Homework 0 for COMS 4772 Fall 2016

Problem 2: (a) False

(b) False.

Problem 3: (a) 5

(b) True.

(c) False.

(d) 2.

(e) $\frac{1}{8}$

Problem 4: (a) 5

(b) $\frac{n-4}{1024}$

(c) 2

(d) $\lambda = \frac{\ln 2}{10^6} \approx 6.9 \times 10^{-7}$

(e) $c = 2$

(f) $\frac{3}{4}$

(g) No, not independent

$\mathbb{E} Y = -\frac{3}{4}$

(h) Yes, they are independent.

$\mathbb{E} X_1 Z = 0$

Problem 5 (a) Pick any $\vec{q} \in \text{conv}(S)$.

So $\exists$ $k \in \mathbb{N}$, $p_1, \dots, p_k \in [0,1]$ with $\sum_{i=1}^{k} p_i = 1$, and $\vec{x}_1, \dots, \vec{x}_k \in S$

such that $\vec{q} = \sum_{i=1}^{k} p_i \vec{x}_i$. (*)

Now fix $\vec{r} \in \mathbb{R}^d$. Then

$$
\begin{aligned}
0 &= \langle \vec{r} - \vec{q}, \vec{q} - \vec{q} \rangle \\
&= \left\langle \vec{r} - \vec{q}, \sum_{i=1}^{k} p_i \vec{x}_i - \vec{q} \right\rangle && \text{by } (*) \\
&= \left\langle \vec{r} - \vec{q}, \sum_{i=1}^{k} p_i (\vec{x}_i - \vec{q}) \right\rangle && \text{since } \sum_{i=1}^{k} p_i = 1 \\
&= \sum_{i=1}^{k} p_i \langle \vec{r} - \vec{q}, \vec{x}_i - \vec{q} \rangle && \text{by linearity.}
\end{aligned}
$$

At least one of the k terms must be non-positive.

Let's say it is the i^{th} term, $p_i \langle \vec{r} - \vec{q}, \vec{x}_i - \vec{q} \rangle \leq 0$.

Since $p_i > 0$, it must be that $\langle \vec{r} - \vec{q}, \vec{x}_i - \vec{q} \rangle \leq 0$.

So $\vec{x}_i$ satisfies the required property. $\boxtimes$

Problem 5 (continued) (b)

First we show $\|\vec{x}-\vec{z}\|_2 \le \Delta$ for any $\vec{x} \in S$.

Since $\vec{z} \in \text{conv}(S)$, we can write it as

$$\vec{z} = \sum_{i=1}^{k} p_i \vec{x}_i' \quad \text{for some } k \in \mathbb{N},\ p_1, \dots, p_k \in (0,1] \text{ with } \sum_{i=1}^{k} p_i = 1,$$
$$\vec{x}_1', \dots, \vec{x}_k' \in S.$$

So
$$\begin{aligned}
\|\vec{x}-\vec{z}\|_2 &= \left\|\vec{x} - \sum_{i=1}^{k} p_i \vec{x}_i'\right\|_2 \\
&= \left\|\sum_{i=1}^{k} p_i (\vec{x}-\vec{x}_i')\right\|_2 \quad \text{since } \sum_{i=1}^{k} p_i = 1 \\
&\le \sum_{i=1}^{k} \|p_i(\vec{x}-\vec{x}_i')\|_2 \quad \text{by triangle inequality} \\
&= \sum_{i=1}^{k} p_i \|\vec{x}-\vec{x}_i'\|_2 \\
&\le \sum_{i=1}^{k} p_i \Delta \quad \text{by defn of } \Delta \\
&= \Delta \quad \text{since } \sum_{i=1}^{k} p_i = 1.
\end{aligned}$$

Now observe that
$$\begin{aligned}
\|\vec{y}-\vec{z}\|_2 &= \left\|\frac{1}{T}\sum_{t=1}^{T} \vec{x}_t - \vec{z}\right\|_2 \\
&= \left\|\frac{1}{T}\sum_{t=1}^{T} (\vec{x}_t - \vec{z})\right\|_2 \\
&= \frac{1}{T}\left\|\sum_{t=1}^{T} (\vec{x}_t - \vec{z})\right\|_2.
\end{aligned}$$

So suffices to prove $\left\|\sum_{t=1}^{T} (\vec{x}_t - \vec{z})\right\|_2 \le \Delta\sqrt{T}$.

$$\left\| \sum_{t=1}^{T} (\vec{x}_t - \vec{z}) \right\|_2^2$$

$$= \left\| \sum_{t=1}^{T-1} (\vec{x}_t - \vec{z}) \right\|_2^2 + 2\left\langle \sum_{t=1}^{T-1} (\vec{x}_t - \vec{z}), \vec{x}_T - \vec{z} \right\rangle + \|\vec{x}_T - \vec{z}\|_2^2$$

$$\leq \left\| \sum_{t=1}^{T-1} (\vec{x}_t - \vec{z}) \right\|_2^2 + 0 + \|\vec{x}_T - \vec{z}\|_2^2 \quad \text{by choice of } \vec{x}_T$$

$$\leq \left\| \sum_{t=1}^{T-1} (\vec{x}_t - \vec{z}) \right\|_2^2 + \Delta^2 \quad \text{by defn of } \Delta.$$

So by induction, $\left\| \sum_{t=1}^{T} (\vec{x}_t - \vec{z}) \right\|_2^2 \leq \Delta^2 T$. $\boxtimes$