Volumes in high-dimensional space

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COMS 4772

Simple volumes

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$$[a, b]^3 \;=\; \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1, x_2, x_3 \in [a, b]\}$$

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- In general,

$$\mathrm{vol}(cA) \;=\; c^d\,\mathrm{vol}(A)\,.$$

Weird facts about the unit ball

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   - By contrast, hypercube $[-1, 1]^d$ has volume 2^d.

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- Therefore

$$(1-\varepsilon)^d \;\le\; e^{-\varepsilon d}$$

fraction of points in B^d have length at most $1-\varepsilon$.

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- So volume outside tropics is at most

$$2(1 - \varepsilon^2)^{d/2} \operatorname{vol}(B^d) \ \leq \ 2e^{-\varepsilon^2 d/2} \operatorname{vol}(B^d) \,.$$

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- For $\varepsilon = \sqrt{2\ln(4d)/d}$, bound is

$$\operatorname{vol}(B^d) \;\leq\; 2\left(\frac{8\ln(4d)}{d}\right)^{d/2} \;\xrightarrow{d\to\infty}\; 0\,.$$