Probability review

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COMS 4772

Linearity of expectation

Random unit vectors

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- Similarly, $\mathbb{E}(X_1 X_2) = \mathbb{E}(X_1 X_2 X_3) = \cdots = 0$.
- Also for any distinct $i_1, i_2, \ldots \in [d]$, $\mathbb{E}(X_{i_1} X_{i_2} \cdots) = 0$.

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  - For any unit vector $\boldsymbol{u} \in S^{d-1}$,

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- E.g., for uniform random unit vector $\boldsymbol{X}$, and any $\boldsymbol{u} \in S^{d-1}$, $\mathbb{E}\,|\langle \boldsymbol{u}, \boldsymbol{X}\rangle| \leq 1/\sqrt{d}$.

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- If X and Y are random variables, then for any scalars $a, b \in \mathbb{R}$,

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- Variance of the sum of *independent* random variables is the sum of the variances.

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- Application to symmetric random walk:

$$\mathbb{P}(|S_n| \geq c\sqrt{n}) \;\leq\; \frac{\mathbb{E}\,|S_n|}{c\sqrt{n}} \;\leq\; \frac{1}{c}\,.$$

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- Further improvements using higher-order moments.

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- **Moment generating function** (mgf): $M_X : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, defined by

$$M_X(\lambda) \;:=\; \mathbb{E}\exp(\lambda X) \;=\; 1 + \lambda\,\mathbb{E}(X) + \frac{\lambda^2}{2}\,\mathbb{E}(X^2) + \frac{\lambda^3}{3!}\,\mathbb{E}(X^3) + \cdots$$

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- Often use logarithm of M_X (a.k.a. *cumulant generating function* or *log mgf*):

$$\psi_X(\lambda) \ := \ \ln M_X(\lambda) \, .$$

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- If ψ_X is finite on interval (λ_1, λ_2) for some $\lambda_1 < 0$ and $\lambda_2 > 0$, then it is infinitely differentiable on the same (open) interval.

Example of (log) mgfs

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Subgaussian random variables

- Many random variables have log mgf $\psi_{X-\mathbb{E}(X)}(\lambda)$ upper-bounded by that of $\mathrm{N}(0, v)$ for some $v > 0$, i.e.,

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  - Improvement over $1/c$ from Markov's and $1/c^2$ from Chebyshev's (except when c is very small).

Hoeffding's inequality

- Suppose X is $[0,1]$-valued r.v. with $\mathbb{E}(X) = \mu$, and Y is $\{0,1\}$-valued r.v. with $\mathbb{E}(Y) = \mu$. Then

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- Tail bound for (sums of) such random variables also called *Hoeffding's inequality*.

Poisson tail bound

- (Centered) Poi(μ) log mgf $\psi_{X-\mu}(\lambda) = \mu(e^{\lambda} - \lambda - 1)$ has

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- Similar to $\text{Bin}(n, p)$ for p close to zero or one (i.e., tossing a very biased coin n times).
  - Variance is small compared to maximal range.

Using variance information

- Let X satisfy $X - \mathbb{E}(X) \leq 1$ and $\mathrm{var}(X) \leq v$. For any $\lambda \geq 0$,

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  - Get tail bound for S as before; called *Bennett's inequality* or *Bernstein's inequality*.

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- So for $p = O(1/n)$, with probability at least $1 - \delta$,

$$\frac{S}{n} - p \;\leq\; O\left(\frac{\log(1/\delta)}{n}\right).$$

Why does this work?

- log mgf bounded by that of Gaussian for λ around zero:

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- Conclusion:

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  - A tighter analysis gets a bound of $k + 2\sqrt{k\ln(1/\delta)} + 2\ln(1/\delta)$.