FACULTY OF ECONOMICS STUDY AIDS 2019

ECT1 Paper 3 Quantitative Methods in Economics

The Faculty Board has agreed to release outline solutions to the 2019 examinations as a study aid for exam revision. They are abridged solutions, and not 'definitive', and should therefore not be considered as an exemplar for 'complete' answers.

Note that the Faculty will not respond to any queries regarding these solutions.

Part I Paper 3 2019

ANSWERS TO SECTION A

1 (a) The base vectors $(1, 0)$ and $(0, 1)$ are mapped to $(1, -1)$ and $(1, 1)$, respectively. This corresponds to 45^o clock-wise rotation combined with expansion by $\sqrt{2}$.

(b) Hence $A^8 = (\sqrt{2})^8 I = 2^4 I.$
We were given

$$\begin{bmatrix} 15 \\ 6 \end{bmatrix} = L^{1215} \begin{bmatrix} m \\ c \end{bmatrix}$$

Multiplying this equation by L from the left yields

$$L \begin{bmatrix} 15 \\ 6 \end{bmatrix} = L^{1216} \begin{bmatrix} m \\ c \end{bmatrix} = 2^{608} \begin{bmatrix} m \\ c \end{bmatrix}$$
$$\begin{bmatrix} 21 \\ -9 \end{bmatrix} = \begin{bmatrix} 2^{608} m \\ 2^{608} c \end{bmatrix}$$

which implies $m = 21 \times 2^{-608}$ and $c = -9 \times 2^{-608}$.

2 The expenditure function is the minimum cost of achieving utility level $\bar{u}$ given prices p. So, it is obtained by solving

$$\min_{x,y} xp_x + yp_y \qquad \text{s.t.} \qquad u(x, y) = \bar{u}$$

The associated Lagrangian is

$$L(x, y, \bar{u}, p_x, p_y, \lambda) = xp_x + yp_y + \lambda(u(x, y) - \bar{u})$$

The envelope theorem for constrained optimisation says

$$\frac{\partial e}{\partial p_x} = \left.\frac{\partial L}{\partial p_x}\right|_{x=x*, y=y^*}$$

The right hand side easily simplifies to $x^*(\bar{u}, p_x, p_y)$, which is nothing but the Hicksian demand for good x.

3 Let the side with length 15 run from (0, 0) to (15, 0) on the horizontal axis, and let (x, y) denote the third vertex of the triangle so the optimisation problem can be written as maximising the altitude of the triangle with base from (0, 0) to (15, 0). Hence we need to solve

$$\max y \qquad \text{s.t.} \quad 2\sqrt{x^2 + y^2} = \sqrt{(x - 15)^2 + y^2}$$

The associated Lagrangian is

$$L(x, y, \lambda) = y + \lambda \left(2\sqrt{x^2 + y^2} - \sqrt{(x - 15)^2 + y^2}\right)$$

whose FOC for x is

$$L_x = \lambda \left[\frac{4x}{2\sqrt{x^2+y^2}} - \frac{(x-15)}{\sqrt{(x-15)^2+y^2)}} \right] = 0$$

which, thanks to the constraint $2\sqrt{x^2+y^2} = \sqrt{(x-15)^2+y^2}$ simplifies to

$$4x = x - 15$$

which, in turn, implies $x^* = -5$, and inserting this value into the constraint gives

$$2\sqrt{25+y^2} = \sqrt{400+y^2}$$

which has two solutions $y^* = \pm 10$.

Hence, the maximum possible area is $(10 \times 15)/2 = 75$.

4 (a) The integral can be visualised as the area bounded by the graph of $f(x) = 1/x$, the x axis, and the vertical line at $x = 1$. The infinite sum can be envisioned the total area of the boxes, all fitted in this area, of dimension $1 \times n$ for $n = 2, 3, \ldots$. Hence, clearly

$$\sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{dx}{x^2}$$

(b) Note that nth summand of the series goes to 0. It is sufficient to show that the series is bounded.

Rewriting the terms as below:

$$\begin{aligned}
-1 < \sum_{n=1}^{\infty} \frac{(-1)^n}{n} &= -1 + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \cdots \\
&= -1 + \frac{1}{2 \times 3} + \frac{1}{4 \times 5} + \frac{1}{6 \times 7} + \cdots \\
&< -1 + \frac{1}{2^2} + \frac{1}{4^2} \\
&< \sum_{n=2}^{\infty} \frac{1}{n^2} \\
&< \int_1^{\infty} \frac{dx}{x^2} = \lim_{B \to \infty} \left[-\frac{1}{B} + 1 \right] = 1
\end{aligned}$$

ANSWERS TO SECTION B

5 (a) $x > S$ and $y < T$. It is straightforward to verify that $\partial u/\partial x > 0$ and $\partial u/\partial y > 0$.

(b) Given c, the indifference curve associate with utility level c is the set of points (x, y) which satisfy

$$A\ln(x - S) - B\ln(T - y) \;=\; c$$

This curve is the graph of the function $y = f(x)$, where

$$y \;=\; T - e^{-c/B}(x - S)^{A/B}$$

Differentiating twice:

$$\frac{d^2y}{dx^2} \;=\; -e^{c/B}\frac{A(A-B)}{B^2}(x - S)^{A/B-2}$$

which is positive since $A < B$. Hence the IC indeed belongs so strictly convex preferences.

(c) The condition $Sp_x < m < Tp_y$ means the domain of the consumer's utility maximisation problem is

$$S < x \leq \frac{m}{p_x} \qquad\qquad 0 \leq y \leq \frac{m}{p_y} < T$$

Note that as x gets close to S, the consumer's utility will certainly go to $-\infty$, because the utility associated with y is bounded on the above domain, and the utility associated with x is given by $A\ln(x - S)$, where $A > 0$. Therefore we can assume that the consumer will definitely choose x greater than or equal to some $\bar{S} > S$, and hence the domain of the problem can be revised as

$$\bar{S} \leq x \leq \frac{m}{p_x} \qquad\qquad 0 \leq y \leq \frac{m}{p_y}$$

which is compact.

Hence a utility maximiser exists. Since u corresponds to strictly convex preferences, the maximiser is unique.

(d) FOC must be satisfied at the unique optimal choice since it is an interior solution:

$$\frac{MU_x}{p_x} = \frac{MU_y}{p_y} \quad\Longrightarrow\quad \frac{A}{p_x(x - S)} = \frac{B}{p_y(T - y)}$$

Combining this equation with the budget identity $xp_x + yp_y = m$, we obtain the optimal demand as

$$x^* = \frac{ATp_y + BSp_x - Am}{p_x(B - A)} \qquad \text{and} \qquad y^* = \frac{ATp_y + BSp_x - Bm}{p_y(B - A)}$$

To test for Giffen behaviour, we look at dx^*/dp_x and dy^*/dp_y:

$$\frac{dx^*}{p_x} = \frac{BS}{B-A} - \frac{A(Tp_y - m)}{(B-A)p_x^2} > 0 \qquad \frac{dy^*}{p_y} = \frac{AT}{A-B} - \frac{B(Sp_x - m)}{(A-B)p_y^2} < 0$$

which means x is a Giffen good.

6 Part (a) is worth half of the total mark for this question; part (b) is worth the other half.

(a) i. If y stands for the monthly interest rate, then $(1+y)^{12} = 1.127$. Using a calculator, we obtain $y \approx 0.01$, which means a monthly interest of 1%.

ii. Let D_t be the debt at the end of month t. So $D_0 = 10^5$, and $D_{60} = 0.8L$, where L is the loan. The debt, from one month to the next satisfies

$$D_t = (1+y)D_{t-1} - m \quad \text{which implies} \quad D_t = (1+y)^t D_0 - m\frac{1-(1+y)^t}{1-(1+y)}$$

and therefore m must satisfy

$$0.8 \times 10^5 = (1.01)^{60} \times 10^5 - m\frac{1-(1.01)^{60}}{1-(1.01)}$$

which implies $m \approx 1245$.

iii. Now, the debt, from one month to the next satisfies

$$D_t = (1.01)D_{t-1} - (1.005)^{t-1}m_0$$

(b) In order to obtain a formula for D_t, denote $a = 1.01$ and $b = 1.005$, and

$$\begin{aligned}
D_t &= aD_{t-1} - b^{t-1}m_0 \\
&= a(aD_{t-2} - b^{t-2}m_0) - b^{t-1}m_0 = a^2D_{t-2} - m_0(ab^{t-2} + b^{t-1}) \\
&= a^2(aD_{t-3} - b^{t-3}m_0) - b^{t-2}m_0 = a^3D_{t-3} - m_0(a^2b^{t-3} + ab^{t-2} + b^{t-1}) \\
&\vdots \\
D_t &= a^tD_0 - m_0(a^{t-1} + a^{t-2}b + \cdots + ab^{t-2} + b^{t-1}) \\
&= a^tD_0 - m_0a^{t-1}\left(\left(\frac{b}{a}\right)^{t-1} + \left(\frac{b}{a}\right)^{t-2} + \cdots + \left(\frac{b}{a}\right) + 1\right) \\
&= a^tD_0 - m_0a^{t-1}\frac{1-(b/a)^t}{1-b/a} \quad = \quad a^tD_0 - m_0\frac{a^t - b^t}{a-b}
\end{aligned}$$

Since $D_0 = L$ and $D_{60} = 0.8L$, and $L = 10^5$, the last equation yields

$$((1.01)^{60} - 0.8) \times 10^5 = m_0\frac{(1.01)^{60} - (1.005)^{60}}{0.005}$$

which implies $m_0 \approx 1087$.

ANSWERS TO SECTION C

7 (a) The marginals are

		X_2 = 0	X_2 = 1	
X_1	0	$^1/_{12}$	$^3/_{12}$	$^4/_{12}$
	1	$^2/_{12}$	$^6/_{12}$	$^8/_{12}$
		$^3/_{12}$	$^9/_{12}$	

They are independent as $P(X_1 = x_1, X_2 = x_2) = P(X_1 = x_1)\,P(X_2 = x_2)$ for $\{x_1, x_2\} \in \{0, 1\}$

(b) The following Table shows joint pmf with same marginals

		X_2 = 0	X_2 = 1	
X_1	0	p	$^4/_{12} - p$	$^4/_{12}$
	1	$^3/_{12} - p$	$^5/_{12} + p$	$^8/_{12}$
		$^3/_{12}$	$^9/_{12}$	

for any $0 \le p \le {}^3/_{12}$.

(c) This is the only restriction the marginals impose on the joint given that probabilities have to be greater than or equal to zero and less than or equal to one.

8 (a) Start from

$$u_n = (1 - u_{n-1}) \times \frac{1}{2} + u_{n-1} \times p$$

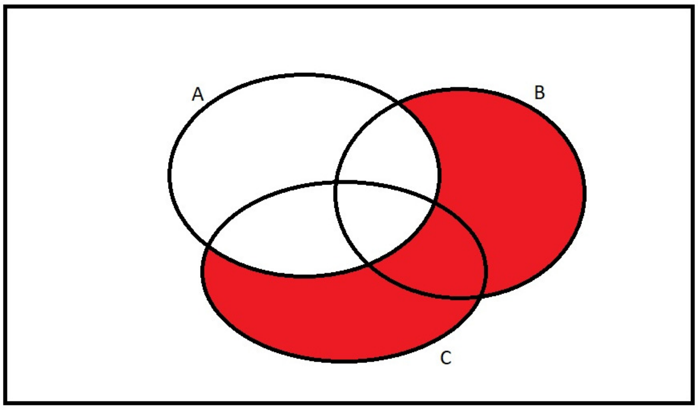
and rearrange. Now we want $u_1 = \frac{1}{2}$ so $\frac{1}{2} = (1 - u_0) \times \frac{1}{2} + u_0 \times p$ means we need $u_0 = 0$. To solve the difference equation try $u_n = Ac^n + B$ then

$$Ac^n + B + \left(\frac{1}{2} - p\right)\left(Ac^{n-1} + B\right) = \frac{1}{2}$$

then $C = -\left(\frac{1}{2} - p\right)$ and $B = \frac{1}{(3-2p)}$ and $A = -B$ and hence

$$u_n = \frac{1 + (-1)^{n-1}\left(\frac{1}{2} - p\right)^n}{3 - 2p}$$

(b) The Venn diagram is



where the shaded area is

$$P\left(A^c \cap (B \cup C)\right)$$

which can be built up as

$$P(B) + P(C) - P(B \cap C) - P(A \cap C) - P(A \cap B) + P(A \cap B \cap C)$$

Define events $A = \{\text{divisible by } 3\}$,$B = \{\text{divisible by } 5\}$,$C = \{\text{divisible by } 7\}$ and $\Omega =$ integers from 1 to 100. Then $P(C) = \frac{20}{100}$, $P(B) = \frac{14}{100}$,$P(B \cap C) = \frac{2}{100}$,$P(A \cap C) = \frac{4}{100}$,$P(A \cap B) = \frac{6}{100}$ and $P(A \cap B \cap C) = \frac{0}{100}$. Hence $P(A^c \cap (B \cup C)) = \frac{22}{100}$

9 Let the observations be $X_1, X_2, \ldots, X_n$. Since the sample size is sufficiently large we may assume $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. This hypothesis test is one tailed (5%) test where we would not reject H_0 if

$$\frac{\bar{X} - 30}{\sqrt{\frac{10000}{100}}} > -1.64$$

or $\bar{X} > 30 - 1.64 \times \sqrt{\frac{10000}{100}} = 13.6$. The power is $1 - Prob\,(Type\,II\,error)$ and

$$\begin{aligned} Prob\,(Type\,II\,error) &= Prob\left(\bar{X} > 13.6 \mid \mu_{\bar{X}} = 26, \sigma_{\bar{X}} = 10\right) \\ &= Prob\left(Z > \frac{13.6 - 26}{10}\right) \\ &= Prob\,(Z > -1.24) = 0.8925 \end{aligned}$$

so the power is 10.75%.

10 (a)

$$
\begin{aligned}
\frac{1}{n}\sum_{i}^{n}(y_i-\bar{y})^2 &= \frac{1}{n}\sum_{i}^{n}(\widehat{\alpha}+\widehat{\beta}x_i+e_i-(\widehat{\alpha}+\widehat{\beta}\bar{x}))^2 \\
&= \frac{1}{n}\sum_{i}^{n}(\widehat{\beta}(x_i-\bar{x})+e_i)^2 \\
&= \frac{1}{n}\sum_{i}^{n}(\widehat{\beta}(x_i-\bar{x}))^2+\frac{1}{n}\sum_{i}^{n}e_i^2+2\frac{1}{n}\sum_{i}^{n}(\widehat{\beta}(x_i-\bar{x})e_i) \\
&= \widehat{\beta}^2\frac{1}{n}\sum_{i}^{n}(x_i-\bar{x})^2+\frac{1}{n}\sum_{i}^{n}e_i^2
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{1}{n}\sum_{i}^{n}e_i^2 &= \frac{1}{n}\sum_{i}^{n}(y_i-\bar{y})^2-\widehat{\beta}^2\frac{1}{n}\sum_{i}^{n}(x_i-\bar{x})^2 \\
&= 50-2^2\times 10=10
\end{aligned}
$$

(b) If there regression equation has a constant then the normal equations will solve for $\widehat{\alpha}$ through condition $\sum_{i=1}^{n}(y_i-\widehat{y}_i)=0$

(c) The simple linear regression model is $y=\alpha+\beta x+\varepsilon$. Let $E[\varepsilon]=\delta$.

Write the new model as $y=\alpha'+\beta' x+\varepsilon'$, where $\alpha'=\alpha+\delta$, $\beta'=\beta$, $\varepsilon'=\varepsilon-\delta$.

$E[\varepsilon']=0$ and the slope parameter β is unchanged.

(d) Only omitting an important variable can cause bias when the omitted variable is correlated with the included explanatory variables.

The assumption of homoskedasticity, or constant conditional variances, plays no role in the proof that OLS estimators are unbiased.

The assumption of homoskedasticity is used to obtain the standard error formulas for the $\widehat{\beta}$ parameters.

A linear transformation of the X variable will result in a different estimate of the population parameter, but the population parameter will also change in value if we redefine X. The OLS estimate of the new population parameter will be unbiased.

ANSWERS TO SECTION D

11 (a) Definition

$$E(X) = \sum_i x_i P(X = x_i)$$

and

$$E(g(X)) = \sum_i g(x_i) P(X = x_i)$$

(b) Using the definition of exponential as a series

$$\begin{aligned} E\left(e^{tX}\right) &= E\left(\sum_{r=0}^{\infty} \frac{X^r t^r}{r!}\right) \\ &= \sum_{r=0}^{\infty} E\left(\frac{X^r t^r}{r!}\right) \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!} E X^r \end{aligned}$$

(c) We have

$$M(t) = E\left(e^{tX}\right) = 1 + tEX + \frac{t^2}{2!}EX^2 + \frac{t^3}{3!}EX^3 + \ldots$$

So

$$\begin{aligned} M(0) &= 1 \\ M'(t) &= EX + \frac{2t}{2!}EX^2 + \frac{3t^2}{3!}EX^3 + \ldots \end{aligned}$$

so $M'(0) = EX$ and

$$M'(t) = EX + tEX^2 + \frac{t^2}{2!}EX^3 + \frac{t^3}{3!}EX^4 + \ldots$$

so

$$M''(t) = EX^2 + \frac{2t}{2!}EX^3 + \frac{3t^2}{3!}EX^4 + \ldots$$

so $M''(0) = EX^2$ and so on.

(d) $M(t) = E\left(e^{tX}\right) = (1-p) \times e^{t.0} + p \times e^{t.1} = pe^t + (1-p)$
Hence $M'(t) = pe^t$ and $M''(t) = pe^t$ and $M'(0) = M''(0) = p$ (and hence $Var(X) = p - p^2 = p(1-p)$).

(e) From definition of Y

$$\begin{aligned} E\left(e^{tY}\right) &= E\left(e^{t(X_1+X_2+\ldots+X_n)}\right) \\ &= E\left(\prod_{j=1}^{n} e^{tX_j}\right) \\ &= \prod_{j=1}^{n} E\left(e^{tX_j}\right) \end{aligned}$$

where the second line follows from properties of exponential function and the third line from properties of expectations of independent variables.

(f) For the binomial Y we have

$$M_Y(t) = \prod_{j=1}^{n} \left(pe^t + (1-p)\right) = \left(pe^t + (1-p)\right)^n$$

hence for EY we have

$$M'(0) = \left[n\left(pe^t + (1-p)\right)^{n-1} pe^t\right]_{t=0} = np$$

and for EY^2

$$\begin{aligned} M''(0) &= \left[n(n-1)\left(pe^t + (1-p)\right)^{n-2}\left(pe^t\right)^2 + n\left(pe^t + (1-p)\right)^{n-1} pe^t\right]_{t=0} \\ &= n(n-1)p^2 + np \end{aligned}$$

Consequently $EY = np$ and $Var(Y) = EY^2 - (EY)^2 = n(n-1)p^2 + np - (np)^2 = -np^2 + np = np(1-p)$ as required

12 (a) The conditional variance Var$(W|E)$ is not constant. We observe that Var$(W|E)$ is increasing in W which makes sense given that the link between E and W is not deterministic. I can elaborate if needed.

(b) From lecture notes

$$\frac{\partial SSR}{\partial \widehat{\alpha}} = \sum_{i=1}^{n} \frac{\partial e_i^2}{\partial \widehat{\alpha}}$$

Breaking this down, first note that by the chain rule

$$\begin{aligned} \frac{\partial e_i^2}{\partial \widehat{\alpha}} &= \frac{\partial e_i^2}{\partial e_i}\frac{\partial e_i}{\partial \widehat{\alpha}} \\ &= 2e_i(-1) \\ &= -2e_i \end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial SSR}{\partial\widehat{\alpha}} &= \textstyle\sum_{i=1}^{n}(-2e_i) \\ &= -2\textstyle\sum_{i=1}^{n} e_i\end{aligned}$$

Set the derivative equal to zero to get the first order condition (FOC)

$$\begin{aligned}\frac{\partial SSR}{\partial\widehat{\alpha}} = 0 &= -2\sum_{i=1}^{n} e_i \\ 0 &= \sum_{i=1}^{n} e_i \\ 0 &= \sum_{i=1}^{n}(Y_i - \widehat{\alpha}) \\ 0 &= \sum_{i=1}^{n} Y_i - \sum_{i=1}^{n}\widehat{\alpha} \\ 0 &= \sum_{i=1}^{n} Y_i - n\widehat{\alpha} \\ n\widehat{\alpha} &= \sum_{i=1}^{n} Y_i \\ \widehat{\alpha} &= \frac{1}{n}\sum_{i=1}^{n} Y_i\end{aligned}$$

The value of α that minimizes deviations is the sample mean of Y

(c) Assuming Gauss Markov assumptions hold and nothing to do with dropping an outlier, less efficient

(d) Starting point

$$\widehat{\beta} = \frac{\sum_{i=1}^{n} X_i^* Y_i}{\sum_{i=1}^{n}(X_i^*)^2} \tag{2}$$

Use the assumption of a linear population regression function and substitute $Y_i = \alpha + \beta X_i + \varepsilon_i$, into (2)

$$\begin{aligned}\widehat{\beta} &= \frac{\sum X_i^*(\alpha + \beta X_i + \varepsilon_i)}{\sum(X_i^*)^2} \\ &= \frac{\sum \alpha X_i^* + \sum \beta X_i^* X_i + \sum X_i^*\varepsilon_i}{\sum(X_i^*)^2} \\ &= \frac{\sum \alpha X_i^*}{\sum(X_i^*)^2} + \frac{\sum \beta X_i^* X_i}{\sum(X_i^*)^2} + \frac{\sum X_i^*\varepsilon_i}{\sum(X_i^*)^2} \\ &= \alpha\frac{\sum X_i^*}{\sum(X_i^*)^2} + \beta\frac{\sum(X_i^*)^2}{\sum(X_i^*)^2} + \frac{\sum X_i^*\varepsilon_i}{\sum(X_i^*)^2}. \\ &= \alpha\frac{0}{\sum(X_i^*)^2} + \beta + \frac{\sum X_i^*\varepsilon_i}{\sum(X_i^*)^2}. \\ \widehat{\beta} &= \beta + \frac{\sum X_i^*\varepsilon_i}{\sum(X_i^*)^2} \qquad (3)\end{aligned}$$

Now take the expectation of $\widehat{\beta}$ in (3)

$$
\begin{aligned}
E(\widehat{\beta}) &= E\left[\beta + \frac{\sum X_i^* \varepsilon_i}{\sum (X_i^*)^2}\right] \\
&= E(\beta) + E\left[\frac{\sum X_i^* \varepsilon_i}{\sum (X_i^*)^2}\right]
\end{aligned}
$$

We use assumption non-stochastic X's, to deal with the denominator in the second term.

$$
E(\widehat{\beta}) = E\big(\beta) + \frac{E(\sum X_i^* \varepsilon_i\big)}{\sum (X_i^*)^2}
$$

Use assumption of non-stochastic X's to move X_i^* through the $E(\cdot)$ operator.

$$
\begin{aligned}
&= \beta + \frac{\sum E\big(X_i^* \varepsilon_i\big)}{\sum (X_i^*)^2} \\
&= \beta + \frac{\sum X_i^* E\big(\varepsilon_i\big)}{\sum (X_i^*)^2}
\end{aligned}
$$

Now we use assumption $E(\varepsilon_i) = 0$.

$$
\begin{aligned}
&= \beta + \frac{\sum X_i^* 0}{\sum (X_i^*)^2} \\
&= \beta
\end{aligned}
$$