Problem Set #2 – PHYS 7127

Bin Xia

1. (15 points) Using the exponential disk model from the book, derive the expressions for the total angular momentum and energy of the disk.

From Eq. (11.1), the surface density distribution for exponential disk model is

$$\Sigma(R) = \Sigma_0 \exp(-R/R_d)$$

From Eq. (11.30), the circular velocity for a thin exponential disk where all stars are assumed to be on circular orbits is

$$V_{c,d}^2(R) = -4\pi G \Sigma_0 R_d y^2 [I_0(y)K_0(y) - I_1(y)K_1(y)],$$

where $y = R/(2R_d)$. The total mass of the disk is

$$\begin{aligned}
M_d &= \int_0^\infty \Sigma 2\pi R dR \\
&= 2\pi \int_0^\infty \Sigma_0 \exp(-R/R_d) R dR \\
&= 2\pi \int_0^\infty \Sigma_0 \exp(-2yR_d/R_d)(2yR_d) d(2yR_d) \\
&= 8\pi \Sigma_0 R_d^2 \int_0^\infty \exp(-2y) y dy \\
&= 8\pi \Sigma_0 R_d^2 \frac{1}{4} \\
&= 2\pi R_d^2 \Sigma_0
\end{aligned}$$

Hence, the total angular momentum is

$$\begin{aligned}
J_d &= \int_0^\infty R(\Sigma 2\pi R dR) V_c(R) \\
&= 2\pi \int_0^\infty \Sigma V_c R^2 dR \\
&= 2\pi \int_0^\infty \Sigma_0 \exp(-R/R_d) \sqrt{-4\pi G \Sigma_0 R_d y^2 [I_0(y)K_0(y) - I_1(y)K_1(y)]} R^2 dR \\
&= 2\pi \int_0^\infty \Sigma_0 \exp(-2yR_d/R_d) \sqrt{4\pi G \Sigma_0 R_d y^2 [I_1(y)K_1(y) - I_0(y)K_0(y)]} (2yR_d)^2 d(2yR_d) \\
&= 2\pi \int_0^\infty \Sigma_0 \exp(-2y) \sqrt{4\pi G \Sigma_0 R_d} \sqrt{y^2 [I_1(y)K_1(y) - I_0(y)K_0(y)]} 8R_d^3 y^2 dy \\
&= 2\pi \Sigma_0 \sqrt{4\pi G \Sigma_0 R_d} 8R_d^3 \int_0^\infty \exp(-2y) \sqrt{y^2 [I_1(y)K_1(y) - I_0(y)K_0(y)]} y^2 dy \\
&= 32\pi^{3/2} \Sigma_0^{3/2} G^{1/2} R_d^{7/2} \int_0^\infty \exp(-2y) \sqrt{y^2 [I_1(y)K_1(y) - I_0(y)K_0(y)]} y^2 dy \\
&= 32\pi^{3/2} \left(\frac{M_d}{2\pi R_d^2} \right)^{3/2} G^{1/2} R_d^{7/2} \int_0^\infty \exp(-2y) \sqrt{y^2 [I_1(y)K_1(y) - I_0(y)K_0(y)]} y^2 dy \\
&= \left\{ 8\sqrt{2} \int_0^\infty \exp(-2y) \sqrt{y^2 [I_1(y)K_1(y) - I_0(y)K_0(y)]} y^2 dy \right\} M_d^{3/2} G^{1/2} R_d^{1/2} \\
&= \alpha M_d^{3/2} G^{1/2} R_d^{1/2},
\end{aligned}$$

where $\alpha \equiv \left\{ 8\sqrt{2} \int_0^\infty \exp(-2y) \sqrt{y^2 [I_1(y)K_1(y) - I_0(y)K_0(y)]} y^2 dy \right\}$ is a constant.

According to the virial theorem, the total energy of the disk satisfies E = -K, where K is the total kinetic energy of the disk, so that

$$
\begin{aligned}
E &= -K \\
&= -\int_0^\infty \frac{1}{2} dmV^2 \\
&= -\int_0^\infty \frac{1}{2}\Sigma 2\pi R dR\left\{-4\pi G\Sigma_0 R_d y^2[I_0(y)K_0(y) - I_1(y)K_1(y)]\right\} \\
&= \int_0^\infty \Sigma_0 \exp(-R/R_d)\pi 4\pi G\Sigma_0 R_d y^2[I_0(y)K_0(y) - I_1(y)K_1(y)]RdR \\
&= 4\pi^2 G\Sigma_0^2 R_d \int_0^\infty \exp(-R/R_d)y^2[I_0(y)K_0(y) - I_1(y)K_1(y)]RdR \\
&= 4\pi^2 G\Sigma_0^2 R_d \int_0^\infty \exp(-2yR_d/R_d)y^2[I_0(y)K_0(y) - I_1(y)K_1(y)]2yR_d d(2yR_d) \\
&= 16\pi^2 G\Sigma_0^2 R_d^3 \int_0^\infty \exp(-2y)y^3[I_0(y)K_0(y) - I_1(y)K_1(y)]dy \\
&= 16\pi^2 G\left(\frac{M_d}{2\pi R_d^2}\right)^2 R_d^3 \int_0^\infty \exp(-2y)y^3[I_0(y)K_0(y) - I_1(y)K_1(y)]dy \\
&= 4G{M_d}^2 R_d^{-1} \int_0^\infty \exp(-2y)y^3[I_0(y)K_0(y) - I_1(y)K_1(y)]dy \\
&= \beta G M_d^2 R_d^{-1},
\end{aligned}
$$

where $\beta \equiv 4\int_0^\infty \exp(-2y)y^3[I_0(y)K_0(y) - I_1(y)K_1(y)]dy$

2. (5 points) What is the corresponding value for the disk's spin parameter? Discuss what is needed in galaxy formation, in addition to tidal torques, to establish such a value.

$$
\begin{aligned}
\lambda_d &\equiv \frac{J_d|E|^{1/2}}{GM_d^{5/2}} \\
&= \frac{\alpha {M_d}^{3/2} G^{1/2} R_d^{1/2}\sqrt{|\beta G M_d^2 R_d^{-1}|}}{GM_d^{5/2}} \\
&= \alpha\sqrt{|\beta|} \\
&\approx 0.426
\end{aligned}
$$

In [ ]:
```
alpha = 1.11
beta = -0.147
lambda_d = alpha * (abs(beta))**0.5
print(f"Hence the corresponding value for the spin parameter of \nan isolated, exponential, infinitesimally th
```

```
Hence the corresponding value for the spin parameter of
an isolated, exponential, infinitesimally thin disk is 0.426.
```

While tidal torques during the linear regime of structure formation set the stage for angular momentum generation, a combination of processes involving, gas dynamics, star formation, feedback mechanisms, mergers, accretion, magnetic fields, and reionization collectively shape the formation and evolution of galaxies in the universe.

3. (5 points) Assume that our disk galaxy formed from a spherical density perturbation of uniform density that experienced turn-around (hint: δta(tta) ≃ 1) at redshift zta. What was the proper size of this sphere at the epoch of turn-around?

For an EdS universe without dark matter, the average density at redshift z_{ta} satisfies

$$
\bar{\rho} = \frac{3H_0^2}{8\pi G}(1+z_{ta})^3
$$

Since the density of the spherical density of perturbation satisfies $\rho = (\delta + 1)\bar{\rho}$,

$$
\rho(z_{ta}) = \frac{3H_0^2}{8\pi G}(1+z_{ta})^3(1+\delta)
$$

The mass of the spherical density perturbation should be equal to the mass of the disk galaxy, Hence,

$$M_d = \rho(z_{ta})\frac{4\pi}{3}r_{ta}^3$$

Therefore, the proper size of this sphere at the epoch of turn-around is given by

$$\begin{aligned}
r_{ta} &= \left[\frac{3M_d}{4\pi\rho(z_{ta})}\right]^{1/3} \\
&= \left[\frac{3M_d}{4\pi\frac{3H_0^2}{8\pi G}(1+z_{ta})^3(1+\delta)}\right]^{1/3} \\
&= \left[\frac{2GM_d}{H_0^2(1+z_{ta})^3(1+\delta)}\right]^{1/3} \\
&= \left[\frac{2GM_d}{H_0^2(1+\delta)}\right]^{1/3}\left(\frac{1}{1+z_{ta}}\right) \\
&\approx \frac{147.7}{1+z_{ta}}\,\mathrm{kpc}
\end{aligned}$$

In [ ]:
```python
from astropy import units as u
from astropy import constants as cs
from astropy.cosmology import WMAP9 as cosmo
import numpy as np
```

In [ ]:
```python
H_0 = cosmo.H(0)
M_d = 1e10 * u.M_sun
R_d = 3 * u.kpc
delta = 5.55 - 1

def r(z):
    """
    the proper size of the spherical density perturbation at the epoch of turn-around
    """
    r = np.cbrt(2*cs.G*M_d/(H_0**2*(1+delta))) / (1+z)
    return r

def rho(z):
    """
    density of the spherical density of perturbation
    """
    rho = 3*H_0**2/(8*np.pi*cs.G) * (1+delta) * (1+z)**3
    return rho
```

In [ ]:
```python
z_ta = 3
r_ta = r(z_ta).to("kpc")
print("The proper size as a function of turn-around redshift is given by:")
print(f"r_ta = {r(0).to('kpc'):.1f}/(1+z_ta) .")
print(f"For example, if the turn-around redshift is {z_ta},\nthe proper size of this sphere at the epoch of tu
```

```
The proper size as a function of turn-around redshift is given by:
r_ta = 147.7 kpc/(1+z_ta) .
For example, if the turn-around redshift is 3,
the proper size of this sphere at the epoch of turn-around is 36.9 kpc.
```

4. (5 points) Under the assumption that disk formation conserves angular momentum, compute the spin parameter of the overdensity at turn-around, as a function of zta. Discuss the implications for disk formation.

Angular momentum and mass are conserved but the binding energy, -E, increases in reverse proportion to its size R. Therefore, the spin parameter of the cloud scales as

$$
\begin{aligned}
\lambda(z_{ta}) &= \lambda_d \left(\frac{r_{ta}}{R_d}\right)^{-1/2} \\
&= \lambda_d \left\{ \left[\frac{2GM_d}{H_0^2(1+\delta)}\right]^{1/3} \left(\frac{1}{1+z_{ta}}\right) / R_d \right\}^{-1/2} \\
&= \lambda_d \left\{ \left[\frac{2GM_d}{H_0^2(1+\delta)R_d^3}\right]^{1/3} \left(\frac{1}{1+z_{ta}}\right) \right\}^{-1/2} \\
&= \lambda_d \left[\frac{H_0^2(1+\delta)R_d^3}{2GM_d}\right]^{\frac{1}{6}} \sqrt{1+z_{ta}}
\end{aligned}
$$

In [ ]:
```
lambda_cofactor = lambda_d * (H_0**2*(1+delta)*R_d**3/(2*cs.G*M_d))**(1/6)
print("The spin paramter of the overdensity at turn-around as a function of z_ta is:")
print(f"lambda(z_ta) = {lambda_cofactor.to('1'):.3f} x (1+z_ta)**0.5 .")
```

```
The spin paramter of the overdensity at turn-around as a function of z_ta is:
lambda(z_ta) = 0.061 x (1+z_ta)**0.5 .
```

The free fall time for the spherical density perturbation is given by $t_{ff} = \sqrt{3\pi/32G\rho}$

In [ ]:
```
print(f"For example, if the turn-around redshift is {z_ta},")
print(f"the density of the spherical density perturbation is {rho(z_ta).to('g/cm**3'):.1e},")
print(f"and the free fall time for the spherical perturbation is given by {np.sqrt(3*np.pi/(32*cs.G*rho(z_ta))
```

```
For example, if the turn-around redshift is 3,
the density of the spherical density perturbation is 3.2e-27 g / cm3,
and the free fall time for the spherical perturbation is given by 1.2e+09 yr,
```

The free fall time is comparable to the age of the universe, and thus rules out the possibility that the disk galaxy could have formed from pure gas clouds with spin parameters expected from tidal torque theory.