Math 131AH – Honors Real Analysis I

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This is math 131AH – Honors Real Analysis I taught by Professor Greene, and our TA is Haiyu Huang. We meet weekly on MWF from 1:00pm – 2:00pm for lectures. There are two textbooks used for the class, *Principles of Mathematical Analysis* by *Rudin* and *Metric Spaces* by *Copson*. You can find other lecture notes at my blog site ductuanvu.wordpress.com/notes/. Please let me know through my email if you spot any mathematical errors/typos.

Contents

§1 Lec 1: Oct 2, 2020

Overview:

- Hmwrk: 30 %
- Midterm 1: 20 %
- Midterm 2: 20 %
- Final: 30 %

§1.1 Introduction

functions $\rightarrow 1, 2, 3, 4, 5, 6, 7 \ldots$
functions defined on $\mathbb{Q}$ with value in $\mathbb{Q}$

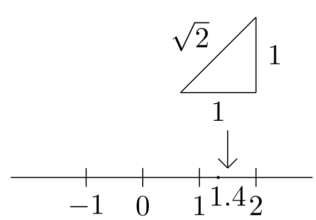
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$$

$a_i \in \mathbb{Q}$ $\;f(x) \in \mathbb{Q}$ if $x \in \mathbb{Q}$. Continuity makes sense.

$$x_0, x \quad x \text{close to } x_0 \implies f(x) \text{ close} f(x_0)$$

polynomials are continuous.
Somthing wrong: $\sqrt{2}$ is missing. What are these numbers that are not $\in \mathbb{Q}$? Choice:

1. Assume everything works and isolate what you need about "real numbers" (most of Rudin chap 1).
2. Construct the real numbers from rational numbers.



Classical argument:

$$x^2 \neq 2 \text{ if } x = \frac{p}{q} \in \mathbb{Q}$$

Proof. Suppose $\left(\frac{p}{q}\right)^2 = 2$
Note: wolog(without loss of generality)

can take $\frac{p}{q} > 0 \;\; p > 0 \; q > 0$

$$\left(\frac{p}{q}\right)^2 = 2$$
$$\frac{p^2}{q^2} = 2$$
$$p^2 = 2q^2$$

Now also wolog, can assume p and q are not <u>both</u> even numbers. But $p^2 = 2q^2$ means p has to be even (p^2 odd if p is odd).

$$p = 2n$$
$$p^2 = 2q^2$$
$$4n^2 = 2q^2$$

So $q^2 = 2n^2$, q is even. But it contradicts the initial assumption, p and q not both even □

Related to: Why functions $\mathbb{Q}$ to $\mathbb{Q}$ not ideal for analysis?
– INFINITE DECIMAL

§2 Lec 2: Oct 5, 2020

§2.1 Mathematical Induction and More on Real Numbers

$P(n) \to 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$, where n is positive numbers.
Math induction: Proof by two steps:

1. Check $P(1)$ is true ✓
2. Assume $P(n)$ is true for all $n \leq N$. Check that
$$P(N+1) \;\text{ is true}$$

Assume $1 + \ldots + N = \frac{N(N+1)}{2}$. Check

$$1 + \ldots + N + (N+1) = \frac{(N+1)(N+1+1)}{2}$$

Induction on k :

$$1^k + 2^k + \ldots + n^k$$

2[nd] illustration:

$$1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r} \quad r \neq 1$$

$r = 1 \implies 1 + r = \frac{1-r^2}{1-r}$

$$\begin{aligned} 1 + r + r^2 + \ldots + r^n + r^{n+1} &= \frac{1 - r^{n+1}}{1 - r} + r^{n+1} \\ &= \frac{1 - r^{n+1} + r^{n+1} - r^{n+2}}{1 - r} \\ &= \frac{1 - r^{n+2}}{1 - r} \end{aligned}$$

$$(1-r)(1+r+\ldots+r^n) = 1 - r^{n+1} \qquad \text{Inspection}$$

$$1 + r + r^2 + \ldots + r^n = \frac{1-r^{n+1}}{1-r}, \quad r \neq 1$$

$|r| < 1$ get inifite sum $\frac{1}{1-r}$

Example 2.1

Prime factors, prime = positive integers (> 1) with no factors except itself and 1, $p = ab$, $a > 1$, $b > 1$

$$2 \quad 3 \quad 5 \quad 7 \quad 11 \quad 13 \quad 17 \quad 19 \quad \ldots$$

Thin out as go along

Theorem 2.2

Every positive integer > 1 is a product of primes.

Proof. Induction: $P(n)$ $n = 2, 3, \ldots$

$$P(2) = 2\checkmark$$

Assume P(n) $\ldots n \leq N$ ($N > 2$). Every integer greater than 1 but smaller than or equal to N as a product of primes. We try to prove: $N+1$ is a product of primes.

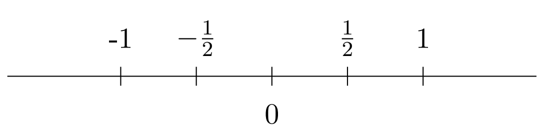
1. $N+1$ is prime: Done $N+1 = N+1$
2. $N+1$ is not a prime
$$N + 1 = a \cdot b \qquad a > 1 \ \ b > 1$$
Induction assumption ($a < N+1$ since $b > 1$), a is a product of primes $a > 1 \implies b < N+1$, b also a product of primes. So, $N+1 = ab$ is a product of primes.

$N+1 = ab$ is a product of prime. ☐

Why does induction work? If $P(n)$ not always true, $P(n)$ look at smallest n where $P(n)$ is false.
$n = 1$ not there $P(1)$ is supposed true (checked already). N_0 smallest one where $P(N_0)$ false $N_0 > 1$. Induction step says that $P(n)$ is true for all $n \leq \underbrace{N_0 - 1}_{>0} \implies P(N_0)$ true ($\times$).
Let's go back to real numbers.

Last time: talked about $\sqrt{2}$ is irrational but $\sqrt{2}$ exists, so we need to enlarge our number system: $\mathbb{Q}$ rational numbers.

$$\frac{p}{q} > \frac{r}{s} \qquad ps > rq \qquad (p, q, r, s > 0)$$



x, y rational $x, y > 0$, $x + y > 0$, $xy > 0$

$x^2 = 2$ no answer in $\mathbb{Q}$. Enlarge number system, $\mathbb{Q} \subset \mathbb{R}$. What should $\mathbb{R}$ be like?

1. $\mathbb{R}$ ought ot have arithmetic like $\mathbb{Q}$

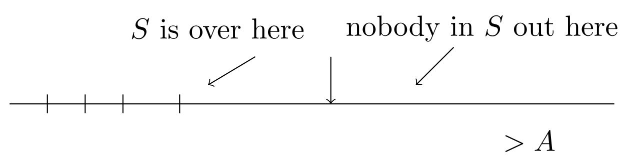
$$x + y \quad xy \quad \frac{x}{y} \quad 0 \quad 1$$

2. $\mathbb{Q} \subset \mathbb{R}$, arithmetic in $\mathbb{R}$ restricted to $\mathbb{Q}$, $\frac{1}{2} + \frac{1}{3}$ in $\mathbb{Q}$ ought to be $\frac{5}{6}$ in $\mathbb{R}$.

3. Order should positive in $\mathbb{Q} \implies$ in $\mathbb{R}$. $\mathbb{R}$ should have an order of its own too, $x \quad y$ positive then $x + y$ pos and xy pos.

4. want to fill in the holes in $\mathbb{Q}$. Want to have Least Upper Bound Property

   $S \subset \mathbb{R}$: An upper bound for S is a number A with property $A \geq x$ if $x \in S$

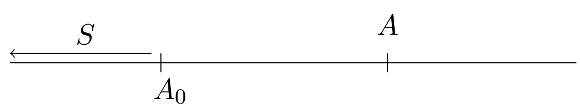


$1, 2, 3, 4, \ldots$ have no upper bound.

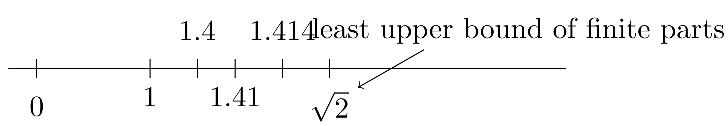
S is bounded above means that some upper bound A exists.

§2.2 Least Upper Bound Property

If S is bounded above ($S \neq \emptyset$) then it has a "least upper bound" where a number A_0 is called the least upper bound of S if A_0 is an upper bound for S & if A is an upper bound for S then $A_0 \leq A$.



Motivation: Think about $\sqrt{2}$



Denote: l.u.b(or supremum)(sequence) $= \sqrt{2}$

Means can define an infinite decimals: least upper bound of successive truncations

$$0.99999\ldots \to 1.0$$

§3 Lec 3: Oct 7, 2020

§3.1 Cauchy Sequence

$\{x_n\}\, x_1, x_2, x_3, \ldots$ values $x_j \in \mathbb{Q} \quad x_j \in \mathbb{R}$

$S \quad x_1, x_i \ldots x_j \in S$

Definition 3.1 — A sequence with values in a set S is a function from positive integers $\{1, 2, 3 \ldots\}$ into S.

Definition 3.2 — A Cauchy sequence is ($\mathbb{Q}$ valued or $\mathbb{R}$ valued) $\{x_i\}$ is sequence s.t. for every $\epsilon > 0$ there is a positive integer N_ϵ s.t.

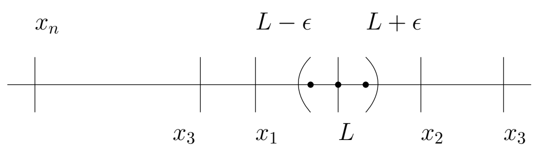
$$|x_i - x_j| < \epsilon \quad \text{if} \quad i, j > N_\epsilon$$

ϵ rational or real (same idea).

Lemma 3.3

If $\{x_j\}$ has a finite limit then it's a Cauchy sequence.

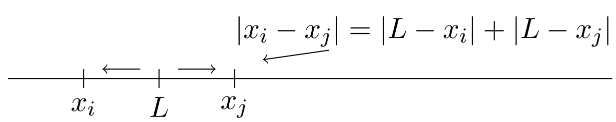
$\{x_i\}$ has L as a limit $\lim x_j = L$ means for every $\epsilon > 0$ then there is an N_ϵ such that $j \geq N_\epsilon, \quad |x_j - L| < \epsilon$



Everybody in $(L - \epsilon, L + \epsilon)$ except a finite number

Proof. Given $\epsilon > 0$, want to find N so that $i, j \geq N \implies |x_i - x_j| < \epsilon$
$|x_i - L|$ small, $|x_j - L|$ small and $\lim x_j = L$.

$$|x_i - x_j| \leq |x_i - L| + |x_j - L|$$



$i, j \geq N_{\frac{\epsilon}{2}}$:

$$|x_i - x_j| \leq \underbrace{|x_i - L|}_{< \frac{\epsilon}{2}} + \underbrace{|x_j - L|}_{< \frac{\epsilon}{2}}$$

Because $\lim x_n = L$, there is an $N_{\frac{\epsilon}{2}}$ s.t. $|L - x_n| < \frac{\epsilon}{2}$ if $n \geq N_{\frac{\epsilon}{2}}$
Get $|x_i - x_j| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ if $i, j \geq N$. Cauchy sequence: there exists number N s.t.

$$|x_i - x_j| < \epsilon \quad \text{if} \quad i, j \geq N$$

$\square$

Cauchy sequence $\implies$ the existence of limit? Yes, for $\mathbb{R}$ valued sequences but NO for $\mathbb{Q}$ valued things.

$\underbrace{\{x_n\}}_{\text{rational numbers}}$ can be Cauchy seq without there being a rationl number L such that $\lim x_j = L$

But allow real L then $\exists L$ s.t. $\lim x_j = L$ if $\{x_j\}$ is Cauchy sequence(no rational limit – since $\sqrt{2}$ is irrational). Because $\mathbb{Q}$ has holes in it! (intuitive idea).

Example 3.4

$1, 1.4, 1.41, 1.414, 1.4142\ldots$ (decimal approx of $\sqrt{2}$) – Cauchy sequence. No – since $\sqrt{2}$ is irrational.

§3.2 Cauchy Completeness of $\mathbb{R}$

If $\{x_j\}, x_j \in \mathbb{R}$ is Cauchy sequence, then $\exists L \in \mathbb{R}$ s.t. $\lim x_j = L$.
"$\mathbb{Q}$ is not Cauchy complete" but $\mathbb{R}$ is. Why does this work?
Need: Least upper bound property. Assume L.U.B Property proof.

Proof. (Cauchy completeness from L.U.B Property)
Hypothesis: $\{x_i\}$ Cauchy seq

1. Prove that $\{x_i\}$ bounded $\impliedby$ $\exists M > 0$ s.t. $|x_i| \le M$ all i.

   Clear if take $\epsilon = 1$ in def. of Cauchy seq $\exists N$ s.t. $|x_i - x_j| < 1$ if $i, j \ge N \implies |x_N - x_j| < 1$ if $j \ge N \implies \quad |x_j| \le |x_N| + 1 \quad j \ge N$

   So, $M = \max(|x_N| + 1, |x_1|, \ldots, |x_{N-1}|$ then $|x_j| \le M$ all j !

   Next stage is to show that a bounded sequence always has a subsequence(tricky!) with a limit. Then if a Cauchy seq has a subseq with limit L, then L is limit of whole seq. (Bolzano – Weierstrass Theorem)

□

§4 Lec 4: Oct 9, 2020

§4.1 Bolzano – Weierstrass Theorem

– implied by Least Upper Bound Property

Theorem 4.1

If $\{x_n\}$ sequence $(x_1, x_2, x_3\ldots)$ that is bounded (means: $\exists M > 0 \ni |x_n| \le M \forall\, n$, then $\exists L$ and a subsequence $\{x_{n_i}\}$ s.t. $\lim x_{n_i} = L$.

Slogan: Every bounded sequence has a convergent subsequence.

Example 4.2

$$1, 2, 1, 2, 1, 2, \ldots$$

The subsequence of the above sequence has either 1 or 2 as the limit.

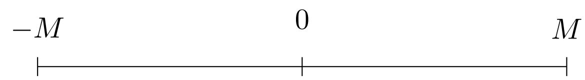
$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots$$

Unbounded sequence – subsequence (limit 1, limit 2, limit 3. . .)

No claim of uniquneness of anything.

Proof – Summer 2008 Analysis Lec 4

Proof. So either $[-M, 0]$ or $[0, M]$ (maybe both) contains x_n for infinitely many n values. If each contained x_n for only finitely many n values X.

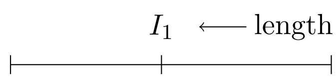


Every x_n is in $[-M, M]$ – $\{x_n\}$ is bounded

$$[-M, M] = [-M, 0] \cup [0, M]$$
$$I_1 = [-M, 0] \quad \text{or} \quad [0, M]$$

where chosen intervalhas x_n for infinitely many n values.
Do this again!

$$I_1 = [a_1, b_1] \qquad |b_1 - a_1| = M$$



left half of I_1, right half of I. Let $I_2 =$ one of halves that contains x_n for infinitely many n values.

$$I_2 = [a_2, b_2] \qquad a_2 < b_2, \quad b_2 - a_2 = \frac{M}{2}$$

Continue

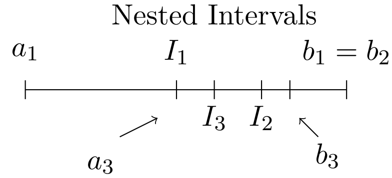
$$I_3 = [a_3, b_3] \qquad a_3 < b_3, \quad b_3 - a_3 = \frac{M}{4}$$

$$\vdots$$

$$I_k = [a_k, b_k] \qquad b_k - a_k = \frac{M}{2^{k-1}}$$

Each I_k contains x_n for infinitely many n values.

Nested Intervals



$$I_{k+1} \subset I_k \subset \ldots \subset I_1 \subset [-M, M]$$

$$a_{k+1} \geq a_k \dots \qquad b_{k+1} \leq b_k \dots$$

Claim $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$

Reason: $\sup a_k \in \bigcap_{k=1}^{\infty} I_k$ where sup = sup of left hand endpoint(=greatest lower bound of bs). l.u.b of a's $\leq b_k$, b_k bigger than or $\geq$ all a's.

$$\alpha = \text{lub a's}$$
$$\alpha \geq a_k \quad \forall k$$
$$\alpha \leq b_k \quad \forall k$$
$$\alpha \in [a_k, b_k]$$

Goal: $\alpha \in \bigcap_{k=1}^{\infty}$. Find a subsequence of $\{x_n\}$ converges to α.
Choose $x_k = x_n$ that belongs to I_k. Can also arrange successively:

$$n_1 < n_2 < n_3 < n_4$$

$x_{n_1} \in I_1 \quad x_{n_2} \in I_2$ can make $n_2 > n_1$ because infinitely possible $x_n's$ in I_2 n value.
Continue to get subsequence, $\{x_{n_k}\}$ subsequence. Claim:

$$\lim_{k \to \infty} x_{n_k} = \infty$$

Reason:

$$\text{dis}(x_{n_k}, \alpha) \leq \text{length of } I_k \quad \alpha \in I_k, \quad x_{n_k} \in I_k$$

which is equivalent to

$$|x_{n_k} - \alpha| \leq \frac{M}{2^{k-1}} \quad \text{given } \epsilon > 0$$

When k is large,

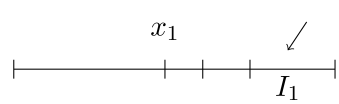
$$\frac{M}{2^{k-1}} < \epsilon$$

So $|x_{n_k} - \alpha| < \epsilon$ □

This argument (or a variant) shows something else:
If $\{x_n\}$ sequence in $[0, 1]$ then there's an $\alpha \in [0, 1]$ with it never happening that

$$x_n = \alpha$$

"The real numbers in $[0, 1]$ are uncountable." (come from the least upper bound property)



I_1 one of $[0, \frac{1}{3}]$ $[\frac{1}{3}, \frac{2}{3}]$ $[\frac{2}{3}, 1]$ such that $x_1 \notin I_1$,

$$[0, \frac{1}{3}] \cap [\frac{1}{3}, \frac{2}{3}] \cap [\frac{2}{3}, 1] = \emptyset$$

$x_1 \notin I_2 \quad I_2 \subset I_1, \quad \& x_1 \notin I_1$. Continue. Get

$$I_1 \supset I_2 \supset I_3 \supset \dots$$

length $I_k = \frac{1}{3^k}$ and I_k is such that $x_1, x_2, x_3 \dots x_k$ are none of the ?n? in I_k. Same as before

$$\exists \alpha \in \bigcap_{\infty}^{k=1} I_k$$

$\alpha =$ sup of set of left hand endpoints of I_k. Claim α cannot be an x_N value. Clear: $x_N \notin I_N$ but $\alpha \in I_n \quad \alpha \in \bigcap_{\infty}^{n=1} I_n$. But contrast:

There is a list of rational numbers in $[0, 1]$

$\frac{p}{q} \qquad p < q$

	2	3	4	5	6	$\cdots$
1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$			
2	-	$\frac{2}{3}$	$\frac{2}{4}$			
3	-	-	$\frac{3}{4}$			
$\vdots$	-	-	$\frac{\sqrt{2}}{2} \in [0, 1] \to$ irrational - no exist			

$[0, 1]$ $\searrow$ not countable

Q is countable

§5 Lec 5: Oct 12, 2020

§5.1 Equivalence Relation

(p.10, Copson – Metric Space)
R set, relation of A and B $(A \times B)$ $\quad (a, b) \in R \quad aRb$
Functions: one b given a – exact one. $(A \to B)$

Example 5.1

$A = B = Q$

aRb or $(a, b) \in R$ if $a > b$

(mother,child)

- (Sara, Sebastian) $\in R$
- (Sara, Alita) $\in R$

Equivalence is a special kind of relation: (on a set $A; B \quad A = B$)

Properties:

1. $aRa \qquad A = Q$
2. $aRb \implies bRa$

3. aRb & bRc then aRc

   Example: $\mathbb{Z}$ $a \sim b$ means $a - b$ is divisible by 5

$$1 \sim 6 \quad 0 \sim 5 \ldots$$

   $a \sim a \quad a - b$ div $\implies b - a$ div. by 5.

If $a - b$ div. by 5, and $b - c$ div by 5, then is $a - c$ div. by 5 true?
Sure, $a - b = 5k, \quad b - c = 5l \implies a - c = 5(k + l)$
"Equivalence classes": set $[a] = \{$ all b such that $aRb\}$
In the example above, $[a] = \{$ all b such that $a - b$ div. by 5$\}$

$$[2] = \{2, 7, -3, 12, -8, \ldots\}$$

$\mathbb{Z}_5$: integer mod 5.

1. $[a]$ $[p]$ either equal or have nothing in common.
2. $a \in [a]$ so is in some equivalence class.

A equivalence relation $\sim$ on $A \leftrightarrow$ a partition of A into subsets which are pairwise disjoint.
Q Cauchy seq. of rational numbers

$$\{x_n\} \sim \{y_n\}$$

means $\lim_{n\to\infty} |x_n - y_n| = 0$. Equivalence relation:

1. $\{x_n\} \sim \{x_n\}$ ($\lim(x_n - x_n) = 0$)
2. $\{x_n\} \sim \{y_n\} \implies \{y_n\} \sim \{x_n\}$
3. $\{x_n\} \sim \{y_n\} \,\&\, \{y_n\} \sim \{z_n\} \implies \{x_n\} \sim \{z_n\}$

Idea: Define a real number to be a (Cauchy seq. of rationals) equivalence class.
Homework: want to check that arithmetic extends to "real numbers"

$$[\{x_n\}] + [\{z_n\}] = [\{x_n + z_n\}]$$

Check that

1. $\{x_n + z_n\}$ is a Cauchy seq.
2. Only depends on equivalence classes.

Want

$$\{x_n\} \sim \{y_n\} \qquad \{z_n\} \sim \{w_n\}$$

then $\{x_n + z_n\} \sim \{y_n + w_n\}$. So,

$$[\{x_n + z_n\}] = [\{y_n + w_n\}]$$

Example 5.2

$\mathbb{Z}_5$

$$[2] + [11] = [2 + 11] = [13]$$

So, $[2+1] \sim [13]([11] = [1])$. Arithmetic (addition) in $\mathbb{Z}_5$ thus makes sense. How about multiplication?

$\frac{[1]}{[a]} \leftarrow$ exists $[a] \neq 0$

$$\frac{[1]}{[2]} = [3] \qquad [2][3] = [6] = [1]$$

Thus, $\mathbb{Z}_5$ is a field.

$\frac{p}{q} \sim \frac{r}{s}, \quad q, s \neq 0$ means $ps = rq$ (when talking about fractions – associate it with equivalence relation). $Q =$ set of equivalences classes. $(\frac{p}{q})$: equivalence classes).
Last time, we proved that Cauchy seq. of real numbers have limits (lub property). Also, no sequence $\{x_n\}$ such that it hits all real numbers in $[0, 1]$ – this is important. Contrast with $Q \cap [0, 1]$, then there is a sequence that hits them all. Refer to the last figure in Lec 4 or `math.ucla.edu/~greene` – Summer 2008.

§6 Lec 6: Oct 14, 2020

Bolzano - Weierstrass:

Every bounded sequence has a convergent subsequence.

And we know about the Least Upper Bound Prop.

§6.1 Applications to Continuous Functions

$f : S \to \mathbb{R}, \quad S \subset \mathbb{R}$

Example 6.1

$$S = [a, b]$$
$$S = \mathbb{R}$$

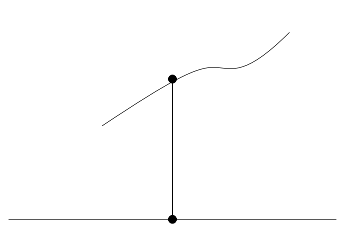
Definition 6.2 — $s_0 \in S, f$ is continuous at s_0 if given $\epsilon > 0, \exists \delta > 0$ s.t.

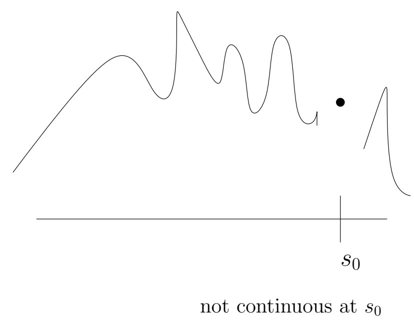
$$|s - s_0| < \delta_\epsilon \implies |f(s) - f(s_0)| < \epsilon$$

Three properties:

$$f : [a, b] \to \mathbb{R}$$
$$f \text{continuous}$$

1. f is bounded on $[a, b]$ means $\exists M$ s.t. for all $x \in [a, b], \quad |f(x)| \leq M$





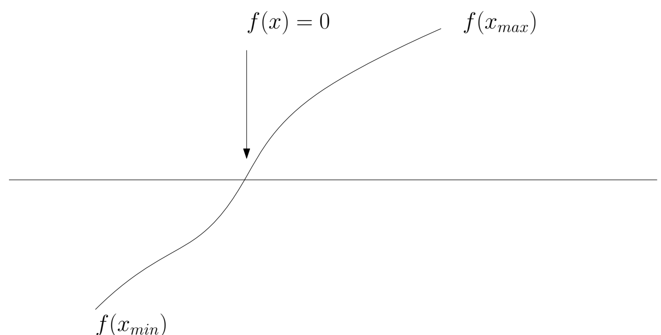
2. There exists $x_{\min}, x_{\max} \in [a,b]$ such that for all $x \in [a,b]$

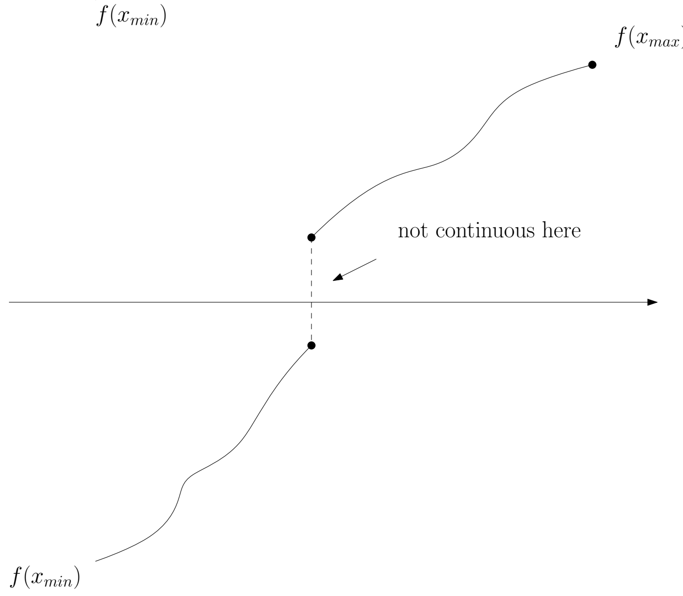
$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

Slogan: f attains its maximum and minimum.

3. If $\alpha, f(x_{\min}) < \alpha < f(x_{\max})$, then $\exists x \in S = [a,b]$ s.t. $f(X) = \alpha$.

"Intermediate Value Theorem" <u>Need</u> the least upper bound prop – "completeness of





real numbers"

Exercise: def of continuity $\{s_n\}$ converges to s_0 $\iff$ if $s_n \to s_0, \quad s_n \in S, s_0 \in S$ then $\{f(s_n)\}$ converges to $f(s_0)$.

Example 6.3

For (3),

$$f(x) = x^2 - 2 \quad \text{on} \quad Q \cap [1,2]$$

Then $f(1) = -1$, $f(2) = 2$, but no rational $x \in [1,2]$ s.t. $f(x) = 0$.

Back to the properties:

1. f is bounded – Think about $|f| \leftarrow$ continuous if f is (exercise).

   $\exists M$ such $|f(x)| \le M$ all $x \in [a,b]$. Suppose no such M exists.

   Try $M = 1, 2, 3, 4, 5, 6, \ldots$. So $\exists x_1 \quad |f(x_1)| > 1$

   $$\begin{gathered} |f(x_2)| > 2 \\ \vdots \\ |f(x_n)| > n \end{gathered}$$

   But Bolzano – Weierstrass: subsequence $\{x_{n_j}\}$ that converges to x_0 say $|f(x_0)| \leftarrow$

   finite number. So $\exists N \ni \quad |f(x_0)| \le N$.

   <u>Now</u> for j large enough

   $$\left|f(x_{n_j}) - f(x_0)\right| < 1$$

   x_{n_j} converges to x_0

   $$\left|f(x_{n_j})\right| < |f(x_0)| + \left|f(x_{n_j} - f(x_0)\right|$$

   So j is large enough that

   $$\underbrace{\left|f(x_{n_j})\right|}_{\ge |n_j| = n_j} \le N + \text{ something less than 1 } \le N$$

2. Attains max and min

   <u>Similar</u>: $\{f(x) : x \in [a,b]\}$ bounded set, has sup where

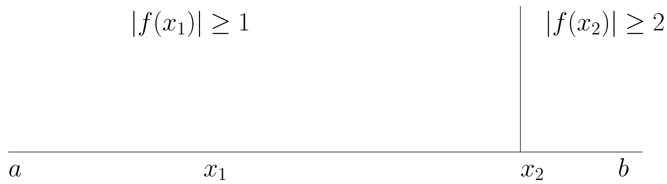
   $$\sup \{f(x) : x \in [a,b]\}$$

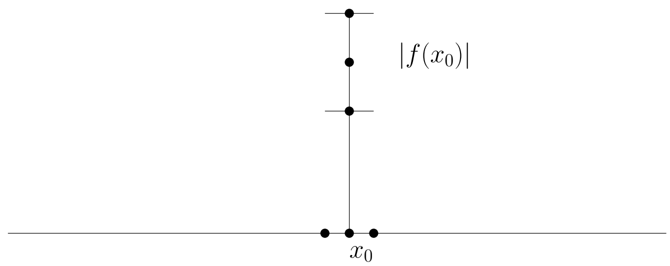
   either in the set of f-values (done if that's true), $\sup f = f(x_0)$.

   OR: $\sup f$ acutally not in the set $\{f(x) : x \in [a,b]\}$

   Now $\{x_{n_j}\}$ converges to $x_0 \in [a,b]$

   Claim 6.1. $f(x_0) = \sup \{f(x) : x \in [a,b]\}$

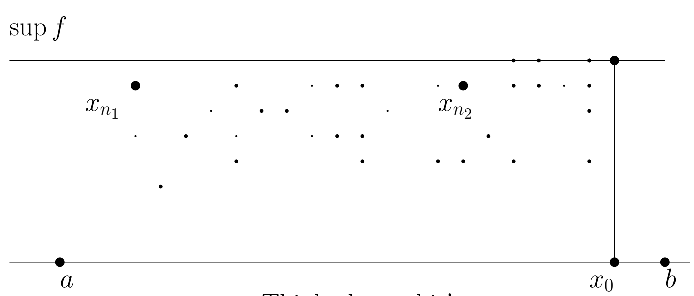




$$f(x_{n_j}) \leq \sup \{f(x) : x \in [a, b]\}$$

and $\lim f(x_{n_j}) = f(x_0) = f(\lim x_{n_j})$. So

$$f(x_0) = \sup f$$



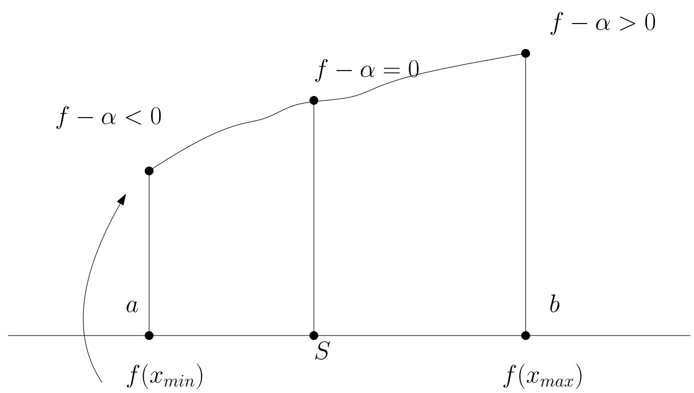
Think about this!

3. $\alpha \in [f(x_{\min}), f(x_{\max})]$ then x such that $f(x) = \alpha$.

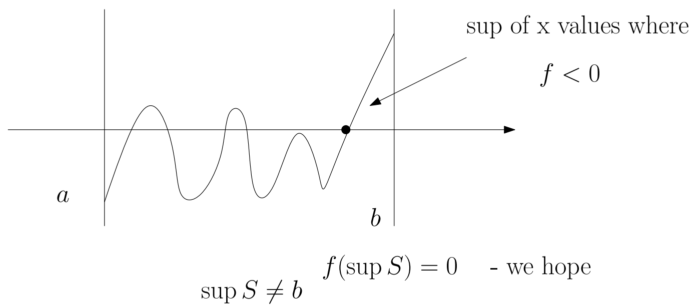
   Proof. Wolog:

$$f(a) < 0 \quad \text{and} \quad f(b) > 0$$

   then $\exists x \in [a, b]$ with $f(x) = 0$.

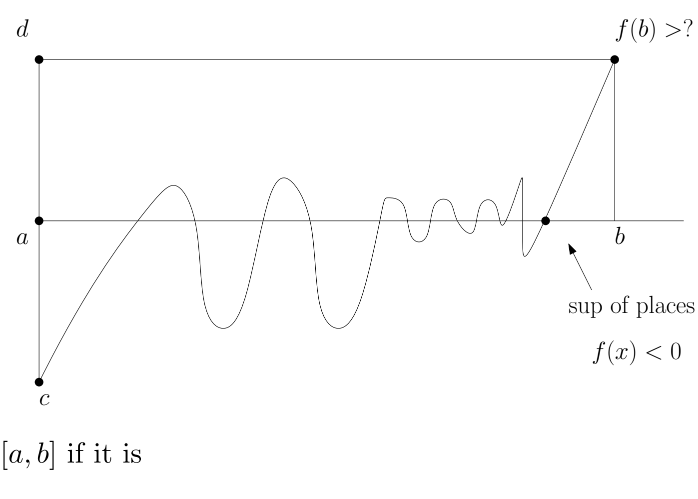


Use l.u.b: Look at $S : \{x : f(x) < 0\}$ and $S \neq \emptyset$ because $f(a) \in S$. Also, S is bounded above – $\exists$ l.u.b for S, $\sup S \in [a, b]$. Hope that $f(\sup S) = 0$.



$\sup S \neq b$ is clear because $f(b) > 0$ so $f(b - \epsilon) > 0$ for small ϵ.

So $\sup S = x_0, \quad a < x_0 < b$. What is $f(x_0)$? If it's negative, then there are slightly bigger $x \in [a_0, b] \ni f(x) < 0$ (continuity). In addition, x_0 cannot be a limit of x with $f(x) < 0$ – $x_0 =$ sup places where $f < 0$. $\square$



f continuous on $[a, b]$ if it is

1. bounded.
2. attains max and min.
3. attains every value between max value and min value.

$f([a, b]) = [c, d]$ where c is min of f and d is max of f.

§7 Dis 1: Oct 1, 2020

Notation:

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3, \ldots\} \\
\mathbb{Z} &= \{0, \pm 1, \pm 2, \ldots\} \\
\mathbb{Q} &= \left\{\frac{p}{q} | p, q \in \mathbb{Z}, q \neq 0\right\} \\
\mathbb{R} &= \text{real numbers} \\
\mathbb{C} &= \{a + bi, \quad a, b \in \mathbb{R}\}
\end{aligned}$$

Set theory:

- $A \subset B$ (or $A \subseteq B$) means $x \in A \implies x \in B$
- $x \in A \cap B$ means $x \in A$ and $x \in B$
- $x \in A \cup B$ means $x \in A$ or $x \in B$
- $x \in A \setminus B \iff x \in A$ and $x \notin B$
- $A = B \iff A \subset B$ and $B \subset A$

§7.1 Induction

Given a sequence of mathematical statement $P(n)$ indexed by $\mathbb{N}$. If $P(1)$ is true and $P(k) \implies P(k+1)$ is true $\forall k \in \mathbb{N}$, then $P(n)$ is true $\forall n \in \mathbb{N}$.

Example 7.1

Prove $\sum_{k=1}^{n}(2k-1) = n^2$ (*) using induction.
Base case $n = 1 : 1 = 1^2$ ✓
Induction step: assume as induction hypothesis that (*) holds

$$\begin{aligned}
\sum_{k=1}^{n+1}(2k-1) &= \sum_{k=1}^{n}(2k-1) + 2(n+1) - 1 \\
&= n^2 + 2n + 1 \\
&= (n+1)^2
\end{aligned}$$

Or we can prove it the following way

$$\begin{aligned}
S &= 1 + 3 + 5 + \ldots + (2n-1) \\
S &= (2n-1) + (2n-3) + \ldots + 3 + 1 \\
2S &= 2n \cdot n \\
S &= n^2
\end{aligned}$$

Example 7.2

$a_{n+1} = \sqrt{2+a_n}, \quad a_1 = 1$. Prove $a_n > 0$ and a_n increasing.
$a_1 > 0$ assume $a_n > 0$, $a_{n+1} = \sqrt{2+a_n} > 0$

$$a_2 = \sqrt{3} \approx 1.732 > 1 = a_1$$

Assume $a_n \leq a_{n+1}$, want to show $a_{n+1} \leq a_{n+2} \impliedby \sqrt{a_n+2} \leq \sqrt{a_{n+1}+2} \impliedby a_n \leq a_{n+1}$

Example 7.3

$(1+x)^n \geq 1+nx$: Bernoulli Inequality
$x \geq -1, \quad n \geq 0$
base case $1 \geq 1$
Assume $(1+x)^n \geq 1+nx$

$$\begin{aligned}(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x) &= 1+(n+1)x+nx^2 \\ &= 1+(n+1)x\end{aligned}$$

Strong Induction:
If $P(1)$ true and $P(1), P(2), \ldots P(k) \implies P(k+1)$ true $\forall k \in \mathbb{N}$ then $P(n)$ holds for all $n \in \mathbb{N}$

Remark 7.4. Induction $\iff$ strong induction

Example 7.5

Every integer greater than 1 is a product of primes.
Assume $2, 3, \ldots, n$ is a product of primes. n+1 is either a prime or a composite, in which case $n+1 = ab, \quad 1 < a, b < n+1$.
By strong induction hypothesis, both a and b are product of primes, hence so is $n+1 = ab$.

Exercise 7.1. Every integer greater than 1 has a prime divisior.

Proof of infinitude of primes by Euclid:

Proof. Assume on the contrary there are finitely many primes $\{p_1, p_2, \ldots, p_k\}$. Define $N = p_1 \ldots p_k + 1 > 1$ and (by above exercise) let p be a prime divisior of N but $p \neq p_j$ for any $1 \leq j \leq k$ otherwise if $p = p_j$ then $p | p_2 \ldots p_k$ also $p | N \implies p | N - p_1 \ldots p_k \implies p | 1$,a contradiction. (no primes divide 1) □

§8 Dis 2: Oct 8, 2020

§8.1 Number System

- $(\mathbb{N}, +, \cdot, <)$: $+ : \mathbb{N} \times \mathbb{N} = \mathbb{N}^2 \to \mathbb{N}$ satisfies commutativity and associativity. Note that 0 is the identity with respect to addition, but $\mathbb{N}$ has no additive inverse.

- $(\mathbb{Z}, +, \cdot, <)$: $(\mathbb{Z}, +)$ is a commutative group (associativity, identity, inverse). $(\mathbb{Z}, \cdot)$ satisfies commutativity, associativity with 1 as mult identity but 2 has no mult inverse.

- $(\mathbb{Q}, +, \cdot, <)$: $(\mathbb{Q}, +)$ and $(\mathbb{Q}, \cdot)$ are commutative group(i). $+$ and $\cdot$ are compatible with distributive law: $a(b+c) = ab + ac$ (ii). Both (i) and (ii) mean $(\mathbb{Q}, +, \cdot)$ is a FIELD. $(Q, <)$ is an ordered set with $<$ satisfying trichotomy and transitivity. $+, \cdot$ are compatible : $y < z \implies x + y < x + z \forall x$, $x > 0, y > 0 \implies xy > 0$. With the above compatibility, $(\mathbb{Q}, +, \cdot, <)$ is an ordered field. Even though $\mathbb{Q}$ is additivity adn multiplicatively complete, $\mathbb{Q}$ is not satisfying in that

  1. $\mathbb{Q}$ is not algebraically closed, $x^2 - 2$ is a polynomial with no root in $\mathbb{Q}$.
  2. $\mathbb{Q}$ is not complete in a metric space: there exists subsets of $\mathbb{Q}$ bounded above but with no least upper bound (supremum), e.g. $A \coloneqq \left\{p \in \mathbb{Q} : p < 0 \text{ or } p^2 < 2\right\}$ and $B = \mathbb{Q} \setminus A$. A contains no largest number and B contains no smallest.

     $$\forall p \in A \exists q \in A \quad q > p$$

     Let $p \in A$. Define $q \coloneqq p - \frac{p^2-2}{p+2} > p$

     $$q^2 - 2 = \left(\frac{2p+2}{p+2}\right)^2 - 2 = \frac{2(p^2-2)}{(p+2)^2} < 0 \implies q^2 < 2$$

     If A has an upper bound α, $\alpha \notin A$: then $\alpha \in B$. It follows that B is the set of all upper bounds for A. Since B contains no smallest number, A has no least upper bound in $\mathbb{Q}$.

Definition 8.1 — S has the least-upper-bound property if $\forall E \subset S$ nonempty, bounded above $\sup E \in S$.

Remark 8.2. $\mathbb{Q}$ does not satisfy the least-upper-bound property.

$(\mathbb{R}, +, \cdot, <)$ there exists an ordered field with the l.u.b property that contains an isomorphic copy of $\mathbb{Q}$.

§8.2 Equivalence Relation

An equivalence relation given $\sim$ on $A \times A$ satisfies

- $x \sim x$ reflexity

- $x \sim y \iff y \sim x$ symmetry

- $x \sim y \cdot y \sim z \implies x \sim z$ transitivy

Example 8.3

$\mathbb{Q}$ Define $\sim$ on $\{(a,b) : a, b \in \mathbb{Z}, b \neq 0\}$ by $(a,b) \sim (c,d)$ if $ad = bc$

$$A = \mathbb{Z}^2 \setminus \{(a,0) : a \in \mathbb{Z}\}$$

$$\begin{aligned} \mathbb{Q} = &\text{ the set of all equivalence classes of A write } \sim \\ &= A/\sim = \{[x] : x \in A\} \end{aligned}$$

In this construction, $\mathbb{Z} \to \mathbb{Q}$, $n \to [(n,1)]$
$+ \setminus \cdot : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$: note that $+$ and $\cdot$ need to be well-defined on $\mathbb{Q}^2$. (need to show $\frac{a}{b} + \frac{c}{d} = \frac{a'}{b'} + \frac{c'}{d'}$ if $\frac{a}{b} \sim \frac{a'}{b'}$ and $\frac{c}{d} \sim \frac{c'}{d'}$.

Example 8.4

$S' = [0,1]/0_m$

Definition 8.5 (Convergent Sequences) — $\{a_n\}_{n\geq 1} \subseteq \mathbb{R}$ is said to be convergent to l if $\forall \epsilon > 0 \quad \exists N(\epsilon) > 0$ s.t. $\forall n \geq N, \quad |a_n - l| < \epsilon$

§9 Dis 3: Oct 13, 2020

§9.1 Equivalence Relation (Cont'd)

Example 9.1

Define $\sim p$ on $\mathbb{Z}$ by $a \sim pb$ if $a - b \in p\mathbb{Z}(p|a-b)$.
$\forall a \exists! b \in \mathbb{Z}, \quad 0 \leq r < p$ s.t. $a = bp + r$.

$$F_p = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}/\sim p = \{[0]_p, [1]_p, [2]_p, \ldots, [p-1]_p\}$$

$[a]_p + [b]_p = [a+b]_p \quad \& \quad [a]_p[b]_p = [ab]_p$

Remark 9.2. $(F_p, +, \cdot)$ is a finite field. F_p cannot be ordered: $1 > 0, 1+1 > 0, \ldots, p-1 > 0$ but $p - 1 = -1$

Example 9.3

$T = \mathbb{R}/\mathbb{Z} \quad a \sim b$ if $ab \in \mathbb{Z}$
$[0,1]/0 \sim 1$
$\forall a \in \mathbb{R}, \quad \exists b = \underbrace{\{a\}}_{\text{fractional part of } a} \in [0,1)$ s.t. $a \sim b$

§9.2 Construction of R via Cauchy Sequences(Cantor)

S = set of rational Cauchy sequences.
$\sim$ on $S : \{x_n\} - \{y_n\}$ if $\lim(x_n - y_n) = 0$ (Q3 – Homework 2)
$Q = S/\sim = \{[\{x_n\}] : \{x_n\} \in S\}$. First we need to define arithmetic on Q.

$$\begin{aligned}
[\{p_n\}] + [\{q_n\}] &= [\{p_n + q_n\}] \\
[\{p_n\}] - [\{q_n\}] &= [\{p_n - q_n\}] \\
[\{p_n\}] \cdot [\{q_n\}] &= [\{p_n q_n\}] \\
[\{p_n\}] \,/\, [\{p_n/q_n\}] &= [\{p_n/q_n\}], \quad [\{q_n\}] \neq 0, = [\{0, 0, 0, \ldots\}]
\end{aligned}$$

$+ : Q \times Q \to Q$. Check well-defined

- $\{x_n\} \cdot \{y_n\}$ cauchy then so is $\{x_n + y_n\}$(Q4)
- $\{x_n\} \sim \{y_n\}$ & $\{z_n\} \sim \{w_n\}$ then $\{x_n + z_n\} \sim \{y_n + w_n\}$ (Q5)
  Commutativity, assoc, identity, $(0 = [\{0, 0, 0, \ldots\}]$, inverse.
- Well-defined: $\{x_n\}, \{y_n\}$ so is $\{x_n y_n\}$ (Q4).
- $\{x_n\} \sim \{y_n\}$ & $\{z_n\} \sim \{w_n\}$ (Q6, Q7)
  comm, assoc, iden, $(1 = [\{1, 1, \ldots, 1\}]$
  mult. inverse (Q9,Q10).
  $<$: trichotomy (Q11), transitivity
  various compatibility (distributivity, etc)
  l.u.b property (Q12)

Note:All the Q used above is assumed to be Q^{hat}

Remark 9.4.

$$\begin{aligned}
Q &\to Q^{\text{hat}} \\
q &\mapsto [q^*] \\
p < q &\iff [p^*] < [q^*]
\end{aligned}$$

Sequences:

- Cauchy seq. are bounded.
- Convergent seq. is Cauchy.
  Theorem: in $\mathbb{R}$, every Cauchy seq. is convergent.

Example 9.5
$a_n = \frac{1}{n}$
$\forall \epsilon > 0 \exists N$ s.t. $\epsilon N > 1$.
$\forall n \geq N \quad \left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$. $\square$