2 \cup \dots \cup A_n)' = A_1' \cap A_2' \cap \dots \cap A_n'$$

$$(A \cap B)' = A' \cup B'$$

If have a sample space  $S$ , and subset of  $S$  are called events. A probability function is a function  $\mathbb{P}$  that assigns a real number each event with three rules:

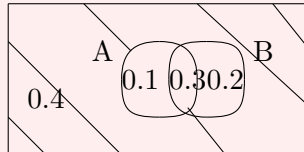
1.  $P(A) \geq 0$
2.  $P(S) = 1$
3.  $A_1, A_2, \dots, A_n$  with  $A_i \cap A_j = \emptyset = \{\}$ , then  $P(A_1 \cup \dots \cup A_n) = P(A_1) + \dots + P(A_n)$



**Example 6.4**

1.1-6 (from the book):  $P(A) = 0.4$ ,  $P(B) = 0.5$ ,  $P(A \cap B) = 0.3$   
Find

- $P(A \cup B) = .1 + .3 + .2 = .6$
- $P(A \cap B)' = .1$
- $P(A' \cap B) = .2$



Note:  $(P, S)$  : probability space on all subsets of  $S$

**Example 6.5**

1.2-5: How many four letter codes can be made from the letters in IOWA if

- Letters may not be repeat:  $4! = 24$  ways.
- Letters may repeat:  $4^4 = 256$  ways.

## §7 | Dis 2: Oct 13, 2020

1.4.16: An urn has 5 balls. One is marked “win” and the other are marked “lose”. You and another player each take balls out one at a time until somebody picks win. You pick first.

W/o replacement:  $P(\text{winning}) = \frac{1}{5} + \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} + \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot 1 = \frac{3}{5}$

With replacement:

$$\begin{aligned}
 P(\text{winning}) &= \frac{1}{5} + \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{1}{5} + \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{1}{5} + \dots \\
 &= \frac{\frac{1}{5}}{1 - \frac{16}{25}} = \frac{5}{9}
 \end{aligned}$$

### §7.1 Conditional Probabilities

$$P(A|B) := \frac{P(A \cap B)}{P(B)}$$

or  $P(A \cap B) = P(B) \cdot P(A|B) = P(A) \cdot P(B|A)$

**Example 7.1**

1.3.7: An urn has 4 balls. 2 are red and 2 are blue. We pull out 2 balls. We are told that at least one is red. What's the probability that they're both red?

$$P(\text{both red}|\text{at least one red}) = \frac{P(\text{both red and at least one red})}{P(\text{at least one red})} = \frac{P(\text{both red})}{P(\text{at least one red})} = \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5}.$$

**§7.2 Bayes's Theorem**

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

**Example 7.2**

1.5-8: Four types of tablets:  $B_1, B_2, B_3, B_4$  with % of sales 0.4, 0.3, 0.2, 0.1 and % tablet needs repair 0.1, 0.05, 0.03, 0.02, respectively. What is the probability that a tablet needing repair is  $B_i$ ?

$$\begin{aligned} P(B_1|\text{need repair}) &= \frac{P(\text{need repair}|B_1) \cdot P(B_1)}{P(\text{need repair})} \\ &= \frac{(0.1)(0.4)}{(0.40)(0.10) + (0.30)(0.05) + (0.20)(0.03) + (0.10)(0.02)} \\ &\approx 63.5\% \end{aligned}$$

$$P(B_2|\text{need repair}) = \frac{(0.30)(0.05)}{0.063} \approx 23.8\%$$

$$P(B_3|\text{need repair}) \approx 9.5\%$$

$$P(B_4|\text{need repair}) \approx 3.2\%$$